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Matrix-valued modified logarithmic Sobolev inequality for sub-Laplacian on SU (2)

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ABSTRACT

We prove that the canonical sub-Laplacian on $SU(2)$ admits a modified log-Sobolev inequality on matrix-valued functions, independent of the matrix sizes. This establishes the first example of a matrix-valued modified log-Sobolev inequality for a sub-Laplacian. We also show that on Lie groups the heat kernel measure p_t at time t satisfies matrix-valued modified log-Sobolev inequality with constants in order $O(t^{-1})$.

1. Introduction

Since the seminal works [28,29] of L. Gross, log Sobolev inequalities have been intensively studied and found rich connections to analysis, geometry, and probability (see

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[30,36] for surveys). In recent decades, log Sobolev inequalities for quantum systems have attracted a lot of attentions in quantum information theory and quantum many-body systems, e.g. [4,12,13,22,34,43]. Motivated by the noncommutative cases, we study modified log Sobolev inequalities for matrix-valued functions for horizontal heat semigroups generated by sub-Laplacians. Such matrix-valued modified log Sobolev inequalities have direct applications to quantum Markov semigroups on matrix algebras.

Recall that a classical Markov semigroup $P_t = e^{t\mathcal{L}} : L^\infty(\Omega,\mu) \to L^\infty(\Omega,\mu)$ on a probability space (Ω,μ) is a semigroup of conservative (unital) positivity preserving maps. We say that P_t satisfies a *logarithmic Sobolev inequality* $\mathrm{LSI}(\lambda)$ if for some $\lambda > 0$

$$\int_\Omega f^2 \log f^2 d\mu - \left(\int_\Omega f^2 d\mu\right) \log \left(\int_\Omega f^2 d\mu\right) \leqslant -\frac{2}{\lambda} \int_\Omega (\mathcal{L}f) f d\mu \tag{1}$$

holds for all real-valued functions $f \in L^2(\Omega,\mu)$ in the domain of the generator $\mathcal{L}$. A different version of (1), called a *modified log Sobolev inequality* $\mathrm{MLSI}(\lambda)$, states that for all positive functions $g \geqslant 0$

$$\int_\Omega g \log g d\mu - \left(\int_\Omega g d\mu\right) \log \left(\int_\Omega g d\mu\right) \leqslant -\frac{1}{2\lambda} \int_\Omega (\mathcal{L}g) \log g d\mu. \tag{2}$$

Recently much progress has been made in extending the MLSI to matrix-valued functions. For instance, H. Li, M. Junge and N. LaRacuente in [31] proved that if a compact Riemannian manifold (M,g) has the Ricci curvature bounded from below by a positive scalar λ, then for all $n \geqslant 1$ and all smooth $n \times n$ positive matrix-valued functions $f : M \to \mathbb{M}_n$

$$\int_M \mathrm{tr}(f \log f) d\mu - \mathrm{tr}\left(\mathbb{E}_\mu f \log \mathbb{E}_\mu f\right) \leqslant -\frac{1}{2\lambda} \int_M \mathrm{tr}\left((\mathrm{id}_{M_n} \otimes \Delta f) \log f\right) d\mu. \tag{3}$$

Here, Δ is the Laplace-Beltrami operator, tr is the standard matrix trace and $\mathbb{E}_\mu f = \int f(x) d\mu(x) \in \mathbb{M}_n$ is the matrix-valued mean with respect to the volume form $d\mu$. This extends the well-known Bakry-Émery theorem [1] to matrix-valued functions. Equation (3) is called a *complete modified log Sobolev inequality* (in short, $\mathrm{CMLSI}(\lambda)$), as it gives a uniform MLSI constant for all matrix-valued functions independent of a matrix size. Later M. Brannan, L. Gao and M. Junge in [11] proved that the heat semigroup on any compact Riemannian manifold satisfies $\mathrm{CMLSI}(\lambda)$, using a finite curvature lower bound λ that is not necessarily positive. In the discrete setting, CMLSIs [22,31] have also been established for finite Markov chains satisfying detail balanced condition, including graph Laplacians on a weighted undirected graph. These results have found applications to quantum Markov semigroups on matrix algebra (see [20, Section 4]).

Despite the progress for the heat semigroups and graph Laplacians, the CMLSI for sub-Laplacians in sub-Riemannian settings is still mostly open. The main obstruction

here is lack of geometric techniques such as curvature-dimension inequalities. We consider M to be an n-dimensional smooth manifold, and $\mathcal{D}$ is a sub-bundle of TM equipped with a metric g and with $\dim \mathcal{D} = k \leqslant n$. Recall that a second order differential operator $\mathcal{L}$ defined on $C^\infty(M)$ is called a *sub-Laplacian* if for every $x \in M$, there is a neighborhood U of x and a collection of smooth vector fields $\{X_0, X_1, ..., X_k\}$ defined on U such that

$$\mathcal{L} = -\sum_{i=1}^{k} X_i^* X_i + X_0,$$

on U and $\{X_1, ..., X_k\}$ are orthonormal with respect to the sub-Riemannian metric g. Here X_i^* is the adjoint of X_i with respect to a smooth probability measure μ. We assume that for $\{X_1, ..., X_k\}$ the Lie algebra generated by these vector fields spans the whole tangent space T_xM at any $x \in M$, and usually this is called a strong Hörmander condition. The operator satisfying a strong Hörmander condition is one of main objects in the study of sub-Riemannian geometry. The literature on such geometry is vast, starting with [46,47] and then covered in [42]. The new direction of introducing curvatures and applications to geometric analysis on sub-Riemannian manifolds has started with [7], while a more analytic and probabilistic description of such sub-Laplacians has been given in [23,24].

In this work, we will study the matrix-valued log Sobolev inequality for the operator $\mathcal{L} = -\sum_{i=1}^{k} X_i^* X_i$ as a generator of Markov semigroup, which adds a new direction with connections to quantum Markov semigroup in the noncommutative setting.

In the scalar-valued cases, P. Ługiewicz and B. Zegarliński [40] proved that on any connected and compact manifolds without boundary, LSI and hence MLSI hold for sub-Laplacians $\mathcal{L} = -\sum_i X_i^* X_i$ satisfying the strong Hörmander condition. One crucial step was the Rothaus lemma in [45],

$$\mathrm{Ent}(|f|^2) \leqslant \mathrm{Ent}(|f - \mathbb{E}_\mu(f)|^2) + \| f - \mathbb{E}_\mu(f) \|^2_{L_2(\mu)},$$

where $\mathrm{Ent}(g) = \int g \log g d\mu$ is the entropy functional. This approach, however, does not apply to CMLSI, because both the Rothaus lemma and LSI always fail for matrix-valued cases as pointed out in [20, Section 7.5]. On the other hand, compared to the result in [11] for the heat semigroup, the difficulty in the sub-Riemannian case stems from the lack of a Ricci curvature bound. Informally, at points of degeneracy of $\mathcal{L}$, the Ricci tensor is not well-defined and might be interpreted as being $-\infty$ in some directions. While there has been a number of results on generalized notions of curvature on some classes of sub-Riemannian manifolds (see e.g. [6–8]), we are not relying on those in this paper. Log Sobolev inequalities have been proven for some sub-Riemannian manifolds including Heisenberg groups and homogeneous spaces by M. Gordina and L. Luo in [25,26], with the techniques relying on particular structure of such groups.

In the current paper, we overcome this issue by relying on the gradient estimate (4) studied by [2,5,19,37,41] in the sub-Riemannian setting. We denote by $\nabla = (X_1, \cdots, X_k)$ the horizontal gradient operator associated to $\mathcal{L}$.

Theorem 1.1 *(Theorem 3.6). Let (M, g) be a Riemannian manifold without boundary, and let μ be a smooth measure on M. Suppose $\mathcal{L} = -\sum_{i=1}^{k} X_i^* X_i$ is a sub-Laplacian, where $\{X_1, ..., X_k\}$ satisfy Hörmander's condition, and denote by $P_t = e^{t\mathcal{L}}$ the symmetric Markov semigroup generated by $\mathcal{L}$. Suppose*

i) *there exist some $t_0 > 0$ and a positive measurable function $C : [0, t_0) \to \mathbb{R}_+$ such that for all $f \in C^\infty(M)$ and $0 \leqslant t < t_0$,*

$$|\nabla P_t f|^2 \leqslant C(t) P_t(|\nabla f|^2), \tag{4}$$

where $|\nabla f|^2 = \sum_i |X_i f|^2$.

ii) $\int_0^{t_0} C(s)ds < \infty$ *is finite.*

Then P_t satisfies CMLSI (λ) *for some positive λ.*

We note that if $C(t) = e^{-2\lambda t}$, condition (4) is exactly Bakry-Émery's (λ, ∞) curvature-dimension inequality. The results by B. Driver and T. Melcher in [19], T. Melcher in [41] and F. Baudoin-M. Bonnefont in [5] show that the gradient estimate (4) holds on SU (2), and on stratified Lie groups including Heisenberg group. In particular, on stratified Lie groups $C(t) \equiv C$ is a constant function. One of our key observations is that the gradient estimate (4) naturally extends to matrix-valued functions. Such an extension holds for diffusive semigroups given by Laplacian or sub-Laplacian but does not hold for noncommutative examples or even discrete spaces such as graph Laplacian.

As a corollary of our main result we obtain a CMLSI for the canonical sub-Laplacian on SU (2). Recall that the Lie algebra $\mathfrak{su}(2)$ is spanned by skew-Hermitian traceless matrices

$$X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, Z = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},$$

whose Lie brackets satisfy $[X, Y] = 2Z, [Y, Z] = 2X$, and $[Z, X] = 2Y$. In particular, it implies that $\{X, Y\}$ satisfies Hörmander's condition.

Corollary 1.2. *The sub-Laplacian $\mathcal{L} = X^2 + Y^2$ on* SU (2) *satisfies a* CMLSI (λ) *for some positive λ.*

The above result is the first ever example of CMLSI obtained for sub-Laplacians. It also has direct implications for the CMLSI constant for quantum Markov semi-

groups. Quantum Markov semigroups are noncommutative generalizations of Markov semigroups, where the underlying function spaces are replaced by matrix algebras or operator algebras. Mathematically, they are continuous semigroup of completely positive trace preserving map. Quantum Markov semigroups satisfy what is known as a GKLS equation or master equation as in [16], and they model the Markovian time evolution of an open quantum system. There is considerable interest in quantum information theory on the convergence rate of quantum Markov semigroup. Thanks to [27,39], it is known that if a quantum Markov semigroup on matrix algebra $\mathbb{M}_m$ is symmetric with respect to the matrix trace, then it admits the following Lindbladian form,

$$S_t = e^{\mathcal{L}t} : \mathbb{M}_m \to \mathbb{M}_m \ , \ \mathcal{L}(\rho) = \sum_{j=1} [a_j, [a_j, \rho]],$$

where a_j are some self-adjoint matrices. As an application of Corollary 1.2, we obtain a uniform lower bound of CMLSI constants for quantum Markov semigroups induced by the sub-Laplacian $\mathcal{L} = X^2 + Y^2$ via Lie algebra representations of $\mathfrak{su}(2)$.

