

A TYPE D ASYMMETRIC SIMPLE EXCLUSION PROCESS GENERATED BY AN EXPLICIT CENTRAL ELEMENT OF $\mathcal{U}_q(\mathfrak{so}_{10})$

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ABSTRACT. The Type D asymmetric simple exclusion process (Type D ASEP) with parameters (q, n, δ) is a two-species interacting particle system exhibiting a drift, where two particles may occupy the same site only if they belong to different species. In previous research by Kuan, Landry, Lin, Park, and Zhou, the Type D ASEP was generated using the quantum Hamiltonian corresponding to central elements from the quantum groups $\mathcal{U}_q(\mathfrak{so}_6)$ and $\mathcal{U}_q(\mathfrak{so}_8)$. We extend this construction to the case of $\mathcal{U}_q(\mathfrak{so}_{10})$. Additionally, we generalize a previously known duality function from Blyschak, Burke, Kuan, Li, Ustilovsky, and Zhou for the Type D ASEP for all n .

1. INTRODUCTION

Originally introduced by Spitzer [17], the asymmetric simple exclusion process (ASEP) has become “the default stochastic model for transport phenomena” [20], having been applied to traffic flow problems [15], growth models [13], vortex models [11], and other biological applications [18, 19]. The *Type D asymmetric simple exclusion process* (*Type D ASEP*) introduced in [12] is a generalization of the ASEP. The state space of the Type D ASEP consists of two species of particles interacting on a one-dimensional lattice, where two particles may occupy the same site as long as they are of different species. The Type D ASEP has three parameters (q, n, δ) , where q is the asymmetry parameter, the positive integer n gives the speed of the drift, and δ specifies the interaction between the two particle species. The generator matrix is given in Section 2.2.

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The procedure connecting quantum groups to a particle system and duality is described in [4]; see also [5, 9, 16]. Previous research [1, 12] uses this procedure to construct Type D ASEP with parameters $n = 3, 4$ from the representation theory of the Type D quantum groups $\mathcal{U}_q(\mathfrak{so}_6)$ and $\mathcal{U}_q(\mathfrak{so}_8)$, respectively. The authors of [12] conjecture that this process can be generalized to $\mathcal{U}_q(\mathfrak{so}_{2n})$.

This paper verifies this claim in the case of $\mathcal{U}_q(\mathfrak{so}_{10})$. The first step toward this is to compute a central element of this quantum group using methods from [12]. Then, a generator of the Type D ASEP with parameter $n = 5$ is constructed from this central element.

In addition, this paper expands on previous results concerning Markov self-duality of the Type D ASEP. In [12], the authors produce a triangular duality function. This paper instead focuses on orthogonal polynomial duality; see [2, 3, 6, 7, 8]. In particular, [1] presents a duality function for Type D ASEP with parameters $n = 3, 4$ and $\delta = 0$. This paper generalizes this result for all n .

The paper is outlined as follows: Section 2 describes the relevant background and notation used. Section 3 states the results: a central element of $\mathcal{U}_q(\mathfrak{so}_{10})$, the construction of the Type D ASEP generator for $n = 5$, and a self-duality function of the Type D ASEP. Section 4 outlines the proofs of these results.

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2. BACKGROUND

2.1. Algebraic Definitions.

Definition 1. The *special orthogonal Lie algebra* $\mathfrak{so}_{2n}$ is defined as

$$\mathfrak{so}_{2n} = \left\{ \begin{bmatrix} A & B \\ C & -A^T \end{bmatrix} \mid A, B, C \in M_{n \times n}(\mathbb{C}), B = -B^T, C = -C^T \right\}.$$

Let $E_{i,j}$ denote the matrix with 1 in entry (i, j) and 0 elsewhere. Then define matrices $E_i, F_i, H_i, 1 \leq i \leq n$ as follows:

$$\begin{aligned} E_n &= E_{n-1, 2n} - E_{n, 2n-1} \\ F_n &= E_{2n-1, n} - E_{2n, n-1} \end{aligned}$$

$$H_n = E_{n-1,n-1} + E_{n,n} - E_{2n-1,2n-1} - E_{2n,2n}$$

and for all $1 \leq i \leq n-1$,

$$E_i = E_{i,i+1} - E_{n+i+1,n+i}$$

$$F_i = E_{i+1,i} - E_{n+i,n+i+1}$$

$$H_i = E_{i,i} - E_{i+1,i+1} - E_{n+i,n+i} + E_{n+i+1,n+i+1}.$$

These matrices generate the fundamental representation of $\mathfrak{so}_{2n}$.

This Lie algebra has rank n and Dynkin diagram D_n , with Cartan matrix (a_{ij}) given by

$$a_{ij} = \begin{cases} 2 & i = j \\ -1 & \{i, j\} = \{n-2, n\} \text{ or } \{k, k+1\}, 1 \leq k \leq n-2 \\ 0 & \text{otherwise} \end{cases}$$

Finally, let L_i denote the linear operator taking a matrix to its i -th diagonal entry. Then the positive roots of $\mathfrak{so}_{2n}$ are $\{L_i + L_j\}_{i < j} \cup \{L_i - L_j\}_{i < j}$. The simple roots are $\alpha_i = L_i - L_{i+1}$ for $1 \leq i \leq n-1$ and $\alpha_n = L_{n-1} + L_n$. In the fundamental representation of $\mathfrak{so}_{2n}$, elements of the Lie algebra act as the underlying matrix on $\mathbb{C}^{2n}$. The fundamental weights are then $\pm L_i$, $1 \leq i \leq n$.

An ordering for the weights

$$L_1 > L_2 > \dots > L_n = -L_n > -L_{n-1} > \dots > -L_1$$

is given in [12].

