Homology, Homotopy and Applications, vol. 25(1), 2023, pp.401–408

LOCALLY DUALIZABLE MODULES ABOUND

JON F. CARLSON AND SRIKANTH B. IYENGAR

(*communicated by Bill Murray*)

Abstract

It is proved that given any prime ideal $\mathfrak{p}$ of height at least 2 in a countable commutative noetherian ring A, there are uncountably many more dualizable objects in the $\mathfrak{p}$-local $\mathfrak{p}$-torsion stratum of the derived category of A than those that are obtained as retracts of images of perfect A-complexes. An analogous result is established dealing with the stable module category of the group algebra of a finite group having sufficient p-rank, over a countable field of positive characteristic p.

1. Introduction

This work is about dualizable objects in tensor triangulated categories arising in commutative algebra and the modular representation theory of finite groups. An object D in a tensor triangulated category $\mathcal{C}$ is *dualizable* if the natural map

$$\mathcal{H}om(D,\mathbb{1})\otimes X \longrightarrow \mathcal{H}om(D,X)$$

is an isomorphism for all X in $\mathcal{C}$. Here $\mathcal{H}om$ and $\otimes$ are the internal function object and product in $\mathcal{C}$, respectively, and $\mathbb{1}$ is the unit of the product.

Consider $\mathrm{D}(\operatorname{Mod} A)$, the derived category of a commutative noetherian ring A, with tensor structure given by the derived tensor product $-\otimes^{\mathbf{L}}-$, the unit is A, and function object is $\operatorname{RHom}_A(-,-)$. The dualizable objects in $\mathrm{D}(\operatorname{Mod} A)$ are precisely the perfect A-complexes, $\operatorname{Perf} A$. These are also the compact objects, and hence $\mathrm{D}(\operatorname{Mod} A)$ is rigid as a compactly generated tensor triangulated category.

We consider also $\operatorname{StMod}(kG)$ the stable module category of a finite group G, with k a field of positive characteristic dividing $|G|$. In this case the tensor structure is given by $-\otimes_k-$ with diagonal G action and the function object is $\operatorname{Hom}_k(-,-)$, again with diagonal G-action; the unit is k with trivial G-action. The dualizable objects are those in $\operatorname{stmod}(kG)$, namely, the kG modules that are stably isomorphic to finite dimensional ones, and hence coincide with the compact objects, so $\operatorname{StMod}(kG)$ is also rigid.

Both $\mathrm{D}(\operatorname{Mod} A)$ and $\operatorname{StMod}(kG)$ admit natural stratifications into "local" triangulated subcategories that determine, to a large extent, their global structure. In

Received Month Day, Year, revised Month Day, Year; published on Month Day, Year.
2020 Mathematics Subject Classification: 13D09 (primary); 18G80, 14F08 (secondary).
Key words and phrases: derived category, dualizable object, stable module category, tensor triangulated category.
Article available at http://dx.doi.org/10.4310/HHA.2023.v25.n1.a1
Copyright © 2023, International Press. Permission to copy for private use granted.

$\mathrm{D}(\operatorname{Mod} A)$ the strata are parameterized by points $\mathfrak{p} \in \operatorname{Spec} A$ and the stratum corresponding to $\mathfrak{p}$ consists of the $\mathfrak{p}$-local and $\mathfrak{p}$-torsion complexes in $\mathrm{D}(\operatorname{Mod} A)$, which is denoted $\varGamma_{\mathfrak{p}} \mathrm{D}(\operatorname{Mod} A)$; see Section 2 for details. There is an analogous stratification $\operatorname{StMod}(kG)$, with parameter space $\operatorname{Proj} H^*(G,k)$; see Section 3. In both cases, the strata are again tensor triangulated subcategories, and it is of interest to understand the dualizable objects in these categories. A noteworthy feature now is that, unless $\mathfrak{p}$ is minimal, there are many more dualizable objects than compact ones, so the strata are not rigid.

