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Hyperboloidal appro quasinormal modes

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Oscillations of black hole spacetimes exhi the bifurcation sphere and spatial infnity. remain regular when evaluated near the event The hyperboloidal approach provides a natura regions smoothly, resulting in a geometric re oscillations, known as quasinormal modes (QN development of the hyperboloidal approach t fat spacetimes, emphasizing both the physic advancements in the feld. By providing a ge hyperboloidal approach ofers an elegant fra black hole oscillations, with implication simulations, stability analysis, and the in signals.

KEYWORDS

hyperboloidal, frequency domain, quasinormal mode operators

1 Introduction

When a black hole (BH) spacetime is perturbed energy of the perturbation towards the BH hori show oscillations that decay exponentially a quasinormal modes (QNMs) [1–4]. Studying these QNMs is centr spectroscopy [5–7] program aims to measure the oscil GW detections and thereby probe the BH geometr [1, 3, 4, 8]. Te dominant quadrupole QNMs have al gravitational waves [9–11], while the detection of highe debates [12–15].

Mathematically, the QNM problem is ofen form where QNM frequencies appear as the eigenvalue However, in their traditional representation, exponentially near the black hole and at spatia acceptable for small perturbations [16]. Reformulating problem using hyperboloidal surfaces—regular from the black hole event horizon to null infnit

[27, 28]. This geometric framework[1] for black-hole perturbations has found many recent applications, which we review in this paper.

2 The traditional approach to QNMs

The Schwarzschild solution, the simplest black-hole solution to Einstein's equations, is given by

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\omega^2, \quad \text{with } f(r) = 1 - \frac{2M}{r},$$

where $d\omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$ is the metric on the unit sphere, and M is the black-hole's mass. Perturbations satisfy the Regge-Wheeler-Zerilli type wave equation

$$(-\partial_t^2 + \partial_{r_*}^2 - V(r))u(t,r_*) = 0, \qquad (1)$$

where $r_* \in (-\infty, \infty)$ is the tortoise coordinate and $V(r)$ is the effective potential. V vanishes at the black-hole horizon and falls off as $1/r^2$ toward spatial infinity.

Solutions to Equation (1) evolve through a transient phase, followed by a ringdown characterized by exponentially damped vibrations (QNMs) [34], and eventually a polynomial, non-oscillatory decay known as the tail [35, 36].

To analyze the QNM phase, one typically considers time-harmonic solutions

$$u(t,r_*) = e^{-i\omega t}R(r_*), \qquad (2)$$

that reduce the wave equation to a Helmholtz equation,

$$\left(\frac{d^2}{dr_*^2} + \omega^2 - V(r)\right)R(r_*) = 0. \qquad (3)$$

Sommerfeld recognized in 1912 that the Helmholtz equation, in stark contrast to the elliptic case, does not admit unique solutions even when we require that the solution vanishes at infinity [37, 38]. To ensure uniqueness, an outgoing radiation condition must be imposed. In the BH context, a Sommerfeld condition applies also near the BH. We therefore impose

$$\lim_{r_* \to \pm\infty}\left(\frac{d}{dr_*} \mp i\omega\right)R(r_*) = 0 \quad \Leftrightarrow \quad R(r_*) \sim e^{\pm i\omega r_*} \text{ as } r_* \to \pm\infty. \qquad (4)$$

It turns out, however, that the boundary conditions (Equation 4) are not sufficient [23, 39]. A more precise notion of purely outgoing solution is needed to uniquely define the QNMs [40]. The formal definition of QNMs followed a different route than the intuitive notion of QNMs as the eigenvalues of a given differential operator.

The time-harmonic Ansatz (Equation 2) is closely related to a Fourier transformation, provides a general formalism ... in the form of initial data caus

QNM applications, ... the Laplace transformation [23, 39, 41, 42] leads to an inhomog spatial differential equation, with a initial data. One must then ensure that $u(t,r)$ remains bounded as $t \to \infty$ in the Laplace form solution $u(t,r)$ results from the convolution o with the source term ... black-hole (BH) inform

The inverse Laplace transformation r a frequency ω with $\Im(\omega) > 0$, where there is only one p of homogeneous solutions $R_\pm(r;\omega)$ to construct the cor function: they must satisfy the boundar (Equation 4) at both ends. Once the Green's function complex ω half-plane, it can be analytically exte region $\Im(\omega) < 0$. The QNMs are then uniquely def Green's functions, or equivalently roots of the Wronskian.

