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DIMENSIONS OF FURSTENBERG SETS AND AN EXTENSION OF BOURGAIN'S PROJECTION THEOREM

PABLO SHMERKIN AND HONG WANG

We show that the Hausdorff dimension of (s,t)-Furstenberg sets is at least $s+\frac{1}{2}t+\epsilon$, where $\epsilon>0$ depends only on s and t. This improves the previously best known bound for $2s<t\le 1+\epsilon(s,t)$, in particular providing the first improvement since 1999 to the dimension of classical s-Furstenberg sets for $s<\frac{1}{2}$. We deduce this from a corresponding discretized incidence bound under minimal nonconcentration assumptions that simultaneously extends Bourgain's discretized projection and sum-product theorems. The proofs are based on a recent discretized incidence bound of T. Orponen and the first author and a certain duality between (s,t) and $\left(\frac{1}{2}t, s+\frac{1}{2}t\right)$-Furstenberg sets.

1. Introduction and main results

1.1. ***Dimension of Furstenberg sets.*** Let $s\in(0,1]$. We say that a set $K\subset\mathbb{R}^2$ is an s-Furstenberg set if for almost all directions $\theta\in S^1$ there is a line ℓ_θ in direction θ such that $\dim_{\mathrm{H}}(K\cap\ell_\theta)\ge s$. Motivated by work of Furstenberg and by the Szemerédi–Trotter theorem in incidence geometry, T. Wolff [1999] posed the problem of estimating the smallest possible dimension $\gamma(s)$ of an s-Furstenberg set. Using elementary geometric arguments, Wolff showed that

$$\gamma(s)\ge\max\left(2s, s+\tfrac{1}{2}\right).$$

Note that both bounds coincide for $s=\frac{1}{2}$. J. Bourgain [2003], building up on work of N. Katz and T. Tao [2001], proved that $\gamma\left(\frac{1}{2}\right)>1+\epsilon$ for some small universal constant $\epsilon>0$; this is much deeper. Much more recently, T. Orponen and the first author [Orponen and Shmerkin 2023] established a similar improvement for $s\in\left(\frac{1}{2},1\right)$:

$$\gamma(s)\ge 2s+\epsilon(s),$$

where $\epsilon(s)>0$ for $s\in\left(\frac{1}{2},1\right)$. For the case $s<\frac{1}{2}$, the first author [Shmerkin 2022] recently obtained a similar improvement, with the significant caveat that it involves only the *packing* dimension of s-Furstenberg sets (which can be larger than Hausdorff dimension). In this paper, as a corollary of our main result we obtain the corresponding Hausdorff dimension improvement:

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Theorem 1.1. *For every $s \in (0,1)$ there is $\epsilon(s) > 0$ such that every s-Furstenberg set K satisfies*

$$\dim_{\mathrm{H}}(K) \geq s + \tfrac{1}{2} + \epsilon(s).$$

Recently, there has been much interest in the following generalization of Furstenberg sets: we say that $K \subset \mathbb{R}^2$ is an (s,t)*-Furstenberg set* if there is a family of lines $\mathcal{L}$ with $\dim_{\mathrm{H}}(\mathcal{L}) \geq t$ such that

$$\dim_{\mathrm{H}}(K \cap \ell) \geq s, \quad \ell \in \mathcal{L}.$$

Since lines are a two-dimensional manifold, the Hausdorff dimension of $\mathcal{L}$ is well-defined. Classical s-Furstenberg sets are, of course, $(s,1)$-Furstenberg sets. The central problem, initiated by U. Molter and E. Rela [2012], is to estimate $\gamma(s,t)$, the smallest possible Hausdorff dimension of (s,t)-Furstenberg sets; this can be seen as a continuous analog of the Szemerédi–Trotter incidence bound. Recent works investigating this problem include [Lutz and Stull 2020; Héra et al. 2022; Di Benedetto and Zahl 2021; Dąbrowski et al. 2022; Fu and Ren 2024]. The best currently known bounds are summarized as follows. Suppose first that $t \leq 1 + \epsilon'(s,t)$ (where $\epsilon'(s,t)$ is a small positive parameter). Then (see [Molter and Rela 2012; Lutz and Stull 2020; Héra et al. 2022; Orponen and Shmerkin 2023])

$$\gamma(s,t) \geq \begin{cases} s+t & \text{if } t \leq s, \\ 2s + \epsilon(s,t) & \text{if } s \leq t \leq 2s - \epsilon'(s,t), \\ s + \frac{1}{2}t & \text{if } 2s - \epsilon'(s,t) \leq t. \end{cases}$$

Suppose now that $t \geq 1 + \epsilon'(s,t)$. Then (see [Fu and Ren 2024])

$$\gamma(s,t) \geq \begin{cases} 2s + t - 1 & \text{if } s + t \leq 2, \\ s + 1 & \text{if } s + t \geq 2. \end{cases}$$

The bounds $s{+}t$, $s{+}1$ are sharp in the respective regimes, but the other bounds are not expected to be sharp. In this article we obtain a small improvement upon the $s{+}\frac{1}{2}t$ bound:

Theorem 1.2. *Given $s \in (0,1]$, $t \in (0,2]$, there is $\epsilon(s,t) > 0$ such that the following holds. Let K be an (s,t)-Furstenberg set. Then*

$$\dim_{\mathrm{H}}(K) \geq s + \tfrac{1}{2}t + \epsilon(s,t).$$

Theorem 1.1 is an immediate corollary, taking $t = 1$.

1.2. ***Discretized incidence estimates and a strengthening of Bourgain's discretized projection and sum-product theorems.*** Theorem 1.2 is a consequence of the following discretized incidence estimate. See Section 2.2 for the definition of (δ, s, C)-sets of points and tubes.

Proposition 1.3. *Given $s \in (0,1)$ and $t \in (s,2]$, there are $\epsilon, \eta > 0$ such that the following holds for small enough δ: Let $P \subset B^2(0,1)$ be a $(\delta, t, \delta^{-\epsilon})$-set. For each $p \in P$, let $\mathcal{T}(p)$ be a $(\delta, s, \delta^{-\epsilon})$-set of tubes through p with $|\mathcal{T}(p)| \geq M$. Then the union $\mathcal{T} = \bigcup_{p \in P} \mathcal{T}(p)$ satisfies*

$$|\mathcal{T}| \geq M\delta^{-(t/2+\eta)}. \tag{1-1}$$

By duality between points and lines (see, e.g., [Orponen and Shmerkin 2023, Theorem 3.2] and the discussion afterwards) we obtain the following corollary of Proposition 1.3 which more closely resembles the Furstenberg set problem. In the statement, $|P|_\delta$ stands for the δ-covering number of P (see Section 2.1).

