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Bus stop spacing with heterogeneous trip lengths and elastic demand

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ABSTRACT

This paper develops models of a bus route in which (i) stop spacing can vary; (ii) trip lengths are heterogeneous; (iii) demand is elastic; and (iv) passengers delay the bus. Since wider spacings make sufficiently long trips faster, and sufficiently short trips slower, they induce long trips and repel short trips. We explore two continuum-approximation models: one with fixed headways and another in which headways depend on the spacing. The pattern of induced/repelled trips means the ridership-maximizing spacing is shorter than the one that maximizes passenger-km traveled. The same pattern also makes the average trip length endogenous to spacing. In the model with endogenous headways, when spacing is very narrow, a rise in spacing can reduce the expected wait time by more than it increases the expected walk time. We draw several lessons for practice and use a discrete simulation to confirm results from the continuous approximation models.

1. Introduction

1.1. Background

In June 2022, the Metropolitan Transportation Authority began removing almost 400 bus stops in the Bronx (about 18% of the prior total) as part of the Bronx Bus Network Redesign (Moloney, 2021). The borough thereby joins the list of places which have, since 2010, systematically eliminated large numbers of bus stops—a practice called “stop consolidation” (El-Geneidy et al., 2006) (or sometimes “stop rebalancing” Miatkowski and Hovenkotter, 2019). These places include Portland, OR (El-Geneidy et al., 2006), San Francisco (Gordon, 2010), Dallas (Macon, 2021), Providence (Miatkowski and Hovenkotter, 2019), Pittsburgh (Blazina, 2020), Dallas (Garnham, 2020) and Cincinnati (Cincinnati Metro, 2022).

What motivates stop consolidation? Each time a bus stops, it loses some time (relative to its path without the stop) to braking, opening/closing doors and then accelerating from a stop. Call this time *stop delay*. Since the stop delay is non-zero, stop removal saves passengers in-vehicle time and could let agencies squeeze more service from a given fleet. The downside of consolidation is that *fewer stops mean wider stop spacings* (distances between consecutive stops), so passengers must cover more ground to and from the bus.

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Table 1
Classification of the literature.

Study	Demand	Heterogeneous trip length	Passenger delay
Vuchic and Newell (1968)	Fixed	Heterogeneous	
Mohring (1972)	Fixed	Uniform	✓
Kikuchi and Vuchic (1982)	Fixed	Uniform	✓
Wirasinghe (1980)	Fixed	Heterogeneous	
Wirasinghe and Ghoneim (1981)	Fixed	Heterogeneous	✓
Kuah and Perl (1988)	Fixed	Heterogeneous	
Chien and Schonfeld (1998)	Fixed	Heterogeneous	
Van Nes and Bovy (2000)	Elastic	Uniform	
Daganzo (2010)	Fixed	Heterogeneous	✓
Tirachini and Hensher (2011)	Fixed	Uniform	✓
Basso et al. (2011)	Elastic	Uniform	✓
Basso and Silva (2014)	Elastic	Uniform	✓
Tirachini (2014)	Fixed	Uniform	✓
Ouyang et al. (2014)	Fixed	Heterogeneous	
Daganzo and Ouyang (2019, Ch. 5)	Fixed	Heterogeneous	

1.2. Contribution

Starting with early studies such as Vuchic and Newell (1968) and Mohring (1972), this tradeoff between access and in-vehicle time has remained at the heart of a large theoretical literature[1] about the agency's choice of stop spacing. This paper contributes to this theoretical literature. To clarify how, it will help to characterize prior studies. Table 1 lists theoretical studies featuring a choice of stop spacing according to three properties:

- *Whether demand is fixed or elastic:* When demand is *fixed*, riders make the same trips regardless of what the stop spacing is. By contrast, *elastic* demand means riders choose whether to ride based on the quality of service. Fixed demand is the more common assumption.
- *Whether trip lengths are uniform or heterogeneous:* About half the studies reviewed feature some heterogeneity of trips lengths. In all such studies listed, this heterogeneity obtains because there are demands among various origin–destination pairs lying at different distances from one another. Other studies consider only the in-vehicle travel time of a trip with a particular length. Usually that length is stated to be an exogenously-given "average" trip length (e.g., Mohring, 1972; Tirachini and Hensher, 2011), but the analysis only winds up considering how stop spacing affects a trip of average length—rather than aggregating over how it affects all the trip lengths which determine the average.
- *Whether passengers cause delays.* There are at least two ways passengers delay the bus. First, passengers impose "boarding" and "alighting" delays when they get on and off the bus. Second, if the bus operates according to an "on call" stopping regime (i.e., if it stops only when people want to alight or board), then as ridership rises the bus stops more often (Kikuchi and Vuchic, 1982).

This paper explores stylized models of a bus route with (i) elastic demand; (ii) heterogeneous trip lengths; and (iii) passenger delays. Note that no studies in Table 1 exhibit this combination[2] of properties. We incorporate (i) and (ii) by introducing a *demand density function* (similar to a probability density function) which gives the density of demand by trip length and falls with the travel time of a given trip length. The ridership in any interval of trip lengths is derived by integrating the demand density function, taking into account the travel time each trip length faces. But the aim of the paper is not only to improve on the realism of existing models nor introduce methodological innovations. Rather, the aims are to demonstrate and deduce the consequences of a general point: *that the choice of stop spacing influences the quantity and lengths of the trips people make.* This is plausible, because (a) passengers choose whether to ride; and (b) certain spacings are superior for serving certain trip lengths (as we show below). By contrast, in studies with fixed demands, the set of trips taken is exogenous; and in studies with uniform trip lengths, the quantity of trips may change with spacings (if demand is elastic) but the lengths of those trips will not.

The models also feature passenger delays, in order to ensure the conclusions are robust. We incorporate passenger delays by introducing a generally-specified *delay function* which rises with ridership and falls with the spacing. As we will see, pairing passenger delays with elastic demands leads to interesting equilibrium phenomenon—as feedback arises between ridership and travel times.

1.3. Plan of paper

The paper proceeds as follows:

[1] Sec. 2 of Tirachini (2014) provides a thorough literature review.

[2] There are three algorithmic studies that have this combination: Dell'Olio et al. (2006), Ibeas et al. (2010) and Alonso et al. (2011). All three show how bi-level optimization algorithms may be used to place bus stops in Santander, Spain when demands between different origin/destination pairs are given by logit models. A difference between those studies and this one is our emphasis on analytical results.

Table 2
Variables, units and definitions.

Variable	Definition	Units
R	Route length	km
x	Trip length	km
s	Stop spacing	km/stop
t	Door-to-door travel time	h
q	Ridership	pax/h
v_b	Bus speed (between stops)	km/h
v_a	Access speed	km/h
B	Fleet size	bus
C	Cycle time (time to complete route)	h
H	Headway (time between buses)	h/bus
Functions		
$\lambda(x,t)$	Demand density	pax/km-h
$T(x,s,q)$	Door-to-door travel time	h
$Q(s)$	Equilibrium ridership/demand	pax/h
$K(s)$	Equilibrium pax-km traveled	pax-km/h
$U(s)$	Equilibrium unit travel time	h/km
$\eta(q,s)$	Delay per km	h/km

- Section 2 sets up a continuum approximation model of an isotropic bus route. It is a *continuum approximation* insofar as some discrete quantities (such as the number of passengers, the number of vehicles) may take non-integer values, and passengers do not necessarily travel integer numbers of spacings. The route is *isotropic* in the sense that conditions are uniform over locations, and the route is characterized by aggregates. For this setting, we show that when spacing rises, the travel time of trips longer (shorter) than a *critical trip length* falls (rises). In turn, because demand is elastic, the number of trips above (below) the critical trip length rises (falls).
- Section 3 extends Section 2 model (which assumes headways are fixed) by supposing there is a fixed number of buses on the route. Thus, headways (and wait times) become endogenous to the choice of spacing, since at wider spacings buses finish the route sooner and can visit each stop more often. This assumption leads to a new result: when spacings are shorter than a threshold, a rise in spacing may cut wait times more quickly than it increases access times. In this case, the door-to-door travel time of *every* trip length declines with a marginal increase in the stop spacing.
- Section 4 derives general lessons from the models of Sections 2 and 3.
- Section 5 uses a simulation to check whether our theoretical results also hold when stops are discrete points, and when passengers travel an integer number of stops.
- Section 6 concludes by drawing practical lessons for optimal system design and offering paths for future research.

2. Model with fixed headways

2.1. Setup

First, we establish the model's setting. Units used are: km for distance, h for time and pax for passengers. The setting is a closed-loop bus route of length R (km). It has a uniform spacing s (km/stop) between any two stops. The route is fully characterized by aggregates (e.g., bus speed, spacing, ridership, etc.). Table 2 lists variables used, their meanings and their units. Throughout the paper, we indicate partial derivatives by way of an integer subscript. Thus, given the function $\eta(q,s)$, η_1 is the partial derivative of η with respect to its first argument, q.

2.2. Travel time

We start out by deriving the travel times for trips of various lengths. The travel time has three components: access time, wait time and in-vehicle time.

Access time is the time passengers spend walking (or otherwise traveling) to and from stops. In deriving the out-of-vehicle time we make assumptions that are standard[3] in the literature. Trip origins and destinations are distributed uniformly along the route. Every passenger boards a bus at the stop closest to their origin and alights at the stop nearest to their destination. Thus, the average passenger travels a total of $s/2$ (km) ($s/4$ to their boarding stop and $s/4$ from their alighting stop). Passengers travel to/from stops at the *access speed*, v_a (km/h). Thus, the expected access time is $s/(2v_a)$, which does not vary with trip length.

Wait time is the time a passenger spends waiting at a stop for a bus to arrive. For routes with reasonably high frequency, passengers can be assumed to not use time tables but rather to show up at the stop and wait. Hence, the expected wait time depends on the route's *headway*, H (h/bus). If passengers arrive at stops randomly, without regard to the schedule, then the average

[3] See the review in Jara-Diaz and Gschwender (2003).

passenger's wait time is $H/2$ (Dial, 1967; Wirasinghe, 1980; Tian et al., 2012; Ansari Esfeh et al., 2021).[4] Regarding the headway, in this section we assume[5]

Assumption A1 (*Fixed Headway*). The headway is fixed at H (h/bus) regardless of the spacing.

