

PERMANENCE PROPERTIES OF F -INJECTIVITY

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ABSTRACT. We prove that F -injectivity localizes, descends under faithfully flat homomorphisms, and ascends under flat homomorphisms with Cohen–Macaulay and geometrically F -injective fibers, all for arbitrary Noetherian rings of prime characteristic. As a consequence, we show that the F -injective locus is open on most rings arising in arithmetic and geometry. As a geometric application, we prove that over an algebraically closed field of characteristic $p > 3$, generic projection hypersurfaces associated to suitably embedded smooth projective varieties of dimension ≤ 5 are F -pure, and hence F -injective. This geometric result is the positive characteristic analogue of a theorem of Doherty.

CONTENTS

1. Introduction	1
2. Definitions and preliminaries	4
3. Localization and descent under faithfully flat homomorphisms	7
4. Ascent under flat homomorphisms with geometrically CMFI fibers	11
5. CMFI homomorphisms and openness of F -injective loci	17
6. Singularities of generic projection hypersurfaces	21
Appendix A. F -rationality descends and implies F -injectivity	24
References	26

1. INTRODUCTION

Let \mathcal{P} be a property of Noetherian local rings, and denote by (R, \mathfrak{m}) a Noetherian local ring. Reasonably well-behaved properties \mathcal{P} usually satisfy the following:

- (I) (Localization) R is \mathcal{P} if and only if $R_{\mathfrak{p}}$ is \mathcal{P} for every prime ideal $\mathfrak{p} \subseteq R$.
- (II) (Descent) If $(R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ is a flat local homomorphism of Noetherian local rings, and S is \mathcal{P} , then R is \mathcal{P} .
- (III) (Ascent/Base change) If $(R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ is a flat local homomorphism of Noetherian local rings, and both R and the closed fiber $S/\mathfrak{m}S$ are \mathcal{P} , then S is \mathcal{P} .
- (IV) (Openness) If (R, \mathfrak{m}) is excellent local, then the locus $\{\mathfrak{q} \in \operatorname{Spec}(R) \mid R_{\mathfrak{q}} \text{ is } \mathcal{P}\}$ is open.

As an illustration, (I)–(IV) hold for many classical properties \mathcal{P} :

2020 *Mathematics Subject Classification*. Primary 13A35; Secondary 13D45, 13H10, 14J17.

Key words and phrases. F -injective ring, local cohomology, base change, generic projection.

RD was supported by an AMS-Simons travel grant.

TM was supported by the National Science Foundation under Grant Nos. DMS-1701622 and DMS-1902616.

\mathcal{P}	(I)	(II) + (III)	(IV)
regular	[Mat89, Thm. 19.3]	[Mat89, Thm. 23.7]	[Mat89, Def. on p. 260]
normal	[Mat89, Def. on p. 64]	[Mat89, Cor. to Thm. 23.9]	[EGAIV ₂ , Prop. 7.8.6(iii)]
reduced	[BouCA, Prop. II.2.17]	[Mat89, Cor. to Thm. 23.9]	[EGAIV ₂ , Prop. 7.8.6(iii)]
complete intersection	[Avr75, Cor. 1]	[Avr75, Thm. 2]	[GM78, Cor. 3.3]
Gorenstein	[Mat89, Thm. 18.2]	[Mat89, Thm. 23.4]	[GM78, Rem. (b) on p. 213]
Cohen–Macaulay	[Mat89, Thm. 17.3(iii)]	[Mat89, Cor. to Thm. 23.3]	[EGAIV ₂ , Prop. 7.8.6(iii)]

In this paper, we are interested in assertions (I)–(IV) for F -singularities. The theory of F -singularities arose in the work of Hochster–Roberts [HR76] and Hochster–Huneke [HH90] in tight closure theory, and in the work of Mehta–Ramanathan [MR85] and Ramanan–Ramanathan [RR85] in the theory of Frobenius splittings. Part of their motivation was to detect singularities of rings of prime characteristic $p > 0$ using the Frobenius homomorphism $F_R: R \rightarrow F_{R*}R$, a starting point for which was Kunz’s characterization of regularity of a Noetherian ring R in terms of the flatness of F_R [Kun69, Thm. 2.1]. The most common classes of F -singularities are related in the following fashion (see Remark A.4 for details):

$$\begin{array}{ccc}
\text{strongly } F\text{-regular} & \implies & F\text{-rational} \\
\Downarrow & & \Downarrow \\
F\text{-pure} & \implies & F\text{-injective}.
\end{array}$$

For three of the four classes of F -singularities listed above, we know that assertions (I)–(IV) hold in some situations:

\mathcal{P}	(I)	(II)	(III*)	(IV)
strongly F -regular	[Has10, Lem. 3.6]	[Has10, Lem. 3.17]	[Has10, Lem. 3.28]	[Has10, Prop. 3.33]
F -rational	Proposition A.3(i)	Proposition A.5	[Vél95, Thm. 3.1]	[Vél95, Thm. 3.5]
F -pure	[HR74, Lem. 6.2]	[HR76, Prop. 5.13]	[Has10, Prop. 2.4(4)]	[Mur21, Cor. 3.5]

Here, strong F -regularity is defined in terms of tight closure as in [Has10, Def. 3.3] for Noetherian rings that are not necessarily F -finite (following Hochster), and (III*) is the following special case of (III):

(III*) (Ascent/Base change for regular homomorphisms) If $(R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ is a flat local homomorphism of Noetherian local rings with geometrically regular fibers, and R is \mathcal{P} , then S is \mathcal{P} .

Property (III) and not just (III*) also holds for F -purity by [SZ13, Prop. 4.8]. In the F -finite setting, where strong F -regularity was first defined by Hochster and Huneke via a splitting condition, (I) follows from [HH94, Thm. 5.5(a)], (II) follows from [HH94, Thm. 5.5(b)], and (IV) follows from [HH94, Thm. 5.9(b)]. Furthermore, (III) holds for F -finite strongly F -regular rings by adapting the argument in [HH94, Thm. 7.3]; see [SZ13, Cor. 4.6].

We note that (III*) for strong F -regularity only holds under the additional assumption that S is excellent, and (I) (resp. (III*)) for F -rationality only holds under the additional assumption that R is the image of a Cohen–Macaulay ring (resp. that R and S are excellent). In all three cases, (IV) is also known to hold for rings essentially of finite type over excellent local rings, but not for arbitrary excellent rings.

For F -injectivity, (I) and (IV) appear in the literature in the F -finite setting [Sch09, Prop. 4.3; QS17, Prop. 3.12], and (II) follows by combining (I) in the F -finite case with [Mur21, Lemma A.3]. On the other hand, (III*) seems to be completely open when the base ring is not assumed to be Cohen–Macaulay. We recall that a ring R of prime characteristic $p > 0$ is

F -injective if, for every maximal ideal $\mathfrak{m} \subseteq R$, the Frobenius action $H_{\mathfrak{m}}^i(F_{R_{\mathfrak{m}}}): H_{\mathfrak{m}}^i(R_{\mathfrak{m}}) \rightarrow H_{\mathfrak{m}}^i(F_{R_{\mathfrak{m}}*}R_{\mathfrak{m}})$ is injective for every i . F -injective rings are related to rings with Du Bois singularities in characteristic zero [Sch09].

The aim of this paper is to address assertions (I)–(IV) for F -injectivity. For (I), we show that F -injectivity localizes for arbitrary Noetherian rings (Proposition 3.3), extending results of Schwede [Sch09, Prop. 4.3] and Hashimoto [Has10, Cor. 4.11]. We then prove (II), i.e., that F -injectivity descends under faithfully flat homomorphisms of Noetherian rings (Theorem 3.8). This latter statement extends results of Hashimoto [Has01, Lem. 5.2; Has10, Lem. 4.6] and the second author [Mur21, Lem. A.3].

Our main theorem resolves (III*) for F -injectivity by proving a more general statement.

Theorem A. *Let $\varphi: (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ be a flat local homomorphism of Noetherian local rings of prime characteristic $p > 0$ whose closed fiber $S/\mathfrak{m}S$ is Cohen–Macaulay and geometrically F -injective over R/\mathfrak{m} . If R is F -injective, then S is F -injective.*

Theorem A, and its non-local version (Corollary 4.2), are new even when φ is regular (i.e. flat with geometrically regular fibers) or smooth. Under the additional hypothesis that R is Cohen–Macaulay, Theorem A is due to Hashimoto [Has01, Cor. 5.7] and Aberbach–Enescu [Ene09, Thm. 4.3]. We note that (III) fails when \mathcal{P} is the property of being F -injective, even if the closed fiber $S/\mathfrak{m}S$ is regular [Ene09, §4]. This indicates that some geometric assumptions on the closed fiber $S/\mathfrak{m}S$ are needed for a version of Theorem A, and consequently, (III), to hold for F -injectivity.

The Cohen–Macaulayness of $S/\mathfrak{m}S$ is used to decompose the local cohomology of S as a tensor product (see §4), and also applies the characterization of Cohen–Macaulay F -injective rings using Frobenius closure essentially due to Fedder and Watanabe [FW89, Rem. 1.9 and Prop. 2.2] to the closed fiber $S/\mathfrak{m}S$ (see Lemma 2.6). If one could show that Theorem A holds without this assumption, then this would also show that F -injectivity deforms (see the proof of [Has01, Lem. 5.9]). The latter remains one of the most important open problems about F -injectivity, and Theorem A provides evidence in support of this problem since we are now able to drop the Cohen–Macaulay assumption on the base ring.

As an application of our main theorem, we prove that F -injectivity is an open condition for most rings that arise in geometric applications, in particular resolving (IV) and answering a question of the second author [Mur21, Rem. 3.6].

Theorem B. *Let R be a ring essentially of finite type over a Noetherian local ring (A, \mathfrak{m}) of prime characteristic $p > 0$, and suppose that A has Cohen–Macaulay and geometrically F -injective formal fibers. Then, the F -injective locus is open in $\mathrm{Spec}(R)$.*

The condition on formal fibers holds if A is excellent. At the same time, we also observe that the F -injective locus need not be open if R is merely locally excellent (see Example 5.9).

Finally, we give a geometric application of our results. Bombieri [Bom73, p. 209] and Andreotti–Holm [AH77, p. 91] asked whether the image of a smooth projective variety of dimension r under a generic projection to \mathbf{P}_k^{r+1} is weakly normal. Greco–Traverso [GT80, Thm. 3.7] proved that this is indeed the case over the complex numbers, and the case in arbitrary characteristic follows from work of Roberts–Zaare–Nahandi [ZN84, Thm. 3.2; RZN84, Thm. 1.1] and Cumino–Maresi [CM81, Thm. 3.8] (see [CGM89, Obs. 2.4(v)]). As an application of our results, we show that generic projection hypersurfaces associated to suitably embedded

smooth varieties of low dimension are in fact F -pure (hence also F -injective) in positive characteristic, which is a stronger condition than being weakly normal [SZ13, Thm. 7.3].

Theorem C. *Let $Y \subseteq \mathbf{P}_k^n$ be a smooth projective variety of dimension $r \leq 5$ over an algebraically closed field k of characteristic $p > r$, such that Y is embedded via the d -uple embedding with $d \geq 3r$. If $\pi: Y \rightarrow \mathbf{P}_k^{r+1}$ is a generic projection and $X = \pi(Y)$, then X is F -pure, and hence F -injective.*

The assumption on characteristic is needed to rule out some exceptional cases in the classification in [Rob75, (13.2)], and the d -uple embedding is used to ensure that Y is appropriately embedded in the sense of [Rob75, §9]. Theorem C is the positive characteristic analogue of a theorem of Doherty [Doh08, Main Thm.], who proved that over the complex numbers, the generic projections X have semi-log canonical singularities. An example of Doherty also shows that generic projections of smooth projective varieties of large dimension are not necessarily F -pure (see Example 6.4).

Notation. All rings will be commutative with identity. If R is a ring of prime characteristic $p > 0$, then the *Frobenius homomorphism* on R is the ring homomorphism

$$\begin{aligned} F_R: R &\longrightarrow F_{R*}R \\ r &\longmapsto r^p. \end{aligned}$$

The notation $F_{R*}R$ is used to emphasize the fact the target of the Frobenius homomorphism has the (left) R -algebra structure given by $a \cdot r = a^p r$. For every integer $e \geq 0$, we denote the e -th iterate of the Frobenius homomorphism by $F_R^e: R \rightarrow F_{R*}^e R$. If $I \subseteq R$ is an ideal, we define the e -th *Frobenius power* $I^{[p^e]}$ to be the ideal $I \cdot F_{R*}^e R \subseteq F_{R*}^e R$ generated by p^e -th powers of elements in I .

