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RESEARCH

Differential Multiplant Monopoly on a Freight Network

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Abstract
In this paper we give a mathematical statement of a differential spatial monopoly wherein transactions are facilitated by a freight transportation network that is based on an underlying graph. The monopolist has a presence at every node, and freight services are available for paths connecting stipulated origin-destination (OD) pairs. The monopolist produces a single homogeneous product that may be manufactured and inventoried, as well as sold, at every node. The dynamics take the form of ordinary differential equations that describe flow conservation. We present the necessary conditions of the monopolist's optimal control problem and observe their interpretation is that marginal revenue equals marginal cost at each node for each instant of continuous time. Existence of an optimal solution to the monopolist's problem is proven and a numerical example solved using two algorithms: one implemented in discrete time and the other in continuous time.

Keywords Spatial monopoly · Network monopoly · Differential game

1 Introduction

In this paper, we consider a differential spatial monopoly involving multiple sales, production, and inventory facilities located at the nodes of a transportation network based on a graph, where $\mathcal{A}$ is the set of arcs and $\mathcal{N}$ is the set of nodes. The monopolist we study has a presence at spatially separated consumption sites (nodes) of a network economy for which the consumption sites are connected by a freight transportation network. The monopolist produces a single homogeneous product that is sold in a conventional retail setting at the consumption sites. In our presentation, for simplicity of notation, we assume every node is a consumption/production/inventory

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site.[1] That is to say, the monopolist has the ability to not only produce and price but also to position, reposition, and hold inventory at all nodes of the transportation network. The market for freight services is assumed to be perfectly competive because freight agents are numerous in that they serve many other clients competing in markets unrelated to the monopolist's output market; thus, the monopolist is a price taker in the market for freight services needed to transport its goods between nodes, but it is not a price taker in the markets that sell its output. In fact, the monopolist sets the allowed consumption levels of its output to its advantage, so that, via market-specific (node-specific) demand functions, it is setting prices for its output. These allocations of output to consumption are intrinsic to the dynamic monopolist we model within the network economy of our interest.

The time scale is long enough for the monopolist to use inventory-based strategies, but short enough that relocation or expansion/contraction of the monopolist's presence cannot occur. We refer to a monopoly possessing the aforementioned features as a differential multiplant monopoly on a freight network (DMMFN). It was first formulated by Friesz (2010) under the assumption of separable variable costs, which is relaxed herein. Friesz (2010) provided neither an analysis/interpretation of necessary conditions nor a proof of existence, both of which are included in this paper for the first time. We also comment that the DMMFN is crudely similar to the model suggest by Dasci and Laporte (2004), which emphasizes travel to stores to purchase consumer goods.

In general, the scholarly literature on dynamic (differential) monopoly is not an energetic field of inquiry at the present, with most papers having appeared roughly 20 to 40 years ago. Among these Gul et al. (1986) is notable for being significantly mathematical and highly cited, although it does not address spatial or network considerations. More recent works dealing with monopoly in a spatial and/or network context include Bensaid and Lesne (1996), Lambertini and Orsini (2007); Zaker (2012), Bensaid and Lesne (1996); and Li (2021).

2 Monopoly in a Network Economy

We imagine a network based on an underlying graph that connects each production node i to each consumption node j by at least one path $p \in \mathcal{P}_{ij}$, where $\mathcal{P}_{ij}$ is the set of paths connecting origin-destination (OD) pair (i, j). A path is comprised of a sequence of arcs for which freight services are available at a fixed tariff r_{ij} for shipment rates s_{ij} of the monopolist's single homogeneous output between OD pair (i, j). The following identity applies:

$$s_{ij} = \sum_{p \in \mathcal{P}_{ij}} h_p \;\; \forall \, (i, j) \in \mathcal{W} \tag{1}$$

where h_p is the departure rate for path p and $\mathcal{W}$ is the set of all OD pairs, while s_{ij} is the departure rate for shipments between $(i, j) \in \mathcal{W}$. As already stated, the firm of

[1] This assumption may easily be relaxed.

interest has a presence at all the nodes of a transportation network, for which there are paths used by shipping agents to satisfy the monopolist's demands for freight services. Time is denoted by the scalar $t \in \Re^1_+$, fixed initial time by $t_0 \in \Re^1_+$, fixed final time by $t_f \in \Re^1_{++}$, with $t_0 < t_f$ so that $t \in [t_0, t_f] \subset \Re^1_+$. There are three sets important to articulating our formulation of differential spatial monopoly; these are as follow: $\mathcal{A}$ for directed arcs, $\mathcal{N}$ for nodes and $\mathcal{W}$ for origin-destination (OD) pairs. Subsets of these sets are formed as is meaningful by using the subscript i for a specific node and ij for a specific OD pair (i, j).Our perspective, in that we are concerned with inventory and shipping decisions, has much in common with the logistics and supply chain literature but is without concern about the arrival times of shipments; rather, we assume all dispatched shipments ultimately reach their destinations and backorders are allowed in the form of negative inventories and virtual OD flows that become real when either output and/or positive inventory allows. Explicit delays of goods enroute can be accomodated using the more complicated mathematical apparatus developed in Friesz and Lin (2024a) and its application to differential spatial monopoly is the subject of separate manuscripts (Friesz and Lin 2024b, c). One may additionally expand the network detail in the model presented herein to reflect shipping routes (paths) in detail. The routing of goods as well as the transport costs incurred in meeting product demand would serve to make the connection to logistics and supply chains complete, since routing is always a logistical consideration. Moreover, by using a bilevel formulation one could design any desired aspects of a logistical network in light of commodity prices (as determined by our model of monopoly) and their fluctuations, thereby providing valuable strategic insight. However, these refinements and extensions are not within the scope of the present paper.

The firm controls production output rates expressed as a vector q, allocations of output to meet demand (consumption) expressed as a vector c, and shipping patterns expressed as a vector s. Inventories I are a vector of state variables determined by the controls. That is:

$$c \in \left(L^2\left[t_0, t_f\right]\right)^{|\mathcal{N}|} \tag{2}$$

$$q \in \left(L^2\left[t_0, t_f\right]\right)^{|\mathcal{N}|} \tag{3}$$

$$s \in \left(L^2\left[t_0, t_f\right]\right)^{|\mathcal{W}|} \tag{4}$$

$$I\left(c, q, s\right) : \left(L^2\left[t_0, t_f\right]\right)^{|\mathcal{N}|} \times \left(L^2\left[t_0, t_f\right]\right)^{|\mathcal{N}|} \times \left(L^2\left[t_0, t_f\right]\right)^{|\mathcal{W}|} \longrightarrow \left(\mathcal{H}^1\left[t_0, t_f\right]\right)^{|\mathcal{N}|} \tag{5}$$

where $L^2[t_0, t_f]$ is the space of square-integrable functions and $\mathcal{H}^1[t_0, t_f]$ is a Sobolev space for the real interval $[t_0, t_f] \in \Re^1_+$.

