Projectivity of the Moduli of Curves

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Abstract: In this expository paper, we show that the Deligne–Mumford moduli space of stable curves is projective over $\mathrm{Spec}(\mathbf{Z})$. The proof we exposit is due to Kollár. Ampleness of a line bundle is deduced from nefness of a related vector bundle via the Ampleness Lemma, a classifying map construction. The main positivity result concerns the pushforward of relative dualizing sheaves on families of stable curves over a smooth projective curve.

Introduction

Let $\overline{\mathcal{M}}_g$ be the moduli stack of stable curves of genus $g \geq 2$ and write $\overline{M}_g$ for its corresponding moduli space. We prove that the moduli of stable curves is projective in the following sense; see Theorem 1.45:

Theorem *The Deligne–Mumford moduli space $\overline{M}_g$ of stable curves of genus $g \geq 2$ is a projective scheme over* $\mathrm{Spec}(\mathbf{Z})$.

In particular, this means that $\overline{M}_g$, which is *a priori* just an algebraic space, is actually a projective scheme over $\mathbf{Z}$. Together with the work of Deligne–Mumford [8] (see also Theorem 0E9C) this means that $\overline{M}_g$ is actually an irreducible projective scheme over $\mathbf{Z}$.

We explain a proof due to Kollár in [21]. Specifically, the task of showing that a certain line bundle on $\overline{M}_g$ is ample is transferred, via Kollár's Ampleness Lemma, to the problem of showing that a related vector bundle is nef on $\overline{M}_g$. Since nefness is a condition that only depends on the behaviour of the vector bundle upon restriction to curves, projectivity is thus reduced to a problem regarding positivity of 1-parameter families of stable curves.

Kollár's method differs from other existing proofs of projectivity of $\overline{M}_g$ in at least two main ways: First, the technique is independent of the methods of Geometric Invariant Theory, on which the proofs of [29, 10, 6] rely. Second, Kollár's criterion does not require one to directly check that a line bundle on the moduli space is ample, in contrast to the approach of Knudsen–Mumford [19, 17, 18]; rather, one only needs to show that some vector bundle on the moduli space is nef. As such, this method has since been used in other settings, such as in the moduli of weighted stable curves [13], of stable varieties [22], and, recently, of K-polystable Fano varieties [5, 33].

An outline of this article is as follows. We set up notation in regards to the moduli of curves in §1, after which we begin in §§2–4 with some material on positivity of sheaves. In §5, we explain Kollár's Ampleness Lemma, see Proposition 1.33. In §6, we prove the main positivity statement: the pushforward of the relative dualizing sheaf of a 1-parameter family of stable curves of genus at least 2 is nef, see Theorem 1.43. Finally, we put everything together in §7 to show that $\overline{M}_g$ is projective over $\mathbf{Z}$ when $g \geq 2$.

Conventions. Throughout, k will denote a field. Following the conventions of the Stacks Project, a *variety* is a separated integral scheme of finite type over a field k and a *curve* is a variety of dimension 1, see Definitions 020D and 0A23. Given a scheme X over k and a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, we write

$$h^i(X, \mathcal{F}) := \dim_k(H^i(X, \mathcal{F})) \quad \text{for all } i \in \mathbf{Z}.$$

We use the Stacks Project [30] as the main technical reference. Results therein are referred to via their four character alphanumeric tags.

1 Stable curves

In this section, we record the definition of the moduli problem in which we are primarily interested, namely that of the moduli space of stable curves. The main references are [8] and Chapter 0DMG.

First we define what we mean by a family of curves. Compare the following with Situation 0D4Z, and with Definitions 0C47, 0C5A, and 0E75. We diverge slightly from the Stacks Project in that we require our families of nodal curves to have geometrically connected fibres. Caution: the closed fibres of a family of nodal curves are *not* curves in the sense of our conventions, as they may be reducible. See Section 0C58 for a discussion on such terminology.

Definition 1.1 Let S be a scheme.

(i) A *family of nodal curves over* S is a flat, proper, finitely presented morphism of schemes $f: X \to S$ of relative dimension 1 such that all geometric fibres are connected and smooth except at possibly finitely many nodes.
(ii) A *family of stable curves over* S is a family of nodal curves such that the geometric fibres have arithmetic genus ≥ 2 and do not contain rational tails or bridges.
(iii) A family of stable curves over S is said to *have genus* g if all geometric fibres have genus g.

Condition (ii) is equivalent to ampleness of the dualizing sheaf, and also finiteness of automorphism groups. See Section 0E73 for details. For the following, see Definition 0E77.

Definition 1.2 For $g \geq 2$, the *moduli stack of stable curves of genus* g is the category $\overline{\mathcal{M}}_g$ fibred in groupoids whose category of sections over a scheme S has objects given by families of stable curves of genus g over S, and morphisms given by isomorphisms of families over S.

The stack $\overline{\mathcal{M}}_g$ is a smooth, proper Deligne–Mumford stack over $\mathrm{Spec}(\mathbf{Z})$, see Theorem 0E9C. Classically, and in many geometric applications such as [12], it is convenient to work with a space rather than the stack. As such, it is useful to extract an algebraic space which is, in some sense, the closest approximation of the stack, obtained by "forgetting" the automorphism groups: this is the notion of a *uniform categorical moduli space* or simply a *moduli space* of a stack, see Definition 0DUG.

Lemma 1.3 *The stack* $\overline{\mathcal{M}}_g$ *admits a uniform categorical moduli space* $f_g: \overline{\mathcal{M}}_g \to \overline{M}_g$ *such that* f_g *is separated, quasi-compact, and a universal homeomorphism.*

Proof The stack $\overline{\mathcal{M}}_g$ has finite inertia by Lemmas 0E7A and 0DSW, so the existence of f_g follows from the Keel–Mori Theorem 0DUT. □

Definition 1.4 The space $\overline{M}_g$ is the *moduli space of curves of genus* g.

Our primary goal is to show that $\overline{M}_g$ is projective over $\mathbf{Z}$, see Theorem 1.45. Thus we must exhibit an ample invertible sheaf on $\overline{M}_g$. We obtain invertible sheaves on the moduli space by taking powers of invertible sheaves on the stack $\overline{\mathcal{M}}_g$, via the following general fact:

Lemma 1.5 *Let $\mathcal{X}$ be an algebraic stack. Assume the inertia $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is finite and let $f\colon \mathcal{X} \to M$ be its moduli space, as in Theorem 0DUT. Then*

$$f^*\colon \operatorname{Pic}(M) \to \operatorname{Pic}(\mathcal{X})$$

is injective. If $\mathcal{X}$ is furthermore quasi-compact, then the cokernel of f^* is annihilated by a positive integer.

Proof For injectivity, note $f_*\mathcal{O}_{\mathcal{X}} \cong \mathcal{O}_M$ as M is initial for morphisms from $\mathcal{X}$ to algebraic spaces and the structure sheaf represents the functor $\operatorname{Hom}(-, \mathbf{A}^1)$. Thus if $\mathcal{N} \in \operatorname{Pic}(M)$ is such that $f^*\mathcal{N} \cong \mathcal{O}_{\mathcal{X}}$, the canonical map $\mathcal{N} \to f_*f^*\mathcal{N} \to \mathcal{O}_M$ is an isomorphism as $\mathcal{N}$ is locally trivial. This further shows that if $\mathcal{N}_1, \mathcal{N}_2 \in \operatorname{Pic}(M)$ are such that there exists an isomorphism $\varphi\colon f^*\mathcal{N}_1 \to f^*\mathcal{N}_2$, then there is a unique isomorphism $\psi\colon \mathcal{N}_1 \to \mathcal{N}_2$ such that $f^*\psi = \varphi$.

We now show that, if $\mathcal{X}$ is furthermore quasi-compact, then there is a positive integer n such that for every $\mathcal{L} \in \operatorname{Pic}(\mathcal{X})$, $\mathcal{L}^{\otimes n} \cong f^*\mathcal{N}$ for some $\mathcal{N} \in \operatorname{Pic}(M)$. For this, we may replace $\mathcal{X}$ by any $\mathcal{X}'$ with a surjective separated étale morphism $h\colon \mathcal{X}' \to \mathcal{X}$ of algebraic stacks inducing isomorphisms on automorphism groups. Indeed, Lemma 0DUV gives the Cartesian square

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow[h]{} & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ M' & \longrightarrow & M \end{array}$$

where M' is the moduli space of $\mathcal{X}'$. If there were $\mathcal{N}' \in \operatorname{Pic}(M')$ such that $h^*\mathcal{L}^{\otimes n} \cong f'^*\mathcal{N}'$, then injectivity of $f'^*\colon \operatorname{Pic}(M') \to \operatorname{Pic}(\mathcal{X}')$ shows that the étale descent datum for $h^*\mathcal{L}^{\otimes n}$ over $\mathcal{X}$ induces an étale descent datum for $\mathcal{N}'$ over M yielding $\mathcal{N} \in \operatorname{Pic}(M)$ as above.

Choose such a cover $h\colon \mathcal{X}' \to \mathcal{X}$ as in Lemma 0DUE: $\mathcal{X}' = \coprod_{i \in I} \mathcal{X}_i$ where each $\mathcal{X}_i$ is a quotient stack $[U_i/R_i]$ in which $(U_i, R_i, s_i, t_i, c_i)$ is a groupoid scheme with U_i and R_i affine, and $s_i, t_i\colon R_i \to U_i$ finite locally free of some constant rank, see Lemmas 0DUM and 03BI. Since $\mathcal{X}$ is quasi-compact, we are reduced to the case where $\mathcal{X}$ is a finite disjoint union of such stacks $\mathcal{X}_i$. Let $f_i\colon \mathcal{X}_i \to M_i$ be the moduli space. If there exists a positive integer n_i annihilating the cokernel of f_i^*, then the least common multiple n of the n_i annihilates the cokernel of f^*.

Thus it suffices to consider the case where $\mathcal{X} = [U/R]$ is as above. By Proposition 06WT, an invertible $\mathcal{O}_{\mathcal{X}}$-module may be represented as a pair $(\mathcal{L}, \alpha)$ consisting of an invertible $\mathcal{O}_U$-module $\mathcal{L}$ together with an isomorphism $\alpha\colon t^*\mathcal{L} \to s^*\mathcal{L}$ of $\mathcal{O}_R$-modules as in Definition 03LI. We claim that if n is the rank of the morphisms $s, t\colon R \to U$, then $(\mathcal{L}^{\otimes n}, \alpha^n)$ is in the image of f^*. Namely, writing $\pi\colon U \to M$, there exists an invertible $\mathcal{O}_M$-module $\mathcal{N}$ and an isomorphism of invertible modules $(\pi^*\mathcal{N}, \alpha_{\mathrm{can}}) \cong (\mathcal{L}^{\otimes n}, \alpha^n)$ on the groupoid (U, R, s, t, c), where α_{can} is the identity map; this makes sense since $\pi \circ t = \pi \circ t$ as maps $R \to M$.

Construct $\mathcal{N}$ as follows. First, if $U = \bigcup U_i$ is any affine open cover, then the $V_i := \pi(U_i)$ together form an affine open cover of M. That the V_i form an open cover follows from the fact that π is the composition of the faithfully flat and finitely presented morphism $U \to \mathcal{X}$ and the universal homeomorphism $\mathcal{X} \to M$: see Lemmas 01UA and 0DUP. That the V_i are affine is because π is integral: see Lemmas 03BJ and 05YU. Next, since $t\colon R \to U$ is finite locally free, Lemma 0BCY constructs an invertible $\mathcal{O}_U$-module $\mathcal{L}' := \mathrm{Norm}_t(s^*\mathcal{L})$ as follows. Let $(\{U_i\}, \{u_{ij}\})$ be a system of cocycles locally defining $\mathcal{L}$, so that $U = \bigcup U_i$ is an affine open cover and $u_{ij} \in \mathcal{O}_U^*(U_i \cap U_j)$ are units. Then $\mathcal{L}'$ is defined by the cocycles $(\{U_i\}, \{u'_{ij}\})$ with $u'_{ij} := \mathrm{Norm}_{t^\sharp}(s^\sharp(u_{ij}))$. Finally, setting $V_i := \pi(U_i)$, Lemma 03BH implies that the u'_{ij} lie in the subgroup $\mathcal{O}_M^*(V_i \cap V_j) \subseteq \mathcal{O}_U^*(U_i \cap U_j)$ of R-invariant units, so $(\{V_i\}, \{u'_{ij}\})$ forms a system of cocycles on M defining an invertible module $\mathcal{N}$.

On the one hand, the construction implies $\mathcal{L}' \cong \pi^*\mathcal{N}$. On the other hand, Lemma 0BCZ yields an isomorphism

$$\mathrm{Norm}_t(\alpha)\colon \mathcal{L}^{\otimes n} \cong \mathrm{Norm}_t(t^*\mathcal{L}) \to \mathrm{Norm}_t(s^*\mathcal{L}) = \mathcal{L}' \cong \pi^*\mathcal{N}.$$

Thus it suffices to show that the diagram of isomorphisms

$$\begin{array}{ccc} t^*\mathcal{L}^{\otimes n} & \xrightarrow[\alpha^n]{} & s^*\mathcal{L}^{\otimes n} \\ {\scriptstyle t^*\mathrm{Norm}_t(\alpha)}\downarrow & & \downarrow{\scriptstyle s^*\mathrm{Norm}_t(\alpha)} \\ t^*\pi^*\mathcal{N} & \overset{\alpha_{\mathrm{can}}}{=\!=\!=} & s^*\pi^*\mathcal{N} \end{array}$$

is commutative. By properties of the norm, the compatibilities of α from

Definition 03LH(1), and the diagram of Lemma 03BH, we have

$$\begin{aligned}\alpha^n &= \mathrm{Norm}_c(c^*\alpha) = \mathrm{Norm}_c(\mathrm{pr}_1^*\alpha \circ \mathrm{pr}_0^*\alpha)\\ &= \mathrm{Norm}_c(\mathrm{pr}_1^*\alpha) \circ \mathrm{Norm}_c(\mathrm{pr}_0^*\alpha) = s^*\,\mathrm{Norm}_s(\alpha) \circ t^*\,\mathrm{Norm}_t(\alpha).\end{aligned}$$

Since $s = t \circ i$ where $i\colon R \to R$ is the inverse, $\mathrm{Norm}_s(\alpha) = \mathrm{Norm}_t(i^*\alpha)$. Therefore

$$s^*\,\mathrm{Norm}_t(\alpha) \circ \alpha^n \circ t^*\,\mathrm{Norm}_t(\alpha)^{-1} = s^*(\mathrm{Norm}_t(\alpha \circ i^*\alpha)).$$

This is the identity since, by Lemma 077Q, $i^*\alpha$ is the inverse of α. □

We now specify some invertible sheaves on $\overline{\mathcal{M}}_g$. By Definition 06TR and Lemma 06WI, the data of such a sheaf $\mathcal{L}$ is the following: for each family of stable curves $X \to S$, an invertible $\mathcal{O}_S$-module $\mathcal{L}(X \to S)$, and for every Cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

an isomorphism of invertible $\mathcal{O}_{S'}$-modules

$$\varphi_g\colon g^*\mathcal{L}(X \to S) \cong \mathcal{L}(X' \to S')$$

such that for every composition of Cartesian squares

$$\begin{array}{ccccc} X'' & \longrightarrow & X' & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ S'' & \xrightarrow{h} & S' & \xrightarrow{g} & S \end{array}$$

the isomorphisms are subject to the cocycle condition

$$\begin{array}{ccc} h^*(g^*\mathcal{L}(X \to S)) & \xrightarrow{h^*\varphi_g} & h^*\mathcal{L}(X' \to S') \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\varphi_h} \\ (gh)^*\mathcal{L}(X \to S) & \xrightarrow{\varphi_{gh}} & \mathcal{L}(X'' \to S''). \end{array}$$

Definition 1.6 For each integer $m \geq 1$, define an invertible sheaf λ_m on $\overline{\mathcal{M}}_g$ as follows. Given a family of stable curves $f\colon X \to S$, let $\omega_{X/S}^{\otimes m}$ be its relative dualizing sheaf, see Definition 0E6Q. This is an invertible $\mathcal{O}_X$-module. Note that the sheaves $f_*\omega_{X/S}^{\otimes m}$ are locally free on S. Set

$$\lambda_m(f\colon X \to S) := \det(f_*\omega_{X/S}^{\otimes m}).$$

Given a Cartesian square as above, we have isomorphisms φ_g given by

$$g^* \det(f_* \omega_{X/S}^{\otimes m}) \cong \det(g^* f_* \omega_{X/S}^{\otimes m}) \to \det(f'_* g'^* \omega_{X/S}^{\otimes m}) \cong \det(f'_* \omega_{X'/S'}^{\otimes m})$$

the functorial base change maps and the fact that the formation of $\omega_{X/S}$ commutes with arbitrary base change, see Lemma 0E6R. Functoriality ensures that these satisfy the required cocycle condition.