Define vectors $v_1, \dots, v_{2n}$ with v_i having a 1 in the i -th coordinate if $1 \leq i \leq n$, a 1 in the $(3n+1-i)$ -th coordinate if $n+1 \leq i \leq 2n$, and zeroes elsewhere. Observe that $v_1, \dots, v_{2n}$ are in the weight spaces of $L_1, \dots, L_n, -L_n, \dots, -L_1$, respectively.

Definition 2. The *quantum group* $\mathcal{U}_q(\mathfrak{so}_{2n})$ is the algebra generated by

$$\{E_i, F_i, q^{H_i} : 1 \leq i \leq n\}$$

with q -deformed relations

$$[E_i, F_i] = \frac{q^{H_i} - q^{-H_i}}{q - q^{-1}}, \quad q^{H_i} E_j = q^{(\alpha_i, \alpha_j)} E_j q^{H_i}, \quad q^{H_i} F_j = q^{-(\alpha_i, \alpha_j)} F_j q^{H_i},$$

along with the Serre relation for every (i, j) with $a_{ij} = -1$ in the Cartan matrix

$$E_i^2 E_j + E_j E_i^2 = (1+q) E_i E_j E_i, \quad F_i^2 F_j + F_j F_i^2 = (1+q) F_i F_j F_i.$$

Here, (α_i, α_j) is the usual dot product. All other pairs of elements commute. Lastly, the coproducts of the generators are

$$\Delta(E_i) = E_i \otimes 1 + q^{H_i} \otimes E_i, \quad \Delta(F_i) = 1 \otimes F_i + F_i \otimes q^{-H_i}, \quad \Delta(q^{H_i}) = q^{H_i} \otimes q^{H_i}.$$

For convenience, in the remainder of the paper, let K_i denote q^{H_i} .

A general definition for the quantum group of a Lie algebra $\mathfrak{g}$ is given in [10], and in Chapter 6 the author introduces a bilinear pairing that will be used in the later construction. Recall that the Borel subalgebras $\mathfrak{b}^\pm$ are the Lie subalgebras generated by $\{E_i, H_i\}$ and $\{F_i, H_i\}$, respectively. Similarly, let $\mathcal{U}_q(\mathfrak{b}^\pm)$ denote the corresponding subalgebras of the quantum group generated by the Borel subalgebras (replacing H_i with q^{H_i}). There is a bilinear pairing $\langle \cdot, \cdot \rangle: \mathcal{U}_q(\mathfrak{b}^-) \times \mathcal{U}_q(\mathfrak{b}^+) \rightarrow \mathbb{Q}(q)$ such that for any linear combinations α, β of the positive simple roots we have

$$\langle q^{H_\alpha}, q^{H_\beta} \rangle = q^{-(\alpha \cdot \beta)} \text{ and } \langle F_i, E_j \rangle = \frac{-\delta_{ij}}{(q - \frac{1}{q})},$$

and all other pairings between generators are zero. The pairing can be computed for products via

$$\langle y, xx' \rangle = \langle \Delta(y), x' \otimes x \rangle, \quad \langle yy', x \rangle = \langle y \otimes y', \Delta(x) \rangle,$$

where $\Delta(xy) = \Delta(x)\Delta(y)$, and $\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle$ is defined to be $\langle x_1, y_1 \rangle \langle x_2, y_2 \rangle$.

2.2. Probabilistic Definitions. Recall that the Type D ASEP consists of two species of particles interacting on a one-dimensional lattice, where at most two particles can occupy a single site, and if so must be of different species. The Type D ASEP has three parameters (q, n, δ) , where $q \neq 1$ is a positive real number, and $n \geq 2$ and $0 \leq \delta \leq n - 2$ are integers. Intuitively, q is the asymmetry parameter, n affects the speed of the drift, and δ affects the interaction between the two species. If the lattice has L sites, then there are 4^L states. A more complete definition of the Type D ASEP is given in [12].¹

First, consider the two-site model. The 16 states are (a, b) , $0 \leq a, b \leq 3$, where each coordinate corresponds to a site, 0 denotes an empty site, 1 denotes a species 1 particle, 2 denotes a species 2 particle, and 3 denotes both a species 1 and species 2 particle. The generator matrix is given by the direct sum of a 4×4

¹Dr. Kuan and Zhengye, two authors of [1, 12], acknowledged a typo in those papers where $\mathfrak{so}_{2n}$ was associated with the parameters $(q, n - 1, \delta)$ rather than (q, n, δ) .

block

$$\mathcal{L}_1 = \begin{bmatrix} * & \frac{(1+q^{-2\delta})(q^{2n-2}-q^{2n-4}+\frac{2}{q^2})}{q^{2\delta}+1} & \frac{(-q^{1-n}+q^{n-1})^2}{q^2} & \frac{q^{2\delta}(1+q^{-2\delta})(q^{2n-2}-q^{2n-4}+\frac{2}{q^2})}{q^{2\delta}+1} \\ q^{-2\delta}(q^{2n}-q^{2n-2}+2) & * & q^{-2n}-q^{2-2n}+2 & (-q^{1-n}+q^{n-1})^2 \\ q^2(-q^{1-n}+q^{n-1})^2 & 2q^2+q^{2-2n}-q^{4-2n} & * & q^{2\delta}(2q^2+q^{2-2n}-q^{4-2n}) \\ q^{2n}-q^{2n-2}+2 & (-q^{1-n}+q^{n-1})^2 & q^{2\delta}(q^{-2n}-q^{2-2n}+2) & * \end{bmatrix}$$