Local cohomology modules are defined by taking injective resolutions in the category of sheaves of Abelian groups on spectra, as is done in [SGA2, Exp. I, Déf. 2.1]. When local cohomology is supported at a finitely generated ideal, this definition matches the Čech or Koszul definitions for local cohomology, even without Noetherian hypotheses [SGA2, Exp. II, Prop. 5].

Acknowledgments. We are grateful to Florian Enescu, Mihai Fulger, Mitsuyasu Hashimoto, Jack Jeffries, János Kollár, Linquan Ma, Lance Edward Miller, Thomas Polstra, Karl Schwede, Kazuma Shimomoto, Kevin Tucker, and Farrah Yhee for helpful discussions. We would especially like to thank Melvin Hochster for insightful conversations and comments on early drafts of this paper. We are grateful to Farrah Yhee for pointing out the reference [Ngu12]. The second author would also like to thank his advisor Mircea Mustața for his constant support and encouragement. We thank the referees for their careful reading of the paper and helpful comments.

2. DEFINITIONS AND PRELIMINARIES

We collect some basic material on F -injective rings and on the relative Frobenius homomorphism, which will be essential in the rest of this paper.

2.1. F -injective and CMFI rings. We start by defining F -injective rings, which were introduced by Fedder [Fed83].

Definition 2.1 [Fed83, Def. on p. 473; Ene00, p. 546]. A Noetherian local ring (R, \mathfrak{m}) of prime characteristic $p > 0$ is F -injective if for all integers $i \geq 0$, the R -module homomorphism

$$H_{\mathfrak{m}}^i(F_R): H_{\mathfrak{m}}^i(R) \longrightarrow H_{\mathfrak{m}}^i(F_{R*}R)$$

is injective. An arbitrary Noetherian ring R of prime characteristic $p > 0$ is F -injective if $R_{\mathfrak{m}}$ is F -injective for every maximal ideal $\mathfrak{m} \subseteq R$. A locally Noetherian scheme X of prime characteristic $p > 0$ is F -injective if there exists an affine open cover $\{\mathrm{Spec}(R_{\alpha})\}_{\alpha}$ of X such that R_{α} is F -injective for all α .

Now let R be a Noetherian k -algebra, where k is a field of characteristic $p > 0$. We say that R is *geometrically F -injective over k* if for every finite purely inseparable extension $k \subseteq k'$, the ring $R \otimes_k k'$ is F -injective. Similarly, if X is a locally Noetherian k -scheme, we say that X is *geometrically F -injective over k* if for every finite purely inseparable extension $k \subseteq k'$, the scheme $X \times_k \mathrm{Spec}(k')$ is F -injective.

Remark 2.2. Our definition of geometric F -injectivity follows [Ene00, p. 546]. The definition in [Has01, Def. 5.3] is a priori stronger, since it asserts that $R \otimes_k k'$ is F -injective for *all* finite extensions $k \subseteq k'$. We will see that these definitions are equivalent (Proposition 4.10(ii)).

We also define the following:

Definition 2.3 [Has01, p. 238; Has10, (4.1)]. A Noetherian ring R of prime characteristic $p > 0$ is *Cohen–Macaulay F -injective* or *CMFI* if R is both Cohen–Macaulay and F -injective. Similarly, a locally Noetherian scheme X of prime characteristic $p > 0$ is *CMFI* if X is Cohen–Macaulay and F -injective.

A Noetherian k -algebra (resp. a locally Noetherian k -scheme) is *geometrically CMFI over k* if it is Cohen–Macaulay and geometrically F -injective over k .

Remark 2.4. Since the notions of “geometrically Cohen–Macaulay” and “Cohen–Macaulay” coincide [Mat89, Rem. on p. 182], a Cohen–Macaulay k -algebra R is geometrically CMFI as in Definition 2.3 if and only if for every finite purely inseparable extension $k \subseteq k'$, the ring $R \otimes_k k'$ is F -injective.

We will use a characterization of CMFI rings in terms of Frobenius closure of ideals.

Definition 2.5 [HH94, (10.2)]. Let R be a ring of prime characteristic $p > 0$. If $I \subseteq R$ is an ideal, then the *Frobenius closure* of I in R is

$$I^F := \{x \in R \mid x^{p^e} \in I^{[p^e]} \text{ for some } e > 0\}.$$

We say that I is *Frobenius closed* if $I = I^F$.

CMFI rings can be characterized in terms of Frobenius closure of ideals generated by systems of parameters.

Lemma 2.6 [FW89, Rem. 1.9 and Prop. 2.2; Has10, Lem. 4.4; QS17, Thm. 3.7 and Cor. 3.9]. *Let (R, \mathfrak{m}) be a Cohen–Macaulay local ring of prime characteristic $p > 0$. Then, the following are equivalent:*

- (i) *The ring R is F -injective.*
- (ii) *Every ideal generated by a system of parameters for R is Frobenius closed.*

- (iii) *There exists an ideal generated by a system of parameters for R that is Frobenius closed.*

Moreover, even if R is not Cohen–Macaulay, we have $(ii) \Rightarrow (i)$.

2.2. The relative Frobenius homomorphism. We recall the definition of the *relative Frobenius homomorphism* [SGA5, Exp. XV, Déf. 3], which is also known as the *Radu–André homomorphism* in the commutative algebraic literature.

Definition 2.7. Let $\varphi: R \rightarrow S$ be a homomorphism of rings of prime characteristic $p > 0$. For every integer $e \geq 0$, consider the co-Cartesian diagram

$$\begin{array}{ccc}
 R & \xrightarrow{F_R^e} & F_{R*}^e R \\
 \varphi \downarrow & & \downarrow \varphi \otimes_R F_{R*}^e R \\
 S & \xrightarrow{\text{id}_S \otimes_R F_R^e} & S \otimes_R F_{R*}^e R \\
 & \searrow F_S^e & \nearrow F_{S/R}^e \\
 & & F_{S*}^e S
 \end{array}$$

(A curved arrow labeled $F_{R*}^e \varphi$ also points from $F_{R*}^e R$ to $F_{S*}^e S$.)

in the category of rings. The e -th *Radu–André ring* is the ring $S \otimes_R F_{R*}^e R$, and the e -th *relative Frobenius homomorphism* associated to φ is the ring homomorphism

$$\begin{aligned}
 F_{S/R}^e: S \otimes_R F_{R*}^e R &\longrightarrow F_{S*}^e S \\
 s \otimes r &\longmapsto s^{p^e} \varphi(r).
 \end{aligned}$$

If $e = 1$, we denote $F_{S/R}^1$ by $F_{S/R}$. We also sometimes denote $F_{S/R}^e$ by F_φ^e .

Remark 2.8. Even when R and S are Noetherian, the Radu–André rings $S \otimes_R F_{R*}^e R$ are not necessarily Noetherian. For these rings to be Noetherian, it suffices, for example, for R to be F -finite¹, or for $R \rightarrow S$ to be F -pure in the sense of Definition 2.10 below [Has10, Lem. 2.14]. See [Rad92, Thm. 7; Dum96, Thm. 4.4; Has01, Lem. 4.2] for more results on the Noetherianity of $S \otimes_R F_{R*}^e R$.

Radu and André used the homomorphism $F_{S/R}$ to give the following characterization of regular homomorphisms, that is, flat ring maps with geometrically regular fibers. Note that setting φ to be the homomorphism $\mathbf{F}_p \rightarrow R$ in the statement below, one recovers Kunz’s characterization of regular rings [Kun69, Thm. 2.1].

Theorem 2.9 [Rad92, Thm. 4; And93, Thm. 1]. *A homomorphism $\varphi: R \rightarrow S$ of Noetherian rings of prime characteristic $p > 0$ is regular if and only if $F_{S/R}$ is flat.*

Note that if the relative Frobenius homomorphism $F_{S/R}$ is flat, then it is automatically faithfully flat since $F_{S/R}$ induces a homeomorphism on spectra [SGA5, Exp. XV, Prop. 2(a)].

One can weaken the faithful flatness of $F_{S/R}$ to obtain the following:

Definition 2.10 [Has10, (2.3) and Lem. 2.5.1]. *A homomorphism $\varphi: R \rightarrow S$ of rings of prime characteristic $p > 0$ is F -pure if $F_{S/R}$ is a pure ring homomorphism.*

¹Recall that a ring R of prime characteristic $p > 0$ is F -finite if the Frobenius map F_R is module-finite.

While F -pure homomorphisms have geometrically F -pure fibers [Has10, Cor. 2.16], the converse does not hold in general [Has10, Rem. 2.17].

We end this section with a version of the relative Frobenius homomorphism for modules.

Lemma 2.11 (cf. [Ene00, p. 557]). *Let $\varphi: R \rightarrow S$ be a homomorphism of Noetherian rings of prime characteristic $p > 0$. Let M be an R -module, and let N_1 and N_2 be S -modules equipped with a homomorphism*

$$\psi: N_1 \longrightarrow F_{S*}^e N_2$$

of S -modules for some $e > 0$. Then, the homomorphism ψ induces a homomorphism

$$\tilde{\psi}: N_1 \otimes_R F_{R*}^e M \longrightarrow F_{S*}^e (N_2 \otimes_R M)$$

$$n \otimes m \longmapsto \psi(n) \otimes m$$

of $(S, F_{R}^e R)$ -bimodules that is functorial in M .*

Proof. We first note that ψ induces an R -bilinear homomorphism

$$N_1 \times F_{R*}^e M \longrightarrow F_{S*}^e (N_2 \otimes_R M)$$

$$(n, m) \longmapsto \psi(n) \otimes m$$

where the left R -module structure on $F_{R*}^e M$ is given by $a \cdot m = a^{p^e} m$. Hence the homomorphism $\tilde{\psi}$ exists as an R -module homomorphism by the universal property of tensor products. This homomorphism $\tilde{\psi}$ is in fact $(S, F_{R*}^e R)$ -bilinear since

$$\tilde{\psi}(n \otimes m) \cdot a = (\psi(n) \otimes m) \cdot a = \psi(n) \otimes ma = \tilde{\psi}(n \otimes ma) = \tilde{\psi}((n \otimes m) \cdot a)$$

$$b \cdot \tilde{\psi}(n \otimes m) = b \cdot (\psi(n) \otimes m) = b^{p^e} \psi(n) \otimes m = \psi(bn) \otimes m = \tilde{\psi}(bn \otimes m) = \tilde{\psi}(b \cdot (n \otimes m))$$

for all $a \in R$ and for all $b \in S$. Functoriality in M follows from the construction of $\tilde{\psi}$ using the universal property of tensor products. \square

Remark 2.12. The construction in Lemma 2.11 yields the relative Frobenius $F_{S/R}$ when we apply it to $M = R$, $N_1 = N_2 = S$ and $\psi = F_S: S \rightarrow F_{S*} S$.

3. LOCALIZATION AND DESCENT UNDER FAITHFULLY FLAT HOMOMORPHISMS

In this section, we prove assertions (I) and (II) for F -injectivity, namely, that F -injectivity localizes (Proposition 3.3) and descends along faithfully flat homomorphisms (Theorem 3.8). We then discuss the behavior of geometric F -injectivity under infinite purely inseparable extensions in §3.4.

3.1. Characterizations of F -injectivity using module-finite algebras. An important ingredient in proving localization and descent of F -injectivity is the following alternative characterization of F -injectivity in terms of module-finite algebras over R , which was pointed out to us by Karl Schwede.

Lemma 3.1. *Let (R, \mathfrak{m}) be a Noetherian local ring of prime characteristic $p > 0$. Fix a filtered direct system $\{\psi_\alpha: R \rightarrow S_\alpha\}_\alpha$ of module-finite homomorphisms such that $F_R: R \rightarrow F_{R*} R$ is the colimit of this directed system. Then, the ring R is F -injective if and only if the R -module homomorphisms*

$$H_{\mathfrak{m}}^i(\psi_\alpha): H_{\mathfrak{m}}^i(R) \longrightarrow H_{\mathfrak{m}}^i(S_\alpha) \tag{1}$$

are injective for all i and for all α .