Our goal will be to show that there is some m such that λ_m descends to an ample invertible sheaf on $\overline{M}_g$.

2 Nakai–Moishezon Criterion for ampleness

In this section, we discuss the Nakai–Moishezon Criterion for ampleness, relating ampleness of an invertible sheaf with positivity of intersection numbers. We directly prove the Criterion for proper algebraic spaces over a field in Proposition 1.10 (compare with [21, Theorem 3.11]); the proof closely follows that of [16, §III.1, Theorem 1], with suitable modifications. Using Lemma 0D3A, one can also formulate a relative version; see, for example, [14, Proposition 2.10].

In the following, we work with proper algebraic spaces over a field. For generalities on algebraic spaces, see Part 0ELT.

We will use numerical intersection theory on spaces as developed in Section 0DN3; see also Section 0BEL and [26, Section 1.1.C] for the situation of varieties. The main construction is the *intersection number* $(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z)$ between a closed subspace $\iota \colon Z \to X$ of positive dimension d and invertible O_X-modules $\mathcal{L}_1, \ldots, \mathcal{L}_d$: this is the coefficient of $n_1 \cdots n_d$ of the numerical polynomial

$$\chi(X, \iota_* O_Z \otimes \mathcal{L}_1^{\otimes n_1} \otimes \cdots \otimes \mathcal{L}_d^{\otimes n_d}) = \chi(Z, \mathcal{L}_1^{\otimes n_1} \otimes \cdots \otimes \mathcal{L}_d^{\otimes n_d}|_Z).$$

See Definition 0EDF.

The Nakai–Moisehzon Criterion relates ampleness with positivity of intersection numbers. To formulate this succinctly, we make a definition. In the following, recall that a separated algebraic space Z is integral if and only if it is reduced and $|Z|$ is irreducible; see Definition 0AD4 and Section 03I7.

Definition 1.7 Let X be a proper algebraic space over k and let $\mathcal{L}$ be an invertible O_X-module. We say that $\mathcal{L}$ *has positive degree* if for every integral closed subspace Z of X of positive dimension d, $(\mathcal{L}^d \cdot Z) > 0$.

Note that the Stacks Project only defines the degree of an invertible sheaf $\mathcal{L}$ either when $\mathcal{L}$ is ample or when $\dim(X) \leq 1$; see Definitions 0BEW and 0AYR. The content of the Nakai–Moishezon Criterion is that if $\mathcal{L}$ has positive degree, then $\mathcal{L}$ is ample. Thus this is *a fortiori* compatible with the conventions of the Stacks Project.

The main technical property we need is permanence of positivity under finite morphisms.

Lemma 1.8 *Let X be a proper algebraic space over k. Let $f \colon Y \to X$ be a finite morphism of algebraic spaces. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. If $\mathcal{L}$ has positive degree, then $f^*\mathcal{L}$ has positive degree.*

Proof This follows from the compatibility of numerical intersection numbers and pullbacks: if $Z \subset Y$ is a proper integral closed subspace of dimension d, then

$$(f^*\mathcal{L}^d \cdot Z) = \deg(Z \to f(Z))(\mathcal{L}^d \cdot f(Z))$$

where $\deg(Z \to f(Z))$ is positive as f is finite; see Lemma 0EDJ. □

The following is the core of the inductive proof of the Criterion:

Lemma 1.9 *Let X be a proper algebraic space over k and let D be an effective Cartier divisor of X. If $\mathcal{O}_X(D)|_D$ is ample, then $\mathcal{O}_X(mD)$ is globally generated for all $m \gg 0$.*

Proof For each $m \geq 0$, there is a short exact sequence

$$0 \to \mathcal{O}_X((m-1)D) \to \mathcal{O}_X(mD) \to \mathcal{O}_X(mD)|_D \to 0.$$

Since $\mathcal{O}_X(D)|_D$ is ample, Serre Vanishing, Lemma 0GFA, gives an integer m_1 such that $H^1(D, \mathcal{O}_X(mD)|_D) = 0$ for $m \geq m_1$. Hence the

$$\rho_m \colon H^1(X, \mathcal{O}_X((m-1)D)) \to H^1(X, \mathcal{O}_X(mD)),$$

arising from the long exact sequence on cohomology are surjective for all $m \geq m_1$, yielding a nonincreasing sequence of nonnegative integers

$$h^1(X, \mathcal{O}_X(mD)) \geq h^1(X, \mathcal{O}_X((m+1)D)) \geq \cdots .$$

There is some $m_2 \geq m_1$ after which the sequence stabilizes, so that, for all $m \geq m_2$, the ρ_m are bijective and the restriction maps

$$H^0(X, \mathcal{O}_X(mD)) \to H^0(D, \mathcal{O}_X(mD)|_D)$$

are surjective. Finally, since $\mathcal{O}_X(D)|_D$ is ample, there exists some m_3 such that $\mathcal{O}_X(mD)|_D$ is generated by its global sections for all $m \geq m_3$.

Let $m_0 := \max(m_2, m_3)$. We show that the evaluation maps

$$H^0(X, O_X(mD)) \otimes_k O_X \to O_X(mD)$$

are surjective for all $m \geq m_0$. We verify this on stalks. For $x \in |X \setminus D|$, a global section defining mD restricts to a unit in $O_X(mD)_x$ and thus generates. So consider $x \in |D|$ and let $\kappa(x)$ be the residue field of D at x; see Definition 0EMW. Since $D \to X$ is a monomorphism, $\kappa(x)$ is also the residue field at x of X by Lemma 0EMX. Consider the diagram

$$\begin{array}{ccc} H^0(X, O_X(mD)) \otimes_k \kappa(x) & \longrightarrow & O_X(mD) \otimes_{O_X} \kappa(x) \\ \downarrow\!\!\downarrow & & \downarrow \simeq \\ H^0(D, O_X(mD)|_D) \otimes_k \kappa(x) & \twoheadrightarrow & O_X(mD)|_D \otimes_{O_D} \kappa(x) \end{array}$$

obtained from the evaluation and restriction maps upon taking the fibre at x. By our choice of m_0, the restriction map on the left is surjective and $O_X(mD)|_D$ is globally generated, so the bottom map is surjective. Since the right map is an isomorphism, commutativity of the diagram implies that the top map is surjective. Nakayama's Lemma then implies that the evaluation map is surjective on the local ring $O_X(mD)_x$. Hence the evaluation map is surjective, meaning $O_X(mD)$ is globally generated. □

Proposition 1.10 (Nakai–Moishezon Criterion) *Let X be a proper algebraic space over k. Let $\mathcal{L}$ be an invertible O_X-module. Then $\mathcal{L}$ is ample on X if and only if $\mathcal{L}$ has positive degree.*

Proof If $\mathcal{L}$ is ample, then X is a scheme, $\mathcal{L}$ is ample in the schematic sense, and $\mathcal{L}$ has positive degree; see Lemmas 0D32 and 0BEV.

Assuming $\mathcal{L}$ has positive degree, we show it is ample. We proceed by induction on $\dim(X)$. When $\dim(X) = 0$, since X is separated, it is a scheme by Theorem 086U, in which case the result is clear. When $\dim(X) = 1$, our assumption simplifies to $\deg(\mathcal{L}) > 0$. Now apply Proposition 09YC to obtain a finite surjective morphism $f: Y \to X$ from a scheme Y. Lemma 1.8 shows that $\deg(f^*\mathcal{L}) > 0$ and so Lemma 0B5X gives ampleness of $f^*\mathcal{L}$. Since f is finite, Lemma 0GFB shows $\mathcal{L}$ is also ample. So we assume that $\dim(X) \geq 2$ and that the Criterion holds for all proper spaces over k of lower dimension.

Step 1. Using Lemmas 0GFB, 0GFA, and 1.8, we may replace X by the reduction of an irreducible component and $\mathcal{L}$ by its restriction to assume that X is integral.

Step 2. We show that some power of $\mathcal{L}$ is effective. As X is integral, the discussion of Section 0ENV shows that $\mathcal{L}$ has a regular meromorphic section s. Consider its sheaf of denominators $\mathcal{I}_1$, the ideal sheaf in $\mathcal{O}_X$ whose sections over $V \in X_{\text{étale}}$ are

$$\mathcal{I}_1(V) := \{f \in \mathcal{O}_X(V) \mid fs \in \mathcal{L}(V)\};$$

compare Definition 02P1. Set $\mathcal{I}_2 := \mathcal{I}_1 \otimes \mathcal{L}^\vee$. Since the formation of the $\mathcal{I}_j$, $j = 1, 2$, is étale local, their properties may be reduced to the schematic case. Thus Lemma 02P0 shows that the $\mathcal{I}_j$ are quasi-coherent sheaves of ideals and the corresponding closed subspaces $Y_j = V(\mathcal{I}_j)$ satisfy $\dim(Y_j) < \dim(X)$. By Lemma 1.8, induction applies so the $\mathcal{L}|_{Y_j}$ are ample. By construction, for each $m \geq 0$, there are exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}_1 \otimes \mathcal{L}^{\otimes m} & \longrightarrow & \mathcal{L}^{\otimes m} & \longrightarrow & \mathcal{L}^{\otimes m}|_{Y_1} & \longrightarrow & 0 \\ & & \| & & & & & & \\ 0 & \longrightarrow & \mathcal{I}_2 \otimes \mathcal{L}^{\otimes (m-1)} & \longrightarrow & \mathcal{L}^{\otimes (m-1)} & \longrightarrow & \mathcal{L}^{\otimes (m-1)}|_{Y_2} & \longrightarrow & 0. \end{array}$$

Serre Vanishing, Lemma 0B5U, gives some $m_0 \geq 0$ such that for all $m \geq m_0$, $H^i(Y_j, \mathcal{L}^{\otimes m}|_{Y_j}) = 0$ for all $i > 0$ and $j = 1, 2$. Thus comparing the long exact sequences in cohomology for the sequences above yields

$$\begin{aligned} h^i(X, \mathcal{L}^{\otimes m}) &= h^i(X, \mathcal{I}_1 \otimes \mathcal{L}^{\otimes m}) \\ &= h^i(X, \mathcal{I}_2 \otimes \mathcal{L}^{\otimes (m-1)}) = h^i(X, \mathcal{L}^{\otimes (m-1)}) \end{aligned}$$

for all $i \geq 2$ and $m \geq m_0$. Hence, for all $m \geq m_0$,

$$N := \sum\nolimits_{i=2}^{\dim(X)} (-1)^i \, h^i(X, \mathcal{L}^{\otimes m})$$

is a constant. By definition of the intersection numbers, the leading coefficient of the numerical polynomial $\chi(X, \mathcal{L}^{\otimes m})$ is $(\mathcal{L}^{\dim X} \cdot X)$ and this is positive by assumption. Thus

$$\chi(X, \mathcal{L}^{\otimes m}) = h^0(X, \mathcal{L}^{\otimes m}) - h^1(X, \mathcal{L}^{\otimes m}) + N \to \infty \quad \text{as } m \to \infty.$$

So $h^0(X, \mathcal{L}^{\otimes m}) \to \infty$ and $\mathcal{L}^{\otimes m}$ is effective for $m \gg 0$. Ampleness is insensitive to powers (see Lemma 01PT), so we may replace $\mathcal{L}$ by $\mathcal{L}^{\otimes m}$ to assume $\mathcal{L} = \mathcal{O}_X(D)$ for some effective Cartier divisor D.

Step 3. By induction, $\mathcal{L}|_D = \mathcal{O}_X(D)|_D$ is ample, so Lemma 1.9 implies $\mathcal{L}^{\otimes m}$ is generated by its global sections for $m \gg 0$. We may replace $\mathcal{L}$ by $\mathcal{L}^{\otimes m}$ to assume that $\mathcal{L}$ is generated by its global sections.

Step 4. Via Lemmas 01NE and 085D, a basis of global sections of $\mathcal{L}$

induces a proper morphism

$$f \colon X \to \mathbf{P}^n_k \quad \text{with } n := h^0(X, \mathcal{L}) - 1$$

such that $f^*\mathcal{O}_{\mathbf{P}^n_k}(1) = \mathcal{L}$. We now claim that f is finite, from which we may conclude: X is then a scheme as f is then representable, and the pullback of an ample by an affine morphism is ample, see Lemmas 03ZQ and 0892. By Lemma 0A4X, it suffices to show that f has discrete fibres. But if there were $y \in \mathbf{P}^n_k$ such that the fibre X_y were positive dimensional, then we would obtain a commutative diagram

$$\begin{array}{ccccc} C & \longrightarrow & X_y & \longrightarrow & X \\ & \overset{\pi}{\searrow} & \downarrow & & \downarrow f \\ & & \operatorname{Spec}(\kappa(y)) & \longrightarrow & \mathbf{P}^n_k \end{array}$$

where the right square is Cartesian, and C is some complete curve in X_y. By commutativity of the diagram, we see that

$$\mathcal{L}|_C = (f^*\mathcal{O}_{\mathbf{P}^n}(1))|_C \simeq \pi^*\mathcal{O}_{\operatorname{Spec}(k(y))} = \mathcal{O}_C.$$

But now we reach a contradiction: on the one hand, $\mathcal{L}$ has positive intersection numbers with C, but on the other hand, by Lemma 0EDK,

$$0 < (\mathcal{L} \cdot C) = \deg_C(\mathcal{L}|_C) = \deg_C(\mathcal{O}_C) = 0,$$

the degree on the right being the usual degree on a curve; see Definition 0AYR. Thus f is a finite morphism, as claimed. □

3 Positivity of invertible sheaves

We next prove some preliminary results about nef invertible sheaves on proper algebraic spaces and about big invertible sheaves on proper schemes over arbitrary fields. See [26] for the theory for varieties over algebraically closed fields.

We start with the definition of nefness.

Definition 1.11 Let X be a proper algebraic space over k. An invertible $\mathcal{O}_X$-module is *nef* if $(\mathcal{L}\cdot C) \geq 0$ for every integral closed subspace $C \subset X$ of dimension 1.

To show that nef invertible sheaves behave well under pullbacks, we show that we may lift curves along surjective morphisms; compare with [16, §I.4, Lemma 1]:

Lemma 1.12 *Let X be a proper algebraic space over k. Let $f\colon Y \to X$ be a surjective morphism of algebraic spaces and let $C \subset X$ be an integral closed subspace of dimension 1. Then there exists an integral closed subspace $C' \subset Y$ of dimension 1 such that $C = f(C')$.*

Proof By the weak version of Chow's Lemma, Lemma 089J, there exists a proper surjective morphism $g\colon Y' \to f^{-1}(C)$ from a scheme Y' projective over k. Taking $\dim(Y') - 1$ general hyperplane sections, we obtain a scheme $C'' \subset Y'$ of dimension 1 mapping onto C, since C'' intersects the fibre over the generic point of C. We can then take $C' \subset Y$ to be one of the irreducible components of $g(C'')$ mapping onto C with reduced induced algebraic space structure. □

Nef invertible sheaves behave well under pullbacks.