3 The Network Monopoly's Extremal Problem

Let us define the inverse demand for our single good at node $i \in \mathcal{N}$ to be $\pi_i(c, t)$ where c_i is the consumption rate and

$$c(t) = (c_i(t) : i \in \mathcal{N}) \tag{6}$$

The firm has the objective of maximizing net profit expressed as revenue less cost and taking the form of an operator acting on production rates, shipment patterns and consumption rates. For simplicity we imagine that the monopolist operates at every node and that every node is a market for the firm's output. That is, the firm's net profit is

$$J_0(c, q, s) = \int_{t_0}^{t_f} e^{-\rho t} \left\{ \sum_{i \in \mathcal{N}} \pi_i(c_i, t) c_i - V(q, t) - \sum_{(i,j) \in \mathcal{W}} r_{ij} s_{ij} - \sum_{i \in \mathcal{N}} \psi_i(I_i, t) \right\} dt \tag{7}$$

where $\rho \in \Re^1_{++}$ is a constant nominal rate of discount, r_{ij} is the fixed exogenous freight rate (tariff) charged per unit of flow s_{ij} for OD pair $(i, j) \in \mathcal{W}$,

$$\psi_i : \mathcal{H}^1[t_0, t_f] \longrightarrow \mathcal{H}^1[t_0, t_f]$$

is the firm's separable inventory cost at node i, and I_i is the inventory/backorder volume at node i. Furthermore, $V(q, t)$ is the monopolist's cost of production as a function of the vector q of outputs at its nodal locations:

$$V(q, t) : \left(L^2[t_0, t_f]\right)^{|\mathcal{N}|} \times \Re^1_+ \longrightarrow \mathcal{H}^1[t_0, t_f]$$

Output at node $i \in \mathcal{N}$ will subsequently be denoted by q_i. In Eq. 7, c_i is the rate of consumption at node i. Our formulation is based on inverse demand functions:

$$\pi_i(c_i, t) : L^2[t_0, t_f] \times \Re^1_+ \longrightarrow \mathcal{H}^1[t_0, t_f]$$

Note that $J(c, q, s)$ is a functional that is completely determined by the controls (c, q, s). The first term of the functional $J(c, q, s)$ in expression Eq. 7 is the monopolist's total revenue; the second term is the monopolist's cost of production; the third term is its shipping cost; and the last term is its inventory or backorder cost.We also impose the terminal time inventory constraints

$$I_i\left(t_f\right) = K_i \quad \forall\, i \in \mathcal{N} \tag{8}$$

where the $K_i \in \Re^1_{++}$ are exogenous. All consumption, production and shipping control variables are non-negative and bounded from above. That is

$$C \geq c \geq 0 \tag{9}$$

$$Q \geq q \geq 0 \tag{10}$$

$$S \geq s \geq 0 \tag{11}$$

where

$$C \in \Re^{|\mathcal{F}|}_{++}$$
$$Q \in \Re^{|\mathcal{F}|}_{++}$$
$$S \in \Re^{|\mathcal{W}|}_{++}$$

are known constant vectors. Constraints Eqs. 9, 10 and 11 are recognized as pure control constraints, while Eq. 8 are terminal conditions for the state space variables.The inventory dynamics, expressing simple flow conservation, obey

$$\frac{dI_i}{dt} = q_i + \sum_{j:(j,i)\in\mathcal{W}} s_{ji} - \sum_{j:(i,j)\in\mathcal{W}} s_{ij} - c_i \quad \forall\, i \in \mathcal{N} \tag{12}$$

$$I_i\left(t_0\right) = I_i^0 \quad \forall\, i \in \mathcal{N} \tag{13}$$

$$I_i\left(t_f\right) = I_i^f \quad \forall i \in \mathcal{N} \tag{14}$$

where every $I_i^0 \in \Re^1_{++}$ and every $I_i^f \in \Re^1_{++}$ are exogenous. We will view the vector of inventories as the the following operator:

$$I\left(c, q, s\right) = \arg\left\{\frac{dI_i}{dt} = q_i + \sum_{j:(j,i)\in\mathcal{W}} s_{ji} - \sum_{j:(i,j)\in\mathcal{W}} s_{ij} - c_i \right. \\ \left. I_i\left(t_0\right) = I_i^0, I_i\left(t_f\right) = K_i, \forall i \in \mathcal{N}\right\} \tag{15}$$

where we have assumed that the dynamics have solutions for all feasible controls. Thus, the monopolist solves the following optimal control problem:

$$\max J_0(c,q,s) = \int_{t_0}^{t_f} \left[e^{-\rho t} \sum_{i \in \mathcal{N}} \pi_i (c_i , t)\, c_i - V(q,t) - \sum_{(i,j) \in \mathcal{W}} r_{ij}\, s_{ij} - \sum_{i \in \mathcal{N}} \psi_i(I_i) \right] dt \quad (16)$$

$$\frac{dI_i}{dt} = q_i + \sum_{j:(j,i) \in \mathcal{W}} s_{ji} - \sum_{j:(i,j) \in \mathcal{W}} s_{ij} - c_i \qquad \forall i \in \mathcal{N} \quad (17)$$

$$I_i (t_0) = I_i^0 \qquad \forall i \in \mathcal{N} \quad (18)$$

$$I_i (t_f) = K_i \qquad \forall i \in \mathcal{N} \quad (19)$$

$$(c, q, s) \in \Omega \quad (20)$$

where

$$\Omega \equiv \{c, q, s : (9), (10), (11)\} \quad (21)$$

In the event the state operator is used to rewrite Eqs. 16-20, we obtain this infinite-dimensional mathematical progamming formulation of the monopolist's problem:

$$\max J_0(c,q,s) = \int_{t_0}^{t_f} e^{-\rho t} \left[\sum_{i \in \mathcal{N}} \pi_i (c_i , t)\, c_i - V(q,t) - \sum_{(i,j) \in \mathcal{W}} r_{ij}\, s_{ij} - \sum_{i \in \mathcal{N}} \psi_i(I_i(c,q,s)) \right] dt \quad (22)$$

$$(c, q, s) \in \Omega \quad (23)$$

This infinite-dimensional mathematical programming form of the problem is useful for the study of the continuous time algorithms for the application studied herein, especially convergence, as documented in Friesz (2010).

4 First Look at Necessary Conditions

In this section we are concerned with the technical chore of expressing the necessary conditions for an optimal solution of the monopolist's optimal control problem. After the necessay conditions have been stated, we will turn to the task of giving them an economic interpretation for certain special cases, as well as the general case. We begin this process by articulating the Hamiltonian for the monopolist's optimal control problem; the relevant Hamiltonian is the following:

$$
\begin{aligned}
H_0 = e^{-\rho t} & \left[\sum_{i \in \mathcal{N}} \pi_i \left(c_i \, , t \right) c_i - \; V(q,t) - \sum_{(i,j) \in \mathcal{W}} r_{ij} \, s_{ij} - \sum_{i \in \mathcal{N}} \psi_i(I_i) \right] \\
& + \sum_{i \in \mathcal{N}} \lambda_i \left[q_i + \sum_{j:(j,i) \in \mathcal{W}} s_{ji} - \sum_{j:(i,j) \in \mathcal{W}} s_{ij} - c_i \right] \\
& + \sum_{i \in \mathcal{N}} \nu_i \cdot e^{-\rho t_f} \left[I_i \left(t_f \right) - K_i \right] \\
& + \sum_{i \in \mathcal{N}} \left[\alpha_i \left(-c_i \right) + \beta_i \left(c_i - C_i \right) \right] \\
& + \sum_{i \in \mathcal{N}} \left[\gamma_i \left(-q_i \right) + \zeta_i \left(q_i - Q_i \right) \right] \\
& + \sum_{(i,j) \in \mathcal{W}} \left[\eta_{ij} \left(-s_{ij} \right) + \mu_{ij} \left(s_{ij} - S_{ij} \right) \right]
\end{aligned}
\tag{24}
$$

where λ_i is the adjoint variable associated with each of the state dynamics Eq. 12, for all $i \in \mathcal{N}$. The adjoint variables obey the adjoint equations and associated terminal-time conditions known as the transversality conditions; these are

$$
\frac{d\lambda_i}{dt} = -\frac{\partial H_0}{\partial I_i} = \frac{\partial \psi_i(I_i)}{\partial I_i} \quad \forall i \in \mathcal{N} \tag{25}
$$

$$
\lambda_i \left(t_f \right) = \frac{\partial}{\partial I_i \left(t_f \right)} v_i \cdot e^{-\rho t_f} \left[I_i \left(t_f \right) - K_i \right] = e^{-\rho t_f} v_i \quad \forall i \in \mathcal{N} \tag{26}
$$