Corollary 1.4. *Given $0 < s < 1$, $s < t$, there are $\epsilon, \eta > 0$ such that the following holds for small enough dyadic δ: Let $\mathcal{L}$ be a $(\delta, t, \delta^{-\epsilon})$-set of lines intersecting $B^2(0,1)$. For each $\ell \in \mathcal{L}$, let $P(\ell)$ be a $(\delta, s, \delta^{-\epsilon})$-set contained in $\ell^{(\delta)}$. Suppose $|P(\ell)|_\delta \geq M$ for all $\ell \in \mathcal{L}$. Then the union $P = \bigcup_{\ell \in \mathcal{L}} P(\ell)$ satisfies*

$$|P|_\delta \geq M\delta^{-(t/2+\eta)}.$$

Note that $M \geq \delta^{-(s-\epsilon)}$ and hence (up to changing the values of η, ϵ) we also have the conclusion $|\mathcal{T}| \geq \delta^{-(s+t/2+\eta)}$. In turn, by [Héra et al. 2022, Lemma 3.3] this yields Theorem 1.2.

For comparison's sake, and because it plays a crucial role in the proof of Proposition 1.3, we recall [Orponen and Shmerkin 2023, Theorem 3.2].

Theorem 1.5. *Given $s \in (0,1)$ and $t \in (s, 2]$, there are $\epsilon, \eta > 0$ such that the following holds for all small enough dyadic δ: Let $\mathcal{T}$ be a $(\delta, t, \delta^{-\epsilon})$-set of dyadic δ-tubes. Assume that for every $T \in \mathcal{T}$ there exists a $(\delta, s, \delta^{-\epsilon})$-set $P(T)$ such that $p \in T$ for all $p \in \mathcal{P}(T)$. Then*

$$\Big|\bigcup_{T \in \mathcal{T}} P(T)\Big| \geq \delta^{-2s-\eta}.$$

The nonconcentration assumption on $\mathcal{T}(p)$ in Proposition 1.3 is quite mild, since M can potentially be much larger than δ^{-s} (and a $\delta^{-\epsilon}$ factor is also allowed). What about P? *Some* nonconcentration is needed, as the following standard example shows: if $P = B(x, r)$ and $\mathcal{T}(p)$ is the set of tubes through p with slopes in a fixed (δ, s, C)-set $\mathcal{S}$, then

$$|\mathcal{T}| \sim |\mathcal{S}|_\delta \cdot \frac{r}{\delta} \sim |\mathcal{T}(p)| \cdot |P|_\delta^{1/2}.$$

A similar estimate holds if P is very dense in the union of a small number of r-balls. The next result asserts that under a minimal single-scale nonconcentration assumption on P that rules out this scenario, there is a gain over the "trivial" estimate of Corollary 2.6:

Theorem 1.6. *Given $s, u \in (0,1)$, there are $\epsilon, \eta > 0$ such that the following holds for small enough dyadic δ: Let $P \subset B^2(0,1)$ be set such that*

$$|P \cap B(x, \delta |P|_\delta^{1/2})|_\delta \leq \delta^u |P|_\delta, \quad x \in B^2(0,1). \tag{1-2}$$

For each $p \in P$, let $\mathcal{T}(p)$ be a $(\delta, s, \delta^{-\epsilon})$-set of dyadic tubes through p with $|\mathcal{T}(p)| \geq M$. Then the union $\mathcal{T} = \bigcup_{p \in P} \mathcal{T}(p)$ satisfies

$$|\mathcal{T}| \geq \delta^{-\eta} M |P|_\delta^{1/2}. \tag{1-3}$$

This result extends Proposition 1.3 (however, the proposition is used in the proof) and due to the minimal nonconcentration assumption it provides new information even when $|P|_\delta \gg \delta^{-1}$. Perhaps more significantly, Theorem 1.6 generalizes Bourgain's celebrated discretized projection theorem [2010, Theorem 3] (or even the refined version with single-scale nonconcentration in [Shmerkin 2023, Theorem 1.7]). Roughly speaking, Bourgain's theorem corresponds to the special "product" case in which the slopes of the tubes in $\mathcal{T}(p)$ are (nearly) independent of p; see Section 5 for the details. This connection between Furstenberg sets and projections is well known; see, e.g., [Oberlin 2014] or [Orponen and Shmerkin 2023, §3.2]. Bourgain's discretized projection theorem is used in the proof of Theorem 3.2 of the latter work (recalled as Theorem 1.5 above), which is in turn used to prove Theorem 1.6, so this does not provide a new proof of the projection theorem. Using a well-known argument of G. Elekes, Theorem 1.6 (or rather its dual formulation below) also easily recovers Bourgain's discretized sum-product theorem [Bourgain 2003, Theorem 0.3; Bourgain and Gamburd 2008, Proposition 3.2]. The details are sketched in Section 5. It is worth noting that although Bourgain's discretized sum-product and projection theorems are closely connected to each other, deducing either from the other takes a substantial amount of work, while they are both rather direct corollaries of Theorem 1.6.

By duality between points and lines (again we refer to [Orponen and Shmerkin 2023, Theorem 3.2] for details), we have the following corollary of Theorem 1.6:

Corollary 1.7. *Given $s, u \in (0,1)$, there are $\epsilon, \eta > 0$ such that the following holds for small enough dyadic δ: Let $\mathcal{T} \subset \mathcal{T}^\delta$ be a set of dyadic tubes such that*

$$|\{T \in \mathcal{T} : T \subset \boldsymbol{T}\}| \le \delta^u |\mathcal{T}|$$

for all $(\delta|\mathcal{T}|^{1/2})$-tubes $\boldsymbol{T}$.

For each $T \in \mathcal{T}$, let $P(T) \subset T$ be a $(\delta, s, \delta^{-\epsilon})$-set with $|P(T)|_\delta \ge M$. Then $P = \bigcup_{T \in \mathcal{T}} P(T)$ satisfies

$$|P| \ge \delta^{-\eta} M |T|^{1/2}.$$

1.3. ***Sketch of proof.*** Many "ϵ-improvements" in discretized geometry are obtained by showing that, in the absence of it, the relevant geometric object has a rigid structure that eventually is shown to contradict some previously known bounds (often involving some other "ϵ-improvement"). This is also the approach we take here. By the simple elementary bounds in Lemmas 2.4–2.5, one obtains the improved bound in Proposition 1.3 unless

$$|P \cap T|_\delta \approx |P|_\delta^{1/2}, \quad T \in \mathcal{T},$$
$$|P|_\delta \approx \delta^{-t}. \tag{1-4}$$

(In this section the notation $\approx$ should be interpreted informally as "up to small powers of δ"). A first ingredient of the proof is showing that (after suitable refinements of P and $\mathcal{T}$) one also gets the desired conclusion unless

$$|P \cap T \cap Q|_\delta \approx |P \cap Q|_\delta^{1/2} \tag{1-5}$$

for all $T \in \mathcal{T}$, all Δ-squares Q intersecting P and a "dense" set of scales $\delta < \Delta < 1$. To see this, we combine the elementary bounds applied to $P \cap Q$ (and a subsystem of tubes passing through $P \cap Q$) with an induction-on-scales mechanism from [Orponen and Shmerkin 2023], recalled as Proposition 2.7 below.