corresponding to the communicating class $\{(3,0), (2,1), (0,3), (1,2)\}$, four 2×2 blocks

$$\mathcal{L}_2 = \begin{bmatrix} * & \frac{q^{1-2n}+q^{2n-1}}{q} \\ q(q^{1-2n}+q^{2n-1}) & * \end{bmatrix}$$

corresponding to the communicating classes

$$\{(1,0), (0,1)\}, \{(2,0), (0,2)\}, \{(3,1), (1,3)\}, \{(3,2), (2,3)\},$$

and four 1×1 blocks with entry 0 corresponding to the communicating classes $\{(0,0)\}, \{(1,1)\}, \{(2,2)\}, \{(3,3)\}$. Here, the diagonal entries are chosen so that the rows sum to 0. To summarize, the generator matrix is

$$(1) \quad \mathcal{L} = \mathcal{L}_1 \oplus \bigoplus_{i=1}^4 \mathcal{L}_2 \oplus \bigoplus_{i=1}^4 [0],$$

with respect to the ordered basis

$$\{(3,0), (2,1), (0,3), (1,2), (1,0), (0,1), (2,0), (0,2), \\ (3,1), (1,3), (3,2), (2,3), (0,0), (1,1), (2,2), (3,3)\}.$$

If there are L sites, then the generator matrix is given by

$$\mathcal{L}^{1,2} + \mathcal{L}^{2,3} + \dots + \mathcal{L}^{L-1,L},$$

where $\mathcal{L}^{x,x+1}$ denotes the matrix acting on lattice sites x and $x+1$.

We now introduce several notations and definitions that will appear in Theorem 3.3. If $x \in \{1, \dots, L\}$ is a lattice site, $i \in 1, 2$, and η is a state, then let η_i^x denote the number of particles of class i at site x in state η (so η_i^x is 0 or 1). Define height functions

$$N_{x-1}^-(\eta_i) = \sum_{i=1}^{x-1} \eta_i^x, \quad N_{x+1}^+(\eta_i) = \sum_{i=x+1}^L \eta_i^x$$

that count the number of particles to the left and right of a particular site.

Define the q -Pochhammer symbol for $a \in \mathbb{R}, k \in \mathbb{N}$ as

$$(a; q)_k := \prod_{i=0}^{k-1} (1 - aq^i),$$

define the q -hypergeometric function ${}_2\varphi_1$ as

$${}_2\varphi_1 \left(\begin{matrix} a, b \\ c \end{matrix}; q, z \right) := \sum_{k=0}^{\infty} \frac{(a; q)_k (b; q)_k}{(c; q)_k} \frac{z^k}{(q; q)_k},$$

and define the q -Krawtchouk polynomials as

$$K_n(q^{-x}; p, c; q) = {}_2\varphi_1 \left(\begin{matrix} q^{-x}, q^{-n} \\ q^{-c} \end{matrix}; q, pq^{n+1} \right).$$

3. RESULTS

First, for ease of reading set some notational shortcuts following the convention in [12]. Set $r = q - \frac{1}{q}$, and let $E_{x_1 x_2 \dots x_k} := E_{x_1} E_{x_2} \dots E_{x_k}$, and similarly for F and K . For example, $r^2 F_{23} E_{23} = (q - \frac{1}{q})^2 F_2 F_3 E_2 E_3$.

Theorem 3.1. *The following element is in the center of $\mathcal{U}_q(\mathfrak{so}_{10})$:*

$$\begin{aligned} & q^8 K_{11223345} + q^6 K_{223345} + q^4 K_{3345} + q^2 K_{45} + K_4 K_5^{-1} + K_4^{-1} K_5 + \frac{1}{q^2} K_{45}^{-1} + \frac{1}{q^4} K_{3345}^{-1} \\ & + \frac{1}{q^6} K_{223345}^{-1} + \frac{1}{q^8} K_{11223345}^{-1} + \frac{r^2}{q} F_4 K_5^{-1} E_4 + \frac{r^2}{q^3} (qF_{34} - F_{43}) K_{35}^{-1} (qE_{43} - E_{34}) \\ & + \frac{r^2}{q^3} F_3 K_{345}^{-1} E_3 + \frac{r^2}{q^7} F_1 K_{1223345}^{-1} E_1 \\ & + \frac{r^2}{q^5} (q^2 F_{234} - qF_{243} - qF_{324} + F_{432}) K_{235}^{-1} (q^2 E_{432} - qE_{342} - qE_{423} + E_{234}) \\ & + \frac{r^2}{q^7} (\boxed{A_1}) K_{1235}^{-1} (\boxed{A_2}) \\ & + \frac{r^2}{q^7} (q^2 F_{123} - qF_{132} - qF_{213} + F_{321}) K_{12345}^{-1} (q^2 E_{321} - qE_{231} - qE_{312} + E_{123}) \\ & + \frac{r^2}{q^5} F_2 K_{23345}^{-1} E_2 + \frac{r^2}{q^7} (qF_{12} - F_{21}) K_{123345}^{-1} (qE_{21} - E_{12}) \\ & - qr^2 (\boxed{A_3}) K_{1234} (\boxed{A_4}) + \frac{r^2}{q^5} (qF_{23} - F_{32}) K_{2345}^{-1} (qE_{32} - E_{23}) \\ & - qr^2 (q^2 F_{532} - qF_{352} - qF_{523} + F_{235}) K_{234} (q^2 E_{235} - qE_{253} - qE_{325} + E_{532}) \\ & - qr^2 (qF_{53} - F_{35}) K_{34} (qE_{35} - E_{53}) - qr^2 F_5 K_4 E_5 - r^4 F_{54} E_{54} \\ & - \frac{r^2}{q} (\boxed{A_5}) K_{123} (\boxed{A_6}) - \frac{r^2}{q} (\boxed{A_7}) K_{23} (\boxed{A_8}) \\ & - \frac{r^2}{q} (q^2 F_{453} - qF_{435} - qF_{534} + F_{345}) K_3 (q^2 E_{354} - qE_{435} - qE_{534} + E_{543}) \\ & - \frac{r^2}{q} F_5 K_4^{-1} E_5 \end{aligned}$$