The maximum principle requires that the Hamiltonian be maximized with respect to the control variables. As such the following Kuhn-Tucker conditions, including complementary slackness for control constraints apply:

$$
\frac{\partial H_0}{\partial c_i} = e^{-\rho t} \frac{\partial \pi_i \left(c_i \, , t \right) c_i}{\partial c_i} - \lambda_i - \alpha_i + \beta_i = 0 \quad \forall i \in \mathcal{N} \tag{27}
$$

$$
\alpha_i c_i = 0 \quad \alpha_i \geq 0 \quad c_i \geq 0 \quad \forall i \in \mathcal{N} \tag{28}
$$

$$
\beta_i \left(c_i - C_i \right) = 0 \quad \beta_i \geq 0 \quad c_i - C_i \leq 0 \quad \forall i \in \mathcal{N} \tag{29}
$$

$$
\frac{\partial H_0}{\partial q_i} = -e^{-\rho t} \frac{\partial V \left(q \, , t \right)}{\partial q_i} + \lambda_i - \gamma_i + \zeta_i = 0 \quad \forall i \in \mathcal{N} \tag{30}
$$

$$\gamma_i q_i = 0 \quad \gamma_i \geq 0 \quad q_i \geq 0 \quad \forall i \in \mathcal{N} \tag{31}$$

$$\zeta_i (q_i - Q_i) = 0 \quad \zeta_i \geq 0 \quad q_i - Q_i \leq 0 \quad \forall i \in \mathcal{N} \tag{32}$$

$$\frac{\partial H_0}{\partial s_{ij}} = -e^{-\rho t} r_{ij} + \lambda_j - \lambda_i - \eta_{ij} + \mu_{ij} = 0 \quad \forall i \in \mathcal{N}, j \in \mathcal{N} : (i,j) \in \mathcal{W} \tag{33}$$

$$\eta_{ij} s_{ij} = 0 \quad \eta_{ij} \geq 0 \quad s_{ij} \geq 0 \quad \forall (i,j) \in \mathcal{W} \tag{34}$$

$$\mu_{ij} (s_{ij} - S_{ij}) = 0 \quad \mu_{ij} \geq 0 \quad s_{ij} - S_{ij} \leq 0 \quad \forall (i,j) \in \mathcal{W} \tag{35}$$

In the above, the α_i and the β_i are dual variables for the lower and upper bounds on the consumption rates c_i, respectively. Similarly, the γ_i and the ζ_i are dual variables for the lower and upper bounds on the production rates q_i. Moreover, the η_{ij} and the μ_{ij} are dual variables for the lower and upper bounds on the shipment rates s_i.To understand the form of Eq. 33, consider the expression

$$M_0 \equiv \sum_{i \in \mathcal{N}} \lambda_i \left[\sum_{j:(j,i) \in \mathcal{W}} s_{ji} - \sum_{j:(i,j) \in \mathcal{W}} s_{ij} \right], \tag{36}$$

which allows the Hamiltonian to be restated as

$$\begin{aligned} H_0 = {} & M_0 + e^{-\rho t} \left[\sum_{i \in \mathcal{N}} \pi_i (c_i, t) c_i - V(q,t) - \sum_{(i,j) \in \mathcal{W}} r_{ij} s_{ij} - \sum_{i \in \mathcal{N}} \psi_i(I_i) \right] \\ & + \sum_{i \in \mathcal{N}} \nu_i \cdot e^{-\rho t_f} \left[K_i - I_i(t_f) \right] \\ & + \sum_{i \in \mathcal{N}} \alpha_i (-c_i) + \beta_i (c_i - C_i) \\ & + \sum_{i \in \mathcal{N}} \left[\gamma_i (-q_i) + \zeta_i (q_i - Q_i) \right] \\ & + \sum_{(i,j) \in \mathcal{W}} \left[\eta_{ij} (-s_{ij}) + \mu_{ij} (s_{ij} - S_{ij}) \right] \end{aligned} \tag{37}$$

Note that M_0 itself may be restated as

$$M_0 = \sum_{i \in \mathcal{N}} \lambda_i \left[\sum_{j:(j,i) \in \mathcal{W}} s_{ji} - \sum_{j:(i,j) \in \mathcal{W}} s_{ij} \right] \tag{38}$$

$$= \sum_{i \in \mathcal{N}} \sum_{j:(j,i) \in \mathcal{W}} \lambda_i s_{ji} - \sum_{i \in \mathcal{N}} \sum_{j:(i,j) \in \mathcal{W}} \lambda_i s_{ij} \tag{39}$$

$$= \sum_{j \in \mathcal{N}} \sum_{i:(i,j) \in \mathcal{W}} \lambda_j s_{ij} - \sum_{i \in \mathcal{N}} \sum_{j:(i,j) \in \mathcal{W}} \lambda_i s_{ij} \quad (40)$$

In deriving Eq. 40, we have taken into account that there are no free indices in M_0, so we are free to swap *i* and *j*, as has been done in the first term of Eq. 40. It is then clear that

$$\frac{\partial M_0}{\partial s_{kl}} = \lambda_l - \lambda_k \quad \forall (k,l) \in \mathcal{W} \quad (41)$$

from which Eq. 33 follows immediately.

4.1 Interpreting the Necessary Conditions for a Special Case

To gain insight into the necessary conditions, we consider the special case of no binding nonnegativity constraints and no binding upper bound constraints on the control variales (c, q, s). That is, we stipulate

$$\begin{pmatrix} C \\ Q \\ S \end{pmatrix} > \begin{pmatrix} c \\ q \\ s \end{pmatrix} > \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad (42)$$

which, by virtue of the complementary slackness conditions Eqs. 28, 31, and 34, requires the following:

$$\alpha_i = \beta_i = \gamma_i = \zeta_i = 0 \quad \forall i \in \mathcal{N} \quad (43)$$

$$\eta_{ij} = \mu_{ij} = 0 \quad \forall i \in \mathcal{N}, j \in \mathcal{N} \text{ and } (i,j) \in \mathcal{W} \quad (44)$$

It then follows from the Kuhn-Tucker identities Eq. 27, 30, and 33 that

$$e^{-\rho t} \frac{\partial \pi_i (c_i, t) c_i}{\partial c_i} = \lambda_i \quad \forall i \in \mathcal{N} \quad (45)$$

$$e^{-\rho t} \frac{\partial V (q, t)}{\partial q_i} = \lambda_i \quad \forall i \in \mathcal{N} \quad (46)$$

$$-e^{-\rho t} r_{ij} + \lambda_j - \lambda_i = 0 \quad \forall i \in \mathcal{N}, j \in \mathcal{N} : (i,j) \in \mathcal{W} \quad (47)$$

From Eqs. 45 and 46, we see immediately that marginal revenue equals marginal cost, in present value terms, at each node where the firm operates:

$$MR_i \equiv e^{-\rho t} \frac{\partial \pi_i \left(c_i\,, t\right) c_i}{\partial c_i} = e^{-\rho t} \frac{\partial V\left(q\,, t\right)}{\partial q_i} \equiv MC_i \quad \forall i \in \mathcal{N} \tag{48}$$

It is clear from Eq. 48 that the monopolist's total marginal revenue equals total marginal cost across all nodes, again in present value terms:

$$MR \equiv e^{-\rho t} \sum_{i \in \mathcal{N}} \frac{\partial \pi_i \left(c_i\,, t\right) c_i}{\partial c_i} = e^{-\rho t} \sum_{i \in \mathcal{N}} \frac{\partial V\left(q\,, t\right)}{\partial q_i} \equiv MC \tag{49}$$

We also note that Eq. 47 may be restated as

$$\lambda_j = \lambda_i + e^{-\rho t} r_{ij} \quad \forall i \in \mathcal{N}, j \in \mathcal{N} : (i, j) \in \mathcal{W} \tag{50}$$