The relation (1-5) can be shown to imply that either one gets the improved bound we are seeking, or P intersects each tube T in a $\left(\delta, \frac{1}{2}t\right)$-set. By (known) elementary arguments $\mathcal{T}$ is a $\left(\delta, s + \frac{1}{2}t\right)$-set of tubes. Hence P is a discretized $\big(\tfrac{1}{2}t, s+\tfrac{1}{2}t\big)$-Furstenberg set. But it follows from Theorem 1.5 that $|P|_\delta \geq \delta^{-t-\eta}$ for some $\eta = \eta\big(\tfrac{1}{2}t, s+\tfrac{1}{2}t\big) > 0$. This contradicts (1-4), showing the impossibility of the rigid configuration described by (1-5) and hence establishing Proposition 1.3. To our knowledge this dual relationship between (s, t) and $\big(\tfrac{1}{2}t, s+\tfrac{1}{2}t\big)$-Furstenberg sets hadn't been noticed before.

We obtain Theorem 1.6 by applying Proposition 1.3 to each scale in a multiscale decomposition of P into "nontrivial Frostman pieces" that was established in [Shmerkin 2023], and is recalled as Theorem 2.10 below. The scales are combined together by another application of Proposition 2.7.

2. Preliminaries

2.1. ***Notation.*** The notation $A \lesssim B$ or $A = O(B)$ stands for $A \leq C \cdot B$ for some constant $C > 0$, and similarly for $A \gtrsim B$ and $A \sim B$. The δ-covering number of a set X (in a metric space) is defined as the smallest number of δ-balls needed to cover X, and is denoted by $|X|_\delta$. The open r-neighbourhood of a set X is denoted by $X^{(r)}$.

2.2. ***(δ, s)-sets of points and tubes.*** Given $r \in 2^{-\mathbb{N}}$, we denote the family of (half-open) dyadic cubes of side-length in $\mathbb{R}^d$ by $\mathcal{D}_r$. The set of cubes in $\mathcal{D}_r$ intersecting a set X is denoted by $\mathcal{D}_r(X)$.

In this article, we work with the following notion of discretization of sets of dimension s:

Definition 2.1 ((δ, s, C)-set). Let $P \subset \mathbb{R}^d$ be a bounded nonempty set, $d \geq 1$. Let $\delta \in 2^{-\mathbb{N}}$, $0 \leq s \leq d$, and $C > 0$. We say that P is a *(δ, s, C)-set* if

$$|P \cap Q|_\delta \leq C \cdot |P|_\delta \cdot r^s, \quad Q \in \mathcal{D}_r(\mathbb{R}^d),\ \delta \leq r \leq 1. \tag{2-1}$$

If $P \subset \mathcal{D}_\delta$ (so P is a family of dyadic cubes), we will abuse notation by identifying P with $\bigcup P$, so it makes sense to speak of (δ, s, C)-sets of dyadic cubes.

We also need to work with discretized families of tubes:

Definition 2.2 (dyadic δ-tubes). Let $\delta \in 2^{-\mathbb{N}}$. A *dyadic δ-tube* is a set of the form

$$\boldsymbol{D}(p) := \{(x, y) : y = ax + b \text{ for some } (a, b) \in p\},$$

where $p \in \mathcal{D}_\delta([-1, 1) \times \mathbb{R})$. The collection of all dyadic δ-tubes is denoted by

$$\mathcal{T}^\delta := \{\boldsymbol{D}(p) : p \in \mathcal{D}_\delta([-1, 1) \times \mathbb{R})\}.$$

A finite collection of dyadic δ-tubes $\{\boldsymbol{D}(p)\}_{p \in \mathcal{P}}$ is called a (δ, s, C)-set if $\mathcal{P}$ is a (δ, s, C)-set.

We remark that a dyadic δ-tube is not exactly a δ-neighbourhood of some line, but the intersection of

$$T = \boldsymbol{D}([a, a+\delta] \times [b, b+\delta])$$

with some fixed bounded set B satisfies $\ell_{a,b}^{(c\delta)} \cap B \subset T \cap B \subset \ell_{a,b}^{(C\delta)}$, where $\ell_{a,b} = \big\{y = \big(a+\tfrac{1}{2}\delta\big)x + \big(b+\tfrac{1}{2}\delta\big)\big\}$ and c, C depend only on B.

An elementary but important observation is that tubes in $\mathcal{T}^\delta$ that intersect a fixed square $p \in \mathcal{D}_\delta$ are parametrized by their slope in a bilipschitz way. In particular, if $\mathcal{T}(p) \subset \mathcal{T}^\delta$ is a family of tubes intersecting $p \in \mathcal{D}_\delta$, then $\mathcal{T}(p)$ is a (δ, s, C)-set if and only if the slopes of tubes in $\mathcal{T}(p)$ form a (δ, s, C') set for $C' \sim C$; see [Orponen and Shmerkin 2023, Corollary 2.12] for the precise statement.

2.3. ***Elementary incidence bounds.*** We now collect some elementary incidence bounds; they correspond in various ways to the lower bound $s + \frac{1}{2}t$ for the dimension of (s, t)-Furstenberg sets. We state our bounds in terms of the following notion:

Definition 2.3. Fix $\delta \in 2^{-\mathbb{N}}$, $s \in [0, 1]$, $C > 0$, $M \in \mathbb{N}$. We say that a pair $(P, \mathcal{T}) \subset \mathcal{D}_\delta \times \mathcal{T}^\delta$ is a (δ, s, C, M)-*nice configuration* if for every $p \in P$ there exists a (δ, s, C)-set $\mathcal{T}(p) \subset \mathcal{T}$ with $|\mathcal{T}(p)| = M$ and such that $T \cap p \neq \varnothing$ for all $T \in \mathcal{T}(p)$.

