RELATIVE VANISHING THEOREMS FOR Q-SCHEMES

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Abstract. We prove the relative Grauert–Riemenschneider vanishing, Kawamata–Viehweg vanishing, and Kollár injectivity theorems for proper morphisms of schemes of equal characteristic zero, solving conjectures of Boutot and Kawakita. Our proof uses the Grothendieck limit theorem for sheaf cohomology and Zariski–Riemann spaces. We also show these vanishing and injectivity theorems hold for locally Moishezon (resp. projective) morphisms of quasi-excellent algebraic spaces admitting dualizing complexes and semianalytic germs of complex analytic spaces (resp. quasi-excellent formal schemes admitting dualizing complexes, rigid analytic spaces, Berkovich spaces, and adic spaces locally of weakly finite type over a field), all in equal characteristic zero.

We give many applications of our vanishing results. For example, we extend Boutot's theorem to all Noetherian $\mathbf{Q}$-algebras by showing that if $R \to R'$ is a cyclically pure map of $\mathbf{Q}$-algebras and R' is pseudo-rational, then R is pseudo-rational. This solves a conjecture of Boutot and affirmatively answers a question of Schoutens. The proof of this Boutot-type result uses a new characterization of pseudo-rationality and rational singularities using Zariski–Riemann spaces. This characterization is also used in the proofs of our vanishing and injectivity theorems and is of independent interest.

Contents

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1. Introduction

Let X be a smooth complex projective variety. Kodaira's vanishing theorem [Kod53, Theorem 2] says that if $\mathscr{L}$ is an ample invertible sheaf on X, then $H^i(X, \omega_X \otimes \mathscr{L}) = 0$ for all $i > 0$. Kodaira's theorem and its generalizations have since become indispensable tools in algebraic geometry over fields of characteristic zero, in particular in birational geometry and the minimal model program (see, e.g., [KMM87; EV92; KM98; Laz04a; Laz04b; Fuj17]).

While the central goal of birational geometry is to study birational equivalences between projective varieties, this often requires working with more general schemes. For example, Hironaka's original proof of resolution of singularities over fields of characteristic zero uses an inductive strategy involving schemes of finite type over quasi-excellent local $\mathbf{Q}$-algebras (see [Hir64, p. 162]). In [dFEM10; dFEM11], de Fernex, Ein, and Mustaţă work with schemes of finite type over formal power series rings to prove Shokurov's ACC conjecture for log canonical thresholds on complex algebraic varieties whose singularities belong to a bounded family (Shokurov's conjecture has since been proved in general [HMX14]). In [Kaw15], Kawakita also works over formal power series rings to prove a special case of Shokurov's ACC conjecture for minimal log discrepancies on smooth threefolds.

A problem in these more general contexts is the lack of Kodaira-type vanishing theorems. One of the most fundamental generalizations of Kodaira's theorem for the minimal model program is the Kawamata–Viehweg vanishing theorem [Kaw82, Theorem 1; Vie82, Theorem I], relative versions of which are known to hold for proper morphisms of varieties over an algebraically closed field of characteristic zero [KMM87, Theorem 1-2-3], or for proper morphisms of complex analytic spaces that are Moishezon, i.e., bimeromorphic to a projective morphism [Nak87, Theorem 3.7]. In particular, this has been an issue in non-Archimedean geometry, where the Kawamata–Viehweg vanishing theorem is only known for proper morphisms to curves [BFJ16, Appendix B; MN15, §5].

Our main result is the following generalization of the Kawamata–Viehweg vanishing theorem to proper morphisms of excellent schemes of equal characteristic zero with dualizing complexes, which resolves conjectures of Boutot [Bou87, Remarque 1 on p. 67] and Kawakita [Kaw15, Conjecture 1.1]. In fact, we also show a version of Kollár's injectivity theorem [Kol86, Theorem 2.2] for klt pairs, which for varieties is due to Kawamata [Kaw85, Theorem 3.2] (see also [EV87, Corollaire 1.11]). Below, a morphism $f\colon X \to Y$ is *maximally dominating* if every generic point of an irreducible component of X maps to a generic point of an irreducible component of Y [ILO14, Exposé II, Définition 1.1.2]. Proper surjective morphisms of integral schemes are maximally dominating.

Theorem A. *Let $f\colon X \to Y$ be a proper maximally dominating morphism of Noetherian schemes of equal characteristic zero such that Y has a dualizing complex $\omega_Y^\bullet$. Let Δ be an effective $\mathbf{Q}$-Weil divisor on X. Suppose one of the following conditions holds:*

(a) *X is regular, Δ has simple normal crossings support, and $\lfloor \Delta \rfloor = 0$.*
(b) *X is normal, (X, Δ) is klt, and Y is locally excellent.*

Denote by ω_X the unique nonzero cohomology sheaf of $f^!\omega_Y^\bullet$ (after possibly applying shifts on each connected component of X) and denote by K_X an associated canonical divisor. Consider a Cartier divisor N on X such that $N \sim_{\mathbf{Q}} K_X + M + \Delta$ for a $\mathbf{Q}$-Cartier divisor M on X.

(i) *Suppose M is f-nef and f-big. Then, we have*
$$R^i f_*\big(\mathcal{O}_X(N)\big) = 0$$
for all $i > 0$.

(ii) *Suppose M is f-semi-ample. Let D be an effective Weil divisor on X for which there exists an integer $n > 0$ such that nM is Cartier and an effective Weil divisor D' on X such that $\mathcal{O}_X(D + D') \simeq \mathcal{O}_X(nM)$. Then, the canonical morphisms*
$$R^i f_*\big(\mathcal{O}_X(N)\big) \longrightarrow R^i f_*\big(\mathcal{O}_X(N + D)\big)$$

induced by the inclusion $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(D)$ *are injective for all* i.

Since Kodaira-type vanishing theorems are false in both positive [Ray78] and mixed characteristic (Totaro; see [BMPSTWW23, Footnote 1]), Theorem A and the methods of this paper yield the most general versions of the Kawamata–Viehweg vanishing theorem possible for proper morphisms of schemes of arbitrary dimension.

We note that on each connected component of X, the exceptional pullback $f^!\omega_Y^\bullet$ of the dualizing complex $\omega_Y^\bullet$ is concentrated in one degree by local duality [Har66, Chapter V, Corollary 6.3] since X is Cohen–Macaulay by assumption in the regular case or by [BK23, Theorem 3.1] in the klt case (see Theorem 9.6 and Remark 9.7).

As far as we are aware, the only previously known cases of Theorem A(i) outside of the context of algebraic varieties or analytic spaces are when $\dim(X) \leq 3$ or $\dim(Y) = 1$. The case when $\dim(X) = 2$ (which also holds in arbitrary characteristic) is essentially due to Lipman [Lip78, Theorem 2.4] when f is generically finite (see [Kol13, Theorem 10.4]). Tanaka [Tan18, §3.1] used Lipman's methods to prove the case when $\dim(X) = 2$ and f is arbitrary (again in arbitrary characteristic). When $\dim(X) = 3$ and f is birational, Bernasconi and Kollár showed that a version of Theorem A(i) holds when the residue fields of Y are perfect fields of characteristic $\notin \{2, 3, 5\}$ [BK23, Theorem 2]. Some cases when Y is the spectrum of a complete DVR are due to Boucksom–Favre–Jonsson [BFJ16, Theorem B.3] and Mustaţă–Nicaise [MN15, Theorem 5.2.3 and Remark 5.3]. Mustaţă and Nicaise also proved a version of Theorem A(ii), again under the assumption that Y is the spectrum of a complete DVR [MN15, Theorem 5.3.1 and Remark 5.4].

To prove Theorem A in the regular case, after replacing Y by $\operatorname{Spec}(\hat{\mathcal{O}}_{Y,y})$ for each $y \in Y$, we can use cyclic covers and log resolutions to reduce to the case when L is f-ample and $\Delta = 0$. We then show the following:

Theorem B. *Let* $f\colon X \to Y$ *be a proper maximally dominating morphism of Noetherian schemes of equal characteristic zero such that* X *is locally pseudo-rational and such that* Y *has a dualizing complex* $\omega_Y^\bullet$. *Denote by* ω_X *the unique nonzero cohomology sheaf of* $f^!\omega_Y^\bullet$ *(after possibly applying shifts on each connected component of* X*). Consider an invertible sheaf* $\mathscr{L}$ *on* X.

(i) *Suppose* $\mathscr{L}$ *is* f*-big and* f*-semi-ample. Then, we have*

$$R^i f_*(\omega_X \otimes_{\mathcal{O}_X} \mathscr{L}) = 0$$

for all $i > 0$.

(ii) *Suppose* $\mathscr{L}$ *is* f*-semi-ample. Let* D *be an effective Weil divisor on* X *for which there exists an integer* $n > 0$ *and an effective Weil divisor* D' *on* X *such that* $\mathcal{O}_X(D + D') \simeq \mathscr{L}^{\otimes n}$. *Then, the canonical morphisms*

$$R^i f_*(\omega_X \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes k}) \longrightarrow R^i f_*\big(\omega_X \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes k}(D)\big)$$

induced by the inclusion $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(D)$ *are injective for all* i *and for all* $k > 0$.

Pseudo-rationality is a characteristic-free version of rational singularities introduced by Lipman and Teissier [LT81] that does not require resolutions of singularities, quasi-excellence, or the existence of dualizing complexes. Note that regular rings are locally pseudo-rational [LT81, §4]. The two statements (i) and (ii) are relative versions of the Grauert–Riemenschneider vanishing theorem [GR70, Satz 2.1] and Kollár's injectivity theorem [Kol86, Theorem 2.2], respectively. We also show dual versions of Theorems A and B analogous to Hartshorne and Ogus's dual formulation [HO74, Proposition 2.2] of the relative Grauert–Riemenschneider vanishing theorem [GR70, Satz 2.3] (see Theorem B*) and Kollár's local version of Kawamata–Viehweg vanishing [Kol11, Corollary 20] (see Theorem 8.2). These dual statements have the advantage of not requiring that Y has a dualizing complex.

Outline of proof. We outline the proof of Theorem B. For simplicity, we consider the case when $f\colon X \to Y$ is a proper surjective morphism of integral schemes. The proof of Theorem B proceeds by approximating the morphism $f\colon X \to Y$ by proper surjective morphisms of varieties over $\mathbf{Q}$, and then deducing the vanishing in Theorem B from the usual statements for varieties over a field of characteristic zero. This approach runs into two major difficulties.

- While we can write f as the limit of proper surjective morphisms $f_\lambda\colon X_\lambda \to Y_\lambda$ of varieties over $\mathbf{Q}$ using the method of relative Noetherian approximation [EGAIV$_3$, §8], we cannot ensure that the X_λ are smooth, even if X is regular.
- Even though direct images behave well under limits by the Grothendieck limit theorem [SGA4$_2$, Exposé VI, Théorème 8.7.3] (see Theorem 3.12), the sheaves ω_X are not known to behave well under limits.

To fix the smoothness of X_λ, we want to replace the inverse system $\{X_\lambda\}_{\lambda\in\Lambda}$ by an inverse system of resolutions of singularities of the X_λ using Hironaka's resolutions of singularities [Hir64, Chapter 0, §3, Main Theorem I]. We were unable to choose resolutions of singularities compatibly as $\lambda \in \Lambda$ varies, so instead we take the inverse system consisting of *all possible* resolutions of singularities of the X_λ.

A major technical difficulty is to show that the resulting inverse limit is a familiar locally ringed space, called the *Zariski–Riemann space* $\mathrm{ZR}(X)$ of X. The Zariski–Riemann space was defined by Zariski for varieties [Zar40, Definition A.II.5; Zar44, §2] and by Nagata for Noetherian separated schemes [Nag63, §3]. Thus, even though our initial interest was to show vanishing theorems for schemes, a surprising and novel aspect of our proof is that we must leave the world of schemes and consider more general locally ringed spaces.

To fix the issues with direct images and limits and with the sheaves ω_X, we use duality to prove statements about local cohomology instead. The reason this works is that while ω_X is not known to behave well under limits, structure sheaves do behave well under limits. Put together, the proof of Theorems A and B proceeds as follows. Below, we concentrate on Theorem B(i).

(I) We replace Y with $\mathrm{Spec}(\hat{\mathcal{O}}_{Y,y})$ to assume that Y is the spectrum of an excellent local domain $(R,\mathfrak{m})$ containing $\mathbf{Q}$.

(II) Using Lipman's local-global duality [Lip78, Theorem on p. 188] (see Lemma 5.2 and Proposition 5.3), we translate the vanishing statements in Theorem B to vanishing statements about the local cohomology modules $H^i_Z(\mathscr{L}^{-1})$ where $Z = f^{-1}(\mathfrak{m})$ as in [HO74, Proposition 2.2].

(III) We prove a new characterization of pseudo-rational rings in equal characteristic zero via Zariski–Riemann spaces (Theorem 6.2), which is of independent interest. This characterization shows that the higher direct images of the structure sheaf under the projection morphism $\mathrm{ZR}(X) \to X$ from the Zariski–Riemann space of X vanish. It then suffices to show that vanishing holds for the composition

$$\mathrm{ZR}(X) \xrightarrow{\pi} X \xrightarrow{f} \mathrm{Spec}(R).$$

See Definition 3.7 for the definition of the Zariski–Riemann space $\mathrm{ZR}(X)$ associated to X.

(IV) Using the method of relative Noetherian approximation [EGAIV$_3$, §8] and Hironaka's resolution of singularities [Hir64, Chapter 0, §3, Main Theorem I], we write the composition in (III) as the limit of morphisms

$$W_{\lambda,p} \xrightarrow{g_{\lambda,p}} X_\lambda \xrightarrow{f_\lambda} \mathrm{Spec}(R_\lambda)$$

of varieties over $\mathbf{Q}$ where the $W_{\lambda,p}$ are smooth (Lemma 4.2). By a version of the Grothendieck limit theorem [SGA4$_2$, Exposé VI, Théorème 8.7.3] for local cohomology modules (Theorem 3.13), the usual vanishing statements for varieties over $\mathbf{Q}$ then imply vanishing and injectivity theorems for Zariski–Riemann spaces (Theorem 5.1) that require no assumptions on the

singularities of X apart from integrality. Using the characterization of pseudo-rational rings in (III), we obtain Theorem B as a consequence.

While the technique of applying the Grothendieck limit theorem on one hand and passing to covers of X that satisfy vanishing theorems on the other have been applied before, as far as we are aware, the idea to pass to the Zariski–Riemann space to show vanishing theorems is new.

We describe two sources of inspiration for the Grothendieck limit theorem and for passing to covers. The inspiration to use the Grothendieck limit theorem comes from Panin's proof of the equicharacteristic case of Gersten's conjecture in algebraic K-theory [Pan03, Theorem A]. Since Gersten's conjecture is a statement about regular local rings, however, Panin was able to use Néron–Popescu desingularization [Pop86, Theorem 2.4; Pop90, p. 45; Swa98, Theorem 1.1] to approximate the regular local rings containing a field k that appear in this special case of Gersten's conjecture with essentially smooth algebras over that field k and stay in the category of rings. By doing so, Panin reduced the equicharacteristic case of Gersten's conjecture to the geometric case already shown by Quillen [Qui73, Theorem 5.11]. This strategy was also applied to related questions in mixed characteristic in [Ska].

Our approach is also related to the theory of big Cohen–Macaulay modules and big Cohen–Macaulay algebras as introduced by Hochster [Hoc73a; Hoc75a; Hoc75b]. In equal characteristic $p > 0$, Hochster and Huneke showed that the integral closure X^+ of X in an algebraic closure of its function field satisfies a Kodaira-type vanishing theorem [HH92, Theorem 1.2]. Thus, the Zariski–Riemann space $\mathrm{ZR}(X)$ plays a similar role in equal characteristic zero that the scheme X^+ does in equal characteristic $p > 0$. Smith later asked whether analogues of the Grauert–Riemenschneider or Kawamata–Viehweg vanishing theorems for X^+ hold in equal characteristic $p > 0$ [Smi97a, Theorem 2.1; Smi97b]. Bhatt answered Smith's question in [Bha12, §7]. This approach was recently applied in mixed characteristic [Bha, Theorem 5.1; TY23, §3.1; BMPSTWW23, §3], where X^+ was shown to satisfy Kodaira-type vanishing theorems.

For rings R essentially of finite type over the complex numbers, Roberts showed that the derived pushforward $\mathbf{R}f_*\mathcal{O}_X$ of the structure sheaf of a resolution of singularities $f\colon X \to \mathrm{Spec}(R)$ is a complex which is Cohen–Macaulay in a suitable sense [Rob80, Corollary on p. 224] (see also [IMSW21, Proposition 4.17]). Our vanishing theorem for Zariski–Riemann spaces (Theorem 5.1) can be seen as a version of Roberts's result that holds for all integral Noetherian local **Q**-algebras. See [IMSW21, Proposition 4.17], which uses Theorems A and B in this paper to show that every excellent **Q**-algebra with a dualizing complex has a maximal Cohen–Macaulay complex that is equivalent to a graded-commutative dg algebra.

Applications. Theorems A and B allow one to extend many results to the setting of excellent rings and schemes of equal characteristic zero that for varieties rely on Kodaira-type vanishing theorems and resolutions of singularities. We describe some examples of these applications.

In joint work with Shiji Lyu [LM, Theorems A and B], we use Theorem A to prove finite generation of the relative adjoint ring for proper morphisms of excellent schemes of equal characteristic zero with dualizing complexes in the vein of [BCHM10, Theorem 1.2]. Our proof uses the approach of Cascini and Lazić [CL12]. We then prove that one can run the relative minimal model program with scaling in the sense of [BCHM10; HM10] in this setting, thereby resolving a question of Kollár [Kol08, (23)]. Using Theorem A and new GAGA theorems for Grothendieck duality and dualizing complexes, we then extend the relative minimal model program with scaling to many categories at once, namely to the categories of quasi-excellent algebraic spaces and formal schemes with dualizing complexes, semianalytic germs of complex analytic spaces, Berkovich spaces, rigid analytic spaces, and adic spaces locally of weakly finite type over a field, all in equal characteristic zero. The case for algebraic spaces of finite type over a field was previously shown by Villalobos-Paz [VP] and the case for complex analytic spaces was previously shown in [Fuj; DHP]. In addition, [BMPSTWW23]

uses Theorem A to establish the minimal model program for threefolds of mixed characteristic for residue characteristics $\notin \{2,3,5\}$, where the characteristic zero fibers are often excellent schemes of equal characteristic zero and are not necessarily of finite type over a field. Lyu and the author of the present paper also extend these results to other categories of spaces [LM, Theorem A^p].

In this paper, using the GAGA-type results obtained in our joint work with Lyu [LM], we can show the following vanishing and injectivity theorems for algebraic spaces, formal schemes, complex analytic spaces, and non-Archimedean analytic spaces.

Theorem A′. *Let $f\colon X \to Y$ be a proper surjective morphism in one of the following categories where X and Y are integral and X is normal:*

- (0) *The category of Noetherian algebraic spaces of equal characteristic zero over a scheme S admitting dualizing complexes.*
- (I) *The category of Noetherian formal schemes of equal characteristic zero admitting c-dualizing complexes.*
- (II) *The category of semianalytic germs of complex analytic spaces.*
- (III) *The category of k-analytic spaces over a complete non-Archimedean field k of characteristic zero.*
- (III′) *The category of rigid k-analytic spaces over a complete non-trivially valued non-Archimedean field k of characteristic zero.*
- (IV) *The category of adic spaces locally of weakly finite type over a complete non-trivially valued non-Archimedean field k of characteristic zero.*

In cases (I) and (IV), we assume that f is projective. In cases (III) and (III′), we assume that either f is projective or that Y is a point.

Let K_X be a canonical divisor on X chosen compatibly with a dualizing complex on Z,[1] *and let Δ be an effective $\mathbf{Q}$-Weil divisor on X. Suppose one of the following conditions holds:*

- (a) *X is regular, Δ has simple normal crossings support, and $\lfloor \Delta \rfloor = 0$.*
- (b) *X is normal and (X, Δ) is klt. In cases (0) and (I), we also assume that Y is quasi-excellent.*

Consider a Cartier divisor N on X such that $N \sim_{\mathbf{Q}} K_X + M + \Delta$ for a $\mathbf{Q}$-Cartier divisor M on X.

- (*i*) *Suppose M is f-nef and f-big. Then, we have*
$$R^i f_*\big(\mathcal{O}_X(N)\big) = 0$$
for all $i > 0$.
- (*ii*) *Suppose M is f-semi-ample and f is locally Moishezon. Let D be an effective Weil divisor on X for which there exists an integer $n > 0$ such that nM is Cartier and an effective Weil divisor D' on X such that $\mathcal{O}_X(D + D') \simeq \mathcal{O}_X(nM)$. Then, the canonical morphisms*
$$R^i f_*\big(\mathcal{O}_X(N)\big) \longrightarrow R^i f_*\big(\mathcal{O}_X(N + D)\big)$$
induced by the inclusion $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(D)$ are injective for all i.

As far as we are aware, in arbitrary dimension, Theorem A′ was previously only known for varieties and for complex analytic spaces. In low dimensions, Theorem A′ was also known for non-Archimedean analytic spaces when $\dim(Y) = 1$. For complex analytic spaces, Theorem A′(*i*) gives an alternative proof of Nakayama's version of the Kawamata–Viehweg vanishing theorem [Nak87, Theorem 3.7], and Theorem A′(*ii*) recovers the special case of Nakayama's version of Kollár's injectivity theorem [Nak87, Theorem 3.10(B)] when f is locally Moishezon. For non-Archimedean analytic spaces, the case when $\dim(Y) = 1$ follows from the work of Boucksom–Favre–Jonsson [BFJ16, Theorem B.3] and Mustaţă–Nicaise [MN15, Theorem 5.2.3, Remark 5.3, Theorem 5.3.1, and

[1] For example, when Z is a variety over k or in cases (II), (III), (III′), and (IV), we can choose K_X so that $\mathcal{O}_{X_{\mathrm{sm}}}((K_X)_{|X_{\mathrm{sm}}}) = \det(\Omega_{X_{\mathrm{sm}}/k})$ where X_{sm} is the smooth locus of X. See [LM, Theorem 24.4(*iii*), Theorem 24.6(*iii*), and Theorem 24.8(*iv*)].

Remark 5.4]. For formal schemes, Smith showed a special case of Kodaira vanishing [Smi17, Theorem 4.2.1] and showed that in general, Kodaira vanishing is false for formal schemes that are smooth and *pseudo*-projective over the complex numbers [Smi17, Proposition 4.3.1].

In §9, we use our vanishing results to study rational singularities. First, we show the following version of Boutot's theorem [Bou87, Théorème on p. 65], which solves a conjecture of Boutot [Bou87, Remarque 1 on p. 67] and gives a complete, positive answer to a question of Schoutens [Sch08, (2) on p. 611]. This result generalizes a result of Schoutens [Sch08, Main Theorem A], who showed that if R' is regular in the statement below, then R is locally pseudo-rational. For the statement below, a ring map $R \to R'$ is *cyclically pure* if $IR' \cap R = I$ for every ideal $I \subseteq R$ [Hoc77, p. 463]. Split or faithfully flat ring maps are cyclically pure [HR74, p. 136].

Theorem C. *Let $R \to R'$ be a cyclically pure map of Noetherian* **Q***-algebras. If R' is locally pseudo-rational, then R is locally pseudo-rational. In particular, if R' is regular, then R is locally pseudo-rational.*

The proof of Theorem C uses techniques from our recent joint work with Charles Godfrey [GM], where we showed that Du Bois singularities descend under cyclically pure maps for rings essentially of finite type over the complex numbers. One key aspect of the proof of Theorem C is that our vanishing theorem for Zariski–Riemann spaces (Theorem 5.1) can be interpreted as an injectivity theorem for the derived pushforward $\mathbf{R}\pi_*\mathcal{O}_{\mathrm{ZR}(X)}$ of the structure sheaf of the Zariski–Riemann space analogous to Kovács and Schwede's injectivity theorem for the 0-th graded piece $\underline{\Omega}^0_X$ of the Deligne–Du Bois complex [KS16, Theorem 3.3].

We also prove the following generalizations of results on rational singularities which were previously known for rings essentially of finite type over a field of characteristic zero:

(1) Pseudo-rationality deforms in equal characteristic zero (Theorem 9.3). This extends a result of Elkik [Elk78, Théorème 5] (for rings essentially of finite type over a field of characteristic zero) and the author [Mur22, Proposition 4.17] (for quasi-excellent local **Q**-algebras). Note that [Mur22, Proposition 4.17] relies on Theorem B of this paper.
(2) Derived splinters and rational singularities coincide for quasi-excellent **Q**-algebras (Theorem 9.5). This extends a theorem of Kovács and Bhatt for rings of finite type over a field of characteristic zero [Kov00, Theorem 3; Bha12, Theorem 2.12].
(3) A criterion for Cohen–Macaulayness of Rees algebras over quasi-excellent local **Q**-algebras with rational singularities (Theorem 9.8). This extends theorems of Sancho de Salas [SdS87, Theorem 1.7] and Lipman [Lip94, Theorems 4.1 and 4.3] for rings essentially of finite type over a field of characteristic zero.
(4) A Briançon–Skoda-type theorem for quasi-excellent **Q**-algebras with rational singularities (Corollary 9.10). This extends a result of Huneke [Hun00, Corollary 4.8] for rings essentially of finite type over a field of characteristic zero, and provides a stronger bound for integral closures of ideals compared to the results in [LT81] which apply in the more general context of pseudo-rational rings of arbitrary characteristic.

We mention that by [BK23, Theorem 3.1], Theorem A also implies that excellent dlt pairs over **Q** satisfy all local rationality properties known to hold for varieties in characteristic zero (see Theorem 9.6).

Additionally, we give an application of our results that is not related to rational singularities.

(5) An adaptation of Hartshorne and Ogus's proof [HO74, Corollary 2.6] that complete local UFD's $(R, \mathfrak{m})$ of dimension ≤ 4 with $R/\mathfrak{m} \simeq \mathbf{C}$ are Gorenstein that removes the algebraizablility condition in [HO74]. The result itself is not new to this paper since Raynaud (unpublished), Danilov [Dan70, Theorem 2], and Boutot [Bou73, Corollaire on p. 693] had given different proofs of the steps in [HO74] that require algebraizability assumptions. Nevertheless, Hartshorne and Ogus's strategy yields an interesting proof of this result using

analytic methods, which was limited to the algebraizable case prior to this paper. See Theorem 10.1.

We mention that Theorem A also implies that the theory of multiplier ideals as developed in [Laz04b, Part Three] is now available for all excellent $\mathbf{Q}$-algebras with dualizing complexes. In [Mur, §4.2], we used this consequence of Theorem A to give a multiplier ideal-theoretic proof of the uniform comparison between symbolic and ordinary powers of ideals in regular $\mathbf{Q}$-algebras due to Ein–Lazarsfeld–Smith [ELS01, Theorem 2.2 and Variant on p. 251] (for smooth $\mathbf{C}$-algebras) and Hochster–Huneke [HH02, Theorem 4.4(a)] (for regular rings of equal characteristic) that does not rely on reduction modulo p or Néron-type desingularization theorems.