Because of Eq. 46, this last expression is equivalent to

$$MC_j = MC_i + e^{-\rho t} r_{ij} \quad \forall i \in \mathcal{N}, j \in \mathcal{N} : (i, j) \in \mathcal{W} \tag{51}$$

Expression Eq. 51 is most easily understood by recognizing that $e^{-\rho t} r_{ij}$ is the marginal transportion cost for OD pair (i, j), whose transportion cost is $e^{-\rho t} r_{ij} s_{ij}$ in present value. It then becomes apparent that Eq. 47, because of its equivalence to Eq. 51, is a statement that the marginal cost of production at the destination (*j*) will equal the marginal cost of production at the origin (*i*) plus the marginal cost of delivery to the destination (*j*) when transportation expenses are given consideration; as such, Eq. 51 is a condition of spatial equilibrium among the monopolist's spatially separated production facilities. Clearly, Eqs. 48 and 51 may be expressed in current value terms upon multiplication by $\exp(\rho t)$:

$$\overline{MR}_i = \overline{MC}_i \quad \forall i \in \mathcal{N} \tag{52}$$

$$\overline{MC}_j = \overline{MC}_i + r_{ij} \quad \forall i \in \mathcal{N}, j \in \mathcal{N} : (i, j) \in \mathcal{W} \tag{53}$$

where the overbar denotes current value.

4.2 Understanding the Necessary Conditions for General Circumstances

Again we appeal to the Kuhn-Tucker identities to gain an interpretation of the necessary conditions for general circumstances. In particular, Eqs. 27, 30, and 33 may be easily restated as

$$\frac{\partial \pi_i \left(c_i\,, t\right) c_i}{\partial c_i} = \lambda_i - \alpha_i + \beta_i \quad \forall i \in \mathcal{N} \tag{54}$$

$$\frac{\partial V_i\,(q\,,t)}{\partial q_i} = \lambda_i - \gamma_i + \zeta_i \quad \forall i \in \mathcal{N} \tag{55}$$

$$\lambda_i + r_{ij} = \lambda_j - \eta_{ij} + \mu_{ij} \quad \forall i \in \mathcal{N}, j \in \mathcal{N} : (i,j) \in \mathcal{W} \tag{56}$$

The entities α_i, γ_i, and η_{ij} are multipliers associated with nonnegativity and will be nonzero only when the rates c_i, q_i, and s_{ij} are respectively zero. They are the non-negative shadow prices associated with each nonnegativity constraint. These shadow prices are governed by the complementary slackness conditions Eqs. 28, 31, and 34. As such, when $c_i = 0$ the marginal cost MC_i is potentially less than when $c_i > 0$. Analogous statements may be made for vanishing production rates and vanishing shipping rates.The entities β_i, ζ_i, and μ_{ij} are shadow prices associated with upper bounds and may be nonzero only when $c_i = C_i$, $\zeta_i = Q_i$, and $\mu_{ij} = S_{ij}$, respectively. As such, when $c_i = C_i$ the marginal cost MC_i is potentially greater than when $c_i < C_i$. Analogous statements may be made about production rates at their upper bounds as well as shipping rates at their upper bounds. These observations along with those of the preceding paragraph make the right-hand sides of expressions Eqs. 54-56 *effective* marginal values that have been adjusted by the relevant shadow prices.

4.3 The Adjoint Equations

The adjoint variables must obey the adjoint equations and the transversality conditions, as is well known for optimal control problems with a terminal-time constraint[2]:

$$\frac{d\lambda_i}{dt} = -\frac{\partial H_0}{\partial I_i} = 0 \quad \forall i \in \mathcal{N} \tag{57}$$

$$\lambda\,(t_f) = \frac{\partial \psi_i(I_i)}{\partial I_i} = v_i \quad \forall i \in \mathcal{N} \tag{58}$$

$$\Longrightarrow \lambda_i\,(t_f) = v_i, \text{ a constant} \quad \forall i \in \mathcal{N} \tag{59}$$

If we assume that inventory holding and backorder costs are zero (or fixed), then a significant simplification results; namely

$$\frac{\partial \psi_i(I_i)}{\partial I_i} = 0 \quad \forall i \in \mathcal{N}$$

and the adjoint variables obey

$$\frac{d\lambda_i}{dt} = -\frac{\partial H_0}{\partial I_i} = 0 \quad \forall i \in \mathcal{N} \tag{60}$$

$$\lambda_i\,(t_f) = v_i \quad \forall i \in \mathcal{N} \tag{61}$$

[2] See, for example, Friesz (2010).

implying that

$$\lambda_i(t) = v_i, \text{ a constant} \quad \forall i \in \mathcal{N}, t \in [t_0, t_f] \tag{62}$$

4.4 Characterizing Shipping Rates via the Maximum Principle

Based on Eq. 41, the Hamiltonian H_0 may be expressed as

$$\begin{aligned} H_0 = & \sum_{i\in\mathcal{N}, j\in\mathcal{N}:(i,j)\in\mathcal{W}} \left(-r_{ij} + \lambda_j - \lambda_i - \eta_{ij} + \mu_{ij}\right) s_{ij} \\ & + e^{-\rho t} \sum_{i\in\mathcal{N}} \pi_i\left(c_i, t\right) c_i - \; V_i(q,t) - \sum_{i\in\mathcal{N}} \psi_i(I_i) \\ & + \sum_{i\in\mathcal{N}} \nu_i\left[K_i - I_i\left(t_f\right)\right] \\ & + \sum_{i\in\mathcal{N}} \left[\alpha_i\left(-c_i\right) + \beta_i\left(c_i - C_i\right)\right] \\ & + \sum_{i\in\mathcal{N}} \left[\gamma_i\left(-q_i\right) + \zeta_i\left(q_i - Q_i\right)\right] \end{aligned} \tag{63}$$

In light of Eq. 63, the maximum principle of optimal control requires, for all $(i, j) \in W$, that the shipping rates obey

$$s_{ij} = \begin{cases} S_i & if \;\; -r_{ij} + \lambda_j - \lambda_i - \eta_{ij} + \mu_{ij} > 0 \\ 0 & if \;\; -r_{ij} + \lambda_j - \lambda_i - \eta_{ij} + \mu_{ij} < 0 \\ s_{ij}^{singular} & if \;\; -r_{ij} + \lambda_j - \lambda_i - \eta_{ij} + \mu_{ij} = 0 \end{cases} \tag{64}$$

for all $i \in \mathcal{N}$ and $j \in \mathcal{N}$ such that$(i, j) \in \mathcal{W}$. Of special interest is that, if $-r_{ij} + \lambda_j - \lambda_i - \eta_{ij} + \mu_{ij} = 0$ occurs only for a finite number of instants of time, then there is no notion of a singular control s_{ij}^{singular}. If there is a dense arc of time for which $-r_{ij} + \lambda_j - \lambda_i - \eta_{ij} + \mu_{ij} = 0$, a singular control must be determinined. The absence or presence of singular controls may be confirmed by numerical solution techniques. We note that, due to Eq. 64, the following holds:

$$\frac{d}{dt}\left(-r_{ij} + \lambda_j - \lambda_i - \eta_{ij} + \mu_{ij}\right) = \frac{d\lambda_j}{dt} - \frac{d\lambda_i}{dt} = \frac{\partial H_0}{\partial I_i} - \frac{\partial H_0}{\partial I_j} = 0 \tag{65}$$

Let us assume that inventory holding and backordering costs are quadratic in own inventory, which is expressed as

$$\psi_i(I_i) = \frac{1}{2} L_i\left(I_i\right)^2, \tag{66}$$

where the $L_i \in \Re^1_+$ for all $i \in \mathcal{N}$ are known constants. Then, from Eq. 65, we have the following:

$$\frac{\partial H_0}{\partial I_i} - \frac{\partial H_0}{\partial I_j} = L_i I_i - L_j I_j = 0 \Longrightarrow I_i = \frac{L_j}{L_i} I_j \tag{67}$$

Thus, we conclude singular shipping rates necessitate constant proportionality of associated inventory levels for a dense arc of time. If that proportionality does not arise, singular shipping rates cannot occur. This is borne out by the numerical example presented in Section 6.