The Laplace approach uniquely defin functions, bypassing the notions of Such definition via this Green's func the Lax-Phillips [44, 45] approach. However, this def allows QNM functions to blow up asymp while black-hole stability demands decay over time, the associated time- singular in the asymptotic regions.

The resolution lies in the global s QNM behavior at asymptotic boundari properties of the Schwarzschild (Equation 1). In Schwarzschild coordinates, $r_* \to \pm\infty$, these limits correspond t i^0 and the bifurcation sphere B. These points connect to past null infinity and the black-hole ... B, and the blow-up of QNM eigenfunctions is accumulation of infinitely many time s

When QNMs are represented on regul slices, they do not exhibit [27, 28, 31], as unbo we discuss in the next section.

3 The hyperboloidal appr QNMs

The singularity of Schwarzschild ti sphere is well-known today, but unde took over four decades [46–49]. Given this singula surprising that QNMs blow up near the up near spatial infinity. Thus, switchi bifurcation sphere does not resolve

Part of the historical confusion a infinite Schwarzschild time for radi same statement is true concerning sp Schwarzschild time for outgoing rad Because this is "reasonable" from a ph widely accepted that QNMs have a sing asymptotic regions.

The first suggestion that outgoing the frequency domain toward null inf and Schutz in a 1975 paper on the stab [50]. Friedman and Schutz recognize the pr

[1] Note that we focus here on the geometric developments around the hyperboloidal framework in asymptotically flat spacetimes. The analytic aspects of QNM regularity beyond the ... of standard time slices were clarified [29, 32, 33], and ... discussed in Section 3.3.

slices where outgoing modes behave asymptotically like [illegible] implying that solutions [illegible] ∞. To make the representation [illegible] hypersurfaces. In a footnote, [illegible] is regular also "if one uses a sp[illegible] asymptotically null."

Schmidt picked up this idea in [illegible] Research Foundation on re[illegible] [21] that QNMs on hyperboloids "are re[illegible] and eigenfunctions." However, [illegible] beyond the 1975 paper.

To understand why it took almost [illegible] essay [26] to the construction of a [illegible] to describe QNMs in asymptoti[illegible][1] [27, 28, 33], we provide a short histor[illegible] coordinates.

3.1 A brief history of hyperboloidal coordinates

The central role of spacetime hyp[illegible] recognized already by [51] [illegible] from 1923 or [52] Dirac's point-form [illegible] from 1949 are hyperboloidal [53] [illegible] 1970s, studies were performed for [54–56] [illegible] and quantum fields [57–60] [illegible] However, these early studies use a time-dependent formulation [illegible] null infinity.

The first hyperboloidal coordinat[illegible] implicit in Penrose's work on [illegible] spacetimes via conformal [61, 62] [illegible] can obtain a hyperboloidal surface [illegible] the Penrose diagram simply by looking at the level sets of Penrose time [63]. In the context of numerical relativity, it was recognized that hyperboloidal time functions that asymptotically approach the retarded time should be beneficial for the computation of gravitational [64, 65] [illegible] hyperboloidal coordinates in black-hole spacetimes were constructed in the context of the analysis of constant mean [66] [illegible] but largely ignored paper by Gowdy in 1981 includes many key elements of the hyperboloidal approach used today in black-hole perturbation [67] [illegible] the height function approach to preserve time-translation symmetry, compactification, fixing null infinity (scri-fixing), hyperboloidal solutions to the wave equation, and the structure of time-harmonic solutions relevant for the frequency domain. These ideas were not picked up by the community at the time.