The last component of travel time, *in-vehicle time*, clearly depends on how far a passenger travels and on how quickly the bus moves. Let v_b (km/h) be the travel time of a bus between stops, so that $1/v_b$ (h) is the time it would take a bus to traverse one kilometer without stopping. Let q (pax/h) be the route's *ridership*. Let $\eta(q,s)$ (h/km) be the *delay function* which gives the time lost to stops per km; it obeys:

Assumption A2 (*Delay Function*). $\eta(q,s)$ has the following properties:

(i) Delay is positive: $\eta(q,s) > 0$ for finite q, s.
(ii) Delay is non-decreasing with ridership: $\eta_1 \geq 0$.
(iii) Delay is convex and decreasing over s: $\eta_2 < 0, \eta_{22} > 0$.
(iv) $\eta \to 0$ as $s \to \infty$.
(v) $\eta \to \infty, \eta_2 \to -\infty$ as $s \to 0$

This specification of delays is intended to be general enough to encompass various theories of passenger delays (e.g., from boarding and alighting or from requesting stops) while capturing some intuitive properties. For example, per (v), as spacing tends to infinity, the bus ceases to stop and so there is no delay. Since $\eta(q,s)$ is given generally, we will give an example[6] of a particular specification with its properties.

Example E1 (*Specification of η Under Assumption A1*). First, suppose each passenger delays the bus by θ (h/pax). It can be shown that the bus picks up and drops off Hq/R pax/km as it moves along the route, so passengers delay the bus by $\theta Hq/R$ h/km. Second, suppose the bus stops at every stop and thereby suffers a *stop delay* of τ (h/stop). In this case, the bus incurs stop delay at a rate of τ/s (h/km). Putting the two delays together, we have

$$\eta(q,s) = \tau/s + H\theta q/R \quad \text{(h/km)} \tag{1}$$

It is easy to verify that $\tau/s + H\theta q/R$ satisfies all five properties of Assumption A2.

The average speed v_b and the delay function $\eta(q,s)$ determine the bus' *unit travel time*: the time it takes to traverse one km. Given q and s, the unit travel time is $1/v_b + \eta(q,s)$. The distance a passenger travels once on the bus is a random variable x (km) that we call the *trip length*. Thus, a trip of length x has an in-vehicle time of $x\left[1/v_b + \eta(q,s)\right]$. As stated earlier, x does not have to be an integer multiple of s.

Adding our three components together, the function

$$T(x,s,q) = \underbrace{H/2}_{\text{wait time}} + \underbrace{s/(2v_a)}_{\text{access time}} + \underbrace{x\left[\frac{1}{v_b} + \eta(q,s)\right]}_{\text{in-vehicle time}} \quad \text{(h)} \tag{2}$$

gives the door-to-door travel time for a trip of length x (km) when the stop spacing is s (km/stop) and the ridership is q (pax/h).

2.3. Demand and equilibrium

Demand for trips of different lengths is endogenous. Let $\lambda(x,t)$ (pax/km-h) be a *demand density function*: it gives the density (per km) of the demand (pax/h) for trips of length x (km) when the travel time of such trips is t (h). Thus, if the ridership is q and spacing is s, so that $t = T(x,s,q)$, then the demand[7] for trips with lengths $x \in [x', x'']$ is

$$\int_{x'}^{x''} \lambda[x, T(x,s,q)]\mathrm{d}x \quad \text{(pax/h)}.$$

The demand density function obeys the following assumption:

[4] See Ansari Esfeh et al. (2021) for a more through review of the relationship between headways and waiting times.

[5] This assumption makes the most sense if buses are allocated across a large network in such a way that headways on the route under analysis do not depend on its stop spacing. Removing stops on one route will let buses complete the route more quickly, but if the saved bus-hours are redistributed across many routes, the change in frequency on any particular route may be small enough to ignore.

[6] Note the paper's results do not depend on this particular account of passenger delays. The paper is not committed to an "all stop" regime nor to a perfectly linear account of boarding/alighting times.

[7] Note we have assumed door-to-door travel time determines demand. It is more accurate to weight access time, in-vehicle time and waiting time differently (Mohring et al., 1987). But doing so adds parameters without qualitatively changing any results, so we have ignored weighting for brevity's sake. Some terms in our expressions could be interpreted in such a way as to bake in a weighting factor. For example, rather than multiplying the access time $1/(2v_a)$ by a weighting coefficient β_{access}, one could think of v_a as equal to the actual access speed divided by β_{access}. Daganzo (2010) does so, writing "We reduce walking time to riding time by using a low walking speed, $w = 2$ km/h...This low value recognizes both the discomfort associated with walking and the natural delays that pedestrians encounter when crossing streets".

Assumption A3 (*Elastic Demand*)**.** For all x, $\lambda(x,t)$ falls with t whenever $\lambda(x,t) > 0$: i.e., $\lambda(x,t) > 0 \Rightarrow \lambda_2(x,t) < 0$.

Since passengers delay the bus, which in turn affects the demand for travel, ridership arises as an equilibrium. The same concept appears – though without the stop spacing element – in Zhang et al. (2020) and Lehe and Pandey (2024). Given a spacing s, the route is in equilibrium at a ridership $q = q_e$ (pax/h) such that

$$q_e = \int_0^\infty \lambda\left[x, T(x, s, q_e)\right] \mathrm{d}x \quad \text{(pax/h)}. \tag{3}$$

This is to say: the travel times created by a ridership q_e invite a ridership of q_e. Since the RHS of (3) is non-negative and non-increasing with q_e, a unique $q_e \geq 0$ solves (3). Since q_e is unique, we can write it as a function of s, $Q(s)$, by the implicit definition[8]

$$Q(s) = \int_0^\infty \lambda[x, T(x, s, Q(s))] \mathrm{d}x \quad \text{(pax/h)}. \tag{4}$$

$Q(s)$ (pax/h) is the rate at which trips start and end in the unique equilibrium that arises when stop spacing is s (km).

Consider now the derivative of $Q(s)$:

$$Q'(s) = \frac{\mathrm{d}}{\mathrm{d}s} \int_0^\infty \lambda[x, T(s, x, Q(s))] \mathrm{d}x \tag{5}$$

$$Q'(s) = \int_0^\infty \lambda_2 \cdot \left[\frac{1}{2v_a} + xQ'(s)\eta_1 + x\eta_2\right] \mathrm{d}x \qquad \text{expanding } T' \text{ from (2)} \tag{6}$$

$$Q'(s) = \frac{\int_0^\infty \lambda_2 \cdot \left(1/2v_a + x\eta_2\right) \mathrm{d}x}{1 - \eta_1 \int_0^\infty x\lambda_2 \mathrm{d}x}. \tag{7}$$

Per Assumptions A2 and A3, the denominator on the RHS of (7) is no smaller than one (because $\lambda_2 \leq 0$ and $\eta_1 \geq 0$). The denominator can be thought of as a *damping factor*: any first-round change in ridership is muted by a change in travel time of the opposite sign. A change in spacing that increases (decreases) demand leads to a rise (fall) in travel times that, in turn, decreases (increases) ridership.

Next, let

$$U(s) := 1/v_b + \eta[Q(s), s] \quad \text{(h/km)} \tag{8}$$

give the unit travel time in the unique equilibrium that arises when the spacing is s. Differentiating yields

$$U'(s) = \eta_1 Q'(s) + \eta_2 \tag{9}$$

$$U'(s) = \frac{\eta_2 + \eta_1/2v_a \cdot \int_0^\infty \lambda_2 dx}{1 - \eta_1 \int_0^\infty x\lambda_2 \mathrm{d}x} \qquad \text{(expanding } Q' \text{ from (7))} \tag{10}$$

Since $\eta_1 \geq 0, \eta_2 < 0, \lambda_2 \leq 0$, it follows that

Proposition P1. *Under Assumptions A1–A3, $U'(s) < 0$.*

Thus, in spite of the feedback and boarding times, an increase in spacing always speeds up the bus (decreases the unit travel time).

2.4. Critical trip length

We now consider how changes in spacings affect trips of different lengths. It is most illustrative to do so graphically. Consider a pair of spacings s_0 and s_1 with $s_1 > s_0$. Now define a pair of lines

$$T_i(x) := T\left[x, s_i, Q(s_i)\right] = s_i/2v_a + H/2 + xU(s_i) \quad \text{for } i = 0, 1. \tag{11}$$

For $i = 0, 1$, $T_i(x)$ gives the travel time of a trip of length x in equilibrium, given a spacing s_i. The slope of $T_i(x)$ is $U(s_i)$, and its intercept is the out-of-vehicle time $H/2 + s_i/2v_a$.

Fig. 1 shows $T_1(x)$ and $T_2(x)$. Per Proposition P1, the unit travel time at the wider s_1 is smaller than at s_0, but the access time (and hence the intercept) is larger. Thus, the two lines necessarily intersect at a positive value of x that we will call the **critical trip length**, $\hat{x}$ (km). A widening $s_0 \to s_1$ will raise (lower) the travel time of all trips shorter (longer) than the critical trip length. By solving $T\left[\hat{x}, s_0, Q(s_0)\right] = T\left[\hat{x}, s_1, Q(s_1)\right]$, we can formalize this result as a proposition:

Proposition P2. *Define*

$$\hat{x} = \frac{1}{2v_a} \cdot \frac{s_1 - s_0}{U(s_0) - U(s_1)} \quad \textit{(km)}. \tag{12}$$

Under Assumptions A1–A3, if $s_0 < s_1$, then

[8] Note the upper bounds of the integrals defining $Q(s)$ and $K(s)$ are ∞. But realistically, we would not expect there to be any trips longer than R (i.e., $\lambda(x, T(x, s, q)) = 0 \ \forall x > R, s \geq 0, q \geq 0$) because there are no transfers to other routes involved in the model. So one could instead write $Q(s)$ and $K(s)$ with R as the integral's upper bound without changing the situation.