Corollary 1.3. *There exists a positive constant λ such that for any Lie algebra homomorphism $\phi : \mathfrak{su}(2) \to i(\mathbb{M}_m)_{s.a.}$ into a matrix algebra $\mathbb{M}_m$, the quantum Markov semigroup*

$$\begin{aligned} &S_t = e^{\mathcal{L}_\phi t} : \mathbb{M}_m \to \mathbb{M}_m, \\ &\mathcal{L}_\phi(\rho) = [\phi(X), [\phi(X), \rho]] + [\phi(Y), [\phi(Y), \rho]] \end{aligned}$$

satisfies CMLSI (λ).

Motivated by the application of L^2-gradient estimate (4) to CMLSI, we also investigate whether L^1-gradient estimate

$$|\nabla P_t(f)| \leqslant C(t) P_t |\nabla(f)| \tag{5}$$

can be used to derive a CMLSI. Let p_t be the heat kernel measure corresponding to the semigroup P_t at time t. In the scalar-valued case, it is known that (5) implies that the corresponding heat kernel measure satisfies a LSI with the constant given by

$$\kappa^{-1} = \int_0^t C(s)^2 ds.$$

This corresponds to the associated Ornstein–Uhlenbeck semigroup $O_s = e^{L_t s}$, where $L_t = -\nabla^* \nabla$ is the symmetric generator on $L^2(dp_t)$. Our last result shows this approach works for CMLSI on Lie groups.

Theorem 1.4 *(Theorem 4.1). Let G be a locally compact Lie group equipped with a left-invariant metric. Let $H_t = e^{\Delta t}$ be the heat semigroup and denote by μ_t the heat kernel*

measure. Then the Ornstein–Uhlenbeck semigroup $O_s = e^{sL_t}$ *satisfies* $\frac{1}{4t}$*-CMLSI, where* $L_t = \nabla^*\nabla$ *is the generator of this semigroup symmetric with respect to the measure* μ_t*.*

The rest of the paper is organized as follows. In Section 2, we review some preliminaries on CMLSI and sub-Laplacians. In Section 3 we prove our main Theorem 3.6 and apply it to SU (2). Section 4 discusses a CMLSI for the heat kernel measures on Lie groups. We end the paper with some discussion on connections of our results to quantum Markov semigroups and list some open questions.

Note 1.5. After submission of this paper, L. Gao, M. Junge, N. LaRacunte and H. Li gave a version of Theorem 1.1 in a preprint [21] using a completely different approach. They showed that a one point ultra-contractivity and a Poincaré inequality imply a CMLSI, which applies to a large class of examples.

Notation. Throughout the paper, we denote by (Ω, μ) a measure space equipped with a probability measure, and by $L^p(\Omega)$ the corresponding L^p-space of complex-valued functions for $1 \leqslant p \leqslant \infty$. Then $\| f \|_p$ is the standard L^p-norm and $\langle f, g\rangle = \int_\Omega \overline{f} g d\mu$ is the L^2-inner product. By $\mathbb{M}_n$ we denote the space of $n \times n$ complex matrices and by tr the standard matrix trace. The identity elements, which is the constant function 1 for $n = 1$ and the identity operator in $\mathbb{M}_n$, is denoted by $\mathbf{1}$, and the identity map between spaces is denoted as id.

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2. Preliminaries

2.1. Logarithmic Sobolev inequalities

We first recall logarithmic Sobolev inequalities for Markov semigroups and its matrix-valued extension. Let (Ω, μ) be a measure space equipped with a probability measure μ. We say that $P : L^\infty(\Omega) \to L^\infty(\Omega)$ is a *Markov map* if

i) $P_t(1) = 1$ (mass conservation);
ii) $P_t(f) \geqslant 0$ if $f \geqslant 0$ (positivity preserving);

A *Markov semigroup* $(P_t)_{t\geqslant 0} : L^\infty(\Omega) \to L^\infty(\Omega)$ is a family of Markov maps satisfying

i) $P_0 = \mathrm{id}$ and $P_s \circ P_t = P_{s+t}$ for $s, t \geqslant 0$ (semigroup property);
ii) For every $f \in L^2$, $P_t f$ converges to f in L^2) as $t \to 0$ (continuity property).

The generator of P_t is given by

$$\mathcal{L}f = \lim_{t\to 0} \frac{P_t f - f}{t} \ , \ P_t = e^{\mathcal{L}t} \ ,$$

with the domain $\mathrm{dom}(\mathcal{L})$ being the space of functions such that the above limit exists. Throughout the paper, we consider the semigroups which are symmetric with respect to a unique invariant measure μ, i.e. $\langle P_t(f), g\rangle_{L^2(\mu)} = \langle f, P_t(g)\rangle_{L^2(\mu)}$ for any $t \geqslant 0$. Namely, each P_t is a symmetric operator on $L^2(\Omega)$ and hence $\int_\Omega P_t(f)d\mu = \int_\Omega f d\mu$. In this case, $\mathcal{L}$ is a negative operator on $L^2(\Omega)$ and P_t is equivalently determined by its Dirichlet form

$$\mathcal{E}(f,g) = \int_\Omega \bar{f}\mathcal{L}(g)d\mu$$

whose domain is $\mathrm{dom}(\mathcal{E}) = \mathrm{dom}((-\mathcal{L})^{1/2})$.

Definition 2.1. For $\lambda > 0$, a Markov semigroup $P_t = e^{t\mathcal{L}}$ is said to satisfy

i) λ-*Poincaré inequality* $\mathrm{PI}(\lambda)$ if for any $f \in \mathrm{dom}(\mathcal{E})$

$$\lambda \,\|\, f - \mathbb{E}_\mu f \,\|_2 \leqslant 2\mathcal{E}(f,f), \tag{6}$$

where $\mathbb{E}_\mu f = \int_\Omega f d\mu$.

ii) λ-*logarithmic Sobolev inequality* $\mathrm{LSI}(\lambda)$ if for any $f \in \mathrm{dom}(\mathcal{E})$,

$$\int_\Omega |f|^2 \log |f|^2 d\mu - \left(\int_\Omega |f|^2 d\mu\right) \log\left(\int_\Omega |f|^2 d\mu\right) \leqslant \frac{2}{\lambda}\mathcal{E}(f,f); \tag{7}$$

iii) λ-*modified logarithmic Sobolev inequality* $\mathrm{MLSI}(\lambda)$ if for any positive $g \in \mathrm{dom}(\mathcal{L})$,

$$\int_\Omega g \log g \ d\mu - \left(\int_\Omega g \ d\mu\right) \log\left(\int_\Omega g \ d\mu\right) \leqslant \frac{1}{2\lambda}\int_\Omega (-\mathcal{L}g)\log g d\mu \ . \tag{8}$$

The λ-Poincaré inequality is equivalent to that the negative generator $(-\mathcal{L})$ has a spectral gap λ. The LSI is an equivalent formulation of hypercontractivity stating that

$$\|P_t\|_{L^2(\Omega)\to L^p(\Omega)} \leqslant 1 \ \text{ for } \ t \leqslant 1 + e^{2\lambda t}.$$

The MLSI is known to describe the *entropy decay*

$$\mathrm{Ent}(P_t g) \leqslant e^{-2\lambda t}\,\mathrm{Ent}(g), \tag{9}$$

where $\mathrm{Ent}(f) = \int_\Omega f \log f d\mu$ is the *entropy functional*. Note that the right side of (8),

$$I(g) := -\int (\mathcal{L}g) \log g \, d\mu = -\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ent}(P_t g)$$

is called the *Fisher information* (also called entropy production), as it is the negative derivative of entropy along the time t.

The exponential entropy decay (9) in particular implies mixing time in L^1 via Pinsker's inequality (cf. [49, Appendix])

$$\frac{1}{2} \| f - 1 \|_1^2 \leqslant \sqrt{\mathrm{Ent}(f)}.$$

It is well-known (cf. [3, Theorem 5.2.1]) that λ-LSI implies λ-MLSI, and they are equivalent if the semigroup is diffusive. Recall that a *carré du champ* operator or the gradient form of $\mathcal{L}$ is

$$\Gamma(f,g) := \frac{1}{2}(\mathcal{L}(fg) - f(\mathcal{L}g) - (\mathcal{L}f)g)$$

or weakly defined as $\langle h, \Gamma(f,g)\rangle = \frac{1}{2}(\mathcal{E}(hf,g) + \mathcal{E}(f,gh) - \mathcal{E}(h,fg))$. We will often use the short notation $\Gamma(f) := \Gamma(f,f)$. P_t is called *diffusive* if the gradient form satisfies the following product rule that

$$\Gamma(fg,h) = f\Gamma(g,h) + \Gamma(f,h)g \, .$$

In this paper, we study MLSIs for matrix-valued functions. This is motivated by the study of quantum Markov semigroups in the noncommutative analysis and quantum information theory. Both LSI and MLSI enjoy the tensorization property, e.g. [3, Section 7.6.3]: if two Markov semigroups T_t and S_t both satisfy (M)LSI, so does their tensor product semigroup $T_t \otimes S_t$. More precisely, if we denote λ_{LSI} (resp. λ_{MLSI}) for optimal constant such (7) (resp. (8)) holds, then tensorization property states that

$$\lambda_{\mathrm{LSI}}(T_t \otimes S_t) = \min\{\lambda_{\mathrm{LSI}}(T_t), \lambda_{\mathrm{LSI}}(S_t)\}$$

and the similar equality holds for λ_{MLSI}.

Nevertheless, in the noncommutative setting when the semigroup describes a quantum system modeled by matrix algebras or operator algebras, tensorization property fails for non-primitive semigroup (non-unique invariant states [11]) and is largely unknown for primitive cases (see [9,35] for positive results on 2-dimensional matrices). It turns out the above tensorization property in noncommutative setting hold for a stronger version of MLSI that is uniform for all its matrix-valued amplification $T_t \otimes \mathrm{id}_{\mathbb{M}_n}$. Here and in the following, we denote by $\mathbb{M}_n$ of $n \times n$ complex matrix algebra and tr for the standard

matrix trace. We say $f : \Omega \to \mathbb{M}_n$ is a matrix-valued density if at each $\omega \in \Omega$, $f(\omega) \geqslant 0$ is positive semi-definite and $\int_\Omega \operatorname{tr}(f) d\mu = 1$.

Definition 2.2. We say a Markov semigroup $T_t : L^\infty(\Omega) \to L^\infty(\Omega)$ satisfies a *complete modified logarithmic Sobolev inequality* CMLSI (λ) with $\lambda > 0$ if

$$\int_\Omega \operatorname{tr}(f \log f) d\mu - \operatorname{tr}\Big(\mathbb{E}_\mu f \log(\mathbb{E}_\mu f)\Big) \leqslant \frac{1}{2\lambda} \int \operatorname{tr}\Big((\mathrm{id}_{\mathbb{M}_n} \otimes \mathcal{L}) f \log f\Big) d\mu \tag{10}$$

for all $n \in \mathbb{N}^+$ and matrix-valued density f.