Lemma 1.13 *Let X be a proper algebraic space over k. Let $f\colon Y \to X$ be a proper morphism of algebraic spaces. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module.*

(i) *If $\mathcal{L}$ is nef, then $f^*\mathcal{L}$ is nef.*
(ii) *If f is surjective and $f^*\mathcal{L}$ is nef, then $\mathcal{L}$ is nef.*

Proof For (i), let $C \subset Y$ be an integral closed subspace of dimension 1. By the projection formula, Lemma 0EDJ, we have

$$(f^*\mathcal{L} \cdot C) = \deg(C \to f(C))(\mathcal{L} \cdot f(C)) \geq 0,$$

where we set $\deg(C \to f(C)) = 0$ by convention if $\dim(f(C)) = 0$.

For (ii), let $C \subset X$ be an integral closed subspace of dimension 1. By Lemma 1.12, there exists an integral closed subspace $C' \subset Y$ such that $C = f(C')$. The projection formula again gives

$$(\mathcal{L} \cdot C) = (\mathcal{L} \cdot f(C')) = \deg(C' \to C)^{-1}(f^*\mathcal{L} \cdot C') \geq 0. \qquad \square$$

Nef invertible sheaves are also well-behaved under field extensions.

Lemma 1.14 *Let X be a proper algebraic space over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then $\mathcal{L}$ is nef if and only if for every field extension $k \subseteq k'$, the pullback of $\mathcal{L}$ to $X \otimes_k k'$ is nef.*

Proof $\Leftarrow$ holds by setting $k = k'$, and hence it suffices to show $\Rightarrow$. By the weak version of Chow's Lemma, Lemma 089J, there exists a proper surjective morphism $g\colon Y \to X$ from a scheme Y proper over k. Since $\mathcal{L}$ is nef, $g^*\mathcal{L}$ is nef by Lemma 1.13, and hence the pullback of $g^*\mathcal{L}$ to $Y \otimes_k k'$ is nef by [14, Lemma 2.18(1)]. Finally, the pullback of $\mathcal{L}$ to $X \otimes_k k'$ is nef by applying Lemma 1.13 again. □

We will need the following result about nef invertible sheaves on curves that are not necessarily integral.

Lemma 1.15 *Let X be a proper scheme of dimension* 1 *over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. If $\mathcal{L}$ is nef, then* $\deg_X(\mathcal{L}) \geq 0$.

Proof When X is integral, the conclusion follows from Lemma 0BEY and the definitions. In general, let $C_1, C_2, \ldots, C_t$ be the irreducible components of X viewed as subschemes of X with the reduced induced subscheme structure. By Lemma 0AYW, we have

$$\deg_X(\mathcal{L}) = \sum_{i=1}^{t} m_i \deg_{C_i}(\mathcal{L}|_{C_i}) \quad \text{for some positive integers } m_i.$$

The integral case gives $\deg_X(\mathcal{L}|_{C_i}) \geq 0$ and thus $\deg_X(\mathcal{L}) \geq 0$. □

We adopt the following definition for big invertible sheaves on proper schemes, following Kollár [21, (i) on pp. 236–237].

Definition 1.16 Let X be a proper scheme over k. An invertible $\mathcal{O}_X$-module $\mathcal{L}$ is *big* if there exists a constant $C > 0$ such that

$$h^0(X, \mathcal{L}^{\otimes n}) > C \cdot n^{\dim(X)} \quad \text{for all sufficiently large } n.$$

By the asymptotic Riemann–Roch Theorem, Proposition 0BJ8, ample invertible sheaves are big. We show that unlike ampleness, the property of being big behaves well under birational morphisms.

Lemma 1.17 *Let $f\colon Y \to X$ be a birational morphism of proper schemes over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module on X. Then $\mathcal{L}$ is big if and only if $f^*\mathcal{L}$ is big.*

Proof Consider the short exact sequence

$$0 \to \mathcal{O}_X \to f_*\mathcal{O}_Y \to Q \to 0.$$

Then $\dim(Q) \leq \dim(X) - 1$ as f is birational, so upon twisting by $\mathcal{L}^{\otimes n}$ and taking global sections, we see that, by [7, Proposition 1.31(a)], there exists a constant $C' > 0$ such that

$$h^0(Y, f^*\mathcal{L}^{\otimes n}) - h^0(X, \mathcal{L}^{\otimes n}) \leq h^0(X, Q \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \leq C' \cdot n^{\dim(X)-1}$$

for all sufficiently large n. Thus $\mathcal{L}$ is big if and only if $f^*\mathcal{L}$ is big. □

Our next goal is to give an alternative characterization of big invertible sheaves on projective varieties. We start with the following result, known as Kodaira's Lemma; see [20, p. 42] and [26, Proposition 2.2.6].

Lemma 1.18 *Let X be a proper scheme over k. Let $\mathcal{L}$ be a big invertible $\mathcal{O}_X$-module. Then for every closed subscheme $Z \subset X$ of dimension less than* $\dim(X)$*, there exists an integer $m > 0$ for which*

$$H^0(X, \mathcal{I}_Z \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m}) \neq 0.$$

Proof Consider the twisted ideal sheaf sequence

$$0 \to \mathcal{I}_Z \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} \to \mathcal{L}^{\otimes n} \to \mathcal{L}^{\otimes n}|_Z \to 0.$$

Since Z is a proper scheme of dimension $< \dim(X)$ over k, there exists a constant $C' > 0$ such that

$$h^0(Z, \mathcal{L}^{\otimes n}|_Z) \leq C' \cdot n^{\dim(Z)}$$

for all sufficiently large n by [7, Proposition 1.31(a)]. Since $\mathcal{L}$ is big,

$$h^0(X, \mathcal{L}^{\otimes m}) > h^0(Z, \mathcal{L}^{\otimes m}|_Z)$$

for some $m > 0$. Taking global sections in the twisted ideal sheaf sequence then gives $H^0(X, \mathcal{I}_Z \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m}) \neq 0$. □

We now prove that a variant of the conclusion in Kodaira's Lemma 1.18 characterizes big invertible sheaves on projective varieties.

Lemma 1.19 *Let X be a projective variety over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then the following are equivalent:*

(i) *$\mathcal{L}$ is big.*
(ii) *For every ample invertible $\mathcal{O}_X$-module $\mathcal{A}$, there exists an integer $m > 0$ for which $H^0(X, \mathcal{A}^{-1} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m}) \neq 0$.*

Proof (i) ⇒ (ii). Let r be sufficiently large so that there are effective Cartier divisors $H_r \in |\mathcal{A}^{\otimes r}|$ and $H_{r+1} \in |\mathcal{A}^{\otimes (r+1)}|$. By Lemma 1.18, there exists an integer $m > 0$ for which $H^0(X, \mathcal{O}_X(-H_{r+1}) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m}) \neq 0$. Since the composition

$$\mathcal{O}_X(-H_{r+1}) \cong \mathcal{A}^{\otimes -(r+1)} \cong \mathcal{O}_X(-H_r) \otimes_{\mathcal{O}_X} \mathcal{A}^{-1} \hookrightarrow \mathcal{A}^{-1}$$

is injective, we then have

$$0 \neq H^0(X, \mathcal{O}_X(-H_{r+1}) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m}) \hookrightarrow H^0(X, \mathcal{A}^{-1} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m}).$$

(ii) ⇒ (i). Let $\mathcal{A}$ be a very ample invertible sheaf on X' and choose an effective Cartier divisor $H \in |\mathcal{A}|$. By (ii), there exists an integer $m > 0$ such that $H^0(X, \mathcal{O}_X(-H) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m}) \neq 0$. We can therefore find an effective Cartier divisor $E \in |\mathcal{O}_X(-H) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m}|$ which satisfies

$$\mathcal{O}_X(E) \cong \mathcal{O}_X(-H) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m} \cong \mathcal{A}^{-1} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m}.$$

By the asymptotic Riemann–Roch Theorem of [7, Proposition 1.31(b)], there exists a constant $C' > 0$ such that for n sufficiently large,

$$h^0(X, \mathcal{L}^{-i} \otimes_{O_X} \mathcal{A}^{\otimes n}) > C' \cdot n^{\dim(X)} \quad \text{for every } i \in \{0, 1, \ldots, m-1\}.$$

Writing $n = m \cdot \lceil n/m \rceil - i$ for $i \in \{0, 1, \ldots, m-1\}$, we then have

$$\begin{aligned} h^0(X, \mathcal{L}^{\otimes n}) &= h^0(X, \mathcal{L}^{-i} \otimes_{O_X} \mathcal{A}^{\otimes \lceil n/m \rceil}(\lceil n/m \rceil E)) \\ &\geq h^0(X, \mathcal{L}^{-i} \otimes_{O_X} \mathcal{A}^{\otimes \lceil n/m \rceil}) \\ &> C' \cdot \lceil n/m \rceil^{\dim(X)} > \frac{C'}{m^{\dim(X)}} \cdot n^{\dim(X)} \end{aligned}$$

and hence choosing $C = C'/m^{\dim(X)}$, we see that $\mathcal{L}$ is big. □

4 Nef locally free sheaves

In this section, we define and study basic properties of nef locally free sheaves; note that these are referred to as *semipositive* in [21]. See [27, Part Two] for the theory for varieties over algebraically closed fields.

First, a definition. Compare with [21, Definition-Proposition 3.3].

Definition 1.20 Let X be a proper algebraic space over k. A finite locally free O_X-module $\mathcal{E}$ is *ample* (resp. *nef*) if $O_{\mathbf{P}(\mathcal{E})}(1)$ is ample (resp. nef) on $\mathbf{P}(\mathcal{E})$ in the sense of Definition 0D31 (resp. Definition 1.11).

Here, $\mathbf{P}(\mathcal{E})$ denotes the projective bundle of one-dimensional quotients of $\mathcal{E}$. In other words, we set

$$\mathbf{P}(\mathcal{E}) := \underline{\mathrm{Proj}}_X(\mathrm{Sym}^\bullet(\mathcal{E})),$$

where $\underline{\mathrm{Proj}}_X$ is defined as in Definition 084C. By Lemma 085D, $\mathbf{P}(\mathcal{E})$ satisfies the following universal property: for an algebraic space $g\colon Y \to X$, giving a morphism $r\colon Y \to \mathbf{P}(\mathcal{E})$ is the same as giving an invertible sheaf $\mathcal{L}$ on Y and a surjective morphism $g^*\mathcal{E} \to \mathcal{L}$. Here, $\mathcal{L} \cong r^*O_{\mathbf{P}(\mathcal{E})}(1)$.

We show that locally free quotients of ample or nef locally free sheaves are ample or nef. See [21, Corollary 3.4(i)].

Lemma 1.21 *Let X be a proper algebraic space over k. Let $\mathcal{E} \to \mathcal{F}$ be a surjection of finite locally free O_X-modules. If $\mathcal{E}$ is ample (resp. nef), then $\mathcal{F}$ is ample (resp. nef).*

Proof The surjection $\mathcal{E} \to \mathcal{F}$ induces a closed embedding $\mathbf{P}(\mathcal{F}) \hookrightarrow \mathbf{P}(\mathcal{E})$ such that $O_{\mathbf{P}(\mathcal{E})}(1)$ restricts to $O_{\mathbf{P}(\mathcal{F})}(1)$ by functoriality of Proj;

see Lemma 085H. The ample case follows from the fact that $\mathbf{P}(\mathcal{E})$ is a projective k-scheme by the assumption that $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is ample, and ampleness is preserved under restriction; see Lemma 01PU. The nef case follows from Lemma 1.13(i). □

We now focus our attention on nef locally free sheaves. First, nef locally free sheaves behave well under pullbacks, as was the case for invertible sheaves in Lemma 1.13.

Lemma 1.22 *Let X be a proper algebraic space over k. Let $f\colon Y \to X$ be a proper morphism of algebraic spaces. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module.*

(i) *If $\mathcal{E}$ is nef, then $f^*\mathcal{E}$ is nef.*
(ii) *If f is surjective and $f^*\mathcal{E}$ is nef, then $\mathcal{E}$ is nef.*

Proof By Lemma 085C, we have a Cartesian diagram

$$\begin{array}{ccc} \mathbf{P}(f^*\mathcal{E}) & \xrightarrow[f']{} & \mathbf{P}(\mathcal{E}) \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

such that $f'^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) \cong \mathcal{O}_{\mathbf{P}(f^*\mathcal{E})}(1)$. Both statements follow from Lemma 1.13 applied to $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$, where for (ii), note that f' is surjective, being the base change of f; see Lemma 03MH. □

Nef locally free sheaves are also well-behaved under field extensions.

Lemma 1.23 *Let X be a proper algebraic space over k. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. Then $\mathcal{E}$ is nef if and only if for every field extension $k \subseteq k'$, the pullback of $\mathcal{E}$ to $X \otimes_k k'$ is nef.*

Proof It suffices to apply Lemma 1.14 to $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ on $\mathbf{P}(\mathcal{E})$. □

To show some other important properties of nef locally free sheaves, we prove the following characterization of nefness. The statement for schemes is known as the Barton–Kleiman Criterion; see [2, p. 437], [27, Proposition 6.1.18], and [21, Definition-Proposition 3.3].

Proposition 1.24 *Let X be a proper algebraic space over k. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. Then the following are equivalent:*

(i) *$\mathcal{E}$ is nef.*
(ii) *For every k-morphism $f\colon C \to X$ from a projective k-scheme C of dimension 1, and for every surjection $f^*\mathcal{E} \to \mathcal{L}$ where $\mathcal{L}$ is invertible, we have* $\deg_C(\mathcal{L}) \geq 0$.

(iii) *For every k-morphism $f : C \to X$ from a regular projective curve C over k, and for every surjection $f^*\mathcal{E} \to \mathcal{L}$ where $\mathcal{L}$ is invertible, we have* $\deg_C(\mathcal{L}) \geq 0$.

If k is algebraically closed, then these conditions are also equivalent to:

(iv) *For every k-morphism $f : C \to X$ from a regular projective curve C over k, and for every ample invertible sheaf $\mathcal{H}$ on C, the locally free sheaf $\mathcal{H} \otimes_{\mathcal{O}_C} f^*\mathcal{E}$ is ample.*

Proof (i) $\Rightarrow$ (ii). Let $f : C \to X$ be a morphism as in (ii), and let $\mathcal{L}$ be an invertible quotient of $f^*\mathcal{E}$ on C. By the universal property of $\mathbf{P}(\mathcal{E})$, we obtain a morphism $r : C \to \mathbf{P}(\mathcal{E})$ such that $\mathcal{L} \cong r^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. By Lemma 1.13(i), $\mathcal{L}$ is nef. We then have $\deg_C(\mathcal{L}) \geq 0$ by Lemma 1.15.

(ii) $\Rightarrow$ (iii). This holds since the morphisms appearing in (iii) are special cases of those appearing in (ii).

(iii) $\Rightarrow$ (i). Let $g : C' \hookrightarrow \mathbf{P}(\mathcal{E})$ be an integral closed subspace of dimension 1. By the weak version of Chow's Lemma, Lemma 089J, there exists a proper surjective morphism $f : C \to C'$ from a scheme C projective over k, and by Lemma 1.12, we may replace C by a closed integral subscheme mapping onto C' to assume that $\dim(C) = 1$. Replacing C by a suitable irreducible component of its normalization, we may also assume that C is regular and integral. Let $\pi : \mathbf{P}(\mathcal{E}) \to X$ be the projection morphism. By the universal property of $\mathbf{P}(\mathcal{E})$, we have a surjection

$$(\pi \circ g \circ f)^*\mathcal{E} \to (g \circ f)^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$$

on C. By (iii) and Lemma 0BEY, the pullback $(g \circ f)^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is nef. Thus $g^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is also nef by Lemma 1.13(ii), and $(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) \cdot C) \geq 0$.