There is a natural functor $\varGamma_{\mathfrak{p}}\colon \mathrm{D}(\operatorname{Mod} A) \to \varGamma_{\mathfrak{p}} \mathrm{D}(\operatorname{Mod} A)$ and the dualizable objects in $\varGamma_{\mathfrak{p}} \mathrm{D}(\operatorname{Mod} A)$ are generated as a thick subcategory by $\varGamma_{\mathfrak{p}} A$; see [**3**, Theorem]. The question arose whether $\varGamma_{\mathfrak{p}} \operatorname{Perf} A$ is dense in the subcategory of local dualizable objects; in other words, whether each dualizable object in $\varGamma_{\mathfrak{p}} \mathrm{D}(\operatorname{Mod} A)$ is a retract of a complex $\varGamma_{\mathfrak{p}} P$, with P a perfect A-complex. The point of this paper is that when A is countable, there are uncountably many, mutually non-isomorphic, indecomposable dualizable objects in $\varGamma_{\mathfrak{p}} \mathrm{D}(\operatorname{Mod} A)$, but at most countably many that are retracts of images of perfect complexes in A; see Theorem 2.1.

There is an analogous description of the dualizable objects in $\varGamma_{\mathfrak{p}} \operatorname{StMod}(kG)$, established in [**4**], and once again it turns that there can be many more dualizable objects in this strata than direct summands of those induced from $\operatorname{stmod}(kG)$, the global dualizable objects; see Theorem 3.1.

Acknowledgments

Our thanks to the referee for positive feedback and suggestions. This work is partly supported by NSF Grant DMS-200985 (SBI).

2. Local algebra

Let A be a commutative noetherian ring and $\mathrm{D}(\operatorname{Mod} A)$ the (full) derived category of A-modules, viewed as a tensor triangulated category, where the product of A-complexes X, Y is the derived tensor product $X \otimes^{\mathbf{L}}_A Y$. The derived category is rigidly compactly generated, with compact objects the perfect A-complexes, that is to say, those that are isomorphic, in $\mathrm{D}(\operatorname{Mod} A)$, to bounded complexes of finitely generated projective modules. We denote this category $\operatorname{Perf} A$. It is the thick subcategory of $\mathrm{D}(\operatorname{Mod} A)$ generated by A.

Given a point $\mathfrak{p} \subseteq \operatorname{Spec} R$ consider the exact functor

$$\varGamma_{\mathfrak{p}}\colon \mathrm{D}(\operatorname{Mod} A) \to \mathrm{D}(\operatorname{Mod} A) \quad \text{where } X \mapsto \mathbf{R}\varGamma_{V(\mathfrak{p})}(X_{\mathfrak{p}})$$

where $\mathbf{R}\varGamma_{V(\mathfrak{p})}(-)$ is the functor representing local cohomology with support in the ideal $\mathfrak{p}$ of A. The image $\varGamma_{\mathfrak{p}} \mathrm{D}(\operatorname{Mod} A)$ consists of precisely the $\mathfrak{p}$-local and $\mathfrak{p}$-torsion complexes. It is a tensor triangulated category in its own right, with product induced from that on $\mathrm{D}(\operatorname{Mod} A)$, unit $\varGamma_{\mathfrak{p}} A$, and function object $\varGamma_{\mathfrak{p}} \operatorname{RHom}_A(-,-)$. It is not rigid, unless $\mathfrak{p}$ is minimal, and there are many more rigid objects than compact ones. See [**3**, Section 4] for proofs of these assertions.

The main result of this section is as follows.

Theorem 2.1. *Let A be a countable commutative noetherian ring and $\mathfrak{p} \in \operatorname{Spec} A$ such that $\operatorname{height} \mathfrak{p} \geqslant 2$. There exist uncountably many mutually non-isomorphic indecomposable dualizable objects in $\varGamma_{\mathfrak{p}} \operatorname{D}(\operatorname{Mod} A)$ none of which is a retract of an object in $\varGamma_{\mathfrak{p}} \operatorname{Perf} A$.*

Remark 2.2. In fact, the argument, which is given towards the end of this section, shows that $\varGamma_{\mathfrak{p}} \operatorname{Perf} A$ contains only countably many isomorphism classes of objects, even allowing for retracts, but that there are uncountably many non-isomorphic indecomposable objects in $\operatorname{Thick}(\varGamma_{\mathfrak{p}} A)$. It is easy to check that the latter subcategory consists of dualizable objects in $\varGamma_{\mathfrak{p}} \operatorname{D}(\operatorname{Mod} A)$. As it happens, these are all the dualizable objects. This is the main result in [**3**]; we do not need this fact here.