Notation. All rings are commutative with identity, and all ring maps are unital. If k is a field, then a *variety over* k is an integral scheme that is separated and of finite type over k.

Intersection products on schemes that are proper over a field are defined using Euler characteristics as in [Kle66, Chapter I; Kle05, Appendix B]. Weil and Cartier divisors are defined as in [EGAIV$_4$, (21.6.2)] and [EGAIV$_4$, Définition 21.1.2], respectively. The two notions coincide on locally Noetherian schemes that are locally factorial [EGAIV$_4$, Théorème 21.6.9(ii)]. A $\mathbf{Q}$-*Weil divisor* (resp. $\mathbf{Q}$-*Cartier divisor*) is a formal $\mathbf{Q}$-linear combination of Weil divisors (resp. Cartier divisors).

We also use the following terminology. A point x in a topological space X is *maximal* if it is the generic point of an irreducible component of X [EGAI$_{\text{new}}$, Chapitre 0, (2.1.1)]. A morphism $X \to Y$ of schemes is *maximally dominating* if every maximal point of X maps to a maximal point of Y [ILO14, Exposé II, Définition 1.1.2].

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2. Preliminaries

2.1. **Excellence and quasi-excellence.** We begin with the notion of excellence. Grothendieck and Dieudonné conjectured that resolutions of singularities exist for all quasi-excellent schemes [EGAIV$_2$, Remarque 7.9.6]. Temkin proved their conjecture for schemes of equal characteristic zero [Tem08, Theorem 1.1].

Definition 2.1 [EGAIV$_2$, Définition 7.8.2 and (7.8.5); Mat80, Definition 34.A]**.** Let R be a ring. We say that R is *excellent* (resp. *quasi-excellent*) if the following conditions (resp. conditions (i), (iii), and (iv) below) are satisfied.

- (i) R is Noetherian.
- (ii) R is universally catenary.
- (iii) R is a *G-ring*, i.e., for every prime ideal $\mathfrak{p} \subseteq R$, the $\mathfrak{p}$-adic completion map $R_{\mathfrak{p}} \to \hat{R}_{\mathfrak{p}}$ has geometrically regular fibers.
- (iv) R is *J-2*, i.e., for every R-algebra S of finite type, the regular locus in $\operatorname{Spec}(S)$ is open.

A locally Noetherian scheme X is *excellent* (resp. *quasi-excellent*) if it admits an open affine covering $X = \bigcup_i \operatorname{Spec}(R_i)$ such that every R_i is excellent (resp. quasi-excellent).

A locally Noetherian scheme X or a ring R is *locally excellent* (resp. *locally quasi-excellent*) if all of its local rings are excellent (resp. quasi-excellent).

Noetherian complete local rings are excellent, and the class of excellent and quasi-excellent schemes are is stable under morphisms locally essentially of finite type [EGAIV$_2$, Scholie 7.8.3; Mat80, (34.A)].

Remark 2.2. While excellence (resp. quasi-excellence) implies local excellence (resp. local quasi-excellence), the converse is false in general [Hoc73b, Example 1]. In fact, by [Mat80, (34.A)], a Noetherian ring is locally excellent (resp. locally quasi-excellent) if and only if it is a universally catenary G-ring (resp. it is a G-ring).

2.2. **Dualizing complexes.** We will also need the notion of a dualizing complex.

Definition 2.3 [Har66, Chapter V, Definition on p. 258; Con00, p. 118]**.** Let X be a locally Noetherian scheme. A *dualizing complex* on X is a complex $\omega_X^\bullet$ in $D^b_{\mathrm{coh}}(X)$ that has finite injective dimension, such that the natural morphism

$$\mathrm{id}_{(-)} \longrightarrow \mathbf{R}\mathscr{H}om_{\mathcal{O}_X}\big(\mathbf{R}\mathscr{H}om_{\mathcal{O}_X}(-,\omega_X^\bullet),\omega_X^\bullet\big)$$

of δ-functors on $D^b_{\mathrm{coh}}(X)$ is an isomorphism.

Remark 2.4. We will cite results about dualizing complexes and Grothendieck duality as we use them. To ensure the existence of dualizing complexes, we recall the following (see [Har66, p. 299]):

(*i*) If X is regular (or more generally, Gorenstein) of finite Krull dimension, then $\mathcal{O}_X$ is a dualizing complex for X. In particular, spectra of Noetherian complete local rings have dualizing complexes.

(*ii*) If $f\colon X \to Y$ is a morphism of finite type between Noetherian schemes, and $\omega_Y^\bullet$ is a dualizing complex for Y, then the exceptional pullback $f^!\omega_Y^\bullet$ is a dualizing complex for X.

We also note that locally Noetherian schemes with dualizing complexes have finite Krull dimension and are universally catenary [Har66, (1) and (2) on p. 300].

2.3. **Relative ampleness conditions.** We define relative ampleness conditions for invertible sheaves and **Q**-Cartier divisors. While most of these definitions exist in the literature, the definitions of f-big is more general than what is usually used. We have made these definitions to facilitate our limit arguments in §4.

Definition 2.5 (see [EGAII, Définitions 4.4.2 and 4.6.1; KMM87, Definitions 0-1-4, 0-3-2, and 0-1-1; Fuj17, §§2.1–2.2])**.** Let $f\colon X \to Y$ be a morphism of schemes, and let $\mathscr{L}$ be an invertible sheaf on X.

(*i*) We say that $\mathscr{L}$ is *f-very ample* if there exists an quasi-coherent $\mathcal{O}_Y$-module $\mathscr{E}$ and a immersion $i\colon X \hookrightarrow \mathbf{P}(\mathscr{E})$ over Y such that $\mathscr{L} \simeq i^*(\mathcal{O}_{\mathbf{P}(\mathscr{E})}(1))$.

(*ii*) Suppose f is quasi-compact. We say that $\mathscr{L}$ is *f-ample* if there exists an affine open cover $Y = \bigcup_i U_i$ such that $\mathscr{L}|_{f^{-1}(U_i)}$ is ample for all i.

(*iii*) We say that $\mathscr{L}$ is *f-generated* if the adjunction morphism $f^*f_*\mathscr{L} \to \mathscr{L}$ is surjective. We say that $\mathscr{L}$ is *f-semi-ample* if there exists an integer $n > 0$ such that $\mathscr{L}^{\otimes n}$ is f-generated.

(*iv*) Suppose that f is a proper maximally dominating morphism. We say that $\mathscr{L}$ is *f-big* if there exists an integer $n > 0$ such that for every maximal point $\eta \in Y$, the pullback $\mathscr{L}_\eta^{\otimes n}$ of $\mathscr{L}^{\otimes n}$ to the fiber X_η induces a rational map

$$\phi_{|\mathscr{L}_\eta^{\otimes n}|}\colon X_\eta \overset{|H^0(X_\eta,\mathscr{L}_\eta^{\otimes n})|}{\dashrightarrow} \mathbf{P}\big(H^0(X_\eta,\mathscr{L}_\eta^{\otimes n})\big)$$

that is generically finite onto its image in the sense of [ILO14, Exposé II, Proposition 1.1.7].

(*v*) Suppose that f is a proper morphism. We say that $\mathscr{L}$ is *f-nef* if $\mathscr{L}|_{f^{-1}(y)}$ is nef for every $y \in Y$, i.e., if for every one-dimensional integral closed subscheme $C \subseteq f^{-1}(y)$, we have

$$\big(\mathscr{L}|_{f^{-1}(y)} \cdot C\big) \geq 0.$$

We can extend these definitions to Cartier divisors L on X by asking that their associated invertible sheaves $\mathcal{O}_X(L)$ satisfy these conditions. If D is a **Q**-Cartier divisor, then we say that D is *f-ample* (resp. *f-semi-ample*, *f-big*, *f-nef*) if some positive integer multiple of D satisfies this property.

Remark 2.6. When X is integral, the definition for f-big in Definition 2.5(*iv*) is equivalent to saying that the volume of $\mathscr{L}_\eta$ is positive for every maximal point $\eta \in Y$ by [Cut14, Theorems 8.2 and 10.7].

2.4. **Rational singularities, pseudo-rational rings, and derived splinters.** We adopt the following definition for rational singularities.

Definition 2.7 (cf. [KKMSD73, pp. 50–51; Kol13, Definition 2.76])**.** Let $(R, \mathfrak{m})$ be a quasi-excellent local $\mathbf{Q}$-algebra. We say that R has *rational singularities* if R is normal and if for every proper birational morphism $f\colon X \to \operatorname{Spec}(R)$ such that X is regular, we have $R^i f_* \mathcal{O}_X = 0$ for all $i > 0$.

Now let Y be a normal locally quasi-excellent locally Noetherian scheme of equal characteristic zero. We say that Y has *rational singularities* if every local ring $\mathcal{O}_{Y,y}$ has rational singularities. If R is a locally quasi-excellent Noetherian ring, we say that R has *rational singularities* if $\operatorname{Spec}(R)$ does.

The condition in Definition 2.7 is not vacuous since resolutions of singularities exist for quasi-excellent local $\mathbf{Q}$-algebras by [Hir64, Chapter I, §3, Main Theorem I(n)]. We check that this definition localizes.

Lemma 2.8. *Let $(R, \mathfrak{m})$ be a quasi-excellent local $\mathbf{Q}$-algebra with rational singularities. Then, $R_\mathfrak{p}$ has rational singularities for every prime ideal $\mathfrak{p} \subseteq R$, and hence $\operatorname{Spec}(R)$ has rational singularities.*

Proof. Since normality is a local condition, it suffices to consider the condition on proper birational morphisms.

Let $f_\mathfrak{p}\colon X_\mathfrak{p} \to \operatorname{Spec}(R_\mathfrak{p})$ be a proper birational morphism from a regular scheme $X_\mathfrak{p}$. We can then find a Cartesian diagram

$$\begin{array}{ccc} X_\mathfrak{p} & \longrightarrow & X \\ {\scriptstyle f_\mathfrak{p}}\downarrow & & \downarrow{\scriptstyle g} \\ \operatorname{Spec}(R_\mathfrak{p}) & \longrightarrow & \operatorname{Spec}(R) \end{array}$$

where the horizontal morphisms are localizing immersions in the sense of [Nay09, Definition 2.7] and g is proper by Nayak's version of Nagata compactification [Nay09, Theorem 4.1]. Set $\overline{X_\mathfrak{p}}$ to be the scheme-theoretic closure of $X_\mathfrak{p}$ in X. Since $X_\mathfrak{p} \hookrightarrow X$ is quasi-compact, the underlying set of $\overline{X_\mathfrak{p}}$ is equal to the set-theoretic closure of $X_\mathfrak{p}$ in X by [EGAI$_{\text{new}}$, Corollaire 6.10.6(1)]. The morphism $\overline{X_\mathfrak{p}} \to \operatorname{Spec}(R)$ is therefore birational.

We now let $\pi_1\colon \tilde{X} \to X$ be a resolution of singularities that is an isomorphism along the set-theoretic image of $X_\mathfrak{p}$ in $\overline{X_\mathfrak{p}}$, which exists by [Hir64, Chapter I, §3, Main Theorem I(n)]. We then obtain the proper birational morphism

$$f \coloneqq (g \circ \pi_1)\colon \tilde{X} \longrightarrow \operatorname{Spec}(R)$$

from the regular scheme $\tilde{X}$ whose base change to $\operatorname{Spec}(R_\mathfrak{p})$ is the morphism $f_\mathfrak{p}$. By assumption, we have $R^i f_* \mathcal{O}_{\tilde{X}} = 0$ for all $i > 0$, and hence $R^i f_{\mathfrak{p}*} \mathcal{O}_{X_\mathfrak{p}} = 0$ for all $i > 0$ by flat base change. □

Even without assuming the existence of resolutions of singularities, we can make the following definition:

Definition 2.9 [LT81, §2]**.** Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d. We say that R is *pseudo-rational* if

- (i) R is normal;
- (ii) R is Cohen–Macaulay;
- (iii) R is *analytically unramified*, i.e., the $\mathfrak{m}$-adic completion $\hat{R}$ of R is reduced; and
- (iv) For every proper birational morphism $f\colon W \to \operatorname{Spec}(R)$ with W normal, denoting the closed fiber by $E = f^{-1}(\{\mathfrak{m}\})$, the natural map

$$H^d_\mathfrak{m}(R) = H^d_{\{\mathfrak{m}\}}\big(\operatorname{Spec}(R), f_*\mathcal{O}_W\big) \xrightarrow{\delta^d_f(\mathcal{O}_W)} H^d_E(W, \mathcal{O}_W)$$

appearing as the edge map in the Leray–Serre spectral sequence for the composition of functors $\Gamma_{\{\mathfrak{m}\}} \circ f_* = \Gamma_E$ is injective.

If X is a locally Noetherian scheme or R is a Noetherian ring, we say that it is *locally pseudo-rational* if all its local rings are pseudo-rational.

3. Limits and local cohomology

In this section, we review some preliminaries on limits of ringed spaces and schemes and their sheaf cohomology. The main new result is that local cohomology is well-behaved under limits of ringed spaces (Theorem 3.13). We set our conventions for limits of spaces in §3.1 and define Zariski–Riemann spaces as an example of such a limit in §3.2. The behavior of sheaf cohomology under limits of ringed spaces is reviewed in §3.3 and we deduce the analogous result for local cohomology as a consequence in §3.4.

3.1. **Limits of ringed spaces and schemes.** We fix our notation for limits of ringed spaces mostly following Fujiwara and Kato [FK18, Chapter 0, §4], which in turn draws on the topos-theoretic formulation of this material in [SGA4$_2$, Exposés VI and VII]. In the scheme-theoretic context, some of this material appears in [EGAIV$_3$, §8].

Setup 3.1 (see [FK18, Chapter 0, §4.1.(e) and §4.2.(a)])**.** Let $\{(X_\lambda, v_{\lambda\mu})\}_{\lambda\in\Lambda}$ be an inverse system of ringed spaces (resp. locally ringed spaces) indexed by a filtered preordered set Λ. By [FK18, Chapter 0, Proposition 4.1.10], the limit

$$X = \varprojlim_{\lambda\in\Lambda} X_\lambda$$

exists in the category of ringed spaces (resp. locally ringed spaces), and is preserved under the forgetful functor from the category of locally ringed spaces to the category of ringed spaces. The underlying topological space of X is the limit of the underlying topological spaces of the X_λ's. The structure sheaf $\mathcal{O}_X$ on X can be described as follows. Denote by $v_{\lambda\mu}\colon X_\mu \to X_\lambda$ the transition morphisms in our inverse system. Note that such a transition morphism $v_{\lambda\mu}$ yields a pullback map

$$v_{\lambda\mu}^{-1}\mathcal{O}_{X_\lambda} \longrightarrow \mathcal{O}_{X_\mu}$$

on structure sheaves, where $v_{\lambda\mu}^{-1}$ is the pullback as Abelian sheaves. Pulling back these maps to X along the canonical projection morphisms $v_\lambda\colon X \to X_\lambda$, we obtain a direct system of Abelian sheaves $\{v_\lambda^{-1}\mathcal{O}_{X_\lambda}\}_{\lambda\in\Lambda}$ on X. The structure sheaf on X can then be described as the colimit

$$\mathcal{O}_X = \varinjlim_{\lambda\in\Lambda} v_\lambda^{-1}\mathcal{O}_{X_\lambda}$$

of this direct system of Abelian sheaves on X. Moreover, by [FK18, Chapter 0, Lemma 4.2.7] this sheaf can be described as a colimit of $\mathcal{O}_X$-modules:

$$\mathcal{O}_X \simeq \varinjlim_{\lambda\in\Lambda} v_\lambda^*\mathcal{O}_{X_\lambda}.$$

Setup 3.2 (see [EGAIV$_3$, §8.2])**.** With notation as in Setup 3.1, suppose the inverse system $\{(X_\lambda, v_{\lambda\mu})\}_{\lambda\in\Lambda}$ lies in the category of schemes. If the transition morphisms $v_{\lambda\mu}\colon X_\mu \to X_\lambda$ are affine for all $\lambda \leq \mu$, then the limit X in the category of locally ringed spaces is a scheme by [EGAIV$_3$, Proposition 8.2.3 and Remarque 8.2.14], and the projection morphisms $v_\lambda\colon X \to X_\lambda$ are affine for all $\lambda \in \Lambda$.

Remark 3.3. Our setup is more general than that in [FK18] since our index sets Λ are only assumed to be filtered and preordered. However, given an inverse system (resp. direct system) indexed by such a filtered preordered set Λ, there always exists a directed set Λ' and an inverse system (resp. direct system) indexed by Λ' using the same objects and morphisms as the original system, together with an initial (resp. final) morphism between the two inverse systems (resp. direct systems) by

[AN82, Theorem 1]. Since this morphism of inverse systems (resp. direct systems) is initial (resp. final), they have the same limit (resp. colimit). We will therefore allow ourselves to index inverse systems and direct systems by filtered preordered sets, and will state results from [FK18] in this generality.

In order for cohomology to behave well with respect to limits, we need to make some additional assumptions on our inverse systems $\{(X_\lambda, v_{\lambda\mu})\}_{\lambda\in\Lambda}$.

Assumptions 3.4. With notation as in Setup 3.1, we will assume the following:

(*a*) For every $\lambda \in \Lambda$, the underlying topological space of X_λ is spectral. Following [Hoc69, p. 43], a topological space Y is *spectral* if it is T_0 and quasi-compact, the quasi-compact open subsets in Y are stable under finite intersection and form an open basis, and every nonempty irreducible closed subset in Y has a generic point.
(*b*) For all $\lambda \leq \mu$, the underlying continuous maps of $v_{\lambda\mu}\colon X_\mu \to X_\lambda$ are quasi-compact.

By [Hoc69, Theorem 7] (see also [FK18, Chapter 0, Theorem 2.2.10(1)]), these assumptions imply that the underlying topological space of X is spectral, and that the underlying continuous maps of the projection morphisms $v_\lambda\colon X \to X_\lambda$ are quasi-compact.

Remark 3.5. If X_λ is a scheme as in Setup 3.2, then the underlying topological space of X_λ is spectral if and only if X_λ is quasi-compact and quasi-separated by [EGAI$_{\text{new}}$, Propositions 2.1.5 and 6.1.12] (see also [FK18, Chapter 0, Example 2.2.2(2)]). Thus, if the X_λ are all Noetherian (or more generally, quasi-compact and quasi-separated) schemes, then both (*a*) and (*b*) hold by [EGAI$_{\text{new}}$, Corollaire 6.1.13 and Proposition 6.1.5(*v*)].

Remark 3.6. A topological space is spectral if and only if it is coherent and sober [FK18, Chapter 0, Remark 2.2.4(1)]. Quasi-compact maps of spectral spaces are called *spectral* in [Hoc69, p. 43].

3.2. **Zariski–Riemann spaces.** An important example of a limit of an inverse system of schemes is the *Zariski–Riemann* space defined by Zariski for varieties [Zar40, Definition A.II.5; Zar44, §2] and by Nagata for Noetherian separated schemes [Nag63, §3].

Definition 3.7 (see [FK06, Definition 5.9; Tem10, §3.2; FK18, Chapter II, Definitions E.2.2 and E.2.3])**.** Let X be a quasi-compact quasi-separated scheme. Denote by AId_X the set of quasi-coherent ideal sheaves $\mathscr{I}$ of finite type on X such that $X - V(\mathscr{I})$ contains all maximal points of X. The *Zariski–Riemann space* of X is the limit

$$\mathrm{ZR}(X) \coloneqq \varprojlim_{\mathscr{I}\in\mathrm{AId}_X} X_{\mathscr{I}}$$

over the inverse system of blowups $X_{\mathscr{I}} \to X$ along $\mathscr{I} \in \mathrm{AId}_X$ in the category of locally ringed spaces. By Remark 3.5 and [Hoc69, Theorem 7], the underlying topological space of $\mathrm{ZR}(X)$ is spectral, and the underlying continuous maps of the projection morphisms $\mathrm{ZR}(X) \to X_{\mathscr{I}}$ are quasi-compact.

We note that the formation of Zariski–Riemann spaces commutes with base change by quasi-compact separated étale morphisms by [Stacks, Tag 087B].

Remark 3.8. In [Tem10, §3.2], Temkin defines the Zariski–Riemann space for integral schemes using all proper birational morphisms $X' \to X$. If X is quasi-compact and quasi-separated, then the limit over such an inverse system coincides with $\mathrm{ZR}(X)$ since all proper birational morphisms can be dominated by a blowup along an ideal sheaf in AId_X by [RG71, Première partie, Corollaire 5.7.12] (see also [Con07, Theorem 2.11]). We have chosen our definition to ensure that our inverse system is indexed by a directed set, instead of a directed category.

3.3. **Sheaf cohomology on limits of ringed spaces.** We will need to understand the behavior of sheaf cohomology on limits of ringed spaces. To do so, we set our notation for sheaves on inverse systems of ringed spaces. Again, much of this material also appears in [SGA4$_2$, Exposés VI and VII] in the language of topos theory.

Setup 3.9 (see [FK18, Chapter 0, §4.4])**.** With notation as in Setup 3.1, for every $\lambda \in \Lambda$, we also fix an $\mathcal{O}_{X_\lambda}$-module $\mathscr{F}_\lambda$, together with morphisms

$$\varphi_{\lambda\mu} \colon v_{\lambda\mu}^* \mathscr{F}_\lambda \longrightarrow \mathscr{F}_\mu$$

of $\mathcal{O}_{X_\mu}$-modules for every $\lambda \leq \mu$, such that $\varphi_{\lambda\nu} = \varphi_{\mu\nu} \circ v_{\mu\nu}^* \varphi_{\lambda\mu}$ whenever $\lambda \leq \mu \leq \nu$. We then have a direct system $\{v_\lambda^* \mathscr{F}_\lambda\}_{\lambda\in\Lambda}$ of $\mathcal{O}_X$-modules whose colimit is the $\mathcal{O}_X$-module

$$\mathscr{F} = \varinjlim_{\lambda\in\Lambda} v_\lambda^* \mathscr{F}_\lambda.$$

We have the following canonical isomorphism from [FK18, Chapter 0, Proposition 4.2.7]:

$$\mathscr{F} \simeq \varinjlim_{\lambda\in\Lambda} v_\lambda^{-1} \mathscr{F}_\lambda. \tag{1}$$

We also make the following definition, which will simplify the statements of some of the results below.

Definition 3.10. Let A be a ring, and consider the ringed space $(\{*\}, A)$ whose underlying topological space is a point and whose structure sheaf is the constant sheaf A. The *category of ringed spaces over A* is the slice category of ringed spaces over $(\{*\}, A)$.

With this notation, we have the following statements about the behavior of sheaf cohomology under limits of ringed spaces, which are special cases of [SGA4$_2$, Exposé VI, Théorème 8.7.3]. The terminology "the Grothendieck limit theorem" is from [Pan03, p. 169].

Theorem 3.11 (The Grothendieck limit theorem [FK18, Chapter 0, Proposition 4.4.1])**.** *Let A be a ring. Let $\{(X_\lambda, v_{\lambda\mu})\}_{\lambda\in\Lambda}$ be an inverse system of spectral ringed spaces over A indexed by a filtered preordered set Λ with quasi-compact transition morphisms, and let*

$$X \coloneqq \varprojlim_{\lambda\in\Lambda} X_\lambda$$

be the inverse limit of this inverse system with canonical projection morphisms $v_\lambda \colon X \to X_\lambda$.

For each $\lambda \in \Lambda$, fix an $\mathcal{O}_{X_\lambda}$-module $\mathscr{F}_\lambda$ on each X_λ, together with morphisms

$$\varphi_{\lambda\mu} \colon v_{\lambda\mu}^* \mathscr{F}_\lambda \longrightarrow \mathscr{F}_\mu$$

of $\mathcal{O}_{X_\mu}$-modules for every $\lambda \leq \mu$, such that $\varphi_{\lambda\nu} = \varphi_{\mu\nu} \circ v_{\mu\nu}^* \varphi_{\lambda\mu}$ whenever $\lambda \leq \mu \leq \nu$. Consider the direct system $\{v_\lambda^* \mathscr{F}_\lambda\}_{\lambda\in\Lambda}$ of $\mathcal{O}_X$-modules whose colimit is the $\mathcal{O}_X$-module

$$\mathscr{F} \coloneqq \varinjlim_{\lambda\in\Lambda} v_\lambda^* \mathscr{F}_\lambda.$$

Then, the canonical map

$$\varinjlim_{\lambda\in\Lambda} H^i(X_\lambda, \mathscr{F}_\lambda) \longrightarrow H^i(X, \mathscr{F})$$

is an isomorphism of A-modules for all $i \geq 0$.

Theorem 3.12 [FK18, Chapter 0, Corollary 4.4.4]**.** *Let $\{(X_\lambda, v_{\lambda\mu})\}_{\lambda\in\Lambda}$ and $\{(Y_\lambda, w_{\lambda\mu})\}_{\lambda\in\Lambda}$ be inverse systems of spectral ringed spaces indexed by a filtered preordered set Λ with quasi-compact transition morphisms, and let*

$$X \coloneqq \varprojlim_{\lambda\in\Lambda} X_\lambda \qquad \text{and} \qquad Y \coloneqq \varprojlim_{\lambda\in\Lambda} Y_\lambda$$

be the inverse limits of these inverse systems with canonical projection morphisms $v_\lambda\colon X \to X_\lambda$ *and* $w_\lambda\colon Y \to Y_\lambda$*, respectively. Consider a system of morphisms* $\{f_\lambda\colon X_\lambda \to Y_\lambda\}_{\lambda\in\Lambda}$ *such that the diagrams*

$$\begin{array}{ccc} X_\mu & \xrightarrow{f_\mu} & Y_\mu \\ {\scriptstyle v_{\lambda\mu}}\downarrow & & \downarrow{\scriptstyle w_{\lambda\mu}} \\ X_\lambda & \xrightarrow{f_\lambda} & Y_\lambda \end{array}$$

commute for all $\lambda \le \mu$*, and set* $f = \varprojlim_{\lambda\in\Lambda} f_\lambda\colon X \to Y$*.*

For each $\lambda \in \Lambda$*, fix an* $\mathcal{O}_{X_\lambda}$*-module* $\mathscr{F}_\lambda$ *on each* X_λ*, together with morphisms*

$$\varphi_{\lambda\mu}\colon v_{\lambda\mu}^*\mathscr{F}_\lambda \longrightarrow \mathscr{F}_\mu$$

of $\mathcal{O}_{X_\mu}$*-modules for every* $\lambda \le \mu$*, such that* $\varphi_{\lambda\nu} = \varphi_{\mu\nu} \circ v_{\mu\nu}^*\varphi_{\lambda\mu}$ *whenever* $\lambda \le \mu \le \nu$*. Consider the direct system* $\{v_\lambda^*\mathscr{F}_\lambda\}_{\lambda\in\Lambda}$ *of* $\mathcal{O}_X$*-modules whose colimit is the* $\mathcal{O}_X$*-module*

$$\mathscr{F} \coloneqq \varinjlim_{\lambda\in\Lambda} v_\lambda^*\mathscr{F}_\lambda.$$

Then, the canonical morphism

$$\varinjlim_{\lambda\in\Lambda} w_\lambda^{-1} R^i f_{\lambda*}(\mathscr{F}_\lambda) \longrightarrow R^i f_*(\mathscr{F})$$

is an isomorphism of $\mathcal{O}_Y$*-modules for all* $i \ge 0$*.*

3.4. **Local cohomology on limits of ringed spaces.** We now show that local cohomology is well-behaved under limits. See [SGA4$_2$, Exposé VI, Corollaire 5.5] and [HO08, Lemma 5.16] for related results.