We show (i) $\Rightarrow$ (iv) assuming that k is algebraically closed. Let $\pi : \mathbf{P}(f^*\mathcal{E}) \to C$ be the projection morphism. We want to show that

$$\mathcal{O}_{\mathbf{P}(\mathcal{H} \otimes_{\mathcal{O}_C} f^*\mathcal{E})}(1) \cong \mathcal{O}_{\mathbf{P}(f^*\mathcal{E})}(1) \otimes_{\mathcal{O}_{\mathbf{P}(f^*\mathcal{E})}} \pi^*\mathcal{H}$$

is ample, where the isomorphism shown holds by definition of relative Proj under the identification $\mathbf{P}(\mathcal{H} \otimes_{\mathcal{O}_C} f^*\mathcal{E}) \cong \mathbf{P}(f^*\mathcal{E})$. Let $Y \subset \mathbf{P}(f^*\mathcal{E})$ be an integral closed subscheme. By the Nakai–Moishezon Criterion (see Proposition 1.10), it suffices to show that

$$\big((\mathcal{O}_{\mathbf{P}(f^*\mathcal{E})}(1) \otimes_{\mathcal{O}_{\mathbf{P}(f^*\mathcal{E})}} \pi^*\mathcal{H})^d \cdot Y\big) > 0$$

where $d = \dim(Y)$. If Y is contained in a closed fibre over C, then this positivity holds since $\mathcal{O}_{\mathbf{P}(f^*\mathcal{E})}(1)$ restricts to $\mathcal{O}_{\mathbf{P}^n}(1)$ on the closed fibre, where $n = \operatorname{rank}(f^*\mathcal{E}) - 1$. Otherwise, it suffices to show that

$$\big((\mathcal{O}_{\mathbf{P}(f^*\mathcal{E})}(1) \otimes_{\mathcal{O}_{\mathbf{P}(f^*\mathcal{E})}} \pi^*\mathcal{H})^d \cdot Y\big) \geq \big((\mathcal{O}_{\mathbf{P}(f^*\mathcal{E})}(1))^{d-1} \cdot \pi^*\mathcal{H} \cdot Y\big)$$

since the right-hand side is positive by the fact that $\pi^*\mathcal{H} \cdot Y$ corresponds to a closed subscheme of dimension $d-1$ contained in a union of closed fibres over C, in which case we can apply the case above. This inequality holds since we can expand the left-hand side by additivity (Lemma 0BER) and then observe that since $f^*\mathcal{E}$ is nef by Lemma 1.22(i), every term involving $\mathcal{O}_{\mathbf{P}(f^*\mathcal{E})}(1)$ is nonnegative by [14, Lemma 2.12], and every term with more than one power of $\pi^*\mathcal{H}$ is zero.

Finally, we show (iv) ⇒ (iii) assuming that k is algebraically closed. Let $f^*\mathcal{E} \to \mathcal{L}$ be a surjection where $\mathcal{L}$ is invertible. Choose an ample invertible sheaf $\mathcal{H}$ on C of degree 1, which exists since k is algebraically closed. Twist this surjection by $\mathcal{H}$. Since the quotient of an ample locally free sheaf is ample by Lemma 1.21, and ample invertible sheaves have positive degree by Lemma 0B5X, we have

$$1 + \deg_C(\mathcal{L}) = \deg_C(\mathcal{H} \otimes_{\mathcal{O}_C} \mathcal{L}) \geq 1$$

where the equality holds by Lemma 0AYX, and the inequality holds by (iv). This shows that $\deg_C(\mathcal{L}) \geq 0$. □

We can now show that nefness is preserved under extensions.

Lemma 1.25 *Let X be a proper algebraic space over k. Let*

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

be a short exact sequence of finite locally free $\mathcal{O}_X$-modules. If $\mathcal{E}'$ and $\mathcal{E}''$ are both nef, then $\mathcal{E}$ is nef.

Proof Let $f\colon C \to X$ be a k-morphism from a regular projective curve C over k, and let $f^*\mathcal{E} \to \mathcal{L}$ be an invertible quotient. By Proposition 1.24, it suffices to show that $\deg_C(\mathcal{L}) \geq 0$.

Denote by $\mathcal{L}'$ the image of $f^*\mathcal{E}'$ in $\mathcal{L}$ and by $\mathcal{L}''$ the quotient sheaf $\mathcal{L}/\mathcal{L}'$. We then have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f^*\mathcal{E}' & \longrightarrow & f^*\mathcal{E} & \longrightarrow & f^*\mathcal{E}'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{L}' & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{L}'' & \longrightarrow & 0 \end{array}$$

where the top row is exact since $\mathcal{E}''$ is locally free, and the bottom row is exact by definition. The sheaf $\mathcal{L}'$ is torsion-free since it is a subsheaf of $\mathcal{L}$, and is therefore locally free since C is regular of dimension 1; see Lemma 0AUW.

First consider the case where $\operatorname{rank}(\mathcal{L}') = 0$, in which case $\mathcal{L}' = 0$ and $\mathcal{L} \to \mathcal{L}''$ is an isomorphism. We then have $\deg_C(\mathcal{L}) = \deg_C(\mathcal{L}'') \geq 0$ by Proposition 1.24 since $\mathcal{E}''$ is nef.

It remains to consider the case where $\operatorname{rank}(\mathcal{L}') = 1$, in which case $\operatorname{rank}(\mathcal{L}'') = 0$. Additivity of Euler characteristics, Lemma 08AA, and the definition of degree, Definition 0AYR, give the first three equations:

$$\begin{aligned}\deg_C(\mathcal{L}) &= \chi(C, \mathcal{L}) - \chi(C, \mathcal{O}_C)\\ &= \chi(C, \mathcal{L}') - \chi(C, \mathcal{O}_C) + \chi(C, \mathcal{L}'')\\ &= \deg_C(\mathcal{L}') + \chi(C, \mathcal{L}'') = \deg_C(\mathcal{L}') + h^0(C, \mathcal{L}'') \geq 0.\end{aligned}$$

The fourth equation follows from Lemma 0AYT as $\mathcal{L}''$ is rank 0, and the final inequality is Proposition 1.24 as $\mathcal{E}'$ is nef. □

Our next goal is to prove that nefness is preserved under various tensor operations. The idea is to use the Barton–Kleiman Criterion, Proposition 1.24, to reduce to the curve case, in which case we will use the following:

Lemma 1.26 *Let C be a regular projective curve over an algebraically closed field k, $\mathcal{E}$ a nef finite locally free $\mathcal{O}_C$-module, and $\mathcal{H}$ an invertible $\mathcal{O}_C$-module of degree $\geq 2g$. Then $\mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{H}$ is globally generated.*

Proof We first show that if $\mathcal{H}$ is an invertible $\mathcal{O}_C$-module such that

$$H^1(C, \mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{H}) \neq 0,$$

then $\deg_C(\mathcal{H}) \leq 2g - 2$. By Serre Duality, Lemma 0FVV, we have

$$H^1(C, \mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{H}) \cong \operatorname{Hom}_{\mathcal{O}_C}(\mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{H}, \omega_C) \neq 0,$$

and we therefore have a nonzero morphism $\mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{H} \to \omega_C$. The image $\mathcal{M}$ of this morphism is torsion-free, hence invertible since C is regular of dimension 1; see Lemma 0AUW. This invertible $\mathcal{O}_C$-module $\mathcal{M}$ satisfies

$$2g - 2 = -2\chi(C, \mathcal{O}_C) = \deg_C(\omega_C) \geq \deg_C(\mathcal{M})$$

since $\mathcal{M}$ is a subsheaf of ω_C. Twisting the surjection $\mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{H} \to \mathcal{M}$ by $\mathcal{H}^{-1}$,

$$\begin{aligned}2g - 2 - \deg_C(\mathcal{H}) &\geq \deg_C(\mathcal{M}) - \deg_C(\mathcal{H})\\ &= \deg_C(\mathcal{M} \otimes_{\mathcal{O}_C} \mathcal{H}^{-1}) \geq 0\end{aligned}$$

where the equality holds by Lemma 0AYX, and the last inequality holds by the nefness of $\mathcal{E}$ and Proposition 1.24.

We now show the statement of the lemma. Let $x \in C$ be a closed point with ideal sheaf $\mathcal{O}(-x)$. We have a short exact sequence

$$0 \to \mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{H}(-x) \to \mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{H} \to \mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{H}|_x \to 0.$$

Using Lemma 0AYX again, we have

$$\deg_C(\mathcal{H}(-x)) = \deg_C(\mathcal{H}) - \deg(\mathcal{O}_C(x)) = \deg_C(\mathcal{H}) - 1 \geq 2g - 1,$$

and hence $H^1(C, \mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{H}(-x)) = 0$ by the previous paragraph. Thus, $\mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{H}$ is globally generated. □

We will also need the following to reduce to the case when the ground field k is of positive characteristic.

Lemma 1.27 *Let Y be a Noetherian scheme, and let $f: X \to Y$ be a proper morphism from an algebraic space X. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. Let $y \in Y$ be a point such that $\mathcal{E}_y$ is ample on the fibre X_y. Then there exists an open neighborhood $V \subseteq Y$ of y such that $\mathcal{E}_{y'}$ is ample on the fibre $X_{y'}$ for every point $y' \in V$.*

Proof Apply Lemma 0D3A to $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ on $\mathbf{P}(\mathcal{E})$. □

Note that the statement analogous to Lemma 1.27 for nefness does not hold as shown by Langer [24, 25] due to examples of Monsky, Brenner, and Trivedi [24, Example 5.3], of Ekedahl, Shepherd-Barron, and Taylor [24, Example 5.6], and of Moret-Bailly [25, §8].

We now prove the following result, originally due to Barton for schemes [2, Proposition 3.5(i)].

Proposition 1.28 *Let X be a proper algebraic space over k. Let $\mathcal{E}$ and $\mathcal{E}'$ be nef finite locally free $\mathcal{O}_X$-modules. Then $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}'$ is nef, as are $\mathcal{E}^{\otimes n}$, $\mathrm{Sym}^n(\mathcal{E})$, $\Gamma^n(\mathcal{E}) := (\mathrm{Sym}^n(\mathcal{E}^\vee))^\vee$, and $\bigwedge^n(\mathcal{E})$ for all $n \geq 0$.*

Proof If $\mathcal{E}$ and $\mathcal{E}'$ are nef, then $\mathcal{G} := \mathcal{E} \oplus \mathcal{E}'$ is nef by Lemma 1.25, and $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}'$ is a locally free quotient of the locally free sheaf $\mathcal{G}^{\otimes 2}$. By Lemma 1.21, it therefore suffices to show that $\mathcal{E}^{\otimes n}$, $\mathrm{Sym}^n(\mathcal{E})$, $\Gamma^n(\mathcal{E})$, and $\bigwedge^n(\mathcal{E})$ are nef. We will denote any such sheaf by $\rho^n(\mathcal{E})$. By Lemma 1.23, we may assume that k is algebraically closed.

Step 1. Proof when $\mathrm{char}(k) > 0$.

Fix a k-morphism $f: C \to X$ from a regular projective curve C over k. Let $\mathcal{L}$ be a quotient invertible sheaf of $\rho^n(\mathcal{E})$, and set $d := \deg_C(\mathcal{L})$. By Proposition 1.24, it suffices to show that $d \geq 0$.

Let $\mathcal{H}$ be an invertible $\mathcal{O}_C$-module of degree $2g$, where g is the genus of C. For every $e > 0$, consider the e-th iterate of the absolute Frobenius morphism $F^e \colon C \to C$, which is a finite morphism of degree p^e. We claim that for every $e > 0$, there is a generic isomorphism

$$(\mathcal{H}^{-n})^{\oplus r} \to F^{e*}\rho^n(f^*\mathcal{E}), \tag{$\star$}$$

where $r := \operatorname{rank}(\rho^n(f^*\mathcal{E}))$. Since $F^{e*}f^*\mathcal{E}$ is nef by Lemma 1.22, the sheaf $F^{e*}f^*\mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{H}$ is globally generated by Lemma 1.26. By choosing $s := \operatorname{rank}(f^*\mathcal{E})$ global sections that form a basis after localizing at the generic point of C, we obtain a morphism $(\mathcal{H}^{-1})^{\oplus s} \to F^{e*}f^*\mathcal{E}$ that induces an isomorphism at the generic point of C. Applying the functor $\rho^n(-)$, we obtain the generic isomorphism

$$\rho^n\big((\mathcal{H}^{-1})^{\oplus s}\big) \to F^{e*}\rho^n(f^*\mathcal{E}).$$

The left-hand side is a direct sum of the sheaves $\mathcal{H}^{-n}$, and hence passing to a direct summand, we obtain a generic isomorphism of the form in ($\star$).

We now show that $d = \deg_C(\mathcal{L}) \geq 0$. Note that $F^{e*}\mathcal{L} \cong \mathcal{L}^{\otimes p^e}$ is a quotient invertible $\mathcal{O}_C$-module of $F^{e*}\rho^n(f^*\mathcal{E})$ and that $\deg_C(F^{e*}\mathcal{L}) = p^e d$ by Lemma 0AYZ. By the previous paragraph, $(\mathcal{H}^{-n})^{\oplus r}$ surjects onto a subsheaf $\mathcal{M}$ of $F^{e*}\mathcal{L}$ that is torsion-free of rank 1, hence invertible since C is regular of dimension 1; see Lemma 0AUW. Twisting the surjection $(\mathcal{H}^{-n})^{\oplus r} \to \mathcal{M}$ by $\mathcal{H}^{\otimes n}$, we see that $\mathcal{M} \otimes_{\mathcal{O}_C} \mathcal{H}^{\otimes n}$ is nef since it is globally generated, and hence

$$\deg_C(\mathcal{M}) = \deg_C(\mathcal{M} \otimes_{\mathcal{O}_C} \mathcal{H}^{\otimes n}) + \deg_C(\mathcal{H}^{-n}) \geq -2gn$$

by Lemma 0AYX and Proposition 1.24. We then have

$$\begin{aligned} p^e d = \deg_C(F^{e*}\mathcal{L}) &= \chi(C, F^{e*}\mathcal{L}) - \chi(C, \mathcal{O}_C) \\ &= \chi(C, \mathcal{M}) - \chi(C, \mathcal{O}_C) + \chi(C, \mathcal{L}/\mathcal{M}) \\ &= \deg_C(\mathcal{M}) + h^0(C, \mathcal{L}/\mathcal{M}) \geq -2gn \end{aligned}$$

where the equalities hold by the additivity of Euler characteristics and the definition of degree; see Lemma 08AA and Definition 0AYR. Since this inequality must hold for all $e > 0$, we see that $d \geq 0$.

Step 2. Proof when $\operatorname{char}(k) = 0$.

It suffices to show that for every k-morphism $f \colon C \to X$ from a regular projective curve C over k, and every invertible quotient $\mathcal{L}$ of $\rho^n(f^*\mathcal{E})$, we have $\deg_C(\mathcal{L}) \geq -n$. Indeed, if $g \colon C' \to C$ is a finite surjective morphism of degree $e > 0$, then

$$e \cdot \deg_C(\mathcal{L}) = \deg_{C'}(g^*\mathcal{L}) \geq -n$$

holds by Lemma 0AYZ. Since this inequality must hold for all $e > 0$, we see that $\deg_C(\mathcal{L}) \geq 0$, and hence $\rho^n(f^*\mathcal{E})$ is nef by Proposition 1.24.