Proof. The homomorphism $H_{\mathfrak{m}}^i(F_R): H_{\mathfrak{m}}^i(R) \rightarrow H_{\mathfrak{m}}^i(F_{R*}R)$ factors as

$$H_{\mathfrak{m}}^i(R) \xrightarrow{H_{\mathfrak{m}}^i(\psi_{\alpha})} H_{\mathfrak{m}}^i(S_{\alpha}) \longrightarrow H_{\mathfrak{m}}^i(F_*R) \quad (2)$$

for every α . Thus, for the direction \Rightarrow , it suffices to note that if the composition (2) is injective, then the first homomorphism $H_{\mathfrak{m}}^i(\psi_{\alpha})$ is also injective. For the direction \Leftarrow , it suffices to take colimits in (2), since local cohomology commutes with filtered colimits. \square

We will also use the following characterization of F -injectivity in terms of homomorphisms on Ext modules. Under F -finiteness hypotheses, this characterization is due to Fedder [Fed83, Rem. on p. 473] for Cohen–Macaulay rings, and is implicit in the proof of [Sch09, Prop. 4.3].

Lemma 3.2 (cf. [Mur21, Lem. A.1]). *Let R be a Noetherian ring of prime characteristic $p > 0$ with a dualizing complex ω_R^{\bullet} . Fix a directed system $\{\psi_{\alpha}: R \rightarrow S_{\alpha}\}_{\alpha}$ of module-finite homomorphisms such that $F_R: R \rightarrow F_{R*}R$ is the colimit of this directed system. Then, the ring R is F -injective if and only if the R -module homomorphisms*

$$\psi_{\alpha}^*: \operatorname{Ext}_R^{-i}(\psi_{\alpha*}S_{\alpha}, \omega_R^{\bullet}) \longrightarrow \operatorname{Ext}_R^{-i}(R, \omega_R^{\bullet}) \quad (3)$$

are surjective for all i and for all α .

Proof. This follows from Lemma 3.1 and Grothendieck local duality [Har66, Cor. V.6.3], since Ext modules [BouA, Prop. X.6.10(b)] (here is where we use the module-finiteness of the S_{α}) and dualizing complexes [Har66, Cor. V.2.3] are compatible with localization. \square

3.2. Localization of F -injectivity. In [Sch09, Def. 4.4], Schwede defines a Noetherian ring R to be F -injective if $R_{\mathfrak{p}}$ is F -injective for every prime ideal $\mathfrak{p} \subseteq R$. We show that Schwede’s definition is equivalent to our definition of F -injectivity (Definition 2.1). This result was shown by Schwede under the additional assumption that R is F -finite [Sch09, Prop. 4.3], and by Hashimoto for the CMFI property [Has01, p. 238; Has10, Cor. 4.11].

Proposition 3.3. *Let R be a Noetherian ring of prime characteristic $p > 0$. The ring R is F -injective if and only if $R_{\mathfrak{p}}$ is F -injective for every prime ideal $\mathfrak{p} \subseteq R$. In particular, if R is F -injective, then for every multiplicative set $S \subseteq R$, the localization $S^{-1}R$ is F -injective.*

Proof. We first prove the if and only if statement. The direction \Leftarrow is true by definition, and hence it suffices to show the direction \Rightarrow . We claim that it suffices to consider the case when R is a complete local ring. Suppose R is F -injective, and let $\mathfrak{p} \subseteq R$ be a prime ideal. Let $\widehat{\mathfrak{m}} \subseteq R$ be a maximal ideal containing \mathfrak{p} , and consider a prime ideal $\mathfrak{q} \subseteq \widehat{R}_{\widehat{\mathfrak{m}}}$ minimal over $\mathfrak{p}\widehat{R}_{\widehat{\mathfrak{m}}}$. Then, the local homomorphism

$$R_{\mathfrak{p}} \longrightarrow (\widehat{R}_{\widehat{\mathfrak{m}}})_{\mathfrak{q}}$$

is faithfully flat with zero-dimensional closed fiber. Since $\widehat{R}_{\widehat{\mathfrak{m}}}$ is F -injective by [Fed83, Rem. on p. 473], the localization $(\widehat{R}_{\widehat{\mathfrak{m}}})_{\mathfrak{q}}$ is also by the Proposition in the complete local case, and hence $R_{\mathfrak{p}}$ is F -injective by [Mur21, Lem. A.3].

It remains to show the direction \Rightarrow when R is complete local, in which case R has a dualizing complex [Har66, (4) on p. 299]. Write $F_R: R \rightarrow F_{R*}R$ as a filtered colimit of module-finite homomorphisms $\psi_{\alpha}: R \rightarrow S_{\alpha}$. The statement now follows from Lemma 3.2, since the surjectivity of the homomorphisms (3) localizes by [BouA, Prop. X.6.10(b)].

The last localization statement now follows because the local rings of $S^{-1}R$ coincide with the local rings of R at the primes of R that do not intersect S . \square

Remark 3.4. As pointed out by Linquan Ma, one can also prove the direction \Rightarrow in Proposition 3.3 in the complete local case using the gamma construction of Hochster–Huneke [HH94, (6.11)]. By the gamma construction and [Mur21, Thm. 3.4(ii)] (see also Remark 5.8), there exists a faithfully flat extension $R \rightarrow R^\Gamma$, such that R^Γ is F -finite, and such that the induced morphism $\mathrm{Spec}(R^\Gamma) \rightarrow \mathrm{Spec}(R)$ is a homeomorphism that identifies F -injective loci. Since the F -injective locus on $\mathrm{Spec}(R^\Gamma)$ is stable under generization by [Sch09, Prop. 4.3], the F -injective locus on $\mathrm{Spec}(R)$ is also stable under generization.

As a consequence of Proposition 3.3, the following properties of F -injective Noetherian rings follow without any F -finiteness hypotheses. See, e.g., [Swa80; Yan85] for the notions of seminormality and weak normality in (iii).

Corollary 3.5. *Let R be a Noetherian, F -injective ring of prime characteristic $p > 0$. We then have the following:*

- (i) R is reduced.
- (ii) R is approximately Gorenstein.
- (iii) R is weakly normal, and in particular, seminormal.
- (iv) If R is in addition local and I is an ideal generated by a regular sequence, then the map

$$\mathrm{id}_{R/I} \otimes_R F_R^e: R/I \longrightarrow R/I \otimes_R F_{R*}^e(R)$$

is injective for all $e > 0$. In particular, I is Frobenius closed.

Proof. For (i) and (iv), the proofs in [SZ13, Rem. 2.6] and [QS17, Prop. 3.11] in the F -finite case apply, since they only use F -finiteness to conclude that F -injectivity localizes (Proposition 3.3). (i) is also shown in [QS17, Lem. 3.11] using the gamma construction.

For (ii), we may assume that R is local since the property of being approximately Gorenstein is local [Hoc77, Def. 1.3]. Since \widehat{R} is F -injective [Fed83, Rem. on p. 473], it is reduced by (i), and hence approximately Gorenstein by [Hoc77, Thm. 1.7]. Thus, R is approximately Gorenstein by [Hoc77, Cor. 2.2].

(iii) is shown in the F -finite case by Schwede [Sch09, Thm. 4.7]. The same proof applies once we know that F -injectivity localizes (Proposition 3.3) and that (i) holds. \square

Remark 3.6. As a consequence of Corollary 3.5(iv), the proof of [QS17, Cor. 3.14] yields the following: if (R, \mathfrak{m}) is a Noetherian local ring, and $I = (x_1, x_2, \dots, x_t)$ is an ideal generated by a regular sequence such that R/x_1R is F -injective, then the Frobenius actions on $H_I^t(R)$ and $H_{\mathfrak{m}}^t(R)$ are injective. In particular, this shows that F -injectivity deforms when R is Cohen–Macaulay (the latter was first shown in [Fed83, Thm. 3.4(1)]).

Globally, localization of F -injectivity implies that F -injectivity can be checked on an affine open cover.

Corollary 3.7. *Let X be a locally Noetherian scheme of prime characteristic $p > 0$. Then the following are equivalent:*

- (i) For every affine open subscheme $\mathrm{Spec}(R)$ of X , the ring R is F -injective.
- (ii) X is F -injective.
- (iii) For every locally closed point $x \in X$, the stalk $\mathcal{O}_{X,x}$ is F -injective.

Furthermore, if X is locally of finite type over a field k of characteristic $p > 0$, then (i)–(iii) are equivalent to $\mathcal{O}_{X,x}$ being F -injective for every closed point $x \in X$.

Proof. Clearly $(i) \Rightarrow (ii)$. If X has an affine open cover $\{R_\alpha\}_\alpha$ such that every R_α is F -injective, then Proposition 3.3 shows that all the stalks of \mathcal{O}_X are F -injective local rings. Thus, $(ii) \Rightarrow (iii)$. Finally, $(iii) \Rightarrow (i)$ follows because if $\text{Spec}(R)$ is an affine open subscheme of X , then any closed point of $\text{Spec}(R)$ is a locally closed point of X .

If X is in addition locally of finite type over a field k , then the set of closed points of X coincide with the set of locally closed points of X , thus proving the equivalence of the last assertion in the statement of the Corollary with (i) – (iii) . \square

3.3. Descent of F -injectivity. We now show that F -injectivity descends along faithfully flat homomorphisms, thereby establishing (II) for the property \mathcal{P} of being F -injective. The corresponding result for the CMFI property was shown by Hashimoto [Has01, Lem. 5.2; Has10, Lem. 4.6].

Theorem 3.8. *Let $\varphi: R \rightarrow S$ be a homomorphism of Noetherian rings of prime characteristic $p > 0$ that is surjective on spectra satisfying the following property: for all prime ideals $\mathfrak{q} \in \text{Spec}(S)$, the induced map $R_{\varphi^{-1}(\mathfrak{q})} \rightarrow S_{\mathfrak{q}}$ is pure (for example if φ is faithfully flat). Then if S is F -injective, so is R .*

Proof. Let $\mathfrak{p} \subseteq R$ be a maximal ideal, and let $\mathfrak{q} \subseteq S$ be a prime ideal that is minimal among primes of S lying over \mathfrak{p} (such a \mathfrak{q} exists because φ is surjective on spectra). By Proposition 3.3, we know that $S_{\mathfrak{q}}$ is F -injective. Since the local homomorphism $R_{\mathfrak{p}} \rightarrow S_{\mathfrak{q}}$ is pure with zero-dimensional closed fiber, we have reduced to the case shown in [Mur21, Lem. A.3]. \square

Our hypothesis on φ in Theorem 3.8 implies that φ is pure. This is because purity is a local condition, and if we choose a prime \mathfrak{q} of S lying over an arbitrary prime \mathfrak{p} of R , then purity of the composition $R_{\mathfrak{p}} \rightarrow S_{\mathfrak{p}} \rightarrow S_{\mathfrak{q}}$ implies $R_{\mathfrak{p}} \rightarrow S_{\mathfrak{p}}$ is also pure. Maps φ satisfying the hypotheses of Theorem 3.8 are also called *strongly pure* in [CGM16, p. 38], and there exist examples of pure maps that are not strongly pure [CGM16, Cor. 5.6.2].

For pure maps that are not necessarily strongly pure, we have the following descent result. This result is optimal given known counterexamples to descent of F -injectivity under pure (or even split) maps [Wat97, Ex. 3.3(1); Ngu12, Ex. 6.6 and Rem. 6.7] that are not quasi-finite.

Proposition 3.9. *Let $\varphi: R \rightarrow S$ be a quasi-finite (e.g., module-finite), pure homomorphism of Noetherian rings of prime characteristic $p > 0$. If S is F -injective, so is R .*

Proof. Let $\mathfrak{m} \subseteq R$ be a maximal ideal. Since quasi-finiteness and purity are preserved under base change and F -injectivity of R can be checked at its maximal ideals, replacing $\varphi: R \rightarrow S$ with its localization $\varphi_{\mathfrak{m}}: R_{\mathfrak{m}} \rightarrow S \otimes_R R_{\mathfrak{m}}$, it suffices to consider the case when R is local.

By [HH95, Lem. 2.2], there exists a maximal ideal $\mathfrak{q} \subseteq S$ lying over \mathfrak{m} such that $R \rightarrow S_{\mathfrak{q}}$ is pure. Since φ is quasi-finite, we also have $\sqrt{\mathfrak{m}S_{\mathfrak{q}}} = \mathfrak{q}$. The statement now follows from [Mur21, Lem. A.3]. \square

3.4. Geometrically CMFI rings and infinite purely inseparable extensions. In the proof of Theorem A, we need to consider how geometrically CMFI rings interact with possibly infinite purely inseparable extensions of the base field.