The first systematic study of the hyperboloidal initial value problem for Einstein equations was initiated by Friedrich in 1983 [68]. Friedrich devised a reformulation of the Einstein equations with respect to a conformally rescaled metric that is regular across null infinity. The conformal field equations are well-suited for the analysis of the asymptotic behavior of Einstein's equations and have led to seminal results such as the nonlinear, semi-global stability of de Sitter-type and Minkowski-type [69, 70] [illegible] developments around conformal field equations [illegible] them numerically [71, 72] are reviewed in [illegible]

[illegible] Gowdy's paper, [illegible] hyperboloidal compactification of Mi[illegible] [illegible] [73] [illegible] hyperboloidal coordinates and numer[illegible] [illegible] 1993 [illegible] essay [illegible] [74, 75] [illegible] [76, 77] [illegible] Gravity [illegible] today [illegible] [illegible] 2008 [illegible] [84] [illegible] height function technique that pres[illegible] [illegible] with an exp[illegible] [illegible] infinity [illegible] [illegible] spacetimes the following [illegible] [illegible] for s[illegible] problems [85–87].

The translation of the hyperboloida[illegible] domain was pr[illegible] [27], where it was demons[illegible] hyperboloidal time functions regula[illegible] in the asymptotic domains. [illegible] [31] [illegible] AdS spacetimes in which spatial slic[illegible] [illegible] [29, 30]. [illegible] analysis of [illegible] black-hole spacetimes using the [illegible] [illegible] [28] [illegible] the basic ideas [illegible] approach [illegible].

3.2 A geometric framework

[illegible] [illegible] a suitable spatial comp[illegible] [illegible] [24] [illegible] [88–94] [illegible]

$$\tau = t + h(r), \tag{5}$$

The time transformation implies a[illegible] in frequency domain [27] [illegible] the time-[illegible] ansatz Equation 2 with respect to the n[illegible] coordinates Equation 5

$$u(t,r) = e^{-i\omega t} R(r) = e^{-i\omega \tau} e^{i\omega h} R(r) = e^{-i\omega \tau} \bar{R}(r).$$

The rescaled radial [illegible] $\bar{R}(r) = e^{i\omega h} R(r)$ is regular bo[illegible] near the event horizon and toward nu[illegible] an explicit example, consider the he[illegible] minimal gauge [95–100]

$$h_{MG}(r) = -r + 2M \log\left(\frac{r}{2M} - 1\right) - 4M \log\left(\frac{r}{2M}\right) = -r - 4M \log\left(\frac{r}{2M}\right)$$

The minimal gauge height function [illegible] asymptotic behavior

$$h_{MG} \to -r \text{ for } r \to \infty, \quad h_{MG} \to +r \text{ for } r \to 2M.$$

The height function [illegible] regularizes the [illegible] asymptotic domains. The [illegible] is directly related to the regulari[illegible]

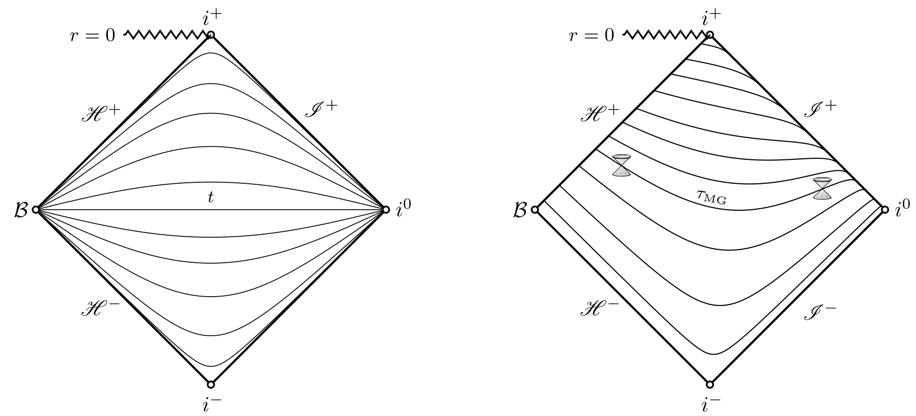

FIGURE 1
Penrose diagrams of the exterior domain in Schwarzschild spacetime contrasts the leve[illegible] hyperboloidal minimal gauge (right panel). Schwarzschild time s[illegible] Minimal gauge slice[illegible] provide a smooth foliation of $\mathscr{H}^+$ and future null infinity, $\mathscr{I}^+$[illegible]

asymptotic boundaries near the horizon and null infinity (see Figure 1). The minimal gauge is unique and its simplicity [illegible] in different setups [31, 102]. [illegible] the minimal gauge was implicitly used [illegible] on QNMs in BH spacetimes [103, 104]. Related hyperboloidal regularization procedures have been suggested by various authors without an explicit [illegible] background of the construction [105–111].