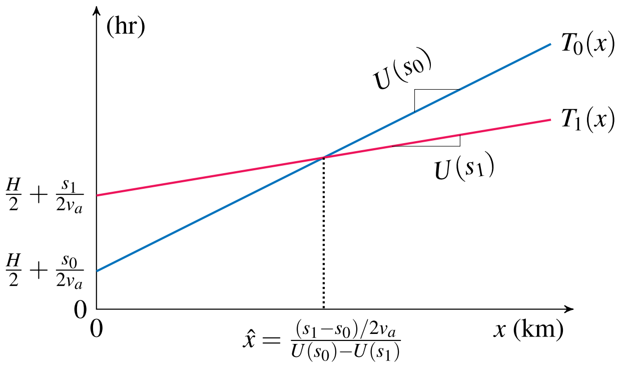


Fig. 1. Travel times by trip length for s_0 and $s_1 > s_0$.

(i) $T\left[x, s_0, Q(s_0)\right] > T\left[x, s_1, Q(s_1)\right]$ *for* $x > \hat{x}$

(ii) $T\left[x, s_0, Q(s_0)\right] < T\left[x, s_1, Q(s_1)\right]$ *for* $x < \hat{x}$

(iii) $\hat{x} > 0$

The critical trip length $\hat{x}$ divides trip lengths that would be harmed by the move $s_0 \to s_1$ from those that would benefit. In turn, since Assumption A3 makes demand depend on travel time, an increase in spacing will "repel" trips with lengths $x < \hat{x}$ (i.e., cause the number of such trips taken to fall) and "induce" trips with lengths $x > \hat{x}$ (i.e., cause their number to rise). By a similar logic, Lehe (2017) shows that imposing a "cordon toll" on downtown car trips repels short trips and induces long ones.

3. Model with fixed fleet size

In addition to saving passengers' in-vehicle time, another claimed advantage of stop consolidation is that it frees up bus-hrs (Saka, 2001; El-Geneidy et al., 2006; Stewart and El-Geneidy, 2016). These saved bus-hrs can be used to various ends. One use is to reduce headways (and hence wait times). In this section, we relax Assumption A1 (Fixed headways) to link stop spacing to headways formally. The novel result is that now Pareto improvements are possible: it is possible for an increase in spacing to make every rider better off.

3.1. Setup

Let B stand for the *fleet size*: the number of buses assigned to the route. Discard Assumption A1 (Fixed headways) and replace it with a new assumption:

Assumption A4 (*Fixed Fleet Size*). B is fixed; the agency does not alter the number of buses as it changes the stop spacing.

Given s and q, the time required for a bus to complete the route is

$$C := R \cdot \left\{ \frac{1}{v_b} + \eta(q, s) \right\} \quad \text{(h)}. \tag{13}$$

Thus, per Little's Law, the headway is

$$H = \frac{C}{B} = \frac{R}{B}\left\{ \frac{1}{v_b} + \eta(q, s) \right\} \quad \text{(bus/h)}, \tag{14}$$

where the quotient R/B (km/bus) is the mean distance between two buses. $\eta(q, s)$ is still assumed to obey A2. As in Section 2, we will give an example specification of η:

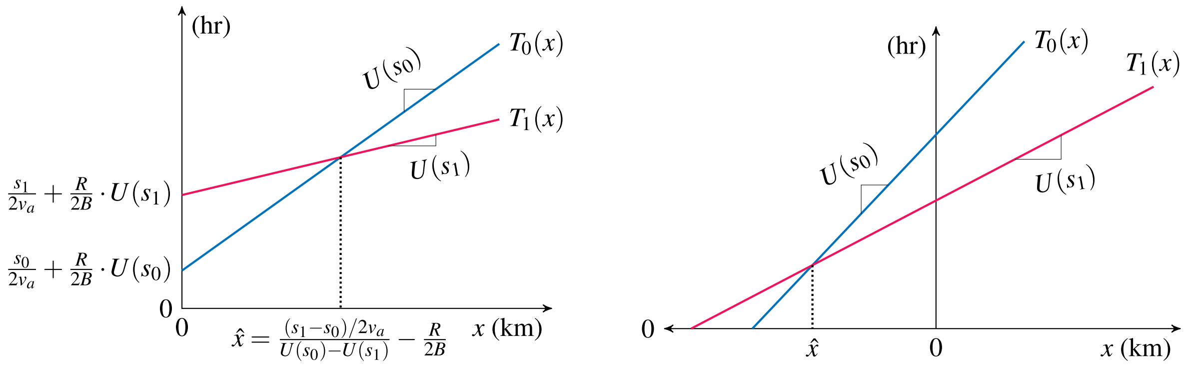
Example E2. Suppose as in Example E1 that the bus is delayed τ (h/stop) by each stop and θ (h/pax) by each passenger, so that $\eta = \tau/s + Hq\theta/R$ but now H is given by (14). Hence,

$$\eta(q, s) = \tau/s + \frac{R}{B}\left\{ \frac{1}{v_b} + \eta(q, s) \right\} q\theta/R = \frac{\tau/s + q\theta/Bv_b}{1 - q\theta/B}$$

The reader may verify that this satisfies all the properties of Assumption A2.

With endogenous headways, travel time is now given by

$$T(x, s, q) := \frac{s}{2v_a} + \left(\frac{R}{2B} + x \right)\left\{ \frac{1}{v_b} + \eta(q, s) \right\} \quad \text{(h)} \tag{15}$$



(a) with positive $\hat{x}$ (b) with negative $\hat{x}$

Fig. 2. Travel times by trip length for s_0 and $s_1 > s_0$ with endogenous headways.

for a trip of length x when ridership is q and spacing is s. The equilibrium demand is still given by the implicit function $Q(s)$ defined by (4), but with the new $T(x, s, q)$. Its derivative is now

$$Q'(s) = \frac{\int_0^\infty \lambda_2 \cdot \left[1/2v_a + (R/2B + x)\eta_2\right] \mathrm{d}x}{1 - \eta_1 \int_0^\infty \lambda_2 \cdot (R/2B + x) \mathrm{d}x}. \tag{16}$$

Plugging this derivative into (9) yields

$$U'(s) = \eta_1 Q'(s) + \eta_2 = \frac{\eta_2 + \eta_1/2v_a \cdot \int_0^\infty \lambda_2 dx}{1 - \eta_1 \int_0^\infty \lambda_2 \cdot (R/2B + x) \mathrm{d}x} \tag{17}$$

As before, since $\eta_1 \geq 0, \eta_2 < 0, \lambda_2 \leq 0$, it follows that

Proposition P3. *Under Assumptions A2, A3, and A4, $U'(s) < 0$.*

3.2. Critical trip length

We now return to the situation covered in Section 2.4: a comparison between a spacing s_0 and a wider one s_1. As before, we define a pair of lines

$$T_i(x) := T\left[x, s_i, Q(s_i)\right] = s_i/2v_a + \left(\frac{R}{2B} + x\right) U(s_i) \quad \text{for } i = 0, 1 \tag{18}$$

each giving the travel time of a trip of length x in the equilibrium with spacing s_i. See Fig. 2(a), which is an update of Fig. 1. As before, the slope of each line is the corresponding unit travel time, and this slope is lower for T_1 than for T_0. Note that the intercept of each line is different: the $H/2$ has been replaced by $U(s_i)/2B$. While the access time is certainly longer at s_1 than at s_0 (i.e., $s_1/2v_a > s_0/2v_a$), the wait time is certainly lower (i.e., $U(s_1)R/2B < U(s_0)R/2B$).

In Fig. 2(a), the out-of-vehicle time (intercept) at s_1 is still higher than at s_0 (just as in Fig. 1). However, nothing in our assumptions keeps the total out-of-vehicle time from being *lower* at the wider spacing. If the widening $s_0 \to s_1$ cuts wait times more than it increases access times, then the intercept of T_1 will exceed T_0's intercept. Fig. 2(b) shows this situation. In this case, the two lines will intersect at a *negative* value of x. Solving $T_1(\hat{x}) = T_0(\hat{x})$ yields a modified version of Proposition P2:

Proposition P4. *Define*

$$\hat{x} = \frac{1}{2v_a} \cdot \frac{s_1 - s_0}{U(s_0) - U(s_1)} - \frac{R}{2B} \quad \textit{(km)} \tag{19}$$

Under Assumptions A2–A4, if $s_0 < s_1$, then:

1. $T\left[x, s_0, Q(s_0)\right] > T\left[x, s_1, Q(s_1)\right] \; \forall x > \hat{x}$
2. $T\left[x, s_0, Q(s_0)\right] < T\left[x, s_1, Q(s_1)\right] \; \forall x < \hat{x}$

Unlike in Proposition P2, this proposition makes no claim that the critical trip length is strictly positive. Obviously, there are no negative trip lengths, so whenever a move $s_0 \to s_1$ yields a negative $\hat{x}$, a "Pareto improvement" occurs: trips of every length experience a lower travel time. This occurs because wait times fall by more than access times rise.

It turns out that such Pareto improvements are only possible when the initial spacing is sufficiently small.

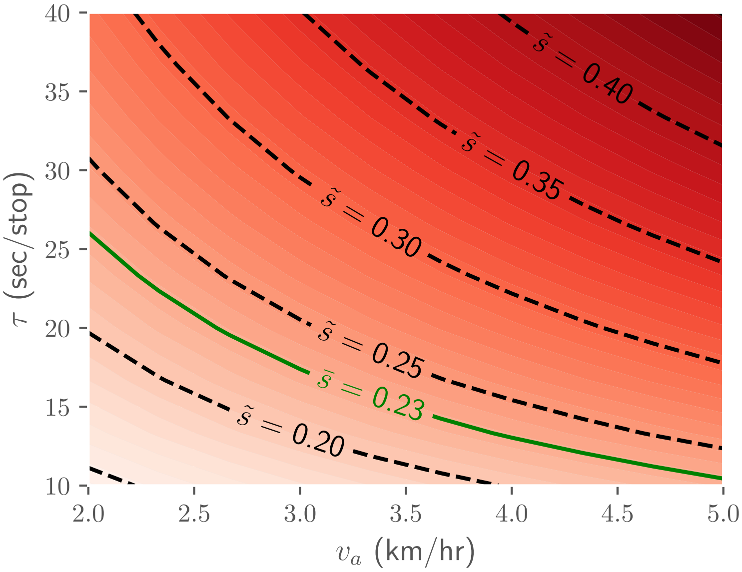


Fig. 3. Contour plot of $\tilde{s}$ from Example E3.