Proposition 3.10 (cf. [Ene00, Prop. 2.21]). *Let (R, \mathfrak{m}) be a geometrically CMFI local k -algebra, where k is a field of characteristic $p > 0$. Let $k \subseteq k'$ be a purely inseparable extension (not necessarily finite). If I is an ideal of R generated by a system of parameters, then $I(R \otimes_k k')$ is Frobenius closed in $R \otimes_k k'$. In particular, if $R \otimes_k k'$ is Noetherian, then $R \otimes_k k'$ is CMFI.*

Proof. Write $k' = \bigcup_{\alpha} L_{\alpha}$ as the directed union of finite purely inseparable subextensions of $k \subseteq k'$. Then, $R \otimes_k k' = \bigcup_{\alpha} R \otimes_k L_{\alpha}$. Since $R \hookrightarrow R \otimes_k L_{\alpha}$ is finite and purely inseparable, the ideal $I(R \otimes_k L_{\alpha})$ is generated by a system of parameters. Moreover, since R is geometrically CMFI, Lemma 2.6 implies the ideal $I(R \otimes_k L_{\alpha})$ is Frobenius closed. We then have

$$(I(R \otimes_k k'))^F = \bigcup_{\alpha} (I(R \otimes_k L_{\alpha}))^F = \bigcup_{\alpha} I(R \otimes_k L_{\alpha}) = I(R \otimes_k k').$$

Here the second and third equalities are straightforward to verify, while the first equality follows from the equality

$$I^{[p^e]}(R \otimes_k k') = \bigcup_{\alpha} I^{[p^e]}(R \otimes_k L_{\alpha}).$$

This shows that $I(R \otimes_k k')$ is Frobenius closed, proving the first assertion of the Proposition.

For the second assertion, suppose $R \otimes_k k'$ is Noetherian. Since $R \hookrightarrow R \otimes_k k'$ is purely inseparable, it is a flat local homomorphism of local rings, and $R \otimes_k k'$ is Cohen–Macaulay by [Mat89, Cor. to Thm. 23.3]. The fact that $R \otimes_k k'$ is F -injective then follows by Lemma 2.6 because $I(R \otimes_k k')$ is a Frobenius closed ideal generated by a system of parameters. \square

4. ASCENT UNDER FLAT HOMOMORPHISMS WITH GEOMETRICALLY CMFI FIBERS

In [Vél95], Vélez showed the following base change result for F -rationality:

Theorem 4.1 [Vél95, Thm. 3.1]. *Let $\varphi: R \rightarrow S$ be a regular homomorphism of locally excellent Noetherian rings of prime characteristic $p > 0$. If R is F -rational, then S is also F -rational.*

Enescu [Ene00, Thm. 2.27] and Hashimoto [Has01, Thm. 6.4] proved that in fact, it suffices to assume that φ has geometrically F -rational fibers.² Our goal is to show the following analogue of their results for F -injectivity.

Theorem A. *Let $\varphi: (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ be a flat local homomorphism of Noetherian local rings of prime characteristic $p > 0$ whose closed fiber $S/\mathfrak{m}S$ is geometrically CMFI over R/\mathfrak{m} . If R is F -injective, then S is F -injective.*

Since F -injectivity localizes (Proposition 3.3), we obtain the following non-local result as a consequence:

Corollary 4.2. *Let $\varphi: R \rightarrow S$ be a flat homomorphism of Noetherian rings of prime characteristic $p > 0$ whose fibers are geometrically CMFI. If R is F -injective, then S is F -injective.*

Remark 4.3. If $\varphi: R \rightarrow S$ is a flat homomorphism of Noetherian rings of prime characteristic $p > 0$ such that φ contracts maximal ideals of S to maximal ideals of R (this holds, for instance, if φ is of finite type and R and S are Jacobson rings [BouCA, Thm. V.3.3]), then F -injectivity ascends under the weaker hypotheses that R is F -injective and *only* the closed fibers of φ are geometrically CMFI. This is because F -injectivity of S is checked at the maximal ideals of S .

Theorem A also implies F -injectivity is preserved under (strict) Henselization.

²Enescu assumes that $B/\mathfrak{m}B \otimes_{A/\mathfrak{m}} F_*^e(A/\mathfrak{m})$ is Noetherian for every $e > 0$, which is used in the proof of [Ene00, Thm. 2.19]. This latter result holds without this Noetherianity assumption by applying Proposition 3.10 to the system of parameters y_1, y_2, \dots, y_d on $B/\mathfrak{m}B$ (cf. [Ene00, Rem. 2.20]).

Corollary 4.4. *If (R, \mathfrak{m}) is a Noetherian local ring of prime characteristic $p > 0$ that is F -injective, then its Henselization R^h and strict Henselization R^{sh} are also F -injective.*

Proof. The closed fiber of $R \rightarrow R^h$ is an isomorphism [EGAIV₄, Thm. 18.6.6(iii)], and the closed fiber of $R \rightarrow R^{sh}$ is a separable algebraic field extension by construction. In either case, the closed fiber is geometrically CMFI, and Theorem A implies R^h and R^{sh} are F -injective. \square

Theorem A is known to fail without geometric assumptions, even if the fibers are regular (see [SZ13] and [Ene09, §4]). Theorem A extends similar base change results due to Hashimoto [Has01, Cor. 5.7] and Aberbach–Enescu [Ene09, Thm. 4.3], which assume that R is Cohen–Macaulay.

The key ingredient to proving Theorem A is the injectivity of the action of relative Frobenius on local cohomology, which we study in §4.2 after reviewing some preliminaries on local cohomology in §4.1. We then prove Theorem A, which we use to analyze the behavior of geometric F -injectivity under arbitrary finitely generated field extensions in §4.3.

4.1. Preliminaries on local cohomology. In order to prove Theorem A, we will need the following preliminary results on local cohomology and flat base change.

Lemma 4.5. *Let $\varphi: (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ be a flat local homomorphism of Noetherian rings and let I be an ideal of R . Let $J = (y_1, y_2, \dots, y_n) \subseteq \mathfrak{n}$ be an ideal of S such that the images of y_1, y_2, \dots, y_n in $S/\mathfrak{m}S$ form a regular sequence on $S/\mathfrak{m}S$.*

- (i) *The sequence y_1, y_2, \dots, y_n forms a regular sequence on S . Moreover, the modules $S/(y_1^t, y_2^t, \dots, y_n^t)$ and $H_J^n(S)$ are flat over R for every integer $t > 0$.*
- (ii) *For every R -module M , there is an isomorphism*

$$H_{IS+J}^i(S \otimes_R M) \simeq H_J^n(S) \otimes_R H_I^{i-n}(M) \quad (4)$$

of S -modules that is functorial in M .

- (iii) *Suppose that R and S are of prime characteristic $p > 0$. Then, the isomorphism (4) for $M = F_{R*}^e R$ is compatible with the relative Frobenius homomorphism, i.e., the diagram*

$$\begin{array}{ccc} H_{IS+J}^i(S \otimes_R F_{R*}^e R) & \xrightarrow{H_{IS+J}^i(F_{S/R}^e)} & H_{IS+J}^i(F_{S*}^e S) \\ \wr & & \wr \\ H_J^n(S) \otimes_R H_I^{i-n}(F_{R*}^e R) & \longrightarrow & F_{S*}^e (H_J^n(S) \otimes_R H_I^{i-n}(R)) \end{array}$$

commutes, where the bottom horizontal homomorphism is that induced by the Frobenius action $H_J^n(S)$ on the local cohomology module $H_J^n(S)$ as in Lemma 2.11.

Proof. For (i), the flatness of $H_J^n(S)$ follows from the flatness of $S/(y_1^t, y_2^t, \dots, y_n^t)$, since the former is a filtered colimit of modules of the latter form. It therefore suffices to note that for every integer $t > 0$, the images of $y_1^t, y_2^t, \dots, y_n^t$ in $S/\mathfrak{m}S$ remain a regular sequence on $S/\mathfrak{m}S$, after which the rest of (i) follows from [HH94, Lem. 7.10(b)].

We now show (ii). Let $x_1, x_2, \dots, x_m \in R$ be a set of generators for I . We will denote by \mathbf{x} and \mathbf{y} the m - and n -tuples (x_1, x_2, \dots, x_m) and (y_1, y_2, \dots, y_n) , respectively. We then have

the following chain of isomorphisms and quasi-isomorphisms of complexes:

$$\begin{aligned} \check{C}^\bullet(\mathbf{y}, \mathbf{x}; S \otimes_R M) &\xleftarrow{\sim} \check{C}^\bullet(\mathbf{y}; S) \otimes_S S \otimes_R \check{C}^\bullet(\mathbf{x}; R) \otimes_R M \\ &\xrightarrow{\sim} \check{C}^\bullet(\mathbf{y}; S) \otimes_R \check{C}^\bullet(\mathbf{x}; M) \\ &\xrightarrow{\text{qis}} H_J^n(S)[-n] \otimes_R \check{C}^\bullet(\mathbf{x}; M). \end{aligned} \quad (5)$$

Here, the first two isomorphisms follow from the definition of the Čech complex [ILL+07, Def. 6.26]. The last quasi-isomorphism follows from (i): The Čech complex $\check{C}^\bullet(\mathbf{y}; S)$ is a complex of flat R -modules quasi-isomorphic to the complex $H_J^n(S)[-n]$ of flat R -modules by the fact that y_1, y_2, \dots, y_n is a regular sequence on S . Note that the quasi-isomorphism $\check{C}^\bullet(\mathbf{y}; S) \rightarrow H_J^n(S)[-n]$ remains a quasi-isomorphism after tensoring by $\check{C}^\bullet(\mathbf{x}; M)$ by [BouA, Cor. 5 to Thm. X.4.3]. By [SGA2, Exp. II, Prop. 5], applying cohomology in (5) yields the desired isomorphism (4), since all involved (quasi-)isomorphisms are functorial in M .

We now show (iii). Setting $M = F_{R*}^e R$ in the situation of (ii), we trace the relative Frobenius homomorphism $F_{S/R}^e$ through the chain of (quasi-)isomorphisms (5):

$$\begin{array}{ccc} \check{C}^\bullet(\mathbf{y}, \mathbf{x}; S \otimes_R F_{R*}^e R) & \xrightarrow{\check{C}^\bullet(\mathbf{y}, \mathbf{x}; F_{S/R}^e)} & \check{C}^\bullet(\mathbf{y}, \mathbf{x}; F_{S*}^e S) \\ \uparrow \wr & & \uparrow \wr \\ (\check{C}^\bullet(\mathbf{y}; S) \otimes_S S \otimes_R \check{C}^\bullet(\mathbf{x}; R)) \otimes_R F_{R*}^e R & \longrightarrow & F_{S*}^e (\check{C}^\bullet(\mathbf{y}; S) \otimes_S S \otimes_R \check{C}^\bullet(\mathbf{x}; R)) \\ \downarrow \wr & & \downarrow \wr \\ \check{C}^\bullet(\mathbf{y}; S) \otimes_R F_{R*}^e \check{C}^\bullet(\mathbf{x}; R) & \longrightarrow & F_{S*}^e (\check{C}^\bullet(\mathbf{y}; S) \otimes_R \check{C}^\bullet(\mathbf{x}; R)) \\ \downarrow \text{qis} & & \downarrow \text{qis} \\ H_J^n(S)[-n] \otimes_R F_{R*}^e \check{C}^\bullet(\mathbf{x}; R) & \longrightarrow & F_{S*}^e (H_J^n(S)[-n] \otimes_R \check{C}^\bullet(\mathbf{x}; R)). \end{array}$$

The last two horizontal maps in the above diagram are induced by the homomorphism of modules in Lemma 2.11 via the Frobenius action on each term of the complex $\check{C}^\bullet(\mathbf{y}; S)$ and on $H_J^n(S)$, respectively. To ensure the diagram commutes, one can trace elements of the individual terms of the complexes through the specific description of the homomorphisms involved. Applying cohomology gives the statement in (iii). \square

4.2. Proof of Theorem A. The key technical result used in the proof of Theorem A is the following:

Proposition 4.6. *Let $\varphi: (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ be a flat local homomorphism of Noetherian local rings of prime characteristic $p > 0$ whose closed fiber $S/\mathfrak{m}S$ is geometrically CMFI over R/\mathfrak{m} . Suppose $J = (y_1, y_2, \dots, y_n)$ is an ideal of S such that the images of y_1, y_2, \dots, y_n in $S/\mathfrak{m}S$ form a system of parameters in $S/\mathfrak{m}S$. For every Artinian R -module M and for every $e > 0$, the homomorphism*

$$\begin{aligned} \frac{S}{J} \otimes_R F_{R*}^e M &\longrightarrow F_{S*}^e \left(\frac{S}{J^{[p^e]}} \otimes_R M \right) \\ (s + J) \otimes m &\longmapsto (s^{p^e} + J^{[p^e]}) \otimes m \end{aligned}$$

from Lemma 2.11 is injective.