We now show that $\deg_C(\mathcal{L}) \geq -n$ for every morphism $f\colon C \to X$ and every quotient invertible sheaf $\mathcal{L}$ of $\rho^n(f^*\mathcal{E})$ as above. Since C is projective over k, there exists a finitely generated $\mathbf{Z}$-algebra $A \subset k$ and a projective morphism $C_A \to \operatorname{Spec}(A)$ such that the diagram

$$\begin{array}{ccc} C & \longrightarrow & C_A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_A} \\ \operatorname{Spec}(k) & \longrightarrow & \operatorname{Spec}(A) \end{array}$$

is Cartesian. Let $\mathcal{H}$ be an invertible sheaf on C of degree 1. By Lemma 0B8W, after possibly enlarging A, we may assume that there exist invertible $\mathcal{O}_{C_A}$-modules $\mathcal{H}_A$ and $\mathcal{L}_A$, and a finite locally free $\mathcal{O}_{C_A}$-module $\mathcal{F}_A$ that pull back to $\mathcal{H}$, $\mathcal{L}$, and $f^*\mathcal{E}$, on C. By Lemma 01ZR and [11, Corollaire 8.5.7], we may also assume that there exists a surjection

$$\rho^n(\mathcal{F}_A) \to \mathcal{L}_A \tag{$\star\star$}$$

that pulls back to $\rho^n(f^*\mathcal{E}) \to \mathcal{L}$ on C. Now by Proposition 1.24, the $\mathcal{O}_C$-module $\mathcal{H} \otimes_{\mathcal{O}_C} f^*\mathcal{E}$ is ample. By Lemma 1.27, after possibly replacing A by a principal localization, we may assume that $\mathcal{H}_A \otimes_{\mathcal{O}_{C_A}} \mathcal{F}_A$ is ample on every fibre of f_A, since it is ample after pulling back to the generic fibre of f_A by applying Lemma 0D2P on $\mathbf{P}(\mathcal{H}_A \otimes_{\mathcal{O}_{C_A}} \mathcal{F}_A)$. Moreover, by generic flatness, Proposition 052A, and Lemma 05F7, we may assume that f_A is flat with one-dimensional fibres.

Let $y \in \operatorname{Spec}(A)$ be a closed point with residue field $\kappa(y)$, and set $C_y := f_A^{-1}(y)$. Since f_A is flat, the invertible $\mathcal{O}_{C_A}$-modules $\mathcal{L}_A$ and $\mathcal{O}_{C_A}$ are flat over A. So, writing η for the generic point of $\operatorname{Spec}(A)$, we have

$$\begin{aligned} \deg_C(\mathcal{L}) = \deg_{C_\eta}(\mathcal{L}_\eta) &= \chi(C_\eta, \mathcal{L}_\eta) - \chi(C_\eta, \mathcal{O}_{C_\eta}) \\ &= \chi(C_y, \mathcal{L}_y) - \chi(C_y, \mathcal{O}_{C_y}) = \deg_{C_y}(\mathcal{L}_y) \end{aligned}$$

where the first equality holds by Lemma 0B59 applied to the field extension $\operatorname{Frac}(A) \subset k$, and the third equality follows from the constancy of Euler characteristics in proper flat families, Lemma 0B9T. By the same argument, $\deg_{C_y}(\mathcal{H}_y) = 1$. Since $\mathcal{H}_y \otimes_{\mathcal{O}_{C_y}} \mathcal{F}_y$ is ample, it is nef, and hence $\mathcal{H}_y^{\otimes n} \otimes_{\mathcal{O}_{C_y}} \rho^n(\mathcal{F}_y)$ is nef by Step 1. Thus, the surjection ($\star\star$)

twisted by $\mathcal{H}_A^{\otimes n}$ and then restricted to C_y implies

$$\begin{aligned}\deg_{C_y}(\mathcal{L}_y) &= \deg_{C_y}(\mathcal{H}_y^{-n} \otimes_{O_{C_y}} \mathcal{H}_y^{\otimes n} \otimes_{O_{C_y}} \mathcal{L}_y) \\ &= -n + \deg_{C_y}(\mathcal{H}_y^{\otimes n} \otimes_{O_{C_y}} \mathcal{L}_y) \geq -n\end{aligned}$$

by Lemma 0AYX and Proposition 1.24, as desired. □

We end this section with a criterion for bigness that will feature in the proof of Lemma 1.32:

Lemma 1.29 *Let X be a projective variety over k and let $\mathcal{L}$ be an invertible O_X-module. Let $\mathcal{F}$ be a finite locally free O_X-module with associated projective bundle $\pi\colon \mathbf{P} \to X$. Assume that*

(i) *$\mathcal{L}$ is nef,*
(ii) *$\mathcal{F}^\vee$ is nef, and*
(iii) *there exists $a \geq 1$ and an ample invertible sheaf $\mathcal{A}$ on X such that*

$$H^0(\mathbf{P}, O_{\mathbf{P}}(a) \otimes_{O_{\mathbf{P}}} \pi^*\mathcal{L} \otimes_{O_{\mathbf{P}}} \pi^*\mathcal{A}^{-1}) \neq 0.$$

Then $\mathcal{L}$ is big and nef.

Proof Set $d := \dim(X)$. By (i) and the asymptotic Riemann–Roch Theorem of [7, Proposition 1.31(b)], it suffices to show that the intersection number $(\mathcal{L}^d)$ is positive. By (iii), we may choose a nonzero morphism

$$O_{\mathbf{P}} \to O_{\mathbf{P}}(a) \otimes_{O_{\mathbf{P}}} \pi^*\mathcal{L} \otimes_{O_{\mathbf{P}}} \pi^*\mathcal{A}^{-1}.$$

Applying the projection formula and rearranging yields a nonzero morphism $\tau\colon \Gamma^a(\mathcal{E}^\vee) \to \mathcal{L} \otimes_{O_X} \mathcal{A}^{-1}$. Since the sheaf on the right-hand side is locally trivial, the image of τ is of the form $\mathcal{I} \otimes_{O_X} (\mathcal{L} \otimes_{O_X} \mathcal{A}^{-1})$ for some coherent sheaf of ideals $\mathcal{I}$. Let $f\colon Y \to X$ be the blowup along $\mathcal{I}$, with exceptional divisor D. Then $f^*\tau$ gives a surjection

$$f^*\Gamma^a(\mathcal{E}^\vee) \twoheadrightarrow \mathcal{M} := f^*\mathcal{L} \otimes_{O_Y} f^*\mathcal{A}^{-1} \otimes_{O_Y} O_Y(-D).$$

By (ii), Proposition 1.28, and Lemma 1.22(i), the sheaf on the left-hand side is nef, hence by Lemma 1.21, $\mathcal{M}$ is also nef. Rearranging gives

$$f^*\mathcal{L} \cong f^*\mathcal{A} \otimes_{O_Y} \mathcal{M} \otimes_{O_Y} O_Y(D).$$

Since f is birational, $\dim(Y) = d = \dim(X)$ and, by Lemma 0BET, $(f^*\mathcal{L}^d) = (\mathcal{L}^d)$ and $(f^*\mathcal{A}^d) = (\mathcal{A}^d)$. In particular, the latter quantity is positive since $\mathcal{A}$ is ample, see Lemma 0BEV. Additivity of intersection numbers, Lemma 0BER, gives

$$(\mathcal{L}^d) = (f^*\mathcal{L}^d) = (f^*\mathcal{A}^d) + \sum\nolimits_{i=1}^d (f^*\mathcal{A}^{d-i} \cdot f^*\mathcal{L}^{i-1} \cdot \mathcal{M}(D)).$$

The latter sum is nonnegative: by additivity and restriction, Lemmas 0BER and 0BEU, the i-th summand is the sum

$$(f^*\mathcal{A}^{d-i} \cdot f^*\mathcal{L}^{i-1} \cdot \mathcal{M}) + (f^*\mathcal{A}^{d-1}|_D \cdot f^*\mathcal{L}^{i-1}|_D) \geq 0$$

of intersection numbers of nef invertible sheaves, and hence each nonnegative by [14, Lemma 2.12]. Therefore $(\mathcal{L}^d) \geq (\mathcal{A}^d) > 0$. □

5 Ampleness Lemma

In this section, we formulate a method for proving ampleness of line bundles of the form $\det(Q)$, where Q is a locally free quotient of a symmetric power of a nef finite locally free sheaf $\mathcal{E}$. The basic method is due to Kollár in [21, Lemmas 3.9 and 3.13], refining an idea of Viehweg [31]. We also include a refinement due to Kovács and Patakfalvi [22].

The idea is as follows: locally, Q is a quotient by a trivial vector bundle, so $\det(Q)$ is locally the pullback of the Plücker bundle under a classifying map to a Grassmannian. Globalize this by passing to its frame bundle to universally trivialize $\mathcal{E}$; the quotient bundle now gives a classifying map to a stack of the form $[\mathbf{G}(N, q)/\mathrm{PGL}_n]$. The Ampleness Lemma 1.33 is then a generalization of the familiar fact that the pullback of an ample sheaf under a finite map is ample.

We begin by constructing frame bundles. Let S be a scheme and let $\mathcal{E}$ be a finite locally free $\mathcal{O}_S$-module of rank n. Let T be a scheme and consider triples $(f \colon T \to S, \mathcal{L}, \psi)$ where

(i) $f \colon T \to S$ is a morphism of schemes,
(ii) $\mathcal{L}$ is an invertible $\mathcal{O}_T$-module, and
(iii) $\psi \colon \mathcal{O}_T^{\oplus n} \to f^*\mathcal{E} \otimes_{\mathcal{O}_T} \mathcal{L}$ is an isomorphism of $\mathcal{O}_T$-modules.

Call two triples $(f, \mathcal{L}, \psi)$ and $(f', \mathcal{L}', \psi')$ over T *equivalent* if $f = f'$ and if there exists an isomorphism $\beta \colon \mathcal{L} \to \mathcal{L}'$ such that $\beta \circ \psi = \psi'$. The *frame functor of* $\mathcal{E}$ is the functor

$$\begin{aligned} \mathrm{Fr}(\mathcal{E}) \colon \mathrm{Sch}^{\mathrm{opp}} &\to \mathrm{Sets} \\ T &\mapsto \{\text{equivalence classes of } (f \colon T \to S, \mathcal{L}, \psi) \text{ as above}\} \end{aligned}$$

with pullbacks under $T' \to T$ defined as expected.

Two important structures: First, projection of $(f \colon T \to S, \mathcal{L}, \psi)$ onto the first factor yields a morphism of functors $\mathrm{Fr}(\mathcal{E}) \to S$. Second, given $f \colon T \to S$, the set of equivalence classes of $(f, \mathcal{L}, \psi)$ admit a simply transitive action of PGL_n via pre-composition on ψ; note this is well-defined since automorphisms of $\mathcal{L}$ are given by scalar multiplication. Therefore $\mathrm{Fr}(\mathcal{E})$ is a functor of PGL_n-sets over S.

Lemma 1.30 *Let S be a scheme. Let $\mathcal{E}$ be a finite locally free O_S-module of rank n and set* $\mathbf{P} := \mathbf{P}(\mathcal{H}om(\mathcal{E}, O_S^{\oplus n}))$. *There exists an effective Cartier divisor* $\mathbf{D} \subset \mathbf{P}$ *such that* $\mathrm{Fr}(\mathcal{E})$ *is represented by the open subscheme*

$$\mathbf{Fr}(\mathcal{E}) = \mathbf{P} \setminus \mathbf{D}.$$

The structure map $\mathbf{Fr}(\mathcal{E}) \to S$ *exhibits this as a* PGL_n*-torsor over* S.

Proof Consider a triple $(f : T \to S, \mathcal{L}, \psi)$ as above. By adjunction, the isomorphism $\psi : O_T^{\oplus n} \to f^*\mathcal{E} \otimes_{O_T} \mathcal{L}$ uniquely determines a surjection $\varphi : f^*\mathcal{H}om(\mathcal{E}, O_S^{\oplus n}) \to \mathcal{L}$. This exhibits $\mathrm{Fr}(\mathcal{E})$ as the subfunctor of the projective bundle $\mathbf{P}$ on which ψ is an isomorphism.

On the other hand, let $\pi : \mathbf{P} \to S$ be the structure map and consider the universal quotient $\varphi_{\mathrm{univ}} : \pi^*\mathcal{H}om(\mathcal{E}, O_S^{\oplus n}) \to O_{\mathbf{P}}(1)$. By adjunction, this yields an injective map $\varphi^{\#}_{\mathrm{univ}} : O_{\mathbf{P}}^{\oplus n} \to O_{\mathbf{P}}(1) \otimes_{O_{\mathbf{P}}} \pi^*\mathcal{E}$ and hence a universal determinant

$$\det(\varphi^{\#}_{\mathrm{univ}}) : O_{\mathbf{P}} \to \det(O_{\mathbf{P}}(1) \otimes_{O_{\mathbf{P}}} \pi^*\mathcal{E}).$$

Let $\mathbf{D}$ be the divisor determined by its vanishing. Then the open subscheme $\mathbf{Fr}(\mathcal{E}) := \mathbf{P} \setminus \mathbf{D}$ represents the functor $\mathrm{Fr}(\mathcal{E})$. □

We call the scheme $\mathbf{Fr}(\mathcal{E})$ the *frame bundle* of $\mathcal{E}$ over S. The torsor structure on the frame bundle induces a classifying map from S to the classifying stack $B\mathrm{PGL}_n$ fitting into a Cartesian diagram

$$\begin{array}{ccc} \mathbf{Fr}(\mathcal{E}) & \longrightarrow & \mathrm{pt} \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ S & \longrightarrow & B\mathrm{PGL}_n \end{array}$$

We now construct lifts of this classifying map to quotient stacks of certain Grassmannians whenever given, additionally, $\alpha : \mathrm{Sym}^d(\mathcal{E}) \to Q$ a finite locally free quotient of rank q, with d some positive integer. The strategy is to pull the quotient back to the frame bundle and take symmetric powers of the *universal trivialization map*

$$\psi_{\mathrm{univ}} := \varphi^{\#}_{\mathrm{univ}}|_{\mathbf{Fr}(\mathcal{E})} : O^{\oplus n}_{\mathbf{Fr}(\mathcal{E})} \to O_{\mathbf{Fr}(\mathcal{E})}(1) \otimes_{O_{\mathbf{Fr}(\mathcal{E})}} \pi^*\mathcal{E}$$

to give PGL_n-equivariant morphisms to $\mathbf{G} := \mathbf{G}(N, q)$, the Grassmannian parameterizing rank q quotients of the module $\mathrm{Sym}^d(\mathbf{Z}^{\oplus n}) \cong \mathbf{Z}^{\oplus N}$

with $N = \binom{n+d-1}{d}$. Note ψ_{univ} is equivariant for the action of PGL_n on $\mathbf{Fr}(\mathcal{E})$, where the action is tautological on the source and trivial on the target; likewise, PGL_n acts on $\mathbf{G}$ via the action on $\mathrm{Sym}^d(\mathbf{Z}^{\oplus n})$ induced by its tautological action.