Here $\mathbb{E}_\mu f = \int_\Omega f d\mu \in \mathbb{M}_n$ is the matrix-valued mean, and $f \log f$ is interpreted as the matrix-valued function that at each point $\omega \in \Omega$, $f \log f(\omega) = f(\omega) \log f(\omega)$ is the pointwise functional calculus of $f(\omega)$ (similarly for $\mathbb{E}_\mu f \log \mathbb{E}_\mu f$). In other words, T_t satisfies CMLSI (λ) if for all $n \geqslant 1$, the matrix-valued semigroup $T_t \otimes \mathrm{id}_{M_n}$ satisfies λ-MLSI. The left-hand side in (10) is the relative entropy of f to its matrix-valued mean $\mathbb{E}_\mu f$ as a constant function

$$D(f||\mathbb{E}_\mu f) := \int_\Omega \operatorname{tr}\Big(f \log f - \mathbb{E}_\mu f \log(\mathbb{E}_\mu f)\Big) d\mu = \int_\Omega D(f(\omega)||\mathbb{E}_\mu f) d\mu(\omega) \,, \tag{11}$$

which is a mixture of classical relative entropy $D(f||g) = \int f \log f - f \log g \, d\mu$ for density function $f, g \in L^1(\Omega)$ and quantum relative entropy $D(\rho||\sigma) = \operatorname{tr}(\rho \ln \rho - \rho \ln \sigma)$ for density operators $\rho, \sigma \in \mathbb{M}_n$. The right-hand side of (10) is again the Fisher information

$$I(f) := -\frac{d}{dt} D\big((\mathrm{id}_{\mathbb{M}_n} \otimes T_t) f||\mathbb{E}_\mu f\big)|_{t=0} = - \int \operatorname{tr}\big((\mathrm{id}_{\mathbb{M}_n} \otimes \mathcal{L}) f \log f\big) d\mu \,.$$

Note that here $\mathbb{E}_\mu f$ as a constant matrix-valued function is invariant under the amplified semigroup $T_t \otimes \mathrm{id}_{\mathbb{M}_n}$. Then (10) is equivalent to the convergence of $T_t f$ to $\mathbb{E}_\mu f$ as an equilibrium state in terms of entropy

$$D((T_t \otimes \mathrm{id}_{\mathbb{M}_n}) f||\mathbb{E}_\mu f) \leqslant e^{-2\lambda t} D(f||\mathbb{E}_\mu f) \,.$$

2.2. Sub-Laplacians

Let (M, g) be a d-dimensional Riemannian manifold without boundary and $H = \{X_i\}_{i=1}^k$ with $k \leqslant d$ be a family of vectors fields. Let $d\mu = \rho d\,\mathrm{vol}$ be a probability measure with smooth density ρ w.r.t. the volume form. Denote $\nabla = (X_1, \cdots, X_k)$ and by X_i^* the adjoint of X_i on $L^2(M, d\mu)$. The sub-Laplacian

$$\mathcal{L} = -\nabla^* \nabla = -\sum_i X_i^* X_i = \sum_i X_i^2 + \mathrm{div}_\mu(X_i) X_i$$

is a symmetric generator on $L^2(M,\mu)$. Here $\mathrm{div}_\mu(X)$ is the divergence of X w.r.t. to μ and $\mathcal{L}$ depends both on the family H and the measure μ. The horizontal *gradient form* is

$$\Gamma(f,g)=\sum_i \langle X_i f, X_i g\rangle \ , \ \Gamma(f):=\Gamma(f,f)=\sum_i |X_i f|^2=|\nabla f|^2$$

It follows from the product rule that Γ is diffusive, and the Fisher information can be rewritten as

$$I(f):=-\int_\Omega (\mathcal{L}f)\log f \ d\mu=\langle \nabla f, \nabla \log f\rangle=\int_\Omega \frac{|\nabla f|^2}{f}d\mu \ ,$$

where we used the chain rule $\nabla(\log f)=f^{-1}(\nabla f)$. Throughout the paper, we will use the short notation $\mathcal{L}f$ for $(\mathcal{L}\otimes \mathrm{id}_{\mathbb{M}_n})f$ for the matrix-valued function f, and similarly for ∇ and $P_t=e^{\Delta t}$. Recall the noncommutative chain rule that for a positive operator A and derivation δ

$$\delta(\log A)=\int_0^\infty (A+s\mathbf{1})^{-1}\delta(A)(A+s\mathbf{1})^{-1}ds \ , \tag{12}$$

where $\mathbf{1}$ is the identity operator. Then for a matrix-valued density f, the entropy production can be rewritten as

$$I(f)=-\int \mathrm{tr}\Big((\mathcal{L}f)\log f\Big)=\langle \nabla f,\nabla \log f\rangle_{\mathrm{tr}}=\int_0^\infty \langle \nabla f,(f+s\mathbf{1})^{-1}(\nabla f)(f+s\mathbf{1})^{-1}\rangle_{\mathrm{tr}}ds \ .$$

Here and in the following, we denote by $\langle\cdot,\cdot\rangle_{\mathrm{tr}}$ the integral-trace inner product for two families of matrix-valued functions (f_i) and (g_i)

$$\langle (f_i),(g_i)\rangle_{\mathrm{tr}}=\sum_{i=1}^n \int \mathrm{tr}(f_i^* g_i)d\mu \tag{13}$$

The identity element $\mathbf{1}\in L^\infty(M,\mathbb{M}_n)$ is the constant function on M of the identity matrix, and $(f+s\mathbf{1})^{-1}$ is the pointwise inverse matrix.

Throughout the paper, we will assume that the family of vector fields of $H=\{X_i\}_{i=1}^k$ satisfies *Hörmander's condition*, that is, at every point $x\in M$ the tangent space at x is spanned by the iterated Lie brackets of X_is

$$T_xM=\mathrm{span}\{[X_{i_1},[X_{i_2},\cdots,[X_{i_{n-1}},X_{i_n}]]],1\leqslant i_1,i_2\cdots i_n\leqslant k\}.$$

By compactness we can assume there is a global constant l_X such that for every point $x\in M$, we only need at most l_Xth iterated Lie bracket in above expression.

It follows that $\ker \mathcal{L} = \{\mathbb{C}1\}$ and μ is the unique invariant measure such that $\int P_t f \, d\mu = \int f \, d\mu$. Moreover, by a celebrated theorem of Hörmander, $\mathcal{L}$ is hypoelliptic. Indeed, we have the following Sobolev-type inequality (see e.g. [40, Lemma 2.1])

$$\|f\|_q \leq C\big(\langle \mathcal{L}f, f\rangle + \|f\|_2^2\big)^{1/2}, \tag{14}$$

where $q = \frac{2dl_X}{dl_X - 2} > 2$. By Varopoulos' Theorem (see [48, Chapter 2]) on the dimension of semigroups, this implies the following ultra-contractivity property for $P_t = e^{\mathcal{L}t}$

$$\|P_t\|_{L^1(M,\mu)\to L^\infty(M,\mu)} \leqslant C' t^{-m/2} \text{ for } 0 < t \leqslant 1, \tag{15}$$

where $m = dl_x$. The Sobolev-type inequality (14) and the ultra-contractivity (15) were used in [40] to prove that every sub-Laplacian $\mathcal{L} = -\sum_i X_i^* X_i$ with Hörmander condition on a compact manifold satisfies LSI (hence equivalently MLSI). Their proof relies on the Rothaus lemma

$$\mathrm{Ent}(f^2) \leqslant \mathrm{Ent}(\mathring{f}^2) + 2\,\|\mathring{f}\|_2^2,$$

where $\mathring{f} = f - \mathbb{E}_\mu f$ is the mean zero part of f, which is a standard tool to improve a defective logarithmic Sobolev inequality to a standard one. Nevertheless, the Rothaus lemma is known to fail for matrix-valued functions [20, Section 7.5], hence such an argument does not apply to the matrix-valued case.

3. Gradient estimates and Fisher information

3.1. Complete gradient estimates

Let (M, g) be a Riemannian manifold. We consider a sub-Laplacian $\mathcal{L} = -\sum_{i=1}^k X_i^* X_i$, where $\{X_1, ..., X_k\}$ satisfy Hörmander's condition. Denote $P_t := e^{\mathcal{L}t}$ and $\nabla(f) := (X_1 f, \cdots, X_k f)$. The key tool in our argument is the following L^2-gradient estimate

$$|\nabla P_t f|^2 \leqslant C(t) P_t(|\nabla f|^2), \ f \in C_c^\infty(M) \tag{16}$$

for some function $C(t)$. In terms of the gradient form, (16) can be rewritten as

$$\Gamma(P_t f, P_t f) \leqslant C(t) P_t \Gamma(f, f). \tag{17}$$

This is closely related to the Bakry-Émery curvature-dimension condition. It was shown in [1] that if the Ricci curvature tensor of (M, g) has a uniform lower bound $\lambda \in \mathbb{R}$, then (17) is satisfied with $C(t) = e^{-2\lambda t}$. In the sub-Riemannian case, although the Ricci tensor should be interpreted as being $-\infty$ at the points where L is degenerate, the gradient estimate (17) might hold for a function $C(t)$ other than the exponential function $e^{-\lambda t}$.

Our first result shows that the gradient estimate (17) automatically extends to matrix-valued functions. We denote by $C_c^\infty(M,\mathbb{M}_n)$ the space of smooth $\mathbb{M}_n$-valued functions with compact support. For $f,g\in C_c^\infty(M,\mathbb{M}_n)$, $f\geqslant g$ means that at each point $x\in M$, $f(x)-g(x)\in(\mathbb{M}_n)_+$.

Proposition 3.1. *Let (M,g) be a Riemannian manifold with a sub-Laplacian operator $\mathcal{L}=-\sum_{i=1}^k X_i^*X_i$ and let $P: L^\infty(M,d\mu)\to L^\infty(M,d\mu)$ be a Markov map preserving μ. Suppose for some constant C and all* **scalar-valued** *functions $f\in C_c^\infty(M)$*

$$\Gamma(Pf,Pf)\leqslant CP\Gamma(f,f)\,.$$

Then

i) *for any family of scalar-valued functions $\{f_1,\cdots,f_n\}\subset C_c^\infty(M)$,*

$$\big[\Gamma(Pf_i,Pf_j)\big]_{i,j=1}^n\leqslant C[P\Gamma(f_i,f_j)]_{i,j=1}^n\,.$$

Here $[\Gamma(Pf_i,Pf_j)]_{i,j}$ and $[P\Gamma(f_i,f_j)]_{i,j}$ are viewed as elements in $C_c^\infty(M,\mathbb{M}_n)$.

ii) *Let $0\leqslant s\leqslant 1$. For any matrix-valued function $f,A,B\in C_c^\infty(M,\mathbb{M}_n)$ with $A,B\geqslant 0$,*

$$\langle\nabla Pf,A^s(\nabla Pf)B^{1-s}\rangle_{\mathrm{tr}}\leqslant C\langle\nabla f,(P^\dagger A)^s(\nabla f)(P^\dagger B)^{1-s}\rangle_{\mathrm{tr}},$$

where $P^\dagger$ is the adjoint map of P on $L^2(M,d\mu)$, and $\langle\cdot,\cdot\rangle_{\mathrm{tr}}$ is the inner product defined by (13). In particular, for any $f=(f_{ij})_{i,j=1}^n\in C_c^\infty(M,\mathbb{M}_n)$,

$$\Gamma(Pf,Pf)\leqslant CP\Gamma(f,f).$$

Here $\Gamma(f,f)=\big[\sum_{l=1}^n\Gamma(f_{li},f_{lj})\big]_{i,j}\in C_c^\infty(M,\mathbb{M}_n)$ and similarly for $\Gamma(Pf,Pf)$.