$$\begin{aligned}
& -\frac{r^4}{q^2}(-qF_{3435} - qF_{5343} + (q^2 + 1)F_{3543})(-qE_{3435} - qE_{5343} + (q^2 + 1)E_{3543}) \\
& -\frac{r^2}{q^3}(\boxed{A_9})K_2(\boxed{A_{10}}) \\
& -\frac{r^2}{q^3}(q^2F_{354} - qF_{435} - qF_{534} + F_{543})K_3^{-1}(q^2E_{453} - qE_{435} - qE_{534} + E_{345}) \\
& -\frac{r^2}{q^3}(qF_{35} - F_{53})K_{34}^{-1}(qE_{53} - E_{35}) - \frac{r^2}{q^3}(\boxed{A_{11}})K_{12}(\boxed{A_{12}}) - \frac{r^4}{q^4}(\boxed{A_{13}})(\boxed{A_{14}}) \\
& -\frac{r^2}{q^5}(\boxed{A_{15}})K_2^{-1}(\boxed{A_{16}}) - \frac{r^2}{q^5}(\boxed{A_{17}})K_{23}^{-1}(\boxed{A_{18}}) \\
& -\frac{r^2}{q^5}(q^2F_{235} - qF_{253} - qF_{325} + F_{532})K_{234}^{-1}(q^2E_{532} - qE_{352} - qE_{523} + E_{235}) \\
& -\frac{r^2}{q^5}(\boxed{A_{19}})K_1(\boxed{A_{20}}) \\
& -\frac{r^4}{q^6}(\boxed{A_{21}})(\boxed{A_{22}}) - \frac{r^2}{q^7}(\boxed{A_{23}})K_{123}^{-1}(\boxed{A_{24}}) - \frac{r^2}{q^7}(\boxed{A_{25}})K_{1234}^{-1}(\boxed{A_{26}}) \\
& -\frac{r^2}{q^7}(\boxed{A_{27}})K_{12}^{-1}(\boxed{A_{28}}) - \frac{r^2}{q^7}(\boxed{A_{29}})K_1^{-1}(\boxed{A_{30}})q^7r^2F_1K_{1223345}E_1 \\
& + q^5r^2(qF_{21} - F_{12})K_{123345}(qE_{12} - E_{21}) + q^5r^2F_2K_{23345}E_2 \\
& + q^3r^2(q^2F_{321} - qF_{231} - qF_{312} + F_{123})K_{12345}(q^2E_{123} - qE_{132} - qE_{213} + E_{321}) \\
& + q^3r^2(qF_{32} - F_{23})K_{2345}(qE_{23} - E_{32}) + q^3r^2F_3K_{345}E_3 + qr^2(\boxed{A_{31}})K_{1235}(\boxed{A_{32}}) \\
& + qr^2(q^2F_{432} - qF_{342} - qF_{423} + F_{234})K_{235}(q^2E_{234} - qE_{243} - qE_{324} + E_{432}) \\
& + qr^2(qF_{43} - F_{34})K_{35}(qE_{34} - E_{43}) + qr^2F_4K_5E_4,
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= q^3F_{1234} - q^2F_{1243} - q^2F_{1324} - q^2F_{2134} + qF_{1432} + qF_{2143} + qF_{3214} - F_{4321} \\
A_2 &= q^3E_{4321} - q^2E_{3421} - q^2E_{4231} - q^2E_{4312} + qE_{2341} + qE_{3412} + qE_{4123} - E_{1234} \\
A_3 &= q^3F_{5321} - q^2F_{3521} - q^2F_{5231} - q^2F_{5312} + qF_{2351} + qF_{3512} + qF_{5123} - F_{1235} \\
A_4 &= q^3E_{1235} - q^2E_{1253} - q^2E_{1325} - q^2E_{2135} + qE_{1532} + qE_{2153} + qE_{3215} - E_{5321} \\
A_5 &= q^4F_{45321} - q^3F_{43521} - q^3F_{45231} - q^3F_{45312} - q^3F_{53421} + q^2F_{34521} + q^2F_{42351} \\
&+ q^2F_{43512} + q^2F_{45123} + q^2F_{52341} + q^2F_{53412} - qF_{23451} - qF_{34512} - qF_{41235} \\
&- qF_{51234} + F_{12345} \\
A_6 &= q^4E_{12354} - q^3E_{12435} - q^3E_{12534} - q^3E_{13254} - q^3E_{21354} + q^2E_{12543} + q^2E_{14325} \\
&+ q^2E_{15324} + q^2E_{21435} + q^2E_{21534} + q^2E_{32154} - qE_{15432} - qE_{21543} - qE_{43215} \\
&- qE_{53214} + E_{54321} \\
A_7 &= q^3F_{4532} - q^2F_{4352} - q^2F_{4523} - q^2F_{5342} + qF_{3452} + qF_{4235} + qF_{5234} - F_{2345} \\
A_8 &= q^3E_{2354} - q^2E_{2435} - q^2E_{2534} - q^2E_{3254} + qE_{2543} + qE_{4325} + qE_{5324} - E_{5432} \\
A_9 &= q^4F_{34532} - (q^3 - q)F_{35342} - (q^3 - q)F_{43532} + q^2F_{34235} + q^2F_{35234} + q^2F_{43523} \\
&+ q^2F_{53423} - qF_{32345} - qF_{43235} - qF_{45323} - qF_{53234} + F_{23453} + (-q^3 - q)F_{34523}
\end{aligned}$$