5 Existence

We rely on a result from Clarke (2013) to establish the existence of a solution to the spatial monopolist's problem.

5.1 Technical Background

Let us consider the following abstract optimal control problem:

$$\min J(u) = \Psi \left[x(t_0), x\left(t_f\right) \right] + \int_{t_0}^{t_f} f_0(x, u, t)\, dt \tag{68}$$

$$\frac{dx}{dt} = g_0(x, t) + \sum_{j=1}^{m} g_j(x(t), t)\, u^j(t) \quad \text{a.e.} \tag{69}$$

$$u(t) \in U(t) \quad \text{a.e.} \tag{70}$$

$$(x(t), t) \in Q \quad \forall t \in [t_0, t_f], (x(t_0), x(t_f)) \in E \tag{71}$$

Clarke (2013) proves the following existence theorem[3] for formulation Eqs. 68-71:

Theorem 1 [Clarke (2013)] If the following regularity conditions are satisfied and there is at least one admissible solution for which $J(u)$ is finite, then there exists an optimal solution to Eqs. 68-71:

(a) *Each* g_j *for* $j = 0, 1, \ldots, m$ *is measurable in t, continuous in x, and has linear growth. That is, there exists a constant B such that*

[3] See, in particular, Theorem 23.11 on page 481.

$$(x,t) \in Q \Longrightarrow |g_j(x,t)| \leq B(1+|x|) \tag{72}$$

where $|x|$ is the Eucliden norm of x;

(b) *For almost every t, the set* $U(t)$ *is closed and convex;*
(c) *The sets E and Q are closed, and* $\Psi\left[x(t_0), x\left(t_f\right)\right]$ *is lower semicontinuous;*
(d) *The integrand* $f_0(x,u,t)$ *is Lebesgue-Borel (LB) measurable*[4] in *t and* (x,u)*, as well as lower semicontinuous in* (x,u)*. Furthermore, the integrand* $f_0(x,\cdot,t)$ *is convex for each* $(x,t) \in Q$ *, and there is a constant* ω_0 *such that*

$$(x,t) \in Q, u \in U(t) \Longrightarrow f_0(x,u,t) \geq \omega_0; \tag{73}$$

(e) *The projection onto E denoted as*

$$\{a \in \Re^n : (a,b) \in E \text{ for some } b \in \Re^n\} \tag{74}$$

is bounded; and
(f) There exists $K_0(t)$ such that

$$u \in U(t) \Longrightarrow \|u\| \leq K_0(t) \text{ for almost every } t \tag{75}$$

5.2 Existence of an Optimal Solution

Let us restate the monopolist's problem Eqs. 16-20 as a minimization problem:

$$\min J_0(c,q,s) = \int_{t_0}^{t_f} e^{-\rho t}\left[V(q,t) + \sum_{(i,j)\in\mathcal{W}} r_{ij}\, s_{ij} + \sum_{i\in\mathcal{N}} \psi_i(I_i) - \sum_{i\in\mathcal{N}} \pi_i(c_i, t)\, c_i\right] dt \tag{76}$$

$$\frac{dI_i}{dt} = q_i + \sum_{j:(j,i)\in\mathcal{W}} s_{ji} - \sum_{j:(i,j)\in\mathcal{W}} s_{ij} - c_i \quad \forall i \in \mathcal{N} \tag{77}$$

$$I_i(t_0) = I_i^0 \qquad \forall i \in \mathcal{N} \tag{78}$$

$$I_i(t_f) = K_i \qquad \forall i \in \mathcal{N} \tag{79}$$

$$(c,q,s) \in \Omega \tag{80}$$

[4] Definition 6.33 of Clarke (2013) defines LB measurability.

We will use Theorem 1 to establish existence for Eqs. 76-80:

Theorem 2 Existence of an optimal solution $(c^*, q^*, s^*) \in \Omega$ for the monopolist. We stipulate:

(i) $|1_n| \leq B(1 + |I|)$ for the linear growth condition;
(ii) $U = \Omega$;
(iii) $(I(t_0), I(t_f)) \in E = N_r(I_0)$ with *r* defined by Eq. 82 for $\phi \geq 1$;
(iv) $V(q, t)$ is LB measurable, lower semicontinuous, and convex;
(v) $\Psi\left[x\left(t_f\right)\right] = \sum_{i\in\mathcal{N}} \nu_i \cdot e^{-\rho t_f}\left[K_i - I_i(t_f)\right]$;
(vi) each $\psi_i(I_i)$ is LB measurable, lower semicontinuous, and convex;
(vii) each $-\pi_i(c_i, t)\, c_i$ is LB measurable, lower semicontinuous, and concave; and
(viii) $f_0(I, c, q, s, t) \geq -\sum_{i\in\mathcal{N}} \pi_i\left(c_i^0, t\right) c_i^0 = \omega_0$ If there is at least one admissible solution of Eqs. 76-80, then there is an optimal solution to the monopolist's optimal control problem Eqs. 76-80.

Proof When the restated monopolist's problem Eqs. 76-80 is considered from the perspective of Theorem 1, the following observations may be made:

(a′): To apply Clarke's result to our problem, we take $g_0(x, t) = 0$. We also take

$$u_i = (c_i,\, q_i,\, s_{ji},\, s_{ij}\ :\ j \in \mathcal{N})$$

so that each g_j will be a vector comprised of ± 1 and 0 entries in order to replicate the monopolist's state (inventory) dynamics. Because, each of the monopolist's control variables are bounded from above and below, inventories will be finite for all $t \in [t_0, t_f]$; let $I_i^{\min}$ represent the minimal inventory for each $i \in \mathcal{N}$ of the monopolist's problem. We select B such that

$$\begin{gathered}|1_n| \leq B\left(1 + \sqrt{\left[I_1^{\min}\right]^2 + \ldots + \left[I_{|\mathcal{N}|}^{\min}\right]^2}\right) \\ \Rightarrow |1_n| \leq B\left(1 + \sqrt{\left[I_1(t)\right]^2 + \ldots + \left[I_{|\mathcal{N}|}(t)\right]^2}\right)\end{gathered} \tag{81}$$

where 1_n is a vector with n entries, all of which are 1, with $|1_n| = \sqrt{n}$ being its Euclidean norm. Naturally, $n = 2|\mathcal{N}| + |\mathcal{W}|$, the number of control variables available to the monopolist. Since each g_j, as noted above, will contain some 0 entries, $|g_j(x, t)| \leq |1_n|$. From (1) we have immediately the linear growth property.

(b′): Our set of control constraints $U \equiv \Omega$ is defined by expression Eq. 21. By observation Ω is closed and convex.
(c′): We note there are no state-space constraints (Q is vacuous). Moreover, $I_i(t_0)$ and $I_i(t_f)$ are fixed for all $i \in \mathcal{N}$. Hence,

$$(I(t_0), I(t_f)) \in E \equiv B_r[I(t_0)]$$

where $E \equiv B_r\left[I\left(t_0\right)\right]$ is a closed ball centered at $I\left(t_0\right)$ with radius r. That radius is determined by

$$r=\phi \sqrt{\left[I_1\left(t_f\right)-I_1\left(t_0\right)\right]^2+\ldots+\left[I_{|\mathcal{N}|}\left(t_f\right)-I_{|\mathcal{N}|}\left(t_0\right)\right]^2} \tag{82}$$

where $\phi \geq 1$. We also note that the terminal cost

$$\Psi\left[x\left(t_f\right)\right]=\sum_{i \in \mathcal{N}} \nu_i \cdot e^{-\rho t_f}\left[K_i-I_i\left(t_f\right)\right] \tag{83}$$

arises from pricing out the terminal inventory constraints using the dual variables v_i for all $i \in \mathcal{N}$, as reflected in the Hamiltonian Eq. 24. By observation Eq. 83 is $\Psi\left[I\left(t_f\right)\right]$, continuous in $I_i\left(t_f\right)$, and has no dependence on any initial inventory level $I_i\left(t_0\right)$. Since $\Psi\left[I\left(t_f\right)\right]$ is continuous, it is lower semicontinuous.

