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Singular fibers of very general Lagrangian fibrations

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Let $\pi : X \to \mathbb{P}^n$ be a (holomorphic) Lagrangian fibration that is very general in the moduli space of Lagrangian fibrations. We conjecture that the singular fibers in codimension one must be semistable degenerations of abelian varieties. We prove a partial result towards this conjecture, and describe an example that provides further evidence.

Keywords: Holomorphic symplectic; Lagrangian fibrations; singular fibers.

Mathematics Subject Classification 2020: 14D06, 14J42, 14K10, 53C26

1. Introduction

An irreducible holomorphic symplectic manifold X that admits a fibration $\pi : X \to \mathbb{P}^n$ by complex tori that are Lagrangian with respect to the holomorphic symplectic structure is known as a (holomorphic) Lagrangian fibration. In [30], the author proved that there are finitely many deformation classes of Lagrangian fibrations $\pi : X \to \mathbb{P}^n$ satisfying certain natural conditions. One of these conditions was that the singular fibers in codimension one should be semistable degenerations of abelian varieties. We will describe explicitly what this means shortly. At this stage, let us just point out that semistable degenerations are "nice" in the following sense: if $\bar{\mathcal{A}}_n$ denotes a toroidal compactification of the moduli space $\mathcal{A}_n$ of abelian varieties, then the boundary of $\bar{\mathcal{A}}_n$ parametrizes semistable degenerations.

The singular fibers of Lagrangian fibrations in codimension one were studied by Matsushita [14, 16] and Hwang and Oguiso [10, 11]. Their classifications include many fibers that are not semistable, because they consider all Lagrangian fibrations, not just very general ones. Matsushita [15, 18] proved that inside the moduli space $\mathcal{M}$ of deformations of X as a complex manifold, there is a hypersurface $\mathcal{H}$ parametrizing Lagrangian fibrations; by *very general* we mean that X corresponds to a very general point of $\mathcal{H}$, i.e. contained in the complement of countably many

Zariski closed subsets. Any Lagrangian fibration can be deformed slightly so that it becomes very general.

Hwang and Oguiso's results imply that if a singular fiber X_t *in codimension one* is semistable then it must be a rank-one semistable degeneration. Explicitly, this means that the normalization of X_t must be a $\mathbb{P}^1$-bundle, or disjoint union of $\mathbb{P}^1$-bundles, over an $(n-1)$-dimensional abelian variety A. Write this as

$$\tilde{X}_t = \coprod_{i=1}^{k} Y_i.$$

Moreover, each Y_i must be a $\mathbb{P}^1$-bundle over A given by projectivizing a topologically trivial rank-two bundle,

$$Y_i \cong \mathbb{P}(\mathcal{O}_A \oplus \mathcal{L}) \to A,$$

where $\mathcal{L} \in \mathrm{Pic}^0 A$. This implies that Y_i has distinguished zero and infinity sections, $(Y_i)_0$ and $(Y_i)_\infty$. The fiber X_t itself is obtained by gluing zero sections to infinity sections, so that

$$(Y_i)_\infty \cong A, \quad \text{is identified with } (Y_{i+1})_0 \cong A$$

(with Y_{k+1} denoting Y_1). In general these identifications $A \cong (Y_i)_\infty \to (Y_{i+1})_0 \cong A$ do *not* take 0 to 0, so X_t itself is *not* a fibration over A. Figures 1 and 2 show examples with $k = 1$ and $k = 3$, respectively.

We expect the following behavior.

Conjecture 1. *Let $\pi : X \to \mathbb{P}^n$ be a very general Lagrangian fibration, and let t be a general point of the hypersurface $\Delta \subset \mathbb{P}^n$ parametrizing singular fibers. Then X_t is a semistable degeneration of abelian varieties.*

For $K3$ surfaces, the conjecture states that a very general elliptic $K3$ surface has only singular fibers of type I_k in Kodaira's classification. This is certainly true; indeed, a very general elliptic $K3$ surface will have only I_1 singular fibers (rational nodal curves), as can be shown by explicitly finding such a surface and showing

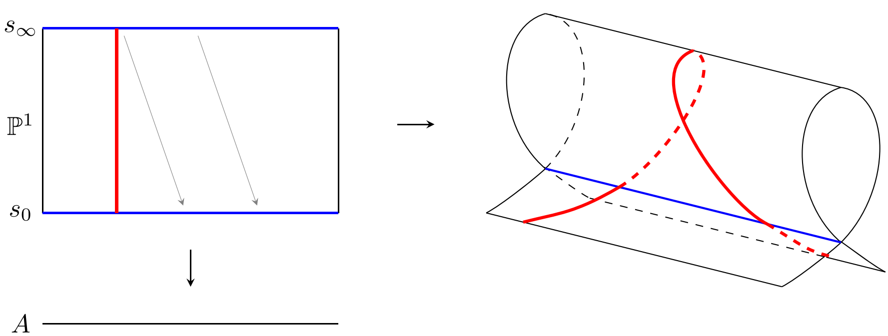

Fig. 1. Rank-one semistable degeneration.

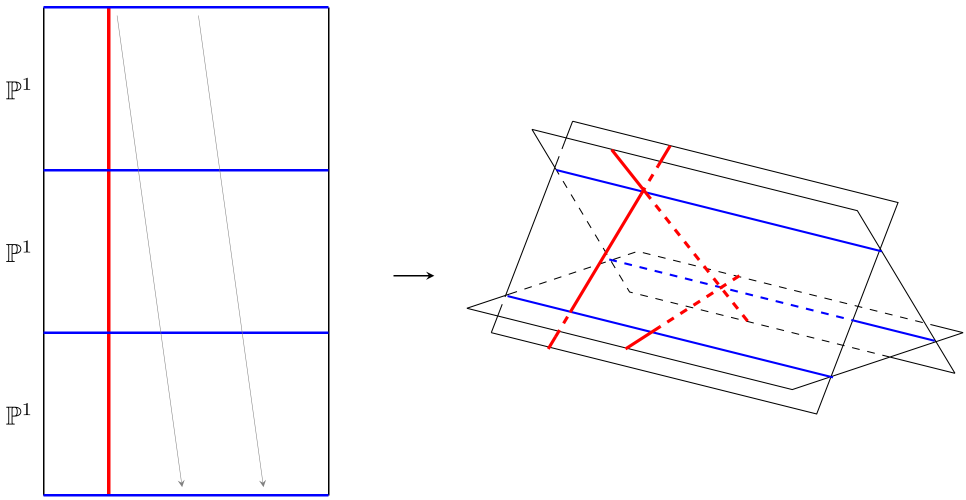

Fig. 2. Reducible semistable degeneration.

that it deforms in a 19-dimensional family. The challenge is to find a proof of this fact that can be extended to higher-dimensional Lagrangian fibrations. Although we do not achieve this, we do find a simple proof of a weaker property of elliptic K3 surfaces that generalizes to give the following theorem.