Lemma 1.31 *Notation as above, there exists a commutative diagram*

$$\begin{array}{ccccc} \mathbf{Fr}(\mathcal{E}) & \xrightarrow[{[\pi^*\alpha]}]{} & \mathbf{G} & \longrightarrow & \mathrm{pt} \\ {\scriptstyle\pi}\downarrow & & \downarrow & & \downarrow \\ S & \xrightarrow{[\alpha]} & [\mathbf{G}/\mathrm{PGL}_n] & \longrightarrow & B\mathrm{PGL}_n \end{array}$$

such that all squares are Cartesian. Moreover, writing $\mathcal{O}_{\mathbf{G}}(1)$ for the Plücker line bundle on $\mathbf{G}$, we have

$$[\pi^*\alpha]^*\mathcal{O}_{\mathbf{G}}(1) \cong \mathcal{O}_{\mathbf{Fr}(\mathcal{E})}(qd) \otimes_{\mathcal{O}_{\mathbf{Fr}(\mathcal{E})}} \pi^* \det(\mathcal{Q}).$$

Proof Pulling back α to $\mathbf{Fr}(\mathcal{E})$ and pre-composing with the d-th symmetric power of the universal trivialization ψ_{univ} gives a surjection

$$\mathrm{Sym}^d(\mathcal{O}_{\mathbf{Fr}(\mathcal{E})}^{\oplus n}) \to \mathcal{O}_{\mathbf{Fr}(\mathcal{E})}(d) \otimes \pi^* \mathrm{Sym}^d(\mathcal{E}) \to \mathcal{O}_{\mathbf{Fr}(\mathcal{E})}(d) \otimes \pi^*\mathcal{Q}.$$

The universal property of $\mathbf{G}$ yields a morphism $[\pi^*\alpha]\colon \mathbf{Fr}(\mathcal{E}) \to \mathbf{G}$ which is PGL_n-equivariant by the description of the actions above, and such that the pullback of the universal quotient bundle on $\mathbf{G}$ is $\mathcal{O}_{\mathbf{Fr}(\mathcal{E})}(d) \otimes \pi^*\mathcal{Q}$. Since the Plücker line bundle is the determinant of the universal quotient, this gives the identification of line bundles. Finally, this data of a PGL_n-torsor over S together with a PGL_n-equivariant morphism to $\mathbf{G}$ is precisely the data of a morphism $[\alpha]\colon S \to [\mathbf{G}/\mathrm{PGL}_n]$ lifting the classifying map for $\mathbf{Fr}(\mathcal{E})$; see Sections 04UI and 04UV. □

The morphism $[\alpha]\colon S \to [\mathbf{G}/\mathrm{PGL}_n]$ is called the *classifying map* of α. The aim is to pull positivity back to $\det(\mathcal{Q})$ via $[\alpha]$ from $\mathcal{O}_{\mathbf{G}}(1)$. This is achieved most directly by asking for $[\alpha]$ to be a quasi-finite morphism of stacks; see Definition 0G2M and compare with [21, Definition 3.8]. Concretely, since S is a scheme, $[\alpha]$ is a representable morphism, so by Lemma 04XD, $[\alpha]$ is quasi-finite if and only if $[\pi^*\alpha]\colon \mathbf{Fr}(\mathcal{E}) \to \mathbf{G}$ is a quasi-finite morphism of schemes. Kovács and Patakfalvi observed in [22, Theorem 5.5] that, when working a field k, it is sufficient to ask for $[\alpha]$ to have finite fibres on $\bar{k}$-points.

The following statement is the heart of the Ampleness Lemma, and is an analogue of the fact that the pullback of an ample line bundle by a generically quasi-finite morphism is big.

Lemma 1.32 *In the situation of Lemma 1.31, assume that*

(i) *S is a normal projective variety over k,*
(ii) *$\mathcal{E}$ is nef, and*
(iii) *there exists a dense open subset $S_0 \subseteq S$ over which the classifying map $[\alpha]$ has finite fibres on $\bar{k}$-points.*

Then $\det(Q)$ *is big and nef. In particular,* $(\det(Q)^{\dim(S)}) > 0$.

Proof We aim to apply Lemma 1.29 with $\mathcal{F} := \mathcal{H}om(\mathcal{E}, \mathcal{O}_S^{\oplus n})$ and $\mathcal{L} := \det(Q)^{\otimes m}$ for some appropriately chosen integer $m > 0$. The first two hypotheses are already satisfied: 1.29(i) is because $\mathcal{L}$ is a tensor power of a determinant of a quotient of a nef sheaf, see Lemma 1.21 and Proposition 1.28; 1.29(ii) is because $\mathcal{F}^\vee \cong \mathcal{E}^{\oplus n}$ is a sum of nef bundles and hence is itself nef by Lemma 1.25.

It remains to arrange for condition 1.29(iii). The construction of the classifying map in Lemma 1.31 gives a rational map $[\pi^*\alpha] : \mathbf{P} \dashrightarrow \mathbf{G}$. Blowing up the ideal sheaf in the image of

$$\bigwedge^q \operatorname{Sym}^d(\mathcal{O}_{\mathbf{P}}^{\oplus n}) \to \mathcal{O}_{\mathbf{P}}(qd) \otimes_{\mathcal{O}_{\mathbf{P}}} \pi^* \det(Q)$$

induced by $\pi^*\alpha \circ \operatorname{Sym}^d(\psi_{\text{univ}})$ yields a birational morphism $b : \mathbf{P}' \to \mathbf{P}$, a morphism $f : \mathbf{P}' \to \mathbf{G}$ resolving $[\pi^*\alpha]$, and an effective Cartier divisor D of $\mathbf{P}'$ such that

$$f^*\mathcal{O}_{\mathbf{G}}(1) = b^*\big(\mathcal{O}_{\mathbf{P}}(qd) \otimes_{\mathcal{O}_{\mathbf{P}}} \pi^* \det(Q)\big) \otimes_{\mathcal{O}_{\mathbf{P}'}} \mathcal{O}_{\mathbf{P}'}(-D).$$

Let $\mathbf{T}$ be the schematic image of $(\pi \circ b, f) : \mathbf{P}' \to S \times_k \mathbf{G}$, and let $\rho : \mathbf{T} \to S$ and $g : \mathbf{T} \to \mathbf{G}$ be the induced morphisms. We claim that g is generically quasi-finite. Identify $\mathbf{Fr}(\mathcal{E})$ as a dense open subscheme of $\mathbf{P}'$ via the birational morphism b. Let $\mathbf{T}_0 \subseteq \mathbf{T}$ be the dense set obtained as the intersection of the image of $\mathbf{Fr}(\mathcal{E})$, which is a dense constructible subset by Chevalley's Theorem 054K, and the open set $S_0 \times_k \mathbf{G}$ with S_0 from hypothesis (iii). Then there is a commutative diagram

$$\begin{array}{ccccc}
\mathbf{Fr}(\mathcal{E})|_{S_0} & & \xrightarrow{[\pi^*\alpha]} & & \mathbf{G} \\
& \searrow & & & \\
& & \mathbf{T}_0 & \xrightarrow{g} & \mathbf{G} \\
& \pi \searrow & \downarrow \rho & & \downarrow \\
& & S_0 & \xrightarrow{[\alpha]} & [\mathbf{G}/\mathrm{PGL}_n]
\end{array}$$

Let $\bar{x} \in \mathbf{G}(\bar{k})$ be a $\bar{k}$-point and let $\bar{y} \in [\mathbf{G}/\mathrm{PGL}_n](\bar{k})$ be its image along the quotient map. Since the outer square is Cartesian by Lemma 1.31, the fibre $(\mathbf{Fr}(\mathcal{E})|_{S_0})_{\bar{x}}$ maps via π to $S_{0,\bar{y}}$. Since $\mathbf{T}_0$ is the image of $\mathbf{Fr}(\mathcal{E})|_{S_0}$ in $S_0 \times_k \mathbf{G}$, this implies that $\mathbf{T}_{0,\bar{x}}$ is contained in the finite set $S_{0,\bar{y}} \times \{\bar{x}\}$.

Thus g has finite fibres on $\bar{k}$-points over $\mathbf{T}_0$, so $\mathbf{T}$ contains a dense set of closed points at which the fibre dimension of g is 0. This latter set is open by Lemma 02FZ. Therefore g is generically quasi-finite.

We may now complete the proof of the Lemma. Since $\mathcal{O}_{\mathbf{G}}(1)$ is ample and g is generically quasi-finite, $g^*\mathcal{O}_{\mathbf{G}}(1)$ is a big invertible sheaf on $\mathbf{T}$. Let $\mathcal{A}$ be any very ample invertible sheaf on S. Then Lemma 1.18 gives

$$H^0(\mathbf{T}, g^*\mathcal{O}_{\mathbf{G}}(m) \otimes_{\mathcal{O}_T} \rho^*\mathcal{A}^{-1}) \neq 0 \quad \text{for some integer } m > 0.$$

Pulling back to $\mathbf{P}'$, multiplying by an equation of the effective divisor D, and then applying the projection formula gives

$$\begin{aligned} 0 &\neq H^0(\mathbf{P}', f^*\mathcal{O}_{\mathbf{G}}(m) \otimes_{\mathcal{O}_{\mathbf{P}'}} b^*\pi^*\mathcal{A}^{-1}) \\ &\subset H^0(\mathbf{P}', b^*(\mathcal{O}_{\mathbf{P}}(qdm) \otimes_{\mathcal{O}_{\mathbf{P}}} \pi^* \det(Q)^{\otimes m} \otimes_{\mathcal{O}_{\mathbf{P}}} \pi^*\mathcal{A}^{-1})) \\ &\cong H^0(\mathbf{P}, (\mathcal{O}_{\mathbf{P}}(qdm) \otimes_{\mathcal{O}_{\mathbf{P}}} \pi^* \det(Q)^{\otimes m} \otimes_{\mathcal{O}_{\mathbf{P}}} \pi^*\mathcal{A}^{-1}) \otimes_{\mathcal{O}_{\mathbf{P}}} b_*\mathcal{O}_{\mathbf{P}'}). \end{aligned}$$

Now $\mathbf{P}$ is normal by hypothesis (i), so the Stein factorization of the birational map b is trivial; see Theorem 03H0. In particular, $b_*\mathcal{O}_{\mathbf{P}'} \cong \mathcal{O}_{\mathbf{P}}$. Setting $\mathcal{L} := \det(Q)^{\otimes m}$ and $a := qdm$, we conclude that

$$H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(a) \otimes_{\mathcal{O}_{\mathbf{P}}} \pi^*\mathcal{L} \otimes_{\mathcal{O}_{\mathbf{P}}} \pi^*\mathcal{A}^{-1}) \neq 0.$$

Thus hypothesis (iii) of Lemma 1.29 is satisfied and it applies to show that $\det(Q)^{\otimes m}$ is big and nef, and so $\det(Q)$ is itself big and nef. □

Proposition 1.33 (Ampleness Lemma) *Let X be a proper algebraic space over k, $\mathcal{E}$ a locally free $\mathcal{O}_X$-module of rank n, d a positive integer, and $\alpha\colon \operatorname{Sym}^d(\mathcal{E}) \to Q$ a locally free quotient of rank q. Assume that*

(i) *$\mathcal{E}$ is nef, and*
(ii) *the classifying map $[\alpha]$ has finite fibres on $\bar{k}$-points.*

Then $\det(Q)$ *is ample on X.*

Proof We aim to apply the Nakai–Moishezon Criterion, Proposition 1.10. Thus we need to show that $\det(Q)$ has positive degree on each integral closed subspace $\iota\colon Y \hookrightarrow X$. Applying Chow's Lemma 088U and normalizing gives a modification $f\colon Y' \to Y$ from a normal projective variety Y'. Compatibility of intersection numbers with pullbacks, Lemma 0EDJ, gives

$$(\det(Q)^{\dim(Y)} \cdot Y) = (\iota^* \det(Q)^{\dim(Y)}) = (f^*\iota^* \det(Q)^{\dim(Y')}).$$

This final quantity is positive by Lemma 1.32: the pullback of $\mathcal{E}$ to Y'

is nef by Lemma 1.22(i), and the classifying map on Y' associated with the pullback of α is the composite

$$[f^*\iota^*\alpha]\colon Y' \xrightarrow{f} Y \xrightarrow{\iota} X \xrightarrow{[\alpha]} [\mathbf{G}/\mathrm{PGL}_n]$$

which generically has finite fibres on $\bar{k}$-points as each of f, i, and $[\alpha]$ do. □

6 Nefness for families of nodal curves

In this section, we prove that $f_*\omega_{S/C}^{\otimes m}$ is nef for all $m \geq 2$ and any family $f\colon S \to C$ of stable curves over a smooth projective curve C over k; see Theorem 1.43. In other words, we show that the corresponding vector bundle on the stack $\overline{\mathcal{M}}_g$ is nef.

Since nefness is insensitive to field extensions by Lemmas 1.14 and 1.23, throughout, we assume our base field k is algebraically closed. Furthermore, all schemes and morphisms appearing will be over k. We will make constant use of the following transitivity property of relative dualizing sheaves: by Lemma 0E30, there is an isomorphism

$$\omega_{S/C} \cong \omega_S \otimes_{\mathcal{O}_S} f^*\omega_C^{-1}.$$

The first positivity result is Proposition 1.36 and it concerns families in which the generic fibre is smooth. This is generalized in Proposition 1.40 to positivity when $\omega_{S/C}$ is twisted up by some sections. Finally, as a general family of stable curves is essentially obtained by glueing generically smooth families along horizontal curves, this gives us the main positivity result in Theorem 1.42.

To begin, we discuss the local structure of nodal families of curves. So let $f\colon S \to C$ be a nodal family of curves over a smooth projective curve C. Consider the closed subset $\mathrm{Sing}(f) \subset S$ of points at which f is not smooth. This has a canonical scheme structure given by the first Fitting ideal of $\Omega^1_{S/C}$; see Section 0C3H.

Lemma 1.34 *Let $f\colon S \to C$ be a family of nodal curves over a smooth projective curve C.*

(i) *If s is an isolated point of* $\mathrm{Sing}(f)$*, then*

$$\mathcal{O}_{S,s}^{\wedge} \cong \mathcal{O}_{C,f(s)}^{\wedge}[[x,y]]/(xy - \pi^n)$$

where π is a uniformizer of $\mathcal{O}_{C,f(s)}^{\wedge}$ and $n \geq 1$.

(ii) *If s is not isolated in* $\mathrm{Sing}(f)$, *then there exists a commutative diagram*

$$\begin{array}{ccccc} S & \longleftarrow & U & \longrightarrow & W \\ {\scriptstyle f}\downarrow & & \downarrow & \swarrow & \\ C & \longleftarrow & V & & \end{array}$$

where $W := V \otimes_k k[u, v]/(uv)$, the morphisms $S \leftarrow U \to W$ and $C \leftarrow V$ are étale, and there is a point $u \in U$ mapping to $s \in S$.

Proof In the isolated case, this follows from Lemma 0CBX, noting that all nodes are split since we assume k is algebraically closed. In the non-isolated case, this follows from Lemma 0CBY. See also 0CDD. □

The isolated points in $\mathrm{Sing}(f)$ as in 1.34(i) are rational double points and can be resolved by repeated blowup. See Section 0BGB and also [1]. Since the singularity is rational, we may harmlessly pass to a resolution of such singularities:

Lemma 1.35 *Let $f : S \to C$ be a family of nodal curves over a smooth projective curve C. Let $b : S' \to S$ be the minimal resolution of the isolated singularities of S. Then $b_*\omega_{S'/C} \cong \omega_{S/C}$.*

Proof There is a canonical morphism $b_*\omega_{S'/C} \to \omega_{S/C}$ obtained by dualizing the map $b^\# : \mathcal{O}_S \to b_*\mathcal{O}_{S'}$. This map is an isomorphism: it is clear on the locus where b is an isomorphism; around the singular points, this follows from Lemma 0BBU. □

We are now ready for the first positivity result, concerning families of nodal curves in which the generic fibre is smooth. In this case, the total space is normal as only the isolated singularities of Lemma 1.34(i) may appear. Compare with [21, Proposition 4.5].

Proposition 1.36 *Let $f : S \to C$ be a family of nodal curves over a smooth projective curve C. If the generic fibre of f is smooth of genus $g \geq 2$, then $f_*\omega_{S/C}^{\otimes m}$ is nef for any $m \geq 2$.*

We first prove Proposition 1.36 under a series of simplifying assumptions in Lemma 1.38, then explain afterward how these assumptions may be removed. The crucial input is the following consequence of Ekedahl's vanishing theorems for surfaces of general type.