$$\begin{aligned}
A_{10} &= q^4 E_{23543} - q^3 E_{23435} - q^3 E_{25343} + q^2 E_{32435} + q^2 E_{32534} + q^2 E_{43253} + q^2 E_{53243} \\
&\quad - q E_{34325} - q E_{53432} + E_{35432} + (-q^3 - q) E_{32543} \\
A_{11} &= q^5 F_{354321} - q^4 F_{354312} - (q^4 - q^2) F_{343521} - (q^4 - q^2) F_{534321} + q^3 F_{342351} \\
&\quad + q^3 F_{352341} + q^3 F_{435231} + q^3 F_{534231} - q^2 F_{323541} - q^2 F_{341235} - q^2 F_{351234} \\
&\quad - q^2 F_{432351} - q^2 F_{435123} - q^2 F_{532341} - q^2 F_{534123} - q^2 F_{543231} + q F_{235431} + q F_{312354} \\
&\quad + q F_{431235} + q F_{531234} + q F_{543123} + (q^3 - q) F_{343512} + (q^3 - q) F_{534312} \\
&\quad + (q^3 + q) F_{354123} + (-q^4 - q^2) F_{354231} - F_{123543} \\
A_{12} &= q^5 E_{123543} - q^4 E_{123435} - q^4 E_{125343} - q^4 E_{213543} + q^3 E_{132435} + q^3 E_{132534} \\
&\quad + q^3 E_{143253} + q^3 E_{153243} + q^3 E_{213435} + q^3 E_{215343} - q^2 E_{134325} - q^2 E_{153432} \\
&\quad - q^2 E_{321435} - q^2 E_{321534} - q^2 E_{432153} - q^2 E_{532143} + q E_{135432} + q E_{343215} \\
&\quad + q E_{534321} + (q^3 + q) E_{321543} + (-q^4 - q^2) E_{132543} - E_{354321} \\
A_{13} &= q^2 F_{235234} + q^2 F_{243523} - q^2 F_{223543} + q^2 F_{324352} + q^2 F_{532432} + (-q^3 - q) F_{234352} \\
&\quad + (-q^3 - q) F_{253432} + (-q^3 - q) F_{325432} + (q^4 + q^2 + 1) F_{235432} \\
A_{14} &= q^2 E_{235234} + q^2 E_{243523} - q^2 E_{223543} + q^2 E_{324352} + q^2 E_{532432} + (-q^3 - q) E_{234352} \\
&\quad + (-q^3 - q) E_{253432} + (-q^3 - q) E_{325432} + (q^4 + q^2 + 1) E_{235432} \\
A_{15} &= q^4 F_{23543} - q^3 F_{23435} - q^3 F_{25343} + q^2 F_{32435} + q^2 F_{32534} + q^2 F_{43253} + q^2 F_{53243} \\
&\quad - q F_{34325} - q F_{53432} + F_{35432} + (-q^3 - q) F_{32543} \\
A_{16} &= q^4 E_{34532} - (q^3 - q) E_{35342} - (q^3 - q) E_{43532} + q^2 E_{34235} + q^2 E_{35234} + q^2 E_{43523} \\
&\quad + q^2 E_{53423} - q E_{32345} - q E_{43235} - q E_{45323} - q E_{53234} + E_{23453} + (-q^3 - q) E_{34523} \\
A_{17} &= q^3 F_{2354} - q^2 F_{2435} - q^2 F_{2534} - q^2 F_{3254} + q F_{2543} + q F_{4325} + q F_{5324} - F_{5432} \\
A_{18} &= q^3 E_{4532} - q^2 E_{4352} - q^2 E_{4523} - q^2 E_{5342} + q E_{3452} + q E_{4235} + q E_{5234} - E_{2345} \\
A_{19} &= q^6 F_{2354321} + q^4 F_{2343512} + q^4 F_{2534312} + (q^4 - q^2) F_{2352341} + (q^4 - q^2) F_{2435231} \\
&\quad - (q^4 - q^2) F_{2235431} + (q^4 - q^2) F_{3243521} + (q^4 - q^2) F_{5324321} - q^3 F_{2341235} \\
&\quad - q^3 F_{2351234} - q^3 F_{2435123} - q^3 F_{2534123} - q^3 F_{3243512} - q^3 F_{3253412} - q^3 F_{4325312} \\
&\quad - q^3 F_{5324312} + q^2 F_{2312354} + q^2 F_{2431235} + q^2 F_{2531234} + q^2 F_{2543123} + q^2 F_{3241235} \\
&\quad + q^2 F_{3251234} + q^2 F_{3432512} + q^2 F_{4325123} + q^2 F_{5324123} + q^2 F_{5343212} - q F_{2123543} \\
&\quad - q F_{3212354} - q F_{3543212} - q F_{4321235} - q F_{5321234} - q F_{5432123} + (-q^3 - q) F_{3254123} \\
&\quad + (q^4 + q^2) F_{2354123} + (q^4 + q^2) F_{3254312} + (-q^5 - q) F_{2354312} + (-q^5 + q) F_{2343521} \\
&\quad + (-q^5 + q) F_{2534321} + (-q^5 + q) F_{3254321} + F_{1235432} \\
A_{20} &= q^6 E_{1235432} - q^5 E_{1234352} - q^5 E_{1253432} - q^5 E_{1325432} + q^4 E_{1235234} + q^4 E_{1243523}
\end{aligned}$$