Proof. We first reduce to the case when M is of finite length. Write M as the union $\bigcup_{\alpha} M_{\alpha}$ of its finitely generated R -submodules M_{α} . Since M is Artinian, the R -modules M_{α} are of finite length. The statement for M then follows from the statement for the M_{α} by taking filtered colimits.

We now assume M is of finite length. We proceed by induction on the length of M . If M has length 1, then $M \simeq R/\mathfrak{m} =: k$ is the residue field of R . It therefore suffices to show

$$\begin{aligned} \frac{S}{J} \otimes_R F_{R*}^e k &\longrightarrow F_{S*}^e \left(\frac{S}{J[p^e]} \otimes_R k \right) \\ s \otimes r &\longmapsto s^{p^e} \otimes r \end{aligned} \tag{6}$$

is injective for every positive integer e . Under the composition of isomorphisms

$$\frac{S}{J[p^f]} \otimes_R (-) \simeq \frac{S}{J[p^f]} \otimes_R \frac{R}{\mathfrak{m}} \otimes_k (-) \simeq \frac{S}{\mathfrak{m}S + J[p^f]} \otimes_k (-)$$

of functors on the category of R -modules annihilated by the maximal ideal \mathfrak{m} for both $f = 0$ and $f = e$, the homomorphism (6) can be identified with the first homomorphism in the composition

$$\begin{aligned} \frac{S}{\mathfrak{m}S + J} \otimes_k F_{k*}^e k &\longrightarrow F_{(S/\mathfrak{m}S)*}^e \left(\frac{S}{\mathfrak{m}S + J[p^e]} \otimes_k k \right) \longrightarrow F_{(S/\mathfrak{m}S)*}^e \left(\frac{S}{\mathfrak{m}S + J[p^e]} \otimes_k F_{k*}^e k \right) \\ s \otimes r &\longmapsto s^{p^e} \otimes r \\ \tilde{s} \otimes \tilde{r} &\longmapsto \tilde{s} \otimes \tilde{r}^{p^e}. \end{aligned} \tag{7}$$

Since $S/\mathfrak{m}S$ is geometrically CMFI over R/\mathfrak{m} , and the ideal $J(S/\mathfrak{m}S)$ is Frobenius closed in $S/\mathfrak{m}S$ by Lemma 2.6, we then get by Proposition 3.10 that $J(S/\mathfrak{m}S \otimes_k F_{k*}^e k)$ is Frobenius closed in $S/\mathfrak{m}S \otimes_k F_{k*}^e k \simeq S/\mathfrak{m}S \otimes_k k^{1/p^e}$. Since

$$\frac{S}{\mathfrak{m}S + J} \otimes_k F_{k*}^e k \simeq \frac{S}{J} \otimes_S \left(\frac{S}{\mathfrak{m}S} \otimes_k F_{k*}^e k \right) \simeq \frac{S/\mathfrak{m}S \otimes_k F_{k*}^e k}{J(S/\mathfrak{m}S \otimes_k F_{k*}^e k)},$$

and similarly,

$$F_{(S/\mathfrak{m}S)*}^e \left(\frac{S}{\mathfrak{m}S + J[p^e]} \otimes_k F_{k*}^e k \right) \simeq F_{(S/\mathfrak{m}S)*}^e \left(\frac{S/\mathfrak{m}S \otimes_k F_{k*}^e k}{J[p^e](S/\mathfrak{m}S \otimes_k F_{k*}^e k)} \right),$$

it follows that the composition in (7) is injective for all $e > 0$. Therefore in particular, the first homomorphism must be injective.

It remains to prove the inductive step, in which case there exist two R -modules M_1, M_2 of length strictly less than that of M , together with a short exact sequence

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0.$$

We then have the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \frac{S}{J} \otimes_R F_{R*}^e M_1 & \longrightarrow & \frac{S}{J} \otimes_R F_{R*}^e M & \longrightarrow & \frac{S}{J} \otimes_R F_{R*}^e M_2 \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & F_{S*}^e \left(\frac{S}{J^{[p^e]}} \otimes_R M_1 \right) & \longrightarrow & F_{S*}^e \left(\frac{S}{J^{[p^e]}} \otimes_R M \right) & \longrightarrow & F_{S*}^e \left(\frac{S}{J^{[p^e]}} \otimes_R M_2 \right) \longrightarrow 0
\end{array}$$

where the rows are exact since $S/J^{[p^t]}$ is flat over R for every $t \geq 0$ by Lemma 4.5(i). The two outer homomorphisms are injective by inductive hypothesis, hence the middle homomorphism is injective by the snake lemma. \square

We are now ready to prove Theorem A.

Proof of Theorem A. Consider the factorization

$$S \xrightarrow{\text{id}_S \otimes_R F_R} S \otimes_R F_{R*} R \xrightarrow{F_{S/R}} F_{S*} S$$

of the Frobenius homomorphism $F_S: S \rightarrow F_{S*} S$, where $F_{S/R}$ is the relative Frobenius homomorphism of Definition 2.7. This factorization induces the factorization

$$H_n^i(S) \xrightarrow{H_n^i(\text{id}_S \otimes_R F_R)} H_n^i(S \otimes_R F_{R*} R) \xrightarrow{H_n^i(F_{S/R})} H_n^i(F_{S*} S) \quad (8)$$

of $H_n^i(F_S)$. To show that S is F -injective, i.e., that $H_n^i(F_S)$ is injective, it suffices to show that the two homomorphisms in (8) are injective.

We start by setting up some notation. Since $S/\mathfrak{m}S$ is Cohen–Macaulay, there exists a sequence of elements $y_1, y_2, \dots, y_n \in S$ whose image in $S/\mathfrak{m}S$ is a system of parameters in $S/\mathfrak{m}S$, hence is also a regular sequence in $S/\mathfrak{m}S$. Now consider the ideal

$$J := (y_1, y_2, \dots, y_n) \subseteq S.$$

Note that $\sqrt{\mathfrak{m}S + J} = \mathfrak{n}$, and hence $H_n^i(S) = H_{\mathfrak{m}S+J}^i(S)$ for all i .

We now show that the first homomorphism in (8) is injective. Applying the functorial isomorphism of Lemma 4.5(ii) to the Frobenius homomorphism $F_R: R \rightarrow F_{R*} R$ and the ideal $I = \mathfrak{m}$ of R , we obtain the commutative diagram

$$\begin{array}{ccc}
H_n^i(S) & \xrightarrow{H_n^i(\text{id}_S \otimes_R F_R)} & H_n^i(S \otimes_R F_{R*} R) \\
\wr & & \wr \\
H_J^n(S) \otimes_R H_{\mathfrak{m}}^{i-n}(R) & \xleftarrow{\text{id}_{H_J^n(S)} \otimes_R H_{\mathfrak{m}}^{i-n}(F_R)} & H_J^n(S) \otimes_R H_{\mathfrak{m}}^{i-n}(F_{R*} R).
\end{array}$$

The horizontal homomorphism in the bottom row is injective by the F -injectivity of R , and then by using the flatness of $H_J^n(S)$ over R in Lemma 4.5(i). By the commutativity of the diagram, we see that the first homomorphism in (8) is injective.

It remains to show that the second homomorphism in (8) is injective. By Lemma 4.5(iii), this homomorphism can be identified with the homomorphism

$$H_J^n(S) \otimes_R F_{R*} H_{\mathfrak{m}}^{i-n}(R) \longrightarrow F_{S*} (H_J^n(S) \otimes_R H_{\mathfrak{m}}^{i-n}(R)) \quad (9)$$

from Lemma 2.11. Since $H_{\mathfrak{m}}^{i-n}(R)$ is Artinian [BS13b, Thm. 7.1.3], Proposition 4.6 implies the homomorphisms

$$\frac{S}{J[p^e]} \otimes_R F_{R*} H_{\mathfrak{m}}^{i-n}(R) \longrightarrow F_{S*} \left(\frac{S}{J[p^{e+1}]} \otimes_R H_{\mathfrak{m}}^{i-n}(R) \right)$$

are injective for every integer $e > 0$ (note that $J[p^e]$ is also generated by elements whose images in $S/\mathfrak{m}S$ form a system of parameters). Taking the colimit over all e , we see that the homomorphism (9) is injective, and hence the second homomorphism in (8) is injective as well. \square

Remark 4.7. In some special cases, one can give simpler proofs of Theorem A.

- (a) If φ is regular (resp. F -pure in the sense of Definition 2.10 and has Cohen–Macaulay closed fiber), then $F_{S/R}$ is faithfully flat by the Radu–André theorem (Theorem 2.9) (resp. pure by definition). The second homomorphism in (8) is therefore injective by the fact that pure ring homomorphisms induce injective maps on Koszul cohomology [HR74, Cor. 6.6], hence on local cohomology [SGA2, Exp. II, Prop. 5].
- (b) When φ is a regular homomorphism, it is also possible to prove Theorem A using Néron–Popescu desingularization [Pop86, Thm. 2.5; Swa98, Cor. 1.3], following the strategy in Vélez’s proof of Theorem 4.1 in [Vél95].

Remark 4.8. The analogue of Theorem A is false for the closely related notion of F -rationality without additional assumptions, since completions of Gorenstein F -rational rings need not be F -rational [LR01, §5], and for completion maps, the closed fiber is even geometrically regular. When R and S are excellent, however, F -rationality does ascend when the base ring R is F -rational and the closed fiber is geometrically F -rational [AE03, Thm. 4.3].

The analogue of Theorem A also fails for F -purity, even for rings of finite type over a field, by Singh’s example of a finite type algebra over a field for which F -regularity/ F -purity does not deform [Sin99, Prop. 4.5].

4.3. Geometrically F -injective rings and finitely generated field extensions. We show that one can take arbitrary finitely generated field extensions in the definition of geometric F -injectivity (Definition 2.1). We start with the following field-theoretic result.

Lemma 4.9 [Stacks, Tag 04KM]. *Let k be a field of characteristic $p > 0$, and let $k \subseteq k'$ be a finitely generated field extension. Then, there is a finite purely inseparable extension $k \subseteq k_1$, such that we have a diagram*

$$\begin{array}{ccc} & k_2 & \\ & \swarrow \quad \searrow & \\ k' & & k_1 \\ & \swarrow \quad \searrow & \\ & k & \end{array}$$

of finitely generated field extensions, where $k_1 \subseteq k_2 := (k' \otimes_k k_1)_{\text{red}}$ is a separable field extension.

We can now show that geometric F -injectivity is preserved under base change by finitely generated field extensions:

Proposition 4.10. *Let R be a Noetherian k -algebra, where k is a field of characteristic $p > 0$.*

- (i) Suppose R is F -injective, and consider a finitely generated separable field extension $k \subseteq k'$. Then, the ring $R \otimes_k k'$ is F -injective.
- (ii) Suppose R is geometrically F -injective over k , and consider a finitely generated field extension $k \subseteq k'$. Then, the ring $R \otimes_k k'$ is F -injective.
- (iii) Suppose R is Cohen–Macaulay and geometrically F -injective over k , and consider a finitely generated field extension $k \subseteq k'$. Then, the ring $R \otimes_k k'$ is Cohen–Macaulay and F -injective.

Proof. (i) follows from Corollary 4.2, since by separability of $k \subseteq k'$, the ring homomorphism $R \otimes_k k \rightarrow R \otimes_k k'$ is regular [EGAIV₂, Prop. 4.6.1].

We now show (ii). Let k_1 and k_2 be as in Lemma 4.9. Since the homomorphism $R \otimes_k k' \rightarrow R \otimes_k k_2$ is faithfully flat by base change, it suffices to show that $R \otimes_k k_2$ is F -injective by Theorem 3.8. Since R is geometrically F -injective and k_1 is a finite purely inseparable field extension, $R \otimes_k k_1$ is F -injective. Moreover, since $k_1 \subseteq k_2$ is a finitely generated separable extension, (i) then implies $R \otimes_k k_2 \simeq (R \otimes_k k_1) \otimes_{k_1} k_2$ is F -injective.