Lemma 1.37 *Suppose that* $\mathrm{char}(k) = p > 0$. *Let S be a smooth projective minimal surface of general type, and D a reduced effective*

Cartier divisor with smooth connected components of genus at least 2 *and* $\mathcal{O}_S(D)|_D \cong \mathcal{O}_D$. *Then for any* $m \geq 2$,

$$h^1(S, \omega_S^{\otimes m}(D)) \begin{cases} = 0 & \text{if } \operatorname{char}(k) \neq 2 \text{ or } m \neq 2, \\ \leq 1 & \text{if } \operatorname{char}(k) = 2 \text{ and } m = 2, \end{cases}$$

Proof From the cohomology of the exact sequence

$$0 \to \omega_S^{\otimes m} \to \omega_S^{\otimes m}(D) \to \omega_S^{\otimes m}(D)|_D \to 0,$$

it suffices to show $h^1(S, \omega_S^{\otimes m}(D)|_D) = 0$ and bound $h^1(S, \omega_S^{\otimes m})$. For the former, the adjunction formula, Lemma 0B4B, gives

$$\omega_S^{\otimes m}(D)|_D \cong \omega_S^{\otimes m}(mD)|_D \cong \omega_D^{\otimes m}.$$

Since the genus of each connected component of D is at least 2,

$$H^1(S, \omega_S^{\otimes m}(D)|_D) \cong H^1(D, \omega_D^{\otimes m}) = 0$$

when $m \geq 2$, by degree reasons, see Lemma 0B90. The bound on $h^1(S, \omega_S^{\otimes m})$ follows from vanishing theorem of Ekedahl [9, Main theorem. (i)]. □

Lemma 1.38 *Proposition 1.36 holds with additional assumptions that*

(i) *the characteristic of* k *is* $p > 0$,
(ii) S *is minimal, and*
(iii) *the genus of* C *is at least 2.*

Proof If $f_*\omega_{S/C}^{\otimes m}$ is not nef, then the Barton–Kleiman Criterion, Proposition 1.24 gives an invertible quotient $\alpha\colon f_*\omega_{S/C}^{\otimes m} \to \mathcal{M}^{-1}$ such that $d := \deg(\mathcal{M}) > 0$. The assumption on the genus of the fibres of f together with (iii) imply that a resolution of S is of general type; see [4, Theorem 1.3]. We now seek a contradiction to Lemma 1.37.

Let $F_C\colon C \to C$ be the absolute Frobenius of C and consider the base change $f'\colon S' \to C$ of f along F_C. This is still a family of nodal curves by Lemma 0C5B. Since smoothness is stable under base change by Lemma 01VB, the generic fibre of f' is also smooth. Since formation of dualizing sheaves commutes with base change, see Lemmas 0B91 and 0E6R,

$$F_C^* f_* \omega_{S/C}^{\otimes m} \cong f'_* g^* \omega_{S/C}^{\otimes m} \cong f'_* \omega_{S'/C}^{\otimes m}$$

where $g\colon S' \to S$ is the projection. Pulling α back by F_C yields a negative quotient $f'_*\omega_{S'/C}^{\otimes m} \to F_C^*\mathcal{M}^{-1}$ of degree $-dp$. Replacing f by

f', we can take $d = \deg(\mathcal{M})$ to be arbitrarily large. Thus we may assume $\mathcal{M} \cong \mathcal{L} \otimes_{\mathcal{O}_C} \omega_C^{\otimes m}$ for some very ample invertible $\mathcal{O}_C$-module $\mathcal{L}$.

Since the generic fibre of $f: S \to C$ is assumed to be smooth, S has only isolated rational double points as in Lemma 1.34(i). A minimal resolution of singularities is obtained by repeated blowups and the resulting exceptional divisor is a chain of projective lines joined along nodes. Thus the minimal resolution of singularities of S will remain a family of nodal curves over C. Therefore, by furthermore using Lemma 1.35, we may replace S by its minimal resolution of its singularities and assume that S is a smooth minimal surface of general type.

Upon rearranging terms of α, we obtain a surjection of sheaves

$$\mathcal{L} \otimes_{\mathcal{O}_C} \omega_C^{\otimes m} \otimes_{\mathcal{O}_C} f_*\omega_{S/C}^{\otimes m} \twoheadrightarrow \mathcal{O}_C.$$

Since C is of dimension 1, we obtain the inequality

$$h^1(C, \mathcal{L} \otimes_{\mathcal{O}_C} \omega_C^{\otimes m} \otimes_{\mathcal{O}_C} f_*\omega_{S/C}^{\otimes m}) \geq h^1(C, \mathcal{O}_C) = g.$$

On the other hand, consider the invertible $\mathcal{O}_S$-module

$$\mathcal{F} := f^*\mathcal{L} \otimes_{\mathcal{O}_S} (f^*\omega_C^{\otimes m} \otimes_{\mathcal{O}_S} \omega_{S/C}^{\otimes m}) \cong f^*\mathcal{L} \otimes_{\mathcal{O}_S} \omega_S^{\otimes m}$$

where we have used transitivity of dualizing sheaves. Since f has relative dimension 1, the Leray spectral sequence, Lemma 01F2, for f and $\mathcal{F}$ degenerates on the E_2-page and yields a short exact sequence

$$0 \to H^1(C, f_*\mathcal{F}) \to H^1(S, \mathcal{F}) \to H^0(C, R^1 f_*\mathcal{F}) \to 0.$$

The projection formula gives $f_*\mathcal{F} \cong \mathcal{L} \otimes_{\mathcal{O}_C} \omega_C^{\otimes m} \otimes_{\mathcal{O}_C} f_*\omega_{S/C}^{\otimes m}$, so this sequence together with the inequality above gives

$$h^1(S, f^*\mathcal{L} \otimes_{\mathcal{O}_S} \omega_S^{\otimes m}) = h^1(S, \mathcal{F}) \geq h^1(C, f_*\mathcal{F}) \geq g \geq 2.$$

Since $\mathcal{L}$ is very ample, we may choose an effective Cartier divisor D in $|f^*\mathcal{L}|$ which is the union of smooth fibres of f. Then $f^*\mathcal{L} \cong \mathcal{O}_S(D)$ yields a contradiction to Lemma 1.37. Therefore $f_*\omega_{S/C}^{\otimes m}$ is nef. □

Proof of Proposition 1.36 We explain how to remove the assumptions (i), (ii), and (iii) of Lemma 1.38.

We may reduce to characteristic $p > 0$ as in Step 2 in the proof of Proposition 1.28. That is, if k were of characteristic 0 and $f_*\omega_{S/C}^{\otimes m}$ had a negative quotient, then choose a finitely generated $\mathbf{Z}$-algebra over which everything is defined. We may then reduce modulo some prime p to yield a contradiction to Lemma 1.38. Thus we may drop assumption (i).

If S were not minimal, consider any (-1)-curve E. Then E is contained in fibres of f since, otherwise, $f|_E : E \to C$ would be a dominant morphism from a curve of genus 0 to a curve of genus $g \geq 2$, which is impossible. So contracting E as in Lemma 0C2N yields a normal projective surface S' a morphism $f' : S' \to C$ such that $f = f' \circ b$, where $b : S \to S'$ is the contraction map. Since $b_* \omega_{S'} \cong \omega_S$, transitivity of relative dualizing sheaves implies $f'_* \omega_{S'/C}^{\otimes m} \cong f_* \omega_{S/C}^{\otimes m}$. Successively contracting (-1)-curves will produce a minimal model $f_{\min} : S_{\min} \to C$ of $f : S \to C$. Induction on the number of contractions gives

$$f_* \omega_{S/C}^{\otimes m} \cong f_{\min,*} \omega_{S_{\min}/C}^{\otimes m}$$

and nefness of the former follows from the nefness of the latter. Thus we may drop both assumptions (i) and (ii) in Lemma 1.38.

Finally, if the genus of C is less than 2. Let $g : C' \to C$ be any finite cover from a smooth projective curve C' of genus at least 2 and let $f' : S' \to C$ be the base change of f. Then, as before, f' is a family of nodal curves with smooth generic fibre and $g^* f_* \omega_{S/C}^{\otimes m} = f'_* \omega_{S'/C}^{\otimes m}$. This is nef by Lemma 1.38. Hence $f_* \omega_{S/C}^{\otimes m}$ is also nef by Lemma 1.22(ii). This completes the proof. □

As a consequence, we obtain the following weak positivity result for $\omega_{S/C}$ on S. See also [21, Corollary 4.6].

Corollary 1.39 *In the situation of Proposition 1.36, let C_t be a section of f. Then $(\omega_{S/C} \cdot C_t) \geq 0$.*

Proof Consider the pushforward along f of the sequence

$$0 \to \omega_{S/C}^{\otimes m}(-C_t) \to \omega_{S/C}^{\otimes m} \to \omega_{S/C}^{\otimes m}|_{C_t} \to 0.$$

We have $R^1 f_*(\omega_{S/C}^{\otimes m}(-C_t)) = 0$ by looking at degrees along fibres, see Lemma 0B90. So this gives a surjection $f_* \omega_{S/C}^{\otimes m} \twoheadrightarrow f_* \omega_{S/C}^{\otimes m}|_{C_t}$. But $f|_{C_t} : C_t \to C$ is an isomorphism, so this is an invertible quotient of degree $(\omega_{S/C} \cdot C_t)$. By the nefness of Proposition 1.36, we conclude that $(\omega_{S/C} \cdot C_t) \geq 0$. □

Towards positivity for general families of stable curves, we need the following generalization of Proposition 1.36, in which the relative dualizing sheaf is twisted up by sections, and where the fibres of f may be of genus 0 or 1. Compare with [21, Proposition 4.7].

Proposition 1.40 *Let $f : S \to C$ be a family of nodal curves over a smooth projective curve C. If the generic fibre of f is smooth, then, for* *any set of pairwise distinct sections $C_1, \dots, C_n$ of f contained in the smooth locus of S,*

$$f_*(\omega_{S/C}^{\otimes m}(a_1C_1 + \cdots + a_nC_n))$$

is nef for any $m \geq 2$ and any $0 \leq a_1, \dots, a_n \leq m$.

Proof Since the C_i avoid the singularities of S, we may reduce to case in which S is smoooth by passing to a minimal resolution of singularities of S using Lemma 1.35. We split the proof into three cases, depending on whether the genus of the generic fibre of f is ≥ 2, 0 or 1. Each case will proceed by induction on $j := \sum a_i$.

Case 1. The generic fibre of f is of genus $g \geq 2$.

Here the base case where each $a_i = 0$ is Proposition 1.36. Assume the claim is proven for $D_j := \sum a_iC_i$; we will prove it for $D_{j+1} := D_j + C_t$ for any index t such that $a_t + 1 \leq m$. Consider the exact sequence

$$0 \to \omega_{S/C}^{\otimes m}(D_j) \to \omega_{S/C}^{\otimes m}(D_{j+1}) \to \omega_{S/C}^{\otimes m}(D_{j+1})|_{C_t} \to 0$$

obtained by twisting sequence for C_t by $\omega_{S/C}^{\otimes m}(D_{j+1})$. Since the C_i are pairwise disjoint, together with transitivity of relative dualizing sheaves and the adjunction formula, we have

$$\begin{aligned}\omega_{S/C}^{\otimes m}(D_{j+1})|_{C_t} &\cong \omega_{S/C}^{\otimes m-a_t-1}|_{C_t} \otimes \left(\omega_S^{\otimes a_t+1}((a_t+1)C_t)|_{C_t} \otimes \omega_{C_t}^{\otimes -a_t-1}\right) \\ &\cong \omega_{S/C}^{\otimes m-a_t-1}|_{C_t}.\end{aligned}$$

Because $a_t + 1 \leq m$, Corollary 1.39 together with Lemma 0BEY shows that this invertible sheaf has non-negative degree on C_t. Also note that $R^1f_*(\omega_{S/C}^{\otimes m}(D_j)) = 0$ due to degree on the fibres; see Lemma 0B90.

Thus applying f_* to the above exact sequence yields an exact sequence

$$0 \to f_*\big(\omega_{S/C}^{\otimes m}(D_j)\big) \to f_*\big(\omega_{S/C}^{\otimes m}(D_{j+1})\big) \to f_*\big(\omega_{S/C}^{\otimes m-a_t-1}|_{C_t}\big) \to 0.$$

The subsheaf is nef by the induction hypothesis, and the quotient sheaf is a nonnegative invertible sheaf on C, as C_t is a section. Thus the extension is nef by Lemma 1.25, completing the induction in this case.

Case 2. The generic fibre of f is of genus $g = 0$.

When $j = \sum a_i \leq 2m - 1$, the sheaf $\omega_{S/C}^{\otimes m}(\sum a_iC_i)$ is negative on fibres of f and hence has vanishing, whence nef, pushforward; these are the base cases. Let $j \geq 2m - 1$ and assume that the claim is true for all divisors of the form $\sum a_iC_i$ with $\sum a_i = j$; we will prove it for $D = C_t + \sum a_iC_i$ for any index t such that $a_t + 1 \leq m$.

We can assume that $(C_t^2) \leq 0$. Indeed, by the Hodge Index Theorem, we may assume that among $C_1, \dots, C_n$, the only section with positive self-intersection is C_1. So if $D = C_1 + \sum a_i C_i$, as $a_1 \leq m < 2m - 1$, there is some index $t \neq 1$ such that $a_t \neq 0$. Thus we may write

$$D = C_1 + \sum a_i C_i = C_t + \sum a_i' C_i$$

with $a_1' := a_1 + 1$, $a_t' := a_t - 1$, and $a_i' := a_i$ for $i \neq 1, t$. Then $\sum a_i = \sum a_i' = j$ and induction will apply to $\sum a_i' C_i$. With this, we see by the adjunction formula as in Case 1, that

$$\omega_{S/C}(C_t)|_{C_t} \cong \mathcal{O}_{C_t} \quad \text{so} \quad (\omega_{S/C} \cdot C_t) = -(C_t^2) \geq 0.$$

From here, induction proceeds as in Case 1.

Case 3. The generic fibre of f is of genus $g = 1$.

To establish the base case and the nonnegativity $(\omega_{S/C} \cdot C_t) \geq 0$, we claim that it suffices to show $\chi(S, \mathcal{O}_S) \geq 0$. Indeed, the canonical bundle formula for elliptic surfaces in [3, Theorem 2] gives

$$\omega_{S/C} \cong f^* \mathcal{M} \otimes_{\mathcal{O}_S} \mathcal{O}_S(F)$$

where F is an effective Cartier divisor supported along fibres of f and $\mathcal{M}$ is an invertible $\mathcal{O}_C$-module with degree $\geq \chi(S, \mathcal{O}_S)$. Thus

$$f_*(\omega_{S/C}^{\otimes m}) \cong \mathcal{M}^{\otimes m} \quad \text{and} \quad (\omega_{S/C} \cdot C_t) \geq (f^* \mathcal{M} \cdot C_t) \geq \chi(S, \mathcal{O}_S)$$

so nefness of $f_*(\omega_{S/C}^{\otimes m})$ and nonnegativity will follow from $\chi(S, \mathcal{O}_S) \geq 0$.

Since $\chi(S, \mathcal{O}_S)$ is a birational invariant, we may in fact assume S is minimal over C. In this case, the effective Cartier divisor F is actually a sum of fibre classes, at least viewed as a **Q**-Cartier divisor: see [3, Bottom of p. 28]. Thus ω_S is a sum of fibre classes and so $(\omega_S^2) = 0$. Noether's Formula then gives

$$12\chi(S, \mathcal{O}_S) = (\omega_S^2) + e(S) = e(S) \geq e(C)e(S_{\bar{\eta}}) = 0,$$

where e denotes ℓ-adic topological Euler characteristic, ℓ any prime different from p, and $S_{\bar{\eta}}$ is the geometric generic fibre of $f: S \to C$. The inequality follows from [23, Lemma 1], and $e(S_{\bar{\eta}}) = 0$ since the generic fibre of f is a smooth curve of genus 1. With this, induction may proceed as in Case 1, and the proof of the Proposition is complete. □

To obtain a positivity result for a general family $f: S \to C$ of stable curves, it remains to consider the non-isolated singularities of Lemma 1.34(ii). Let D be the subscheme of 1-dimensional components of $\mathrm{Sing}(f)$, and call it the *double locus* of S. The following explains how a general family of nodal curves is obtained by glueing nodal families with only double points along the double locus:

Lemma 1.41 *Let $f\colon S \to C$ be a family of nodal curves over a smooth projective curve C. Let $\nu\colon S^\nu \to S$ be the normalization. Then S^ν is a disjoint union of nodal families of curves over C with smooth generic fibre and $\omega_{S^\nu/S} \cong \mathcal{O}_{S^\nu}(-D^\nu)$ where $D^\nu := \nu^{-1}(D)$.*

Proof Let $s \in D$ be a point of the double locus of S and consider the diagram of Lemma 1.34(ii):

$$\begin{array}{ccccc} S & \longleftarrow & U & \longrightarrow & W \\ {\scriptstyle f}\downarrow & & \downarrow & \swarrow & \\ C & \longleftarrow & V & & \end{array}$$

Since the morphisms $S \leftarrow U \to W$ are étale and normalization commutes with smooth base change by Lemma 03GV, there are étale morphisms $S^\nu \leftarrow U^\nu \to W^\nu$. Since $C \leftarrow V$ is also étale, V is smooth, so the same Lemma gives

$$W^\nu = V \otimes_k (k[u] \times k[v]) \to V \otimes_k k[u,v]/(uv) = W.$$

In particular, W^ν is smooth. As the morphisms from U^ν are étale, we conclude that S^ν, locally around s, is the disjoint union of two families of nodal curves over C with smooth generic fibre. Since this is true for all $s \in D$, S^ν itself is a disjoint union of families of nodal curves over C with smooth generic fibre.