(d′): Since the costs of production, shipping, and inventory holding are positive, it is clear that the integrand of Eq. 76 obeys

$$\begin{aligned} f_0(I, c, q, s, t) \equiv V(q, t)+ & \sum_{(i, j) \in \mathcal{W}} r_{i j} s_{i j}+\sum_{i \in \mathcal{N}} \psi_i\left(I_i\right) \\ & -\sum_{i \in \mathcal{N}} \pi_i\left(c_i, t\right) c_i \geq-\sum_{i \in \mathcal{N}} \pi_i\left(c_i, t\right) c_i \geq \\ & -\sum_{i \in \mathcal{N}} \pi_i\left(c_i^0, t\right) c_i^0 \equiv \omega_0, \text{ a constant, } \end{aligned} \tag{84}$$

where $c^0=\left(c_i^0: i \in \mathcal{N}\right)$ is the global minimizer of $-\sum_{i \in \mathcal{N}} \pi_i\left(c_i, t\right) c_i$ subject to $C_i \geq c_i \geq 0$ for all $i \in \mathcal{N}$. This demonstrates the inequality

$$f_0(x, u, t) \geq \omega_0 \tag{85}$$

required for the application of Theorem 1. Moreover, the integrand $f_0(x, u, t)$ will be Lebesgue-Borel measurable in t and $(x, u)=(I, c, q, s)$, as well as lower semicontinuous in $(x, u)=(I, c, q, s)$, if we make these quite mild assumptions: $V(q, t)$, along with $\psi_i(I_i)$ and $\pi_i\left(c_i, t\right) c_i$ for all $i \in \mathcal{N}$, are individually LB measurable and lower semicontinuous. The convexity of $f_0(I, \cdot, \cdot, \cdot, t)$ is assured by assuming the $\psi_i(I_i)$ are convex for each $i \in \mathcal{N}$.

(e′): By virtue of how we constructed $E \equiv B_r\left[I\left(t_0\right)\right]$, any projection onto it is bounded.

(f′): We note that $u \equiv(c, q, s) \in \Omega \equiv U$ is bounded for almost all $t \in\left[t_0, t_f\right]$ because Ω includes fixed bounds on all control variables. That is, there exists $K_0 \in \Re_{++}^1$ such that for

$$\|u\| = \sqrt{\sum_{i=1}^{|\mathcal{N}|} (c_i)^2 + \sum_{i=1}^{|\mathcal{N}|} (q_i)^2 + \sum_{(i.j)=1}^{|\mathcal{W}|} (s_{ij})^2} \leq K_0 \tag{86}$$

for almost every $t \in [t_0, t_f]$.

Hence, all the conditions for the application of Theorem 1 are satisfied and existence is proven.

6 Discrete Time Approximation

We note that Eqs. 16-20 may be solved in a number of ways, although direct appeal to the necessary conditions is unlikely to be successful for general networks due to the large number variables. Furthermore, since there are no time shifts and the state dynamics are linear in the formulation proposed above, time discretization in conjunction with finite dimensional mathematical programming is especially appealing. To this end we construct a discrete time approximation of Eqs. 16-20:

$$J(c, q, s) = \sum_{k=0}^{N} e^{-\rho t_k} \left\{ \sum_{i \in \mathcal{N}} \pi_i \; [c_i (t_k)] \, c_i (t_k) - \; V [q (t_k)] \right.$$
$$\left. - \sum_{(i,j) \in \mathcal{W}} r_{ij} \, s_{ij} (t_k) - \sum_{i \in \mathcal{N}} \psi_i [I_i (t_k)] \right\} \tag{87}$$

$$I_i (t_k) - I_i \, (t_{k-1}) = \; q_i (t_k) + \sum_{(j,i) \in \mathcal{W}} s_{ji} (t_k) - \sum_{(i,j) \in \mathcal{W}} s_{ij} (t_k) - c_i \, (t_k) \quad \forall \, i \, \in \mathcal{N}, \forall k \in [1, N] \tag{88}$$

where we define

$$t_k = t_0 + k \Delta t$$

and N is the number of discretizations, defined by

$$N = \frac{t_f - t_0}{\Delta t}$$

$$I_i (t_0) = I_i^0 \quad \forall \, i \, \in \mathcal{N} \tag{89}$$

Table 1 From-To Array Representing Example Network

Arc Name	From Node	To Node	Shipping Variable
1	1	2	s_1
2	1	3	s_2
3	2	3	s_3
4	2	4	s_4
5	3	5	s_5

Table 2 Controls and states for example

Controls					States			
c_1	c_2	c_3	c_4		I_1	I_2	I_3	I_4
q_1	q_2	q_3	q_4					
s_1	s_2	s_3	s_4	s_5				

Furthermore

$$\begin{aligned} c^k &= (c_i\,(t_k) : i \in \mathcal{N}) \\ q^k &= (q_i\,(t_k) : i \in \mathcal{N}) \\ s^k &= (s_{ij}\,(t_k) : (i\,,j) \in \mathcal{W}) \\ c &= \left(c^k : k \in [1, N]\right) \\ q &= \left(q^k : k \in [1, N]\right) \\ s &= \left(s^k : k \in [1, N]\right) \end{aligned}$$

so that we may write

$$C \geq c \geq 0 \tag{90}$$

$$Q \geq q \geq 0 \tag{91}$$

$$S \geq s \geq 0 \tag{92}$$

6.1 Numerical Example

By assuming the monopolist's variable costs are additive and separable we are able to adapt the numerical example found in Friesz (2010) to the present model formuation for DMMFN. To that end, let us consider a network of 5 arcs and 4 nodes for which the single firm of interest has activities located at each node $i = 1, 2, 3, 4$. We assume that the production cost function has the form

$$V\,(q) = \sum_{i \in \mathcal{N}} V_i(q_i) \tag{93}$$

Consumption of the firm's output potentially occurs at every node; this consumption may be of local output or of imported output as the network topology permits. Table

1 depicts the 4 node, 5 arc network of interest as a from-to array. The time interval of interest is [0, 10]; that is $t_0 = 0$ and $t_f = 10$. Before time discretization there are 13 controls and 4 state variables associated with this example; these are listed in Table 2. At time $t_0 = 0$, the initial inventory at each node is

$$I_1(0) = 5 \tag{94}$$

$$I_2(0) = 3 \tag{95}$$

$$\begin{aligned} I_3(0) &= 2 \\ I_4(0) &= 0 \end{aligned} \tag{96}$$

In addition, we impose the condition that no backordering is allowed by any firm at any node at the terminal time $t_f = 10$. That is

$$I_i(10) = 0 \text{ for } i = 1, 2, 3, 4 \tag{97}$$

The inventory dynamics are the following flow balance equations:

$$\begin{aligned} \frac{dI_1}{dt} &= q_1 - s_1 - s_2 - c_1 \\ \frac{dI_2}{dt} &= q_2 + s_1 - s_3 - s_4 - c_2 \\ \frac{dI_3}{dt} &= q_3 + s_2 + s_3 - s_5 - c_3 \\ \frac{dI_4}{dt} &= q_4 + s_4 + s_5 - c_4 \end{aligned}$$