Theorem 2. *Let $\pi : X \to \mathbb{P}^n$ be a very general Lagrangian fibration, and let t be a general point of the hypersurface $\Delta \subset \mathbb{P}^n$ parametrizing singular fibers. Then X_t is either*

(1) *an étale quotient of an n-dimensional abelian variety* (*such a fiber is singular because it will have multiplicity greater than one*),
(2) *a semistable degeneration of abelian varieties, or*
(3) *a fibration over an $(n-1)$-dimensional abelian variety A by singular elliptic curves of Kodaira type II* (*a cuspidal rational curve*), *III* (*two rational curves meeting at a tacnode*), *or IV* (*three rational curves meeting at a point*), *up to an étale cover.*

Of course, we believe that only the second case is possible. Theorem 2 has been obtained independently by Lehn [12, Theorem 5.7] as a corollary of his generalization of Voisin's results on deformations of Lagrangian submanifolds to deformations of Lagrangian normal crossing subvarieties. Our proof is more direct: a standard argument shows that a very general Lagrangian fibration must have Picard number $\rho(X) = 1$, and this places restrictions on the structure of the discriminant hypersurface $\pi^{-1}(\Delta)$ from which the theorem follows.

Note that it is *not* true that the singular fibers in codimension two (or higher) must be semistable. For example, let $\pi : X \to \mathbb{P}^n$ be a Beauville–Mukai system [4, 19], i.e. the compactified relative Jacobian of a complete linear system of curves on a

$K3$ surface. There will be cuspidal curves in codimension two in this linear system, and their compactified Jacobians will not be semistable; see Sawon [28] for details. Moreover, we can make this Lagrangian fibration very general by deforming it in a 20-dimensional family, inside the 21-dimensional moduli space of deformations of X, while preserving the existence of these non-semistable singular fibers in codimension two.

In Sec. 2, we recall some facts about singular fibers of elliptic $K3$ surfaces. In Sec. 3, we extend these results to higher-dimensional Lagrangian fibrations, thereby proving our main result, Theorem 2. Section 4 contains a number of illustrative examples. Using one of these examples, we show that Conjecture 1 is true when X is of $K3^{[n]}$-type (Proposition 9).

2. K3 Surfaces

We begin by recalling Kodaira's classification of singular fibers of elliptic surfaces.

Theorem 3 (see Barth *et al.* [2]). *A non-multiple singular fiber of a minimal elliptic surface must be one of the following types*:

- I_k, $k \geq 1$, *a cycle of* k *rational curves with intersection matrix given by the Cartan matrix of the affine Dynkin diagram of type* $\tilde{A}_{k-1}$ (*with* I_1 *a nodal rational curve*),
- II, *a cuspidal rational curve*,
- III, *two rational curves meeting at a tacnode*,
- IV, *three rational curves meeting at a single point*,
- I_k^*, $k \geq 0$, $k+5$ *rational curves with intersection matrix given by* $\tilde{D}_{k+4}$,
- II^*, *nine rational curves with intersection matrix given by* $\tilde{E}_8$,
- III^*, *eight rational curves with intersection matrix given by* $\tilde{E}_7$,
- IV^*, *seven rational curves with intersection matrix given by* $\tilde{E}_6$.

A multiple singular fiber must be of type mI_0, *a smooth elliptic curve with multiplicity* $m \geq 2$, *or of type* mI_k *with* $k \geq 1$, *i.e. type* I_k *with multiplicity* $m \geq 2$.

Let $S \to \mathbb{P}^1$ be an elliptic $K3$ surface, with fiber class $F \in \mathrm{H}^{1,1}(S) \cap \mathrm{H}^2(S,\mathbb{Z})$. If we deform S so that F remains algebraic, i.e. of type $(1,1)$, then the resulting $K3$ surface will still be elliptic. This means that elliptic $K3$ surfaces are codimension one inside the 20-dimensional moduli space of all $K3$ surfaces. Moreover, a very general elliptic $K3$ surface will have Picard number $\rho = 1$; note that it is non-projective and does not admit a section (or even a multi-valued rational section).

Lemma 4. *Let* $S \to \mathbb{P}^1$ *be a very general elliptic K3 surface. Then every singular fiber of* S *is reduced and irreducible. Thus by Kodaira's classification, every singular fiber is either of type* I_1 (*nodal rational curve*) *or type* II (*cuspidal rational curve*).

Proof. The Néron–Severi group of a non-projective elliptic surface is spanned by the fiber class F and the irreducible components of the singular fibers. Suppose there are k singular fibers $S_1, \ldots, S_k$ with $m_1, \ldots, m_k$ irreducible components, respectively. Obviously some linear combination of the components of S_i is linearly equivalent to F, so each S_i really contributes $m_i - 1$ additional independent classes to the Néron–Severi group. Thus the rank of the Néron–Severi group is

$$\rho(S) = 1 + \sum_{i=1}^{k} (m_i - 1).$$

Recall that the Shioda–Tate formula [31] for an elliptic surface with a section is

$$\rho(S) = 2 + \mathrm{rank} MW(\pi) + \sum_{i=1}^{k} (m_i - 1),$$

where $\mathrm{MW}(\pi)$ is the Mordell–Weil group of sections of $\pi : S \to \mathbb{P}^1$. Our formula above is essentially a degenerate version of the Shioda–Tate formula for non-projective surfaces. Since a very general elliptic $K3$ surface has $\rho(S) = 1$, we immediately see that $m_i = 1$ for all i, i.e. every singular fiber of S is irreducible.

Multiple fibers contribute nontrivially to Kodaira's formula for the canonical bundle of an elliptic surface, but $K3$ surfaces have trivial canonical bundles, and thus they cannot have any multiple fibers. Thus, every singular fiber of S is reduced. □

Question. Although it does not follow from the above arguments, very general elliptic $K3$ surfaces actually have only nodal rational curves as singular fibers. This can be proved by explicitly constructing such a $K3$ surface and then showing that there are 19-parameters describing its deformations. Is there a deformation argument that eliminates cuspidal rational curves? Given an elliptic fibration over a disc with a cuspidal rational curve over 0, we can deform it to a fibration with two nodal rational curves. However, we need a global argument to show that such a deformation fits in to an elliptic $K3$ surface.

3. Higher Dimensions

We want to extend the ideas of the previous section to higher-dimensional Lagrangian fibrations, i.e. fibrations $\pi : X \to \mathbb{P}^n$ where X is an irreducible holomorphic symplectic manifold and the general fiber is an n-dimensional abelian variety (see Sawon [23] and the papers cited therein). We start by describing Hwang and Oguiso's [10, 11] Kodaira-type classification of general singular fibers. By [10, Proposition 3.1(2)] the discriminant locus $\Delta \subset \mathbb{P}^n$ parametrizing singular fibers is a hypersurface. Roughly speaking, the main observation of Hwang and Oguiso is that for a general singular fiber X_t, above a general point $t \in \Delta$, there is a residual $(n-1)$-dimensional abelian variety present and the fiber is degenerating in the one remaining dimension. To state this precisely, let V be a component of the reduction $(X_t)_{\mathrm{red}}$ of X_t, and let $\hat{V}$ be its normalization. Then the Albanese map $\hat{V} \to \mathrm{Alb}(\hat{V})$ is a fiber bundle with fiber either $\mathbb{P}^1$ or an elliptic curve. The image of such a fiber in $(X_t)_{\mathrm{red}}$ is called a *characteristic leaf*; if two (or more) leaves meet they form a *characteristic curve*, and once we add in multiplicities coming from the multiplicities of the components of X_t we obtain a *characteristic* 1-*cycle*.