In [27], it was shown that the time transformation with a suitable rescaling to a regular [illegible] QNMs. The rescaling takes into account the asymptotic behavior of the QNM eigenfunctions toward the BH and toward infinity. The resulting equations have short-range potentials suitable for compactification of the exterior black region from the radial coordinate $r\in[r_h,\infty)$ — or equivalently $(-\infty,+\infty)$ — into a compact domain $\sigma\in[\sigma_h,\sigma_{\mathscr{I}^+}]$. This rescaling is related to the conformal compactification of black-hole spacetimes.

The external boundary conditions (Equation 4) are automatically satisfied in terms of a radially compact hyperboloidal coordinate (τ,σ) [27, 100], when the underlying function $\bar{R}(\sigma)$ is regular at the black-hole horizon (σ_h) and future null infinity $(\sigma_{\mathscr{I}^+})$. Thus, we no longer need to impose boundary conditions to the wave equation by hand. The boundary condition is replaced by a regularity condition on $\bar{R}(\sigma)$ in the entire domain $[\sigma_h,\sigma_{\mathscr{I}^+}]$. In practical terms, one derives the regularity condition at the boundary directly from the differential equation. When formulating the problem in (τ,σ), the resulting differential equation [illegible] equivalent to Helmholtz Equation 3 [illegible] a generic form [28, 99, 100]:

$$\left(\alpha_2(\sigma)\frac{d^2}{d\sigma^2}+\alpha_1(\sigma)\frac{d}{d\sigma}+\alpha_0(\sigma)\right)\bar{R}(\sigma)=0, \qquad (6)$$

with coefficients α_2, α_1 and α_0 depending on the [illegible] for the hyperboloidal height function [illegible] of the above equation is that it is [illegible] equation, i.e., its $\alpha_2(\sigma)\propto(\sigma-\sigma_{\mathscr{I}^+})(\sigma-\sigma_h)$ [illegible] and therefore $\alpha_2(\sigma_h)=\alpha_2(\sigma_{\mathscr{I}^+})=0$. Hence, at the boundaries [illegible]

Equation 6 provides [illegible] directly [illegible] with the r[illegible] its [illegible] positivity [illegible] serving as boun[illegible] singular [illegible] solution. From the spacetime [illegible] $\alpha_2(\sigma_h)=\alpha_2(\sigma_{\mathscr{I}^+})=0$ [illegible] at the boundaries, the light cones p[illegible] domain [illegible] modes [illegible] [27, 100]. The finite behaviour of $\bar{R}(\sigma)$ [illegible] function [illegible] important [illegible] combined [illegible] hyperboloi[illegible] below [illegible] features [illegible] eigenfunctions [illegible] numer[illegible] problems relevant to black-hole phy[illegible] astronomy.

3.3 From geometry to analys[illegible]

The hyperboloidal framework regul[illegible] Helmholtz Equation 3. In fact, any bounde[illegible] satisfying the singular Equation 6 automatically f[illegible] fulfils the Sommerfeld condition (Equation 4) [illegible] think that bounded solutions exist [illegible] However, we saw that the conditions in Equation 4 [illegible] underlying solution [illegible] are necessary, but not sufficient to [illegible] uniquely. In the traditional formul[illegible] grow asymptotically. The complex plan[illegible] might contain regions [illegible] Equation 4 [illegible] differential equation [illegible] but contaminated by unwanted solut[illegible] boundaries. Removing the asymptotic [illegible] directly into these unphysical solu[illegible] half-plane $\mathrm{Im}(\omega)<0$ [28, 31].