We assume the inverse demands at each node take the following form:

$$\pi_1(c_1, t) = 4(11 - c_1) \exp\left(\frac{t}{40}\right) \tag{98}$$

$$\pi_2(c_2, t) = 3(11 - c_2) \exp\left(\frac{t}{30}\right) \tag{99}$$

$$\pi_3(c_3, t) = 3.5(11 - c_3) \exp\left(\frac{t}{35}\right) \tag{100}$$

$$\pi_4(c_4, t) = 2.5(11 - c_4) \exp\left(\frac{t}{25}\right) \tag{101}$$

The individual terms of the production cost function Eq. 93 are the following:

$$V_1(q_1) = 2\frac{q_1^2}{2} \tag{102}$$

$$V_2(q_2) = 0.7\frac{q_2^2}{2} \tag{103}$$

$$V_3(q_3) = 1.3\frac{q_3^2}{2} \tag{104}$$

$$V_4(q_4) = 4\frac{q_4^2}{2} \tag{105}$$

We assume the holding costs are

$$\psi_1(I_1) = \frac{I_1^2}{2} \tag{106}$$

$$\psi_2(I_2) = 10\frac{I_2^2}{2} \tag{107}$$

$$\psi_3(I_3) = 3\frac{I_3^2}{2} \tag{108}$$

$$\psi_4(I_4) = 4\frac{I_4^2}{2} \tag{109}$$

We assume that the freight tariffs for each arc are the following:

$$r_1 = 5 \tag{110}$$

$$r_2 = 2 \tag{111}$$

$$r_3 = 3 \tag{112}$$

$$r_4 = 2 \tag{113}$$

$$r_5 = 4 \tag{114}$$

Fig. 1 Inventory Dynamics

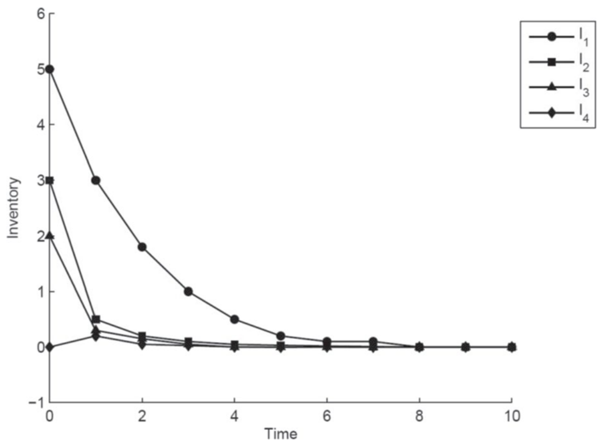


Fig. 2 Production Rate

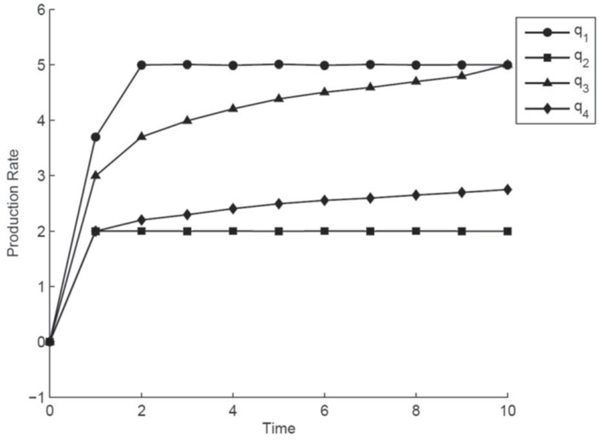


Fig. 3 Consumption Rate

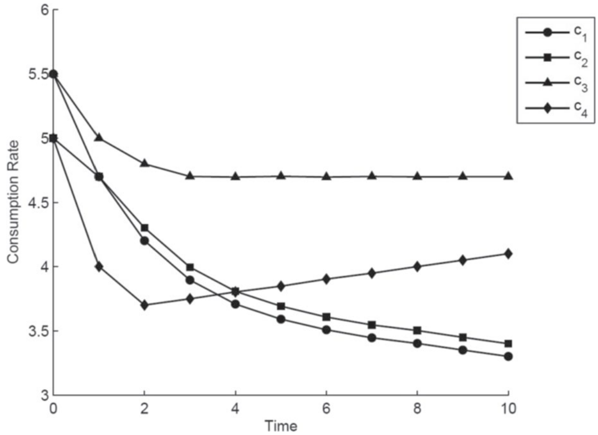
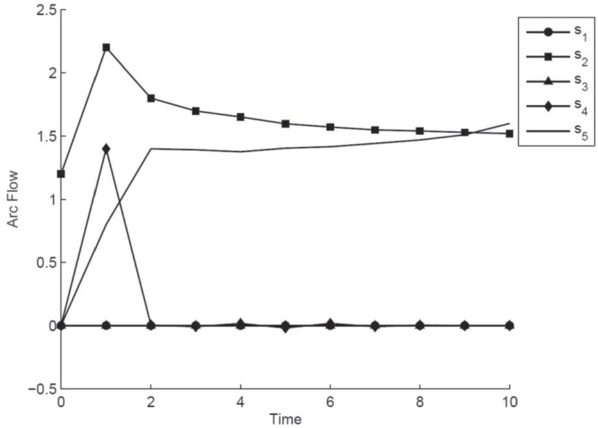


Fig. 4 Freight Flow



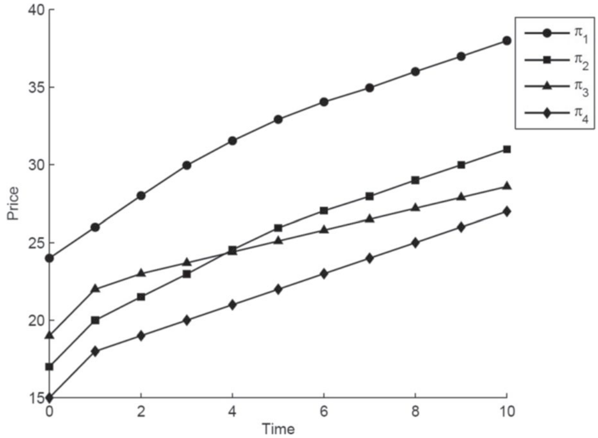


Fig. 5 Price

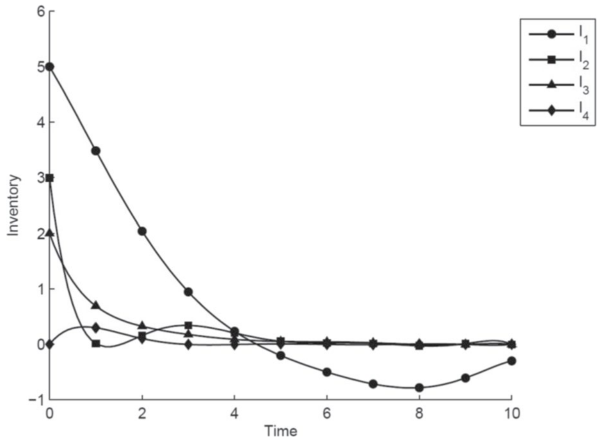


Fig. 6 Inventory Dynamics

We impose the following upper bounds on control variables:

$$C = \begin{pmatrix} 5 \\ 10 \\ 10 \\ 5 \end{pmatrix}, Q = \begin{pmatrix} 5 \\ 2 \\ 5 \\ 5 \end{pmatrix}, S = \begin{pmatrix} 10 \\ 10 \\ 10 \\ 10 \end{pmatrix}$$

Nonnegativity is also enforced. The individual firms' criterion functional is easily stated by substitution of the above information into Eq. 87.