Hwang and Oguiso gave a classification of these characteristic 1-cycles.

Theorem 5 ([10, 11]). *Let X_t be a general singular fiber with multiplicity m. Then all characteristic* 1-*cycles $\sum_i m_i C_i$ in X_t are isomorphic, and the* 1-*cycle $\sum_i \frac{m_i}{m} C_i$ is either*

(1) *a smooth elliptic curve,*
(2) *one of the singular elliptic fibers of Kodaira's classification, as in Theorem 3, or*
(3) *of type A_∞, i.e. a* 1-*cycle $\sum_{i\in\mathbb{Z}} C_i$ consisting of a chain of infinitely many rational curves C_i, with C_i meeting C_j if and only if $j = i \pm 1$.*

A singular fiber that is a product of an $(n-1)$-dimensional abelian variety and a singular elliptic curve from Kodaira's classification will have characteristic 1-cycle of the same type as the singular elliptic curve. For example, the product of an $(n-1)$-dimensional abelian variety and an elliptic curve of type I_2 will have characteristic 1-cycle of type I_2; such a singular fiber will have two irreducible components. But singular fibers need not be products, and so it is possible for an irreducible singular fiber to have a reducible characteristic 1-cycle, as illustrated in Fig. 3.

Remark. Unlike elliptic $K3$ surfaces, the singular fibers of Lagrangian fibrations can have multiplicities. Hwang and Oguiso [11] showed that various values up to and including six are possible, depending on the type of the (characteristic 1-cycle of the) singular fiber. In particular, it is possible for the (reduced) characteristic 1-cycle of a *singular* fiber to be a *smooth* elliptic curve if the fiber has multiplicity $m > 1$. Some examples will be given in Sec. 4.

Remark. Of course, a singular fiber cannot have infinitely many components. Thus a characteristic 1-cycle of type A_∞ must "wrap around" the singular fiber. An example of a reduced and irreducible singular fibre with characteristic 1-cycle of type A_∞ will be given in Sec. 4.2.

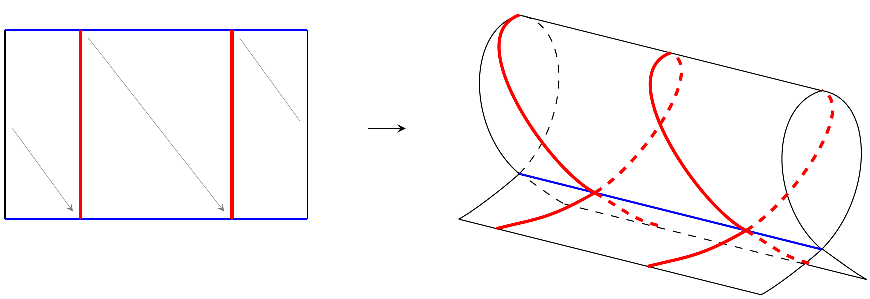

Fig. 3. Irreducible singular fiber with characteristic 1-cycle of type I_2.

Let $\pi : X \to \mathbb{P}^n$ be a Lagrangian fibration with $n \geq 2$. Let $F \in \mathrm{H}^{n,n}(X) \cap \mathrm{H}^{2n}(X, \mathbb{Z})$ be the fiber class and let $L \in \mathrm{H}^{1,1}(X) \cap \mathrm{H}^2(X, \mathbb{Z})$ be the pullback of a hyperplane in $\mathbb{P}^n$. Inside the moduli space $\mathcal{M}$ of deformations of X as a complex manifold, there is a hypersurface $\mathcal{H}_F$ parametrizing deformations such that F remains algebraic (see Voisin [33]) and there is also a hypersurface $\mathcal{H}_L$ parametrizing deformations such that L remains algebraic. In fact, these hypersurfaces are identical and Matsushita [15, 18] proved that they parametrize deformations of X that remain Lagrangian fibrations. In a family of deformations of X, the Picard number is upper semicontinuous; Oguiso [20] proved that the Picard number jumps up on a dense countable subset. More importantly for us, a family parametrized by $\mathcal{M}' \subset \mathcal{M}$ of codimension k cannot contain only deformations of X with Picard numbers $> k$. In particular, a family parametrized by a hypersurface in $\mathcal{M}$ must contain deformations with Picard numbers 1 (or 0). It follows that a very general Lagrangian fibration will have Picard number $\rho(X) = 1$, the Néron–Severi group will be generated by the non-ample divisor L, and X will be non-projective. In addition, it will not admit a (multi-valued rational) section; otherwise it would be projective by a criterion of Campana (see [21, Proposition 3.2]).

Our main result, Theorem 2, follows directly from the following proposition: the three cases for the characteristic 1-cycle correspond directly to the three cases for the structure of the singular fiber in Theorem 2.

Proposition 6. *Let $\pi : X \to \mathbb{P}^n$ be a very general Lagrangian fibration, and let t be a general point of the discriminant locus $\Delta \subset \mathbb{P}^n$. Then the (reduced) characteristic 1-cycle of the singular fiber X_t is either*

(1) *a smooth elliptic curve,*
(2) *a singular elliptic curve of Kodaira type I_k with $k \geq 1$, or of type A_∞, or*
(3) *a singular elliptic curve of Kodaira type II, III, or IV.*

Remark. In the first case, X_t must have multiplicity $m = 2, 3, 4$, or 6 (not 1 or it would be a smooth fiber). In the second case, we must have $m = 1$ for type I_k with k odd, and $m = 1$ or 2 for type I_k with k even and type A_∞. In the third case, we must have $m = 1$ or 5 for type II, $m = 1$ or 3 for type III, and $m = 1$ or 2 for type IV (see Hwang and Oguiso [11]).

Before presenting the proof of Proposition 6, we state a generalization of the Shioda–Tate formula to higher-dimensional fibrations by abelian varieties.