Theorem 3.13. *Let* A *be a ring. Let* $\{(X_\lambda, v_{\lambda\mu})\}_{\lambda\in\Lambda}$ *be an inverse system of spectral ringed spaces over* A *indexed by a filtered preordered set* Λ *with quasi-compact transition morphisms, and let*

$$X \coloneqq \varprojlim_{\lambda\in\Lambda} X_\lambda$$

be the inverse limit of this inverse system with canonical projection morphisms $v_\lambda\colon X \to X_\lambda$*.*

For each $\lambda \in \Lambda$*, fix an* $\mathcal{O}_{X_\lambda}$*-module* $\mathscr{F}_\lambda$ *on each* X_λ*, together with morphisms*

$$\varphi_{\lambda\mu}\colon v_{\lambda\mu}^*\mathscr{F}_\lambda \longrightarrow \mathscr{F}_\mu$$

of $\mathcal{O}_{X_\mu}$*-modules for every* $\lambda \le \mu$*, such that* $\varphi_{\lambda\nu} = \varphi_{\mu\nu} \circ v_{\mu\nu}^*\varphi_{\lambda\mu}$ *whenever* $\lambda \le \mu \le \nu$*. Consider the direct system* $\{v_\lambda^*\mathscr{F}_\lambda\}_{\lambda\in\Lambda}$ *of* $\mathcal{O}_X$*-modules whose colimit is the* $\mathcal{O}_X$*-module*

$$\mathscr{F} \coloneqq \varinjlim_{\lambda\in\Lambda} v_\lambda^*\mathscr{F}_\lambda.$$

Fix $\alpha \in \Lambda$*, a quasi-compact open subset* $U_\alpha \subseteq X_\alpha$*, and a closed subset* $Z_\alpha \subseteq U_\alpha$*. For each* $\lambda \ge \alpha$*, set* $U_\lambda = v_{\alpha\lambda}^{-1}(U_\alpha)$ *and* $Z_\lambda = X_\lambda - U_\lambda$*. Then, the canonical map*

$$\varinjlim_{\lambda\ge\alpha} H^i_{Z_\lambda}(X_\lambda, \mathscr{F}_\lambda) \longrightarrow H^i_Z(X, \mathscr{F})$$

is an isomorphism of A*-modules for all* $i \ge 0$*.*

Proof. By Excision [Gro67, Proposition 1.3], we may replace X by U to assume that Z is closed in X. Consider $\lambda, \mu \in \Lambda$ such that $\lambda \le \mu$. By [Ive86, Functoriality II.9.7] applied to the maps

$v_\lambda\colon X \to X_\lambda$, $v_\mu\colon X \to X_\mu$, and $v_{\lambda\mu}\colon X_\mu \to X_\lambda$, there is a commutative diagram

$$\begin{array}{ccccccc}
\cdots \longrightarrow & H^i_{Z_\lambda}(X_\lambda, \mathscr{F}_\lambda) & \longrightarrow & H^i(X_\lambda, \mathscr{F}_\lambda) & \longrightarrow & H^i\big(U_\lambda, \mathscr{F}_\lambda|_{U_\lambda}\big) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & H^i_{Z_\mu}(X_\mu, v_{\lambda\mu}^{-1}\mathscr{F}_\lambda) & \longrightarrow & H^i(X_\mu, v_{\lambda\mu}^{-1}\mathscr{F}_\lambda) & \longrightarrow & H^i\big(U_\mu, (v_{\lambda\mu}^{-1}\mathscr{F}_\lambda)|_{U_\mu}\big) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & H^i_{Z_\mu}(X_\mu, \mathscr{F}_\mu) & \longrightarrow & H^i(X_\mu, \mathscr{F}_\mu) & \longrightarrow & H^i\big(U_\mu, \mathscr{F}_\mu|_{U_\mu}\big) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & H^i_Z(X, v_\mu^{-1}\mathscr{F}_\mu) & \longrightarrow & H^i(X, v_\mu^{-1}\mathscr{F}_\mu) & \longrightarrow & H^i\big(U, (v_\mu^{-1}\mathscr{F}_\mu)|_U\big) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & H^i_Z(X, \mathscr{F}) & \longrightarrow & H^i(X, \mathscr{F}) & \longrightarrow & H^i\big(U, \mathscr{F}|_U\big) & \longrightarrow \cdots
\end{array}$$

of Abelian groups with exact rows, where the vertical arrows in the top and third rows are induced by pulling back along $v_{\lambda\mu}$ and v_μ and the vertical arrows in the second and bottom rows are obtained from the maps

$$v_{\lambda\mu}^{-1}\mathscr{F}_\lambda \longrightarrow v_{\lambda\mu}^*\mathscr{F}_\lambda \xrightarrow{\varphi_{\lambda\mu}} \mathscr{F}_\mu$$

and the description of $\mathscr{F}$ as a colimit of Abelian sheaves as in (1). The commutative diagram is in fact a commutative diagram of A-modules by the argument in [FK18, Chapter 0, §4.3.(c)]. Now taking colimits over all $\mu \geq \lambda \geq \alpha$, the middle and right arrows in the diagram yield isomorphisms of A-modules by Theorem 3.11. The five lemma [CE56, Chapter I, Proposition 1.1] then implies the desired isomorphisms. □

Part I. Relative vanishing and injectivity theorems

4. Approximating morphisms of schemes

As outlined in §1, the idea in our proof of Theorem B is to approximate the morphism $f\colon X \to Y$ by morphisms of varieties over **Q**. Since this approximation construction takes up the bulk of the proof of Theorem B, we state and prove it separately below. In §4.1, we prove that many ampleness conditions on invertible sheaves behave well under limits. We prove our approximation result (Lemma 4.2) in §4.2.

4.1. **Relative ampleness conditions and limits.** We prove that all ampleness conditions defined in Definition 2.5 (except for f-nefness) behave well under limits.

Lemma 4.1 (cf. [EGAIV$_3$, Lemme 8.10.5.2])**.** *Let $\{S_\lambda\}_{\lambda\in\Lambda}$ be an inverse system of quasi-compact quasi-separated schemes with affine transition morphisms and limit S. Let $f_\alpha\colon X_\alpha \to Y_\alpha$ be a morphism of S_α-schemes of finite presentation for some $\alpha \in \Lambda$, and let $\mathscr{L}_\alpha$ be an invertible sheaf on X_α. For every $\lambda \geq \alpha$, let*

$$f_\lambda\colon X_\lambda \longrightarrow Y_\lambda$$

be the base change of f_α along $S_\lambda \to S_\alpha$, and denote by $\mathscr{L}_\lambda$ the pullback of $\mathscr{L}_\alpha$ to X_λ. Denote by $f\colon X \to Y$ the limit of the morphisms f_λ, and denote by $\mathscr{L}$ the pullback of $\mathscr{L}_\alpha$ to X.

- *(i) If $\mathscr{L}$ is f-very ample (resp. f-ample), then there exists an index $\lambda \in \Lambda$ such that $\mathscr{L}_\mu$ is f_μ-very ample (resp. f_μ-ample) for all $\mu \geq \lambda$.*
- *(ii) Suppose f_α is quasi-separated. If $\mathscr{L}$ is f-generated (resp. f-semi-ample), then there exists an index $\lambda \in \Lambda$ such that $\mathscr{L}_\mu$ is f_μ-generated (resp. f_μ-semi-ample) for all $\mu \geq \lambda$.*

(*iii*) *Suppose f is a proper morphism of schemes. If $\mathscr{L}$ is f-big, then there exists an index $\lambda \in \Lambda$ such that $\mathscr{L}_\mu$ is f_μ-big for all $\mu \geq \lambda$.*

Proof. The statement (*i*) follows from [EGAIV$_3$, Lemme 8.10.5.2], since very ampleness (resp. ampleness) is stable under base change [EGAII, Propositions 4.4.10(*iii*) and 4.6.13(*iii*)].

For (*ii*), after replacing $\mathscr{L}_\alpha$ by a positive integer power, it suffices to show the f-generated case. For all $\mu \geq \alpha$, the pullback of the morphism $f_\mu^* f_{\mu *}\mathscr{L}_\mu \to \mathscr{L}_\mu$ is

$$f^* w_\mu^* f_{\mu *}\mathscr{L}_\mu = v_\mu^* f_\mu^* f_{\mu *}\mathscr{L}_\mu \longrightarrow v_\mu^*\mathscr{L}_\mu$$

where $w_\mu \colon Y \to Y_\mu$ and $v_\mu \colon X \to X_\mu$ are the canonical projection morphisms. The colimit of these morphisms is the adjunction morphism $f^* f_*\mathscr{L} \to \mathscr{L}$, since the left adjoint f^* commutes with colimits and then by applying Theorem 3.12 together with the isomorphism [FK18, Chapter 0, Proposition 4.2.7] that allows us to replace w_μ^{-1} with w_μ^*. Now since $f_{\mu *}\mathscr{L}_\mu$ is quasi-coherent [EGAI$_{\text{new}}$, Proposition 6.7.1], we can apply [EGAIV$_3$, Corollaire 8.5.7] to say that there exists $\lambda \in \Lambda$ such that

$$f_\mu^* f_{\mu *}\mathscr{L}_\mu \longrightarrow v_\mu^*\mathscr{L}_\mu$$

is surjective for all $\mu \geq \lambda$, as required.

We now show (*iii*). We first note that the f_μ are proper for large enough μ [EGAIV$_3$, Théorème 8.10.5(*xii*)]. Moreover, by [EGAIV$_3$, Proposition 8.4.2(*a*)(*i*)], the morphisms $Y \to Y_\mu$ and $X \to X_\mu$ induce bijections on maximal points for large enough μ. Note that Y has only finitely many maximal points by the quasi-compactness of S. Thus, choosing $\lambda \geq \alpha$ large enough we may assume that the morphisms $X_\mu \to Y_\mu$ are proper and maximally dominating for all $\mu \geq \lambda$, and we may replace Y_μ by the spectra $\operatorname{Spec}(\kappa(\eta_\mu))$ of its residue fields at maximal points to assume that the Y_μ are spectra of fields k_μ with colimit k.

Now let $n > 0$ be an integer such that $\mathscr{L}^{\otimes n}$ induces a generically finite morphism onto its image. Then, there exists an open subset $U \subseteq \mathbf{P}_k(H^0(X, \mathscr{L}^{\otimes n}))$ such that the rational map

$$\phi_{|\mathscr{L}^{\otimes n}|} \colon X \xrightarrow{\quad |H^0(X,\mathscr{L}^{\otimes n})| \quad} \mathbf{P}_k\big(H^0(X, \mathscr{L}^{\otimes n})\big)$$

induced by $\mathscr{L}^{\otimes n}$ restricts to a finite morphism over U. By [EGAIV$_3$, Corollaire 8.6.4, Théorème 8.8.2(*i*), and Corollaire 8.8.2.5], [EGAII, (4.1.3) and (4.2.10)], and flat base change, after possibly replacing λ by a larger index, we may assume there exists an open subset $U_\lambda \subseteq \mathbf{P}_{k_\lambda}(H^0(X_\lambda, \mathscr{L}_\lambda^{\otimes n}))$ such that $v_\lambda^{-1}(U_\lambda) = U$, and such that the rational map $\phi_{|\mathscr{L}^{\otimes n}|}$ restricted to U is the base change of the rational maps

$$\phi_{|\mathscr{L}_\mu^{\otimes n}|} \colon X_\mu \xrightarrow{\quad |H^0(X_\mu,\mathscr{L}_\mu^{\otimes n})| \quad} \mathbf{P}_{k_\mu}\big(H^0(X_\mu, \mathscr{L}_\mu^{\otimes n})\big)$$

restricted to $U_\mu = v_{\lambda\mu}^{-1}(U_\lambda)$ for all $\mu \geq \lambda$. Moreover, we may assume that the maps $\phi_{|\mathscr{L}_\mu^{\otimes n}|}$ restricted to U_λ are finite for all $\mu \geq \lambda$ by [EGAIV$_3$, Théorème 8.10.5(*x*)]. We therefore see that $\phi_{|\mathscr{L}_\mu^{\otimes n}|}$ induces a generically finite morphism onto its image for all $\mu \geq \lambda$, and hence $\mathscr{L}_\mu$ is f_μ-big for all $\mu \geq \lambda$. □

4.2. **The approximation lemma.** We now show our main approximation result.

Lemma 4.2. *Let $\mathbf{k}$ be a Noetherian ring and let $(R, \mathfrak{m})$ be a integral Noetherian local $\mathbf{k}$-algebra. Consider a proper surjective morphism $f \colon X \to \operatorname{Spec}(R)$ from an integral scheme X. Write R as the colimit*

$$R \simeq \varinjlim_{\lambda \in \Lambda} R_\lambda \tag{2}$$

of a direct system of sub-$\mathbf{k}$-algebras of finite type indexed by a directed set Λ and partially ordered by inclusion. We then have the following:

(*i*) *There exists* $\alpha \in \Lambda$ *and a proper surjective morphism* $f'_\alpha \colon X'_\alpha \to \operatorname{Spec}(R_\alpha)$ *from a reduced scheme* X'_α *for which the diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & \operatorname{Spec}(R) \\ {\scriptstyle v'_\alpha}\downarrow & & \downarrow \\ X'_\alpha & \xrightarrow{f'_\alpha} & \operatorname{Spec}(R_\alpha) \end{array}$$

is Cartesian. Moreover, α *can be chosen such that denoting by* $f'_\lambda \colon X'_\lambda \to \operatorname{Spec}(R_\lambda)$ *the base change of* f'_α *to* $\operatorname{Spec}(R_\lambda)$*, there exist integral closed subschemes* $X_\lambda \subseteq X'_\lambda$ *for all* $\lambda \geq \alpha$ *such that the following hold:*

- *Setting* $\mathfrak{m}_\lambda \coloneqq \mathfrak{m} \cap R_\lambda$*, we have*

$$\dim(X) \leq \dim\bigl(X_\lambda \otimes_{R_\lambda} (R_\lambda)_{\mathfrak{m}_\lambda}\bigr)$$

  for all $\lambda \geq \alpha$*.*
- *The limit of the morphisms*

$$f_\lambda \colon X_\lambda \hookrightarrow X'_\lambda \xrightarrow{f'_\lambda} \operatorname{Spec}(R_\lambda)$$

  with transition morphisms $v_{\lambda\mu} \colon X_\mu \to X_\lambda$ *is the morphism* $f \colon X \to \operatorname{Spec}(R)$*.*

(*ii*) *Let* $\mathscr{L}$ *be an invertible sheaf on* X*. Then, after possibly replacing* α *with a larger index, we can write*

$$\mathscr{L} \simeq v_\alpha^* \mathscr{L}_\alpha$$

for an invertible sheaf $\mathscr{L}_\alpha$ *on* X_α*, where* $v_\alpha \colon X \to X_\alpha$ *are the canonical projection morphisms. Moreover, if* $\mathscr{L}$ *is* f*-very ample (resp.* f*-ample,* f*-semi-ample,* f*-big), then we may assume that the invertible sheaves* $\mathscr{L}_\lambda \coloneqq v_{\alpha\lambda}^* \mathscr{L}_\alpha$ *are* f_λ*-very ample (resp.* f_λ*-ample,* f_λ*-semi-ample,* f_λ*-big) for all* $\lambda \geq \alpha$*.*

(*iii*) *For each* $\lambda \geq \alpha$*, the inverse system*

$$\bigl\{g_{\lambda,p} \colon W_{\lambda,p} \longrightarrow X_\lambda\bigr\}_{p \in P_\lambda} \tag{3}$$

of all projective birational morphisms from integral schemes $W_{\lambda,p}$ *that are separated and of finite type over* $\mathbf{k}$ *such that* $f_\lambda \circ g_{\lambda,p}$ *is projective is nonempty and indexed by a directed set* P_λ*. Moreover, if projective resolutions of singularities (resp. normalizations, Macaulayfications) exist for all integral schemes that are separated and of finite type over* $\mathbf{k}$*, then we may assume that the schemes* $W_{\lambda,p}$ *are regular (resp. normal, Cohen–Macaulay).*

(*iv*) *Consider the set*

$$J = \bigsqcup_{\lambda \in \Lambda} P_\lambda$$

with the preorder where $(\lambda, p) \leq (\mu, q)$ *if and only* $\lambda \leq \mu$ *and the morphism* $g_{\mu,q}$ *fits into a commutative diagram*

$$\begin{array}{ccccc} W_{\mu,q} & \xrightarrow{g_{\mu,q}} & X_\mu & \longrightarrow & \operatorname{Spec}(R_\mu) \\ \downarrow & & {\scriptstyle v_{\lambda\mu}}\downarrow & & \downarrow \\ W_{\lambda,p} & \xrightarrow{g_{\lambda,p}} & X_\lambda & \longrightarrow & \operatorname{Spec}(R_\lambda). \end{array}$$

Then, the set J *is filtered, and the morphism* $\pi \colon \operatorname{ZR}(X) \to X$ *of locally ringed spaces from the Zariski–Riemann space of* X *is the limit of the inverse systems* (3) *as* $\lambda \in \Lambda$ *also varies.*

Remark 4.3. Let $\mathbf{k}$ be a quasi-excellent Noetherian $\mathbf{Q}$-algebra, in which case projective resolutions of singularities exist by [Tem08, Theorem 1.1]. In this case, given $f\colon X \to \operatorname{Spec}(R)$ as above, we have the commutative diagram

$$\begin{array}{ccccc} \operatorname{ZR}(X) & \xrightarrow{\pi} & X & \xrightarrow{f} & \operatorname{Spec}(R) \\ \downarrow & & \downarrow{\scriptstyle v_\lambda} & & \downarrow \\ W_{\lambda,p} & \xrightarrow{g_{\lambda,p}} & X_\lambda & \xrightarrow{f_\lambda} & \operatorname{Spec}(R_\lambda) \end{array}$$

of locally ringed spaces, where all but $\operatorname{ZR}(X)$ are Noetherian $\mathbf{k}$-schemes and the schemes in the bottom row are integral schemes that are separated and of finite type over $\mathbf{k}$ with $W_{\lambda,p}$ regular, such that the morphisms in the top row are the limits of the morphisms in the bottom row. We can also localize the R_λ at $\mathfrak{m}_\lambda = \mathfrak{m} \cap R_\lambda$ without affecting the inverse limit (since the inverse systems satisfy the same universal property) in order to assume that the R_λ are local (although the schemes in the bottom row of the diagram above are now *essentially* of finite type over $\mathbf{k}$).

The necessary normalizations exist in (*iii*) when $\mathbf{k}$ is a Nagata ring in the sense of [Mat80, Definition 31.A], and the necessary Macaulayfications exist in (*iii*) when $\mathbf{k}$ is CM-quasi-excellent in the sense of [Čes21, Definition 1.2] by [Čes21, Remark 1.4, Theorem 5.3, and Remark 5.4].

Proof of Lemma 4.2. We construct the morphisms in (*i*). The first part of the statement follows from [EGAIV$_3$, Théorème 8.8.2(*ii*)]. By [EGAIV$_3$, Proposition 8.7.2 and Théorème 8.10.5(*vi*),(*xii*)], after possibly replacing α by a larger index, we may assume that X'_λ is reduced and that f'_λ is proper and surjective for all $\lambda \geq \alpha$. Denote by η and η_λ the generic points of Y and Y_λ, respectively. By transitivity of fibers [EGAI$_{\text{new}}$, Corollaire 3.4.9] and applying [EGAIV$_2$, (4.4.1)], the generic fibers $f_\lambda'^{-1}(\eta_\lambda)$ are also irreducible.

Next, we show that we can replace the morphisms f'_λ by some $f_\lambda\colon X_\lambda \to \operatorname{Spec}(R_\lambda)$ for integral closed schemes $X_\lambda \subseteq X'_\lambda$. For each $\lambda \geq \mu \geq \alpha$, we will construct the following commutative diagram:

$$\begin{array}{ccccc} X & = & X & \xrightarrow{f} & \operatorname{Spec}(R) \\ {\scriptstyle v_\mu}\downarrow & & \downarrow{\scriptstyle v'_\mu} & & \downarrow \\ X_\mu & \hookrightarrow & X'_\mu & \xrightarrow{f'_\mu} & \operatorname{Spec}(R_\mu) \\ {\scriptstyle v_{\lambda\mu}}\downarrow & & \downarrow{\scriptstyle v'_{\lambda\mu}} & & \downarrow \\ X_\lambda & \hookrightarrow & X'_\lambda & \xrightarrow{f'_\lambda} & \operatorname{Spec}(R_\lambda). \end{array} \tag{4}$$

Here, the squares in the right column are Cartesian. The scheme X_λ is the scheme-theoretic closure of $f_\lambda'^{-1}(\eta_\lambda)$, which coincides with the set-theoretic closure with reduced scheme structure since X'_λ is reduced [EGAI$_{\text{new}}$, Corollaire 6.10.6]. We define X_μ in a similar fashion. These schemes X_λ and X_μ are irreducible by [EGAI$_{\text{new}}$, Chapitre 0, Proposition 2.1.13], hence integral. The morphisms in the rightmost column induce bijections on generic points for all $\mu \geq \lambda \geq \alpha$, since all rings in the direct system (2) are domains. Thus, by transitivity of scheme-theoretic images [EGAI$_{\text{new}}$, Proposition 6.10.3], morphisms in the leftmost column exist in a way that makes the squares in left column commute. Now for every $\lambda \geq \alpha$, consider the composition

$$f_\lambda\colon X_\lambda \hookrightarrow X'_\lambda \xrightarrow{f'_\lambda} \operatorname{Spec}(R_\lambda).$$

This morphism is proper since it is the composition of proper morphisms. The base change of f_λ to the generic point is

$$f_\lambda^{-1}(\eta_\lambda) \xrightarrow{\sim} f_\lambda'^{-1}(\eta_\lambda) \longrightarrow \operatorname{Spec}\big(\kappa(\eta_\lambda)\big), \tag{5}$$

since X_λ is the set-theoretic closure of $f_\lambda'^{-1}(\eta_\lambda)$ in X'_λ, and hence X_λ and X'_λ are isomorphic over the generic point η_λ. Thus, we see that the morphism f_λ is surjective with irreducible generic fiber.

By [EGAIV$_3$, Lemme 13.1.2], the squares

$$\begin{array}{ccc} X & \xrightarrow{f} & \operatorname{Spec}(R) \\ {\scriptstyle v_\lambda}\downarrow & & \downarrow \\ X_\lambda & \xrightarrow{f_\lambda} & \operatorname{Spec}(R_\lambda) \end{array}$$

are Cartesian for all $\lambda \geq \alpha$. We therefore see that $f\colon X \to \operatorname{Spec}(R)$ satisfies the universal property for the limit of the morphisms f_λ.

Finally, to ensure that

$$\dim(X) \leq \dim\big(X_\lambda \otimes_{R_\lambda} (R_\lambda)_{\mathfrak{m}_\lambda}\big)$$

for all $\lambda \geq \alpha$, we choose a maximal chain

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_{\dim(X)} = X \tag{6}$$

of irreducible closed subsets in X, which exists since $\dim(X) < \infty$ by [EGAIV$_2$, Corollaire 5.6.6]. Since limits commute with fiber products and since

$$R \simeq \varinjlim_{\lambda \geq \alpha} (R_\lambda)_{\mathfrak{m}_\lambda}$$

by [EGAI$_{\text{new}}$, Chapitre 0, Proposition 6.1.6(ii)], we have

$$X \simeq \varprojlim_{\lambda \geq \alpha} \big(X_\lambda \otimes_{R_\lambda} (R_\lambda)_{\mathfrak{m}_\lambda}\big)$$

since they satisfy the same universal property. Next, [EGAIV$_3$, Proposition 8.6.3] says that the partially ordered set of closed subschemes in X is the colimit of the partially ordered sets of closed subschemes in $X_\lambda \otimes_{R_\lambda} (R_\lambda)_{\mathfrak{m}_\lambda}$ as $\lambda \in \Lambda$ varies. Thus, after possibly replacing α by a larger index, we may assume that the chain (6) is the preimage of a chain of closed subsets in

$$X_\alpha \otimes_{R_\alpha} (R_\alpha)_{\mathfrak{m}_\alpha}.$$

Moreover, since the chain (6) is a chain of *strict* inclusions of *irreducible* closed subsets, we can apply [EGAIV$_3$, Proposition 8.6.3] again to say that after possibly replacing α by a larger index, we have a chain

$$Z_{0,\alpha} \subsetneq Z_{1,\alpha} \subsetneq \cdots \subsetneq Z_{\dim(X),\alpha} = X_\alpha \otimes_{R_\alpha} (R_\alpha)_{\mathfrak{m}_\alpha}$$

with strict inclusions whose preimage in X is the chain (6), and that each closed subset in this chain is irreducible [EGAIV$_3$, Proposition 8.4.2(a)(i)]. Since this chain of inclusions must base change to a chain of strict inclusions in X, we see that the preimage of this chain in

$$X_\lambda \otimes_{R_\lambda} (R_\lambda)_{\mathfrak{m}_\lambda} \simeq X_\alpha \otimes_{R_\alpha} (R_\lambda)_{\mathfrak{m}_\lambda}$$

is still a chain of closed subsets with strict inclusions for each $\lambda \geq \alpha$, which are still irreducible by [EGAIV$_3$, Proposition 8.4.2(a)(i)].

We now show (ii). By [EGAIV$_3$, Théorème 8.5.2(ii)], after possibly replacing α with a larger index, there exists a coherent sheaf $\mathscr{L}'_\alpha$ on X'_α such that $v'^*_\alpha \mathscr{L}'_\alpha \simeq \mathscr{L}$. By [EGAIV$_3$, Proposition 8.5.5], after possibly replacing α by a larger index again, we may assume that the inverse image $\mathscr{L}'_\lambda \coloneqq v'^*_{\alpha\lambda} \mathscr{L}'_\alpha$ on X'_λ is invertible for all $\lambda \geq \alpha$. We now set

$$\mathscr{L}_\lambda \coloneqq \mathscr{L}'_\lambda|_{X_\lambda},$$

which is invertible and satisfies $\mathscr{L} \simeq v'^*_\alpha \mathscr{L}'_\alpha \simeq v^*_\alpha \mathscr{L}_\alpha$ by the commutativity of the squares in the left column of (4).

We now show that if $\mathscr{L}$ is f-very ample (resp. f-ample, f-semi-ample, f-big), then we may assume the same holds for $\mathscr{L}_\lambda$ for all $\lambda \geq \alpha$. This holds for $\mathscr{L}'_\lambda$ instead of $\mathscr{L}_\lambda$ by Lemma 4.2. Restricting $\mathscr{L}'_\lambda$ to X'_λ preserves these properties in each case by [EGAII, Propositions 4.4.10(i *bis*) and 4.6.13(i *bis*)] for f-very ample and f-ample, [CT20, Lemma 2.11(i)] and its proof for f-generated

and f-semi-ample, and the fact that $X_\lambda \hookrightarrow X'_\lambda$ induces an isomorphism over the generic point η_λ by construction for f-big (see (5)).