We record a simple observation.

Lemma 2.3. *When A is a countable noetherian ring, there are only countably many isomorphism classes of objects in $\operatorname{mod} A$, and, more generally, only countably many isomorphism classes in $\operatorname{D}^b(\operatorname{mod} A)$.*

Proof. Any finitely generated A-module occurs as a cokernel of a map $A^m \to A^n$, for each nonnegative integers m, n, and each such map is given by a matrix of size $m \times n$ with coefficients in A. Since A is countable, there are only countably many such matrices, which justifies the claim about $\operatorname{mod} A$. The one about complexes follows because in $\operatorname{D}(\operatorname{Mod} A)$ any complex with finitely generated homology is isomorphic to one of the form

$$0 \longrightarrow M_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0\,,$$

where M_n is in $\operatorname{mod} R$ and each F_i is a finite free A-module. $\square$

The result below is also well-known; see [**9**, p. 500].

Lemma 2.4. *Let A be a commutative noetherian local ring that is complete with respect to $\mathfrak{m}$, its maximal ideal. If $\dim A \geqslant 2$, then there exists an uncountable collection of distinct prime ideals in A, each of height one. In the same vein, there is an uncountable collection of elements $\{a_u\}_{u \in U}$ with $\sqrt{a_u} \neq \sqrt{a_v}$ for $u \neq v$.*

Proof. Assume to the contrary that there are only countable many prime ideals $\{\mathfrak{p}_i\}_{i \geqslant 1}$ of height one. Since each non-invertible element in A is contained in a height one prime, by Krull's Principal Ideal theorem, it follows that $\mathfrak{m} \subseteq \cup_i \mathfrak{p}_i$. Since A is complete, it has the countable prime avoidance property; see, for instance, [**6, 10**]. We deduce that $\mathfrak{m} \subseteq \mathfrak{p}_i$ for some i, contradicting the hypothesis that $\dim A \geqslant 2$.

Because every prime ideal of height one is minimal over a principal ideal, and the radical of principal ideal is a finite intersection of prime ideals, the second part of the assertion follows from the first. $\square$

Proof of Theorem 2.1. One has that $\varGamma_{\mathfrak{p}} \operatorname{D}(\operatorname{Mod} A) \simeq \varGamma_{\mathfrak{p}} \operatorname{D}(\operatorname{Mod} A_{\mathfrak{p}})$, so replacing A by its localization at $\mathfrak{p}$ we can suppose it is local, say with maximal ideal $\mathfrak{m}$, with $\dim A \geqslant 2$. Let $\widehat{A}$ be the $\mathfrak{m}$-adic completion of A. A key input in the arguments presented below is that the natural map of rings $A \to \operatorname{Ext}^*_A(\mathbf{R}\Gamma_{\mathfrak{m}} A, \mathbf{R}\Gamma_{\mathfrak{m}} A)$ factors through the completion map $A \to \widehat{A}$ and yields an isomorphism

$$\widehat{A} \xrightarrow{\cong} \operatorname{Ext}^*_A(\mathbf{R}\Gamma_{\mathfrak{m}} A, \mathbf{R}\Gamma_{\mathfrak{m}} A).$$

Hence, derived Morita theory yields the Greenlees-May adjoint equivalence

$$\operatorname{Thick}_{\widehat{A}}(\widehat{A}) \underset{\operatorname{RHom}_A(\mathbf{R}\Gamma_{\mathfrak{m}} A, -)}{\overset{\mathbf{R}\Gamma_{\mathfrak{m}} A \otimes^{\mathbf{L}}_{\widehat{A}} -}{\rightleftarrows}} \operatorname{Thick}_A(\mathbf{R}\Gamma_{\mathfrak{m}} A)\,.$$

It is also helpful that $\operatorname{RHom}_A(\mathbf{R}\Gamma_{\mathfrak{m}}, -) \cong \mathbf{L}\Lambda^{\mathfrak{m}}(-)$, the left derived functor of $\mathfrak{m}$-adic completion. See the discussion around [**3**, (4.2), (4.3)].