Lemma 2.4. *Let $(P, \mathcal{T})$ be a (δ, s, C, M)-nice configuration. Then for any δ-tube T (not necessarily in $\mathcal{T}$),*

$$|\mathcal{T}| \gtrsim C^{-1/s} \cdot |T \cap P|_\delta \cdot M.$$

Proof. We may assume P is δ-separated. Fix $p \in T \cap P$. Since $\mathcal{T}(p)$ is a (δ, s, C)-set, there is a subset $\mathcal{T}'(p) \subset \mathcal{T}(p)$ such that $|\mathcal{T}'(p)| \geq \frac{1}{2}|\mathcal{T}(p)| = \frac{1}{2}M$ and each tube $T' \in \mathcal{T}'(p)$ makes an angle $\gtrsim C^{-1/s}$ with the direction of T. In turn, this implies that the sets $\mathcal{T}'(p)$, $p \in T \cap P$ have overlap $\lesssim C^{1/s}$. This gives the claim. □

Lemma 2.5. *Let $(P, \mathcal{T})$ be a (δ, s, C, M)-nice configuration. Suppose $|T \cap P| \leq K$ for all $T \in \mathcal{T}$. Then*

$$|\mathcal{T}| \geq K^{-1} \cdot |P| \cdot M.$$

Proof. We have

$$|P| \cdot M = \sum_{p \in P} |\mathcal{T}_p| = \sum_{T \in \mathcal{T}} |\{p : T \in \mathcal{T}_p\}| \leq \sum_{T \in \mathcal{T}} |T \cap P| \leq |\mathcal{T}| \cdot K.$$ □

Corollary 2.6. *Let $(P, \mathcal{T})$ be a (δ, s, C, M)-nice configuration. Then*

$$|\mathcal{T}| \gtrsim C^{-1/s} \cdot |P|^{1/2} \cdot M.$$

Moreover, $\mathcal{T}$ contains a $\big(\delta, s + \frac{1}{2}t, \log(1/\delta)^{O(1)} C^{1/s}\big)$-set of δ-tubes.

Proof. If $|T \cap P| \geq |P|^{1/2}$ for some $T \in \mathcal{T}$, we apply Lemma 2.4, otherwise we apply Lemma 2.5. In any case we get the first claim.

Let $\mathcal{L}$ be the set of lines corresponding to tubes in $\mathcal{T}$ (or, equivalently, the δ-neighbourhood of the central lines of tubes in T). It follows from the first claim and two dyadic pigeonholings that the Hausdorff content of $\mathcal{L}$ satisfies

$$\mathcal{H}^{s+t/2}_\infty(\mathcal{L}) \gtrsim \log(1/\delta)^{-O(1)} C^{-1/s}.$$

See, e.g., the proof of [Héra et al. 2022, Lemma 3.3] or [Orponen et al. 2024, Lemma 3.5, in particular, equation (3.9)]. The conclusion then follows from the discrete version of Frostman's lemma [Fässler and Orponen 2014, Proposition A.1] (which is stated in $\mathbb{R}^3$ but works just as well in the space of lines). □

2.4. ***A multiscale incidence bound.*** We next recall [Orponen and Shmerkin 2023, Proposition 5.2]. Fix two dyadic scales $0 < \delta < \Delta \leq 1$ and families $P_0 \subset \mathcal{D}_\delta$ and $\mathcal{T}_0 \subset \mathcal{T}^\delta$. For $Q \in \mathcal{D}_\Delta$ and $\boldsymbol{T} \in \mathcal{T}^\Delta$, we define

$$P_0 \cap Q = \{p \in P_0 : p \subset Q\} \quad \text{and} \quad \mathcal{T}_0 \cap \boldsymbol{T} := \{T \in \mathcal{T}_0 : T \subset \boldsymbol{T}\}.$$

We also write

$$\mathcal{D}_\Delta(P_0) = \{Q \in \mathcal{D}_\Delta : P_0 \cap Q \neq \varnothing\},$$
$$\mathcal{T}^\Delta(\mathcal{T}_0) = \{\boldsymbol{T} \in \mathcal{T}^\Delta : \mathcal{T}_0 \cap \boldsymbol{T} \neq \varnothing\}.$$

In the next proposition, for $\Delta \in 2^{-\mathbb{N}}$ and $Q \in \mathcal{D}_\Delta$, the map $S_Q : \mathbb{R}^2 \to \mathbb{R}^2$ is the homothety that maps Q to the square $[0, 1)^2$, and $S_Q(P_0) = \{S_Q(p) : p \in P_0\}$. Furthermore, the notation $A \lessapprox_\delta B$ stands for $A \leq \log(1/\delta)^C B$ for a constant $C > 0$, and likewise for $A \approx_\delta B$.

Proposition 2.7 [Orponen and Shmerkin 2023, Proposition 5.2]. *Fix dyadic numbers* $0 < \delta < \Delta \leq 1$. *Let* $(P_0, \mathcal{T}_0)$ *be a* (δ, s, C, M)*-nice configuration. Then there exist sets* $P \subset P_0$ *and* $\mathcal{T}(p) \subset \mathcal{T}_0(p)$, $p \in P$, *such that defining* $\mathcal{T} = \bigcup_{p \in P} \mathcal{T}(p)$ *the following hold*:

(i) $|\mathcal{D}_\Delta(P)| \approx_\delta |\mathcal{D}_\Delta(P_0)|$ *and* $|P \cap Q| \approx_\delta |P_0 \cap Q|$ *for all* $Q \in \mathcal{D}_\Delta(P)$.

(ii) $|\mathcal{T}(p)| \gtrapprox_\delta |\mathcal{T}_0(p)| = M$ *for* $p \in P$.

(iii) $(\mathcal{D}_\Delta(P), \mathcal{T}^\Delta(\mathcal{T}))$ *is* $(\Delta, s, C_\Delta, M_\Delta)$*-nice for some* $C_\Delta \approx_\delta C$ *and* $M_\Delta \geq 1$.

(iv) *For each* $Q \in \mathcal{D}_\Delta(P)$ *there exist* $C_Q \approx_\delta C$, $M_Q \geq 1$, *and a family of tubes* $\mathcal{T}_Q \subset \mathcal{T}^{\delta/\Delta}$ *such that* $(S_Q(P \cap Q), \mathcal{T}_Q)$ *is* $(\delta/\Delta, s, C_Q, M_Q)$*-nice.*

Furthermore, the families $\mathcal{T}_Q$ *can be chosen so that*

$$\frac{|\mathcal{T}_0|}{M} \gtrapprox_\delta \frac{|\mathcal{T}^\Delta(\mathcal{T})|}{M_\Delta} \cdot \left(\max_{Q \in \mathcal{D}_\Delta(P)} \frac{|\mathcal{T}_Q|}{M_Q} \right). \tag{2-2}$$

2.5. ***Uniformization.*** Next, we recall a basic lemma asserting the existence of large uniform subsets. See, e.g., [Orponen and Shmerkin 2023, Lemma 7.3] for the proof.