Next, we show (*iii*). For each $\lambda \in \Lambda$, consider the inverse system

$$\left\{g_{\lambda,p'} \colon W_{\lambda,p'} \longrightarrow X_\lambda\right\}_{p' \in P'_\lambda} \tag{7}$$

of all projective birational morphisms where the $W_{\lambda,p'}$ are integral schemes that are separated and of finite type over $\mathbf{k}$. The inverse system (3) is nonempty since it contains the identity and is indexed by a subset of AId_{X_λ} since every projective birational morphism to X_λ is a blowup [EGAIII$_1$, Corollaire 2.3.7]. This inverse system is coinitial with the inverse system (3) by Chow's lemma [EGAII, Corollaire 5.6.2].

If projective resolutions of singularities (resp. normalizations, Macaulayfications) exist for all integral schemes that are separated and of finite type over $\mathbf{k}$, then the inverse system of morphisms in (3) is coinitial with the subsystem consisting of morphisms from regular (resp. normal, Cohen–Macaulay) schemes $W_{\lambda,p}$. This proves the "moreover" statement.

Finally, it remains to show (*iv*). Set

$$J' = \bigsqcup_{\lambda \in \Lambda} P'_\lambda$$

with the preorder where $(\lambda, p') \leq (\mu, q')$ if and only $\lambda \leq \mu$ and the morphism $g_{\mu,q'}$ fits into a commutative diagram

$$\begin{array}{ccccc} W_{\mu,q'} & \xrightarrow{g_{\mu,q'}} & X_\mu & \longrightarrow & \mathrm{Spec}(R_\mu) \\ \downarrow & & \downarrow{\scriptstyle v_{\lambda\mu}} & & \downarrow \\ W_{\lambda,p'} & \xrightarrow{g_{\lambda,p'}} & X_\lambda & \longrightarrow & \mathrm{Spec}(R_\lambda). \end{array}$$

By the argument in (*iii*), the two inverse systems

$$\begin{aligned} &\left\{g_{\lambda,p} : W_{\lambda,p} \longrightarrow X_\lambda\right\}_{(\lambda,p) \in J} \\ &\left\{g_{\lambda,p'} \colon W_{\lambda,p'} \longrightarrow X_\lambda\right\}_{(\lambda,p') \in J'} \end{aligned} \tag{8}$$

are coinitial. It therefore suffices to show that J' is filtered and that the morphism $\pi \colon \mathrm{ZR}(X) \to X$ is the limit of the morphisms in (8).

To show that J' is filtered, let (λ_1, p'_1) and (λ_2, p'_2) be two indices in J'. Since Λ is directed, there exists $\mu \in \Lambda$ such that $\lambda_1 \leq \mu$ and $\lambda_2 \leq \mu$. We now claim we can construct a commutative diagram of the form

$$\begin{array}{ccccc} & & W'_{\lambda_1,p'_1} & \hookrightarrow & W_{\lambda_1,p'_1} \times_{X_{\lambda_1}} X_\mu \\ & \nearrow & & & \searrow^{\mathrm{pr}_2} \\ W_{\mu,q'} & & & & X_\mu \\ & \searrow & & & \nearrow_{\mathrm{pr}_2} \\ & & W'_{\lambda_2,p'_2} & \hookrightarrow & W_{\lambda_2,p'_2} \times_{X_{\lambda_2}} X_\mu \end{array}$$

where the composition $W_{\mu,q'} \to X_\mu$ is projective and birational, and where $W_{\mu,q'}$ is integral. We set W'_{λ_1,p'_1} to be the closure of the inverse image of the open set in X_μ over which the second projection $\mathrm{pr}_2 \colon W_{\lambda_1,p'_1} \times_{X_{\lambda_1}} X_\mu \to X_\mu$ is an isomorphism, and similarly for W'_{λ_2,p'_2}. Since $W'_{\lambda_1,p'_1} \to X_\mu$ is a projective and birational morphism from an integral scheme by construction, it is the blowup along some ideal $\mathscr{I}_1 \in \mathrm{AId}_{X_\mu}$ by [EGAIII$_1$, Corollaire 2.3.7], and similarly $W'_{\lambda_2,p'_2} \to X_\mu$ is the blowup along some ideal $\mathscr{I}_2 \in \mathrm{AId}_{X_\mu}$. We can therefore consider the blowup $W_{\mu,q'} \to X_\mu$ along $\mathscr{I}_1\mathscr{I}_2$, which factors uniquely through W'_{λ_1,p'_1} and W'_{λ_2,p'_2} by the universal property of blowups [Stacks, Tag 0806]. Note that $W_{\mu,q'}$ is integral by [EGAII, Proposition 8.1.4(i)].

It remains to show that the limit of the morphisms in (8) is indeed the morphism $\pi\colon \mathrm{ZR}(X) \to X$. We claim that the limit of the inverse system

$$\left\{g_{\lambda,p'} \times_{X_\lambda} \mathrm{id}_X \colon W_{\lambda,p'} \times_{X_\lambda} X \longrightarrow X_\lambda \times_{X_\lambda} X\right\}_{(\lambda,p')\in J'} \tag{9}$$

coincides with the limit of the inverse system (8). This follows since the squares

$$\begin{array}{ccc} W_{\lambda,p'} \times_{X_\lambda} X & \xrightarrow{g_{\lambda,p'}\times_{X_\lambda}\mathrm{id}_X} & X_\lambda \times_{X_\lambda} X \\ \downarrow & & \downarrow{\scriptstyle v_\lambda} \\ W_{\lambda,p'} & \xrightarrow{\quad g_{\lambda,p'}\quad} & X_\lambda \end{array}$$

are Cartesian, and hence the limits of the two inverse systems (8) and (9) satisfy the same universal property.

We now show that the limit of the inverse system (9) is indeed the morphism $\pi\colon \mathrm{ZR}(X) \to X$. It suffices to show that the the inverse system (9) is coinitial with the inverse system defining $\mathrm{ZR}(X)$. Let $g_{\lambda,p'}\colon W_{\lambda,p'} \to X_\lambda$ be morphism in (8). As before, we know that $g_{\lambda,p'}$ is the blowup along some ideal $\mathscr{I} \in \mathrm{AId}_{X_\lambda}$ by [EGAIII$_1$, Corollaire 2.3.7]. We then have the commutative diagram

$$\begin{array}{ccc} X_{v_\lambda^{-1}\mathscr{I}\cdot\mathcal{O}_X} & \xrightarrow{\pi_{v_\lambda^{-1}\mathscr{I}\cdot\mathcal{O}_X}} & X \\ \downarrow & & \downarrow{\scriptstyle v_\lambda} \\ W_{\lambda,p'} & \xrightarrow{\quad g_{\lambda,p'}\quad} & X_\lambda \end{array}$$

by the universal property of blowups [Stacks, Tag 0806], where the top horizontal arrow is the blowup along $v_\lambda^{-1}\mathscr{I}\cdot\mathcal{O}_X$. Note that $v_\lambda^{-1}\mathscr{I}\cdot\mathcal{O}_X \in \mathrm{AId}_X$. By the universal property of fiber products, we see that $X_{v_\lambda^{-1}\mathscr{I}\cdot\mathcal{O}_X}$ factors through the base change of $g_{\lambda,p'}$.

Conversely, suppose $\pi_{\mathscr{I}}\colon X_{\mathscr{I}} \to X$ is an admissible blowup. Then, by [EGAIV$_3$, Théorème 8.8.2(ii)] (here we use the Noetherianity of X to say that the blowup $\pi_{\mathscr{I}}$ is finitely presented), there exists an index $\alpha \in \Lambda$ and a morphism $h'_\alpha\colon W'_\alpha \to X_\alpha$ for which the diagram

$$\begin{array}{ccc} X_{\mathscr{I}} & \xrightarrow{\pi_{\mathscr{I}}} & X \\ \downarrow & & \downarrow{\scriptstyle v_\alpha} \\ W'_\alpha & \xrightarrow{h'_\alpha} & X_\alpha \end{array}$$

is Cartesian. For each $\lambda \geq \alpha$, denote by $h'_\lambda\colon W'_\lambda \to X_\lambda$ the base change of h'_α to X_λ. By [EGAIV$_3$, Proposition 8.7.2 and Théorème 8.10.5(i),($xiii$)], for large enough $\lambda \geq \alpha$, the scheme W'_λ is reduced, the morphism h'_λ is projective, and the restriction of h'_λ to an open subset U_λ of X_λ induces an isomorphism. Denote by ξ and ξ_λ the generic points of X and X_λ, respectively. By transitivity of fibers [EGAI$_{\mathrm{new}}$, Corollaire 3.4.9] and [EGAIV$_2$, (4.4.1)], the generic fibers $h_\lambda'^{-1}(\xi_\lambda)$ are also irreducible. Now let W_λ be the scheme-theoretic closure of $h_\lambda'^{-1}(\xi_\lambda)$, which coincides with the set-theoretic closure with reduced scheme structure since W'_λ is reduced [EGAI$_{\mathrm{new}}$, Corollaire 6.10.6]. The scheme W_λ is irreducible by [EGAI$_{\mathrm{new}}$, Chapitre 0, Proposition 2.1.13], hence integral. Now consider the composition

$$h_\lambda\colon W_\lambda \hookrightarrow W'_\lambda \xrightarrow{h'_\lambda} X_\lambda.$$

This morphism is projective since it is the composition of projective morphisms, and is birational since its restriction to U_λ is still an isomorphism. By [EGAIV$_3$, Lemme 13.1.2], the square

$$\begin{array}{ccc} X_{\mathscr{I}} & \xrightarrow{\pi_\lambda} & X \\ \downarrow & & \downarrow{\scriptstyle v_\lambda} \\ W_\lambda & \xrightarrow{h_\lambda} & X_\lambda \end{array}$$

is Cartesian. Thus, the inverse system (9) is coinitial with the inverse system defining $\mathrm{ZR}(X)$, and hence their limits coincide. □

5. Relative vanishing and injectivity theorems for Zariski–Riemann spaces

Our goal in this section is to prove the following relative vanishing and injectivity theorem for Zariski–Riemann spaces. This theorem is stated using local cohomology following the dual formulation of Grauert–Riemenschneider vanishing due to Hartshorne and Ogus [HO74, Proposition 2.2]. This dual formulation allows us to prove these statements for Zariski–Riemann spaces. This statement also has the advantage of not requiring the existence of dualizing complexes or canonical sheaves ω_X, which are not known to behave well under limits.

Theorem 5.1. *Let $(R,\mathfrak{m})$ be an integral Noetherian local $\mathbf{Q}$-algebra and set $Y \coloneqq \operatorname{Spec}(R)$. Let $f\colon X \to Y$ be a proper surjective morphism from an integral scheme X. Set $Z = f^{-1}(\{\mathfrak{m}\})$, and denote by $\pi\colon \mathrm{ZR}(X) \to X$ the canonical projection morphism from the Zariski–Riemann space of X.*

Consider an invertible sheaf $\mathscr{L}$ on X.

(i) Suppose $\mathscr{L}$ is f-big and f-semi-ample. Then, we have

$$H^i_{\pi^{-1}(Z)}\big(\mathrm{ZR}(X), \pi^*\mathscr{L}^{-1}\big) = 0$$

for all $i < \dim(X)$.

(ii) Suppose $\mathscr{L}$ is f-semi-ample. Let D be an effective Weil divisor on X for which there exists an integer $n > 0$ and an effective Weil divisor D' on X such that $\mathcal{O}_X(D + D') \simeq \mathscr{L}^{\otimes n}$. Then, the canonical morphisms

$$H^i_{\pi^{-1}(Z)}\big(\mathrm{ZR}(X), \pi^*\big(\mathscr{L}^{-k}(-D)\big)\big) \longrightarrow H^i_{\pi^{-1}(Z)}\big(\mathrm{ZR}(X), \pi^*\mathscr{L}^{-k}\big)$$

induced by the inclusion $\mathcal{O}_X(-D) \hookrightarrow \mathcal{O}_X$ are surjective for all i and for all $k > 0$.

In §5.1, we prove that for Noetherian schemes X, vanishing and injectivity can be stated in terms of higher direct images and ω_X or in terms of local cohomology modules (Proposition 5.3). This will be used after reducing to the case of varieties over $\mathbf{Q}$ to prove Theorem 5.1 and will also be used later to prove Theorems A and B. The key ingredient for showing the two formulations are equivalent is a combination of Grothendieck local duality and Grothendieck duality for proper morphisms, which is called the *local-global duality of Lipman* in [HHK98, p. 283]. We then prove Theorem 5.1 in §5.2 using our approximation results from §4.

5.1. Lipman's local-global duality.

We prove that Theorems B and B* are equivalent when dualizing complexes exist. The key ingredient is the following duality statement due to Lipman [Lip78]. See [Har66, Definition on p. 276] for the notion of a normalized dualizing complex used below. Hartshorne and Ogus give a different approach using formal duality in the proof of [HO74, Proposition 2.2]. If $\mathscr{L}$ is a locally free sheaf of finite rank on a ringed space X, the *dual* of $\mathscr{L}$ is $\mathscr{L}^\vee \coloneqq \mathscr{H}om_{\mathcal{O}_X}(\mathscr{L}, \mathcal{O}_X)$.

Lemma 5.2 (see [Lip78, Theorem on p. 188])**.** *Let $f\colon X \to \operatorname{Spec}(R)$ be a proper morphism where $(R,\mathfrak{m})$ is a Noetherian local ring with a normalized dualizing complex $\omega_R^\bullet$. Set $Z = f^{-1}(\{\mathfrak{m}\})$, and let $\mathscr{L}$ be a locally free sheaf of finite rank on X. Then, there is a quasi-isomorphism*

$$\mathbf{R}\Gamma_Z(X, \mathscr{L}^\vee) \simeq \operatorname{Hom}_R\big(\mathbf{R}f_*(\omega_X^\bullet \otimes_{\mathcal{O}_X} \mathscr{L}), E\big),$$

functorial in $\mathscr{L}$, where $\omega_X^\bullet = f^!\omega_R^\bullet$ and where E is the injective hull of the residue field of R. In particular, if X is Cohen–Macaulay of pure dimension n, we have an isomorphism

$$H^i_Z(X, \mathscr{L}^\vee) \simeq \operatorname{Hom}_R\big(R^{n-i}f_*(\omega_X \otimes_{\mathcal{O}_X} \mathscr{L}), E\big)$$

for every i, where ω_X denotes the unique cohomology sheaf of $f^!\omega_Y^\bullet$.

Proof. We follow the proof in [Lip78, Theorem on p. 188], keeping track of morphisms $\mathscr{L} \to \mathscr{M}$ of locally free sheaves of finite rank along the way. We have the commutative diagram

$$\begin{array}{ccccc} \mathbf{R}f_*(\omega_X^\bullet \otimes \mathscr{L}) & \xleftarrow{\sim} & \mathbf{R}f_*\,\mathscr{H}om_{\mathcal{O}_X}(\mathscr{L}^\vee, \omega_X^\bullet) & \xrightarrow{\sim} & \operatorname{Hom}_R(\mathbf{R}f_*\mathscr{L}^\vee, \omega_R^\bullet) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{R}f_*(\omega_X^\bullet \otimes \mathscr{M}) & \xleftarrow{\sim} & \mathbf{R}f_*\,\mathscr{H}om_{\mathcal{O}_X}(\mathscr{M}^\vee, \omega_X^\bullet) & \xrightarrow{\sim} & \operatorname{Hom}_R(\mathbf{R}f_*\mathscr{M}^\vee, \omega_R^\bullet) \end{array} \tag{10}$$

where the horizontal arrows are isomorphisms induced by the isomorphism of functors

$$\mathbf{R}\mathscr{H}om_{\mathcal{O}_X}(\mathscr{L}^\vee, -) \xrightarrow{\sim} - \otimes_{\mathcal{O}_X} \mathscr{L}$$

coming from [Har66, Chapter V, Corollary 6.3] and the evaluation at 1 map in the left square, and are induced by Grothendieck duality [Har66, Chapter VII, Corollary 3.4(*c*); Con00, Theorem 3.4.4] in the right square. Since $\mathbf{R}f_*\mathscr{L}^\vee$ is quasi-coherent with coherent cohomology (by the assumption that f is proper [EGAIII$_1$, Théorème 3.2.1]), we can apply local duality [Har66, Chapter V, Corollary 6.3] to obtain the commutative diagram

$$\begin{array}{ccc} \mathbf{R}\Gamma_{\mathfrak{m}}(R, \mathbf{R}f_*\mathscr{L}^\vee) & \xrightarrow{\sim} & \operatorname{Hom}_R\big(\operatorname{Hom}_R(\mathbf{R}f_*\mathscr{L}^\vee, \omega_R^\bullet), E\big) \\ \uparrow & & \uparrow \\ \mathbf{R}\Gamma_{\mathfrak{m}}(R, \mathbf{R}f_*\mathscr{M}^\vee) & \xrightarrow{\sim} & \operatorname{Hom}_R\big(\operatorname{Hom}_R(\mathbf{R}f_*\mathscr{M}^\vee, \omega_R^\bullet), E\big) \end{array} \tag{11}$$

where the horizontal arrows are isomorphisms. Using the isomorphism of functors $\mathbf{R}\Gamma_{\mathfrak{m}} \circ \mathbf{R}f_* \simeq \mathbf{R}\Gamma_Z$, we can identify the objects in the left column with $\mathbf{R}\Gamma_Z(X, \mathscr{L}^\vee)$ and $\mathbf{R}\Gamma_Z(X, \mathscr{M}^\vee)$, respectively. We can then combine the diagram we have obtained so far to obtain the following commutative diagram:

$$\begin{array}{ccccc} \mathbf{R}\Gamma_Z(X, \mathscr{L}^\vee) & \xrightarrow{\sim} & \operatorname{Hom}_R\big(\operatorname{Hom}_R(\mathbf{R}f_*\mathscr{L}^\vee, \omega_R^\bullet), E\big) & \xrightarrow{\sim} & \operatorname{Hom}_R\big(\mathbf{R}f_*(\omega_X^\bullet \otimes \mathscr{L}), E\big) \\ \uparrow & & \uparrow & & \uparrow \\ \mathbf{R}\Gamma_Z(X, \mathscr{M}^\vee) & \xrightarrow{\sim} & \operatorname{Hom}_R\big(\operatorname{Hom}_R(\mathbf{R}f_*\mathscr{M}^\vee, \omega_R^\bullet), E\big) & \xrightarrow{\sim} & \operatorname{Hom}_R\big(\mathbf{R}f_*(\omega_X^\bullet \otimes \mathscr{M}), E\big). \end{array}$$

Here, the left square is (11) with the identification $\mathbf{R}\Gamma_{\mathfrak{m}} \circ \mathbf{R}f_* \simeq \mathbf{R}\Gamma_Z$ made above, and the right square is obtained from (10) and applying $\operatorname{Hom}_R(-, E)$ which has no higher Ext modules since E is injective. Finally, the "in particular" statement follows from the first statement after taking i-th cohomology, since in this case $\omega_X \simeq \omega_X^\bullet[-n]$ by local duality [Har66, Chapter V, Corollary 6.3]. □

We now show that Theorems B and B* are equivalent when dualizing complexes exist.

Proposition 5.3. *Let $f\colon X \to Y$ be a proper morphism of Noetherian schemes, and suppose that X is Cohen–Macaulay and equidimensional and that Y has a dualizing complex $\omega_Y^\bullet$. Denote by ω_X the unique cohomology sheaf of $f^!\omega_Y^\bullet$ (after possibly applying shifts on each connected component of X).*

Consider an invertible sheaf $\mathscr{L}$ on X and fix $y \in Y$. Denote by $f_y\colon X_y \to \operatorname{Spec}(\mathcal{O}_{Y,y})$ the base change of f along $\operatorname{Spec}(\mathcal{O}_{Y,y}) \to Y$, by $\mathscr{L}_y$ the pullback of $\mathscr{L}$ to X_y, and set $Z_y = f_y^{-1}(y)$. For all i, we have the following:

(i) $R^i f_{y*}(\omega_{X_y} \otimes_{\mathcal{O}_{X_y}} \mathscr{L}_y) = 0$ if and only if $H^{\dim(X_y)-i}_{Z_y}(X_y, \mathscr{L}_y^{-1}) = 0$.

(*ii*) *Let D be an effective Cartier divisor on X, and denote by D_y the pullback of D to X_y. Then, the canonical morphism*

$$R^i f_{y*}(\omega_{X_y} \otimes_{\mathcal{O}_{X_y}} \mathscr{L}_y) \longrightarrow R^i f_{y*}\big(\omega_{X_y} \otimes_{\mathcal{O}_{X_y}} \mathscr{L}_y(D_y)\big)$$

induced by the inclusion $\mathcal{O}_{X_y} \to \mathcal{O}_{X_y}(D_y)$ is injective if and only if the canonical morphism

$$H^{\dim(X_y)-i}_{Z_y}\big(X_y, \mathscr{L}_y^{-1}(-D_y)\big) \longrightarrow H^{\dim(X_y)-i}_{Z_y}(X_y, \mathscr{L}_y^{-1})$$

induced by the inclusion $\mathcal{O}_{X_y}(-D_y) \to \mathcal{O}_{X_y}$ is surjective.

Proof. Since all statements are local by flat base change, we may replace Y with $\operatorname{Spec}(\mathcal{O}_{Y,y})$ to assume that Y is the spectrum of a Noetherian local ring $(R, \mathfrak{m})$ with a dualizing complex, since dualizing complexes localize [Har66, Chapter V, Corollary 2.3]. After translating the dualizing complex, we may assume it is normalized.

We first consider (*i*). Let E denote the injective hull of the residue field of R. Since $\operatorname{Hom}_R(-, E)$ is faithfully exact [Ish64, Corollary 3.2(2)], we see that $R^i f_*(\omega_X \otimes_{\mathcal{O}_X} \mathscr{L}) = 0$ if and only if

$$\operatorname{Hom}_R\big(R^i f_*(\omega_X \otimes_{\mathcal{O}_X} \mathscr{L}), E\big) = 0.$$

By Lemma 5.2, this is equivalent to $H^{\dim(X)-i}_Z(X, \mathscr{L}^{-1}) = 0$.

We now consider (*ii*). Since $\operatorname{Hom}_R(-, E)$ is faithfully exact [Ish64, Corollary 3.2(2)], the morphism

$$R^i f_*(\omega_X \otimes_{\mathcal{O}_X} \mathscr{L}) \longrightarrow R^i f_*\big(\omega_X \otimes_{\mathcal{O}_X} \mathscr{L}(D)\big)$$

is injective if and only if

$$\operatorname{Hom}_R\big(R^i f_*\big(\omega_X \otimes_{\mathcal{O}_X} \mathscr{L}(D)\big), E\big) \longrightarrow \operatorname{Hom}_R\big(R^i f_*(\omega_X \otimes_{\mathcal{O}_X} \mathscr{L}), E\big)$$

is surjective. By Lemma 5.2, this is equivalent to the surjectivity of

$$H^{\dim(X)-i}_Z\big(X, \mathscr{L}^{-1}(-D)\big) \longrightarrow H^{\dim(X)-i}_Z(X, \mathscr{L}^{-1}). \qquad \square$$

5.2. **Relative vanishing and injectivity theorem for Zariski–Riemann spaces.** In this subsection, we prove our relative vanishing and injectivity theorem for Zariski–Riemann spaces using our approximation results in §4. As outlined in §1, the idea is to approximate the local cohomology modules in question by approximating the morphism $f\colon X \to Y$ by a morphism of $\mathbf{Q}$-varieties. We will show later that this vanishing descends to X using relative vanishing for the canonical morphism $\pi\colon \operatorname{ZR}(X) \to X$ from the Zariski–Riemann space associated to X (Theorem 6.2).

Proof of Theorem 5.1. For (*ii*), since the map $\mathcal{O}_X \to \mathcal{O}_X(D)$ factors the map $\mathcal{O}_X \to \mathcal{O}_X(D + D')$, we can replace D by $D + D'$ to assume that $\mathcal{O}_X(D) \simeq \mathscr{L}^{\otimes n}$, and in particular, we may assume that D is Cartier.

We now proceed in a sequence of steps.

Step 1. *It suffices to show that for morphisms $f\colon X \to Y$ fitting into a Cartesian diagram*

$$\begin{array}{ccc} X_y & \xrightarrow{f_y} & \operatorname{Spec}(\mathcal{O}_{Y,y}) \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where $f\colon X \to Y$ is a morphism of varieties over $\mathbf{Q}$, X is smooth, Y is affine, and $y \in Y$ is a point, we have

$$H^i_{f_y^{-1}(\{y\})}(X_y, \mathscr{L}_y^{-1}) = 0$$

*for all $i < \dim(X_y)$ for (*i*) and the morphisms*

$$H^i_{f_y^{-1}(\{y\})}\big(X_y, \mathscr{L}_y^{-k}(-D)\big) \longrightarrow H^i_{f_y^{-1}(\{y\})}(X_y, \mathscr{L}_y^{-k})$$

*are surjective for all i and all $k > 0$ for (*ii*), where $\mathscr{L}_y$ is the pullback of $\mathscr{L}$ to X_y.*

By Lemma 4.2 and Remark 4.3 applied to $f\colon X \to \operatorname{Spec}(R)$ and $\mathbf{k} = \mathbf{Q}$, we have the commutative diagram

$$\begin{array}{ccccc} \operatorname{ZR}(X) & \xrightarrow{\pi} & X & \xrightarrow{f} & \operatorname{Spec}(R) \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{W}_{\lambda,p} & \xrightarrow{\tilde{g}_{\lambda,p}} & \tilde{X}_\lambda & \xrightarrow{\tilde{f}_\lambda} & \operatorname{Spec}\big((R_\lambda)_{\mathfrak{m}_\lambda}\big) \\ \downarrow & & \downarrow & & \downarrow \\ W_{\lambda,p} & \xrightarrow{g_{\lambda,p}} & X_\lambda & \xrightarrow{f_\lambda} & \operatorname{Spec}(R_\lambda) \end{array} \tag{12}$$

of locally ringed spaces, where all but $\operatorname{ZR}(X)$ are Noetherian schemes of equal characteristic zero, the bottom squares are Cartesian, and where the schemes in the bottom row are varieties over $\mathbf{Q}$ with $W_{\lambda,p}$ smooth, such that the morphisms in the top row are the limits of the morphisms in the bottom row. As in the proof of Lemma 4.2(*i*) and in Remark 4.3, the inverse limits of the morphisms in the middle and bottom row satisfy the same universal property, and hence both limit to the morphisms in the top row. By Lemma 4.2(*ii*), we also have that

$$\mathscr{L} \simeq v_\lambda^* \mathscr{L}_\lambda \simeq \varinjlim_{\lambda \in \Lambda} v_\lambda^* \mathscr{L}_\lambda$$

for f_λ-big and f_λ-semi-ample (resp. f_λ-semi-ample) invertible sheaves $\mathscr{L}_\lambda$, where $v_\lambda\colon X \to X_\lambda$ are the canonical projection morphisms. We denote by $\tilde{\mathscr{L}}_\lambda$ the pullback of $\mathscr{L}_\lambda$ to $\tilde{X}_\lambda$, which is $\tilde{f}_\lambda$-semi-ample by [CT20, Lemma 2.12(*i*)], and is also $\tilde{f}_\lambda$-big in situation (*i*) by the fact that the base change $R_\lambda \to (R_\lambda)_{\mathfrak{m}_\lambda}$ does not affect generic fibers. We also know that $g_{\lambda,p}^* \mathscr{L}_\lambda$ is $(f_\lambda \circ g_{\lambda,p})$-semi-ample by [CT20, Lemma 2.11], and is also $(f_\lambda \circ g_{\lambda,p})$-big in situation (*i*) by the fact that $g_{\lambda,p}$ induces a birational morphism along the generic fiber of f_λ. The same reasoning applies to $\tilde{g}_{\lambda,p}^* \tilde{\mathscr{L}}_\lambda$.