$$\begin{aligned}
& + q^4 E_{1324352} + q^4 E_{1532432} - (q^4 - q^2) E_{1223543} - q^3 E_{1235423} - q^3 E_{2134235} \\
& - q^3 E_{2135234} - q^3 E_{2143523} - q^3 E_{2153423} - q^3 E_{2354213} - q^3 E_{3214352} \\
& - q^3 E_{3215342} - q^3 E_{4321532} - q^3 E_{5321432} + q^2 E_{2352134} + q^2 E_{2435213} + q^2 E_{3243521} \\
& + q^2 E_{5324321} - q E_{2343521} - q E_{2534321} - q E_{3254321} + (q^3 - q) E_{2123543} \\
& + (q^4 + q^2) E_{2134352} + (q^4 + q^2) E_{2135423} + (q^4 + q^2) E_{2153432} + (q^4 + q^2) E_{3215432} \\
& + (-q^5 - q) E_{2135432} + E_{2354321} \\
A_{21} = & -q^3 F_{12341235} + q^3 F_{12354312} - q^3 F_{12534123} + q^3 F_{13212354} - q^3 F_{13253412} \\
& + q^3 F_{13543212} + q^3 F_{14321235} - q^3 F_{14325312} + q^3 F_{15321234} + q^3 F_{15432123} \\
& + q^3 F_{21235431} - q^3 F_{21352341} - q^3 F_{21435231} - q^3 F_{32143521} \\
& - q^3 F_{53214321} - (q^4 + q^2) F_{12235431} - (q^4 + q^2) F_{11235432} + (q^4 + q^2) F_{12352341} \\
& + (q^4 + q^2) F_{12354123} + (q^4 + q^2) F_{12435231} + (q^4 + q^2) F_{13243521} + (q^4 + q^2) F_{15324321} \\
& + (q^4 + q^2) F_{21343521} + (q^4 + q^2) F_{21534321} + (q^4 + q^2) F_{32154321} \\
& - q(q^2 + 1)^2 F_{12343521} - q(q^2 + 1)^2 F_{12534321} - q(q^2 + 1)^2 F_{13254321} \\
& - q(q^4 + q^2 + 1) F_{21354321} + (q^6 + q^4 + q^2 + 1) F_{12354321} \\
A_{22} = & -q^3 E_{12341235} + q^3 E_{12354312} - q^3 E_{12534123} + q^3 E_{13212354} - q^3 E_{13253412} \\
& + q^3 E_{13543212} + q^3 E_{14321235} - q^3 E_{14325312} + q^3 E_{15321234} + q^3 E_{15432123} \\
& + q^3 E_{21235431} - q^3 E_{21352341} - q^3 E_{21435231} - q^3 E_{32143521} - q^3 E_{53214321} \\
& - (q^4 + q^2) E_{12235431} - (q^4 + q^2) E_{11235432} + (q^4 + q^2) E_{12352341} + (q^4 + q^2) E_{12354123} \\
& + (q^4 + q^2) E_{12435231} + (q^4 + q^2) E_{13243521} + (q^4 + q^2) E_{15324321} + (q^4 + q^2) E_{21343521} \\
& + (q^4 + q^2) E_{21534321} + (q^4 + q^2) E_{32154321} - q(q^2 + 1)^2 E_{12343521} \\
& - q(q^2 + 1)^2 E_{12534321} - q(q^2 + 1)^2 E_{13254321} - q(q^4 + q^2 + 1) E_{21354321} \\
& + (q^6 + q^4 + q^2 + 1) E_{12354321} \\
A_{23} = & q^4 F_{12354} - q^3 F_{12435} - q^3 F_{12534} - q^3 F_{13254} - q^3 F_{21354} + q^2 F_{12543} + q^2 F_{14325} \\
& + q^2 F_{15324} + q^2 F_{21435} + q^2 F_{21534} + q^2 F_{32154} - q F_{15432} - q F_{21543} - q F_{43215} \\
& - q F_{53214} + F_{54321} \\
A_{24} = & q^4 E_{45321} - q^3 E_{43521} - q^3 E_{45231} - q^3 E_{45312} - q^3 E_{53421} + q^2 E_{34521} + q^2 E_{42351} \\
& + q^2 E_{43512} + q^2 E_{45123} + q^2 E_{52341} + q^2 E_{53412} - q E_{23451} - q E_{34512} - q E_{41235} \\
& - q E_{51234} + E_{12345} \\
A_{25} = & q^3 F_{1235} - q^2 F_{1253} - q^2 F_{1325} - q^2 F_{2135} + q F_{1532} + q F_{2153} + q F_{3215} - F_{5321} \\
A_{26} = & q^3 E_{5321} - q^2 E_{3521} - q^2 E_{5231} - q^2 E_{5312} + q E_{2351} + q E_{3512} + q E_{5123} - E_{1235}
\end{aligned}$$