Proof. We use the standard bra-ket notation $|h\rangle$ for a vector in $\mathbb{C}^n$ and $\langle h|$ for the dual vector. Let $\{|i\rangle\}_{i=1}^n$ be an orthonormal basis of $\mathbb{C}^n$. For i), we have for any vector $|h\rangle=\sum_i h_i|i\rangle\in\mathbb{C}^n$,

$$\begin{aligned}\langle h|\big[\Gamma(Pf_i,Pf_j)\big]|h\rangle=&\sum_{i,j}\bar h_ih_j\Gamma(Pf_i,Pf_j)=\Gamma(Pf_h,Pf_h)\\ \leqslant&CP\Gamma(f_h,f_h)=C\sum_{i,j}\bar h_ih_jP\Gamma(f_i,f_j)=C\langle h|[P\Gamma(f_i,f_j)]|h\rangle,\end{aligned}$$

where $f_h=\sum_i h_if_i\in C^\infty(M)$ and the inequality holds pointwise for each $x\in M$. For ii), we write $A^s=\sum_{i,j}A^s_{i,j}(x)|i\rangle\langle j|$ and $B^{1-s}=\sum_{k,l}B^{1-s}_{k,l}(x)|k\rangle\langle l|$. Note that $A^s_{i,j}(x)$ is the coefficient function for A^s not the s-power of $A_{i,j}(x)$.

For $f = \sum_{i,j} f_{i,j}|i\rangle\langle j|$ we have that for each $x \in M$

$$\begin{aligned}
&\operatorname{tr}\Big(\big(\nabla Pf(x)\big)^* A^s(x)\big(\nabla Pf(x)\big)B^{1-s}(x)\Big)\\
=&\sum_m \sum_{i,j,k,l} A^s_{i,j}(x)B^{1-s}_{k,l}(x)\langle l|(X_m Pf(x))^*|i\rangle\langle j|(X_m Pf(x))|k\rangle\\
=&\sum_m \sum_{i,j,k,l} A^s_{i,j}(x)B^{1-s}_{k,l}(x)\overline{X_m Pf_{i,l}(x)}X_m Pf_{j,k}(x) \qquad (18)\\
=&\sum_{i,j,k,l} A^s_{i,j}(x)B^{1-s}_{k,l}(x)\Gamma(Pf_{i,l}, Pf_{j,k})(x)\\
\leqslant& C\sum_{i,j,k,l} A^s_{i,j}(x)\overline{B^{1-s}_{l,k}}(x)P\Gamma(f_{i,l}, f_{j,k})(x) \qquad (19)
\end{aligned}$$

where the equality (18) follows from the fact that the evaluation $\langle i|\cdot|j\rangle$ is a linear functional, hence commutes with ∇ and P. Since B and B^{1-s} are pointwise positive in $C^\infty(M, \mathbb{M}_n)$, we have $B^{1-s}_{k,l}(x) = \overline{B^{1-s}_{l,k}}(x)$ for every x. Then the inequality (19) follows from the assumption that

$$\big[\Gamma(Pf_{i,l}, Pf_{j,k})\big]_{il,jk} \leqslant \big[P\Gamma(f_{i,l}, f_{j,k})\big]_{il,jk}$$

and $\big[A^s_{i,j}\overline{B^{1-s}_{l,k}}\big]_{il,jk}$ is a positive matrix in terms of indices (il, jk). Let

$$P(f)(x) = \int_M f(y)dm(x,y)$$

be the kernel representation. The adjoint is then $P^\dagger(f)(y) = \int_M f(x)dm(x,y)$. Integrating (19) over M, we have

$$\begin{aligned}
\big\langle(\nabla Pf), A^s(\nabla Pf)B^{1-s}\big\rangle_{\mathrm{tr}} =& \int_M \operatorname{tr}\Big(\big(\nabla Pf(x)\big)^* A^s(x)\big(\nabla Pf(x)\big)B^{1-s}(x)\Big)d\mu(x)\\
\leqslant& C\int_M \sum_{i,j,k,l} A^s_{i,j}(x)\overline{B^{1-s}_{l,k}}(x)P\Gamma(f_{i,l}, f_{j,k})(x)d\mu(x)\\
=& C\int_M \sum_{i,j,k,l} P^\dagger(A^s_{i,j}\overline{B^{1-s}_{l,k}})(y)\Gamma(f_{i,l}, f_{j,k})(y)d\mu(y)\\
=& C\sum_{i,j,k,l}\int_M \Big(\int_M A^s_{i,j}(x)B^{1-s}_{k,l}(x)dm(x,y)\Big)\Gamma(f_{i,l}, f_{j,k})(y)d\mu(y)\\
=& C\int_M \Big(\int_M \operatorname{tr}\big((\nabla f(y))^* A^s(x)(\nabla f(y))B^{1-s}(x)\big)dm(x,y)\Big)d\mu(y)
\end{aligned}$$

$$\begin{aligned}&\leqslant C\int_M \operatorname{tr}\Big((\nabla f(y))^*(P^\dagger A(y))^{1-s}(\nabla f(y))(P^\dagger B(y))^s\Big)d\mu(y)\\&= C\langle \nabla f, (P^\dagger A)^s(\nabla f)(P^\dagger B)^{1-s}\rangle_{\mathrm{tr}},\end{aligned}$$

where the last inequality used the Lieb concavity in [38] for the function

$$(A,B)\mapsto \operatorname{tr}(K^*A^sKB^{1-s})$$

is joint concave for $(A,B)\in \mathbb{M}_m$. The last assertion follows from choosing $s=0$ or $s=1$. □

Lemma 3.2. *Under the assumptions of Proposition 3.1, if in addition $[P,\mathcal{L}]=0$, then for any matrix-valued density ρ*

$$I(P^\dagger\rho)\leqslant CI(\rho)\ . \tag{20}$$

Proof. Recall that for a matrix-valued function $f\in C^\infty(M,\mathbb{M}_n)$

$$\nabla f=(X_if)_{i=1}^k\in\bigoplus_{i=1}^k L^\infty(M,\mathbb{M}_n)\cong L^\infty(M,\oplus_{i=1}^k\mathbb{M}_n)\ ,$$

where the inner product is defined as

$$\langle\nabla f,\nabla g\rangle_{\mathrm{tr}}=\sum_{i=1}^k\int_M \operatorname{tr}\Big((X_if)^*(x)(X_ig)(x)\Big)d\mu(x)\ .$$

Given a matrix-valued density function ρ, define the operator

$$\begin{aligned}&M_\rho: L^2(M,\oplus_{i=1}^k\mathbb{M}_n)\longrightarrow L^2(M,\oplus_{i=1}^k\mathbb{M}_n)\ ,\\&M_\rho\left((f_i)_{i=1}^k\right):=\left(\int_0^1\rho^sf_i\rho^{1-s}ds\right)_{i=1}^k\ ,\\&K_\rho=\nabla^*M_\rho\nabla: L^2(M,\mathbb{M}_n)\to L^2(M,\mathbb{M}_n).\end{aligned}$$

For simplicity, we assume that $\rho\in C^\infty(M)$ satisfies $\mu_2\mathbf{1}\leqslant\rho\leqslant\mu_1\mathbf{1}$ for some $0<\mu_1<\mu_2$. Then $\mu_1\,\mathrm{id}\leqslant M_\rho\leqslant\mu_2\,\mathrm{id}$ is a bounded positive operator on $L^2(M,\oplus_{i=1}^k\mathbb{M}_n)$. In this case, K_ρ has the same domain and kernel as $L=-\nabla^*\nabla$. Then Proposition 3.1 implies that

$$P^\dagger K_\rho P\leqslant CK_{P^\dagger\rho} \tag{21}$$

as a positive operator on $L^2(M,\mathbb{M}_n)$. Note that here $K_{P^\dagger\rho}$ is a positive operator because the adjoint $P^\dagger$ is also a positive map. Thus we can define the inverse K_ρ^{-1} as a densely

defined operator on the support $\ker(L)^\perp = \{f - \mathbb{E}_\mu f \mid f \in L^2(M, \mathbb{M}_n)\}$. Then, since $P(\ker(L)^\perp) \subset \ker(L)^\perp$ (namely, $P(\mathbf{1}) = \mathbf{1}$) we have that the following equivalence from [44, Lemma 2]

$$\begin{aligned} & P^\dagger K_\rho P \leqslant C K_{P^\dagger \rho} && (22)\\ \iff & K_{P^\dagger\rho}^{-1/2} P^\dagger K_\rho P K_{P^\dagger\rho}^{-1/2} \leqslant C && (23)\\ \iff & \| K_{P^\dagger\rho}^{-1/2} P^\dagger K_\rho^{1/2} \| \leqslant \sqrt{C} \\ \iff & K_\rho^{1/2} P^\dagger K_{P^\dagger\rho}^{-1} P K_\rho^{1/2} \leqslant C \\ \iff & P K_{P^\dagger\rho}^{-1} P^\dagger \leqslant C K_\rho^{-1}. && (24) \end{aligned}$$

Recall that we can represent $M_\rho(f)$ as the double operator integral

$$M_\rho(f) = \int_0^1 \rho^s f \rho^{1-s} ds = \int_0^\infty \int_0^\infty \frac{\lambda - \mu}{\log \lambda - \log \mu} dE_\lambda(\rho)(f) dE_\mu(\rho) \,,$$

where $E_\lambda(\rho)$ is the spectral resolution of ρ. Then

$$M_\rho^{-1}(f) = \int_0^\infty \int_0^\infty \frac{\log \lambda - \log \mu}{\lambda - \mu} dE_\lambda(\rho)(f) dE_\mu(\rho) = \int_0^\infty (\rho + s\mathbf{1})^{-1} f (\rho + s\mathbf{1})^{-1} ds.$$

Therefore by the noncommutative chain rule (12)

$$K_\rho(\ln \rho) = \nabla^* M_\rho \nabla(\ln \rho) = \nabla^* M_\rho M_\rho^{-1} \nabla(\rho) = -\mathcal{L}(\rho).$$

Then for mean zero elements $-\mathcal{L}\rho \in \ker(L)^\perp$ we have

$$\begin{aligned} & \langle (-\mathcal{L}\rho), K_\rho^{-1}(-\mathcal{L}\rho) \rangle_{\mathrm{tr}} \\ =& \langle (-\mathcal{L}\rho), K_\rho^{-1} K_\rho(\ln \rho) \rangle_{\mathrm{tr}} \\ =& \langle -\mathcal{L}\rho, \ln \rho \rangle_{\mathrm{tr}} \\ =& I(\rho) \,. \end{aligned}$$

Similarly, since $[P, -\mathcal{L}] = 0$

$$\begin{aligned} & \langle (-\mathcal{L}\rho), P K_{P^\dagger\rho}^{-1} P^\dagger (-\mathcal{L}\rho) \rangle \\ =& \langle -\mathcal{L}P^\dagger\rho, K_{P^\dagger\rho}^{-1}(-\mathcal{L}P^\dagger\rho) \rangle \\ =& I(P^\dagger\rho). \end{aligned}$$

Thus by (22) we have

$$I(P^\dagger \rho) \leqslant C I(\rho)\ .$$

That completes the proof. □

We now show that the Fisher information inequality in Lemma 3.2 is generally sufficient to derive an CMLSI.