Theorem 7 ([21, Theorem 1.1]). *Let $\varphi : X \to Y$ be a proper surjective morphism with rational section O, whose generic fiber $A := X_\eta$ is an abelian variety defined over the field $K = \mathbb{C}(Y)$ with origin O. Assume further that X and Y have only $\mathbb{Q}$-factorial rational singularities, φ is equi-dimensional in codimension one, and $h^1(X, \mathcal{O}_X) = h^1(Y, \mathcal{O}_Y)$. Write $\Delta = \cup_{i=1}^k \Delta_i$ for the decomposition into irreducible components of the discriminant divisor $\Delta \subset Y$, and assume that $\varphi^{-1}(\Delta_i) \subset X$ consists of m_i irreducible components. Then the Mordell–Weil group*

$MW(\varphi)$, i.e. the group $A(K)$ of K-rational points of A (equivalently, the group of rational sections of φ), is a finitely generated abelian group of rank

$$\mathrm{rank} MW(\varphi) = \rho(X) - \rho(Y) - \mathrm{rank} NS(A_K) - \sum_{i=1}^{k}(m_i - 1).$$

Proof of Proposition 6. We apply Oguiso's theorem to a Lagrangian fibration $\pi : X \to \mathbb{P}^n$. In this case, X and $Y = \mathbb{P}^n$ are both smooth, Lagrangian fibrations are equi-dimensional, and $h^1(X, \mathcal{O}_X) = h^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = 0$. The theorem assumes the existence of a (rational) section. In [26], the author proved that there is a hypersurface in $\mathcal{H}_F = \mathcal{H}_L$ parametrizing Lagrangian fibrations that admit sections; in other words, Lagrangian fibrations that admit sections are codimension two in $\mathcal{M}$. A very general Lagrangian fibration that admits a section will therefore have Picard number $\rho(X) = 2$. In addition, the existence of a section implies that X is projective and hence the Néron–Severi group $NS(A_K)$ must have rank at least one. The Shioda–Tate formula,

$$\mathrm{rank} MW(\pi) = 2 - 1 - \mathrm{rank} NS(A_K) - \sum_{i=1}^{k}(m_i - 1),$$

then forces $\mathrm{rank} MW(\pi) = 0$, $\mathrm{rank} NS(A_K) = 1$, and $m_i = 1$ for all i. If we consider instead a Lagrangian fibration $\pi : X \to \mathbb{P}^n$ that does not admit any (multi-valued rational) section, then there is an obvious analogue of the formula in which $\mathrm{rank} MW(\pi)$ and $\mathrm{rank} NS(A_K)$ both vanish and

$$\rho(X) = \rho(\mathbb{P}^n) + \sum_{i=1}^{k}(m_i - 1).$$

For a very general Lagrangian fibration $\rho(X) = 1$ and again we find that $m_i = 1$ for all i. Summarizing, we have proved that if the discriminant divisor $\Delta \subset \mathbb{P}^n$ of a very general Lagrangian fibration $\pi : X \to \mathbb{P}^n$ decomposes into irreducible components as $\Delta = \bigcup_{i=1}^{k} \Delta_i$, then $\pi^{-1}(\Delta_i)$ is irreducible for all i.

Next, let t be a general point of $\Delta_i \subset \Delta$. The statement above does *not* imply that the fiber X_t is irreducible. It is possible that the fibre X_t has several irreducible components $Y_1, \ldots, Y_l$ that are all contained in the single irreducible divisor $\pi^{-1}(\Delta_i)$, because of "monodromy" permuting the components as we move around a loop in Δ_i starting and ending at t, avoiding the codimension one subset $\Delta_{i0} \subset \Delta_i$ parametrizing non-general singular fibers. However, we see that this monodromy representation

$$\pi_1(\Delta_i \backslash \Delta_{i0}, t) \to \mathrm{Sym}_l,$$

must act transitively on the set of components of X_t. This implies that the components Y_j must all have the same multiplicity. Referring to Theorem 5, we can conclude that the (reduced) characteristic 1-cycle of X_t must belong to one of the three cases of the proposition. The other possibilities in Hwang and Oguiso's classification are singular elliptic curves of Kodaira type I_m^* with $m \geq 0$, II^*, III^*, and IV^*, but these all have components with different multiplicities, which is not allowed. $\square$

4. Examples

In this section, we describe the singular fibers in codimension one for several examples of Lagrangian fibrations.

4.1. *Multiple fibers with smooth reduction*

In [11, Example 6.2], Hwang and Oguiso described several examples where the singular fiber has smooth reduction, which is equivalent to the (reduced) characteristic 1-cycle being a smooth elliptic curve. Their examples are local Lagrangian fibrations, but the same constructions easily extend to give global compact Lagrangian fibrations. For instance, let us extend Example 6.2(v) which has singular fibers of multiplicity 2 and characteristic 1-cycle an arbitrary smooth elliptic curve E_1 with equation $y^2 = x^3 + ax + b$. Let E_2, E_3, and E_4 be arbitrary smooth elliptic curves with coordinates t, z, and s, respectively, and let p_2 be a 2-torsion point on E_3. Then define X to be the quotient of $E_1 \times E_2 \times E_3 \times E_4$ by the fixed-point-free involution

$$g^* : ((x, y), t, z, s) \mapsto ((x, -y), -t, z + p_2, s).$$

The symplectic form $\sigma := \frac{dx \wedge dt}{y} + dz \wedge ds$ on $E_1 \times E_2 \times E_3 \times E_4$ is preserved by g^* and therefore descends to X. The projection

$$\begin{aligned} X &\to (E_2/\pm 1) \times E_4 \cong \mathbb{P}^1 \times E_4, \\ [((x, y), t, z, s)] &\mapsto (\pm t, s), \end{aligned}$$

makes X into a Lagrangian fibration. The singular fibers sit above $q_2 \times E_4$, where q_2 is a 2-torsion point in E_2 (fixed by ± 1), and they look like the hyperelliptic surface

$$E_1 \times E_3/((x, y), z) \sim ((x, -y), z + p_2),$$

with multiplicity 2.

This example is an isotrivial Lagrangian fibration: all the smooth fibers are isomorphic to $E_1 \times E_3$. We can construct a non-isotrivial example by replacing $E_1 \times E_2$ by a certain elliptic $K3$ surface S. Specifically, choose for S an elliptic $K3$ surface admitting a symplectic involution τ which acts as ± 1 on each fiber and as ± 1 on the base $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. The only fibers of $S \to \mathbb{P}^1$ that are fixed by τ are those above 0 and ∞; we assume these are smooth. We can now modify the example above by defining X to be the quotient of $S \times E_3 \times E_4$ by the fixed-point-free involution

$$g^* : (q, z, s) \mapsto (\tau(q), z + p_2, s).$$

The symplectic form $\sigma := \sigma_S + dz \wedge ds$ descends to X and the projection to $\mathbb{P}^1/\pm 1 \times E_4$ makes X into a Lagrangian fibration. The singular fibers sit above $\{0\} \times E_4$ and $\{\infty\} \times E_4$ and look like the hyperelliptic surfaces

$$S_0 \times E_3/(w_0, z) \sim (-w_0, z + p_2) \quad \text{and} \quad S_\infty \times E_3/(w_\infty, z) \sim (-w_\infty, z + p_2),$$

with multiplicity 2, where S_0 and S_∞ are the elliptic fibers of $S \to \mathbb{P}^1$ above 0 and ∞, respectively, with coordinates w_0 and w_∞.