To show Theorem 5.1(*i*), the hypothesis in Step 1 implies

$$H^i_{(\tilde{f}_\lambda \circ \tilde{g}_{\lambda,p})^{-1}(\{\mathfrak{m}_\lambda\})}\big(\tilde{W}_{\lambda,p}, \tilde{g}_{\lambda,p}^* \tilde{\mathscr{L}}_\lambda^{-1}\big) = 0$$

for all $i < \dim(X_\lambda \otimes_{R_\lambda} (R_\lambda)_{\mathfrak{m}_\lambda})$ since $\tilde{g}_{\lambda,p}^* \tilde{\mathscr{L}}_\lambda^{-1}$ is $(\tilde{f}_\lambda \circ \tilde{g}_{\lambda,p})$-big and $(\tilde{f}_\lambda \circ \tilde{g}_{\lambda,p})$-semi-ample. Since

$$\dim(X) \leq \dim\big(X_\lambda \otimes_{R_\lambda} (R_\lambda)_{\mathfrak{m}_\lambda}\big)$$

for all $\lambda \in \Lambda$ (see Lemma 4.2(*i*)), this implies in particular that the vanishing holds for all $i < \dim(X)$. Taking colimits over all $(\lambda, p) \in J$, Theorem 3.13 implies

$$H^i_{\pi^{-1}(Z)}\big(\operatorname{ZR}(X), \pi^* \mathscr{L}^{-1}\big) = H^i_{(f\circ\pi)^{-1}(\{\mathfrak{m}\})}\big(\operatorname{ZR}(X), \pi^* \mathscr{L}^{-1}\big) = 0$$

for all $i < \dim(X)$, where the colimit of the inverse images of the sheaves $\tilde{g}_{\lambda,p}^* \tilde{\mathscr{L}}_\lambda$ on $\operatorname{ZR}(X)$ is $\pi^* \mathscr{L}$ by the commutativity of the diagram (12).

It remains to show Theorem 5.1(*ii*). By [Stacks, Tag 0B8W(3)], we can find $\alpha \in \Lambda$ such that $\mathcal{O}_X(-D) \simeq v_\alpha^{-1} \mathscr{I}_\alpha \cdot \mathcal{O}_X$ for an ideal sheaf $\mathscr{I}_\alpha \subseteq \mathcal{O}_{X_\alpha}$. After replacing X_α by the blowup of X_α along $\mathscr{I}_\alpha$, we may assume that $\mathscr{I}_\alpha$ is invertible. Let D_α be the effective Cartier divisor corresponding to $\mathscr{I}_\alpha$. By [EGAIV$_3$, Corollaire 8.5.2.5], after possibly replacing α by a larger index and D_α by its pullback (which exist since the $v_{\alpha\lambda}$ are surjective morphisms of integral Noetherian schemes [EGAIV$_4$, Proposition 21.4.5(*iii*)]), we may assume that $\mathcal{O}_{X_\alpha}(-D_\alpha) \simeq \mathscr{L}_\alpha^{\otimes -n}$. We can therefore write the injection $\mathscr{L}^{-k}(-D) \hookrightarrow \mathscr{L}^{-k}$ as the colimit of the injections

$$v_\lambda^*\big(\mathscr{L}_\lambda^{-k}(-D_\lambda)\big) \hookrightarrow v_\lambda^*\big(\mathscr{L}_\lambda^{-k}\big).$$

Denoting by $\tilde{D}_\lambda$ the restriction of D_λ to $\tilde{X}_\lambda$, we see that

$$H^i_{(\tilde{f}_\lambda \circ \tilde{g}_{\lambda,p})^{-1}(\{\mathfrak{m}_\lambda\})}\big(\tilde{W}_{\lambda,p}, \tilde{g}_{\lambda,p}^* \tilde{\mathscr{L}}_\lambda^{-k}(-\tilde{D}_\lambda)\big) \longrightarrow H^i_{(\tilde{f}_\lambda \circ \tilde{g}_{\lambda,p})^{-1}(\{\mathfrak{m}_\lambda\})}\big(\tilde{W}_{\lambda,p}, \tilde{g}_{\lambda,p}^* \tilde{\mathscr{L}}_\lambda^{-k}\big)$$

is surjective for all i since $\tilde{g}_{\lambda,p}^*\tilde{\mathscr{L}}_\lambda$ is f-semi-ample by [CT20, Lemma 2.11]. Taking colimits over all $(\lambda,p)\in J$, Theorem 3.13 implies

$$H^i_{(f\circ\pi)^{-1}(\{\mathfrak{m}\})}\big(\mathrm{ZR}(X),\pi^*\big(\mathscr{L}^{-k}(-D)\big)\big)\longrightarrow H^i_{(f\circ\pi)^{-1}(\{\mathfrak{m}\})}\big(\mathrm{ZR}(X),\pi^*\mathscr{L}^{-k}\big),$$

is surjective for all i, where the colimit of the inverse images of the sheaves $g_{\lambda,p}^*(\mathcal{O}_{X_\lambda}(-D_\lambda))$ on $\mathrm{ZR}(X)$ is $\pi^*\mathcal{O}_X(-D)$ by the commutativity of the diagram (12).

Step 2. *Conclusion of proof.*

We start by proving the special case of Theorem 5.1(i) stated in Step 1. By Proposition 5.3(i) and flat base change, it suffices to show that

$$R^if_*(\omega_X\otimes_{\mathcal{O}_X}\mathscr{L})=0$$

for all $i>0$. Consider the Stein factorization

$$X\xrightarrow{f'}Y'\xrightarrow{g}Y$$

of f. We can replace $f\colon X\to Y$ by $f'\colon X\to Y'$ to assume that Y is normal, since the relative normalization morphism g is affine and hence $R^if'_*(\omega_X\otimes_{\mathcal{O}_X}\mathscr{L})$ vanishes if and only if $R^if_*(\omega_X\otimes_{\mathcal{O}_X}\mathscr{L})$ does. By flat base change and the fact that the formation of ω_X is compatible with ground field extensions [Har66, Chapter V, Corollary 3.4(a)], it suffices to show that denoting by

$$f_{\mathbf{C}}\colon X_{\mathbf{C}}\longrightarrow Y_{\mathbf{C}}$$

the base change of f along the field extension $\mathbf{Q}\subseteq\mathbf{C}$, we have

$$R^if_{\mathbf{C}*}(\omega_{X_{\mathbf{C}}}\otimes_{\mathcal{O}_{X_{\mathbf{C}}}}\mathscr{L})=0$$

for all $i>0$. We note that $f_{\mathbf{C}}$ is maximally dominating by the flatness of $\mathbf{Q}\subseteq\mathbf{C}$ [ILO14, Exposé II, Proposition 1.1.5]. Since $Y_{\mathbf{C}}$ is normal, it is the disjoint union of normal varieties. We claim we may work one irreducible component at a time to assume that $f_{\mathbf{C}}$ is a projective surjective morphism of complex varieties. Note that $\mathscr{L}_{\mathbf{C}}$ is $f_{\mathbf{C}}$-semiample by [CT20, Lemma 2.12(i)]. By transitivity of fibers [EGAI$_{\mathrm{new}}$, Corollaire 3.4.9], the compatibility of maps induced by linear systems and flat base change [EGAII, (4.2.10)], and the fact that generically finite morphisms are stable under flat base change (combine [ILO14, Exposé II, Proposition 1.1.5] and the characterization of generically finite morphisms in [ILO14, Exposé II, Proposition 1.1.7]), we see that the restriction of large enough powers of $\mathscr{L}_{\mathbf{C}}$ induce generically finite morphisms on each fiber of $f_{\mathbf{C}}$ over the maximal points of Y. We therefore see that $\mathscr{L}_{\mathbf{C}}$ is $f_{\mathbf{C}}$-big. Now the required vanishing holds in situation (i) by relative Kawamata–Viehweg vanishing for complex algebraic varieties [KMM87, Theorem 1-2-3].

It remains to show Theorem 5.1(ii) holds in the special case stated in Step 1. By Proposition 5.3(ii), flat base change, and the fact that the formation of ω_X is compatible with ground field extensions [Har66, Chapter V, Corollary 3.4(a)], it suffices to show that denoting by

$$f_{\mathbf{C}}\colon X_{\mathbf{C}}\longrightarrow Y_{\mathbf{C}}$$

the base change of f along the field extension $\mathbf{Q}\subseteq\mathbf{C}$, the morphisms

$$R^if_*(\omega_{X_{\mathbf{C}}}\otimes_{\mathcal{O}_{X_{\mathbf{C}}}}\mathscr{L}_{\mathbf{C}}^{\otimes k})\longrightarrow R^if_*\big(\omega_{X_{\mathbf{C}}}\otimes_{\mathcal{O}_{X_{\mathbf{C}}}}\big(\mathscr{L}^{\otimes k}(D)\big)_{\mathbf{C}}\big)$$

are injective for all i and all $k>0$, where $\mathscr{L}_{\mathbf{C}}^{\otimes k}$ and $(\mathscr{L}^{\otimes k}(D))_{\mathbf{C}}$ are the pullbacks of $\mathscr{L}^{\otimes k}$ and $\mathscr{L}^{\otimes k}(D)$ to $X_{\mathbf{C}}$, respectively. This statement holds by Fujino's version of Kollár's injectivity theorem for simple normal crossings pairs [Fuj17, Theorem 5.6.1]. □

6. Rational singularities via Zariski–Riemann spaces

In this section, we prove a new characterization of rational singularities and pseudo-rational rings via Zariski–Riemann spaces in equal characteristic zero. An advantage of this characterization is that we do not need resolutions of singularities, quasi-excellence, or the existence of dualizing complexes. This characterization is a version of the characterizations of rational singularities for varieties over fields of characteristic zero due to Lipman and Teissier [LT81, $(iv)'$ on p. 102 and Corollary of (iii) on p. 107], Lipman [Lip94, Lemma 4.2], and Kovács [Kov00, Theorem 1]. We have modeled the formulation of our characterization after the characterization of Du Bois singularities due to Godfrey and the author of the present paper [GM, Theorem 2.3], where the 0-th graded piece of the Deligne–Du Bois complex $\underline{\Omega}^0_X$ takes the role of $\mathbf{R}\pi_*\mathcal{O}_{\mathrm{ZR}(X)}$.

We start with the following result for pseudo-rationality.

Lemma 6.1. *Let $(R,\mathfrak{m})$ be a Noetherian local ring of dimension d, and set $X \coloneqq \operatorname{Spec}(R)$. Denote by $\pi\colon \mathrm{ZR}(X) \to X$ the canonical projection from the Zariski–Riemann space of X. For every i, the map*

$$H^i_{\{\mathfrak{m}\}}(R) \xrightarrow{\delta^i_\pi} H^i_{\pi^{-1}(\{x\})}\big(\mathrm{ZR}(X), \mathcal{O}_{\mathrm{ZR}(X)}\big) \tag{13}$$

is injective if and only if for every proper birational morphism $W \to X$, the map

$$H^i_{\{\mathfrak{m}\}}(R) \xrightarrow{\delta^i_f} H^i_{f^{-1}(\{x\})}\big(W, \mathcal{O}_W\big)$$

is injective.

In particular, R is pseudo-rational if and only if δ^d_π is injective and R is normal, Cohen–Macaulay, and analytically unramified.

Proof. By Theorem 3.13, we have a factorization

$$H^i_{\{\mathfrak{m}\}}(R) \xrightarrow{\delta^i_f} H^i_{f^{-1}(\{\mathfrak{m}\})}(W, \mathcal{O}_W) \longrightarrow H^i_{(f\circ g)^{-1}(\{\mathfrak{m}\})}(\tilde{W}, \mathcal{O}_{\tilde{W}}) \longrightarrow H^i_{\pi^{-1}(\{\mathfrak{m}\})}\big(\mathrm{ZR}(X), \mathcal{O}_{\mathrm{ZR}(X)}\big)$$

of $\delta^i_\pi(\mathcal{O}_X)$ for every proper birational morphism $f\colon W \to X$, where $g\colon \tilde{W} \to W$ is a projective birational morphism such that $f \circ g$ is a blowup of X. Such a morphism g exists by Chow's lemma [EGAII, Théorème 5.6.1] and the fact that every projective birational morphism is a blowup [EGAIII$_1$, Corollaire 2.3.7].

Note that $\Rightarrow$ follows since if the composition δ^i_π is injective, then δ^i_f is injective. For $\Leftarrow$, the maps

$$H^i_{\{\mathfrak{m}\}}(R) \to H^i_{(f\circ g)^{-1}(\{\mathfrak{m}\})}(\tilde{W}, \mathcal{O}_{\tilde{W}})$$

are injective for every $\tilde{W}$ as constructed above. We can therefore take colimits and apply Theorem 3.13 to see that δ^i_π is injective. □

We are now ready to show our characterization of rational singularities and pseudo-rational rings. Below, we will freely use the fact that the formation of $\mathrm{ZR}(X)$ commutes with base change by quasi-compact separated étale morphisms [Stacks, Tag 087B], and hence $\mathrm{ZR}(X) \times_X X_y \simeq \mathrm{ZR}(X_y)$ where $X_y \coloneqq \operatorname{Spec}(\mathcal{O}_{X,y})$.

Theorem 6.2. *Let X be a Noetherian scheme. Denote by $\pi\colon \mathrm{ZR}(X) \to X$ the canonical projection morphism from the Zariski–Riemann space of X. Consider the following conditions:*

- (i) *The natural morphism $\mathcal{O}_X \to \mathbf{R}\pi_*\mathcal{O}_{\mathrm{ZR}(X)}$ is a quasi-isomorphism.*
- (ii) *The natural morphism $\mathcal{O}_X \to \mathbf{R}\pi_*\mathcal{O}_{\mathrm{ZR}(X)}$ admits a left inverse in the derived category of $\mathcal{O}_X$-modules.*

(*iii*) *For every point* $y \in X$, *setting* $X_y \coloneqq \operatorname{Spec}(\mathcal{O}_{X,y})$ *and* $\pi_y \colon \operatorname{ZR}(X_y) \to X_y$, *the natural morphism*

$$H^i_{\{y\}}(X_y, \mathcal{O}_{X_y}) \xrightarrow{\delta^i_{\pi_y}} H^i_{\pi_y^{-1}(\{y\})}\big(\operatorname{ZR}(X_y), \mathcal{O}_{\operatorname{ZR}(X_y)}\big)$$

is injective for every i.

(*iv*) X *is locally pseudo-rational.*

(*v*) X *is locally quasi-excellent and has rational singularities.*

Then, we have the following implications:

$$(i) \Longrightarrow (ii) \Longrightarrow (iii) \Longleftarrow (iv) \Longleftrightarrow (v)$$

equal characteristic zero (over $(i) \Rightarrow (ii)$)

equal characteristic zero + locally quasi-excellent (over $(iv) \Leftrightarrow (v)$)

equal characteristic zero + locally analytically unramified (over $(iii) \Leftarrow (iv)$)

Proof. The implication (*i*) ⇒ (*ii*) follows by definition of a quasi-isomorphism, and the implication (*ii*) ⇒ (*iii*) holds because the morphism δ^d_π also admits a left inverse.

We now show (*iv*) ⇒ (*iii*). Injectivity for $i < \dim(X_y)$ holds since pseudo-rational implies Cohen–Macaulay (see Definition 2.9) and hence

$$H^i_{\{y\}}(X_y, \mathcal{O}_{X_y}) = 0$$

for all $i < \dim(X_y)$. For $i = \dim(X_y)$, we apply Lemma 6.1.

We now show (*iii*) ⇒ (*i*) in equal characteristic zero. By Noetherian induction and the fact that quasi-isomorphisms can be checked locally, it suffices to show that if $y \in X$ is a point and $\mathcal{O}_X \to \mathbf{R}\pi_*\mathcal{O}_{\operatorname{ZR}(X)}$ is a quasi-isomorphism at η for every proper generization $\eta \rightsquigarrow y$, then $\mathcal{O}_X \to \mathbf{R}\pi_*\mathcal{O}_{\operatorname{ZR}(X)}$ is a quasi-isomorphism at y. Replacing X by $\operatorname{Spec}(\mathcal{O}_{X,y})$, we may assume that X is the spectrum of a local $\mathbf{Q}$-algebra $(R, \mathfrak{m})$ such that $\mathcal{O}_X \to \mathbf{R}\pi_*\mathcal{O}_{\operatorname{ZR}(X)}$ is a quasi-isomorphism away from $\{\mathfrak{m}\}$.

For every $i < \dim(X)$, we know the morphisms

$$H^i_{\{y\}}(X, \mathcal{O}_X) \xrightarrow{\delta^i_\pi} H^i_{\pi^{-1}(\{y\})}\big(\operatorname{ZR}(X), \mathcal{O}_{\operatorname{ZR}(X)}\big)$$

are isomorphisms by combining (*iii*) and the fact that the modules on the right vanish by Theorem 5.1(*i*) applied to $\mathscr{L} = \mathcal{O}_X$. Moreover, this morphism is also an isomorphism for $i = \dim(X)$ since it is injective by (*iii*) and surjective by [LT81, Remark (*b*) on p. 103]. By the long exact sequence on local cohomology associated to the exact triangle

$$\mathcal{O}_X \longrightarrow \mathbf{R}\pi_*\mathcal{O}_{\operatorname{ZR}(X)} \longrightarrow C^\bullet \xrightarrow{+1}$$

where $C^\bullet \coloneqq \operatorname{Cone}(\mathcal{O}_X \to \mathbf{R}\pi_*\mathcal{O}_{\operatorname{ZR}(X)})$, we therefore see that $\mathbf{H}^i_{\{y\}}(X, C^\bullet) = 0$ for all i. Now consider the long exact sequence

$$\cdots \longrightarrow \mathbf{H}^i_{\{y\}}(X, C^\bullet) \longrightarrow \mathbf{H}^i(X, C^\bullet) \longrightarrow \mathbf{H}^i\big(X - \{y\}, C^\bullet\big) \longrightarrow \cdots.$$

Since $\mathbf{H}^i(X - \{y\}, C^\bullet) = 0$ for all i by the inductive hypothesis, we see that $\mathbf{H}^i(X, C^\bullet) = 0$ for all i as well, i.e., (*i*) holds.

We now show (*iii*) ⇒ (*iv*) in equal characteristic zero if the local rings of X are analytically unramified. First, (*ii*) implies the injection $\mathcal{O}_X \hookrightarrow f_*\mathcal{O}_W$ is an isomorphism for every finite birational morphism $f \colon W \to X$, and hence X is normal. Next, X is Cohen–Macaulay since (*iii*) implies we have the injections

$$H^i_{\{y\}}(X_y, \mathcal{O}_{X_y}) \overset{\delta^i_{\pi_y}}{\hookrightarrow} H^i_{\pi_y^{-1}(\{y\})}\big(\operatorname{ZR}(X_y), \mathcal{O}_{\operatorname{ZR}(X_y)}\big) = 0$$

where the vanishing holds by Theorem 5.1(i) applied to $\mathscr{L} = \mathcal{O}_X$. Finally, injectivity at $i = d$ holds by Lemma 6.1.

It remains to show (iv) $\Leftrightarrow$ (v) in equal characteristic zero if X is locally quasi-excellent. We have (iv) $\Leftarrow$ (i) (quasi-excellent reduced rings are analytically unramified by [Mat89, Corollary to Theorem 23.9]), and hence it suffices to show (i) $\Leftrightarrow$ (v). But this follows from Theorem 3.12 since the inverse system of blowups $W \to X_y$ with W regular is coinitial with the inverse system defining $\mathrm{ZR}(X_y)$ by resolution of singularities [Hir64, Chapter I, §3, Main Theorem I(n)] since the composition of blowups is also a blowup [RG71, Première partie, Lemme 5.1.4] (see also [Con07, Lemma 1.2]). □

Since regular local rings are pseudo-rational [LT81, §4], we see that (i) holds for regular rings in equal characteristic zero. This relative vanishing statement also holds for varieties of dimension ≤ 4 over algebraically closed fields of arbitrary characteristic using a recent result of Kovács [Kov, Theorem 8.6]. The result below will not be used in the sequel.

Theorem 6.3. *Let X be a locally pseudo-rational excellent Noetherian scheme. Assume that X is of equal characteristic zero or a quasi-projective variety of dimension ≤ 4 over an algebraically closed field of arbitrary characteristic. Denote by $\pi\colon \mathrm{ZR}(X) \to X$ the canonical projection morphism from the Zariski–Riemann space of X. Then, we have*

$$R^i\pi_*\mathcal{O}_{\mathrm{ZR}(X)} = 0$$

for all $i > 0$.

Proof. In equal characteristic zero, the desired vanishing holds by Theorem 6.2. We give an alternative proof here in parallel with the dimension ≤ 4 case. By Theorem 3.12, it suffices to show that for a cofinal subset of ideals $\{\mathscr{I}_\lambda\}_{\lambda\in\Lambda} \subseteq \mathrm{AId}_X$, the blowup $\pi_{\mathscr{I}_\lambda}\colon X_{\mathscr{I}_\lambda} \to X$ along $\mathscr{I}_\lambda$ satisfies $R^i\pi_{\mathscr{I}_\lambda *}\mathcal{O}_{X_{\mathscr{I}_\lambda}} = 0$ for all $i > 0$.

Let $X_{\mathscr{I}} \to X$ be a blowup where $\mathscr{I} \in \mathrm{AId}_X$. Since $X_{\mathscr{I}}$ is an excellent scheme in the equal characteristic zero case [EGAIV$_2$, Proposition 7.8.6(i)] or a quasi-projective variety in the dimension ≤ 4 case, there exists a birational blowup $\tilde{X} \to X_{\mathscr{I}}$ such that $\tilde{X}$ is regular in the equal characteristic zero case [Tem08, Theorem 1.1] or such that $\tilde{X}$ is normal and Cohen–Macaulay in the dimension ≤ 4 case [Bro83, Corollary 1.8]. By [RG71, Première partie, Lemme 5.1.4] (see also [Con07, Lemma 1.2]), the composition

$$\tilde{\pi}\colon \tilde{X} \longrightarrow X_{\mathscr{I}} \longrightarrow X$$

can be written as the blowup of X along a coherent ideal sheaf $\mathscr{I}_\lambda \in \mathrm{AId}_X$ contained in $\mathscr{I}$. We have $R^i\tilde{\pi}_*\mathcal{O}_{\tilde{X}} = 0$ by [CR15, Theorem 1.1] in equal characteristic zero case and by [Kov, Theorem 8.6] in the dimension ≤ 4 case. □

Remark 6.4. Only the vanishing statements proved so far are used in the proof of Theorem C. If one is interested in the proof of Theorem C, they can proceed directly to reading §9.1.

7. Relative vanishing and injectivity theorems for schemes

We are now ready to prove the following dual version of Theorem B. Theorem B* below implies Theorem B by Proposition 5.3.

Theorem B*. *Let $f\colon X \to Y$ be a proper maximally dominating morphism of locally Noetherian schemes of equal characteristic zero such that X is locally pseudo-rational.*

Consider an invertible sheaf $\mathscr{L}$ on X. For every $y \in Y$, denote by $f_y\colon X_y \to \mathrm{Spec}(\mathcal{O}_{Y,y})$ the base change of f along $\mathrm{Spec}(\mathcal{O}_{Y,y}) \to Y$, denote by $\mathscr{L}_y$ the pullback of $\mathscr{L}$ to X_y, and set $Z_y = f_y^{-1}(y)$.

(i) *Suppose $\mathscr{L}$ is f-big and f-semi-ample. Then, we have*

$$H^i_{Z_y}(X_y, \mathscr{L}_y^{-1}) = 0$$

for every $y \in Y$ and for all $i < \dim(X_y)$.

(*ii*) *Suppose $\mathscr{L}$ is f-semi-ample. Let D be an effective Weil divisor on X for which there exists an integer $n > 0$ and an effective Weil divisor D' such that $\mathcal{O}_X(D + D') \simeq \mathscr{L}^{\otimes n}$. THen, the canonical morphisms*

$$H^i_{Z_y}\big(X_y, \mathscr{L}_y^{-k}(-D_y)\big) \longrightarrow H^i_{Z_y}(X_y, \mathscr{L}_y^{-k})$$

induced by the inclusion $\mathcal{O}_{X_y}(-D_y) \hookrightarrow \mathcal{O}_{X_y}$ are surjective for every $y \in Y$ for all i and all $k > 0$, where D_y is the pullback of D to X_y.

Proof. For (*ii*), since the map $\mathcal{O}_X \to \mathcal{O}_X(D)$ factors the map $\mathcal{O}_X \to \mathcal{O}_X(D + D')$, we can replace D by $D + D'$ to assume that $\mathcal{O}_X(D) \simeq \mathscr{L}^{\otimes n}$, and in particular, we may assume that D is Cartier.

We proceed in a sequence of steps.

Step 1. *Reduction to the case when X and Y are integral.*

Consider the Stein factorization

$$X \xrightarrow{f'} Y' \xrightarrow{g} Y$$

of f. By Incomparability applied to the finite morphism g, the points in Y' lying over maximal points of Y must be maximal, and hence f' is maximally dominating. Since Y' is normal, it is the disjoint union of integral normal schemes. We claim we may work with one connected component of X and Y' at a time to assume that f is a surjective morphism of integral schemes. For each $y \in Y$, we have a decomposition

$$f_y^{-1}(\{y\}) = \bigsqcup_{y' \in g^{-1}(\{y\})} f_{y'}'^{-1}(\{y'\}) \tag{14}$$

into connected components by [EGAIII$_1$, Corollaire 4.3.3]. By the Mayer–Vietoris sequence (see [Har75, Proof of Proposition 4.2]), we then have a decomposition of functors

$$H^i_{Z_y}(X_y, -) \simeq \bigoplus_{y' \in g^{-1}(\{y\})} H^i_{Z_{y'}}(X_{y'}, -)$$

where $X_{y'} \coloneqq X \times_{Y'} \operatorname{Spec}(\mathcal{O}_{Y',y'})$ and $Z_{y'} = f_{y'}'^{-1}(\{y'\})$. Since each $f_{y'}'^{-1}(\{y'\})$ lies in a unique connected component of $X \times_{Y'} \operatorname{Spec}(\mathcal{O}_{Y',y'})$, we can use Excision [Gro67, Proposition 1.3] to replace X and Y by the connected components of X and Y' to assume that both X and Y are integral. We note that $\mathscr{L}$ is f'-semi-ample by [CT20, Lemma 2.10], and in case (*i*) is f'-big since the decomposition (14) also holds for maximal points $y \in Y$.