Theorem 3.3. *Let (M, g) be a Riemannian manifold, $L = -\sum_{i=1}^k X_i^* X_i$ is a sub-Laplacian and $P_t = e^{tL}$ is a symmetric Markov semigroup on $L^2(M, \mu)$ with the infinitesimal generator L. Suppose there exists a positive function $C : [0, \infty) \to \mathbb{R}_+$ such that*

i) *for all scalar-valued functions $f \in \mathrm{dom}(\Gamma)$ and $t \geqslant 0$,*

$$\Gamma(P_t f, P_t f) \leqslant C(t) P_t \Gamma(f, f) \tag{25}$$

ii) $\kappa := \int_0^\infty C(s) ds < \infty$.

Then P_t satisfies CMLSI (λ) *with $\lambda = \frac{1}{2\kappa}$.*

Proof. Assumption i) allows us to apply Proposition 3.1 and Lemma 3.2 to see that for any bounded and invertible matrix-valued density function ρ,

$$I(P_t \rho) \leqslant C(t) I(\rho)\ .$$

Then, using the identity

$$I(P_t \rho) = -\frac{d}{dt} \left(D(P_t \rho || \mathbb{E}_\mu \rho) \right)$$

and $P_0(\rho) = \rho$, $\lim_{t \to \infty} P_t(\rho) = \mathbb{E}_\mu \rho$,

$$\begin{aligned} &\int_\Omega \mathrm{tr}(\rho \log \rho - \mathbb{E}_\mu \rho \log \mathbb{E}_\mu \rho) d\mu = D(\rho || \mathbb{E}_\mu \rho) \\ &= \int_0^\infty I(P_t \rho) dt \leqslant \int_0^\infty C(t) I(\rho) dt = \kappa I(\rho). \end{aligned}$$

This proves CMLSI for bounded and invertible ρ. The general case follows by approximation, see e.g. [11, Appendix]. □

Theorem 3.3 can be regarded as a matrix-valued extension of Bakry-Émery's approach. Indeed, in [1] they proved that if Ricci $\geqslant \lambda$ then for a scalar-valued function

$$\Gamma(P_t f, P_t f) \leqslant e^{-2\lambda t} P_t \Gamma(f, f),$$

and therefore

$$I(P_t f) \leqslant e^{-2\lambda t} I(f).$$

Thus P_t satisfies CMLSI (λ), where

$$\lambda = \frac{2}{\kappa} = 2\left(\int_0^\infty e^{-2\lambda t} dt\right)^{-1}.$$

This approach works if and only if $e^{-2\lambda t}$ is integrable, in other words, the lower curvature bound λ is positive. Recently this method has been extended to a non-positive curvature regime, where

$$I(P_t f) \leqslant e^{2\lambda t} I(f) \text{ for some } \lambda > 0,$$

and the upper bound is not integrable. The idea is to use L^∞-mixing time defined for $0 < \varepsilon < 1$

$$t(\varepsilon) := \inf\{t > 0 : \|P_t - \mathbb{E}_\mu\|_{L^1(\Omega) \to L^\infty(\Omega)} \leqslant \varepsilon\}.$$

The L^∞-mixing time approach was used in [18] to prove an LSI via hypercontractivity. It was proved in [11] that at time $t(\varepsilon)$, for any matrix-valued density ρ the relative entropy (11) from $P_{t(\varepsilon)}\rho$ to the mean $\mathbb{E}_\mu \rho$ is at most ε times the initial relative entropy, i.e.

$$D(P_{t(\varepsilon)}\rho || \mathbb{E}_\mu \rho) \leqslant \varepsilon D(\rho || \mathbb{E}_\mu \rho)\,. \tag{26}$$

Using this, we have the following version of Theorem 3.3.

Theorem 3.4. *Let (M, g) be a Riemannian manifold with a sub-Laplacian $\mathcal{L} = -\sum_{i=1}^k X_i^* X_i$ and let $P_t = e^{t\mathcal{L}}$ be the symmetric Markov semigroup generated by L. Suppose*

i) *for some $0 < \varepsilon < 1$, the L^∞-mixing time is finite*

$$t(\varepsilon) := \inf\{t > 0 : \|P_t - \mathbb{E}_\mu\|_{L^1(\Omega) \to L^\infty(\Omega)} \leqslant \varepsilon\} < \infty.$$

ii) *there exists a positive function $C : [0, t(\varepsilon)) \to \mathbb{R}_+$ such that for all scalar-valued functions $f \in \mathrm{dom}(\Gamma)$ and $0 \leqslant t \leqslant t(\varepsilon)$,*

$$\Gamma(P_t f, P_t f) \leqslant C(t) P_t \Gamma(f,f) \tag{27}$$

and $\kappa_\varepsilon := \int_0^{t(\varepsilon)} C(s)ds$ *is finite.*

Then P_t *satisfies* CMLSI (λ_ε) *with* $\lambda_\varepsilon = \dfrac{(1-\varepsilon)}{2\kappa_\varepsilon}$.

Proof. The proof is similar to the proof of Theorem 3.3. Note that by (26), we have

$$D(\rho||\mathbb{E}_\mu\rho) - D(P_{t(\varepsilon)}\rho||\mathbb{E}_\mu\rho) \geqslant (1-\varepsilon)D(\rho||\mathbb{E}_\mu\rho).$$

Then

$$\begin{aligned}(1-\varepsilon)D(\rho||\mathbb{E}_\mu\rho) &\leqslant D(\rho||\mathbb{E}_\mu\rho) - D(P_{t(\varepsilon)}\rho||\mathbb{E}_\mu\rho)\\ &= \int_0^{t(\varepsilon)} I(P_t\rho)dt\\ &\leqslant \int_0^{t(\varepsilon)} C(t)I(\rho)dt = \kappa I(\rho),\end{aligned}$$

which proves the claim. □

The L^∞-mixing time $t(\varepsilon)$ is finite whenever the semigroup satisfies a Poincaré inequality (PI) and ultracontractivity.

$$\|P_t - \mathbb{E}_\mu\|_{L^2(\Omega,\mu)\to L^2(\Omega,\mu)} \leqslant e^{-\gamma t} \tag{PI(γ)}$$

$$\|P_{t_0}\|_{L^1(\Omega,\mu)\to L^\infty(\Omega,\mu)} \leqslant C \text{ for some } t_0 \geqslant 0. \tag{Ultracontractivity}$$

Indeed, for $t \geqslant t_0$

$$\begin{aligned}&\|P_t - \mathbb{E}_\mu\|_{L^1(\Omega,\mu)\to L^\infty(\Omega,\mu)}\\ &\leqslant \|P_{\frac{t_0}{2}}\|_{L^1\to L^2} \cdot \|P_{t-t_0} - \mathbb{E}_\mu\|_{L^2\to L^2} \cdot \|P_{\frac{t_0}{2}}\|_{L^2\to L^\infty}\\ &\leqslant \|P_{t_0}\|_{L^1\to L^\infty} \cdot \|P_{t-t_0} - \mathbb{E}_\mu\|_{L^2\to L^2} \leqslant Ce^{-\gamma(t-t_0)}.\end{aligned}$$

Therefore, $t(\varepsilon) \leqslant \frac{\log C - \log\varepsilon}{\gamma} + t_0$.

We now show that under Hörmander's condition, the gradient estimate (17) for any short period of time t is sufficient for CMLSI. We recall a lemma from [22, Lemma 2.1 & Lemma 2.2].

Lemma 3.5. *Let $\rho, \sigma \in L^\infty(M, \mathbb{M}_n)$ be two matrix-valued density functions. Denote the multiplication operator $M_\rho^{-1} : L^\infty(M, \mathbb{M}_n) \to L^\infty(M, \mathbb{M}_n)$ and the associate weighted L^2-norm as follows*

$$M_\rho^{-1}(f) = \int_0^\infty (\rho + s\mathbf{1})^{-1} f (\rho + s\mathbf{1})^{-1} ds$$

$$\| f \|_{\rho^{-1}}^2 = \langle f, f \rangle_{\rho^{-1}} := \langle f, M_\rho^{-1} f \rangle_{\mathrm{tr}} = \int_0^\infty \mathrm{tr}\left(f^*(\rho + s\mathbf{1})^{-1} f (\rho + s\mathbf{1})^{-1} \right) ds \, .$$

Then,

i) *if $\rho \leqslant C\sigma$, $C \| f \|_{\rho^{-1}} \geqslant \| f \|_{\sigma^{-1}}$*
ii) *$D(\rho || \mathbb{E}_\mu \rho) \leqslant \| \rho - \mathbb{E}_\mu \rho \|_{\mathbb{E}_\mu \rho^{-1}}$*

Proof. The proof is almost identical to the finite-dimensional matrix case in [22]. □

Theorem 3.6. *Let (M, g) be a Riemannian manifold with μ being a smooth probability measure. Let $\mathcal{L} = -\sum_{i=1}^k X_i^* X_i$ be a sub-Laplacian and denote by $P_t = e^{t\mathcal{L}}$ the symmetric Markov semigroup generated by $\mathcal{L}$. Suppose*

i) *$\{X_i\}_{i=1}^k$ satisfies Hörmander's condition.*
ii) *there exists some $0 < t_0 < 1$ and a positive function $C : [0, t_0) \to \mathbb{R}_+$ such that for all scalar-valued functions $f \in \mathrm{dom}(\Gamma)$ and $0 \leqslant t \leqslant t_0$,*

$$\Gamma(P_t f, P_t f) \leqslant C(t) P_t \Gamma(f, f) \tag{28}$$

and $\kappa := \int_0^{t_0} C(s) ds$ is finite.

Then P_t satisfies CMLSI.

Proof. By Varopoulos' dimension condition, we have that for any positive scalar function f

$$P_t f \leqslant \| P_t f \|_\infty \, \mathbf{1} \leqslant C t^{-m} \| f \|_1 \, \mathbf{1} = C t^{-m} \mathbb{E}_\mu f \, , \; 0 < t \leqslant 1$$

for some C and m. Then $Ct^{-m}\mathbb{E}_\mu - P_t$ is a positive map from $L^\infty(M)$ to $L^\infty(M)$. Note that any positive map onto $L^\infty(M)$ is automatically completely positive, because positivity in $L^\infty(M, \mathbb{M}_n)$ is a.s. pointwise positive. Then, for any matrix-valued density ρ

$$P_t\rho \leqslant Ct^{-m}(\mathbb{E}_\mu\rho)\mathbf{1}\ ,\ 0 < t \leqslant 1$$

as matrix-valued functions in $L^\infty(M, \mathbb{M}_n)$, where the right-hand side stands for the constant function of the value $\mathbb{E}_\mu\rho = \int_M \rho\ d\mu$. For a matrix-valued density ρ such that $D(\rho||\mathcal{E}_\mu\rho) < \infty$, we denote $\rho_t := P_t\rho$. Then for any $0 < t \leqslant 1$

$$\begin{aligned} D(\rho_t||\mathbb{E}_\mu\rho) &\leqslant \|\rho_t - \mathbb{E}_\mu\rho\|_{\mathbb{E}_\mu\rho^{-1}} && (29)\\ &\leqslant \frac{1}{\gamma}\ \|\nabla\rho_t\|_{\mathbb{E}_\mu\rho^{-1}} && (30)\\ &\leqslant \frac{t^m}{C\gamma}\ \|\nabla\rho_t\|_{\rho_t^{-1}} = \frac{t^m}{C\gamma}I(\rho_t). && (31) \end{aligned}$$