Note that a theorem of Hwang [9] asserts that the base of a Lagrangian fibration of an irreducible holomorphic symplectic manifold must be isomorphic to projective space, if it is smooth. However, in the examples above X is not an *irreducible* holomorphic symplectic manifold; indeed, it admits an étale cover that is a product, $E_1 \times E_2 \times E_3 \times E_4$ and $S \times E_3 \times E_4$, respectively. Consequently, bases that are not isomorphic to $\mathbb{P}^2$ can arise.

4.2. *Beauville–Mukai systems*

Recall the construction of the Beauville–Mukai integrable system [4, 19]. Let S be a $K3$ surface containing a smooth genus n curve C. Then C moves in an n-dimensional linear system, $|C| \cong \mathbb{P}^n$; denote by $\mathcal{C} \to \mathbb{P}^n$ the corresponding family of curves. If S is very general in the sense that its Néron–Severi group $NS(S)$ is generated over $\mathbb{Z}$ by $[C]$ then every curve in the family $\mathcal{C}$ is reduced and irreducible. This means that we can apply the Altman and Kleiman construction [1] of the compactified relative Jacobian to obtain $X := \overline{\mathrm{Jac}}^d(\mathcal{C}/\mathbb{P}^n)$. We can also regard X as a Mukai moduli space of stable sheaves on S with Mukai vector

$$v = (0, [C], d + 1 - n).$$

Here, we think of an element of X as a torsion sheaf $\iota_* L$, where $\iota : C \hookrightarrow S$ is the embedding of a curve into the K3 surface and L is a degree d line bundle (or more generally, a rank one torsion-free sheaf) on C; then $\iota_* L$ is stable in the sense of Simpson [32]. This latter point of view shows that X admits a holomorphic symplectic structure, and $X \to \mathbb{P}^n$ is therefore a Lagrangian fibration. Moreover, the definition of X as a Mukai moduli space makes sense even when $NS(S) \ncong \mathbb{Z}.[C]$, and provided the Mukai vector v is primitive and a general polarization of S is chosen, we will once again obtain a (smooth, compact) Lagrangian fibration $X \to \mathbb{P}^n$.

For a very general (polarized) $K3$ surface, a general codimension one singular curve in the family $\mathcal{C}$ will have a single node. This can be proved by studying the Beauville–Mukai system and using properties of Lagrangian fibrations (see Sawon [28, Lemma 2.4]); a special case of Chen [5, Lemma 3.1]. Let C be such a curve, with normalization $\tilde{C}$ of genus $n - 1$, and C obtained by identifying two points p

and $q \in \tilde{C}$. The normalization of the compactified Jacobian $\overline{\mathrm{Jac}}^d C$ of C is a $\mathbb{P}^1$-bundle over the Jacobian $\mathrm{Jac}^d \tilde{C}$ of $\tilde{C}$, and the compactified Jacobian $\overline{\mathrm{Jac}}^d C$ itself is obtained by identifying the 0- and ∞-sections of this $\mathbb{P}^1$-bundle via a translation

$$0\text{-section} \cong \mathrm{Jac}^d \tilde{C} \overset{\otimes \mathcal{O}(p-q)}{\longrightarrow} \mathrm{Jac}^d \tilde{C} \cong \infty\text{-section}.$$

Since the characteristic leaves are the images of the $\mathbb{P}^1$-fibers, the type of the characteristic 1-cycle therefore depends on whether or not $\mathcal{O}(p-q)$ is a torsion line bundle in $\mathrm{Jac}^0 \tilde{C}$.

Lemma 8. *The characteristic* 1*-cycle of a general singular fiber of a Beauville–Mukai system constructed from a very general K3 surface is of type* A_∞*.*

Remark. A local example of a Lagrangian fibration with general fibers of type A_∞ was described by Matsushita (see [10, Proposition 4.13]).

Proof. Consider first the genus $n = 2$ case, studied in detail in Sawon [25]. The discriminant locus $\Delta \subset \mathbb{P}^2$ is an irreducible curve of degree 30 with 72 cusps and 324 nodes. We can write it as a disjoint union

$$\Delta = (\Delta \backslash \Delta_0) \cup \Delta_0 = (\Delta \backslash \Delta_0) \cup \Delta_c \cup \Delta_{nn},$$

where $\Delta \backslash \Delta_0$ parametrizes curves with one node, Δ_c parametrizes curves with one cusp, and Δ_{nn} parametrizes curves with two nodes. Note that Δ_c is the set of cusps of Δ and Δ_{nn} is the set of nodes of Δ. Consider the normalization $\tilde{\mathcal{C}} \to (\Delta \backslash \Delta_0)$ of the family of nodal curves $\mathcal{C} \to (\Delta \backslash \Delta_0)$. Because the cuspidal curves over Δ_c also have normalizations of genus one, this family extends to a family $\tilde{\mathcal{C}} \to (\Delta \backslash \Delta_0) \cup \Delta_c$ of smooth genus one curves. The family of line bundles $\mathcal{O}(p-q)$ on the curves $\tilde{C}$ then defines a section Z of the relative Jacobian

$$\mathrm{Jac}^0(\tilde{\mathcal{C}}/(\Delta \backslash \Delta_0)),$$

over $\Delta \backslash \Delta_0$. As we approach a cuspidal curve, $p \to q$, so this section extends to a section, which we still denote by Z, of the relative Jacobian

$$\mathrm{Jac}^0(\tilde{\mathcal{C}}/(\Delta \backslash \Delta_0) \cup \Delta_c).$$

Moreover, the section Z intersects the zero section Z_1, i.e. takes the value $\mathcal{O}$, precisely above Δ_c, because $\mathcal{O}(p-q) = \mathcal{O}(p-p) = \mathcal{O}$ only for a cuspidal curve. In particular, the section Z is not equal to the zero section Z_1.

Fix $k \geq 2$ and consider the multi-section Z_k of $\mathrm{Jac}^0(\tilde{\mathcal{C}}/(\Delta \backslash \Delta_0) \cup \Delta_c)$ parametrizing line bundles on the curves $\tilde{C}$ of order precisely k. The section Z cannot be contained in Z_k, because Z intersects the zero section Z_1 whereas Z_k and Z_1 are disjoint. Therefore Z and Z_k intersect transversally at finitely many points, possibly zero. This means that $\mathcal{O}(p-q)$ is k-torsion for only finitely many curves $\tilde{C}$ in the family $\tilde{\mathcal{C}} \to (\Delta \backslash \Delta_0) \cup \Delta_c$. Taking the union over all k, we see that $\mathcal{O}(p-q)$ is torsion for at most countably many curves $\tilde{C}$. It follows that for a curve C corresponding to a general point in $\Delta\backslash\Delta_0$, with a single node and normalization $\tilde{C}$, $\mathcal{O}(p-q)$ will not be torsion in $\mathrm{Jac}^0\tilde{C}$, and therefore the compactified Jacobian $\overline{\mathrm{Jac}}^d C$ will be of type A_∞, i.e. the characteristic 1-cycle will consist of an infinite chain of rational curves.