Step 2. *Conclusion of proof.*

By Step 1, we may assume that X and Y are integral. Since the desired vanishing is a local condition, we may replace Y by $\operatorname{Spec}(\mathcal{O}_{Y,y})$ and X by X_y. Denote by $\pi\colon \operatorname{ZR}(X) \to X$ the canonical projection morphism from the Zariski–Riemann space of X. By Theorem 5.1, we know that

$$H^i_{\pi^{-1}(Z)}\big(\operatorname{ZR}(X), \pi^*\mathscr{L}^{-1}\big) = 0$$

for all $i < \dim(X)$ and that

$$H^i_{\pi^{-1}(Z)}\big(\operatorname{ZR}(X), \pi^*\big(\mathscr{L}^{-k}(-D)\big)\big) \longrightarrow H^i_{\pi^{-1}(Z)}\big(\operatorname{ZR}(X), \pi^*\mathscr{L}^{-k}\big)$$

is surjective for all i, respectively. Next, since $\mathcal{O}_X \to \pi_*\mathcal{O}_{\operatorname{ZR}(X)}$ is an isomorphism and

$$R^j\pi_*\mathcal{O}_{\operatorname{ZR}(X)} = 0$$

for all $j > 0$ by Theorem 6.2, the projection formula [EGAIII$_1$, Chapitre 0, Proposition 12.2.3] combined with the Grothendieck spectral sequence [Gro67, Corollary 5.6] imply the desired vanishing and surjectivity on X. □

8. Relative vanishing and injectivity theorems for klt pairs

Our goal in this section is to deduce our version of the Kawamata–Viehweg vanishing theorem and Kollár's injectivity theorem for klt pairs (Theorem A) from Theorem B. To do so, we establish a covering lemma in §8.1. We then prove the dual version of Theorem A for normal crossings pairs (Theorem 8.2) in §8.2, and then prove Theorem A itself in §8.3.

8.1. **A covering lemma.** We start with the following version of Kawamata's covering lemma (cf. [Kaw81, Theorem 17]).

Lemma 8.1 (cf. [EV92, Lemma 3.19])**.** *Let X be an integral regular scheme projective over an integral Noetherian local $\mathbf{Q}$-algebra $(R, \mathfrak{m})$. Let*

$$D = \sum_{j=1}^{r} D_j$$

be a reduced simple normal crossings divisor and let $N_1, N_2, \ldots, N_r$ be positive integers. Then, there exists a finite surjective morphism $\tau\colon W \to X$ from a regular integral scheme W such that

(a) *We have $\tau^* D_j = N_j \cdot (\tau^* D_j)_{\mathrm{red}}$ for every $j \in \{1, 2, \ldots, r\}$.*
(b) *$\tau^* D$ is a simple normal crossings divisor.*
(c) *The degree of τ divides some power of $\prod_{j=1}^{r} N_j$.*

Proof. We adapt the proof of [EV92, Lemma 3.19]. Write

$$D = \sum_{j=1}^{r} D_j$$

as a sum of regular divisors that are possibly disconnected. We construct W inductively. It therefore suffices to prove the case when $N_1 > 1$ and $N_2 = N_3 = \cdots = N_r = 1$.

Let $f\colon X \to \operatorname{Spec}(R)$ be the structure morphism for X. Let $\mathscr{A}'$ be an f-very ample invertible sheaf on X, which exists since f is projective. Then, there exists an integer $n > 0$ such that $\mathscr{A}'^{\otimes N}(-D_1)$ is f-generated by [EGAII, Proposition 2.6.8(i)]. Letting $m > 0$ be an integer such that $N_1 \mid (n+m)$ and setting

$$\mathscr{A} \coloneqq \mathscr{A}'^{\otimes (n+m)/N_1},$$

we see that $\mathscr{A}^{\otimes N_1}(-D_1)$ is f-very ample by [EGAII, Proposition 4.4.8].

We now construct divisors $H_1, H_2, \ldots, H_{\dim(X)}$ as follows. Since $\mathscr{A}^{\otimes N_1}(-D_1)$ is f-very ample, there is a closed immersion $i\colon X \hookrightarrow \mathbf{P}^N_R$ over R such that $i^*\mathcal{O}(1) \simeq \mathscr{A}^{\otimes N_1}(-D_1)$. We now choose $\dim(X)$ general irreducible divisors

$$H_1, H_2, \ldots, H_{\dim(X)} \in \left|\mathscr{A}^{\otimes N_1}(-D_1)\right|$$

as the vanishing of sections in $H^0(\mathbf{P}^N_R, \mathcal{O}(1))$ such that $D + \sum_i H_i$ is a simple normal crossings divisor using the Bertini theorem in [BMPSTWW23, Theorem 2.17 and Remark 2.18]. Since there are $\dim(X) + 1$ divisors in the set $\{H_1, H_2, \ldots, H_{\dim(X)}, D_1\}$, we know that

$$\left(\bigcap_{i=1}^{\dim(X)} H_i\right) \cap D_1 = \emptyset. \tag{15}$$

Let $\tau_i\colon W_i \to X$ be the cyclic cover obtained by taking the N_1-th root out of $H_i + D_1$, which satisfies $\mathcal{O}_X(H_i + D_1) \simeq \mathscr{A}^{\otimes N_1}$ by [EV92, (3.5)]. Then, (a) and (c) are satisfied by construction, but W_i may be singular over $H_i \cap D_1$ and $\tau_i^*(D)$ may not have simple normal crossings over $H_i \cap D_1$. To fix this, let W be the normalization of

$$W_1 \times_X W_1 \times_X \cdots \times_X W_{\dim(X)}.$$

To show that W is regular and that the pullback of D to W is a simple normal crossings divisor, we describe an alternative inductive construction of W. Let $W^{(\nu)}$ be the normalization of

$$W_1 \times_X W_1 \times_X \cdots \times_X W_\nu$$

and consider the composition

$$\tau^{(\nu)} \colon W^{(\nu)} \longrightarrow W_1 \times_X W_1 \times_X \cdots \times_X W_\nu \longrightarrow X.$$

Outside of the singular locus of $W^{(\nu)}$, we see that $W^{(\nu+1)}$ is obtained from $W^{(\nu)}$ by taking the N_1-th root out of

$$\tau^{(\nu)*}(H_{\nu+1} + D_1) = \tau^{(\nu)*}(H_{\nu+1}) + N_1 \cdot (\tau^{(\nu)*} D_1)_{\mathrm{red}}.$$

This is the same as taking the N_1-th root out of $\tau^{(\nu)*}(H_{\nu+1})$ by [EV92, Remark 3.3(b) and Corollary 3.11]. Since $\tau^{(\nu)*}(H_{\nu+1})$ has no singularities, [EV92, Lemma 3.15] implies the singularities of $W^{(\nu+1)}$ lie over the singularities of $W^{(\nu)}$, and hence inductively over $H_1 \cap D_1$. However, since W is independent of the numbering of the H_i, the singularities of W in fact lie over

$$\bigcap_{i=1}^{\dim(X)} (H_i \cap D_1) = \left(\bigcap_{i=1}^{\dim(X)} H_i \right) \cap D_1 = \emptyset$$

by the choice of the H_i in (15). □

8.2. **The Kawamata–Viehweg vanishing theorem and Kollár's injectivity theorem for normal crossings pairs.** Before proving Theorem A, we first prove the Kawamata–Viehweg vanishing theorem and Kollár's injectivity theorem for normal crossings divisors on regular schemes of equal characteristic zero. We follow the proof in [KM98, Theorem 2.64].

In the statement below, (*i*) is a generalization of [Kol11, Corollary 20], and (*ii*) is a (dual) version of [Kaw85, Theorem 3.2; EV87, Corollaire 1.11]. Theorem 8.2 below implies the regular case of Theorem A by Proposition 5.3.

Theorem 8.2. *Let $f \colon X \to Y$ be a proper maximally dominating morphism of locally Noetherian schemes of equal characteristic zero such that X is regular.*

Let Δ be a $\mathbf{Q}$-divisor on X with normal crossings support such that $\lfloor \Delta \rfloor = 0$. Consider a divisor L on X such that $L \sim_{\mathbf{Q}} M + \Delta$ for a $\mathbf{Q}$-divisor M on X. For every $y \in Y$, denote by $f_y \colon X_y \to \operatorname{Spec}(\mathcal{O}_{Y,y})$ the base change of f along $\operatorname{Spec}(\mathcal{O}_{Y,y}) \to Y$, denote by L_y the pullback of L to X_y, and set $Z_y = f_y^{-1}(y)$.

(*i*) *Suppose M is f-nef and f-big. Then, we have*

$$H^i_{Z_y}\big(X_y, \mathcal{O}_{X_y}(-L_y)\big) = 0$$

for every $y \in Y$ and for all $i < \dim(X_y)$.

(*ii*) *Suppose M is f-semi-ample. Let D be an effective divisor on X for which there exists an integer $n > 0$ such that nM is Cartier and an effective divisor D' on X such that $\mathcal{O}_X(D + D') \simeq \mathcal{O}_X(nM)$. Then, the canonical morphisms*

$$H^i_{Z_y}\big(X_y, \mathcal{O}_{X_y}(-L_y - D_y)\big) \longrightarrow H^i_{Z_y}\big(X_y, \mathcal{O}_{X_y}(-L_y)\big)$$

induced by the inclusion $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(D)$ are surjective for every $y \in Y$ for all i, where D_y is the pullback of D to X_y.

Proof. We proceed in a sequence of steps.

Step 1. *It suffices to show that when f is surjective, X is integral, and $Y = \operatorname{Spec}(R)$ for an excellent local domain $(R, \mathfrak{m})$ with a dualizing complex $\omega_R^\bullet$, setting $Z = f^{-1}(\{\mathfrak{m}\})$, we have*

$$H^i_Z\big(X, \mathcal{O}_X(-L)\big) = 0$$

for all $i < \dim(X)$ *for* (*i*) *and*

$$H^i_Z\big(X, \mathcal{O}_X(-L-D)\big) \longrightarrow H^i_Z\big(X, \mathcal{O}_X(-L)\big)$$

are surjective for all i *for* (*ii*).

The desired vanishing and surjectivity are local conditions, and hence we can fix $y \in Y$. We first claim we may replace f by its base change $\hat{f}_y \colon \hat{X}_y \to \operatorname{Spec}(\hat{\mathcal{O}}_{Y,y})$ along the morphism $\operatorname{Spec}(\hat{\mathcal{O}}_{Y,y}) \to Y$ for each $y \in Y$. Note that the pullback $\hat{\Delta}_y$ of Δ_y to $\hat{X}_y$ satisfies $\lfloor \hat{\Delta} \rfloor = 0$ and has normal crossings support by applying [EGAIV$_2$, Lemme 7.9.3.1] to each stratum of an étale cover of (X, Δ) where the pullback of Δ has simple normal crossings support. Here we note that maximal ideals extend to maximal ideals for étale local maps of local rings [EGAIV$_4$, Théorème 17.6.1], and hence taking étale covers is compatible with completion. The pullback of M to $\hat{X}_y$ is $\hat{f}_y$-nef [Kee03, Lemma 2.20(1)] for (*i*) and $\hat{f}_y$-semi-ample [CT20, Lemma 2.12] for (*ii*). The morphism $\hat{f}_y$ is proper and maximally dominating by flat base change [ILO14, Exposé II, Proposition 1.1.5], the ring $\hat{\mathcal{O}}_{Y,y}$ is excellent by [EGAIV$_2$, Scholie 7.8.3(*iii*)], and the vanishing on local cohomology for f descends from that on $\hat{f}$ by faithfully flat base change [HO08, Theorem 6.10].

To show the pullback of M to $\hat{X}_y$ is $\hat{f}_y$-big for (*i*), let $n > 0$ be an integer such that nM induces a generically finite morphism on every generic fiber of f. Then, by transitivity of fibers [EGAI$_{\text{new}}$, Corollaire 3.4.9], the compatibility of maps induced by linear systems and flat base change [EGAII, (4.2.10)], and the fact that generically finite maps are stable under flat base change (combine [ILO14, Exposé II, Proposition 1.1.5] and the characterization of generically finite morphisms in [ILO14, Exposé II, Proposition 1.1.7]), we see that the pullback of nM to $\hat{X}_y$ induces generically finite morphisms along the generic fibers of $\hat{f}_y$.

Finally, we repeat the argument of Step 1 of the proof of Theorem B* to reduce to the case when X and Y are integral. Note that the excellence of $\mathcal{O}_{Y,y}$ is not lost by [EGAIV$_2$, Scholie 7.8.3].

Step 2. *The vanishing (resp. surjectivity) in Step 1 holds when* Δ *has simple normal crossings support and* M *is* f*-big and* f*-semi-ample (resp.* f*-semi-ample).*

By Theorem B*, it suffices to reduce to the case when $\Delta = 0$. The idea is to induce on the number of components in Δ, which we do by showing the following more general result:

Claim 8.2.1. *Let* X *be an integral regular scheme projective over a Noetherian local domain* $(R, \mathfrak{m})$ *containing* **Q**. *Let* L *be a Cartier divisor on* X *such that*

$$L \sim_{\mathbf{Q}} M + \sum_{j=1}^{r} a_j D_j,$$

where the D_j *are regular (possibly disconnected) divisors,* $\sum_j D_j$ *is a simple normal crossings divisor, and the* a_j *are rational numbers in* $[0, 1)$. *Then, there is a finite surjective morphism* $p \colon W \to X$ *from a regular integral scheme* W *and a divisor* M_W *on* W *such that* $M_W \sim_{\mathbf{Q}} p^*M$ *and such that* $\mathcal{O}_X(-L)$ *is a direct summand of* $p_*\mathcal{O}_W(-M_W)$.

Proof of Claim 8.2.1. We proceed by induction on r. Write $a_1 = b/m$, where m is a positive integer. By Lemma 8.1 applied to $\sum_j D_j$, $N_1 = m$, and $N_2 = N_3 = \cdots = N_r = 1$, there exists a finite surjective morphism $p_1 \colon X_1 \to X$ such that $p_1^*D_1 \sim mD'$ for some divisor D' on X_1. Moreover, each $p_1^*D_j$ is regular and $\sum_j p_1^*D_j$ is a simple normal crossings divisor. By [KM98, Theorem 2.64, Step 1] (see also [EV92, Corollary 3.11]), the canonical morphism $\mathcal{O}_X \to p_{1*}\mathcal{O}_{X_1}$ splits, and hence $\mathcal{O}_X(-L) \to p_{1*}\mathcal{O}_{X_1}(-p_1^*L)$ also splits.

Now D_1 corresponds to a section of $\mathcal{O}_{X_1}(mD')$, and hence we can take the associated m-th cyclic cover $p_2 \colon X_2 \to X_1$ as in [KM98, Definition 2.50] (see also [EV92, (3.5)]). Then, [KM98, Lemma

2.51] (see also [EV92, Lemma 3.15(b)]) implies that X_2 is regular, the $p_2^*D_j$ are regular, and $\sum_j p_1^*D_j$ is a simple normal crossings divisor. We have the decompositions

$$p_{2*}\mathcal{O}_{X_2} = \bigoplus_{\ell=0}^{m-1} \mathcal{O}_{X_1}(-\ell D'),$$
$$p_{2*}\mathcal{O}_{X_2}(-p_2^*p_1^*L + b\,p_2^*D') = \bigoplus_{\ell=0}^{m-1} \mathcal{O}_{X_1}\big(-p_1^*L + (b-\ell)D'\big).$$

The $\ell = b$ summand shows that $\mathcal{O}_{X_1}(-p_1^*L)$ is a direct summand of $p_{2*}\mathcal{O}_{X_2}(-p_2^*p_1^*L + b\,p_2^*D')$.

We now have the $\mathbf{Q}$-linear equivalence

$$p_2^*p_1^*L - b\,p_2^*D' \sim_{\mathbf{Q}} p_2^*p_1^*M + \sum_{j=2}^{r} a_j\,p_2^*p_1^*D_j,$$

which satisfies the hypotheses of Claim 8.2.1. By the inductive hypothesis, there exists a finite surjective morphism $W \to X_2$ satisfying the conclusion of Claim 8.2.1 for X_2. The composition $W \to X_2 \to X$ then satisfies the conclusion of Claim 8.2.1 for X. □

We now construct a finite surjective morphism $p\colon Z \to X$ as in Claim 8.2.1 to prove the special case in Step 2.

To prove the vanishing in Step 2, we have an injection

$$H^i_Z\big(X, \mathcal{O}_X(-L)\big) \hookrightarrow H^i_{p^{-1}(Z)}\big(W, \mathcal{O}_W(-M_W)\big).$$

Since $M_W \sim_{\mathbf{Q}} p^*M$, it is $(f\circ p)$-big (by [LM, Lemma 5.10]) and $(f\circ p)$-semi-ample (by [CT20, Lemma 2.11(i)]). Since regular local rings are pseudo-rational [LT81, §4], we see the right-hand side vanishes by Theorem B*(i).

To prove the surjectivity in Step 2, we have the commutative diagram

$$\begin{array}{ccc} H^i_Z\big(X, \mathcal{O}_X(-L-D)\big) & \longrightarrow & H^i_Z\big(X, \mathcal{O}_X(-L)\big) \\ \big\downarrow\big\uparrow & & \big\downarrow\big\uparrow \\ H^i_{p^{-1}(Z)}\big(W, \mathcal{O}_W(-M_W - p^*D)\big) & \longrightarrow & H^i_{p^{-1}(Z)}\big(W, \mathcal{O}_W(-M_W)\big) \end{array}$$

where the surjective arrows pointing upwards are induced by the projection coming from the split injection in Claim 8.2.1. Since $M_W \sim_{\mathbf{Q}} p^*M$, it is f-semi-ample by [CT20, Lemma 2.11(i)]. Since regular local rings are pseudo-rational [LT81, §4], we see the bottom horizontal arrow is surjective by Theorem B*(ii). The commutativity of the diagram implies the top horizontal arrow is also surjective.

Step 3. *Conclusion of proof for (i).*

We first find a log resolution $g\colon \tilde{X} \to X$ on which we can write

$$g^*M \sim_{\mathbf{Q}} A + G$$

where A is an $(f\circ g)$-ample $\mathbf{Q}$-divisor on $\tilde{X}$ and G is an effective $\mathbf{Q}$-divisor on $\tilde{X}$ such that $G \cup \mathrm{Exc}(g) \cup g_*^{-1}\Delta$ has simple normal crossings support and the coefficients on G are arbitrarily small. Applying Chow's lemma [EGAII, Théorème 5.6.1] and then taking a log resolution using [Hir64, Chapter I, §3, Main Theorem I(n)], we can find a projective log resolution $g_1\colon X_1 \to X$ of the pair (X, Δ) such that $f\circ g_1$ is projective. Then, we know that g_1^*M is $(f\circ g_1)$-big (by [LM, Lemma 5.10]) and $(f\circ g_1)$-nef (by [Kee03, Lemma 2.17(1)]). By Kodaira's lemma [LM, Corollary 5.9], we

can write $g_1^*M \sim_{\mathbf{Q}} A + E$ where A is an $(f \circ g_1)$-ample $\mathbf{Q}$-divisor and E is an effective $\mathbf{Q}$-divisor. Let $g_2 \colon \tilde{X} \to X_1$ be a log resolution of $(X_1, E + g_1^*\Delta)$ and consider the composition

$$g \colon \tilde{X} \xrightarrow{g_2} X_1 \xrightarrow{g_1} X.$$

Then, we have $g_2^*M \sim_{\mathbf{Q}} g_2^*A + g_2^*E$ and $g_2^*E \cup \operatorname{Exc}(g) \cup g_*^{-1}\Delta$ has simple normal crossings support. Since g_2 is constructed as a blowup of X_1 along regular centers, there exists an effective g_2-exceptional $\mathbf{Q}$-divisor F such that $-F$ is g-ample. After possibly replacing F by a small rational multiple, we therefore see that $g^*M - F$ is $(f \circ g)$-ample by [EGAII, Proposition 4.6.13(ii)] and $g_2^*E \cup F \cup \operatorname{Exc}(g) \cup g_*^{-1}(\Delta)$ is a $\mathbf{Q}$-divisor with simple normal crossings support. Finally, for every integer $k \geq 0$, we can write

$$g^*M \sim_{\mathbf{Q}} \frac{1}{k+1}(k\,g^*M + g^*M - F) + \frac{1}{k+1}F$$

where $k\,g^*M + g^*M - F$ is $(f \circ g)$-ample by the openness of the relative ample cone [Kee03, Theorem 3.9; Kee18, Theorem E2.2]. By taking k large, we can therefore set $A = \frac{1}{k+1}(k\,g^*M + g^*M - F)$ and $G = \frac{1}{k+1}F$ to assume that the coefficients on G are arbitrarily small.

Now let $\omega_Y^\bullet$ be a dualizing complex on Y, which exists by [EGAIV$_2$, Scholie 7.8.3(iii)]. Let ω_X and $\omega_{\tilde{X}}$ be canonical sheaves constructed by taking the unique cohomology sheaves of $f^!\omega_Y^\bullet$ and $(f \circ g)^!\omega_Y^\bullet$, respectively, and let K_X and $K_{\tilde{X}}$ be associated canonical divisors (see [LM, Definition 6.2]). We need to show that

$$H^i_Z\big(X, \mathcal{O}_X(-L)\big) = 0$$

for all $i < \dim(X)$, which by Proposition 5.3 is implied by

$$R^i f_*\big(\omega_X(L)\big) = 0$$

for all $i > 0$.

As in [Kol13, Notation 2.6], write

$$K_{\tilde{X}} + g_*^{-1}\Delta \sim_{\mathbf{Q}} g^*(K_X + \Delta) + \sum_i b_i G_i$$

where the coefficients $b_i \in \mathbf{Q}$ satisfy $b_i > -1$ for all i since the pair (X, Δ) is klt [Kol13, Corollary 2.13 and Proposition 2.15] and the G_i are g-exceptional. We then have

$$\begin{aligned} K_{\tilde{X}} + A + g_*^{-1}\Delta + G + \sum_i \big(\lceil b_i \rceil - b_i\big)G_i &\sim_{\mathbf{Q}} g^*(K_X + \Delta) + A + G + \sum_i \lceil b_i \rceil G_i \\ &\sim_{\mathbf{Q}} g^*K_X + g^*L + \sum_i \lceil b_i \rceil G_i. \end{aligned}$$

We therefore see that the divisor

$$\tilde{L} \coloneqq \big(g^*K_X - K_{\tilde{X}}\big) + g^*L + \sum_i \lceil b_i \rceil G_i$$

satisfies

$$\tilde{L} \sim_{\mathbf{Q}} A + g_*^{-1}\Delta + G + \sum_i \big(\lceil b_i \rceil - b_i\big)G_i$$

which is the sum of the $(f \circ g)$-ample $\mathbf{Q}$-divisor A and an effective $\mathbf{Q}$-divisor with simple normal crossings support and coefficients in $[0, 1)$. Since A is also g-ample by [EGAII, Proposition 4.6.13(v)], Step 2 and Proposition 5.3(i) imply

$$R^i(f \circ g)_*\big(\omega_{\tilde{X}}(\tilde{L})\big) = 0 \qquad \text{and} \qquad R^i g_*\big(\omega_{\tilde{X}}(\tilde{L})\big) = 0 \tag{16}$$

for all $i > 0$.

We are now ready to show the theorem in case (*i*). Consider the Leray spectral sequence

$$E_2^{i,j} = R^i f_*\big(R^j g_*\big(\omega_{\tilde{X}}(\tilde{L})\big) \Rightarrow R^{i+j}(f\circ g)_*\big(\omega_{\tilde{X}}(\tilde{L})\big).$$

By (16), the E_2 page of this spectral sequence is concentrated in the row $j=0$. We therefore have an isomorphism

$$R^i f_*\big(g_*\big(\omega_{\tilde{X}}(\tilde{L})\big)\big) \simeq R^i(f\circ g)_*\big(\omega_{\tilde{X}}(\tilde{L})\big) = 0$$

for all $i>0$ using (16) for the vanishing on the right-hand side. Now by definition of $\tilde{L}$ and the projection formula [EGAIII$_1$, Chapitre 0, Proposition 12.2.3], we have

$$g_*\big(\omega_{\tilde{X}}(\tilde{L})\big) \simeq \omega_X(L)\otimes_{\mathcal{O}_X} g_*\left(\mathcal{O}_{\tilde{X}}\left(\sum_i \lceil b_i\rceil G_i\right)\right) \simeq \omega_X(L)$$

since the G_i are g-exceptional (see [Laz04a, Example 2.1.16]). We therefore have

$$R^i f_*\big(\omega_X(L)\big) \simeq R^i f_*\big(g_*\big(\omega_{\tilde{X}}(\tilde{L})\big)\big) = 0.$$

Step 4. *Conclusion of proof for* (*ii*).

Let $g\colon \tilde{X}\to X$ be a log resolution of (X,Δ). As in Step 3, the divisor

$$\tilde{L} \coloneqq \big(g^*K_X - K_{\tilde{X}}\big) + g^*L + \sum_i \lceil b_i\rceil G_i$$

satisfies

$$\tilde{L} \sim_{\mathbf{Q}} g^*M + g_*^{-1}\Delta + \sum_i \big(\lceil b_i\rceil - b_i\big)G_i$$

which is the sum of the $\mathbf{Q}$-divisor g^*M, which is $(f\circ g)$-semi-ample by [CT20, Lemma 2.11(*i*)] and satisfies $\mathcal{O}_{\tilde{X}}(g^*D+g^*D') \simeq \mathcal{O}_{\tilde{X}}(n\,g^*M)$, and an effective $\mathbf{Q}$-divisor with simple normal crossings support and coefficients in $[0,1)$. Now applying Step 2 and Proposition 5.3(*ii*) on $\tilde{X}$, the canonical morphisms

$$R^i(f\circ g)_*\big(\omega_{\tilde{X}}(\tilde{L})\big) \longrightarrow R^i(f\circ g)_*\big(\omega_{\tilde{X}}(\tilde{L}+g^*D)\big)$$

are injective for all i. Since g^*M is also g-semi-ample (by [CT20, Lemma 2.10(*i*)]) and g-big, Step 2 and Proposition 5.3(*i*) imply

$$R^i g_*\big(\omega_{\tilde{X}}(\tilde{L})\big) = 0$$

for all $i>0$. The same argument using the Leray spectral sequence as in Step 3 implies that the canonical morphisms

$$R^i f_*\big(\omega_X(L)\big) \longrightarrow R^i f_*\big(\omega_X(L+D)\big)$$

are injective for all i. □

8.3. **The Kawamata–Viehweg vanishing theorem and Kollár's injectivity theorem for klt pairs.** We can now prove Theorem A.