Here (29) uses part ii) of Lemma 3.5, (30) uses the spectral gap γ and (31) uses part i) of Lemma (3.5) and $\rho_t = P_t\rho \leqslant Ct^{-m}E_\mu\rho$. Denote $h(t) = D(\rho_t||\mathbb{E}_\mu\rho)$. Then the above inequality shows that

$$h(t) \leqslant -\frac{t^m}{C\gamma}h'(t)\ ,\ 0 < t \leqslant 1$$

Consider the differential equation

$$s'(t) = -\frac{t^m}{C\gamma}s(t),$$

whose solution is

$$s(t) = e^{-\frac{t^{m+1}}{\gamma C(m+1)}}s(0).$$

Thus by Grönwall's inequality

$$D(\rho_t||\mathbb{E}_\mu(\rho)) = h(t) \leqslant e^{-\frac{t^{m+1}}{\gamma C(m+1)}}h(0) = D(\rho||\mathbb{E}_\mu(\rho))\ ,\ 0 < t \leqslant 1\ .$$

This implies that for any matrix-valued density ρ

$$D(\rho||\mathbb{E}_\mu(\rho)) - D(\rho_{t_0}||\mathbb{E}_\mu(\rho)) \geqslant (1 - e^{-\frac{t^{m+1}}{\gamma C(m+1)}})D(\rho||\mathbb{E}_\mu(\rho)).$$

Then using assumption (28),

$$\begin{aligned} (1 - e^{-\frac{t^{m+1}}{\gamma C(m+1)}})D(\rho||\mathbb{E}_\mu(\rho)) &\leqslant (D(\rho||\mathbb{E}_\mu(\rho)) - D(\rho_{t_0}||\mathbb{E}_\mu(\rho))\\ &\leqslant \int_0^{t_0} I(\rho_t)dt \end{aligned}$$

$$\begin{aligned}&\leqslant \int_0^{t_0} C(t)I(\rho)dt\\ &=\kappa I(\rho)\end{aligned}$$

which proves CMLSI (λ) with

$$\lambda = \kappa^{-1}(1 - e^{-\frac{t^{m+1}}{\gamma C(m+1)}}) . \quad \square$$

3.2. Applications to the sub-Laplacian on SU (2)

We now apply our results to the canonical sub-Laplacian on the special unitary group SU (2). Recall the skew-Hermitian matrices

$$X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, Z = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

span its Lie algebra over $\mathbb{R}$. The group $G = \mathrm{SU}(2)$ is isomorphic to 3-sphere $\mathbb{S}^3$ via the following parametrization

$$G = \{cI + xX + yY + zZ : c^2 + x^2 + y^2 + z^2 = 1, x, y, x, c \in \mathbb{R}\}.$$

The Lie algebra is $\mathfrak{g} = \mathrm{span}_{\mathbb{R}}\{X, Y, Z\}$ with Lie bracket rules as

$$[X, Y] = 2Z \ , [Y, Z] = 2X \ , [Z, X] = 2Y \ . \tag{32}$$

Suppose that $\mathfrak{g}$ is equipped with the left-invariant metric such that $\{X, Y, Z\} \subset \mathfrak{g}$ forms an orthonormal basis, the corresponding Laplacian (Casimir operator) is

$$\Delta = X^2 + Y^2 + Z^2.$$

It is known that SU (2) has the constant Ricci curvature 2. Then by the complete Bakry-Émery theorem [31, Theorem 3.4] the heat semigroup $T_t = e^{-\Delta t}$ satisfies CMLSI (2).

We are interested in the sub-Riemannian setting. The canonical sub-Riemannian structure is given by $H = \{X, Y\}$ as a generating set of $\mathfrak{g}$ since $[X, Y] = 2Z$ satisfies Hörmander's condition. The horizontal sub-Laplacian is

$$\mathcal{L} = X^2 + Y^2.$$

The semigroup $P_t = e^{\mathcal{L}t}$ on SU (2) has been intensively studied. In particular, Baudoin and Bonnefont in [5] proved that for all $p > 1$, the following L^p-gradient estimate holds for all scalar-valued function f,

$$\Gamma(P_t f, P_t f)^{\frac{p}{2}} \leqslant C_p e^{-2pt} P_t(\Gamma(f,f)^{\frac{p}{2}}), \tag{33}$$

where C_p is a constant depending on p. Applying our Theorem 3.3 for $p = 2$, we obtain

Theorem 3.7. *The semigroup $P_t = e^{\mathcal{L}t}$ generated by the sub-Laplacian $\mathcal{L} = X^2 + Y^2$ satisfies* CMLSI $\left(\frac{C_2}{8}\right)$, *where C_2 is the constant in (33) for $p = 2$.*

We note that this is the first example that CMLSI is obtained for sub-Laplacians. The CMLSI constant for sub-Laplacians has direct applications to quantum Markov semigroup via its representations. The representation theory of SU(2) gives the well-known spin structure of quantum mechanics, where any irreducible representation of SU(2) is indexed by an integer $m \in \mathbb{N}^+$. Let $\phi_m : \mathfrak{su}(2) \to i(\mathbb{M}_m)_{\text{sa}}$ be the Lie algebra homomorphism induced by the m-th irreducible representation, and let $\{|j\rangle | j = 1, \cdots, m\}$ be the orthonormal basis consisting of eigenfunctions $\eta_m(Z)$. Denote $X_m := \phi_m(X)$, and similarly for Y_m and Z_m. Under the normalization of (32),

$$\begin{aligned}
X_m|j\rangle &= \sqrt{(j-1)(m-j+1)}\,|j-1\rangle - \sqrt{(j+1)(m-j-1)}\,|j+1\rangle,\\
Y_m|j\rangle &= i\sqrt{(j-1)(m-j+1)}\,|j-1\rangle + i\sqrt{(j+1)(m-j-1)}\,|j+1\rangle,\\
Z_m|j\rangle &= (m-2j+1)\,|j\rangle.
\end{aligned}$$

We consider the quantum Markov semigroup

$$\begin{aligned}
&S_t = e^{\mathcal{L}_m t} : \mathbb{M}_m \to \mathbb{M}_m\ ,\\
&\mathcal{L}_m(\rho) = [X_m, [X_m, \rho]] + [Y_m, [Y_m, \rho]].
\end{aligned}$$

This semigroup can be viewed as a representation of classical Markov semigroup $P_t : L^\infty(G) \to L^\infty(G)$. Indeed, let $\pi_m : G \to \mathbb{M}_m$ be the mth irreducible representation of $G = \text{SU}(2)$. The following transference diagram holds

$$\begin{array}{ccc}
L^\infty(G, \mathbb{M}_m) & \xrightarrow{P_t \otimes \mathrm{id}_{\mathbb{M}_m}} & L^\infty(G, \mathbb{M}_m)\\
\alpha \uparrow & & \alpha \uparrow\\
\mathbb{M}_m & \xrightarrow{\quad S_t \quad} & \mathbb{M}_m,
\end{array} \tag{34}$$

where the map α is given by

$$\begin{aligned}
&\alpha : \mathbb{M}_m \to L^\infty(G, \mathbb{M}_m)\ ,\\
&\alpha(\rho)(g) = \pi_m(g)(\rho)\pi_m(g)^*\ .
\end{aligned}$$

Note that α is an algebra homomorphism, which is an embedding of the matrix algebra $\mathbb{M}_m$ into the matrix-valued functions $L^\infty(G, \mathbb{M}_m)$ on G. Under this embedding, the quantum Markov semigroup S_t is exactly the restriction of the matrix-valued extension

$P_t \otimes \mathrm{id}_{\mathbb{M}_m}$ on the image of $\alpha(\mathbb{M}_m)$. Such a transference relation holds for any (projective) unitary representation, which yields the following dimension-free CMLSI estimate.

Theorem 3.8. *Let $S_t = e^{\mathcal{L}_m t} : \mathbb{M}_m \to \mathbb{M}_m$, $\mathcal{L}_m(\rho) = [X_m, [X_m, \rho]] + [Y_m, [Y_m, \rho]]$ be the transference semigroups above. Then for all $m \geqslant 1$, S_t satisfies* CMLSI $\left(\frac{C_2}{8}\right)$.

Proof. This follows from the diagram (34) as in [20, Section 4]. □

The above example was first studied by Li, Junge and LaRacuente [31] with a dimension-dependent CMLSI constant. It is clear from the diagram (34) that CMLSI constant for matrix-valued functions in $L^\infty(G, \mathbb{M}_m)$ is crucial for the transference to quantum Markov semigroup on matrix algebras, while only the LSI or MLSI for the scalar-valued functions is not enough.

4. Ornstein–Uhlenbeck semigroup on Lie groups

In this section, we discuss matrix-valued modified Log-Sobolev constants for the heat kernel measures on Lie groups. Before the matrix-valued case, we first review a standard approach for scalar-valued functions. Let (M, g, d_g) be a Riemannian manifold with the volume form d_g and let ∇ and $\Delta = \nabla \cdot \nabla$ be the gradient operator and Laplace-Beltrami operator respectively. Denote by $H_t = e^{\Delta t}$ the heat semigroup. Recall that Bakry and Émery in [1] proved that $\mathrm{Ricci}(M) \geqslant \lambda$ for some $\lambda \in \mathbb{R}$ if and only if the following $p = 1$ gradient estimate

$$|\nabla H_t f| \leqslant C(t) H_t(|\nabla f|) \ , \ \text{for all } f \in C_c^\infty(M), t > 0 \tag{35}$$

holds for $C(t) = e^{-\lambda t}$. Let μ_t^x be the heat kernel measure at some point $x \in M$ such that

$$H_t f(x) = \int_M f d\mu_t^x \ .$$

It is well known that the gradient estimate (35) for $p = 1$ implies the LSI for μ_t^x

$$\int f^2 \log f^2 d\mu_t^x - \left(\int f^2 d\mu_t^x\right) \log \left(\int f^2 d\mu_t^x\right) \leqslant 2t \int |\nabla f|^2 d\mu_t^x \tag{36}$$

with the constant $\kappa = \int_0^t C(s)^2 ds$. This further implies an MLSI, namely, that for $g \geqslant 0$

$$\int g \log g d\mu_t^x - \left(\int g d\mu_t^x\right) \log \left(\int g d\mu_t^x\right) \leqslant t \int \frac{|\nabla g|^2}{g} d\mu_t^x. \tag{37}$$

In terms of the semigroup, (36) and (37) are equivalent to respectively hypercontractivity and exponential entropy decay for the Ornstein–Uhlenbeck semigroup

$$P_s = e^{L_{x,t}s}, L_{x,t} = \nabla^* \nabla,$$

where ∇^* is the adjoint of ∇ in $L^2(d\mu_t^x)$ and $L_{x,t}$ is symmetric with respect to the measure μ_t^x.