The same argument works for higher genus curves. For a very general $K3$ surface S, Galati and Knutsen proved that in the n-dimensional linear system $|C| \cong \mathbb{P}^n$ there will exist a curve with a cusp and $n-2$ nodes [7, Theorem 1.1]. Moreover, they showed that the $n-2$ nodes may be smoothed independently, producing an $(n-2)$-dimensional family of curves with single cusps. In other words, $\Delta\backslash\Delta_0$ parametrizes curves with one node, and there again exists a non-empty codimension one subset $\Delta_c \subset \Delta$ (i.e. codimension two in $|C|$) parametrizing cuspidal curves. The proof for $n > 2$ then proceeds as above. $\square$

Remark. The union $\bigcup_{k\geq 1} Z_k$ of all the multi-sections Z_k is dense in $\mathrm{Jac}^0(\tilde{\mathcal{C}}/(\Delta\backslash\Delta_0)\cup\Delta_c)$. It follows that the section Z in the above proof will intersect infinitely many of the Z_k. At these points of intersection $\mathcal{O}(p-q)$ will be k-torsion and therefore the compactified Jacobian $\overline{\mathrm{Jac}}^d C$ will be of type I_k, i.e. the characteristic 1-cycle will consist of k rational curves forming a cycle. Thus the Beauville–Mukai system contains singular fibers corresponding to characteristic 1-cycles of type I_k for infinitely many $k \geq 2$, along with type A_∞ for the general singular fiber. However, for a given $k \geq 2$ we cannot guarantee that type I_k will occur; part of the problem is that $(\Delta\backslash\Delta_0)\cup\Delta_c$ is only quasi-projective, so an anticipated intersection of Z and Z_k might only occur over the boundary. In addition, there will be no singular fibers corresponding to characteristic 1-cycles of type I_1, as these would require $p = q$, which as we saw only occurs for cuspidal curves over Δ_c.

Using this example, we can show that Conjecture 1 is true when X is of $\mathrm{K3}^{[n]}$-type.

Proposition 9. *Let $\pi : X \to \mathbb{P}^n$ be a very general Lagrangian fibration such that X is deformation equivalent to $S^{[n]}$ for a K3 surface S. Then the general singular fiber X_t is a semistable degeneration of abelian varieties.*

Proof. According to Markman [13, Theorem 1.5], the Lagrangian fibration $\pi : X \to \mathbb{P}^n$ must be bimeromorphic to a Tate–Shafarevich twist of a Beauville–Mukai system. In fact, we can strengthen this: because $\pi : X \to \mathbb{P}^n$ is a very general Lagrangian fibration, we can assume it has Picard number one. This means that it won't have any non-isomorphic birational models, and we conclude that it is isomorphic to a Tate–Shafarevich twist of a Beauville–Mukai system. We can also assume that the Beauville–Mukai system is constructed from a very general $K3$ surface. A Tate-Shafarevich twist is a kind of compactified torsor: it is locally isomorphic as a fibration, and a torsor over the smooth fibers. In particular, $\pi : X \to \mathbb{P}^n$ will have the same general singular fibers as the Beauville–Mukai system, which as we have seen above are semistable degenerations of abelian varieties. $\square$

Remark. One expects similar results for the other known irreducible holomorphic symplectic manifolds, the generalized Kummer varieties and O'Grady's examples. However, as far as the author knows, the analogue of Markman's theorem has not been established in these other cases. For the deformation type of generalized Kummer varieties, one expects a very general Lagrangian fibration to be a Tate–Shafarevich twist of a Debarre integrable system [6] or some variation of it (see Wieneck [34]). The general singular fibers of these fibrations are semistable, but determining their precise structure is something that the author will return to in future work.

4.3. *Hilbert schemes of elliptic $K3$ surfaces*

Next we consider an elliptic $K3$ surface $S \to \mathbb{P}^1$. Assume $S \to \mathbb{P}^1$ admits a section, but is otherwise very general in the sense that it contains exactly 24 nodal rational curves as singular fibers, above the points $p_1, \dots, p_{24} \in \mathbb{P}^1$. The Hilbert scheme $\mathrm{Hilb}^n S$ of n points on S is an irreducible holomorphic symplectic manifold (see Beauville [3]), and the elliptic fibration on S induces a Lagrangian fibration

$$\mathrm{Hilb}^n S \to \mathrm{Sym}^n S \to \mathrm{Sym}^n \mathbb{P}^1 = \mathbb{P}^n,$$

where the first map is the Hilbert–Chow morphism.

The fiber over a general point of $\mathrm{Sym}^n \mathbb{P}^1$, given by an n-tuple $\{x_1, \dots, x_n\}$ of *distinct* points on $\mathbb{P}^1$, is isomorphic to the product $S_{x_1} \times \cdots \times S_{x_n}$ of the corresponding elliptic fibers of $S \to \mathbb{P}^1$. The singular fibers occur over the hyperplanes

$$\Delta_i := \{\{x_1, \dots, x_n\} \in \mathrm{Sym}^n \mathbb{P}^1 \,|\, x_j = p_i \text{ for some } j\},$$

for $i = 1, \dots, 24$, and over the "big diagonal"

$$\Delta_0 := \{\{x_1, \dots, x_n\} \in \mathrm{Sym}^n \mathbb{P}^1 \,|\, x_j = x_k \text{ for some } j \text{ and } k\}.$$

Consider the former; moreover, without loss of generality, let $\{x_1, \dots, x_n\}$ be a general point of Δ_1 with $x_1 = p_1$, $x_j \neq p_i$ for all $j \geq 2$ and i, and $x_j \neq x_k$ for all j and k. Then the singular fiber over $\{x_1, \dots, x_n\}$ will be isomorphic to

$$S_{x_1} \times S_{x_2} \times \cdots \times S_{x_n} \cong S_{p_1} \times S_{x_2} \times \cdots \times S_{x_n},$$

where S_{p_1} is a nodal rational curve and $S_{x_2}, \dots, S_{x_n}$ are smooth elliptic curves. This is semistable and the characteristic 1-cycle is clearly of type I_1.