Proposition P5. *Under Assumptions A2, A3, and A4, there exists some $\tilde{s} > 0$ such that*

$$s < \tilde{s} \implies \frac{\mathrm{d}}{\mathrm{d}s} T\left[x, s, Q(s)\right] < 0 \quad \forall x \geq 0. \tag{20}$$

The proof is given in Appendix. This is to say that there exists some spacing so narrow that a marginal rise in spacing will benefit passengers of every trip length, because it will cut headways more than it raises access times. $\tilde{s}$ is a sort of "threshold": when $s < \tilde{s}$, the stops are so close together that removing stops reduces wait times faster than it increases access times.

While the idea of a Pareto improvement may seem fanciful, it turns out that $\tilde{s}$ may plausibly be in the neighborhood of actual average stop spacings in certain places, as the example below demonstrates:

Example E3. For the sake of illustration, this example ignores changes in demand due to a marginal change in spacing. Hence, we can treat buses as incurring a fixed stop delay of τ (h/stop). The delay function becomes $\eta(q, s) = \tau/s$ (h/km), and travel time is given by

$$T(x, s, q) = \frac{s}{2v_a} + \left(\frac{R}{2B} + x\right)\left\{\frac{1}{v_b} + \frac{\tau}{s}\right\} \quad \text{(h)}.$$

(Here the time lost to boarding/alighting may be considered included in v_b). Differentiating yields

$$\mathrm{d}T/\mathrm{d}s = 1/2v_a - (R/2B + x)\tau/s^2.$$

Since $\mathrm{d}T/\mathrm{d}s$ falls with x, if $\mathrm{d}T/\mathrm{d}s < 0$ at $x = 0$ then $\mathrm{d}T/\mathrm{d}s < 0$ for all $x > 0$ (i.e., then all trip lengths see travel time fall from a marginal increase in spacing). At $x = 0$, $\mathrm{d}T/\mathrm{d}s = 1/2v_a - (R/2B)\tau/s^2 < 0$ if and only if $s < \tilde{s}$, where

$$\tilde{s} = \sqrt{\tau v_a R/B} \quad \text{(km/stop)}.$$

We now apply this formula to Route 8 of the Chicago Transit Authority over a range of plausible v_a and τ values. Using the *gtfs-segments* Python package (Devunuri and Lehe, 2024), we find Route 8 has an average spacing of $\bar{s} = 230$ (m) and a ratio $R/B = 3.65$. Table 2 of Alves et al. (2020) shows walk speeds varying between 2.88 to 4.8 km/h, so consider $v_a \in [2, 5]$ (km/h). Metropolitan Transportation Authority (2022) assumes $\tau = 20$ (s/stop) for the Queens Bus Network Redesign, so consider $\tau \in \left[10/3600, 40/3600\right]$ (h/stop). Fig. 3 uses the formula $\tilde{s} = \sqrt{\tau v_a R/B}$ to draw contours of $\tilde{s}$ in (v_a, τ) space. Since $\tilde{s}$ exceeds the true average ($\bar{s} = 0.23$) for many (v_a, τ) combinations, it is plausible that marginal increases in spacing could yield Pareto improvements on Route 8.

4. General lessons

This section derives two lessons which apply under either assumption about headways.

4.1. Rivalrous metrics

The fact that stop consolidation induces sufficiently long trips, and repels trips shorter than these, makes *ridership* and *passenger-km traveled* rivalrous. By "rivalrous" we mean that one can only be achieved by sacrificing the other. Both are performance metrics

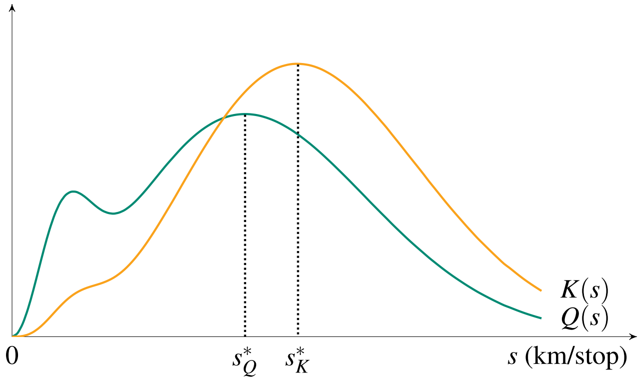


Fig. 4. Ridership and travel over a range of spacings.

used for US transit agencies (Talley, 1988), who report both annually to the country's National Transit Database. Let

$$K(s) := \int_0^\infty x\lambda[x, T\{x, s, Q(s)\}]\mathrm{d}x \quad \text{(pax-km/h)} \tag{21}$$

give the passenger-km traveled in equilibrium, given s. Fig. 4 shows hypothetical examples of $Q(s)$ and $K(s)$. They are drawn falling to zero as $s \to 0$ and as $s \to \infty$. This is sensible: the travel time of a trip of any length becomes infinite at these limits.

Observe that the curves are drawn such that $K(s)$ achieves its global maximum at a larger spacing than $Q(s)$. This ordering is not accidental but can be proven.

Proposition P6. *Letting s_Q^* be the largest spacing at which Q achieves its global maximum and s_K^* the smallest spacing at which K achieves its global maximum, it is true that $s_K^* \geq s_Q^*$.*

(The proof is in the Appendix.)

We can actually say something stronger than Proposition P6 under most circumstances.

Proposition P7. *If at $s = s_Q^*$, there are trips of more than one length being made, then $s_K^* > s_Q^*$.*

(The proof is in the Appendix.)

Thus, unless there is only one trip length being made, ridership will be maximized at a strictly smaller spacing than passenger-km traveled.

4.2. Changing average trip length

Models with differing features (e.g., Kikuchi and Vuchic, 1982; Kuah and Perl, 1988; Chien and Schonfeld, 1998) derive an optimal stop spacing which scales with either the square root of an average trip length or else with the square root of a linear transform of average trip length. Here, the average trip length in an equilibrium with spacing s is

$$\bar{x}(s) := K(s)/Q(s) \quad \text{(km)}. \tag{22}$$

Its derivative is

$$\mathrm{d}\bar{x}(x)/\mathrm{d}s = \frac{1}{Q(s)}\left[K'(s) - \bar{x}(s)Q'(s)\right]. \tag{23}$$

There is no reason to expect that the term in square brackets will be zero, so in general $\bar{x}$ should change with s. In the following example, $\bar{x}(s)$ is necessarily rising.

Example E4. Suppose that A1 (fixed headway) holds and that the demand density function has an exponential form

$$\lambda(x, t) := f(x)\exp(-\beta t), \tag{24}$$

where $\beta > 0$ measures the sensitivity of demand to travel time and $f(x)$ (pax/km-h) is the maximum possible demand density at x. Expanding K' and Q' from (23), and noting $T_2 = 1/2v_a + xU'(s)$ (per A1), yields

$$\bar{x}'(s) = \frac{1}{2v_a Q(s)}\left[\int_0^\infty x\lambda_2 \mathrm{d}x - \bar{x}\int_0^\infty \lambda_2 \mathrm{d}x\right] + \frac{U'(s)}{Q(s)}\left[\int_0^\infty x^2\lambda_2 \mathrm{d}x - \bar{x}\int_0^\infty x\lambda_2 \mathrm{d}x\right]. \tag{25}$$

Since $\lambda(x, t) = f(x)\exp(-\beta t)$ implies $\lambda_2 = -\beta\lambda$, the integrands become

$$\bar{x}'(s) = -\frac{\beta}{2v_a Q(s)}\left[\int_0^\infty x\lambda \mathrm{d}x - \bar{x}\int_0^\infty \lambda \mathrm{d}x\right] - \frac{\beta U'(s)}{Q(s)}\left[\int_0^\infty x^2\lambda \mathrm{d}x - \bar{x}\int_0^\infty x\lambda \mathrm{d}x\right] \tag{26}$$

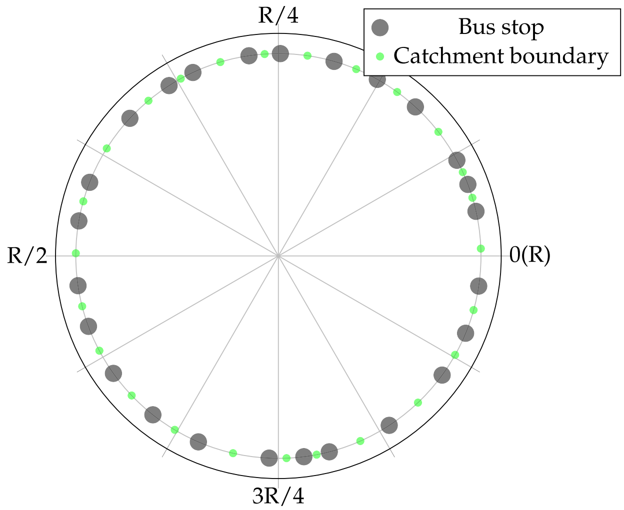


Fig. 5. Bus network with heterogeneous spacings.

which reduces to

$$\bar{x}'(s) = -\frac{\beta U'(s)}{Q(s)}\left[\int_0^\infty x^2\lambda \mathrm{d}x - \bar{x}\int_0^\infty x\lambda \mathrm{d}x\right] \tag{27}$$

$$\bar{x}'(s) = -\beta U'(s)\left[E(x^2) - (\bar{x})^2\right], \tag{28}$$

where $E(x^2)$ is the expected value of x^2. The difference $E\left(x^2\right) - (\bar{x})$ is simply the variance of x, which is positive. Hence, since $U'(s) < 0$, $\bar{x}(s)$ is rising.