Proof of Theorem A. For (*ii*), since the map $\mathcal{O}_X\to\mathcal{O}_X(D)$ factors the map $\mathcal{O}_X\to\mathcal{O}_X(D+D')$, we can replace D by $D+D'$ to assume that $\mathcal{O}_X(D)\simeq\mathcal{O}_X(nM)$, and in particular, we may assume that D is Cartier.

Let $g\colon\tilde{X}\to X$ be a log resolution of the pair (X,Δ) such that $\mathrm{Exc}(g)\cup g_*^{-1}\Delta$ has simple normal crossings support. Since (X,Δ) is klt, we can write

$$K_{\tilde{X}} + g_*^{-1}\Delta \sim_{\mathbf{Q}} g^*(K_X+\Delta) + \sum_i a_i E_i$$

as in [Kol13, Notation 2.6], where the coefficients $a_i \in \mathbf{Q}$ satisfy $a_i > -1$ for all i, the E_i are exceptional, and $\lfloor g_*^{-1}\Delta \rfloor = 0$. We then have

$$K_{\tilde{X}} + g^*M + g_*^{-1}\Delta + \sum_i \big(\lceil a_i \rceil - a_i\big)E_i \sim_{\mathbf{Q}} g^*(K_X + M + \Delta) + \sum_i \lceil a_i \rceil E_i$$
$$\sim_{\mathbf{Q}} g^*N + \sum_i \lceil a_i \rceil E_i.$$

We therefore see that the divisor

$$\tilde{N} \coloneqq g^*N + \sum_i \lceil a_i \rceil E_i$$

satisfies

$$\tilde{N} \sim_{\mathbf{Q}} K_{\tilde{X}} + g^*M + g_*^{-1}\Delta + \sum_i \big(\lceil a_i \rceil - a_i\big)E_i.$$

In case (i), we have now realized $\tilde{N}$ as $\mathbf{Q}$-linearly equivalent to the sum of $K_{\tilde{X}}$, the $(f \circ g)$-big (by [LM, Lemma 5.10]) and $(f \circ g)$-nef (by [Kee03, Lemma 2.17(1)]) $\mathbf{Q}$-divisor g^*M, and an effective $\mathbf{Q}$-divisor with simple normal crossings support and coefficients in $[0, 1)$. Since g^*M is also g-nef (by the projection formula [Kle05, Proposition B.16]) and g-big, Theorem 8.2(i) and Proposition 5.3(i) imply

$$R^i(f \circ g)_*\big(\mathcal{O}_{\tilde{X}}(\tilde{N})\big) = 0 \qquad \text{and} \qquad R^i g_*\big(\mathcal{O}_{\tilde{X}}(\tilde{N})\big) = 0 \tag{17}$$

for all $i > 0$.

We now consider the Leray spectral sequence

$$E_2^{i,j} = R^i f_*\big(R^j g_*\big(\mathcal{O}_{\tilde{X}}(\tilde{N})\big)\big) \Rightarrow R^{i+j}(f \circ g)_*\big(\mathcal{O}_{\tilde{X}}(\tilde{N})\big).$$

By the vanishing $R^i g_*(\mathcal{O}_{\tilde{X}}(\tilde{N})) = 0$, the E_2 page of this spectral sequence is concentrated in the row $j = 0$. We therefore have a natural isomorphism

$$R^i f_*\big(g_*\big(\mathcal{O}_{\tilde{X}}(\tilde{N})\big)\big) \simeq R^i(f \circ g)_*\big(\mathcal{O}_{\tilde{X}}(\tilde{N})\big) = 0$$

for all $i > 0$ using (17) for the vanishing on the right-hand side. Now by definition of $\tilde{N}$ and the projection formula [EGAIII$_1$, Chapitre 0, Proposition 12.2.3], we have

$$g_*\big(\mathcal{O}_{\tilde{X}}(\tilde{N})\big) \simeq \mathcal{O}_X(N) \otimes_{\mathcal{O}_X} g_*\left(\mathcal{O}_{\tilde{X}}\left(\sum_i \lceil a_i \rceil E_i\right)\right) \simeq \mathcal{O}_X(N)$$

since X is normal and the E_i are g-exceptional (see [Laz04a, Example 2.1.16]). We therefore have

$$R^i f_*\big(\mathcal{O}_X(N)\big) \simeq R^i f_*\big(g_*\big(\mathcal{O}_{\tilde{X}}(\tilde{N})\big)\big) = 0.$$

In case (ii), we have now realized $\tilde{N}$ as $\mathbf{Q}$-linearly equivalent to the sum of $K_{\tilde{X}}$, the $(f \circ g)$-semi-ample (by [CT20, Lemma 2.11(i)]) $\mathbf{Q}$-divisor g^*M that satisfies $\mathcal{O}_{\tilde{X}}(g^*D) \simeq \mathcal{O}_{\tilde{X}}(n\, g^*M)$, and an effective $\mathbf{Q}$-divisor with simple normal crossings support and coefficients in $[0, 1)$. Theorem 8.2(ii) and Proposition 5.3(ii) imply the canonical morphisms

$$R^i(f \circ g)_*\big(\mathcal{O}_{\tilde{X}}(\tilde{N})\big) \longrightarrow R^i(f \circ g)_*\big(\mathcal{O}_{\tilde{X}}(\tilde{N} + g^*D)\big) \tag{18}$$

are injective for all i. Since g^*M is also g-semi-ample (by [CT20, Lemma 2.10(i)]) and g-big, Theorem 8.2 and Proposition 5.3(i) imply

$$R^i g_*\big(\mathcal{O}_{\tilde{X}}(\tilde{N})\big) = 0$$

for all $i > 0$. The same argument using the Leray spectral sequence as in the previous paragraph implies that the canonical morphisms

$$R^i f_*\big(\mathcal{O}_X(N)\big) \longrightarrow R^i f_*\big(\mathcal{O}_X(N + D)\big)$$

are injective for all i. □

Part II. Applications

9. Rational singularities

In this section, we apply Theorem 5.1 to study rational singularities. An interesting aspect of our proofs is that we can use the Zariski–Riemann space to replace resolutions of singularities when they may not exist.

We start by proving our version of Boutot's theorem (Theorem C) in §9.1. In §9.2, we prove that pseudo-rationality deforms in equal characteristic zero. We then show a version of Kempf's criterion for rational singularities and pseudo-rational rings and show that derived splinters and rings with rational singularities coincide for quasi-excellent $\mathbf{Q}$-algebras. In the last two subsections, we recall Bernasconi and Kollár's result that says that dlt pairs satisfy local rationality properties, and prove a criterion for the Cohen–Macaulayness of Rees algebras and a Briançon–Skoda theorem for pseudo-rational rings. All these results extend results known for rings essentially of finite type over a field of characteristic zero to the context of (quasi-)excellent $\mathbf{Q}$-algebras.

9.1. **Boutot's theorem.** In this subsection, we prove our version of Boutot's theorem for pseudo-rationality in equal characteristic zero (Theorem C). This solves a conjecture of Boutot [Bou87, Remarque 1 on p. 67] and answers a question of Schoutens [Sch08, (2) on p. 611] in the affirmative.

We have modeled the proof of Theorem C after the proof of the Boutot-type theorem for Du Bois singularities due to Godfrey and the author [GM, Theorem A]. The key insight is that $\mathbf{R}\pi_*\mathcal{O}_{\mathrm{ZR}(X)}$ can play the role the 0-th graded piece of the Deligne–Du Bois complex $\underline{\Omega}^0_X$ did in our results for Du Bois singularities.

We start by showing the following version of [GM, Proposition 3.1]. As before, we will freely use the fact that the formation of $\mathrm{ZR}(X)$ commutes with base change by quasi-compact separated étale morphisms [Stacks, Tag 087B].

Proposition 9.1. *Let $f\colon W \to X$ be a surjective morphism between integral locally Noetherian schemes of equal characteristic zero. Suppose that for every $y \in X$, setting $X_y \coloneqq \mathrm{Spec}(\mathcal{O}_{X,y})$ and denoting by $f_y\colon W_y \to X_y$ the base change of f along $X_y \to X$, the natural morphism*

$$H^i_{\{y\}}(X_y, \mathcal{O}_{X_y}) \longrightarrow \mathbf{H}^i_{\{y\}}(W_y, \mathbf{R}f_{y*}\mathcal{O}_{W_y})$$

is injective. If Y satisfies the condition in Theorem 6.2(iii), then X does also.

Proof. By the functoriality of $\mathrm{ZR}(-)$ for dominant morphisms between integral schemes, we have the commutative diagram

$$\begin{array}{ccc} \mathrm{ZR}(X_y) & \longleftarrow & \mathrm{ZR}(W_y) \\ {\scriptstyle \pi_{X_y}}\downarrow & & \downarrow{\scriptstyle \pi_{W_y}} \\ X_y & \xleftarrow{f_y} & W_y. \end{array}$$

Applying $\mathbf{H}^i_{\{y\}}(X_y, -)$, we obtain the commutative diagram

$$\begin{array}{ccc} \mathbf{H}^i_{\{y\}}\big(X_y, \mathbf{R}\pi_{X_y*}\mathcal{O}_{\mathrm{ZR}(X_y)}\big) & \longrightarrow & \mathbf{H}^i_{\{y\}}\big(X_y, \mathbf{R}(f_y \circ \pi_{W_y})_*\mathcal{O}_{\mathrm{ZR}(W_y)}\big) \\ \uparrow & & \uparrow\wr \\ H^i_{\{y\}}(X_y, \mathcal{O}_{X,y}) & \hookrightarrow & \mathbf{H}^i_{\{y\}}(X_y, \mathbf{R}f_{y*}\mathcal{O}_{W_y}) \end{array}$$

where the right vertical arrow is an isomorphism by Theorem 6.2, and the bottom horizontal arrow is an injection by hypothesis. By the commutativity of the diagram, we see that the left vertical arrow is an injection. Finally, since

$$\mathbf{H}^i_{\{y\}}\big(X_y, \mathbf{R}\pi_{X_y*}\mathcal{O}_{\mathrm{ZR}(X_y)}\big) \simeq H^i_{\pi^{-1}_{X_y}\{y\}}\big(\mathrm{ZR}(X_y), \mathcal{O}_{\mathrm{ZR}(X_y)}\big),$$

we see that X_y is pseudo-rational by Theorem 6.2. □

We can now prove the following Boutot-type theorem for pseudo-rationality. More precisely, we show that the condition in Theorem 6.2(*iii*) descends for morphisms of the type below. Case (*i*) for varieties is due to Kovács [Kov00, Theorem 1] and the case when X is excellent is due to Bernasconi and Kollár [BK23, Proposition 2.14]. Cases (*ii*)–(*iv*) for varieties is due to Boutot [Bou87, Théorème on p. 65]. The author of the present paper previously proved that case (*iii*) holds in arbitrary characteristic as well [Mur22, Proposition 4.20].

Theorem 9.2. *Let $f\colon W \to X$ be a surjective morphism between integral locally Noetherian schemes of equal characteristic zero. Assume one of the following holds:*

(*i*) *The natural morphism $\mathcal{O}_X \to \mathbf{R}f_*\mathcal{O}_W$ admits a left inverse in $D(X)$.*
(*ii*) *f is affine, and for every affine open subset $U \subseteq X$, the $\mathcal{O}_X(U)$-module map $\mathcal{O}_X(U) \to \mathcal{O}_W(f^{-1}(U))$ is pure.*
(*iii*) *f is faithfully flat.*
(*iv*) *f is partially pure at every $y \in X$ in the sense that there is a $w \in W$ such that $f(w) = y$ and the map $\mathcal{O}_{X,y} \to \mathcal{O}_{W,w}$ is pure* [CGM16, p. 38].

If Y satisfies the condition in Theorem 6.2(iii), then X does also.

Proof. Any of the hypotheses above implies the hypothesis in Proposition 9.1 after possibly replacing X and W by maps on spectra of local rings. See the proof of [GM, Theorem 3.2]. □

In [Sch08, (2) on p. 611], Schoutens asked the following: Given a cyclically pure map $R \to R'$ of **Q**-algebras, if R' is locally pseudo-rational, then is R locally pseudo-rational? Schoutens showed that in this situation, if R' is regular, then R is locally pseudo-rational [Sch08, Main Theorem A].

We answer Schoutens's question in the affirmative by showing Theorem C. This result gives a new proof of Schoutens's theorem [Sch08, Main Theorem A].

Proof of Theorem C. By [Has10, Corollary 3.12], R is Noetherian and normal and $R \to R'$ is pure. By Theorem 9.2(*ii*), we know that R satisfies the condition in Theorem 6.2(*iii*). By Theorem 6.2, it therefore suffices to show that $R_{\mathfrak{p}}$ is analytically unramified for every prime ideal $\mathfrak{p} \subseteq R$.

Let $\mathfrak{p} \subseteq R$ be a prime ideal. Then, there exists a maximal ideal $\mathfrak{m} \subseteq R'$ such that the map $R_{\mathfrak{p}} \to R'_{\mathfrak{m}}$ is pure by [HH95, Lemma 2.2], and hence the map $(R_{\mathfrak{p}})^\wedge \to (R'_{\mathfrak{m}})^\wedge$ on completions is pure [And18, Lemme A.2.2]. Since $R'_{\mathfrak{m}}$ is analytically unramified, the completion $(R'_{\mathfrak{m}})^\wedge$ is reduced, and hence $(R_{\mathfrak{p}})^\wedge$ is also reduced.

The "in particular" statement holds since if R' is regular, then it is locally pseudo-rational [LT81, §4]. □

9.2. **Deformation for pseudo-rationality.** We now prove that pseudo-rationality deforms. This result is due to Elkik [Elk78] when R is essentially of finite type over a field of characteristic zero, and to the author [Mur22] when R is a quasi-excellent **Q**-algebra. Here, the key insight is that the Zariski–Riemann space is functorial enough to act as a replacement for the embedded resolutions of singularities used in [Elk78; Mur22].

Theorem 9.3 (cf. [Elk78, Théorème 5; Mur22, Proposition 4.17])**.** *Let $(R, \mathfrak{m})$ be a Noetherian local **Q**-algebra and let $t \in \mathfrak{m}$ be a nonzerodivisor. If R/tR is pseudo-rational, then R is pseudo-rational.*

Proof. First, R is normal by [Sey72, Proposition I.7.4] and Cohen–Macaulay by [Mat89, Theorem 17.3(*ii*)]. Next, t maps to a nonzerodivisor in $\hat{R}$, and $\hat{R}/t\hat{R} \simeq (R/tR)^\wedge$ is reduced, and hence $\hat{R}$ is reduced by [EGAIV$_2$, Proposition 3.4.6].

Set $X \coloneqq \operatorname{Spec}(R)$ and $X_t \coloneqq \operatorname{Spec}(R/tR)$. By Lemma 6.1, it remains to show that

$$H^d_{\mathfrak{m}}(R) \xrightarrow{\delta^d_\pi} H^d_{\pi^{-1}(\{\mathfrak{m}\})}\big(\operatorname{ZR}(X), \mathcal{O}_{\operatorname{ZR}(X)}\big)$$

is injective. Set $\mathrm{ZR}(X)_t \coloneqq \mathrm{ZR}(X) \times_X X_t$ and $\pi'_t\colon \mathrm{ZR}(X_t) \to X_t$. We claim we have a commutative diagram

$$\begin{array}{ccc} \mathrm{ZR}(X_t) & & \\ & \searrow & \\ & & \mathrm{ZR}(X)_t \longrightarrow \mathrm{ZR}(X) \\ & \swarrow_{\pi'_t} & \downarrow{\scriptstyle \pi_t} \qquad \downarrow{\scriptstyle \pi} \\ & & X_t \hookrightarrow X \end{array}$$

in the category of locally ringed spaces, where the square is Cartesian. By definition of limits, it suffices to show that for every morphism $f\colon W \to X$ in the inverse system defining $\mathrm{ZR}(X)$, there exists a morphism $f'_t\colon W'_t \to X_t$ in the inverse system defining $\mathrm{ZR}(X_t)$ such that the diagram

$$\begin{array}{ccc} W'_t & & \\ & \searrow & \\ & & W_t \longrightarrow W \\ & \swarrow_{f'_t} & \downarrow{\scriptstyle f_t} \qquad \downarrow{\scriptstyle f} \\ & & X_t \hookrightarrow X \end{array}$$

commutes. First, choose an irreducible component $(W_t)_0$ of W_t dominating X_t. Since X_t is normal, f is an isomorphism in codimension 1, and hence $(W_t)_0$ maps birationally to X_t. We can then apply Chow's lemma [EGAII, Corollaire 5.6.2] to find a projective birational morphism $W'_t \to (W_t)_0$ such that the composition $W'_t \to X_t$ is projective birational, which proves the claim.

Next, consider the commutative diagram

$$\begin{array}{ccccc} 0 & & 0 & & \\ \downarrow & & \downarrow & & \\ H^{d-1}_{\mathfrak{m}}(R/tR) & \hookrightarrow & H^{d-1}_{\pi_t^{-1}(\{\mathfrak{m}\})}\big(\mathrm{ZR}(X)_t, \mathcal{O}_{\mathrm{ZR}(X)_t}\big) & \longrightarrow & H^{d-1}_{\pi_t'^{-1}(\{\mathfrak{m}\})}\big(\mathrm{ZR}(X_t), \mathcal{O}_{\mathrm{ZR}(X_t)}\big) \\ \downarrow & & \downarrow & & \\ H^d_{\mathfrak{m}}(R) & \xrightarrow{\delta^d_\pi} & H^d_{\pi^{-1}(\{\mathfrak{m}\})}\big(\mathrm{ZR}(X), \mathcal{O}_{\mathrm{ZR}(X)}\big) & & \\ {\scriptstyle t\cdot -}\downarrow & & \downarrow{\scriptstyle t\cdot -} & & \\ H^d_{\mathfrak{m}}(R) & \xrightarrow{\delta^d_\pi} & H^d_{\pi^{-1}(\{\mathfrak{m}\})}\big(\mathrm{ZR}(X), \mathcal{O}_{\mathrm{ZR}(X)}\big) & & \\ \downarrow & & \downarrow & & \\ 0 & & 0 & & \end{array}$$

with exact columns where the left half is obtained from [Ive86, Functoriality II.9.7]. The top left horizontal arrow is injective since the composition in the top row is injective by Lemma 6.1, where we use the fact that the edge maps in Definition 2.9(*iv*) behave well under composition of morphisms [Smi97a, Proposition 1.12]. The columns are exact at the top by the fact that R is Cohen–Macaulay and by Theorem 5.1(*i*).

Now suppose there exists an element $0 \neq \eta \in \ker(\delta^d_\pi)$. Since every element in $H^d_{\mathfrak{m}}(R)$ is annihilated by a power of t, after multiplying η by a power of t we may assume that $t\eta = 0$, in which case η lies in the image of $H^{d-1}_{\mathfrak{m}}(R/tR)$ in the left column. The commutativity of the diagram implies that the composition

$$H^{d-1}_{\mathfrak{m}}(R/tR) \longrightarrow H^d_{\pi^{-1}(\{\mathfrak{m}\})}\big(\mathrm{ZR}(X), \mathcal{O}_{\mathrm{ZR}(X)}\big)$$

is injective. Since $\eta \in \ker(\delta_\pi^q)$ by assumption, this shows that $\eta = 0$, a contradiction. □

9.3. **Kempf's criterion.** We show that Kempf's criterion [KKMSD73, Proposition on p. 50] holds for quasi-excellent schemes of equal characteristic zero with dualizing complexes. This gives a proof of Kempf's criterion in equal characteristic zero independent of [Kov, Theorem 8.6].

The trace morphism below comes from the adjunction $\mathbf{R}f_* \dashv f^!$ in Grothendieck duality [Har66, Chapter VI, Corollary 3.4(*b*); Con00, Lemma 3.4.3].

Proposition 9.4 (cf. [KKMSD73, Proposition on p. 50; Lip94, Lemma 4.2])**.** *Let $f\colon X \to Y$ be a proper birational morphism of Noetherian schemes of equal characteristic zero such that X is regular and such that Y has a dualizing complex $\omega_Y^\bullet$. Denote by ω_X the unique cohomology sheaf of $f^!\omega_Y^\bullet$ (after possibly applying shifts on each connected component of X).*

The following are equivalent:

(*i*) *Y is normal and $R^i f_*\mathcal{O}_X = 0$ for all $i > 0$.*
(*ii*) *Y is Cohen–Macaulay and the trace morphism $f_*\omega_X \to \omega_Y$ is an isomorphism.*

Proof. By [Lip94, Lemma 4.2], even without characteristic zero hypotheses, (*i*) holds if and only if the trace morphism $\mathbf{R}f_*\omega_X^\bullet \to \omega_Y^\bullet$ is a quasi-isomorphism.

We want to show that $\mathbf{R}f_*\omega_X^\bullet \to \omega_Y^\bullet$ is a quasi-isomorphism if and only if (*ii*) holds. Since X is regular, after possibly applying shifts on each connected component of X, $\omega_X^\bullet$ is concentrated in one degree. Moreover, $R^i f_*\omega_X = 0$ for all $i > 0$ by Theorem B(*i*) applied to $\mathscr{L} = \mathcal{O}_X$. We therefore see that $\mathbf{R}f_*\omega_X^\bullet \to \omega_Y^\bullet$ is a quasi-isomorphism if and only if $f_*\omega_X \to \omega_Y$ is an isomorphism and $\omega_Y^\bullet$ is concentrated in one degree. But the condition that $\omega_Y^\bullet$ is concentrated in one degree is equivalent to Y being Cohen–Macaulay by local duality [Har66, Chapter V, Corollary 6.3]. □

9.4. **Derived splinters.** We show that Kovács's splitting criterion for rational singularities [Kov00, Theorem 3] holds for quasi-excellent schemes of equal characteristic zero. The direction ⇒ gives an independent proof of [Kov, Theorem 8.7] in equal characteristic zero. Recall that a scheme S is a *derived splinter* if for every proper surjective morphism $f\colon X \to S$, the natural morphism $\mathcal{O}_S \to \mathbf{R}f_*\mathcal{O}_X$ splits in the derived category of coherent sheaves on S [Bha12, Definition 1.3].

Theorem 9.5 (cf. [Kov00, Theorem 3; Bha12, Theorem 2.12])**.** *Let S be a quasi-excellent Noetherian scheme of equal characteristic zero. Then, S is a derived splinter if and only if S has rational singularities.*

Proof. ⇒. Let $\pi\colon \mathrm{ZR}(S) \to S$ denote the canonical projection morphism from the Zariski–Riemann space of S. For every admissible blowup $\pi_{\mathscr{I}}\colon S_{\mathscr{I}} \to S$ and every $s \in S$, setting $S_s \coloneqq \operatorname{Spec}(\mathcal{O}_{S,s})$ and $\pi_{\mathscr{I},s}\colon S_{s,\mathscr{I}_s} \to S_s$, we know that the morphism

$$H^i_{\{s\}}\big(S_s, \mathcal{O}_{S_s}\big) \xrightarrow{\delta^i_{\pi_{\mathscr{I},s}}} H^i_{\pi^{-1}_{\mathscr{I},s}(\{s\})}\big(S_{s,\mathscr{I}_s}, \mathcal{O}_{S_{s,\mathscr{I}_s}}\big)$$

is injective for every i since it admits a left inverse. Taking colimits over all $\mathscr{I} \in \mathrm{AId}_S$ and aplying Theorem 3.13, we see that the morphism

$$H^i_{\{s\}}\big(S_s, \mathcal{O}_{S_s}\big) \xrightarrow{\delta^i_{\pi_s}} H^i_{\pi_s^{-1}(\{s\})}\big(\mathrm{ZR}(S_s), \mathcal{O}_{\mathrm{ZR}(S_s)}\big)$$

is injective for every i. By Theorem 6.2, this shows S has rational singularities.

⇐. Let $g\colon Y \to S$ be a proper surjective morphism. We want to show that $\mathcal{O}_S \to \mathbf{R}g_*\mathcal{O}_Y$ splits in the derived category of coherent sheaves on S. By Chow's lemma [EGAII, Corollaire 5.6.2], we may assume that g is projective. By homogeneous prime avoidance [Bou72, Chapter III, §1, no. 4, Proposition 8], we can take repeated hyperplane sections of Y to assume that g generically finite.

By the Raynaud–Gruson flattening theorem [RG71, Première partie, Théorème 5.2.2], we can then find a commutative square that fits into a diagram

$$\begin{array}{ccccc} & & Y'' & \longrightarrow & Y \\ & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ W & \xrightarrow{\mu} & Y' & \xrightarrow{b} & S \end{array}$$

where b is a blowup of S, h is the strict transform of g along b, and h is finite flat. We take $\mu\colon W \to Y'$ to be a resolution of singularities, which exists by [Tem08, Theorem 1.1]. Now consider the corresponding commutative diagram

$$\begin{array}{ccccc} & & \mathbf{R}(b\circ h)_*\mathcal{O}_{Y''} & \longleftarrow & \mathbf{R}g_*\mathcal{O}_Y \\ & & \uparrow\downarrow & & \uparrow \\ \mathbf{R}(b\circ\mu)_*\mathcal{O}_W & \longleftarrow & \mathbf{R}b_*\mathcal{O}_{Y'} & \longleftarrow & \mathcal{O}_S. \end{array}$$

Since we are in equal characteristic zero and Y' is normal, there is a section of $\mathcal{O}_{Y'} \to h_*\mathcal{O}_{Y''} \simeq \mathbf{R}h_*\mathcal{O}_{Y''}$ coming from the trace map [Bha12, Example 2.1]. By definition of rational singularities, the composition $\mathcal{O}_S \to \mathbf{R}(b\circ\mu)_*\mathcal{O}_W$ is a quasi-isomorphism, and in particular, splits. Thus, $\mathcal{O}_S \to \mathbf{R}b_*\mathcal{O}_{Y'}$ splits, which shows that $\mathcal{O}_S \to \mathbf{R}g_*\mathcal{O}_Y$ splits by the commutativity of the diagram. □

9.5. **Rationality of dlt pairs.** In [BK23], Bernasconi and Kollár proved the following theorem without characteristic assumptions by assuming that certain vanishing theorems hold and that thrifty log resolutions exist. Since the required vanishing statements hold in equal characteristic zero by Theorem A(i) and thrifty log resolutions are known to exist by [Tem18], we obtain the following unconditional result. We note Theorem 9.6 below was already noted in [BK23, p. 2857] as a consequence of Theorem A(i) in this paper.