Next consider a general point $\{x_1, \dots, x_n\}$ of Δ_0; without loss of generality, assume $x_1 = x_2$, $x_j \neq x_k$ otherwise, and $x_j \neq p_i$ for all j and i. For simplicity, we first consider the $n = 2$ case. Write E for the elliptic curve S_{x_1}. The Hilbert scheme $\mathrm{Hilb}^2 S$ is obtained from $\mathrm{Sym}^2 S$ by blowing up the diagonal; thus each point of the diagonal in $\mathrm{Sym}^2 S$ is replaced by a $\mathbb{P}^1$. The singular fiber over $\{x_1 = x_2\} \in \Delta_0$ is therefore isomorphic to the union of $\mathrm{Sym}^2 E$ and a $\mathbb{P}^1$-bundle over D, where $D \cong E$

is the diagonal in $\mathrm{Sym}^2 E$. Now $\mathrm{Sym}^2 E$ is also a $\mathbb{P}^1$-bundle over E, because we have the Abel–Jacobi map

$$\mathrm{Sym}^2 E \longrightarrow \mathrm{Pic}^2 E \cong E,$$

taking a degree two divisor to its corresponding line bundle. The diagonal D is a 4-valued section of this $\mathbb{P}^1$-bundle; for instance, it intersects the fiber above $\mathcal{O}(2y_0) \in \mathrm{Pic}^2 E$ in the four points $2y_1 \in D \subset \mathrm{Sym}^2 E$ where $y_1 - y_0$ is 2-torsion in E. Instead, one could observe that the composition of the maps

$$E \cong D \hookrightarrow \mathrm{Sym}^2 E \to \mathrm{Pic}^2 E,$$

takes y to $\mathcal{O}(2y)$, i.e. is multiplication by 2. Putting everything together, we see that the singular fiber over $\{x_1 = x_2\}$ will have characteristic 1-cycle of type I_0^*, i.e. a central $\mathbb{P}^1$ with four other $\mathbb{P}^1$s attached in a $\tilde{D}_4$ configuration.

For $n > 2$ we simply take the product of the above with the smooth $(n-2)$-dimensional abelian variety $S_{x_3} \times \cdots \times S_{x_n}$, so the characteristic 1-cycle is still of type I_0^*. In particular, the singular fibers over Δ_0 are never semistable. However, we will see in Sec. 4.4 that they become semistable after a small deformation of the Lagrangian fibration.

Thinking of $\mathbb{P}^n$ as parametrizing degree n polynomials, Δ_0 is precisely the hypersurface parametrizing polynomials with a repeated root. A resultant argument then shows that Δ_0 has degree $2(n-1)$, but it seems that one should attach a multiplicity to Δ_0 because it parametrizes non-semistable singular fibers. We will say more about this in Sec. 4.4.

4.4. *Deforming to a very general Lagrangian fibration*

We will now show that the Hilbert scheme of an elliptic $K3$ surface can be deformed, as a Lagrangian fibration, to a Beauville–Mukai integrable system. Under this deformation, the non-semistable singular fibers of the Hilbert scheme become semistable fibers of the Beauville–Mukai system.

Let S be an elliptic $K3$ surface admitting a section D, and assume that the Néron–Severi group is generated by D and the fiber F. In what follows we will often abbreviate $\mathrm{H}^0(S, \mathcal{O}(D))$ as $\mathrm{H}^0(S, D)$, etc.

Lemma 10. *For $n \geq 2$ the fixed part of $nF + D$ is D and the movable part is nF. Thus*

$$|nF + D| \cong |nF| \cong \mathbb{P}^n.$$

We can identify this with $\mathrm{Sym}^n \mathbb{P}^1$ because each of the n fibers moves in a pencil.

Proof. The fact that D is the fixed part of $nF + D$ is proved in Sec. 2.7.4 of Saint-Donat [22]. The lemma then follows from:

$$\mathrm{H}^0(S, nF + D) \cong \mathrm{H}^0(S, nF) = \mathrm{H}^0(S, \pi^*\mathcal{O}(n)) \cong \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(n)) \cong \mathbb{C}^{n+1},$$

where $\pi : S \to \mathbb{P}^1$ is the elliptic fibration. □

By the local Torelli theorem for $K3$ surfaces there exists a hypersurface in the moduli space of all deformations of S such that the class $nF + D$ stays algebraic. Moreover, choosing a general one-parameter deformation in this hypersurface gives a family of $K3$ surfaces $\mathcal{S} \to \Delta$ over a disc $\Delta \subset \mathbb{C}$ with $S_0 = S$ and such that S_t has Picard number one and Néron–Severi group generated by $nF + D$ for very general $t \in \Delta$. In particular, $nF + D$ will be ample on S_t and the general element of $|nF + D|$ will be a smooth curve of genus n. (Explicitly, this family could be given by deforming S in the direction corresponding to $(2 - n)F + D$ under the isomorphism

$$\mathrm{H}^1(S, T) \cong \mathrm{H}^1(S, \Omega^1) \cong \mathrm{H}^{1,1}(S),$$

induced by the isomorphism $T \cong \Omega^1$ coming from interior product with the holomorphic symplectic form σ.)

Next, we define a relative moduli space $\mathcal{M}/\Delta$ of sheaves on $\mathcal{S}/\Delta$ by defining M_t to be the Mukai moduli space of stable sheaves on S_t with Mukai vector

$$v = (0, [nF + D], 1 - n).$$

A general element of M_t will look like $\iota_* L$, where $\iota : C \hookrightarrow S_t$ is the inclusion of a curve C in the linear system $|nF + D|$ and L is a degree zero line bundle on C. (In general, for a degree d line bundle L on a curve C of genus g, the Mukai vector of $\iota_* L$ is $(0, [C], d + 1 - g)$.)

Proposition 11. *The relative moduli space $\mathcal{M}/\Delta$ is a family of Lagrangian fibrations over $\mathbb{P}^n_\Delta$. Specifically*

(1) *for very general t, M_t is a Beauville–Mukai integrable system*,
(2) *for $t = 0$, M_0 is birational to* $\mathrm{Hilb}^n S_0$ *with Lagrangian fibration induced by the original elliptic fibration on $S_0 = S$.*

Moreover, the birational map in part 2 commutes with the Lagrangian fibrations,

$$\begin{tikzcd} M_0 \arrow[rr, dashed] \arrow[dr] & & \mathrm{Hilb}^n S_0 \arrow[dl] \\ & \mathbb{P}^n. & \end{tikzcd}$$

Proof. For very general t, M_t is a Beauville–Mukai system by definition; so there is nothing to prove in part 1. When $t = 0$, the linear system $|nF + D|$ has D as a base locus. Thus every curve C in the linear system $|nF + D|$ looks like $F_1 + \cdots + F_n + D$, where F_i are fibers of the elliptic fibration $S \to \mathbb{P}^1$. Assume that the F_i are smooth and distinct. A degree zero line bundle L on $F_1 + \cdots + F_n + D$ is given by line bundles $L_1, \ldots, L_{n+1}$ on $F_1, \ldots, F_n$, and D, respectively, plus isomorphisms $(L_i)_{p_i} \cong (L_{n+1})_{p_i}$ at $p_i = F_i \cap D$, for $1 \leq i \leq n$. Stability of $\iota_* L$ implies that the L_i all have degree zero (see Sec. 4.2 of Sawon [29] for an explicit calculation in a similar example). The only degree zero line bundle on $D \cong \mathbb{P}^1$ is the trivial line bundle, whereas degree zero line bundles on F_i are parametrized by $\mathrm{Pic}^0 F_i \cong F_i$. Therefore the fiber of $M_0 \to |nF + D|$ over the point $F_1 + \cdots + F_n + D$ is isomorphic to

$$\mathrm{Pic}^0 F_1 \times \cdots \times \mathrm{Pic}^0 F_n \cong F_1 \times \cdots \times F_n.$$

Similarly, the elliptic fibration $S_0 \to \mathbb{P}^1$ induces a Lagrangian fibration

$$\mathrm{Hilb}^n S_0 \to \mathrm{Sym}^n \mathbb{P}^1 \cong \mathbb{P}^n,$$

whose fiber over a general point $F_1 + \cdots + F_n \in |nF| = \mathrm{Sym}^n \mathbb{P}^1 \cong \mathbb{P}^n$ is also isomorphic to $F_1 \times \cdots \times F_n$. Since these Lagrangian fibrations both admit global sections, they must be birational.