When average trip length changes with spacing, the application of a formula with an average trip length will chase a "moving target". After applying the formula, the average trip length itself will update. The following example illustrates.

Example E5. Consider an instance of Section 2 model with route length $R = 10$ (km), headway $H = 1/6$ (h), access speed $v_a = 3$ (km/h) and demand density

$$\lambda(x,t) = 1000\exp(-5t)\,\text{for}\,x \in (0,10). \tag{29}$$

The delay function $\eta(q,s)$ takes the functional from Example E1 with $\theta = 7/3600$ (h/pax), $\tau = 20/3600$ (h/stop) and $H = 1/6$.

Suppose the initial spacing is only $s_0 = 0.15$ (km/stop). At this spacing, we can calculate that the average trip length in equilibrium is $\bar{x}_0 = 2.56$ (km) and the ridership is 1664 (pax/h).

Suppose the operator wants to optimize spacing, but they treat demand as fixed and apply the formula for optimal spacing from Kikuchi and Vuchic (1982, Eq. 22): $s^* = \sqrt{2\bar{x}v_a\tau}$. Given $\bar{x}_0 = 2.56$, this formula gives $s^* = \sqrt{2\cdot 2.56\cdot 3\cdot 20/3600} = 0.29$ (km/stop). The widening from 0.15 to 0.29 km/stop yields a critical trip length $\hat{x} = 1.5$ (km), repelling trips below this length and inducing ones above. Thus, after the widening, the new average trip length is $\bar{x}_1 = 3.03$ and the new ridership is 1804 (pax/h).

5. Discrete simulation

This section sets up a simulation that takes two steps toward realism:

(i) It drops the continuous approximation context. Stops, origins and destinations are at particular locations. Passengers travel an integer number of stops to their before alighting.
(ii) It drops the uniform spacing assumption. Now the distances among stops are heterogeneous.

The idea behind undertaking the simulation is to see whether the simulation reproduces results from the above derivations. While the route does not have one single stop spacing, s, the average stop spacing, $\bar{s}$, can still be used to characterize the route. We posit that the relationships in Propositions P6, P7 and P5 should hold reasonably well when we replace s with $\bar{s}$ to account for heterogeneous stop spacings.

5.1. Setup

The setting is similar to the setting in Sections 2 and 3. Buses run along a ring of length R (km) with a speed of v_b (km/h) between stops. They stop at every stop on the route and lose τ (h/stop) at each stop. Each passenger delays the bus by θ_b (hour/pax) while boarding and θ_a (hour/pax) while alighting. Unlike Sections 2 and 3, *stop spacing is not equal along the whole route* and a transit network cannot be uniquely described using the stop spacing, s. See Fig. 5. The bus route is defined by a set of real numbers giving the locations of all the stops on the route, $\mathcal{L}$. If there are N stops along a route, the average stop spacing is R/N (km).

$Q_o(\mathcal{L})$ and $Q_d(\mathcal{L})$ are the sets of number of trips originating and ending at different stops, i.e., the ith element, $Q_o^i(\mathcal{L})$ (pax/h), is the number of trips originating at stop $i \in \mathcal{L}$. A trip's door-to-door travel time depends on its origin and length. Let $l \in [0, R]$ (km) be the origin for a trip of length x (km). The door-to-door travel time, given $\mathcal{L}$, is

$$\begin{aligned} T_l(x, \mathcal{L}) = {} & \frac{\|l - \mathrm{OS}\|}{v_b} + \frac{\|\mathrm{OS} - \mathrm{DS}\|}{v_b} + \tau \cdot \#\{\mathrm{OS} - \mathrm{DS}\} \\ & + H\theta_b \cdot \#Q_o\{\mathrm{OS} - \mathrm{DS}\} + H\theta_a \cdot \#Q_d\{\mathrm{OS} - \mathrm{DS}\} + \frac{\|x + l - \mathrm{DS}\|}{v_a} + \frac{H}{2} \quad \text{(h)}, \end{aligned} \tag{30}$$

where H is the headway, OS, DS $\in \mathcal{L}$ are origin and destination bus stops, $\|a - b\|$ is the distance between points a and b, $\#\{\mathrm{OS} - \mathrm{DS}\}$ is the number of bus stops between OS and DS, $\#Q_o\{\mathrm{OS} - \mathrm{DS}\}$ is the number of passengers boarding and $\#Q_d\{\mathrm{OS} - \mathrm{DS}\}$ is the number of passengers alighting between OS and DS:

$$\#Q_o\{\mathrm{OS} - \mathrm{DS}\} = \sum_{i=\mathrm{OS}}^{\mathrm{DS}} \left[Q_o^i(\mathcal{L})\right], \quad \#Q_d\{\mathrm{OS} - \mathrm{DS}\} = \sum_{i=\mathrm{OS}}^{\mathrm{DS}} \left[Q_d^i(\mathcal{L})\right]. \tag{31}$$

The headway, H (h), is fixed for the fixed headway model. When the fleet size is fixed at B, the headway is endogenous. Recall N is the number of stops (i.e., the number of elements in $\mathcal{L}$). Thus $R/v_b + N\tau + \theta_b \sum Q_o(\mathcal{L}) + \theta_a \sum Q_d(\mathcal{L})$ is the cycle time, and the headway is $H = \left[R/v_b + N\tau + \theta_b \sum Q_o(\mathcal{L}) + \theta_a \sum Q_d(\mathcal{L})\right]/B$.

5.2. Equilibrium

Recall the demand density function $\lambda(x, t)$ gave the demand density over trip lengths. Here, assume that every point on the route is the same as far as demand is concerned, and let $\gamma(x, t)$ (pax/km^2-h) give the *demand-density at every point on the route* for a trip of length x (km), given travel time t. At origin l, the demand for trips with lengths $x \in [x', x'']$ is

$$\int_{x'}^{x''} \gamma\left[x, T_l(x, \mathcal{L})\right] \mathrm{d}x \quad \text{(pax/km-h)}.$$

Thus the total demand for trips of lengths $x \in \left[x', x''\right]$ can be found by integrating over all origins:

$$\int_0^R \int_{x'}^{x''} \gamma\left[x, T_l(x, \mathcal{L})\right] \mathrm{d}x\mathrm{d}l \quad \text{(pax/h)}.$$

Given $\mathcal{L}$, we need to calculate $Q_o(\mathcal{L})$ and $Q_d(\mathcal{L})$ at equilibrium. Passengers walk to their nearest stop, hence, zone served by a stop is demarcated by the midpoints between the stop and its two adjacent stops, as shown in Fig. 5. Let those points be b_{-i} and b_{+i}. The, the demand originating from stop i is

$$Q_o^i(\mathcal{L}) = \int_{b_{-i}}^{b_{+i}} \int_{b_{+i}-l}^{R+b_{-i}+l} \gamma\left[x, T_l(x, \mathcal{L})\right] \mathrm{d}x\mathrm{d}l \quad \text{(pax/h)}. \tag{32}$$

The limits $\left\{b_{-i}, b_{+i}\right\}$ give the catchment area of stop i. If a trip is too short, it will not be served by transit as both the origin and destination will be in the catchment area of the same stop. Thus, the minimum allowable trip length is $b_{-i} - l$. Similarly, the maximum allowable trip length is $R + b_{+i} - l$. This is different from the CA model where we integrate over all trips from length 0 to R. Similarly, the demand alighting at stop i is

$$Q_d^i(\mathcal{L}) = \int_{b_{+i}}^{R+b_{-i}} \int_{R+b_{-i}-l}^{R+b_{+i}-l} \gamma\left[x, T_l(x, \mathcal{L})\right] \mathrm{d}x\mathrm{d}l \quad \text{(pax/h)}. \tag{33}$$

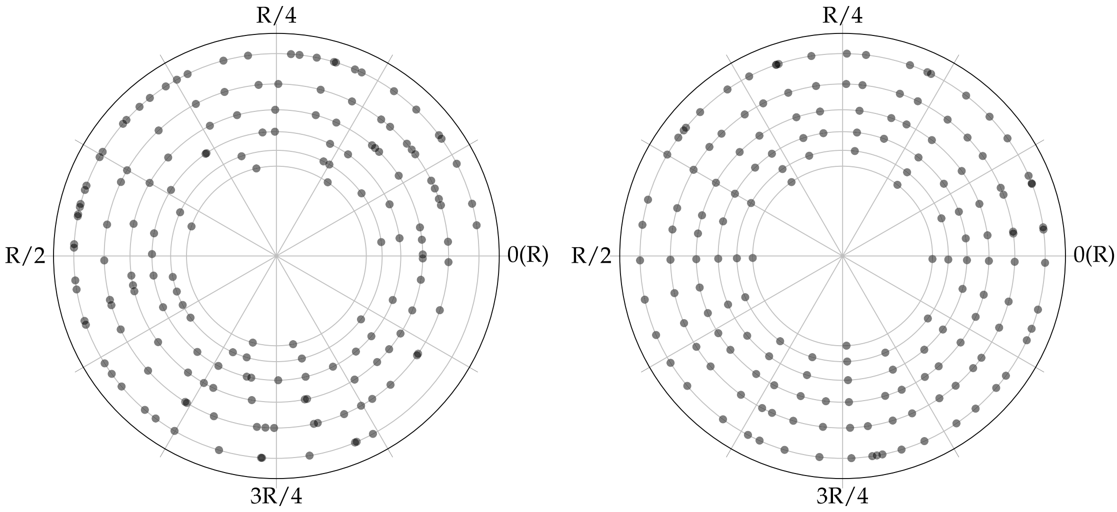
To calculate the number of passengers alighting at stop i, we need to integrate over all trips originating outside the catchment area of stop i, with trip lengths such that the trip ends inside the catchment of stop i. Simultaneously solving (32) and (33) for all $i \in \mathcal{L}$ will give us the equilibrium values of $Q_o(\mathcal{L})$ and $Q_d(\mathcal{L})$, i.e., we need to solve 100 non-linear simultaneous equations for a route with 50 stops. We solve these equations iteratively. We start with an initial guess for $Q_o(\mathcal{L})$ and $Q_d(\mathcal{L})$. Find the function $T_l(x, \mathcal{L})$ for all $l \in \mathcal{L}$. Then, using (32) and (33), we calculate the new values of $Q_o(\mathcal{L})$ and $Q_d(\mathcal{L})$. We repeat this process until the values of $Q_o(\mathcal{L})$ and $Q_d(\mathcal{L})$ converge to a fixed point.