Let $\{\mathfrak{p}_u\}_u$ be the uncountable collection of prime ideals in $\widehat{R}$ supplied by Lemma 2.4. For each $\mathfrak{p}_u$ let K_u denote the Koszul complex on $\mathbf{R}\Gamma_{\mathfrak{m}} R$ on a minimal generating set for the ideal $\mathfrak{p}_u$ of $\widehat{R}$. The complex K_u has the following properties:

(1) Each K_u is dualizable in $\Gamma_{\mathfrak{m}}\operatorname{D}(\operatorname{Mod} A)$;

(2) One has $K_u \ncong K_v$ for $u \neq v$;

(3) Each K_u is indecomposable.

Indeed (1) holds because $\mathbf{R}\Gamma_{\mathfrak{m}} A$, being the unit of the product on $\Gamma_{\mathfrak{m}}\operatorname{D}(\operatorname{Mod} A)$ is dualizable, and K_u is finitely built from $\mathbf{R}\Gamma_{\mathfrak{m}} A$.

Under the adjoint equivalence above, $\mathbf{R}\Gamma_{\mathfrak{m}} A$ is mapped to $\mathbf{L}\Lambda^{\mathfrak{m}} A$, so K_u is mapped to the Koszul complex on $\widehat{A}$ on the chosen minimal generating set for $\mathfrak{p}_u$. It follows that the kernel I_u of the natural map

$$\widehat{A} \longrightarrow \operatorname{Ext}^*_A(K_u, K_u) \cong \operatorname{Ext}^*_{\widehat{A}}(\mathbf{L}\Lambda^{\mathfrak{m}} K_u, \mathbf{L}\Lambda^{\mathfrak{m}} K_u)$$

satisfies $\sqrt{I_u} = \mathfrak{p}_u$. Since the $\{\mathfrak{p}_u\}_u$ are distinct, (2) follows. Moreover, the Koszul complex over $\widehat{A}$ on any minimal generating set for the ideal $\mathfrak{p}$ is indecomposable in $\operatorname{D}(\operatorname{Mod}\widehat{A})$, by [**1**, Proposition 4.7]. Consequently, K_u is indecomposable in $\operatorname{D}(\operatorname{Mod} A)$. This justifies (3).

Since the collection $\{K_u\}_u$ is uncountable, to complete the proof we have to verify that there are only finitely many isomorphism classes of indecomposable direct summands of $\Gamma_{\mathfrak{m}} P$, with P a perfect A-complex. To see this, note that since $\widehat{A}$ is complete, $\operatorname{Thick}_{\widehat{A}}(\widehat{A})$ is a Krull-Schmidt category, and hence so is $\operatorname{Thick}_A(\mathbf{R}\Gamma_{\mathfrak{m}} A)$, by the equivalence above. It remains to recall that there are only countably many isomorphism classes of perfect A-complexes, by Lemma 2.3. □

3. Finite groups

Let G be a finite group and k a field of positive characteristic p, where p divides the order of G. The stable category $\operatorname{StMod}(kG)$ of kG-modules modulo projective modules is a tensor triangulated category, there the product is $-\otimes_k -$ with the diagonal G-action, and the unit is the trivial kG-module k. The compact objects are the modules equivalent to finitely generated modules. They form a thick subcategory denoted $\operatorname{stmod}(kG)$.

The cohomology ring $\operatorname{H}^*(G,k) = \operatorname{Ext}^*_{kG}(k,k)$ is a finitely generated graded k-algebra and $\operatorname{Ext}^*_{kG}(M,N)$ is a finitely generated module over $\operatorname{H}^*(G,k)$, for all M, N

in $\operatorname{mod} kG$. The projectivized spectrum $V_G(k) = \operatorname{Proj} \mathrm{H}^*(G,k)$ is the collection of homogeneous prime ideals in $H^*(G,k)$, except the the maximal one, $H^{\geqslant 1}(G,k)$.