The proof of (36) and (37) from (35) is now standard (cf. [2, Theorem 6.1]). Indeed, (37) follows from the function inequality at point $x \in M$,

$$H_t(g \log g) - H_t(g) \log H_t(g) \leqslant \frac{\kappa}{2} H_t \left(\frac{\nabla g}{g} \right). \tag{38}$$

Fix $t > 0$, and for $g \in C_c^\infty(M)$ we define

$$G : [0,t] \to C^\infty(M) \ , \ G(s) = H_s(H_{t-s} g \log H_{t-s} g).$$

Note that

$$H_t(g \log g) = G(t) \ , H_{t-s} g \log H_{t-s} g = G(0) \ .$$

Then

$$H_t(g \log g) - H_t g \log H_t g = G(t) - G(0) = \int_0^t \partial_s G(s) ds \ .$$

The derivative is given by

$$\begin{aligned}
\partial_s G(s) =& \Delta H_s \left(H_{t-s} g \log H_{t-s} g \right) - H_s \left((\Delta H_{t-s}) g \log H_{t-s} g \right) - H_s (\Delta H_{t-s} g) \\
=& H_s \left(\Delta (H_{t-s} g \log H_{t-s} g) - (\Delta H_{t-s} g) \log H_{t-s} g - \Delta H_{t-s} g \right) \\
=& H_s \left(\Delta (H_{t-s} g \log H_{t-s} g) - (\Delta H_{t-s} g) \log H_{t-s} g - \Delta H_{t-s} g \right) \\
=& 2 H_s \left(\nabla H_{t-s} g \cdot \nabla (\log H_{t-s} g) \right) \\
=& 2 H_s \left(\frac{|\nabla H_{t-s} g|^2}{H_{t-s} g} \right) \ ,
\end{aligned}$$

where we used the product rule for $\Delta = \nabla \cdot \nabla$

$$\Delta(f \log f) = \Delta(f) \log f + f \Delta(\log f) + 2 \nabla f \nabla (\log f) \ .$$

Here comes the step using the gradient estimate for $p = 1$ which gives that for any $s > 0$,

$$\frac{|\nabla H_s g|^2}{H_s g} \leqslant C(s)^2 \frac{H_s(|\nabla g|)^2}{H_s g} \leqslant C(s)^2 H_s(\frac{|\nabla g|^2}{g}), \tag{39}$$

where the second inequality uses the joint convexity of the bivariate function $(x, y) \mapsto \frac{x^2}{y}$. Then (38) follows immediately from the semigroup property

$$H_t(g\log g) - H_t g \log H_t g \leqslant 2\int_0^t H_s\left(\frac{|\nabla H_{t-s}g|^2}{H_{t-s}g}\right) ds \leqslant 2C(s)^2 \int_0^t H_{t-s}H_s\left(\frac{|\nabla g|^2}{g}\right) ds$$
$$= 2\left(\int_0^t C(s)^2 ds\right) H_t\left(\frac{|\nabla g|^2}{g}\right).$$

To summarize, the key consequence the gradient estimate (35) for $p = 1$ used above is that

$$|\nabla H_t f| \leqslant C(t) H_t(|\nabla f|) \;\Rightarrow\; \frac{|\nabla H_t g|^2}{H_t g} \leqslant C(t)^2 H_t\left(\frac{|\nabla g|^2}{g}\right). \tag{40}$$

This approach can be used to recover Gross' LSI for the Ornstein–Uhlenbeck semigroup on $\mathbb{R}^n$, as well as the Heisenberg group in [2] in the sub-Riemannian setting, for which the gradient estimate (35) for $p = 1$ was proved by H.-Q. Li in [37] with $C(t) \cong C$ being constant.

Given Theorem 3.6 using the gradient estimate for $p = 2$, it is natural to consider whether the gradient estimate for $p = 1$ can be used to tackle the matrix-valued case. However, there are two obstructions. First, it is not clear whether $p = 1$ gradient estimate extends to matrix-valued cases as in Proposition 3.1. Note that for a matrix A, the absolute value $|A|$ should be interpreted as $\sqrt{A^*A}$ using functional calculus for positive operators. For a matrix-valued function $f \in C_c^\infty(M)$, we have the gradient vector $\nabla f = (X_1 f, \cdots, X_n f)$ and $|\nabla f| = \sqrt{\sum_i (X_i f)^*(X_i f)}$. Secondly, we also lack of (40) in the matrix-valued case as an analog of Lemma 3.2 for $p = 2$. These observations make it unclear whether (35) implies a CMLSI for the Ornstein–Uhlenbeck semigroup.

In the following, we shall show that the above approach partially works for matrix-valued case if we have the exact commutation relation

$$\nabla H_t f = e^{-\lambda t} H_t \nabla f. \tag{41}$$

Equation (41) is also called an intertwining relation. It was first introduced by Carlen and Maas in [14] as a sufficient condition to derive the entropic Ricci curvature condition. Here we present a different use of it. The natural example of spaces for which (41) holds are groups. Let G be a Lie group equipped with a left-invariant metric. Let $X_1, \cdots, X_n$ be a O.N.B of the Lie algebra of left invariant vector field. It is well known that the Casimir element $\Delta = \sum_i X_i^2$, which is also the Laplace-Beltrami operator, is a central element for the Lie algebra. Thus for the gradient operator $\nabla = (X_1, \cdots, X_n)$

$$\nabla\Delta f = \Delta \nabla f \;,\; \nabla H_t f = H_t \nabla f \;,\; \forall f \in C_c^\infty(M), \tag{42}$$

which is an intertwining relation (41) with $\lambda = 0$.

Theorem 4.1. *Let G be a locally compact Lie group equipped with a left-invariant metric. Let $H_t = e^{\Delta t}$ be the heat semigroup and denote p_t as the heat kernel measure (for the identity element). Then for any positive matrix-valued function $g \in C_c^\infty(G, M_n)$*

$$\operatorname{tr} H_t(g \log g) - \operatorname{tr}(H_t g \log H_t g) \leqslant 2t \operatorname{tr} H_t\Big(\int_0^\infty (\nabla g)(g + r\mathbf{1})^{-1}(\nabla g)(g + r\mathbf{1})^{-1} dr \Big) ,$$

where $\mathbf{1}$ is the constant matrix-valued function of identity operator. In particular, the Ornstein–Uhlenbeck semigroup $P_s = e^{L_t s}$ satisfies $\frac{1}{4t}$-CMLSI, where $L_t = \nabla^* \nabla$ is the generator symmetric on $L^2(dp_t)$.

Proof. Let $g \in C_c^\infty(G, M_n)$ be a matrix-valued function on G and $g(x) \geqslant 0$ for all $x \in G$. For the ease of notation, we write $H_t g := (H_t \otimes \mathrm{id}_{M_n})g$ and $\Delta g = (\Delta \otimes \mathrm{id}_{M_n})g$. Fix $t > 0$ and define the function

$$G : [0, t] \to C^\infty(M) \ , \ G(s) = \operatorname{tr}\Big(H_s(H_{t-s} g \log H_{t-s} g) \Big).$$

Then

$$\operatorname{tr} H_t(g \log g) - \operatorname{tr}(H_t g \log H_t g) = G(t) - G(0) = \int_0^t \partial_s G(s) ds \ .$$

Recall that for a smooth scalar function $\phi : (a, b) \to \mathbb{R}$ on a interval (a, b) and matrix-valued function $s \mapsto g(s)$ with $\operatorname{spec} g(s) \subset (a, b)$,

$$\frac{d}{ds} \operatorname{tr}(\phi(g(s))) = \operatorname{tr}(\phi'(g(s)) g'(s)) \ . \tag{43}$$

The derivative with respect to s is given by

$$\begin{aligned} \partial_s G(s) =& \Delta H_s\Big(\operatorname{tr}\left(H_{t-s} g \log H_{t-s} g\right) \Big) + H_s\Big(\operatorname{tr}(-\Delta H_{t-s} g (\log H_{t-s} g + 1)) \Big) \\ =& H_s\Big(\operatorname{tr}\left(\Delta(H_{t-s} g \log H_{t-s} g) - \Delta(H_{t-s} g) \log H_{t-s} g - \Delta(H_{t-s} g) \right) \Big) \ , \end{aligned}$$

where we used (43) with $\phi(x) = x \log x$ and the fact that $H_t \operatorname{tr} = \operatorname{tr} H_t$ commute (more precisely, $H_t \operatorname{tr} = \operatorname{tr}(H_t \otimes \mathrm{id}_{M_n})$). By the product rule for $\Delta = \nabla \cdot \nabla$,

$$\Delta(f \log f) = \Delta(f) \log f + f \Delta(\log f) + 2 \nabla f \nabla(\log f).$$

Using the noncommutative derivative,

$$\nabla(\log f) = \int_0^\infty (f + r\mathbf{1})^{-1}(\nabla f)(f + r\mathbf{1})^{-1} dr,$$

$$\nabla\big((f+r\mathbf{1})^{-1}\big) = -(f+r\mathbf{1})^{-1}(\nabla f)(f+r\mathbf{1})^{-1},$$

we have

$$\begin{aligned}\Delta(\log f) =& \nabla\cdot\Big(\int_0^\infty (f+r\mathbf{1})^{-1}(\nabla f)(f+r\mathbf{1})^{-1}\Big)\\ =& \int_0^\infty (f+r\mathbf{1})^{-1}(\Delta f)(f+r\mathbf{1})^{-1}dr\\ &- 2\sum_i \int_0^\infty (f+r\mathbf{1})^{-1}(X_i f)(f+r\mathbf{1})^{-1}(X_i f)(f+r\mathbf{1})^{-1}dr\end{aligned}$$

Then

$$\begin{aligned}&\operatorname{tr}\Big(\Delta(f\log f) - (\Delta f)\log f - \Delta f\Big)\\ =&\operatorname{tr}\Big(f\Delta(\log f) + 2\nabla f\nabla(\log f) - \Delta f\Big)\\ =&\operatorname{tr}\Big(\int_0^\infty f(f+r\mathbf{1})^{-1}(\Delta f)(f+r\mathbf{1})^{-1}dr\Big)\\ &- 2\sum_i \operatorname{tr}\Big(\int_0^\infty f(f+r\mathbf{1})^{-1}(X_i f)(f+r\mathbf{1})^{-1}(X_i f)(f+r\mathbf{1})^{-1}dr\Big)\\ &+ 2\sum_i \operatorname{tr}\Big(\int_0^\infty (X_i f)(f+r\mathbf{1})^{-1}(X_i f)(f+r\mathbf{1})^{-1}dr\Big)\\ &- \operatorname{tr}\Big(\Delta f\Big).\end{aligned} \tag{44}$$

The first and the last term cancel out

$$\operatorname{tr}\Big(\int_0^\infty f(f+r\mathbf{1})^{-1}(\Delta f)(f+r\mathbf{1})^{-1}dr\Big) = \operatorname{tr}\Big(\big(\int_0^\infty f(f+r\mathbf{1})^{-1}(f+r\mathbf{1})^{-1}dr\big)\Delta f\Big) = \operatorname{tr}\Big(\Delta f\Big),$$

where we used the tracial property $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ and the integral identity

$$\int_0^\infty (A+r\mathbf{1})^{-1}A(A+r\mathbf{1})dr = \mathbf{1}$$

for a positive operator A. Also, the second term is always negative. Note that for positive f, $(X_i f)^* = X_i f$ is self-adjoint. Then for any $s > 0$,

$$\begin{aligned}
&\operatorname{tr}\Big(f(f+r\mathbf{1})^{-1}(X_if)(f+r\mathbf{1})^{-1}(X_if)(f+r\mathbf{1})^{-1}\Big)\\
=&\operatorname{tr}\Big(f^{1/2}(f+r\mathbf{1})^{-1}(X_if)(f+r\mathbf{1})^{-1}(X_if)(f+r\mathbf{1})^{-1}f^{1/2}\Big)\\
=&\operatorname{tr}\Big(\big(f^{1/2}(f+r\mathbf{1})^{-1}(X_if)(f+r\mathbf{1})^{-\frac{1}{2}}\big)\big((f+r\mathbf{1})^{-\frac{1}{2}}(X_if)(f+r\mathbf{1})^{-1}f^{1/2}\big)\Big)\geqslant 0.
\end{aligned}$$