The left panel in Figure 2 shows two solutions to Equation 6 [illegible] which are bounded in the entire exte[illegible] $\sigma=0$ [illegible] $\sigma=1$ [illegible] horizon [illegible] solution [illegible] $\omega_{\rm QNM}$ [illegible] These solutio[illegible]

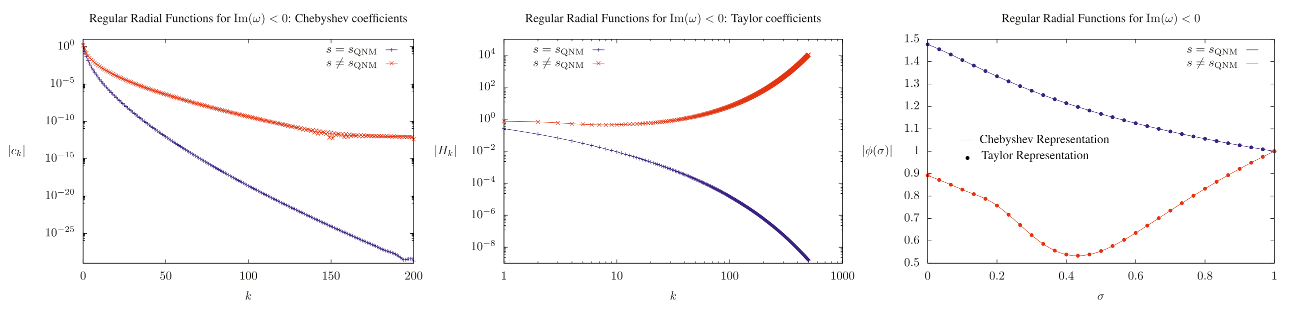

FIGURE 2
Solutions to hyperboloidal radial equation (left panel) (ω). Bounded solutions exist in ... whether QNM or not. The continuous lines were obtained with a spectral method code based on a Chebyshe... result from a Taylor representation of the solution as power series around the horizon ... the solutions with a higher degree of regularity, heuristically verified via the faster decay of $|c_k|$ (middle panel) or the asymptotic decay of $|H_k|$ of the Taylor coefficients

are obtained with two different numerical approaches. The solid line results from solving the ODE with a Chebyshev collocation point spectral method [112], where the dotted ... using Leaver's Taylor expansion ... domain hyperboloidal formulation [28, 39] ... [113]. Both strategies yield the same results. At first glance, there is nothing special in the behavior of the solutions that allows us to distinguish a QNM from a non-QNM eigenfunction. Indeed, it is even possible to specify a hyperboloidal initial data such that the corresponding time evolution has a ... damped oscillation [28].

What distinguishes a QNM from a non-QNM ... their regularity class. By studying the convergence rate of their discrete numerical representation, ... belong to different regularity classes. Figure 2 ... shows the Chebyshev coefficients from the ... point spectral method. These coefficients decay exponentially for C^{Ω} analytic functions and algebraically for C^k functions. We observe an intermediary decay ... class of these functions ... C^k and C^{Ω}. The Chebyshev coefficients for QNM eigenfunctions decay faster than for non-QNM eigenfunctions, indicating QNM eigenfunctions belong to a better regularity class. A similar conclusion arises from the Taylor expansion coefficients (right panel ...). $|H_k|$ decays asymptotically. For non-QNM functions $|H_k|$ grows asymptotically. Even though the series does not converge absolutely, it converges conditionally due to oscillations in (H_k) ... of Figure 2). These conclusions are formalized by interpreting QNMs as a formal eigenvalue problem of the generator of time translations for a null foliation, acting on an appropriate Hilbert space [28, 38, 114–116], where QNM eigenfunctions belong to the Gevrey 2 regularity class.

4 Applications

The QNM problem plays a fundamental role in gravitational wave astronomy. The ... faces three main challenges: (i) ...

... frequencies ... approached by the GW de... line ... dynamics ... which corresponds to ... hyperbo... tools to tack... challenges.

4.1 QNM excitation factors

Even though ... (i) mainly co... for the excitation ... QNMs depend on the particular initial ... the dynamics. This perturbation ... power-law tail decay. Determining ... always been challenging due to the ... modes at the horizon and spatial ... common approach to avoid the infinities ... sphere when calculating integrals ... $R(-\infty, \infty)$ to deform the integration ... plane [119–121].