The support variety $V_G(M)$ of a finitely generated module M is defined as the collection of those ideals that contain the annihilator of $\operatorname{Ext}^*_{kG}(M,M)$ in $\mathrm{H}^*(G,k)$. Support varieties for infinite dimensional kG-modules are introduced in [**5**]; see also [**2**]. These are subsets of $V_G(k)$ that are not necessarily closed.

As in the previous section, for each $\mathfrak{p}$ in $V_G(k)$, there exists an exact functor

$$\varGamma_{\mathfrak{p}}\colon \operatorname{StMod}(kG) \longrightarrow \operatorname{StMod}(kG)$$

whose image is the subcategory consisting of all kG-modules whose support is contained in V. Then, $\varGamma_{\mathfrak{p}} \operatorname{StMod}(kG)$ is again tensor triangulated, with tensor product inherited from $\operatorname{StMod}(kG)$. The unit is $\varGamma_{\mathfrak{p}} k$, and the function object is $\varGamma_{\mathfrak{p}} \operatorname{Hom}_k(-,-)$. There are numerous equivalent ways to characterize dualizable modules in this category; see [**4**]. The full subcategory of dualizable modules in $\varGamma_{\mathfrak{p}} \operatorname{StMod}(kG)$ form a thick triangulated subcategory, and $\varGamma_{\mathfrak{p}} M$ is dualizable for each M in $\operatorname{stmod}(kG)$. The theorem below and its proof are similar to Theorem 2.1.

Theorem 3.1. *Let k be a countable field of characteristic p, and G an elementary abelian p-group of rank $\geqslant 3$. Let $\mathfrak{p}$ be a closed point in $V_G(k)$. There exists an uncountable collection of mutually non-isomorphic, indecomposable dualizable modules in $\varGamma_{\mathfrak{p}} \operatorname{StMod}(kG)$, none of which is a direct summand of $\varGamma_{\mathfrak{p}} M$ for M in* $\operatorname{stmod}(kG)$.

Proof. Since the dualizable objects form a thick subcategory, it suffices to prove that there is an uncountable collection of mutually non-isomorphic, indecomposable objects in the thick category generated by $\varGamma_{\mathfrak{p}} \operatorname{stmod}(kG)$ that are not retracts of the images of the finite dimensional ones. This has nothing to do with tensor triangulated structure on $\varGamma_{\mathfrak{p}} \operatorname{StMod}(kG)$, and we are free to choose any coalgebra structure on kG that is convenient.

We may assume $G = H \times \langle z \rangle$ where H is elementary abelian of rank $r-1$ and $z^p = 1$. That is, we can assume that the ideal $\mathfrak{p}$ is the radical of the restriction to a subalgebra $k[z]/(z^p) \in kG$, the inclusion into kG being a π-point associated to $\mathfrak{p}$ in the language of [**8**]. The choice of the complementary subalgebra kH is somewhat arbitrary. That is, we can choose kH to be the subalgebra generated by any collection $x_1, \dots, x_{r-1}$ in $\operatorname{rad}(kG)$ such that the images of $z, x_1, \dots, x_{r-1}$ in $\operatorname{rad}(kG)/\operatorname{rad}^2(kG)$ form a k-basis. Then $kG \cong kC \otimes kH$ where C is generated by the unit $1+z$ and H is generated by $1+x_1, \dots, 1+x_{r-1}$. This is an isomorphism of k-algebras, but not generally as Hopf algebras.

Keeping in mind that $\varGamma_{\mathfrak{p}} = \varGamma_{V(\mathfrak{p})}$, from [**7**, Proposition 5.2] we see that

$$R := \underline{\operatorname{End}}_{kG}(\varGamma_{\mathfrak{p}} k) = \underline{\operatorname{Hom}}_{kG}(\varGamma_{\mathfrak{p}} k, \varGamma_{\mathfrak{p}} k) \cong \underline{\operatorname{Hom}}_{kG}(\varGamma_{\mathfrak{p}} k, k) \cong \prod_{i=0}^{\infty} \mathrm{H}^i(H,k)\,,$$

as an additive group. The product is the obvious one, except that, if p is odd, then any two elements of odd degree multiply to zero. Hence the endomorphism ring is a commutative local ring that, modulo a nilpotent ideal, is the completion of a polynomial ring of degree $r-1$. In particular its Krull dimension is $r-1 \geqslant 2$.