Definition 2.8. Let $N \geq 1$, and let

$$\delta = \Delta_N < \Delta_{N-1} < \cdots < \Delta_1 \leq \Delta_0 = 1$$

be a sequence of dyadic scales. We say that a set $P \subset [0, 1)^2$ is $(\Delta_j)_{j=1}^N$*-uniform* if there is a sequence $(K_j)_{j=1}^N$ such that $|P \cap Q|_{\Delta_j} = K_j$ for all $j \in \{1, \dots, N\}$ and all $Q \in \mathcal{D}_{\Delta_{j-1}}(P)$.

Lemma 2.9. *Given* $P \subset [0, 1)^2$ *and a sequence* $\delta = \Delta_N < \Delta_{N-1} < \cdots < \Delta_1 \leq \Delta_0 = 1$ *of dyadic numbers*, $N \geq 1$, *there is a* $(\Delta_j)_{j=1}^N$*-uniform set* $P' \subset P$ *such that*

$$|P'|_\delta \geq (4N^{-1} \log(1/\delta))^{-N} |P|_\delta. \tag{2-3}$$

Note that if the number N of scales is independent of δ, then the lower bound on $|P'|_\delta$ can be simplified as

$$|P'|_\delta \geq C_N^{-1} \cdot \log(1/\delta)^{-C_N} \cdot |P|_\delta,$$

with C_N independent of δ.

2.6. ***A multiscale decomposition.*** To conclude this section, we recall the multiscale decomposition into "Frostman pieces" of uniform sets that satisfy a single-scale nonconcentration assumption provided by [Shmerkin 2023, Theorem 4.1].

Theorem 2.10. *For every $u > 0$ and $\epsilon > 0$ there are $\xi = \xi(u) > 0$ and $\tau = \tau(\epsilon) > 0$ such that the following holds for all sufficiently small $\rho \le \rho_0(\epsilon)$ and $n \ge n_0(\rho, \epsilon)$: Let P be a $(\rho^j)_{j=1}^n$-uniform set and write $\delta = \rho^n$. Suppose*

$$|P \cap B(x, \delta |P|_\delta^{1/2})|_\delta \le \delta^u |P|_\delta \quad \textit{for all } x. \tag{2-4}$$

Then there exists a collection of dyadic scales

$$\delta = \Delta_N < \Delta_{N-1} < \cdots < \Delta_1 < \Delta_0 = 1, \quad N \le N_0(\epsilon),$$

each of which is a power of ρ, and numbers $\alpha_0, \dots, \alpha_{N-1} \in [0, 2]$ such that, defining $\lambda_j = \Delta_{j+1}/\Delta_j$, the following hold:

(i) *For each j and each $Q \in \mathcal{D}_{\Delta_j}(P)$,*

$$|P \cap Q \cap B(x, r\Delta_j)|_{\Delta_{j+1}} \le \lambda_j^{-\epsilon} \cdot r^{\alpha_j} \cdot |P \cap Q|_{\Delta_{j+1}}$$

for all $x \in B^2(0, 1)$ and all $r \in [\lambda_j, 1]$.

(ii) $\sum\{\alpha_j \log(1/\lambda_j) : \lambda_j \le \delta^\tau\} \ge \log |P|_\delta - 2\epsilon \log(1/\delta)$.

(iii) $\sum\{\log(1/\lambda_j) : \alpha_j \in [\xi, d - \xi]$ *and* $\lambda_j \le \delta^\tau\} \ge \xi \log(1/\delta)$.

This is just (a slightly weaker version of) [Shmerkin 2023, Theorem 4.1], although stated using different notation: the measure μ there corresponds to normalized Lebesgue measure on $P(\delta)$ (or the union of δ-squares intersecting δ); ρ corresponds to 2^{-T} in [Shmerkin 2023]; the scales Δ_j correspond to both 2^{-TA_i} and 2^{-TB_i}. The last two claims only concern scale intervals $[\Delta_{j+1}, \Delta_j]$ corresponding to $[2^{-TB_i}, 2^{-TA_i}]$, while for $[\Delta_{j+1}, \Delta_j] = [2^{-TB_{i+1}}, 2^{-TA_i}]$ we simply take $\alpha_j = 0$.

3. Proof of Proposition 1.3

In this section we prove Proposition 1.3. Note that there is no loss of generality in assuming that P is a union of dyadic δ-squares and the tubes in $\mathcal{T}(p)$ are dyadic δ-tubes intersecting p.

The parameter ϵ should be thought of as being much smaller than η (and will be chosen after η). Both ϵ, η will ultimately be chosen in terms of s, t only. We let $N = \lceil \eta^{-1} \rceil$, so that $N = N(s, t)$. Let $\rho = \delta^{1/N}$; without loss of generality, ρ is dyadic.

In the course of this proof, $A \lessapprox B$ stands for $A \le C(s, t)\delta^{-C(s,t)\epsilon} B$. Likewise, a (δ, u)-set is a (δ, u, C)-set for $C \lessapprox 1$.

Replacing P by its $(\rho^j)_{j=1}^N$-uniformization (given by Lemma 2.9), we may assume that P is $(\rho^j)_{j=1}^N$-uniform. Note that the uniformization is still a (δ, t)-set.

We will construct sequences $P_j \subset \mathcal{D}_\delta$, $\mathcal{T}_j \subset \mathcal{T}^\delta$, $j = 0, 1, \dots, N$, with the following properties:

(a) P_j is $(\rho^j)_{j=1}^N$-uniform.

(b) $P_{j+1} \subset P_j$, $P_0 \subset P$ and $|P_{j+1}| \gtrapprox |P_j|$, $|P_0| \gtrapprox |P|$. In particular, P_N is a (δ, t)-set.

(c) $\mathcal{T}_0(p) = \mathcal{T}(p)$, $\mathcal{T}_{j+1}(p) \subset \mathcal{T}_j(p)$ and $M_{j+1} \sim |\mathcal{T}_{j+1}(p)| \gtrapprox |\mathcal{T}_j(p)|$ for $p \in P_{j+1}$ and some M_{j+1}. In particular, each $\mathcal{T}(p)$, $p \in P_N$, is a (δ, t)-set.