The hyperboloidal formalism offers ... to determine such excitation factor... behavior of the QNM eigenfunctions ... of Leaver's continued fraction ... defined on hyperboloidal hypersurf... development of the Leaver method ... excitation factors for ... hyperboloidal [28, 39]. While Leaver's meth... a Taylor expansion around the hori... hyperboloidal functions, the strat... solve a linear partial differential ... excitation amplitude as an unknown ... [122], or alternatively via the so-ca... [123] ...

The hyperboloidal formalism is also ... BH superradiant ... spectral ... [124–126] ... stability of the QNMs

4.2 Quadratic QNMs

Since GR is a nonlinear theory, BH spectroscopy must also account for nonlinear perturbations [21, 121–136]. The quadratic coupling of first-order solutions dictates the dynamics at second order in perturbation theory [137–139]. When formulated in the standard approach, the blow-up of QNM eigenfunctions at the bifurcation sphere and spatial infinity imposes severe challenges to second-order studies, both at theoretical and numerical levels. The hyperboloidal framework for black-hole perturbation theory beyond linear order becomes indispensable [23, 130, 140], for time-domain as well as for studies in the frequency domain [13, 141].

4.3 QNM instability and the pseudospectrum

Apart from his groundbreaking work on QNMs [34], Vishveshwara also highlighted that the QNM spectra is very sensitive to small modifications in the black-hole potential [142, 143]. At the same time, the QNM spectra destabilisation was also observed by Nollert [144] and Nollert and Price [145]. The phenomenon's impact in the BH spectroscopy programme has been largely overlooked over the past decades. Only recently has the challenge (iii) gained greater attention [11, 12, 146–155].

Small modifications in oscillatory frequencies for wave equations result in minor spectral responses only if the wave operators are self-adjoint. However, the flow of GWs into the BH and out into the wave zone places BH perturbation theory within the framework of non-self-adjoint operators. The successful application of non-self-adjoint operator theory to gravitational systems was only made possible by the hyperboloidal approach to black-hole perturbations [11, 12] (see also Ref. [156] for an alternative approach akin to “complex scaling”). In this approach, one can use the mathematical formalism of pseudospectra [157] to study QNM spectral instability [11, 12] and perform a non-modal analysis [151] that a traditional mode analysis might overlook. Since the breakthrough offered by the hyperboloidal framework, the analysis of QNM pseudospectra has been performed in several different contexts, from astrophysically relevant scenarios to applications in the gauge-gravity duality [158–167].

5 Discussion

The hyperboloidal approach to QNMs offers a geometric regularization of black-hole perturbations. By connecting the regular oscillations near BHs with those observed far away, this method bypasses the problematic divergences inherent in the traditional approach at the bifurcation sphere and spatial infinity.

With hindsight, the hyperboloidal approach relies on a simple coordinate transformation that resolves the asymptotic singularity of the standard time [84]. It is astonishing that it took decades for relativists to adopt regular coordinates to describe black-hole perturbations. We suspect that part of the confusion arose from the asymptotic behavior of the QNMs. It is not widely appreciated that the standard …

is singular at infinity with respect t… scale wave phenomena demand coordi… … characterizes that hyp… … gravitational wave propagation problems. … in the hyperboloid… … breakthroughs in the stu… … of the … perturb… … QNM pseudospectrum 4… has demonstrated that the hyperbolo… to non-relativistic … [16, 18] … broadening … applicability.

From a numerical perspective, find… among the many ways to construct hyp… particularly for high-precision and … Exploring gauge condit… algorithms to leverage advanced co… be essential for practical applica… perturbations and including the nu… Einstein equations along hyperboloi… [169, 170].

Much of the current work has focuse… vacuum spacetimes. The formalism for… theory is fully developed for spheri… the same concepts are also valid … The fort… hyperboloidal approach is versatile… settings, including those with dife… non-vacuum spacetimes. Developing th… for applying this framework to a broad…

Author contributions

RP: Writing – original draft, Writin… Writing – original draft, Writing – rev…

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Conflict of interest

The author… research … absence of any commercial or financia… construed as a potential conflict of i…

Publisher's note

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