For each $\zeta \in R$, we define the module K_ζ to be the third object in the triangle

$$K_\zeta \longrightarrow \varGamma_{\mathfrak{p}} k \xrightarrow{\zeta} \varGamma_{\mathfrak{p}} k \longrightarrow .$$

Evidently, K_ζ is in the thick subcategory generated by $\varGamma_{\mathfrak{p}} \operatorname{stmod}(kG)$. Let $\mathfrak{a}_{\mathfrak{p}}(K_\zeta)$ be the annihilator of the R-module $\underline{\operatorname{Hom}}_{kG}(\varGamma_{\mathfrak{p}} k, K_\zeta))$. Then we recall [**7**, Theorem 7.6] that the radical of $\mathfrak{a}_{\mathfrak{p}}(K_\zeta)$ coincides with the radical of the ideal generated by ζ. Thus, we have that if ζ and γ are elements in R that generate ideals with different radicals, then K_ζ is not isomorphic to K_γ.

When M is a finite dimensional module, the R-module $\underline{\operatorname{End}}_{kG}(\varGamma_{\mathfrak{p}} M)$ is finitely generated. Consequently, $\varGamma_{\mathfrak{p}} M$ has only a finite number of indecomposable summands, as otherwise, $\underline{\operatorname{End}}_{kG}(\varGamma_{\mathfrak{p}} M)$ would have an infinite number of idempotents. It follows from the Lemma 2.3 that the collection of modules K_ζ that can be direct summands of modules of the form $\varGamma_V M$ for M finitely generated, is countable. Since $\dim R \geqslant 2$ it remains to recall from Lemma 2.4 that it has an uncountable number of elements ζ having mutually distinct radicals. □

Example 3.2. Suppose $p = 2$ and that G is elementary abelian of order 8. We use the notation of the previous proof. We write $kG = kH \otimes kC$ where $kH = k[x, y]/(x^2, y^2)$ and $kC = k[z]/(z^2)$. Here the variety V is the point corresponding to the inclusion $kC \hookrightarrow kG$. Choose $\zeta \in \underline{\operatorname{Hom}}_{kG}(\varGamma_V k, \varGamma_V k)$ to have the form $\zeta = (0, \zeta_1, \zeta_2, \dots)$ where $\zeta_i \in \mathrm{H}^i(H, k)$. Assume that $\zeta_1 \neq 0$. Because $\zeta_1 \neq 0$, we have that the restriction to kH of K_ζ is a direct sum $\sum_{i=0}^{\infty} U_i$ where $U_i \cong kH$. Choose $u_i \in U_i$ to be a kH-generator, for all i. With some calculation, it can be shown that the action of z is give by a formula

$$zu_i = \sum_{j=0}^{i} (\alpha_j x + \beta_j y) u_{i-j}$$

where for each j, the elements $\alpha_j, \beta_j \in k$, depend on the choices of ζ_ℓ for $\ell \leqslant j$. With some slight adjustment in the proof, Theorem 3.1 tells us that if k is countable, then there is an uncountable collection of such elements ζ such that the resulting modules K_ζ are mutually non-isomorphic and not isomorphic to a direct summand of any $\varGamma_V M$ for any $M \in \operatorname{stmod}(kG)$.

General finite groups

We end with the following result, extending Theorem 3.1 to any finite group.

Theorem 3.3. *Let k be a countable field and G a finite group. Suppose that $\mathfrak{p}$ be a closed point in $V_G(k)$ that is contained in $\operatorname{res}^*_{G,E}(V_G(k))$ for some elementary abelian p-subgroup E having rank $\geqslant 3$. There exists an uncountable collection of mutually non-isomorphic, indecomposable dualizable modules in $\varGamma_{\mathfrak{p}} \operatorname{StMod}(kG)$, none of which is a direct summand of $\varGamma_{\mathfrak{p}} M$ for M in $\operatorname{stmod}(kG)$.*