(d) For each j, any $Q \in \mathcal{D}_{\rho^j}(P_j)$ and any δ-tube T,

$$|\mathcal{T}| \gtrapprox M \cdot |P_j|_{\rho^j}^{1/2} \cdot |T \cap P_j \cap Q|_\delta.$$

Recall that $|\mathcal{T}(p)| \geq M$ for each $p \in P$. Pigeonhole a dyadic number M_0 such that $|\mathcal{T}(p)| \sim M_0$ for all $p \in P_0' \subset P$ with $|P_0'| \gtrapprox |P|$. We let P_0 be the $(\rho^j)_{j=1}^N$-uniformization of P_0' given by Lemma 2.9. Then P_0 is a (δ, t)-set, and we take $\mathcal{T}_0(p) = \mathcal{T}(p)$ for $p \in P_0$.

Once $P_j, \mathcal{T}_j$ are defined, we let P'_{j+1}, $\mathcal{T}'_{j+1}$ be the objects provided by Proposition 2.7 applied to $(P_j, \mathcal{T}_j)$ at scale $\Delta = \rho^{j+1}$. It follows from Proposition 2.7(i) and the regularity of P_j that $P'_{j+1} \subset P_j$ and $|P'_{j+1}| \gtrapprox P_j$. Pigeonhole a number M_{j+1} such that $|\mathcal{T}'_{j+1}(p)| \sim M_{j+1}$ for all $p \in P''_{j+1} \subset P'_{j+1}$, where $|P''_{j+1}| \gtrapprox |P'_{j+1}|$. Finally, let P_{j+1} be the $(\rho^j)_{j=1}^N$-uniformization of P''_{j+1} and $\mathcal{T}_{j+1}(p) = \mathcal{T}'_{j+1}(p)$ for $p \in P_{j+1}$.

Properties (a)–(b) hold by construction. Property (c) follows from Proposition 2.7(ii). To see property (d), let C_Δ, M_Δ, $\mathcal{T}_Q$, C_Q, M_Q be the objects provided by Proposition 2.7(iii)–(iv). By Corollary 2.6,

$$|\mathcal{T}^\Delta(\mathcal{T}'_{j+1})| \gtrapprox M_\Delta \cdot |P_{j+1}|^{1/2},$$

and by Lemma 2.4 and rescaling,

$$|\mathcal{T}_Q| \gtrapprox M_Q \cdot |T \cap P_j \cap Q|_\delta.$$

Putting these facts together with (2-2), we see that property (d) holds.

We pause to observe that $(\mathcal{P}_N, \mathcal{T}_N)$ is a (δ, s, C_N, M_N)-nice configuration, where $C_N \lessapprox 1$ and $M_N \gtrapprox M$.

We now consider several cases. Suppose first that there are $T \in \mathcal{T}_N$, $j \in \{1, \ldots, N\}$, and $Q \in \mathcal{D}_{\rho^j}(P_N)$ such that

$$|T \cap P_N \cap Q|_\delta \geq \delta^{-2\eta} \cdot |P_N \cap Q|_\delta^{1/2}.$$

Then, by (d), the uniformity of P_N, and the fact that $P_j \supset P_N$ is a (δ, t)-set, we see that (1-1) holds if ϵ is small enough in terms of s, t, η.

Hence, we assume from now on that

$$|T \cap P_N \cap Q|_\delta \leq \delta^{-2\eta} \cdot |P_N \cap Q|_\delta^{1/2} \tag{3-1}$$

for $j \in \{1, \ldots, N\}$, $Q \in \mathcal{D}_{\rho^j}(P_N)$, and $T \in \mathcal{T}_N$.

We consider two further subcases. Suppose first that for at least half of the squares p in P_N, at least half of the tubes $T \in \mathcal{T}_N(p)$ satisfy

$$|T \cap P_N|_\delta \leq \delta^{2\eta} \cdot |P_N|_\delta^{1/2}.$$

Then Lemma 2.5 (applied to suitable restrictions of P_N and $\mathcal{T}_N$) yields (1-1).

We can then assume that

$$|T \cap P_N|_\delta \geq \delta^{2\eta} \cdot |P_N|_\delta^{1/2} \quad \text{for all } T \in \mathcal{T}'_N, \tag{3-2}$$

where $\mathcal{T}'_N = \bigcup_{p \in P'_N} \mathcal{T}'(p)$, with $|P'_N| \geq \frac{1}{2}|P_N|$ and $|\mathcal{T}'_N(p)| \geq \frac{1}{2}|\mathcal{T}_N(p)|$. It then follows from (3-1) that, for any $Q \in \mathcal{D}_{\rho^n}(P_N)$ and $T \in \mathcal{T}'_N$,

$$\begin{aligned} |T \cap P_N \cap Q|_\delta &\leq \delta^{-4\eta}(|P_N|_\delta^{-1/2}|P_N \cap Q|_\delta^{1/2})|T \cap P_N|_\delta \\ &\leq \delta^{-5\eta}(\rho^j)^{t/2}|T \cap P_N|_\delta, \end{aligned}$$

using that P_N is a (δ, t)-set and taking ϵ sufficiently small in terms of η, s, t.

Recalling that $N = \lceil \eta^{-1} \rceil$, we deduce that, for any r-ball B_r with $\rho^{j+1} \leq r < \rho^j$,

$$|T \cap P_N \cap B_r|_\delta \lesssim \delta^{-5\eta}(\rho^j)^{t/2}|T \cap P_N|_\delta \leq \delta^{-6\eta} r^{t/2}|T \cap P_N|_\delta,$$

so that $T \cap P_N$ is a $(\delta, \frac{1}{2}t, \delta^{-7\eta})$-set for each $T \in \mathcal{T}'$.

Now, taking ϵ small enough in terms of s, t, η, we deduce from Corollary 2.6 that $\mathcal{T}'$ contains a $(\delta, s + \frac{1}{2}t, \delta^{-\eta})$-set $\mathcal{T}''$. Then $\{T \cap P_N : T \in \mathcal{T}''\}$ satisfies the assumptions of Theorem 1.5 (with $\frac{1}{2}t$ in place of s and $s + \frac{1}{2}t$ in place of t), provided that η and δ are taken small enough in terms of s, t only. Applying Theorem 1.5, we conclude that $|P_N|_\delta > \delta^{-t-3\eta}$ (again assuming $\eta = \eta(s, t)$ is small enough). The first claim of Corollary 2.6 then yields (1-1).

4. Proof of Theorem 1.6

By iterating Proposition 2.7, we obtain the follow multiscale version.