$$\begin{aligned}
A_{27} &= q^5 F_{123543} - q^4 F_{123435} - q^4 F_{125343} - q^4 F_{213543} + q^3 F_{132435} + q^3 F_{132534} \\
&\quad + q^3 F_{143253} + q^3 F_{153243} + q^3 F_{213435} + q^3 F_{215343} - q^2 F_{134325} - q^2 F_{153432} \\
&\quad - q^2 F_{321435} - q^2 F_{321534} - q^2 F_{432153} - q^2 F_{532143} + q F_{135432} + q F_{343215} \\
&\quad + q F_{534321} + (q^3 + q) F_{321543} + (-q^4 - q^2) F_{132543} - F_{354321} \\
A_{28} &= q^5 E_{354321} - q^4 E_{354312} - (q^4 - q^2) E_{343521} - (q^4 - q^2) E_{534321} + q^3 E_{342351} \\
&\quad + q^3 E_{352341} + q^3 E_{435231} + q^3 E_{534231} - q^2 E_{323541} - q^2 E_{341235} - q^2 E_{351234} \\
&\quad - q^2 E_{432351} - q^2 E_{435123} - q^2 E_{532341} - q^2 E_{534123} - q^2 E_{543231} + q E_{235431} \\
&\quad + q E_{312354} + q E_{431235} + q E_{531234} + q E_{543123} + (q^3 - q) E_{343512} + (q^3 - q) E_{534312} \\
&\quad + (q^3 + q) E_{354123} + (-q^4 - q^2) E_{354231} - E_{123543} \\
A_{29} &= q^6 F_{1235432} - q^5 F_{1234352} - q^5 F_{1253432} - q^5 F_{1325432} + q^4 F_{1235234} + q^4 F_{1243523} \\
&\quad + q^4 F_{1324352} + q^4 F_{1532432} - (q^4 - q^2) F_{1223543} - q^3 F_{1235423} - q^3 F_{2134235} \\
&\quad - q^3 F_{2135234} - q^3 F_{2143523} - q^3 F_{2153423} - q^3 F_{2354213} - q^3 F_{3214352} - q^3 F_{3215342} \\
&\quad - q^3 F_{4321532} - q^3 F_{5321432} + q^2 F_{2352134} + q^2 F_{2435213} + q^2 F_{3243521} + q^2 F_{5324321} \\
&\quad - q F_{2343521} - q F_{2534321} - q F_{3254321} + (q^3 - q) F_{2123543} + (q^4 + q^2) F_{2134352} \\
&\quad + (q^4 + q^2) F_{2135423} + (q^4 + q^2) F_{2153432} + (q^4 + q^2) F_{3215432} + (-q^5 - q) F_{2135432} \\
&\quad + F_{2354321} \\
A_{30} &= q^6 E_{2354321} + q^4 E_{2343512} + q^4 E_{2534312} + (q^4 - q^2) E_{2352341} + (q^4 - q^2) E_{2435231} \\
&\quad - (q^4 - q^2) E_{2235431} + (q^4 - q^2) E_{3243521} + (q^4 - q^2) E_{5324321} - q^3 E_{2341235} \\
&\quad - q^3 E_{2351234} - q^3 E_{2435123} - q^3 E_{2534123} - q^3 E_{3243512} - q^3 E_{3253412} - q^3 E_{4325312} \\
&\quad - q^3 E_{5324312} + q^2 E_{2312354} + q^2 E_{2431235} + q^2 E_{2531234} + q^2 E_{2543123} + q^2 E_{3241235} \\
&\quad + q^2 E_{3251234} + q^2 E_{3432512} + q^2 E_{4325123} + q^2 E_{5324123} + q^2 E_{5343212} - q E_{2123543} \\
&\quad - q E_{3212354} - q E_{3543212} - q E_{4321235} - q E_{5321234} - q E_{5432123} + (-q^3 - q) E_{3254123} \\
&\quad + (q^4 + q^2) E_{2354123} + (q^4 + q^2) E_{3254312} + (-q^5 - q) E_{2354312} + (-q^5 + q) E_{2343521} \\
&\quad + (-q^5 + q) E_{2534321} + (-q^5 + q) E_{3254321} + E_{1235432} \\
A_{31} &= q^3 F_{4321} - q^2 F_{3421} - q^2 F_{4231} - q^2 F_{4312} + q F_{2341} + q F_{3412} + q F_{4123} - F_{1234} \\
A_{32} &= q^3 E_{1234} - q^2 E_{1243} - q^2 E_{1324} - q^2 E_{2134} + q E_{1432} + q E_{2143} + q E_{3214} - E_{4321}.
\end{aligned}$$

This element acts as $q^{10} + q^6 + q^4 + q^2 + 2 + \frac{1}{q^2} + \frac{1}{q^4} + \frac{1}{q^6} + \frac{1}{q^{10}}$ times the identity in the fundamental representation of $\mathcal{U}_q(\mathfrak{so}_{10})$.

Using this central element, the method in [4] can be applied to obtain a Markov process. The following theorem states that the resulting process is indeed the Type D ASEP with $n = 5$.

Theorem 3.2. *Let C denote the central element of $\mathcal{U}_q(\mathfrak{so}_{10})$ from Theorem 3.1. Let H denote the action of $\Delta(C)$ on $\mathbb{C}^{10} \otimes \mathbb{C}^{10}$, so that H is a 100×100 matrix. Define the quantum Hamiltonian $\hat{H} = H - \Lambda \cdot \text{Id}$, where*

$$\Lambda = q^{12} + q^6 + q^4 + q^2 + 2 + \frac{1}{q^2} + \frac{1}{q^4} + \frac{1}{q^6} + \frac{1}{q^{12}}.$$

There exist linearly independent eigenvectors g_0, g_1, g_2, g_3 of $\hat{H}$ with eigenvalue 0 such that if G_δ is the diagonal matrix with entries given by g_δ , then removing certain states from $G_\delta^{-1} \hat{H} G_\delta$ results in the generator of the two-site Type D ASEP with parameters $(q, 5, \delta)$, for $\delta = 0, 1, 2, 3$.