For $\omega_{S^\nu/S}$, since ν is a finite morphism, its relative dualizing sheaf is characterized by the formula

$$\nu_*\omega_{S^\nu/S} = \mathcal{H}om_{\mathcal{O}_S}(\nu_*\mathcal{O}_{S^\nu}, \mathcal{O}_S);$$

see Section 0FKW. Evaluation at 1 yields an injection $\nu_*\omega_{S^\nu/S} \to \mathcal{O}_S$ whose image is an ideal sheaf $\mathcal{I}$ of a subscheme supported on D. In fact, this is the ideal sheaf of D. To see this, since formation of the evaluation map commutes with flat pullback (see Lemmas 0C6I and 02KH), using the local structure of S around $s \in D$ above, it suffices to show that, for

$$R^\nu := k[u] \times k[v] \leftarrow k[u,v]/(uv) =: R$$

we have $I := \mathrm{Hom}_R(R^\nu, R) = (u, v)$. Indeed, R^ν is generated as an R-module by $(1,0)$ and $(0,1)$, and they are annihilated by v and u, respectively, so any R-module map $\varphi\colon R^\nu \to R$ must be of the form

$$\varphi((1,0)) = \alpha u \quad \text{and} \quad \varphi((0,1)) = \beta v \quad \text{for some } \alpha, \beta \in R.$$

Furthermore, this shows that the image of I under the ring extension $R \to R^\nu$ is the ideal of the two preimages of the node. Hence we conclude that $\nu_*\omega_{S^\nu/S} \cong \mathcal{I}$ the ideal sheaf of D in S, and so $\omega_{S^\nu/S} \cong \mathcal{O}_{S^\nu}(-D^\nu)$ is the ideal sheaf of D^ν in S^ν. □

With the notation above, we have the following intermediate result:

Proposition 1.42 *Let $f\colon S \to C$ be a family of stable curves over a smooth projective curve C. Assume that*

(i) *the double curve D is a union of sections of f, and*
(ii) *its preimage $D^\nu := \nu^{-1}(D)$ is a union of sections of $f^\nu\colon S^\nu \to C$.*

Then $f_*\omega_{S/C}^{\otimes m}$ is nef for any $m \geq 2$.

Proof By transitivity of relative dualizing sheaves and Lemma 1.41, $\nu^*\omega_{S/C} \cong \omega_{S^\nu/C}(D^\nu)$. Thus pulling $\omega_{S/C}^{\otimes m}$ back to S^ν and tensoring with the subscheme sequence for D^ν yields

$$0 \to \omega_{S^\nu/C}^{\otimes m}((m-1)D^\nu) \to \nu^*(\omega_{S/C}^{\otimes m}) \to \omega_{S^\nu/C}^{\otimes m}(mD^\nu)|_{D^\nu} \to 0.$$

Since D^ν is an effective Cartier divisor of S^ν, the adjunction formula, Lemma 0AA4, together with hypothesis (ii) gives

$$\omega_{S^\nu/C}(D^\nu)|_{D^\nu} \cong \omega_{D^\nu/C} \cong \mathcal{O}_{D^\nu}.$$

Applying ν_* to the short exact sequence yields an exact sequence on S:

$$0 \to \nu_*\big(\omega_{S^\nu/C}^{\otimes m}((m-1)D^\nu)\big) \to \omega_{S/C}^{\otimes m} \otimes_{\mathcal{O}_S} \nu_*\mathcal{O}_{S^\nu} \to \mathcal{O}_D^{\oplus 2} \to 0.$$

Since the preimage of the antidiagonal $\mathcal{O}_D$ along the map $\nu_*\mathcal{O}_{S^\nu} \to \mathcal{O}_D^{\oplus 2}$ is $\mathcal{O}_S$, there is a short exact sequence

$$0 \to \nu_*\big(\omega_{S^\nu/C}^{\otimes m}((m-1)D^\nu)\big) \to \omega_{S/C}^{\otimes m} \to \mathcal{O}_D \to 0.$$

Now push down to C. Write $f^\nu := f \circ \nu\colon S^\nu \to C$. Since the fibres of f are stable curves, the fibres of f^ν are stable pointed curves, so $R^1 f^\nu_*(\omega_{S^\nu/C}^{\otimes m}((m-1)D^\nu)) = 0$ for all $m \geq 2$. The relative Leray spectral sequence for f^ν, Lemma 0734, shows that $R^1 f_*\nu_*(-)$ is a subsheaf of $R^1 f^\nu_*(-)$. Thus applying f_* to the preceding short exact sequence yields

$$0 \to f^\nu_*\big(\omega_{S^\nu/C}^{\otimes m}((m-1)D^\nu))\big) \to f_*\omega_{S/C}^{\otimes m} \to f_*\mathcal{O}_D \to 0.$$

The term on the left is nef by Lemma 1.41 together with (ii) and Proposition 1.40; by (i), the sheaf $f_*\mathcal{O}_D$ is isomorphic to the sum of copies of $\mathcal{O}_C$. Thus $f_*\omega_{S/C}^{\otimes m}$ is an extension of a direct sum of non-negative line bundles by a nef bundle, and hence nef by Lemma 1.25. □

Putting everything together now gives the main positivity result.

Theorem 1.43 *Let $f\colon S \to C$ be a family of stable curves over a smooth projective curve C. Then $f_*\omega_{S/C}^{\otimes m}$ is nef for any $m \geq 2$.*

Proof In order to apply Proposition 1.42 to f, we need to arrange for the components of the double curve D and its preimage D^ν in the normalization $\nu\colon S^\nu \to S$ to be sections over C. So let C' be any such component and form the Cartesian diagram

$$\begin{array}{ccc} S' & \longrightarrow & S \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ C' & \xrightarrow{\;g\;} & C \end{array}$$

By Lemma 0E76, f' is still a family of stable curves. Moreover, the inverse image of C' is now a section over C'. Since $g^* f_*\omega_{S/C}^{\otimes m} \cong f'_*\omega_{S'/C'}^{\otimes m}$, by Lemma 1.22(ii), we may replace f by f'. Repeating this for every component of D and D^ν, we may arrange for hypotheses (i) and (ii) of Proposition 1.42 to be verified, upon which we may conclude. □

7 Projectivity of the moduli of curves

Finally, we put everything together to show that the Deligne–Mumford moduli space $\overline{M}_g$ of stable curves is projective over $\operatorname{Spec}(\mathbf{Z})$.

The first step is to show that for a family of curves $f\colon X \to S$ over an algebraically closed field k whose moduli map has finite fibres, there is some m such that λ_m pulls back to an ample invertible sheaf on S. In fact, $m = 6$ works by using the fact that tri-canonically embedded stable curves are projectively normal and are determined by their quadratic equations, see [28, Corollary on p. 58]. In the following, we argue directly and only show that $m = 3d$ work for all sufficiently large d, perhaps depending on the family f.

Lemma 1.44 *Let $f\colon X \to S$ be a family of stable curves of genus $g \geq 2$ over an algebraically closed field k. If the moduli map $[f]\colon S \to \overline{\mathcal{M}}_g$ is of finite type and has finite fibres on k-points, then $[f]^*\lambda_{3d} = \det(f_*\omega_{X/S}^{\otimes 3d})$ is ample on S for all $d \gg 0$.*

Proof We apply the Ampleness Lemma 1.33 to the multiplication map

$$\mu_d\colon \operatorname{Sym}^d(f_*\omega_{X/S}^{\otimes 3}) \to f_*\omega_{X/S}^{\otimes 3d}.$$

We choose d sufficiently large so that

(i) the fibres of f are determined by their degree d equations in their tri-canonical embedding, and
(ii) μ_d is surjective.

To see that this is possible, note that by Lemma 0E8X, $\omega_{X/S}^{\otimes 3}$ is f-very ample, so there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\iota} & \mathbf{P} \\ & \searrow^{f} & \downarrow \pi \\ & & S \end{array} \qquad \text{where } \mathbf{P} := \mathbf{P}(f_*\omega_{X/S}^{\otimes 3})$$

and ι is a closed immersion in which the fibres of $f \colon X \to S$ are embedded as tri-canonical curves of degree $6g-6$. Thus (i) is satisfied for any $d \geq 6g-6$, as can be seen by taking joins with disjoint codimension 3 linear spaces; see for example [28, Theorem 1].

As for (ii), let $\mathcal{I}$ be the ideal sheaf of X in $\mathbf{P}$ and consider the sequence

$$0 \to \mathcal{I} \otimes_{\mathcal{O}_\mathbf{P}} \mathcal{O}_\mathbf{P}(d) \to \mathcal{O}_\mathbf{P}(d) \to \iota_*\omega_{X/S}^{\otimes 3d} \to 0.$$

Then μ_d is the direct image under π of the surjection. Now S is Noetherian as it is of finite type over $\overline{\mathcal{M}}_g$, which is of finite presentation over $\mathrm{Spec}(\mathbf{Z})$ (see Lemmas 0DSS and 0E9B). Thus relative Serre Vanishing, Lemma 02O1, applies to give a d_0 such that

$$R^1\pi_*(\mathcal{I} \otimes_{\mathcal{O}_\mathbf{P}} \mathcal{O}_\mathbf{P}(d)) = 0 \quad \text{for all } d \geq d_0$$

whence (ii) is satisfied for any $d \geq d_0$.

Choose any $d \geq \max(6g-6, d_0)$ and set $\mu := \mu_d$. We now verify the hypotheses of the Ampleness Lemma 1.33. The basic positivity is given by Theorem 1.43, ensuring that $f_*\omega_{X/S}^{\otimes 3}$ is nef. To understand the classifying map, fix a closed point $0 \in S$ and set $V := H^0(X_0, \omega_{X_0/k}^{\otimes 3})$. For each closed point $s \in S$, choose an isomorphism

$$\varphi_s \colon V \xrightarrow{\cong} H^0(X_s, \omega_{X_s/k}^{\otimes 3})$$

to view X_s as being embedded in $\mathbf{P}V$. We obtain maps

$$\mu_{0,s} \colon \mathrm{Sym}^d(V) \xrightarrow{\mathrm{Sym}^d(\varphi_s)} \mathrm{Sym}^d(H^0(X_s, \omega_{X_s/k}^{\otimes 3})) \xrightarrow{\mu|_s} H^0(X_s, \omega_{X_s/k}^{\otimes 3d})$$

whose kernel is the space of degree d equations defining X_s in $\mathbf{P}V$. Up to the action of $\mathrm{PGL}(V)$ on the source, $\mu_{0,s}$ is independent of the choice of isomorphism φ_s. Since $R^1 f_*\omega_{X/S}^{\otimes 3} = 0$ by Lemma 0E8X, the base change maps on direct images are isomorphisms by Lemma 0D2M. Therefore the classifying map of Lemma 1.31 is identified with the map

$$[\mu]\colon S \to [\mathbf{G}(\mathrm{Sym}^d(V), q)/\mathrm{PGL}(V)] \quad \text{where } q := (6d-1)(g-1),$$

which sends a closed point s of S to the $\mathrm{PGL}(V)$ equivalence class K_s of $\ker(\mu_{0,s})$. Now condition (i) from our choice of d implies that for any two closed points $s, s' \in S$,

$$K_s = K_{s'} \quad \text{if and only if} \quad X_s \cong X_{s'}$$

meaning $[\mu]$ has finite fibres on k-points if and only if $[f]\colon S \to \overline{\mathcal{M}}_g$ does. Thus the Ampleness Lemma 1.33 applies to show that $f_*\omega_{X/S}^{\otimes 3d}$ is ample. □

Theorem 1.45 *The moduli space $\overline{M}_g$ of stable curves of genus $g \geq 2$ is projective over* $\mathrm{Spec}(\mathbf{Z})$.

Proof Since $\overline{\mathcal{M}}_g$ is quasi-compact, by Lemma 0E9B, Lemma 1.5 allows us to choose an integer n such that the invertible sheaf $\lambda_m^{\otimes n}$ descends to an invertible sheaf $\mathcal{L}_m$ on $\overline{M}_g$ for all m. We show that there exists some m such that $\mathcal{L}_m$ is ample over $\mathrm{Spec}(\mathbf{Z})$.

By Lemmas 0E7A and 1.3, $\overline{\mathcal{M}}_g$ is a Deligne–Mumford stack with a moduli space, so [32, Proposition 2.6] shows there exists a scheme S and a finite surjective morphism $\varphi\colon S \to \overline{\mathcal{M}}_g$. We have a diagram

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & \overline{\mathcal{M}}_g \\ & {\scriptstyle\pi}\searrow & \downarrow{\scriptstyle f} \\ & & \overline{M}_g \end{array}$$

We claim that $\pi := f \circ \varphi$ is a finite surjective morphism of algebraic spaces. Indeed, π is the composition of a finite surjective map φ with a universal homeomorphism f (see Theorem 0DUT), so π is surjective with discrete fibres. By Lemma 0A4X, finiteness of π will now follow from properness of π. Since $\overline{\mathcal{M}}_g$ is proper over $\mathrm{Spec}(\mathbf{Z})$ by Theorem 0E9C, the same is true for both S and $\overline{M}_g$, by Lemmas 0CL7 and 0DUZ, respectively. Hence π is proper by Lemma 04NX.

By Lemma 0GFB, $\mathcal{L}_m$ is ample on $\overline{M}_g$ over $\mathrm{Spec}(\mathbf{Z})$ if and only if $\pi^*\mathcal{L}_m$ is ample on S over $\mathrm{Spec}(\mathbf{Z})$. Thus it suffices to show that there exists some m such that $\pi^*\mathcal{L}_m = \varphi^*\lambda_m^{\otimes n}$ is ample over $\mathrm{Spec}(\mathbf{Z})$. Let p be a prime number and let S_p be the base change of S along

$\mathrm{Spec}(\overline{\mathbf{F}}_p) \to \mathrm{Spec}(\mathbf{Z})$. The restriction $\varphi_p \colon S_p \to \overline{\mathcal{M}}_g$ of φ to S_p is finite and satisfies

$$\varphi_p^* \lambda_m^{\otimes n} = \varphi^* \lambda_m^{\otimes n}|_{S_p} = \pi^* \mathcal{L}_m|_{S_p}.$$

By Lemma 1.44, we may choose d_p such that $\varphi_p^* \lambda_{3d}^{\otimes n}$ is ample for all $d \geq d_p$. Now Lemma 0D2N gives an open neighbourhood U_p of p in $\mathrm{Spec}(\mathbf{Z})$ over which $\varphi^* \lambda_{3d}^{\otimes n}$ is ample. By quasi-compactness, there exists a finite set of primes P such that $\mathrm{Spec}(\mathbf{Z}) = \bigcup_{p \in P} U_p$. Then $\pi^* \mathcal{L}_m$ is ample over $\mathrm{Spec}(\mathbf{Z})$ for any $m = 3d$ with $d \geq \max(d_p \mid p \in P)$. □

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