Once we have the equilibrium, we can calculate the total demand simply by summing over all stops:

$$Q(\mathcal{L}) = \sum_{i \in \mathcal{L}} Q_o^i(\mathcal{L}) = \sum_{i \in \mathcal{L}} Q_d^i(\mathcal{L}) \quad \text{(pax/h)}. \tag{34}$$

Passenger-km demanded is calculated by multiplying x to the integrand of (32) and (33) and then summing over all stops:

$$K(\mathcal{L}) = \sum_{i \in \mathcal{L}} \int_{b_{-i}}^{b_{+i}} \int_{b_{+i}-l}^{R+b_{-i}+l} x \cdot \gamma\left[x, T_l(x, \mathcal{L})\right] \mathrm{d}x\mathrm{d}l \quad \text{(pax-km/h)}. \tag{35}$$



(a) Random generation (b) Consolidation

Fig. 6. Example networks for different simulations.

5.3. Varying spacings

The route is now described by particular locations, but we would like to check if earlier Propositions about s hold up in this more realistic setting. We do so by interpreting s as the *average* stop spacing and evaluating $Q(s)$ and $K(s)$ as the average spacing varies.

There are two ways to vary average spacing. The first way is what we call "random generation". For each N of stops considered (and thus each average spacing considered), we randomly generate N locations of different stops and then calculate ridership and passenger-km traveled. This is to say $\mathcal{L}$ is generated from scratch for each average spacing. The second way we call "consolidation". It involves generating an initial $\mathcal{L}$ randomly and then removing one stop at a time without changing the locations of other stops. At each iteration, the stop chosen for removal is the stop responsible for the narrowest extant spacing on the route.

Fig. 6 shows example networks generated using each method. The stop locations are plotted along concentric circles[9] to show the relationship between different networks. Average spacing is increasing as we go closer to the center of the circle. As the figures show, random generation produces completely different routes as spacing rises, and consolidation shows a particular route being gradually pruned.

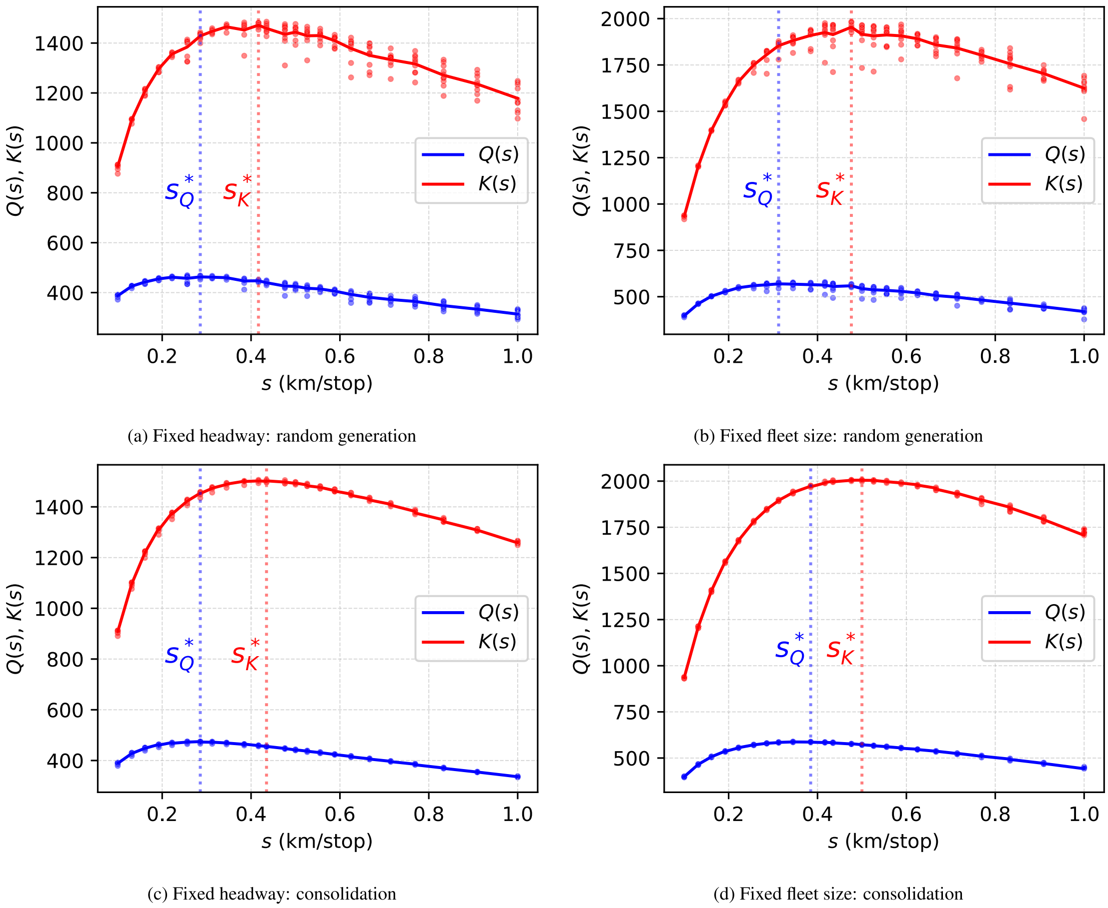
5.4. Results

We now present the results[10] of the simulations. The simulations are conducted with $v_a = 3$ (km/h), $v_b = 30$ (km/h), $R = 10$ (km), and $\gamma(x,t) = 25\exp(-4t)$ pax/km^2-h. The walk speed v_a is taken from Alves et al. (2020). We set the stop delay $\tau = 20/3600$ (h/stop), which is used in the Queens Bus Network Redesign by the Metropolitan Transportation Authority (Metropolitan Transportation Authority, 2022). Note some other studies have lower values (e.g., Glick and Figliozzi (2017) estimate about 16 s in Portland). For boarding and alighting delays, we choose $\theta_b = 5/3600$ (hour/pax), and $\theta_a = 2/3600$ (hour/pax), which are in the middle of the range reported by Rajbhandari et al. (2003).

For the fixed headway model, we set $H = 1/6$ (h). For the fixed fleet size model, we set $B = 6$. We evaluate $Q(\mathcal{L})$ and $K(\mathcal{L})$ over a range of s from .1 to 1 (km/stop) (i.e., from $N = 100$ to $N = 10$). We cover this range ten different times for each model (fixed headway vs. fixed fleet size) and for each method (random generation vs. consolidation). Fig. 7 plots the resulting demand and passenger-km by spacing. The results of particular simulations are drawn as (semi-transparent) dots, and the average values, grouped by s, are connected together as a line.

The simulations with random generation show more variance in ridership and passenger-km when stop spacing is large. This is because different networks with same stop spacing can vary a lot in how those stops are distributed, and hence in how trips are made. The simulations with consolidation show less variance because all networks start with a small spacing and are very similar; these networks are then pruned to increase the average spacing. Since the networks are similar to begin with, the resulting networks are also similar and the ridership and passenger-km do not vary much.

[9] Note that all networks have the same route. The circumference does not represent the route length.
[10] https://github.com/UTEL-UIUC/heterogeneous-trip-lengths hosts the code used to run the simulation.



(a) Fixed headway: random generation

(b) Fixed fleet size: random generation

(c) Fixed headway: consolidation

(d) Fixed fleet size: consolidation

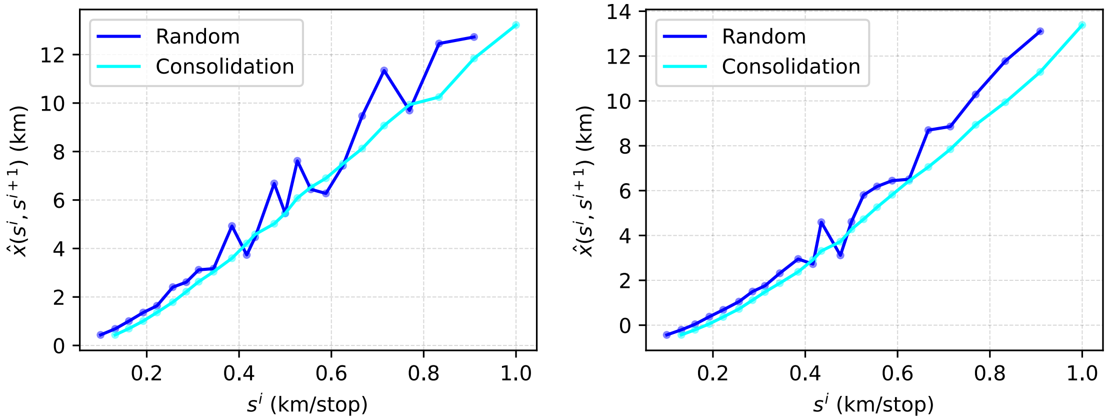
Fig. 7. Q and K with $\bar{s}$.

In Fig. 7, the vertical lines labeled s_Q^* and s_K^* show the ridership and passenger-kilometers maximizing spacings for each set of simulations. As expected from the discussion above, all the simulations show that s_K^* substantially larger than s_Q^*. We can compute these optimum spacings analytically for the demand function and parameters used in the simulation. For the fixed H model, the simulation with random generation gives $s_Q^* = 0.29$ (km/stop) and $s_K^* = 0.42$ (km/stop); the consolidation simulation gives $s_Q^* = 0.29$ (km/stop) and $s_K^* = 0.43$ (km/stop); the analytical solution gives $s_Q^* = 0.32$ (km/stop) and $s_K^* = 0.44$ (km/stop). This means that the difference between the simulated and analytical s_Q^* is only 10%. Similarly, the passenger-km maximizing spacing only has an error of 3%. This can partly be explained by the fact that the analytical solution allows for the number of stops to take any real value, while the simulations only have a natural number of stops. For our 10 km route, 35 stops maximize ridership for both random generation and consolidation, while analytically the optimal number of stops is 31.68. Similarly for maximizing passenger-km, random generation gives 24 stops and consolidation gives 23 stops, while the analytical solution says 22.55 stops.