Theorem 9.6 (see [BK23, Theorem 3.1])**.** *Let X be an excellent normal Noetherian scheme of equal characteristic zero with a dualizing complex $\omega_X^\bullet$. Let Δ be an effective $\mathbf{R}$-Weil divisor on X such that (X,Δ) is dlt. Then, we have the following:*

- (i) *X is Cohen–Macaulay.*
- (ii) *Let D be a Weil divisor on X such that $D+\Delta_D$ is $\mathbf{R}$-Cartier for some $0 \le \Delta_D \le \Delta$. Then, $\mathcal{O}_X(D)$ is Cohen–Macaulay.*
- (iii) *For every reduced Weil divisor $B \subseteq \lfloor\Delta\rfloor$, the pair (X,B) is rational in the sense of* [Kol13, Definition 2.80]*. In particular, X has rational singularities.*
- (iv) *Log canonical centers of (X,Δ) are normal and have rational singularities.*

Proof. We can apply [BK23, Theorem 3.1] since Grauert–Riemenschneider vanishing holds in this setting by Theorem A(i), and thrifty log resolutions exist in this setting by [Tem18, Theorem 1.1.6]. Recall that if X is normal and $D \subseteq X$ is a reduced Weil divisor, then a resolution $f\colon Y \to X$ is *thrifty* if $(Y, f_*^{-1}D)$ is snc (i.e., Y is regular and $f_*^{-1}D$ has simple normal crossings), f is an isomorphism over the generic point of every stratum of $\mathrm{snc}(X,D)$, and f is an isomorphism at the generic point of every stratum of $(Y, f_*^{-1}D)$ (see [Kol13, Definition 2.79]). □

Remark 9.7. For the statement of Theorem A, we need to know that if X is an excellent normal Noetherian scheme of equal characteristic zero with a dualizing complex $\omega_X^\bullet$, and there exists an effective $\mathbf{Q}$-Weil divisor such that (X,Δ) is klt, then X is Cohen–Macaulay. The proof of Theorem 9.6 in [BK23] only needs Theorem A in the regular case. We may therefore use the regular case of Theorem A to deduce that klt pairs are Cohen–Macaulay in the proof of the klt case of Theorem A. Note that the fact that X is Cohen–Macaulay is not used in the proof that the regular case of Theorem A implies the klt case (see §8.3).

9.6. **Cohen–Macaulayness of Rees algebras and a Briançon–Skoda theorem for rational singularities.** Finally, we give a criterion for when Rees algebras and associated graded rings of graded sequences of ideals are Cohen–Macaulay. These results are essentially due to Sancho de Salas [SdS87] and Lipman [Lip94], who proved the Cohen–Macaulayness of $G_{\mathbf{F}^{(e)}}$ and $R_{\mathbf{F}^{(e)}}$ assuming certain vanishing theorems hold for the projection morphism $f\colon X \to \operatorname{Spec}(R)$. In particular, Sancho de Salas noted the relevant vanishing theorems hold for rings of finite type over $\mathbf{C}$ [SdS87, Theorem 2.8(a)]. We can prove the following statement for quasi-excellent rings thanks to our version of the Grauert–Riemenschneider vanishing theorems (Theorem B*(i)). See also [SdS87, Theorem 1.4 and Corollary 1.6; Lip94, Theorems 4.1 and 4.3] for partial converses to these results.

Theorem 9.8 (see [SdS87, Theorem 1.7; Lip94, Theorems 4.1 and 4.3])**.** *Let $(R, \mathfrak{m})$ be a Noetherian local $\mathbf{Q}$-algebra of dimension d. Let $\mathbf{F} \coloneqq (F_i)_{i=0}^{\infty}$ be a graded sequence of ideals in R such that $F_0 = R$. Suppose that*

$$R_{\mathbf{F}} \coloneqq \bigoplus_{i=0}^{\infty} F_i t^i$$

is Noetherian of dimension $d+1$. Set $X \coloneqq \operatorname{Proj}_R(R_{\mathbf{F}})$ with canonical projection $f\colon X \to \operatorname{Spec}(R)$.

(i) If R is Cohen–Macaulay and X is locally pseudo-rational, then

$$G_{\mathbf{F}^{(e)}} \coloneqq \bigoplus_{n=1}^{\infty} F_{en}/F_{e(n+1)}$$

is Cohen–Macaulay for some $e > 0$.

(ii) If R and X are locally pseudo-rational, then

$$R_{\mathbf{F}^{(e)}} \coloneqq \bigoplus_{n=1}^{\infty} F_{en} t^n$$

is Cohen–Macaulay and generated in degree 1 for some $e > 0$.

Proof. In either case, note that X is Cohen–Macaulay by definition of pseudo-rationality.

For (i), we apply [Lip94, Theorem 4.3]. It suffices to show that

$$H^i_{f^{-1}(\{\mathfrak{m}\})}(X, \mathcal{O}_X) = 0$$

for all $i < d$. This vanishing holds by Theorem B*(i).

For (ii), we apply [Lip94, Theorem 4.1]. It suffices to show that

$$\mathcal{O}_{\operatorname{Spec}(R)} \longrightarrow \mathbf{R}f_*\mathcal{O}_X$$

is a quasi-isomorphism. Since R and X are locally pseudo-rational, Theorem 6.2 implies that both the composition

$$\mathcal{O}_{\operatorname{Spec}(R)} \longrightarrow \mathbf{R}f_*\mathcal{O}_X \xrightarrow{\sim} \mathbf{R}\pi_*\mathcal{O}_{\operatorname{ZR}(X)}$$

and the second morphism in this composition are quasi-isomorphisms in the derived category of $\mathcal{O}_{\operatorname{Spec}(R)}$-modules, where $\pi\colon \operatorname{ZR}(X) \to \operatorname{Spec}(X)$ is the canonical projection. Since the composition $\operatorname{ZR}(X) \to X \to \operatorname{Spec}(R)$ is isomorphic to the projection morphism $\operatorname{ZR}(\operatorname{Spec}(R)) \to \operatorname{Spec}(R)$ by definition of the Zariski–Riemann space, we see that $\mathcal{O}_{\operatorname{Spec}(R)} \to \mathbf{R}f_*\mathcal{O}_X$ is a quasi-isomorphism. □

This yields the following characterization of rational singularities in equal characteristic zero, which for rings essentially of finite type over a field is due to Lipman [Lip94].

Corollary 9.9 (see [Lip94, p. 149])**.** *Let $(R, \mathfrak{m})$ be a quasi-excellent Noetherian local $\mathbf{Q}$-algebra. Then, R has rational singularities if and only if there exists an ideal $I \subseteq R$ such that $R[It]$ is Cohen–Macaulay and $\operatorname{Proj}_R(R[It])$ is regular.*

Proof. $\Rightarrow$. Let $X \to \operatorname{Spec}(R)$ be any projective resolution of singularities, which exists by [Hir64, Chapter I, §3, Main Theorem I(n)]. Then, $X \simeq \operatorname{Proj}_R(R[Jt])$ for some ideal $J \subseteq R$ by [EGAIII$_1$, Corollaire 2.3.7]. By Theorem 9.8(ii), $R[J^e t]$ is Cohen–Macaulay for some $e > 0$. Since

$$\operatorname{Proj}_R\big(R[Jt]\big) \simeq \operatorname{Proj}_R\big(R[J^e t]\big),$$

setting $I = J^e$ works.

$\Leftarrow$. By [Lip94, Theorem 4.1], we know that $\mathcal{O}_{\operatorname{Spec}(R)} \to \mathbf{R}f_*\mathcal{O}_X$ is a quasi-isomorphism. This implies R has rational singularities since the definition of rational singularities is independent of the resolution of singularities chosen [Mur22, Remark 4.13]. □

We also obtain the following version of the Briançon–Skoda theorem for rings with rational singularities. This statement strengthens the Briançon–Skoda theorem shown by Lipman and Teissier for pseudo-rational rings in arbitrary characteristic [LT81, Theorem 2.1]. The statement below is due to Huneke when R is essentially of finite type over a field of characteristic zero [Hun00].

Corollary 9.10 (cf. [Hun00, Corollary 4.8])**.** *Let R be a normal locally excellent Noetherian $\mathbf{Q}$-algebra with rational singularities. Then, for every ideal $I \subseteq R$ of analytic spread ℓ, we have*

$$\overline{I^n} \subseteq I^{n-\ell}$$

for all $n \geq \ell$.

Proof. Since the question is local, we can localize at every maximal ideal in R to assume that R is local. By [Hun00, Corollary 4.8], it suffices to show there exists an ideal $J \subseteq R$ such that $\operatorname{Proj}_R(R[Jt])$ is regular and $R[Jt]$ is Cohen–Macaulay. This condition is equivalent to R having rational singularities by Corollary 9.9. □

10. Complete local UFDs of dimension ≤ 4 with residue field $\mathbf{C}$ are Gorenstein

In this section, we give another application of our vanishing theorems that is not related to rational singularities. In [Sam61, p. 17], Samuel asked whether every UFD is Cohen–Macaulay. While the answer is no in general [Ber67, Proposition on p. 655] (see also [Lip75, §4] for a survey), all complete local UFDs $(R, \mathfrak{m})$ such that $R/\mathfrak{m} \simeq \mathbf{C}$ are S_3. This result is due to Raynaud (unpublished), Danilov [Dan70, Theorem 2], Boutot [Bou73, Corollaire on p. 693], and Hartshorne–Ogus (when R is algebrizable) [HO74, Theorem 2.5]. Hartshorne and Ogus showed that in dimensions ≤ 4, such rings are in fact Gorenstein [HO74, Corollary 2.6].

We remove the algebrizability condition in Hartshorne and Ogus's proof using our vanishing theorems. Hartshorne and Ogus's proof uses resolutions of singularities [Hir64] and the relative Grauert–Riemenschneider vanishing theorem [GR70, Satz 2.3].

Theorem 10.1 (see [HO74, Corollary 2.6])**.** *Let $(R, \mathfrak{m})$ be a complete local UFD of dimension ≤ 4 such that $R/\mathfrak{m} \simeq \mathbf{C}$. Then, R is Gorenstein.*

Proof. Let $f\colon Z \to \operatorname{Spec}(R)$ be a resolution of singularities, and set $M^i := R^i f_* \mathcal{O}_Z$ for every i. By [HO74, Proposition 2.4], we have $M^1 = 0$. Replacing Hartshorne and Ogus's dual version of the Grauert–Riemenschneider vanishing theorem [HO74, Proposition 2.2] by Theorem B*(i) in the proof of [HO74, Theorem 2.3], we then see that R satisfies S_3. Since $\dim(R) \leq 4$, we see that R satisfies Hartshorne and Ogus's condition C [HO74, Definition 1.7], i.e., for every prime ideal $\mathfrak{p} \subseteq R$, we have

$$\operatorname{depth}(R_\mathfrak{p}) \geq \min\left\{\dim(R_\mathfrak{p}), \frac{1}{2}\dim(R_\mathfrak{p}) + 1\right\}.$$

This implies R is Gorenstein by [HO74, Corollary 1.8]. □

11. Extensions to other categories

In this section, we prove Theorem A′, which extends our main vanishing and injectivity theorem to other categories of spaces. While most objects and notions appearing in the statement of Theorem A′ were either defined or checked to be compatible with existing definitions and relative GAGA in [LM, §§23–25], the notion of bigness in [LM, Defintion 25.5] requires projectivity assumptions to apply relative GAGA theorems. We therefore define relative bigness below before proving Theorem A′ in the next subsection.

Remark 11.1. The projectivity assumption on f is necessary in cases (I), (III), (III′), and (IV) since as far as we aware, appropriate versions of Chow's lemma for proper bimeromorphic morphisms in these categories do not exist in the literature. See [AT19, §1.6] and [Duc21, Théorème 6.6]. We have written our proof of Theorem A′ so that if such a version of Chow's lemma becomes available in one of these categories, then Theorem A′ would also be true in that category.

11.1. **Relatively big invertible sheaves and locally Moishezon morphisms.** In this subsection, we define relatively big invertible sheaves and locally Moishezon morphisms in the categories of spaces appearing in Theorem A′.

To be able to state Theorem A′(i) without projectivity assumptions, we need to define relative bigness without relying on GAGA or the existence of a relatively ample invertible sheaf. Instead, we adapt Nakayama's definition of relative bigness for complex-analytic spaces [Nak87, Definition on p. 568], which uses relative Proj. We define relative Proj as in [Stacks, Tag 084C] for algebraic spaces, [Nak04, Chapter II, §1.b] for complex analytic spaces, [Duc21, 5.4] for Berkovich spaces, [Con06, Definition 2.3.3] for rigid-analytic spaces, and [Zav, Definition 6.7] for adic spaces. All these constructions are compatible with relative GAGA over affinoid subdomains in the base, since locally, relative Proj is a closed subscheme of a projective space over an affinoid space.

We now define relative bigness for invertible sheaves and Cartier divisors. We restrict to the case of morphisms between integral schemes for simplicity. The idea is to use functorial properties of Proj to discuss bigness instead of generic fibers as Nakayama does in the complex-analytic case in [Nak87, Definition on p. 568] (see also [Kol22, p. 1666]), since generic fibers to do not always exist in the categories we are interested in.

Definition 11.2. Let $f\colon X \to Y$ be a proper morphism such that X is integral. Consider an invertible sheaf $\mathscr{L}$ on X. For each integer $n > 0$, consider the adjunction morphism $f^*f_*\mathscr{L}^{\otimes n} \to \mathscr{L}$. Let $\mathfrak{b} \subseteq \mathcal{O}_X$ be the coherent ideal sheaf such that this adjunction morphism factors as

$$f^*f_*\mathscr{L}^{\otimes n} \twoheadrightarrow \mathfrak{b} \cdot \mathscr{L}^{\otimes n} \hookrightarrow \mathscr{L}^{\otimes n}.$$

Taking symmetric algebras and relative Proj, we obtain the commutative diagram

$$\begin{array}{ccccc} \mathrm{Bl}_{\mathfrak{b}} X & \hookrightarrow & \mathbf{P}_X(f^*f_*\mathscr{L}^{\otimes n}) & \xrightarrow{\sim} & X \times_Y \mathbf{P}_Y(f_*\mathscr{L}^{\otimes n}) \\ & \searrow^{\pi} \swarrow & & & \downarrow{\scriptstyle \mathrm{pr}_2} \\ & X & \dashrightarrow & \mathbf{P}_Y(f_*\mathscr{L}^{\otimes n}) & \longrightarrow Y \end{array} \tag{19}$$

where the composition in the bottom row is f. We say that $\mathscr{L}$ is *f-big* if there exists an integer $n > 0$ such that the dashed arrow in the diagram above is *generically finite* onto the closure of its image, i.e., is finite away from a nowhere dense set in the closure of its image. We extend this definition to Cartier divisors L on X by asking that its associated invertible sheaf $\mathcal{O}_X(L)$ is f-big. If D is a **Q**-Cartier divisor, then we say that D is *f-big* if some positive integer multiple of D is f-big.

We note the partially defined map $X \dashrightarrow \mathbf{P}_Y(f_*\mathscr{L}^{\otimes n})$ in (19) is meromorphic in the sense of Remmert [Rem57, Def. 15] (see also [Pet94, Definition 1.7]) in the complex analytic case and is meromorphic in the sense of Morrow and Rosso [MR23, Definition 3.2] in the non-Archimedean case.

Remark 11.3. Suppose f as in Definition 11.2 is projective. Then, we see that $X \dashrightarrow \mathbf{P}_Y(f_*\mathscr{L}^{\otimes n})$ is generically finite if and only if for every affinoid subdomain $U \subseteq Y$, this map is the relative analytification of a generically finite rational map of schemes over U. As a result, we see that our definition of relative bigness coincides with our previous definition in [LM, Definition 25.5] for projective morphisms, and hence is compatible with GAGA.

We define (locally) Moishezon morphisms as follows. Our definition is an adaptation of the definition for complex analytic spaces in [Moi74, Definition 2; Kol22, Definition 11].

Definition 11.4. Let $f\colon X \to Y$ be a proper morphism. We say that f is *Moishezon* if the morphism f is bimeromorphic (over Y) to a projective morphism. We say that f is *locally Moishezon* if for every point $y \in Y$, there exists an affinoid subdomain $U \subseteq Y$ containing y such that the morphism $f^{-1}(U) \to U$ is Moishezon.

We now show that the existence of an f-big invertible sheaf implies that f is locally Moishezon.

Lemma 11.5. *Let $f\colon X \to Y$ be a proper morphism such that X is integral. Consider an f-big invertible sheaf $\mathscr{L}$ on X. For all $n > 0$ such that the dashed arrow in (19) is generically finite onto the closure of its image, let f^p be the finite part of the Stein factorization*

$$\mathrm{Bl}_{\mathfrak{b}}\, X \longrightarrow X^p \xrightarrow{f^p} \mathbf{P}_Y(f_*\mathscr{L}^{\otimes n})$$

of the morphism $\mathrm{Bl}_{\mathfrak{b}}\, X \to \mathbf{P}_Y(f_*\mathscr{L}^{\otimes n})$. Then, the composition

$$X^p \xrightarrow{f^p} \mathbf{P}_Y(f_*\mathscr{L}^{\otimes n}) \longrightarrow Y$$

is a locally projective morphism bimeromorphic (over Y) to f. In particular, f is locally Moishezon.

Proof. Let $n > 0$ be an integer such that the dashed arrow in (19) is generically finite onto the closure of its image. The morphism $\mathrm{Bl}_{\mathfrak{b}}\, X \to \mathbf{P}_Y(f_*\mathscr{L}^{\otimes n})$ is then generically finite onto its image. Now consider the Stein factorization of this morphism

$$\mathrm{Bl}_{\mathfrak{b}}\, X \longrightarrow X^p \longrightarrow \mathbf{P}_Y(f_*\mathscr{L}^{\otimes n}),$$

which exists for algebraic spaces by [Stacks, Tag 0A1B], for semianalytic germs of complex analytic spaces by applying [GR84, 10.6.1] to a representative of this morphism, for Berkovich spaces by [Ber90, Proposition 3.3.7], for rigid analytic spaces by [BGR84, Proposition 9.6.3/5], and for adic spaces locally of weakly finite type over a field by [Man23, Theorem 3.9]. The morphism $X^p \to \mathbf{P}_Y(f_*\mathscr{L}^{\otimes n})$ is finite, and the morphism $\mathrm{Bl}_{\mathfrak{b}}\, X \to X^p$ is surjective and bimeromorphic. We therefore see that the morphism

$$X^p \xrightarrow{\phi} \mathbf{P}_Y(f_*\mathscr{L}^{\otimes n}) \longrightarrow Y$$

is a locally projective morphism in the sense that for every $y \in Y$, there exists an affinoid subdomain $U \subseteq Y$ containing y such that $X^p \times_Y U \to U$ is projective. Thus, f is locally Moishezon. □

11.2. **Proof of Theorem A′.** We can now prove Theorem A′.

Proof of Theorem A′. Since the statement is local on Y, we may assume that Y is affinoid. We will also be able to replace Y by smaller affinoid subdomains during the proof below.

We first show that Theorem A′ holds when f is projective. The case for algebraic spaces follows by flat base change [Stacks, Tag 073K] applied to an étale cover of Y. For the other cases, by the GAGA-type results in [LM, §§23–25], we know that f is the analytification of a projective morphism of schemes, and that the hypotheses in Theorem A′ are compatible under the GAGA correspondence. Since the vanishing and injectivity statements on the scheme side hold by Theorem A, the compatibility of analytification with higher direct images [EGAIII$_1$, Proposition 5.1.2; AT19, Theorem C.1.1; Poi10, Théorème A.1; Köp74, Folgerung 6.6; Hub07, Corollary 6.4] shows that the desired vanishing and injectivity statements are preserved under analytification.

For the locally Moishezon case, we first reduce case (III) to case (III′). Note that in both cases, Y is a point. Let $k \subseteq K_r$ be a field extension where K_r is a complete non-trivially valued non-Archimedean field such that $X \hat{\otimes}_k K_r$ is a strictly K_r-analytic space (such a K_r exists as in the proof of [Ber90, Proposition 2.2.4]). Since coherent cohomology is compatible with the field extension $k \subseteq K_r$ by [Ber90, Proposition 2.1.2(*ii*)] (see the proof of [Ber90, Proposition 3.3.5]), we can detect the desired vanishing and injectivity statements after base change to K_r. Note that the formation of ω_X is compatible with base change to K_r [Ber93, Proposition 3.3.3(*ii*)]. Bigness is compatible with this extension since blowups and relative Proj are compatible with ground field extensions (locally, they are defined as the scheme-theoretic notions on affinoid subdomains, which are compatible with base change). Nefness is compatible with this extension because proper Berkovich curves are always projective [dJ95, Proposition 3.2 and Remark 3.3; Duc, Théorème 3.7.2], and hence are algebraizations of projective curves, to which we we can apply the scheme-theoretic result in [Kee03, Lemma 2.18(1)]. Finally, the comparison between (Berkovich) strictly K_r-analytic spaces and rigid K_r-analytic spaces in [Ber93, Theorem 1.6.1] is compatible with coherent cohomology [Ber93, p. 37], the formation of ω_X (by definition on the smooth locus, which is preserved by [Ber90, Proposition 3.3.1(*iii*)]), and both bigness (since finite morphisms are by [Ber90, Proposition 3.3.2]) and nefness (since coherent cohomology, and hence the computation of Euler characteristics, is compatible as above). This shows we may reduce case (III) to case (III′).

It remains to reduce to the case when f is projective in cases (0), (II), and (III′), where in the last case we assume that Y is a point. We claim that after possibly replacing Y with an affinoid subdomain, we can construct a commutative diagram

$$\begin{array}{ccccc} & & \tilde{X} & & \\ & & \downarrow{\scriptstyle \mu} & & \\ & & \mathrm{Bl}_{\mathfrak{b}} X & & \\ & \swarrow^{\pi} & & \searrow^{\phi} & \\ X & & & & X^p \\ & \searrow_{f} & & \swarrow_{f^p} & \\ & & Y & & \end{array} \tag{20}$$

of proper morphisms, where π, ϕ, and μ are bimeromorphic, f^p is projective, and $\pi \circ \mu$ is a projective log resolution of (X, Δ) such that $f^p \circ \phi \circ \mu$ is projective. The bottom square exists in case (*i*) by Lemma 11.5, and exists in case (*ii*) by the locally Moishezon assumption. To construct the desired projective log resolution, we first apply Chow's lemma for bimeromorphic/birational morphisms, and then take a projective log resolution. Projective log resolutions exist in these categories by [AHV77, Theorem 5.3.1; Sch99, Theorem 3.2.3 and Remark on p. 327; Tem18, Theorem 1.1.13]. For algebraic spaces, Chow's lemma (without birationality assumptions) holds by [Stacks, Tag 088U]. For semianalytic germs of complex analytic spaces, we can apply the complex-analytic version of Chow's lemma [Moi74, Par. 2; Hir75, Corollary 2] to a representative of ϕ. Finally, for rigid analytic spaces, we can apply Chow's lemma for proper Moishezon rigid analytic spaces [Con10, Corollary 4.1.2] since Y is a point.

Next, we construct appropriate divisors on $\tilde{X}$ to which we will apply the projective case of Theorem A′ proved above. As in the proof of Theorem A in §8.3, we can write

$$K_{\tilde{X}} + (\pi \circ \mu)^* M + (\pi \circ \mu)_*^{-1} \Delta + \sum_i \big(\lceil a_i \rceil - a_i \big) E_i \sim_{\mathbf{Q}} (\pi \circ \mu)^* N + \sum_i \lceil a_i \rceil E_i,$$

in which case setting

$$\tilde{N} \coloneqq (\pi \circ \mu)^* N + \sum_i \lceil a_i \rceil E_i,$$

we have

$$\tilde{N} \sim_{\mathbf{Q}} K_{\tilde{X}} + (\pi\circ\mu)^*M + (\pi\circ\mu)_*^{-1}\Delta + \sum_i \big(\lceil a_i\rceil - a_i\big)E_i.$$

Since $(\pi\circ\mu)^*M$ is $(\pi\circ\mu)$-big and $(\pi\circ\mu)$-nef by the projection formula, we know that

$$R^i(\pi\circ\mu)_*\big(\mathcal{O}_{\tilde{X}}(\tilde{N})\big) = 0$$

for all $i>0$ by the projective case of (*i*).

For (*i*), we know that $(\pi\circ\mu)^*M$ is $(f^p\circ\phi\circ\mu)$-nef and $(f^p\circ\phi\circ\mu)$-big by the projection formula, the normality of X, and the commutativity of the diagram (20). We therefore see that

$$R^i(f\circ\pi\circ\mu)_*\big(\mathcal{O}_{\tilde{X}}(\tilde{N})\big) = R^i(f^p\circ\phi\circ\mu)_*\big(\mathcal{O}_{\tilde{X}}(\tilde{N})\big) = 0$$

for all $i>0$ by the projective case of (*i*). We then consider the Leray spectral sequence

$$E_2^{i,j} = R^if_*\big(R^j(\pi\circ\mu)_*\big(\mathcal{O}_{\tilde{X}}(\tilde{N})\big)\big) \Rightarrow R^{i+j}(f\circ\pi\circ\mu)_*\big(\mathcal{O}_{\tilde{X}}(\tilde{N})\big).$$

By the vanishing $R^i(\pi\circ\mu)_*(\mathcal{O}_{\tilde{X}}(\tilde{N})) = 0$, the E_2 page of this spectral sequence is concentrated in the row $j=0$. We therefore have a natural isomorphism

$$R^if_*\big((\pi\circ\mu)_*\big(\mathcal{O}_{\tilde{X}}(\tilde{N})\big)\big) \simeq R^i(f\circ\pi\circ\mu)_*\big(\mathcal{O}_{\tilde{X}}(\tilde{N})\big) = 0$$

for all $i>0$. Now by definition of $\tilde{N}$ and the projection formula, we have

$$(\pi\circ\mu)_*\big(\mathcal{O}_{\tilde{X}}(\tilde{N})\big) \simeq \mathcal{O}_X(N)\otimes_{\mathcal{O}_X}(\pi\circ\mu)_*\left(\mathcal{O}_{\tilde{X}}\left(\sum_i\lceil a_i\rceil E_i\right)\right) \simeq \mathcal{O}_X(N)$$

since X is normal and the E_i are $(\pi\circ\mu)$-exceptional (cf. [Laz04a, Example 2.1.16]). We therefore have

$$R^if_*\big(\mathcal{O}_X(N)\big) \simeq R^if_*\big((\pi\circ\mu)_*\big(\mathcal{O}_{\tilde{X}}(\tilde{N})\big)\big) = 0.$$

For (*ii*), since the map $\mathcal{O}_X\to\mathcal{O}_X(D)$ factors the map $\mathcal{O}_X\to\mathcal{O}_X(D+D')$, we can replace D by $D+D'$ to assume assume that $\mathcal{O}_X(D)\simeq\mathcal{O}_X(nM)$, and in particular, we may assume that D is Cartier. The projective case implies that the canonical morphisms

$$R^i(f\circ\pi\circ\mu)_*\big(\mathcal{O}_{\tilde{X}}(\tilde{N})\big) \longrightarrow R^i(f\circ\pi\circ\mu)_*\big(\mathcal{O}_{\tilde{X}}\big(\tilde{N}+(\pi\circ\mu)^*D\big)\big)$$

are injective for all i since $(\pi\circ\mu)^*M$ is $(f\circ\pi\circ\mu)$-semi-ample (by the projection formula, the normality of X, and the commutativity of the diagram (20)) and $\mathcal{O}_{\tilde{X}}((\pi\circ\mu)^*D)\simeq\mathcal{O}_{\tilde{X}}(n(\pi\circ\mu)^*M)$. The same argument using the Leray spectral sequence as in the previous paragraph then implies that the canonical morphisms

$$R^if_*\big(\mathcal{O}_X(N)\big) \longrightarrow R^if_*\big(\mathcal{O}_X(N+D)\big)$$

are injective for all i. □

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