Finally, it is a general fact that a birational map $\phi : X \dashrightarrow X'$ between Lagrangian fibrations that takes a general fiber isomorphically to a fiber must commute with the fibrations. To prove this, recall that a birational map between irreducible holomorphic symplectic manifolds induces a correspondence between divisors; it is enough to show that this correspondence preserves the divisors inducing the Lagrangian fibrations on X and X'. Consider the pullback H of a general hyperplane under $\pi : X \to \mathbb{P}^n$. It contains a dense open subset $U \subset H$ that is a union of fibers that don't meet the indeterminacy locus of ϕ. The corresponding divisor H' on X' is the closure of $\phi(U)$. But the image $\phi(U)$ will be a union of fibers of $\pi' : X' \to \mathbb{P}^n$, and H' will therefore be the pullback of a hypersurface. Finally, $\dim|H'| = \dim|H| = n + 1$ implies that H' is the pullback of a hyperplane, completing the proof. (This last part can also be proved without assuming that the bases of the Lagrangian fibrations are isomorphic to $\mathbb{P}^n$: see Matsushita [17, Corollary 2].) $\square$

Remark. The identification of smooth fibers of M_0 with those of $\mathrm{Hilb}^n S_0$ easily extends to some singular fibers. For example, suppose that F_1 is a nodal rational curve. Then we should allow L_1 to be a rank one torsion-free sheaf on F_1. In other words, we should replace $\mathrm{Pic}^0 F_1$ by the compactified Jacobian $\overline{\mathrm{Pic}}^0 F_1$. However, $\overline{\mathrm{Pic}}^0 F_1$ is isomorphic to F_1, as required. The case when some fibers coincide, for example $F_1 = F_2$, appears to be more complicated, even for F_1 smooth.

It is not clear if $M_0 \dashrightarrow \mathrm{Hilb}^n S_0$ extends to an isomorphism; more likely it involves some modification of special fibers. Example 1 of Matsushita [17], which arises from performing Mukai flops on Lagrangian $\mathbb{P}^n$s contained in fibers, demonstrates how this could happen.

Remark. For each smooth fiber, we identified $\mathrm{Pic}^0 F$ with F. This identification is fixed by taking $\mathcal{O}_F$ to the basepoint in F given by the intersection $F \cap D$ with the section D. Alternatively, we could work with the moduli space $M(0, [C], 1)$ whose general element looks like $\iota_* L$ for a degree n line bundle L. Then stability will force L to come from degree one line bundles on $F_1, \ldots, F_n$ and a degree zero line bundle on D. The fiber over $F_1 + \cdots + F_n + D$ would then be

$$\mathrm{Pic}^1 F_1 \times \cdots \times \mathrm{Pic}^1 F_n \cong F_1 \times \cdots \times F_n$$

and this isomorphism is canonical: we don't even need a basepoint in each fiber.

Corollary 12. *The Hilbert scheme* $\mathrm{Hilb}^n S_0$ *can be deformed to a Beauville–Mukai integrable system, as a Lagrangian fibration over* $\mathbb{P}^n$*.*

Proof. This is achieved by a birational modification of the relative moduli space $\mathcal{M}/\Delta$. We showed that the fibre M_0 above $t = 0$ is birational to $\mathrm{Hilb}^n S_0$. Birational holomorphic symplectic manifolds correspond to non-separated points in the moduli space (see Huybrechts [8, Theorem 4.6]), so we can replace M_0 by $\mathrm{Hilb}^n S_0$ in the family. Moreover, the divisors inducing the Lagrangian fibrations on M_t for $t \neq 0$ specialize to the divisor inducing the Lagrangian fibration on M_0. As we saw in the proof of Proposition 11, under the birational map this corresponds to the divisor inducing the Lagrangian fibration on $\mathrm{Hilb}^n S_0$. Therefore after the birational modification of $\mathcal{M}/\Delta$, we still have a family of Lagrangian fibrations over $\mathbb{P}^n_\Delta$. □

Assume that S is very general, so that it contains exactly 24 nodal rational curves as singular fibers. For $n = 2$, the Lagrangian fibration on $\mathrm{Hilb}^2 S_0$ has a discriminant locus consisting of 24 lines and a conic, with the lines tangent to the conic; the singular fibers above the lines are semistable, whereas those above the conic are not. A Beauville–Mukai integrable system will have a discriminant locus of degree 30 (see Sawon [24]), and general singular fibers will be semistable. Thus the corollary demonstrates how non-semistable fibers can deform to semistable fibers under a small deformation of the Lagrangian fibration. In addition, it suggests that the conic should be counted with multiplicity three, so that the degree of the discriminant locus is preserved under deformations.

For general n, a Beauville–Mukai integrable system over $\mathbb{P}^n$ will have a discriminant locus of degree $6(n+3)$ (see Sawon [24]), and general singular fibers will be semistable. The discriminant locus of the Lagrangian fibration on $\mathrm{Hilb}^n S_0$ consists of 24 hyperplanes and a degree $2n-2$ hypersurface R. The latter is the image of the "big" diagonal under the projection

$$\mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \to \mathrm{Sym}^n \mathbb{P}^1 \cong \mathbb{P}^n$$

and its degree can be calculated by identifying $\mathbb{P}^n$ with the space of homogeneous polynomials f of degree n; R is then given by the discriminant of f, i.e. the resultant of f and f', which has degree $2n-2$. The singular fibers above the hyperplanes are semistable, whereas those over R are not. Indeed, the behavior is similar to the $n = 2$ case: over a general point of R, the singular fiber will look like the product of $n-2$ elliptic curves and a singular fiber over the conic in the $n = 2$ case. Thus we once again see how non-semistable fibers can deform to semistable fibers under a small deformation of the Lagrangian fibration. Moreover, we expect that R should again be counted with multiplicity three, so that the total degree of the discriminant locus,

$$24 + 3(2n - 2) = 6(n + 3),$$

is preserved under the deformation.

Remark. One expects that a similar example can be constructed from generalized Kummer varieties. Namely, if A is a two-dimensional complex torus admitting an elliptic fibration then its generalized Kummer variety $K_n(A)$ admits a Lagrangian fibration (see Sawon [27, Sec. 3.2]). This Lagrangian fibration should be birational to another constructed from a moduli space of rank zero sheaves on A, and then a deformation of A should induce a deformation of $K_n(A)$ to a Debarre system [6].

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