Finally (iii) follows from (ii) and the fact that Cohen–Macaulayness is preserved under base change by finitely generated field extensions [Mat89, Rem. on p. 182]. \square

5. CMFI HOMOMORPHISMS AND OPENNESS OF F -injective loci

After reviewing some basic material on CMFI homomorphisms in §5.1, we prove Theorem B, which says that F -injective locus is open in many cases, including in the setting of (IV). We also give an example of a locally excellent ring for which the F -injective locus is not open (Example 5.9). In §5.3, we use Theorems A and B to show that F -injectivity of a graded ring can be detected by localizing at the irrelevant ideal.

5.1. Definition and properties of CMFI homomorphisms. We begin by defining CMFI homomorphisms and proving some basic properties about them.

Definition 5.1 (cf. [Has01, Def. 5.4]). Let $\varphi: R \rightarrow S$ be a flat homomorphism of Noetherian rings. We say that φ is F -injective if, for every prime ideal $\mathfrak{p} \subseteq R$, the fiber $S \otimes_R \kappa(\mathfrak{p})$ of φ over \mathfrak{p} is geometrically F -injective over $\kappa(\mathfrak{p})$. We say that φ is *Cohen–Macaulay F -injective* or *CMFI* if φ is Cohen–Macaulay (i.e. all fibers of φ are Cohen–Macaulay) and F -injective. A morphism $f: X \rightarrow Y$ of locally Noetherian schemes of prime characteristic $p > 0$ is *CMFI* if f is flat with Cohen–Macaulay and geometrically F -injective fibers.

As in Remark 2.4, a flat homomorphism φ is CMFI if and only if the fibers of φ are geometrically CMFI.

Example 5.2. Since regular homomorphisms are CMFI, if (R, \mathfrak{m}) is an excellent local ring (or more generally, a local G -ring), then the canonical map $R \rightarrow \hat{R}$ is CMFI.

We show that the classes of F -injective and CMFI homomorphisms satisfy nice properties.

Lemma 5.3 (cf. [Has01, Cor. 5.6]). *Let $\varphi: R \rightarrow S$ and $\psi: S \rightarrow T$ be homomorphisms of Noetherian rings.*

- (i) *If φ is F -injective (resp. CMFI), then every base change of φ along a homomorphism essentially of finite type is F -injective (resp. CMFI).*
- (ii) *If ψ is faithfully flat and $\psi \circ \varphi$ is F -injective (resp. CMFI), then φ is F -injective (resp. CMFI).*

(iii) If φ is F -injective (resp. CMFI) and ψ is CMFI, then $\psi \circ \varphi$ is F -injective (resp. CMFI).

Analogous results hold for F -injective and CMFI morphisms of schemes.

Proof. We first show (i). By [EGAIV₂, Lem. 7.3.7], it suffices to note that if R is a geometrically F -injective (resp. geometrically CMFI) k -algebra, then $R \otimes_k k'$ is also geometrically F -injective (resp. geometrically CMFI) for every finitely generated field extension $k \subseteq k'$. This property holds by Proposition 4.10(ii) (resp. Proposition 4.10(iii)).

We next prove (ii). Since $\psi \circ \varphi$ is flat and ψ is faithfully flat, it follows that φ is flat. Consider $\mathfrak{p} \in \text{Spec}(R)$ and consider a finite, pure inseparable field extension $k := \kappa(\mathfrak{p}) \subseteq k'$. Since $T \otimes_k k'$ is F -injective (resp. CMFI) by assumption, it follows that $S \otimes_k k'$ is F -injective (resp. CMFI) by Theorem 3.8 (resp. [Has10, Lem. 4.6]).

For the proof of (iii), consider $\mathfrak{p} \in \text{Spec}(R)$ and consider a finite, purely inseparable field extension $k := \kappa(\mathfrak{p}) \subseteq k'$. The induced map

$$\psi \otimes_k \text{id}_{k'} : S \otimes_k k' \longrightarrow T \otimes_k k'$$

is CMFI by (i) since $S \rightarrow S \otimes_k k'$ is module-finite. By Corollary 4.2 (resp. [Ene09, Thm. 4.3]), we find that $T \otimes_k k'$ is F -injective (resp. CMFI). \square

As a consequence, the class of Noetherian rings with geometrically F -injective or geometrically CMFI formal fibers is closed under taking essentially of finite type ring homomorphisms.

Corollary 5.4. *Let R be a Noetherian ring of prime characteristic $p > 0$, and let S be an essentially of finite type R -algebra. If R has geometrically F -injective (resp. geometrically CMFI) formal fibers, then so does S .*

Proof. By [EGAIV₂, Cor. 7.4.5], it suffices to show that the property “geometrically F -injective” (resp. “geometrically CMFI”) satisfies the assertions (P_I), (P_{II}), and (P_{III}) from [EGAIV₂, (7.3.4)], and the assertion (P_{IV}) from [EGAIV₂, (7.3.6)]. First, (P_{III}) holds since fields are CMFI. Next, (P_I) and (P_{II}) hold by Lemmas 5.3(iii) and 5.3(ii), respectively. Finally, (P_{IV}) holds by Proposition 4.10(ii) (resp. Proposition 4.10(iii)). \square

We also find that F -injectivity often interacts well with tensor products:

Corollary 5.5. *Let k be a perfect field of characteristic $p > 0$. If R is a CMFI k -algebra, then for any essentially of finite type F -injective k -algebra S , the tensor product $R \otimes_k S$ is F -injective.*

Proof. Since k is perfect, the k -algebra map $k \rightarrow R$ is CMFI. Thus, by essentially of finite type base change (Lemma 5.3(i)), $S \rightarrow R \otimes_k S$ is also CMFI. Then $R \otimes_k S$ is F -injective by ascent of F -injectivity under CMFI homomorphisms (Corollary 4.2) because S is F -injective by hypothesis. \square

5.2. Openness of the F -injective locus. We first prove a result on the behavior of F -injective loci under CMFI morphisms.

Proposition 5.6. *Let $f: X \rightarrow Y$ be a CMFI morphism of locally Noetherian schemes of prime characteristic $p > 0$. Let $\text{FI}(X)$ (resp. $\text{FI}(Y)$) be the locus of points in X (resp. Y) at which X (resp. Y) is F -injective. We then have the following:*

- (i) $f^{-1}(\text{FI}(Y)) = \text{FI}(X)$.
- (ii) If the F -injective locus of Y is open, then the F -injective locus of X is open.

(iii) If f is faithfully flat and quasi-compact, then $\mathrm{FI}(X)$ is open if and only if $\mathrm{FI}(Y)$ is open.

Proof. (ii) follows from (i) by continuity of f , and (iii) follows from (ii) by [EGAIV₂, Cor. 2.3.12] because f is quasi-compact and faithfully flat. Thus, it suffices to prove (i). It suffices to show that for all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is F -injective if and only if $\mathcal{O}_{Y,f(x)}$ is F -injective. This follows from Theorems A and 3.8 since the flat local homomorphism

$$\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{Y,f(x)}$$

induced by f is CMFI by [Has10, Cor. 4.11]. \square

Remark 5.7. Proposition 5.6(i) fails for morphisms with geometrically CMFI fibers without the flatness hypothesis. Indeed, the canonical map

$$\mathrm{Spec}(k[x]/(x^2)) \longrightarrow \mathrm{Spec}(k[x])$$

has geometrically CMFI fibers and $k[x]$ is F -injective, but $k[x]/(x^2)$ is not since it is not reduced (Corollary 3.5(i)). However, in this case the F -injective locus of $k[x]/(x^2)$ is still open, albeit empty.

The next result affirmatively answers a question of the second author [Mur21, Rem. 3.6].

Theorem B. *Let R be a ring essentially of finite type over a Noetherian local ring (A, \mathfrak{m}) of prime characteristic $p > 0$, and suppose that A has Cohen–Macaulay and geometrically F -injective formal fibers. Then, the F -injective locus is open in $\mathrm{Spec}(R)$.*

We note that the hypotheses on the formal fibers of A are satisfied when A is excellent, or more generally, a G -ring.

Proof. Let $A \rightarrow \widehat{A}$ be the completion of A at \mathfrak{m} , and let Λ be a p -basis for $\widehat{A}/\mathfrak{m}\widehat{A}$ as in the gamma construction of Hochster–Huneke (see [HH94, (6.11)] or [Mur21, Constr. 3.1]). For every cofinite subset $\Gamma \subseteq \Lambda$, consider the commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & \widehat{A} & \longrightarrow & \widehat{A}^\Gamma \\ \downarrow & & \downarrow & & \downarrow \\ R & \xrightarrow{\pi} & R \otimes_A \widehat{A} & \xrightarrow{\pi^\Gamma} & R \otimes_A \widehat{A}^\Gamma \end{array}$$

where the squares are co-Cartesian. By the gamma construction [Mur21, Thm. 3.4(ii)], there exists a cofinite subset $\Gamma \subseteq \Lambda$ such that π^Γ induces a homeomorphism on spectra identifying F -injective loci. Since $R \otimes_A \widehat{A}^\Gamma$ is F -finite, the F -injective locus of $\mathrm{Spec}(R \otimes_A \widehat{A}^\Gamma)$ is open by [Mur21, Lem. A.2], and so, the F -injective locus of $R \otimes_A \widehat{A}$ is also open. Since an essentially of finite type base change of a faithfully flat CMFI morphism is faithfully flat and CMFI by Lemma 5.3(i), it follows that $\pi: R \rightarrow R \otimes_A \widehat{A}$ is a faithfully flat CMFI homomorphism of Noetherian rings. Thus, the F -injective locus of R is open by Proposition 5.6(iii). \square

Remark 5.8. Let (A, \mathfrak{m}, k) be a Noetherian complete local ring of prime characteristic $p > 0$. In the proof of [Mur21, Thm. 3.4(ii)], it is stated that the following property holds when \mathcal{P} is the property “ F -injective”:

($\Gamma 3$) For every local ring B essentially of finite type over A , if B is \mathcal{P} , then there exists a cofinite subset $\Gamma_1 \subseteq \Lambda$ such that $B^\Gamma := B \otimes_A A^\Gamma$ is \mathcal{P} for every cofinite subset $\Gamma \subseteq \Gamma_1$.

In [Mur21], the proof of (F3) for F -injectivity is incorrect as stated: the residue field of B is not necessarily a finite extension of the residue field k of A . The proof of the fact that (F3) holds for F -injectivity is still a straightforward adaptation of the proof of [EH08, Lem. 2.9(b)]: it suffices to work with vector spaces over the residue field l of B instead of vector spaces over the residue field k of A .

Even though Theorem B shows that the F -injective locus of a ring which is essentially of finite type over an excellent local ring is open, we now use a construction of Hochster [Hoc73] to show that the F -injective locus of an arbitrary locally excellent Noetherian ring of prime characteristic is not necessarily open. Here by a *locally excellent* Noetherian ring R , we mean a ring such that for every prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$, the localization $R_{\mathfrak{p}}$ is excellent.

Example 5.9. Let (R, \mathfrak{m}) be the local ring of a closed point on an affine variety over an algebraically closed field of characteristic $p > 0$, such that R is not F -injective. For all $n \in \mathbb{N}$, let R_n be a copy of R with maximal ideal $\mathfrak{m}_n = \mathfrak{m}$. Let $R' := \bigotimes_{n \in \mathbb{N}} R_n$, where the infinite tensor product is taken over k , and consider the ring

$$T := S^{-1}R',$$

where $S = R' \setminus (\bigcup_n \mathfrak{m}_n R')$. Then, T is a locally excellent Noetherian ring whose F -injective locus is not open by applying [Hoc73, Prop. 2] when $\mathcal{P} = “F$ -injective”.

For a specific example of a local ring (R, \mathfrak{m}) of an affine variety that is not F -injective, let k be an algebraically closed field of characteristic 2 and let

$$R = k[x^2, xy, y]_{(x^2, xy, y)} \subseteq k[x, y]_{(x, y)}.$$

Then, R is not weakly normal [Sch09, Rem. 3.7(iv)], hence also not F -injective by Corollary 3.5(iii).