6.2 Solution by Discrete Time Approximation

We solve the monopolist's problem corresponding to the details presented in Section 6.1 using a discrete time approximation with $N = 10$ equal time steps. In our calculations we allowed GAMS/MINOS to solve the specific instance of Eq. 87 through Eq. 92 corresponding to the data we have given. The solution time for this example is

Fig. 7 Production Rate

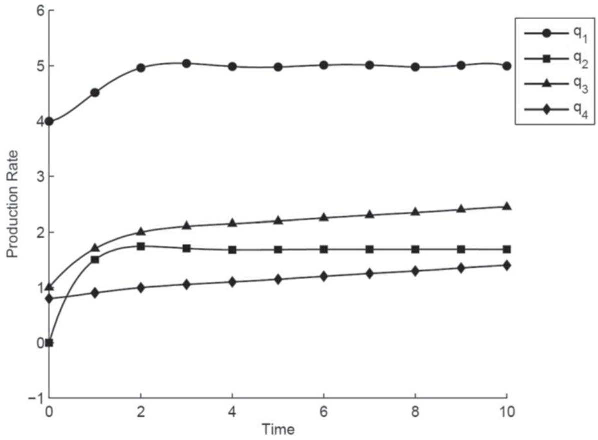


Fig. 8 Consumption Rate

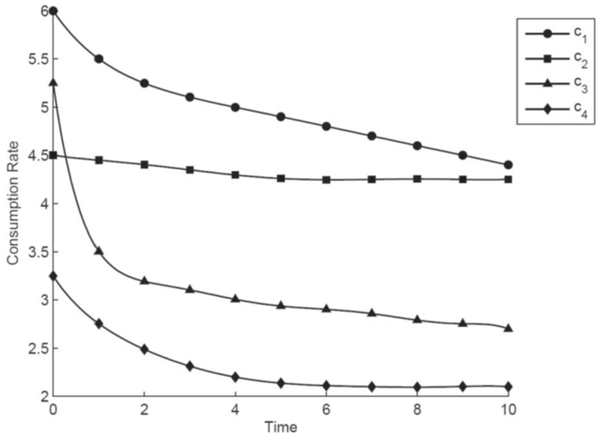
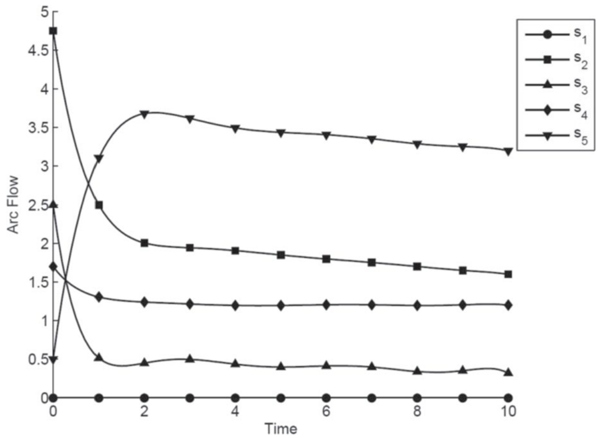


Fig. 9 Freight Flow



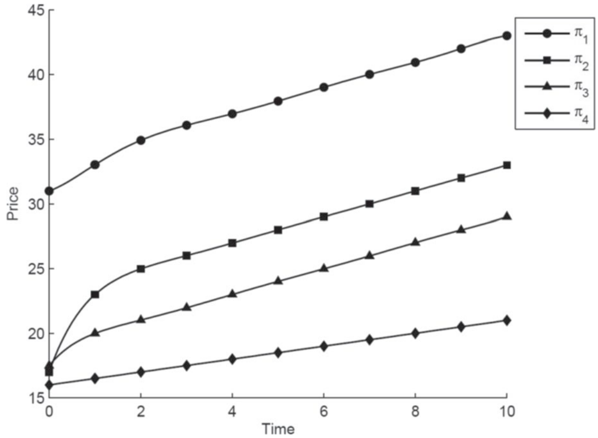


Fig. 10 Price

approximately 2 cpu seconds on a Pentium® 4 single-processor computer. The results are presented in Figs. 1, 2, 3, 4 and 5.

6.3 Solution by Continuous Time Gradient Projection

We also solved the example using the continuous time gradient projection method, as presented in Friesz (2010). We calculate 40 values of the gradients and then construct a 6-th order polynomial approximation of each as a smooth function of time. The algorithm is implemented in MATLAB and the solution time for the example presented is approximately 10 cpu seconds on a Pentium® 4 single-processor computer. The results are shown in Figs. 6, 7, 8, 9 and 10. The gradient projection algorithm is articulated below in terms of the state and control vectors specific to the example problem:

Gradient Projection Algorithm

Step 0. Initialization. Set $k = 0$ and pick $c_i^0(t)$, $q_i^0(t)$ and $s_i^0(t)$ for $i = 1, 2, 3, 4$.
Step 1. Find State Trajectory. Using current controls, solve the state initial value problem

$$\begin{array}{ll} \frac{dI_1}{dt} = q_1^k - s_1^k - s_2^k - c_1^k & I_1(0) = 5 \\ \frac{dI_2}{dt} = q_2^k + s_1^k - s_3^k - s_4^k - c_2^k & I_2(0) = 3 \\ \frac{dI_3}{dt} = q_3^k + s_2^k + s_3^k - s_5^k - c_3^k & I_3(0) = 2 \\ \frac{dI_4}{dt} = q_4^k + s_4^k + s_5^k - c_4^k & I_4(0) = 0 \end{array}$$

and call the solution $I_1^k(t), I_2^k(t), I_3^k(t)$ and $I_4^k(t)$.
Step 2. Find Adjoint Trajectory. Using current controls and states, solve the adjoint final value problem

$$\begin{array}{ll} (-1)\frac{d\lambda_1}{dt} = \exp(-\rho t)(I_1^k) & \lambda_1(10) = vI_1^k(10) \\ (-1)\frac{d\lambda_2}{dt} = \exp(-\rho t)(10I_2^k) & \lambda_2(10) = vI_2^k(10) \\ (-1)\frac{d\lambda_3}{dt} = \exp(-\rho t)(3I_3^k) & \lambda_3(10) = vI_3^k(10) \\ (-1)\frac{d\lambda_4}{dt} = \exp(-\rho t)(4I_4^k) & \lambda_4(10) = vI_4^k(10) \end{array}$$

picking the dual variable *v* to enforce zero inventory at the terminal time; call the solution $\lambda_1^k(t), \lambda_2^k(t), \lambda_3^k(t)$ and $\lambda_4^k(t)$.

Step 3. Find Gradient. Using current controls, states and adjoints, calculate

$$\nabla_u J(u^k) = \frac{\partial H\left(I^k, u^k, \lambda^k\right)}{\partial u} = \frac{\partial f_0(I^k, u^k)}{\partial u} + \left(\lambda^k\right)^T \frac{\partial f\left(I^k, u^k\right)}{\partial u}$$

where

$$u = \begin{pmatrix} c^k \\ q^k \\ s^k \end{pmatrix}$$

and $H(I, u, \lambda, t)$ is the relevant Hamiltonian for this problem.

Step 4. Update and Apply Stopping Test. For a suitably small step size θ_k, update according to

$$u^{k+1} = P_U\left[u^k - \theta_k \nabla J\left(u^k\right)\right]$$

where P_U denotes the minimum norm projection of the vector *u* onto the feasible set *U*. If an appropriate stopping test is satisfied, declare

$$u^*(t) \approx u^{k+1}(t)$$

Otherwise set $k = k + 1$ and go to Step 1.

7 Conclusions

We have shown that the version of dynamic network monopoly addressed herein combines explicit dynamics, nonlinearity, and computational tractability. Its solutions have a marginal-revenue-equals-marginal-cost interpretation, just as one would intuitively expect. Moreover, we have proven solutions exist under plausible, checkable regularity conditions. We also found the model to be highly computable, easily yielding numerical solutions using both discrete-time and continuous-time solution schemes.

Author Contributions Friesz is responsible for 100% of the paper now being submitted. In particular, he developed the model, proved the theorem on existence, derived the necessary conditions, and adapted the dataset from another application.

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Data Availability No datasets were generated or analysed during the current study.

Declarations

Competing Interests The authors declare no competing interests.

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