For the fixed B model, the simulation with random generation maximizes ridership when spacing is 0.31 (km/stop), consolidation simulation maximizes it at 0.38 (km/stop), and analytically the optimal spacing is 0.41 (km/stop). This error is much larger at about 25% for random generation and 7% for consolidation. For maximizing passenger-km, the spacings are 0.48, 0.50, and 0.53, with random generation, consolidation, and analytically; yielding an error of 10%. Hence, the analytical solution consistently overestimates the optimal spacing by a small amount. The simulations do confirm that s_K^* is always greater than s_Q^*, as proved in Proposition P7.

Fig. 8 plots the critical trip length, $\hat{x}(s^i, s^{i+1})$, between consecutive stop spacings s^i and s^{i+1} for our simulation. The critical trip length is increasing for both fixed headway and fixed fleet size models, and for both random generation and consolidation methods. Plot 9(b) shows that the critical trip length is negative when s^i is very small. A negative $\hat{x}$ indicates that the travel time of trips of all lengths is decreasing as the spacing increases, resulting in a Pareto improvement consistent with Proposition P5. The Pareto improving threshold is around $\bar{s} \approx 150$ (m), for both the random generation and consolidation methods.

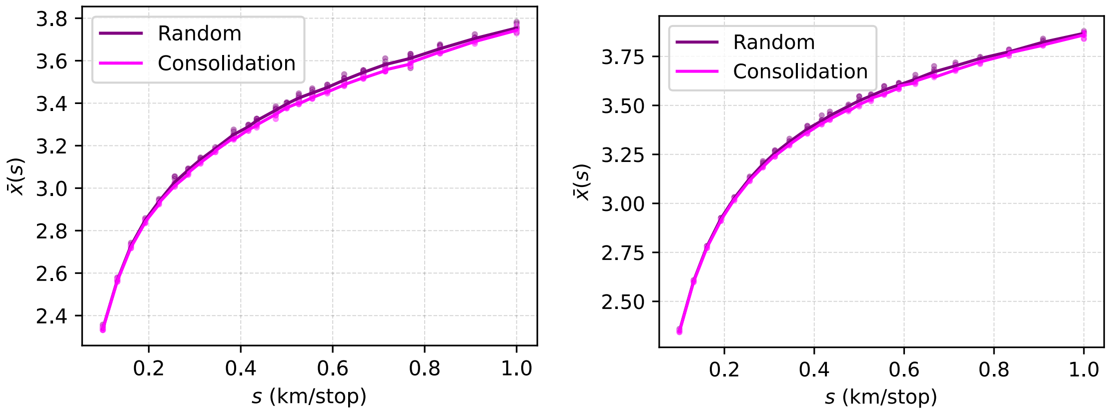
We also plot the average trip length for all four simulations in Fig. 9. As proved in Example E4, the simulation confirms that the average trip length is always increasing for exponential demand, for both fixed headway and fixed fleet size models, and for both random generation and consolidation methods. While this simulation can always be extended to include more realistic features, we only use it to show that the general lessons hold even without assumptions of continuous approximation.



(a) Fixed headway

(b) Fixed fleet size

Fig. 8. Critical trip length: $\hat{x}(s^i, s^{i+1})$.



(a) Fixed headway

(b) Fixed fleet size

Fig. 9. Average trip lengths.

6. Conclusion

This paper has explored macroscopic, continuum-approximation models of a bus route with the following features: (i) stop spacing is a design variable; (ii) heterogeneous trip lengths; (iii) elastic demand; (iv) passenger delays. These features are analyzed in a model with fixed headways (Section 2) and another with endogenous headways (Section 3). We then produced a simulation which confirmed key results of the analysis for both models (Section 5).

The paper's models greatly simplify reality. There are no discussions of bus size or crowding inside the bus, financial costs or fares, road congestion, line spacing or transfers (since there is only one route). The models could be made more realistic by incorporating these elements. Still, the paper derives useful points which seem broadly plausible:

(i) An increase in spacing generally (exception discussed below) results in a "critical trip length", such that travel times of trips above the critical trip length rise and those below it fall (Propositions P2 and P4).
(ii) This pattern of travel times rising and falling can induce trips longer than the critical trip length and repel trips shorter than the critical trip length. Consequently, the stop spacing that maximizes passenger-km traveled is at least as large as the one that maximizes ridership (Proposition P6) and almost certainly strictly larger (Proposition P7). This is true across both models.
(iii) The average trip length almost certainly changes with spacing (Section 4.2). Thus, optimality formulae which depend on an average trip length need to be "handled with care". An agency which optimizes spacings for its current trips will face a "moving target", insofar as the spacing influences what trips are made.

(iv) If headways fall with spacing (Section 3), and if the spacing is smaller than a threshold $\tilde{s}$, then a rise in spacing may reduce expected wait time more than it increases expected access time, which lowers the travel time of every trip length (Proposition P5).

We conclude with three directions the authors believe to be especially promising extensions of the work.

6.1. Case studies

Future work could test the paper's insights against real-life cases studies. The literature on stop spacing includes not only theoretical studies but also case studies with precise levels of detail collected from real cities and surveys (e.g., Furth and Rahbee, 2000; El-Geneidy et al., 2006; Li and Bertini, 2009; Stewart and El-Geneidy, 2016; Wu et al., 2022). By modeling various stop spacings with a travel demand model, it may be possible to obtain estimates of the $Q(s)$ and $K(s)$ functions. Or, if surveys were collected before and after a stop consolidation, the data may be analyzed to check if the average trip length changed.

6.2. Optimization

The paper's mode of analysis has been mainly positive rather than normative—that is, to derive certain broad descriptions of how the world works under declared assumptions, rather than to propose solutions. Still, the work has some obvious implications for optimal system design. One application where elasticity and trip length heterogeneity would matter is the design of "hierarchical" bus systems. The model of Daganzo and Ouyang (2019, Ch. 5) mentioned in the introduction is part of a larger exposition about the advantages of having local and express routes to serve passengers of different lengths. Incorporating elastic demand into such a model would probably show that bifurcating service between local and express routes probably also induces shorter trips (onto locals) and longer trips (onto express routes). The overall implications of that fact are ambiguous.

Another line of inquiry might involve mode choice. We have made demand elastic but not been specific about what riders do if they do not ride the bus. In reality, travelers choose walking, cycling, driving and a proliferating menu of "micromobility" modes. Since driving tends to have a higher fixed cost but cover ground faster, a rough view would suggest wider spacings tend to draw would-be drivers on longer trips; and, conversely, that narrower spacings draw would-be walkers taking shorter trips. One declared motivation for subsidizing bus systems is to reduce car traffic, and so such logic ought to bear on system design.

6.3. Other types of passenger heterogeneity

Recall from the introduction that there is now a trend toward bus stop consolidation. The trend reflects a consensus that many cities have erred too far on the side of reducing access time at the expense of in-vehicle time. This perception is due in part to simple comparison: some US cities have much narrower mean spacings than others (Pandey et al., 2021; Devunuri et al., 2024). Also, case studies analyzing stop spacings in real cities (Furth and Rahbee, 2000; Li and Bertini, 2009; Stewart and El-Geneidy, 2016) tend to recommend consolidation. Stewart and El-Geneidy (2016) suggests removing about 2000 of Montreal's stops (23% of the total).

It does not take long to remove stops and change bus schedules, but the consolidation trend has been gradual, and one reason is that agencies face considerations mostly ignored in the theoretical work on stop spacing—specifically, heterogeneous passengers. Berez (2015) (a survey of best practices for stop consolidation in the US) reports: "All planners interviewed for this study said that concerns related to elderly and disabled passengers were among the primary objections to bus stop consolidation". To address these concerns, the model in this paper could be extended to incorporate heterogeneity of access speed. Instead of all passengers having a universal v_a, there would be a bivariate distribution of trip length and access speed. Passengers with mobility concerns (such as the elderly and disabled) would be characterized by low values of v_a. But age and disability are not the only source of heterogeneity: mean urban walking speeds vary substantially among countries (Levine and Norenzayan, 1999), and a few pedestrian studies have plotted continuous distributions of walking speeds for particular contexts (e.g., Chandra and Bharti, 2013).

One consequence of access speed heterogeneity is that there will not be a singular critical trip length. The critical trip length will be a *function* of access speed, rather than a value. The sets of trips that gain and lose travel time following a change in spacing will compose two-dimensional regions of a (v_a, x) plane, rather than intervals of x. Fig. 10 illustrates what this would look like for the fixed headway model. Sufficiently slow walkers with *any* trip length will see door-to-door travel time rise from $s_0 \rightarrow s_1$. So wider spacing favors longer trips *and* fast walkers (or, equivalently, people who do not mind walking). In turn, we ought to expect that the current spacing attracts people with certain access speeds.

CRediT authorship contribution statement

Ayush Pandey: Writing – review & editing, Writing – original draft, Visualization, Software, Investigation, Formal analysis. **Lewis J. Lehe:** Writing – review & editing, Writing – original draft, Project administration, Investigation.

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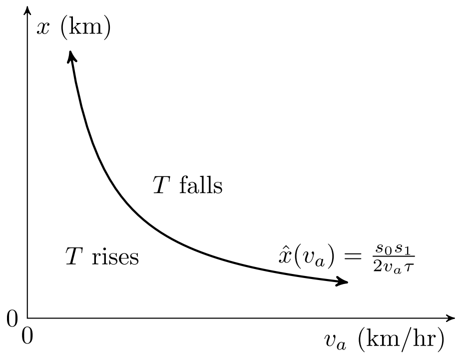

Fig. 10. Regions of the (v_a, x) plane where travel time rises and falls from a rise in spacing $s_0 \rightarrow s_1$.