Since strong F -regularity, F -rationality, and F -purity all imply F -injectivity (Remark A.4), these loci in $\operatorname{Spec}(T)$ are also not open by applying [Hoc73, Prop. 2] to these properties \mathcal{P} .

5.3. F -injectivity of graded rings. As a consequence of Theorems A and B, one can show that the irrelevant ideal of a Noetherian graded ring over a field controls the behavior of F -injectivity.

Recall that if $R = \bigoplus_{n=0}^{\infty} R_n$ is a graded ring such that $R_0 = k$ is a field, then $k^* := k \setminus \{0\}$ acts on R as follows: any $c \in k^*$ induces a ring automorphism

$$\lambda_c: R \longrightarrow R,$$

where λ_c maps a homogeneous element $t \in R$ of degree n to $c^n t$. Moreover, if k is infinite, then an ideal I of R is homogeneous if and only if I is preserved under this action of k^* . In other words, I is homogeneous if for all $c \in k^*$, we have $\lambda_c(I) = I$. With these preliminaries, we have the following result:

Theorem 5.10. *Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a Noetherian graded ring such that $R_0 = k$ is a field of characteristic $p > 0$. Let $\mathfrak{m} := \bigoplus_{n=1}^{\infty} R_n$ be the irrelevant ideal. Then, the following are equivalent:*

- (i) R is F -injective.
- (ii) For every homogeneous prime ideal \mathfrak{p} of R , the localization $R_{\mathfrak{p}}$ is F -injective.
- (iii) $R_{\mathfrak{m}}$ is F -injective.

Proof. (i) \Rightarrow (ii) follows from Proposition 3.3, while (ii) \Rightarrow (iii) is trivial. To complete the proof of the theorem, it suffices to show (iii) \Rightarrow (i).

Let $K := k(t)$, where t is an indeterminate, and let $R_K := K \otimes_k R$. Since $k \hookrightarrow K$ is regular and faithfully flat, the inclusion $R \hookrightarrow R_K$ is also regular and faithfully flat. Furthermore, because the irrelevant maximal ideal η of R_K is expanded from the irrelevant maximal ideal \mathfrak{m} of R , it follows that $R_{\mathfrak{m}} \hookrightarrow (R_K)_{\eta}$ is faithfully flat with geometrically regular closed fiber. Thus, by Theorem A, the ring $(R_K)_{\eta}$ is F -injective. If we can show that R_K is F -injective, then R will be F -injective by Theorem 3.8.

Theorem B shows that the F -injective locus of R_K is open because R_K is of finite type over K . Let I be the radical ideal defining the complement of the F -injective locus. Since the non- F -injective locus is preserved under automorphisms of R , we see that I is stable under the action of K^* . It follows that I is a homogeneous ideal since K is infinite. Since η is in the complement of the closed set defined by I , this forces I to equal R_K . Thus, R_K is F -injective, completing the proof of the Theorem. \square

6. SINGULARITIES OF GENERIC PROJECTION HYPERSURFACES

Bombieri [Bom73, p. 209] and Andreotti–Holm [AH77, p. 91] asked whether the image of a smooth projective variety of dimension r under a generic projection to \mathbf{P}_k^{r+1} is weakly normal. As an application of our results on F -injectivity, we show that in low dimensions, we have the following stronger result:

Theorem C. *Let $Y \subseteq \mathbf{P}_k^n$ be a smooth projective variety of dimension $r \leq 5$ over an algebraically closed field k of characteristic $p > r$, such that Y is embedded via the d -uple embedding with $d \geq 3r$. If $\pi: Y \rightarrow \mathbf{P}_k^{r+1}$ is a generic projection and $X = \pi(Y)$, then X is F -pure, and hence F -injective.*

Here when we say X is F -pure, we mean that all local rings of X are F -pure (and not that X is globally Frobenius split). Theorem C is the positive characteristic analogue of a theorem of Doherty [Doh08, Main Thm.], who proved that over the complex numbers, the image X of the generic projection has semi-log canonical singularities.

The proof of Theorem C largely follows that of [Doh08, Main Thm.], which checks that the singularities on generic projections are Du Bois after reduction modulo p using Fedder’s criterion [Fed83, Prop. 2.1]. Compared to Doherty’s proof, the main differences are that we must be careful about what possibilities occur in Roberts’s classification [Rob75, (13.2)] for fixed prime characteristics, and we also cannot use Doherty’s characteristic zero proof that the pinch point is Du Bois in (1a). We also need to use Corollary 5.5 instead of [Doh08, Thm. 3.9].

We will use the following result, the characteristic zero analogue of which was implicit in the proof of [Doh08, Main Thm.].

Lemma 6.1 (cf. [Sch09, Prop. 4.8]). *Let X be a reduced locally Noetherian scheme of prime characteristic $p > 0$. Suppose that X can be written as the union of two closed Cohen–Macaulay subschemes Y_1 and Y_2 of the same dimension. If Y_1 , Y_2 , and $Y_1 \cap Y_2$ are all F -injective, then X is F -injective.*

Schwede proves Lemma 6.1 under the additional hypothesis that X is F -finite. While we will only need the F -finite case in the sequel, the proof in [Sch09] applies in the non- F -finite case as well, once we know that Corollary 3.7 holds.

Proof of Theorem C. By [RZN84, Thm. 1.1], the generic projection $\pi: Y \rightarrow X$ is finite and birational. Thus, X is a hypersurface, and hence also Gorenstein. It therefore suffices to show that X has F -injective singularities by [Fed83, Lem. 3.3] (see also Remark A.4).

Since F -injectivity is unaffected under taking completions [Fed83, Rem. on p. 473], it suffices to show that for every closed point $x \in X$, the completion $\widehat{\mathcal{O}}_{X,x}$ of the local ring at x is F -injective. In [Rob75, (13.2)], Roberts provides a list of possibilities for the isomorphism classes of these complete local rings, which we treat individually. We label each case according to the notation in [Rob75].

- (0) The variety X could analytically locally have simple normal crossings at x , i.e.,

$$\widehat{\mathcal{O}}_{X,x} \simeq \frac{k[[z_1, z_2, \dots, z_{r+1}]]}{(z_1 z_2 \cdots z_d)}$$

for some $d \in \{1, 2, \dots, r+1\}$. This ring is F -pure by Fedder's criterion (see [Doh08, p. 2411]).

- (1a) When $r \geq 2$ and using the assumption that $p \neq 2$, the variety X could have a pinch point at x , i.e.,

$$\widehat{\mathcal{O}}_{X,x} \simeq \frac{k[[z_1, z_2, \dots, z_{r+1}]]}{(z_r^2 - z_1^2 z_{r+1})}.$$

This ring is F -pure by Fedder's criterion (see [BS13a, p. 160]).

- (1b) When $r \geq 4$ and using the assumption that $p \neq 3$, we could have

$$\widehat{\mathcal{O}}_{X,x} \simeq \frac{k[[z_1, z_2, \dots, z_{r+1}]]}{(z_r^3 + \Phi_4 + \Phi_5)}$$

where

$$\begin{aligned} \Phi_4(z_1, z_2, z_3, z_r, z_{r+1}) &= z_1^2 z_3 z_r - z_1^3 z_{r+1} + 2z_2 z_3 z_r^2 - 3z_1 z_2 z_r z_{r+1}, \\ \Phi_5(z_1, z_2, z_3, z_r, z_{r+1}) &= z_1^2 z_3^2 z_r - z_1 z_2^2 z_3 z_{r+1} - z_2^3 z_{r+1}^2. \end{aligned}$$

This ring is F -pure by Fedder's criterion (see [Doh08, p. 2412]).

The remaining cases are analytically locally unions of two hypersurfaces. By Lemma 6.1, it suffices to show that the hypersurfaces and their intersections are F -injective.

- (2a) When $r \geq 3$ and using the assumption that $p \neq 2$, the variety X could analytically locally be the union of a hyperplane Y_1 and a pinch point Y_2 at x , i.e.,

$$\widehat{\mathcal{O}}_{X,x} \simeq \frac{k[[z_1, z_2, \dots, z_{r+1}]]}{z_1(z_r^2 - z_2^2 z_{r+1})}.$$

The hyperplane Y_1 is regular, hence F -injective, and both Y_2 and the intersection $Y_1 \cap Y_2$ are pinch points, hence are F -injective by (1a).

- (2b) When $r = 5$ and using the assumption that $p \neq 3$, the variety X could analytically locally be the union of a hyperplane Y_1 and a hypersurface Y_2 of the form in (1b), i.e.,

$$\widehat{\mathcal{O}}_{X,x} \simeq \frac{k[[z_1, z_2, \dots, z_{r+1}]]}{z_1(z_r^3 + \Psi_4 + \Psi_5)}$$

where with notation as in (1b), we set $\Psi_i = \Phi_i(z_2, z_3, z_4, z_r, z_{r+1})$ for $i \in \{4, 5\}$. The hyperplane Y_1 is regular, hence F -injective, and both Y_2 and $Y_1 \cap Y_2$ are of the form in (1b), hence are F -injective.

- (2c) When $r = 5$ and using the assumption that $p \neq 2$, the variety X could analytically locally be the union of two pinch points Y_1 and Y_2 at x , i.e.,

$$\widehat{\mathcal{O}}_{X,x} \simeq \frac{k[[z_1, z_2, \dots, z_{r+1}]]}{(z_r^2 - z_1^2 z_{r+1})(z_{r-2}^2 - z_2^2 z_{r-1})}.$$

Both Y_1 and Y_2 are F -injective by (1a). To show that $Y_1 \cap Y_2$ is F -injective, it suffices to show that the local ring

$$\left(\frac{k[z_1, z_r, z_{r+1}]_{(z_1, z_r, z_{r+1})}}{(z_r^2 - z_1^2 z_{r+1})} \otimes_k \frac{k[z_2, z_3, \dots, z_{r-2}, z_{r-1}]_{(z_2, z_3, \dots, z_{r-2}, z_{r-1})}}{(z_{r-2}^2 - z_2^2 z_{r-1})} \right)_{(z_1, z_2, \dots, z_{r+1})}$$

is F -injective, since F -injectivity is unaffected by completion [Fed83, Rem. on p. 473]. Each factor of the tensor product is CMFI by the fact that their completions are F -injective by using (1a). Thus, the tensor product is F -injective by Corollary 5.5. Finally, the further localization of the tensor product is F -injective since F -injectivity localizes by Proposition 3.3.

- (3) When $r \geq 4$ and using the assumption that $p \neq 2$, the variety X could analytically locally be the union of a simple normal crossing divisor Y_1 and a pinch point Y_2 at x , i.e.,

$$\widehat{\mathcal{O}}_{X,x} \simeq \frac{k[[z_1, z_2, \dots, z_{r+1}]]}{z_1 z_2 (z_r^2 - z_3^2 z_{r+1})}.$$

The hypersurfaces Y_1 and Y_2 are F -injective by (0) and (1a), respectively. As in (2c), to show that $Y_1 \cap Y_2$ is F -injective, it suffices to show that the local ring

$$\left(\frac{k[z_1, z_2]_{(z_1, z_2)}}{(z_1 z_2)} \otimes_k \frac{k[z_3, z_4, \dots, z_{r+1}]_{(z_3, z_4, \dots, z_{r+1})}}{(z_r^2 - z_3^2 z_{r+1})} \right)_{(z_1, z_2, z_3, z_4, \dots, z_{r+1})}$$

is F -injective. The inner tensor product ring is F -injective by Corollary 5.5 since each factor is CMFI by the fact that the completions of these factors are CMFI via (0) and (1a), respectively. Finally, as in (2c), a further localization does not affect F -injectivity.

- (4) When $r = 5$ and using the assumption that $p \neq 2$, the variety X could analytically locally be the union of a simple normal crossing divisor Y_1 and a pinch point Y_2 at x , i.e.,

$$\widehat{\mathcal{O}}_{X,x} \simeq \frac{k[[z_1, z_2, \dots, z_{r+1}]]}{z_1 z_2 z_3 (z_r^2 - z_4^2 z_{r+1})}.$$

For this, the same reasoning as in (3) applies. \square

On the other hand, we can adapt an example of Doherty [Doh08, Cor. 4.7 and Ex. 4.8] to show that generic projections of large dimension cannot be F -pure.