Proof. We use the induction functor $\operatorname{StMod}(kE) \to \operatorname{StMod}(kG)$ that takes a kE-module M to $M^{\uparrow G} = kG \otimes_{kE} M$. The restriction to a kE-module of the idempotent module $\varGamma_{\mathfrak{p}} k$ is still an idempotent module, and a support variety argument establishes that, in the stable category, it has the form

$$(\Gamma_{\mathfrak{p}} k)_{\downarrow E} \cong \sum \Gamma_{\mathfrak{q}} k$$

where the sum is over the finite collection of closed points $\mathfrak{q} \in V_E(k)$ such that $\mathrm{res}^*_{G,E}(\mathfrak{q}) = \mathfrak{p}$. Then by Frobenius reciprocity, we have that

$$\Gamma_{\mathfrak{p}} k \otimes (k_{\downarrow E})^{\uparrow G} \cong ((\Gamma_{\mathfrak{p}} k)_{\downarrow E})^{\uparrow G} \cong \sum (\Gamma_{\mathfrak{q}} k)^{\uparrow G}.$$

As a consequence, the modules $(\Gamma_{\mathfrak{q}} k)^{\uparrow G}$ are dualizable.

Let $\mathfrak{q}$ denote any one of the points with $\mathrm{res}^*_{G,E}(\mathfrak{q}) = \mathfrak{p}$. Because the induction functor is exact, for ζ in $\mathrm{Hom}_{kE}(\Gamma_{\mathfrak{p}} k, \Gamma_{\mathfrak{p}} k)$, the module $K_\zeta^{\uparrow G}$ is also dualizable. Moreover, by the Mackey Theorem, any such module has at most a finite number of indecomposable direct summands. Now the theorem follows from the fact that, by Theorem 3.1, there is an uncountable number of such modules and they are in $\Gamma_{\mathfrak{p}} \operatorname{StMod}(kG)$. On the other hand, the thick subcategory obtained by taking the idempotent completion of $\Gamma_{\mathfrak{p}} \operatorname{stmod}(kG)$ has only a countable number of indecomposable objects. $\square$

References

[**1**] H. Altmann, E. Grifo, J. Montaño, W. T. Sanders, and T. Vu, *Lower bounds on projective levels of complexes*, J. Alg. **491** (2017), 343–356.

[**2**] D. J. Benson, S. B. Iyengar, and H. Krause, *Local cohomology and support for triangulated categories*, Ann. Scient. Éc. Norm. Sup. (4) **41** (2008), 575–621.

[**3**] D. J. Benson, S. B. Iyengar, H. Krause, and J. Pevtsova, *Local dualisable objects in local algebra*, proceeding of the Abel Symposium, 2022, Springer, to appear.

[**4**] D. J. Benson, S. B. Iyengar, H. Krause, and J. Pevtsova, *Locally dualisable modular representations and local regularity*, (in preparation).

[**5**] D. J. Benson, J. F. Carlson and J. Rickard, *Complexity and varieties for infinitely generated modules*, II, Math. Proc. Cam. Phil. Soc. **120** (1996), 597–615.

[**6**] L. Burch, *Codimension and analytic spread*, Proc. Cam. Phil. Soc. **72** (1972), 369-–373.

[**7**] J. F. Carlson, *Idempotent modules, locus of compactness and local supports*, Proc. Southeastern Lie Theory Workshop Series, Proc. Symp. Pure Math. 2024 (to appear).

[**8**] E. Friedlander and J. Pevtsova, Π-*supports for modules for finite group schemes*, Duke Math J. **139** (2007), 317-368.

[**9**] W. Hassler and R. Wiegand, *Extended Modules*, J. Commut. Algebra, **1** (2009), 481–505.

[**10**] R. Y. Sharp and P. Vámos, *Baire's category theorem and prime avoidance in complete local rings*, Arch. Math. (Basel), **44** (1985), 243—248.

Jon F. Carlson jfc@math.uga.edu

Department of Mathematics, University of Georgia, Athens, GA 30602, USA

Srikanth B. Iyengar srikanth.b.iyengar@utah.edu

Department of Mathematics, University of Utah, Salt Lake City, UT 84112, USA