Corollary 4.1. *Fix $N \geq 2$ and dyadic numbers*

$$0 < \delta = \Delta_N < \Delta_{N-1} < \cdots < \Delta_1 < \Delta_0 = 1.$$

Let $(P_0, \mathcal{T}_0)$ be a (δ, s, C, M)-nice configuration. Then there exists a set $P \subset P_0$ such that the following hold:

(i) *$|\mathcal{D}_{\Delta_j}(P)| \approx_\delta |\mathcal{D}_{\Delta_j}(P_0)|$ and $|P \cap Q| \approx_\delta |P_0 \cap Q|$ for all $j \in [1, N]$ and all $Q \in \mathcal{D}_{\Delta_j}(P)$.*

(ii) *For each $j \in [0, N-1]$ and each $Q \in \mathcal{D}_{\Delta_j}(P)$ there exist $C_Q \approx_\delta C$, $M_Q \geq 1$, and a family of tubes $\mathcal{T}_Q \subset \mathcal{T}^{\Delta_{j+1}/\Delta_j}$ such that $(S_Q(P \cap Q), \mathcal{T}_Q)$ is $(\Delta_{j+1}/\Delta_j, s, C_Q, M_Q)$-nice.*

Furthermore, the families $\mathcal{T}_Q$ can be chosen so that if $Q_j \in \mathcal{D}_{\Delta_j}(P)$, then

$$\frac{|\mathcal{T}_0|}{M} \gtrapprox_\delta \prod_{j=0}^{N-1} \frac{|\mathcal{T}_{Q_j}|}{M_{Q_j}}. \tag{4-1}$$

All the constants implicit in the $\approx_\delta$ notation are allowed to depend on N.

Proof. We proceed by induction in N. The case $N = 2$ follows from Proposition 2.7. Suppose the claim holds for N and let us verify it for $N + 1$. First apply Proposition 2.7 with $\delta = \Delta_{N+1}$ and $\Delta = \Delta_N$. Let P', $\mathcal{T}$ be the resulting objects. Property (ii) holds (at the moment for P') for $j = N$ thanks to Proposition 2.7(iv). We then apply the inductive assumption to $(\mathcal{D}_{\Delta_N}(P'), \mathcal{T}^{\Delta_N}(\mathcal{T}))$, which is legitimate by Proposition 2.7(iii). This yields a set $P'' \subset \mathcal{D}_{\Delta_N}$; we define

$$P = \bigcup_{Q \in \mathcal{D}_{\Delta_N}(P'')} P' \cap Q.$$

This ensures that $S_Q(P \cap Q)$, when viewed at scale Δ_{j+1}/Δ_j, equals P'' for $j < N$ and P' for $j = N$, and so property (ii) holds for all $j \in [0, N]$. Finally, (4-1) holds thanks to (2-2) and the inductive assumption. □

Proof of Theorem 1.6. We will eventually take $\eta = \eta(s, u)$ and $\epsilon = \epsilon(\eta, s, u) \ll \eta$. Fix $\rho = \rho(\epsilon) \in (0, 1)$ small enough that the conclusion of Theorem 2.10 holds, and

$$\frac{\log(4 \log(1/\rho))}{\log(1/\rho)} < \epsilon.$$

Then, for $\delta = \rho^n$, the $(\rho^j)_{j=1}^n$-uniformization P' of P given by Lemma 2.9 satisfies $|P'| \geq \delta^\epsilon |P|$. We assume from now on that $\delta = \rho^n$, where n is taken sufficiently large in terms of all other parameters. Take $\epsilon < \min\left(\frac{1}{2}u, \frac{1}{2}\eta\right)$. Then P' satisfies (1-2) with $\frac{1}{2}u$ in place of u, and the conclusion (1-3) for P' implies it for P with $\frac{1}{2}\eta$ in place of η. Hence we assume from now on that P is $(\rho^j)_{j=1}^n$-uniform.

We apply Theorem 2.10 to P. Let $\xi = \xi(u)$, $\tau = \tau(\epsilon)$ be the numbers provided by the theorem, and let $(\Delta_j)_{j=0}^N$ and $(\alpha_j)_{j=1}^N$ be the scales and exponents corresponding to P.

Let $\lambda_j = \Delta_{j+1}/\Delta_j$. We apply Corollary 4.1 to $(P, \mathcal{T})$ and the scales (Δ_j). Let $P' \subset P$ be the resulting set. Since $N \leq N_0(\epsilon)$, the notation $A \lessapprox_\delta B$ in the statement of Corollary 4.1 translates to $A \leq \log(1/\delta)^{C(\epsilon)} B$; in particular, $A \leq \delta^{\epsilon\tau/2} B$ if δ is small enough in terms of ϵ. With these remarks, the $(\Delta_j)_{j=1}^N$-uniformity of P, Corollary 4.1(i) and Theorem 2.10(i) show that if $Q \in \mathcal{D}_{\Delta_j}(P')$, then $S_Q(P' \cap Q)$ is a $(\lambda_j, \alpha_j, \lambda_j^{-\epsilon})$-set whenever $\lambda_j \leq \delta^\tau$.

Let

$$\mathcal{N} = \{j : \lambda_j \leq \delta^\tau,\ \alpha_j \notin [\xi, 2 - \xi]\},$$
$$\mathcal{G} = \{j : \lambda_j \leq \delta^\tau,\ \alpha_j \in [\xi, 2 - \xi]\}.$$

(Here $\mathcal{N}$ stands for "normal" and $\mathcal{G}$ for "good" scales.) It follows from Corollary 4.1(ii) combined with Corollary 2.6 and Proposition 1.3 that, for $Q \in \mathcal{D}_{\Delta_j}(P')$,

$$j \in \mathcal{N} \quad \Longrightarrow \quad |\mathcal{T}_Q| \geq (1/\log\delta)^{C(\epsilon,s)} \cdot M_Q \cdot |S_Q(P \cap Q)|_{\lambda_j}^{1/2},$$
$$j \in \mathcal{G} \quad \Longrightarrow \quad |\mathcal{T}_Q| \geq (1/\log\delta)^{C(\epsilon,s)} \cdot M_Q \cdot \lambda_j^{-(\alpha_j/2 - \eta)},$$

where $\eta = \eta(\xi, s) = \eta(u, s) > 0$. It is indeed possible to take a value of η uniformly over $t \in [\xi, 2 - \xi]$ because the value of η in Proposition 1.3 is robust under perturbations of t. Note that since $\tau = \tau(\epsilon)$, if δ is small enough in terms of ϵ, and $j \in \mathcal{G}$, then $\lambda_j \leq \delta^\tau$ is small enough that Proposition 1.3 is indeed applicable. It follows from Theorem 2.10(i) that