Proposition 6.2 (cf. [Doh08, Cor. 4.7]). *Let $Y \subseteq \mathbf{P}_k^n$ be a smooth projective variety of dimension 30 over an algebraically closed field of characteristic $p > 0$ such that Ω_Y^1 is nef. If $\pi: Y \rightarrow \mathbf{P}_k^{31}$ is a generic projection and $X = \pi(Y)$, then X is not F -pure.*

Proof. The proof of [Doh08, Cor. 4.7] shows that in this setting, there are closed points $x \in X$ with Hilbert–Samuel multiplicity at least $2^5 = 32$. Now let $f \in k[x_1, x_2, \dots, x_{31}]$ be a local defining equation of X at x , and let $\mathfrak{m}_x \subseteq k[x_1, x_2, \dots, x_{31}]$ be the maximal ideal defining $x \in X$. By the multiplicity condition, we have $f \in \mathfrak{m}_x^{32}$. Thus, we have

$$f^{p-1} \in \mathfrak{m}_x^{32(p-1)} \subseteq \mathfrak{m}_x^{31(p-1)+1} \subseteq \mathfrak{m}_x^{[p]},$$

where the second inclusion follows from the pigeonhole principle [HH02, Lem. 2.4(a)]. Fedder's criterion [Fed83, Prop. 2.1] therefore implies that X is not F -pure at x . \square

Remark 6.3. The calculation in [Doh08, Thm. 4.5] implies that the hypersurface X in Proposition 6.2 is also not semi-log canonical.

Abelian varieties give examples of smooth projective varieties satisfying the hypothesis of Proposition 6.2, since they have nef cotangent bundles. We give some other examples of such varieties.

Example 6.4. Over the complex numbers, there are many explicit examples of smooth projective varieties Y with ample cotangent bundle Ω_Y^1 (see [Laz04, Constrs. 6.3.36–6.3.39]). In positive characteristic, we know of two methods to produce such examples.

- (a) Fix an integer $r > 0$ and an integer $n \geq 3r - 1$. Following the proof of [Laz04, Constr. 6.3.42], general complete intersections of $(n - r)$ hypersurfaces of sufficiently large degree in a product of n curves of genus ≥ 2 are r -dimensional smooth projective varieties with ample cotangent bundles.
- (b) Fix an integer $r > 0$ and an integer $n \geq 2r$. By Xie's proof of Debarre's ampleness conjecture [Xie18, Thm. 1.2], general complete intersections $X \subseteq \mathbf{P}_k^n$ of $(n - r)$ hypersurfaces of sufficiently large degree are r -dimensional smooth projective varieties with ample cotangent bundles.

By Proposition 6.2, applying either construction when $r = 30$ yields examples of smooth projective varieties whose generic projections are not F -pure.

APPENDIX A. F -rationality descends and implies F -injectivity

We address the relationship between F -rationality and F -injectivity and the descent property (II) for F -rationality. We start by defining F -rationality.

Definition A.1 [FW89, Def. 1.10; HH90, Def. 2.1; HH94, Def. 4.1]. Let R be a Noetherian ring. A sequence of elements $x_1, x_2, \dots, x_n \in R$ is a *sequence of parameters* if, for every prime ideal $\mathfrak{p} \subseteq R$ containing (x_1, x_2, \dots, x_n) , the images of x_1, x_2, \dots, x_n in $R_{\mathfrak{p}}$ are part of a system of parameters in $R_{\mathfrak{p}}$. An ideal $I \subseteq R$ is a *parameter ideal* if I can be generated by a sequence of parameters in R .

Now let R be a Noetherian ring of prime characteristic $p > 0$. We say that R is *F -rational* if every parameter ideal in R is tightly closed in the sense of [HH90, Def. 3.1].

We now prove the following lemma, which allows us to spread out a system of parameters in a localization to a sequence of parameters in the whole ring.

Lemma A.2 (cf. [QS17, Proof of Prop. 6.9]). *Let R be a Noetherian ring, and consider a prime ideal $\mathfrak{p} \subseteq R$ of height t . Let $I \subseteq R_{\mathfrak{p}}$ be an ideal generated by a system of parameters. Then, there exists a sequence of parameters $x_1, x_2, \dots, x_t \in \mathfrak{p}$ such that $I = (x_1, x_2, \dots, x_t)R_{\mathfrak{p}}$.*

Proof. Note that when $\text{ht } \mathfrak{p} = t = 0$, we do not have anything to prove. So assume $t \geq 1$. Write $I = (a_1, a_2, \dots, a_t)R_{\mathfrak{p}}$, where t is the height of \mathfrak{p} and $a_i \in R$ for every i . Note that \mathfrak{p} is minimal over (a_1, a_2, \dots, a_t) . Let J be the \mathfrak{p} -primary component of (a_1, a_2, \dots, a_t) in R . Then, we have $I = JR_{\mathfrak{p}}$ and $\text{ht } J = t$.

We now claim that there exist elements $b_1, b_2, \dots, b_t \in J^2$ such that setting $x_i = a_i + b_i$, the sequence x_1, x_2, \dots, x_t is a sequence of parameters in R . Note that for each i , the ideal

$(a_i) + J^2$ has height t because it is sandwiched between the two ideals J^2 and J , both of which have height t . For $i = 1$, we have

$$(a_1) + J^2 \not\subseteq \bigcup_{\mathfrak{q} \in \text{Min}_R(R)} \mathfrak{q}$$

by prime avoidance. Thus, by a theorem of Davis [Kap74, Thm. 124], there exists $b_1 \in J^2$ such that

$$x_1 := a_1 + b_1 \notin \bigcup_{\mathfrak{q} \in \text{Min}_R(R)} \mathfrak{q}.$$

For every $1 < i \leq t$, the same method implies there exist $b_i \in J^2$ such that

$$x_i := a_i + b_i \notin \bigcup_{\text{Min}_R(R/(x_1, x_2, \dots, x_{i-1}))} \mathfrak{q}.$$

By construction, x_1, x_2, \dots, x_t generates an ideal of height t in R , and therefore it is a sequence of parameters by [Bos13, §2.4, Prop. 13], since the images of x_1, x_2, \dots, x_t continue to generate an ideal of height t in $R_{\mathfrak{p}}$ for any prime \mathfrak{p} that contains these elements. Since $(x_1, x_2, \dots, x_t)R_{\mathfrak{p}} \subseteq I$ and $I = (x_1, x_2, \dots, x_t)R_{\mathfrak{p}} + I^2$, Nakayama's lemma implies $I = (x_1, x_2, \dots, x_t)R_{\mathfrak{p}}$ (see [Mat89, Cor. to Thm. 2.2]). \square

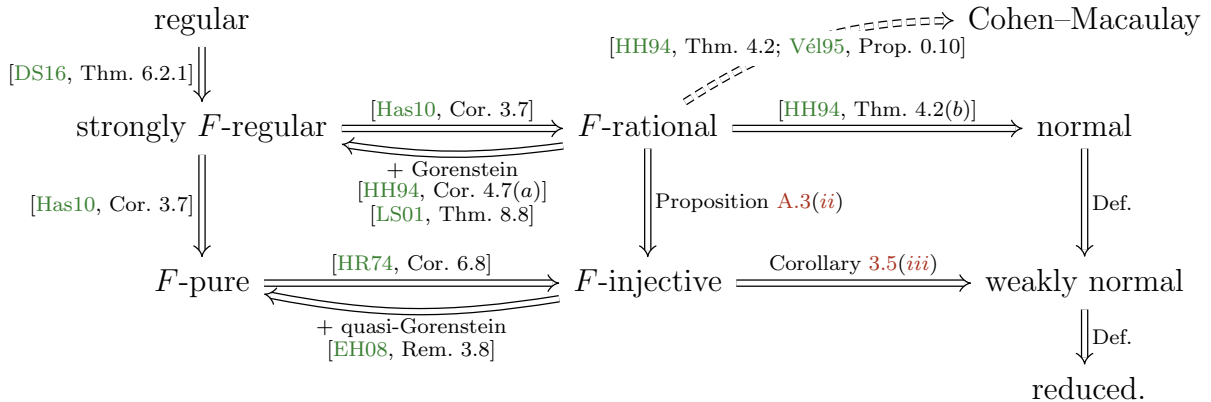
The following result shows that F -rational rings are F -injective and are even CMFI under mild assumptions.

Proposition A.3. *Let R be an F -rational Noetherian ring of prime characteristic $p > 0$. We then have the following:*

- (i) *If R is the homomorphic image of a Cohen–Macaulay ring or is locally excellent, then $R_{\mathfrak{m}}$ is F -rational for every maximal ideal $\mathfrak{m} \subseteq R$, and R is Cohen–Macaulay.*
- (ii) *R is F -injective.*

Proof. (i) is shown in [HH94, Thm. 4.2; Vél95, Prop. 0.10]. We now show (ii). By Lemma 2.6, it suffices to show that every ideal $I \subseteq R_{\mathfrak{m}}$ generated by a system of parameters in $R_{\mathfrak{m}}$ is Frobenius closed. By Lemma A.2, there exists a sequence of parameters $x_1, x_2, \dots, x_t \in \mathfrak{m}$ such that $I = (x_1, x_2, \dots, x_t)R_{\mathfrak{m}}$. Since R is F -rational, we see that (x_1, x_2, \dots, x_t) is tightly closed in R , hence also Frobenius closed in R . Since Frobenius closure commutes with localization [QS17, Lem. 3.3], we see that I is Frobenius closed in $R_{\mathfrak{m}}$. \square

Remark A.4. For Noetherian rings of prime characteristic $p > 0$, the relationship between F -injectivity and other classes of singularities can be summarized as follows:



Here, strong F -regularity is defined in terms of tight closure [Has10, Def. 3.3], following Hochster. The dashed implication holds if either R is the homomorphic image of a Cohen–Macaulay ring, or is locally excellent. We note that the implications with citations to the present paper may be well-known to experts.

Finally, for the sake of completeness, we show that F -rationality descends for arbitrary Noetherian rings. This is shown in [Vél95] assuming the existence of test elements.

Proposition A.5 (cf. [Vél95, (6) on p. 440]). *Let $\varphi: R \rightarrow S$ be a faithfully flat homomorphism of Noetherian rings of prime characteristic $p > 0$. If S is F -rational, then R is F -rational.*

Proof. Let I be a parameter ideal in R . We first claim that $\varphi(I)S$ is a parameter ideal in S . Let $\mathfrak{q} \subseteq S$ be a prime ideal containing $\varphi(I)S$, and let $\mathfrak{p} \subseteq R$ be the contraction of \mathfrak{q} in R . Since I is a parameter ideal in R , we can write $I = (x_1, x_2, \dots, x_t)$, where the images of x_1, x_2, \dots, x_t in $R_{\mathfrak{p}}$ can be completed to a system of parameters $x_1, x_2, \dots, x_t, x_{t+1}, \dots, x_d$ for $R_{\mathfrak{p}}$. Since the induced map $R_{\mathfrak{p}} \rightarrow S_{\mathfrak{q}}$ is also faithfully flat, the images of $x_1, x_2, \dots, x_t, x_{t+1}, \dots, x_d$ in $S_{\mathfrak{q}}$ form part of a system of parameters of $S_{\mathfrak{q}}$. We therefore see that $\varphi(I)S$ is indeed a parameter ideal in S .

Returning to the proof of the Proposition, let I be a parameter ideal in R . Since $\varphi(R^{\circ}) \subseteq S^{\circ}$ by the faithful flatness of φ , we have that

$$\varphi(I)S \subseteq \varphi(I_R^*)S \subseteq (\varphi(I)S)_S^* = \varphi(I)S,$$

where the last equality holds by the F -rationality of S and the fact that $\varphi(I)S$ is a parameter ideal. Since the leftmost and rightmost ideals are equal, we therefore have $\varphi(I)S = \varphi(I_R^*)S$. Contracting the expansions back to R , we then get $I = I_R^*$. \square

Remark A.6. Let \mathcal{P} be the property of Noetherian local rings being F -rational. In Proposition A.5 we showed that \mathcal{P} satisfies (II), and Véléz showed that \mathcal{P} satisfies (IV) [Vél95, Thm. 3.5]. Moreover, \mathcal{P} also satisfies (I) when R is the homomorphic image of a Cohen–Macaulay ring (Proposition A.3(i)), and (III*) holds when R and S are locally excellent [Vél95, Thm. 3.1]. However, to the best of our knowledge, it is not known if the localization of an F -rational local ring is always F -rational for arbitrary Noetherian rings, or if an arbitrary F -rational Noetherian ring is Cohen–Macaulay.

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