Applying the above estimate to $f=H_{t-s}g$, we see that

$$\begin{aligned}
&\operatorname{tr}\big(\Delta(H_{t-s}g\log H_{t-s}g)-\Delta(H_{t-s}g)\log H_{t-s}g-\Delta(H_{t-s}g)\big)\\
\leqslant&2\sum_i\operatorname{tr}\Big(\int_0^\infty(X_iH_{t-s}g)(H_{t-s}g+r\mathbf{1})^{-1}(X_iH_{t-s}g)(H_{t-s}g+r\mathbf{1})^{-1}dr\Big)\\
=&2\operatorname{tr}\Big(\int_0^\infty(\nabla H_{t-s}g)(H_{t-s}g+r\mathbf{1})^{-1}(\nabla H_{t-s}g)(H_{t-s}g+r\mathbf{1})^{-1}dr\Big)\\
\overset{(1)}{=}&2\operatorname{tr}\Big(\int_0^\infty(H_{t-s}\nabla g)(H_{t-s}g+r\mathbf{1})^{-1}(H_{t-s}\nabla g)(H_{t-s}g+r\mathbf{1})^{-1}dr\Big)\\
\overset{(2)}{\leqslant}&2H_{t-s}\operatorname{tr}\Big(\int_0^\infty(\nabla g)(g+r\mathbf{1})^{-1}(\nabla g)(g+r\mathbf{1})^{-1}dr\Big).
\end{aligned}$$

Here, (1) uses the commutation relation $\nabla H_tg=H_t\nabla g$, and (2) uses the joint convexity of the trace function (see cf. [44, Eq. (7)])

$$(A,B)\in(M_n)_+\times(M_n)_+\mapsto\operatorname{tr}\Big(\int_0^\infty A(B+r\mathbf{1})^{-1}A(B+r\mathbf{1})^{-1}dr\Big).$$

Finally, using the semigroup property we see that

$$\operatorname{tr}H_t(g\log g)-\operatorname{tr}(H_tg\log H_tg)=\int_0^t\partial_sG(s)ds \tag{45}$$

$$\begin{aligned}
\leqslant&2\int_0^tH_sH_{t-s}\operatorname{tr}\Big(\int_0^\infty(\nabla g)(g+r\mathbf{1})^{-1}(\nabla g)(g+r\mathbf{1})^{-1}dr\Big)ds\\
=&2\int_0^tH_t\operatorname{tr}\Big(\int_0^\infty(\nabla g)(g+r\mathbf{1})^{-1}(\nabla g)(g+r\mathbf{1})^{-1}dr\Big)ds\\
=&2t\operatorname{tr}H_t\Big(\int_0^\infty(\nabla g)(g+r\mathbf{1})^{-1}(\nabla g)(g+r\mathbf{1})^{-1}dr\Big).
\end{aligned} \tag{46}$$

To see that this implies a CMLSI, we note that $H_t f(e) = \mathbb{E}_{p_t} f$ at the identity element e and the Fisher information for a positive matrix-valued function $g \in C_c^\infty(G, M_n)$ is

$$I(g) = \int_G \operatorname{tr}\Big(\int_0^\infty (\nabla g)(g + r\mathbf{1})^{-1}(\nabla g)(g + r\mathbf{1})^{-1} dr \Big) dp_t.$$

Then (45) at the identity element $x = e$ gives

$$\begin{aligned} D(g||\mathbb{E}_{p_t} g) &= \int \operatorname{tr}(g \log g) - \operatorname{tr}(\mathbb{E}_{p_t} g \log \mathbb{E}_{p_t} g) dp_t \\ &= \operatorname{tr} H_t(g \log g) - \operatorname{tr}(H_t g \log H_t g)|_{x=e} \\ &\leqslant 2t \operatorname{tr} H_t \Big(\int_0^\infty (\nabla g)(g + r\mathbf{1})^{-1}(\nabla g)(g + r\mathbf{1})^{-1} dr \Big)|_{x=e} \\ &= 2t \int_G \operatorname{tr}\Big(\int_0^\infty (\nabla g)(g + r\mathbf{1})^{-1}(\nabla g)(g + r\mathbf{1})^{-1} dr \Big) dp_t \\ &= 2t I(g) \,. \end{aligned}$$

That completes the proof. □

Remark 4.2. Comparing to the scalar-valued case (37), our constant differs by a factor of 2. This is because for a scalar-valued function f, we can find (44) explicitly

$$\begin{aligned} &- 2\Big(\int_0^\infty f(f + r\mathbf{1})^{-3} |\nabla f|^2 dr \Big) + 2\Big(\int_0^\infty (f + r\mathbf{1})^{-2} |\nabla f|^2 dr \Big) \\ =& - f^{-1}|\nabla f|^2 + 2f^{-1}|\nabla f|^2 = f^{-1}|\nabla f|^2 \,, \end{aligned}$$

which gains a factor of 2 compared to the matrix-valued case.

5. Final discussion

We end our discussion with some remarks on connection to the noncommutative setting and extendability of our results.

The assertion (i) in the key Proposition 3.1 is closely related to the Γ_2-*condition* introduced by M. Junge and Q. Zeng [33] for noncommutative symmetric Markov semigroup. In particular, they showed that if a Riemannian manifold has its Ricci curvature bounded below by λ, then the heat semigroup T_t satisfies

$$[\Gamma(T_t f_i, T_t f_j)]_{i,j} \leqslant e^{-2\lambda t} [T_t \Gamma(f_i, f_j)]_{i,j} \,. \tag{47}$$

Part i) in Proposition 3.1 extends this to the sub-Laplacian case for a single Markov map P. Although (47) gives a matrix-valued version of Bakry-Émery's curvature-dimension condition, it is not clear whether (47) in the matrix-valued case or noncommutative setting implies a MLSI. On the other hand, part ii) of Proposition 3.1 for matrix-valued function f

$$\langle \nabla T_t f, A^s(\nabla T_t f)B^{1-s}\rangle_{\mathrm{tr}} \leqslant e^{-2\lambda t}\langle \nabla f, (T_t^\dagger A)^s(\nabla f)(T_t^\dagger B)^{1-s}\rangle_{\mathrm{tr}}, \tag{48}$$

is more related to *complete gradient estimates* studied by Wirth and Zhang [51]. Their *complete gradient estimates* were inspired by the *entropic Ricci curvature lower bound* introduced in the work of E. Carlen and J. Mass [14], and N. Datta and C. Rouzé in [17], which are powerful tools in deriving modified log Sobolev inequalities for quantum Markov semigroups [10,11]. It is clear that (48) is more general than (47) by choosing $A = B$ and $s = 0$, and they coincide when reduced to scalar-valued functions. They differ in matrix-valued cases because the matrix multiplication is not commutative, and there are various ways to multiply a matrix A by X, such as AX, XA and A^sXA^{1-s}. Proposition 3.1 basically shows that for sub-Laplacians, (47) and (48) are equivalent and both reduce to Bakry-Émery's curvature dimension condition for scalar-valued functions

$$\Gamma(T_t f, T_t f) \leqslant e^{-2\lambda t} T_t \Gamma(f, f)\ . \tag{49}$$

One can consider whether the above equivalence holds beyond the sub-Laplacian case. Indeed, the equivalence between (47) and (49) holds with for all symmetric Markov semigroup with the same argument as used in Proposition 3.1. The equivalence between (47) and (48) is more involved. To formulate (48) for a general symmetric generator $P_t = e^{Lt} : L^\infty(\Omega) \to L^\infty(\Omega)$, one needs a densely defined closed derivation operator $\delta : L^2(\Omega) \to \mathcal{H}$ such that $L = -\delta^*\delta$, where $\mathcal{H}$ is a Hilbert module or more specifically L^2-space. It was proved in a preprint by M. Junge, E. Ricard and D. Shlyahktenko [32] that every symmetric (quantum) Markov semigroup admits such a derivation on a noncommutative L^2-space $L^2(\mathcal{M})$ of some von Neumann algebra (see also the more recent preprint [50] for the strongly continuous semigroup). The proof for part ii) of Proposition 3.1 works as long as the range of δ commutes with the functions in $L^\infty(\Omega)$, which is obvious for sub-Laplacians because the range of $\nabla = (X_1, \cdots, X_k)$ can be viewed as the diagonal matrices in $C^\infty(M, \mathbb{M}_k)$. Nevertheless, in a private communication Melchior Wirth pointed out to us that this is unlikely to hold for quantum Markov semigroups.

Our main Theorem 3.6 states that any sub-Laplacian $\mathcal{L} = -\sum_{i=1}^k X_i^* X_i$ generated by a collection of vector fields satisfying (i) Hörmander's condition and (ii) the gradient estimates

$$\Gamma(P_t f, P_t f) \leqslant C(t) P_t \Gamma(f, f) \tag{50}$$

for some integrable function $C(t)$, satisfies a CMLSI. Up to the writing of this paper, such estimates were obtained for $\mathrm{SU}(2)$ in [5] and for nilpotent Lie groups in [41] including Heisenberg group in [19]. An interesting question is to explore whether (50) holds for a larger class of sub-Laplacians on Lie groups, which by our results will imply a CMLSI and a dimension-free estimate for the corresponding quantum Markov semigroups as discussed in Section 3.2. On the other hand, such results were achieved by Gao, Junge, LaRacunte and Li in a preprint [21], which proves that one point ultra-contractivity plus spectral gap implies CMLSI. Their results in particular give CMLSI for sub-Laplacians of any Hörmander system on a compact manifold.

The intertwining relation

$$\nabla P_t f = e^{-\lambda t} P_t \nabla f \tag{51}$$

is obviously a stronger condition than Bakry-Émery's curvature-dimension inequality

$$|\nabla P_t f| \leqslant C(t) P_t(|\nabla f|)\ , \quad \text{for any } f \in C_c^\infty(M), t > 0, \tag{52}$$

hence can be used to derive an CMLSI for the semigroup P_t itself. Indeed, the Ornstein–Uhlenbeck semigroup on $\mathbb{R}^n$ satisfies $\nabla P_t f = e^{-t} P_t \nabla f$ with $\lambda = 1$, which is an alternative approach to a CMLSI for the Gaussian measure in [31]. In the sub-Riemannian setting, (51) is never satisfied because of lack of curvature lower bound. So our method here does not apply to Ornstein–Uhlenbeck semigroup on Heisenberg group studied in [2,37], for which the standard (scalar-valued) log Sobolev inequality can be obtained from (52). In that sense, finding a CMLSI for the sub-Laplacian Ornstein–Uhlenbeck semigroup on the Heisenberg group remains an open question. For the more applications of intertwining relations to CMLSIs in the noncommutative setting, see [10,11,14,15,17,31,51,52].

Data availability

No data was used for the research described in the article.

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