$$|S_Q(P \cap Q)|_{\lambda_j} \geq \lambda_j^{\epsilon - \alpha_j}.$$

Combining these facts with the conclusion (4-1) and the trivial bound $|\mathcal{T}_Q| \geq M_Q$ (applied at scales outside $\mathcal{N} \cup \mathcal{G}$), we obtain

$$\frac{|\mathcal{T}|}{M} \geq \Bigl(\prod_{j \in \mathcal{N}} \lambda_j^{\epsilon/2} \lambda_j^{-\alpha_j/2}\Bigr)\Bigl(\prod_{j \in \mathcal{G}} \lambda_j^{-\eta} \lambda_j^{-\alpha_j/2}\Bigr) \geq \delta^\epsilon \Bigl(\prod_{j \in \mathcal{N} \cup \mathcal{G}} \lambda_j^{-\alpha_j/2}\Bigr)\Bigl(\prod_{j \in \mathcal{G}} \lambda_j^{-\eta}\Bigr) \geq \delta^{3\epsilon} \cdot |P|_\delta^{1/2} \cdot \delta^{-\xi\eta},$$

using Theorem 2.10(ii)–(iii) for the last inequality. Taking $\epsilon < \frac{1}{6}\xi\eta$, this gives the claim with $\frac{1}{2}\xi\eta$ in place of η. □

5. Connection with Bourgain's projection theorem

We conclude by showing how Theorem 1.6 has as corollaries both Bourgain's discretized projection and sum-product theorems. We start with the former. Let $\Pi_s(x, y) = x - sy$.

Theorem 5.1 (Bourgain's discretized projection theorem [2010]; see also [Shmerkin 2023]). *Given $s, u \in (0, 1)$ there are $\epsilon, \eta > 0$ such that the following hold: Let $P \subset B^2(0, 1)$ satisfy*

$$|P \cap B(x, \delta|P|_\delta^{1/2})|_\delta \le \delta^u |P|_\delta, \quad x \in B^2(0, 1).$$

Let $S \subset [1, 2]$ be a $(\delta, s, \delta^{-\epsilon})$-set. Then there is $s \in S$ such that

$$|\Pi_s(P')|_\delta \ge \delta^{-\eta} \cdot |P|_\delta^{1/2} \quad \textit{for all } P' \subset P,\ |P'|_\delta \ge \delta^\epsilon |P|_\delta.$$

Proof. The argument is standard. Suppose the claim does not hold. Hence, for each $s \in S$ there is a set $P_s \subset P$ with $|P_s|_\delta \ge \delta^\epsilon |P|_\delta$ such that

$$|\Pi_s(P_s)|_\delta \le \delta^{-\eta} \cdot |P|^{1/2}.$$

The set $X = \{(p, s) : p \in P_s\}$ has size $\ge \delta^\epsilon |P||S|$; hence there is a set $P' \subset P$ with $|P'|_\delta \gtrsim \delta^\epsilon |P|_\delta$ such that $|S_p| \gtrsim \delta^\epsilon |S|$, where $S_p = \{s : p \in P_s\}$.

Given a tube $T = \boldsymbol{D}([a, a+\delta] \times [b, b+\delta])$ (recall Definition 2.2), we let $\sigma(T) = [a, a+\delta]$ be the corresponding slope interval. For each $p \in P'$ let $\mathcal{T}_p$ be the set of dyadic tubes through p such that $\sigma(T) \cap S_p \ne \varnothing$.

If $\epsilon < \frac{1}{2}u$, then P' still satisfies the single-scale nonconcentration assumption (with $\frac{1}{2}u$ in place of u). Since S_p is a $(\delta, s, O(\delta^{-2\epsilon}))$-set, so is $\mathcal{T}_p$. Also, $|\mathcal{T}_p| \gtrsim \delta^\epsilon |S|$. Hence if $\epsilon > 0$ is small enough in terms of s, u, we can apply Theorem 1.6 to obtain (for $\mathcal{T} = \bigcup_{p \in P'} \mathcal{T}_p$)

$$|\mathcal{T}| \gtrsim \delta^{-\eta'} \cdot \delta^\epsilon |S| \cdot |P'|_\delta^{1/2} \ge \delta^{2\epsilon - \eta'} \cdot |S| \cdot |P|_\delta^{1/2},$$

where $\eta' > 0$ depends on s, u only. Taking $\eta' > 4\epsilon$, this implies that there is $s \in S$ such that there are $\gtrsim \delta^{-\eta'/2} \cdot |P|_\delta^{1/2}$ tubes $T \in \mathcal{T}$ with $s \in \sigma(T)$. All of these tubes intersect P_s by construction. We conclude that

$$|\Pi_s P_s|_\delta \gtrsim \delta^{-\eta'/2} |P|_\delta^{1/2}$$

which is a contradiction if we take $\eta = \frac{1}{3}\eta'$. □

We turn to the discretized sum-product problem. We have the following corollary of Theorem 1.6:

Corollary 5.2. *Given $s, u \in (0, 1)$, there are $\epsilon, \eta > 0$ such that the following holds for all small enough δ: Let $A, B_1, B_2 \subset [1, 2]$ satisfy the following: A is a $(\delta, s, \delta^{-\epsilon})$-set, and B_1, B_2 satisfy the single-scale nonconcentration bound*

$$|B_i \cap [a, a+\delta|B_i|_\delta]|_\delta \le \delta^u |B_i|, \quad a \in [1, 2].$$

Then

$$|A + B_1|_\delta |A \cdot B_2|_\delta \ge \delta^{-\eta} |A|_\delta |B_1|_\delta^{1/2} |B_2|_\delta^{1/2}. \tag{5-1}$$

Taking $A = B_1 = B_2$, one immediately recovers Bourgain's discretized sum-product theorem, even under the weak nonconcentration assumption of [Bourgain and Gamburd 2008, Proposition 3.2].

To prove the corollary, consider (as in [Elekes 1997]) the set $P = (A + B_1) \times (A \cdot B_2)$. For each $(b_1, b_2) \in B_1 \times B_2$, the set P intersects the line $\ell_{b_1,b_2} = \{x + b_1, b_2 x : x \in \mathbb{R}\}$ in an affine copy of A. Using that $(b_1, b_2) \to \ell_{b_1,b_2}$ is bilipschitz, it is routine to verify that this configuration satisfies the assumptions of Corollary 1.7, with $2u$ in place of u (considering for each (b_1, b_2) the δ-dyadic tube that contains $\ell_{b_1,b_2} \cap [0, 2]^2$). See, e.g., [Dąbrowski et al. 2022, §6.3] for details of adapting Elekes' argument to the discretized setting. The conclusion of Corollary 1.7 is then precisely (5-1).

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