Appendix

Proof of Proposition P5. Note that

$$\frac{\mathrm{d}T\{x,s,Q(s)\}}{\mathrm{d}s} = \frac{1}{2v_a} + \left[\frac{R}{2B} + x\right]\frac{\mathrm{d}\eta}{\mathrm{d}s}. \tag{A.1}$$

As s increases, in-vehicle and wait times decrease, and access time increases. Since the decrease in in-vehicle time depends on trip length, we can write the maximum change in travel time as

$$\max_{x\geq 0} \frac{\mathrm{d}T\{x,s,Q(s)\}}{\mathrm{d}s} = \frac{1}{2v_a} + \frac{R}{2B}\frac{\mathrm{d}\eta}{\mathrm{d}s}. \tag{A.2}$$

At $s = 0$, any increase in stop spacing makes the delay fall from infinity to a finite value. Hence,

$$\frac{\mathrm{d}\eta}{\mathrm{d}s} = -\infty \implies \max \left.\frac{\mathrm{d}T}{\mathrm{d}s}\right|_{s=0} < 0. \tag{A.3}$$

Next, η_2 is increasing and approaches zero as $s \rightarrow \infty$. Using the intermediate value theorem, we write,

$$\exists \quad s_c > 0 \quad \text{such that} \quad \eta_2 = -\frac{1}{2v_a}\frac{1}{R/2B}. \tag{A.4}$$

When $s \in [0, s_c]$

$$\eta_2 \leq -\frac{1}{2v_a}\frac{1}{R/2B} \tag{A.5}$$

$$\implies \frac{1}{2v_a} + \left(\frac{R}{2B} + x\right)\eta_2 < 0 \qquad \forall x \geq 0. \tag{A.6}$$

Since $\lambda_2 < 0$, $\eta_1 \geq 0$, and $\int_0^\infty \lambda_2 (R/2B + x)\eta_2 < 0$, we can use (16) to show that

$$Q' = \frac{\int_0^\infty \lambda_2 \cdot \left[1/2v_a + (R/2B + x)\eta_2\right]\mathrm{d}x}{1 - \int_0^\infty \lambda_2 \cdot (R/2B + x)\eta_1 \mathrm{d}x} > 0. \tag{A.7}$$

Substituting $Q'\eta_1 > 0$ in $\mathrm{d}\eta/\mathrm{d}s = Q'\eta_1 + \eta_2$, we show that

$$\left.\frac{\mathrm{d}\eta}{\mathrm{d}s}\right|_{s\in[0,s_c]} > \eta_2. \tag{A.8}$$

This means that $\mathrm{d}\eta/\mathrm{d}s$ is trapped above η_2 for $s \in [0, s_c]$. Hence, it crosses $-\frac{1}{2v_a}\frac{1}{R/2B}$ before η_2. Consequently,

$$\left.\frac{\mathrm{d}\eta}{\mathrm{d}s}\right|_{s=s_c} > -\frac{1}{2v_a}\frac{1}{R/2B} \implies \max \left.\frac{\mathrm{d}T}{\mathrm{d}s}\right|_{s=s_c} > 0. \tag{A.9}$$

Using intermediate Value Theorem, (A.3) and (A.9), we know that there is a spacing where $\mathrm{d}\eta/\mathrm{d}s$ crosses $-\frac{1}{2v_a}\frac{1}{R/2B}$. Let $\tilde{s} \in [0, s_c]$ be the smallest such spacing, then

$$\max \left.\frac{\mathrm{d}T}{\mathrm{d}s}\right|_{\forall s<\tilde{s}} < 0. \tag{A.10}$$

Hence,

$$s < \tilde{s} \implies \frac{\mathrm{d}T\{x,s,Q(s)\}}{\mathrm{d}s} < 0 \quad \forall x \geq 0. \quad \square \tag{A.11}$$

Proof of P6. Consider some spacing s_0 which is smaller than s_Q^*. To prove the theorem, we will show that $K(s_Q^*)$ is necessarily larger than $K(s_0)$. There are essentially two cases.

Case 1: *Assumption A4 holds and the increase $s_0 \to s_Q^*$ yields a negative $\hat{x}$.*
This is the Pareto improvement case. The move $s_0 \to s_Q^*$ must not lower passenger-km traveled, because no trips are repelled.

Case 2: *Otherwise.* Either Assumption A1 holds (so $\hat{x}$ is necessarily positive) or else Assumption A4 holds but s_0 is such that $s_0 \to s_Q^*$ yields a positive $\hat{x}$. Since $\hat{x} > 0$, the move $s_0 \to s_Q^*$ repels trips shorter than $\hat{x}$ and induces trips longer than $\hat{x}$. Thus, the total difference in ridership, $\Delta Q := Q(s_Q^*) - Q(s_0)$, and total difference in passenger-km traveled, $\Delta K := K(s_Q^*) - K(s_0)$, can each be written as the sum of a weakly negative integral and a weakly positive one:

$$\Delta Q = \underbrace{\int_0^{\hat{x}} \lambda\big[x, T(x, s_1, Q_1)\big] - \lambda\big[x, T(x, s_0, Q_0)\big]\,\mathrm{d}x}_{Q^- \le 0} + \underbrace{\int_{\hat{x}}^{\infty} \lambda\big[x, T(x, s_1, Q_1)\big] - \lambda\big[x, T(x, s_0, Q_0)\big]\mathrm{d}x}_{Q^+ \ge 0} \tag{A.12}$$

$$\Delta K = \underbrace{\int_0^{\hat{x}} x\Big\{\lambda\big[x, T(x, s_1, Q_1)\big] - \lambda\big[x, T(x, s_0, Q_0)\big]\Big\}\,\mathrm{d}x}_{K^- \le 0} + \underbrace{\int_{\hat{x}}^{\infty} x\Big\{\lambda\big[x, T(x, s_1, Q_1)\big] - \lambda\big[x, T(x, s_0, Q_0)\big]\Big\}\,\mathrm{d}x}_{K^+ \ge 0}. \tag{A.13}$$

Indicating the terms by the labels below them, note $\hat{x}Q^- \le K^-$ and $\hat{x}Q^+ \le K^+$. Thus,

$$\hat{x}\Delta Q = \hat{x}Q^- + \hat{x}Q^+ \le K^- + K^+ = \Delta K \tag{A.14}$$

By construction, s_Q^* is the largest value of s that achieves the maximum value of Q and so $\Delta Q \ge 0$. Per (A.14), $\Delta Q \ge 0 \implies \Delta K \ge 0$.

In either Case 1 or Case2, passenger-km traveled is at least as large at s_Q^* as at any narrower spacing. Thus, s_K^* cannot be any smaller than s_Q^*. □

Proof of Proposition P7. To begin, we will establish that under either Assumptions A1 or A4, an infinitesimal rise in spacing at s_Q^* will result in a critical trip length $\hat{x}$.

- First, suppose Assumption A1 holds. The derivative

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=s_Q^*} T\Big[x, s, Q(s)\Big] = 1/2v_a + xU'(s) \tag{A.15}$$

is positive (negative) for x smaller than (larger than) $\hat{x} = -1/2v_a U'(s) > 0$.

- Second, suppose that Assumption A4 holds. Now we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=s_Q^*} T\Big[x, s, Q(s)\Big] = 1/2v_a + \left(\frac{R}{2B} + x\right)U'(s). \tag{A.16}$$

Per Proposition P5, if s is larger than the threshold $\tilde{s}$ then this derivative will be positive (negative) for x smaller than (larger than) some strictly positive critical trip length $\hat{x}$. We can assume that s_Q^* is larger than this $\tilde{s}$. Otherwise, the spacing could be slightly increased and no trips would be repelled, in which case s_Q^* would not the largest spacing at which $Q(s)$ is maximized.

In either case there exists a strictly positive $\hat{x}$ dividing trips repelled from induced. Hence, the derivatives $Q'(s_Q^*)$ and $K'(s_Q^*)$ can be divided between repelled and induced trips:

$$Q'(s_Q^*) = \underbrace{\int_0^{\hat{x}} \lambda_2\Big[x, T\Big\{x, s_Q^*, Q(s_Q^*)\Big\}\Big]T_2\Big\{x, s_Q^*, Q(s_Q^*)\Big\}\mathrm{d}x}_{Q^- < 0} + \underbrace{\int_{\hat{x}}^{\infty} \lambda_2\Big[x, T\Big\{x, s_Q^*, Q(s_Q^*)\Big\}\Big]T_2\Big\{x, s_Q^*, Q(s_Q^*)\Big\}\mathrm{d}x}_{Q^+ > 0}. \tag{A.17}$$

$$K'(s_Q^*) = \underbrace{\int_0^{\hat{x}} x\lambda_2\Big[x, T\Big\{x, s_Q^*, Q(s_Q^*)\Big\}\Big]T_2\Big\{x, s_Q^*, Q(s_Q^*)\Big\}\mathrm{d}x}_{K^- < 0} + \underbrace{\int_{\hat{x}}^{\infty} x\lambda_2\Big[x, T\Big\{x, s_Q^*, Q(s_Q^*)\Big\}\Big]T_2\Big\{x, s_Q^*, Q(s_Q^*)\Big\}\mathrm{d}x}_{K^+ > 0}. \tag{A.18}$$

If there are trips being made at s_Q^* with more than one length, some must have lengths other than $\hat{x}$. If all were smaller (larger) than $\hat{x}$, then $Q'(s_Q^*)$ would be negative (positive). But the first-order condition on a maximum requires $Q'(s_Q^*) = 0$. So there must be trips made both shorter and larger than $\hat{x}$: i.e., $Q^- < 0$ and $Q^+ > 0$. It follows from $Q^- < 0$ that $\hat{x}Q^- < K^-$ and from $Q^+ > 0$ that $K^+ > \hat{x}Q^+$. Thus,

$$\hat{x}Q'(s) = \hat{x}Q^- + \hat{x}Q^+ < K^- + K^+ = K'(s). \tag{A.19}$$

Again, since s_Q^* is a maximum, $Q'(s_Q^*) = 0$ and so $K'(s_Q^*) > 0$: i.e., K is rising at $s = s_Q^*$. Since Proposition P6 shows $s_K^* \ge s_Q^*$, it must be that $s_K^* > s_Q^*$. □

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