In addition, we prove the following generalization of Theorem 3.1 from [1]:

Theorem 3.3. *The Type D ASEP with $\delta = 0$ is self-dual with respect to the self-duality function*

$$D_{\alpha_1, \alpha_2}^L(\eta, \xi) = D_{\alpha_1}^L(\eta_1, \xi_1) \cdot D_{\alpha_2}^L(\eta_2, \xi_2)$$

where

$$D_{\alpha_i}^L(\xi_i, \eta_i) = \prod_{x=1}^L K_{\eta_i^x} \left(q^{-2\xi_i^x}, p_i^x(\xi_i, \eta_i), 1, q^2 \right)$$

and

$$p_i^x(\xi_i, \eta_i) = \alpha_i^{-1} q^{-2(N_{x-1}^-(\xi_i) - N_{x+1}^+(\eta_i)) + 2x - 2}.$$

Note that here, α_1 and α_2 are not roots of $\mathfrak{so}_{10}$; they are parameters that depend on the reversible measures explained in [1].

4. PROOFS

The proofs of the first two theorems were assisted by a computer. The code used can be found at <https://github.com/e-rohr/Type-D-ASEP>

4.1. Proof of Theorem 3.1. We find the central element using [14, Lemma 3.1], as was done in [12] for $\mathcal{U}_q(\mathfrak{so}_6)$ and $\mathcal{U}_q(\mathfrak{so}_8)$. The lemma is restated here for convenience:

Lemma 4.1. *For each weight μ of the fundamental representation of a Lie algebra $\mathfrak{g}$, let v_μ be a vector in the weight space. Suppose q is not a root of unity, and 2μ is always in the root lattice of $\mathfrak{g}$. Let $e_{\mu\lambda}$ and $f_{\lambda\mu}$ be products of E_i 's and F_i 's in $\mathcal{U}_q(\mathfrak{g})$, respectively, such that $e_{\mu\lambda}$ sends v_λ to v_μ and $f_{\lambda\mu}$ sends v_μ to v_λ . If $e_{\mu\lambda}^*$*

and $f_{\lambda\mu}^*$ are the corresponding dual elements under $\langle \cdot, \cdot \rangle$, and ρ is half the sum of the positive roots of $\mathfrak{g}$, then

$$(2) \quad \sum_{\mu \geq \lambda} q^{(\mu-\lambda, \mu)} q^{-(2\rho, \mu)} e_{\mu\lambda}^* q^{H_{-\mu-\lambda}} f_{\lambda\mu}^*$$

is a central element of $\mathcal{U}_q(\mathfrak{g})$.

Several terms in (2) are straightforward to compute; the computations are explained in [12, Section 2.1.1]. In particular:

$$q^{(\mu-\lambda, \mu)} = \begin{cases} q^2 & \lambda = -\mu \\ 1 & \lambda = \mu \\ q & \text{otherwise,} \end{cases} \quad q^{(-2\rho, \mu)} = \begin{cases} q^{2i-2n} & \mu = L_i \\ q^{2n-2i} & \mu = -L_i, \end{cases}$$

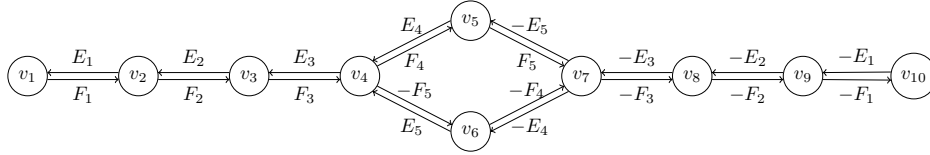
$$H_{-2L_i} = H_{n-1} - H_n - 2 \sum_{j=i}^{n-1} H_j,$$

and if $i < j$,

$$H_{L_i-L_j} = \sum_{k=i}^{j-1} H_k.$$

We can add or subtract these last two equations to find any $H_{\pm L_i \pm L_j}$.

Next, we find $e_{\mu\lambda}$ and $f_{\lambda\mu}$. The following diagram shows the actions of E_i and F_i on $\{v_1, \dots, v_{2n}\}$ for $n = 5$. Recall that the vectors $v_1, \dots, v_{2n}$ belong to the weight spaces of $L_1, \dots, L_n, -L_n, \dots, -L_1$, respectively.



For example, for $\mu = L_3$ and $\lambda = -L_2$, we have $e_{\mu\lambda} = -E_3E_4E_5E_3E_2$ and $f_{\lambda\mu} = F_2F_3F_5F_4F_3$. Note that there is no difference between taking the top and bottom path because E_4, E_5 commute and F_4, F_5 commute.

The procedure for obtaining the dual elements $e_{\mu\lambda}^*$ and $f_{\lambda\mu}^*$ is described in depth in [12] and summarized here. Given $e_{\mu\lambda}$ of the form $E_{x_1} \dots E_{x_k}$, create the set of elements $\{E_{\sigma(x_1)} \dots E_{\sigma(x_k)}\}_\sigma$, where σ ranges through all possible permutations. Some of these elements may be linearly dependent due to the relations in the quantum group. Thus, take a basis $e_1 = e_{\mu\lambda}, e_2, \dots, e_m$ of the span of these

elements, and create a corresponding basis $f_1, \dots, f_m$, where each f_i is the same as e_i but with F_j replacing each E_j . Form a matrix M of q -pairings such that $M_{ij} = \langle e_i, f_j \rangle$. Then $e_{\mu\lambda}^*$ is the dot product of the first row of M^{-1} with the column vector $(f_1, \dots, f_m)$. Obtaining $f_{\lambda\mu}^*$ is similar.

The calculation for $\mathcal{U}_q(\mathfrak{so}_{10})$ is done with the aid of a computer. The same code could be used for calculating a central element of $\mathcal{U}_q(\mathfrak{so}_{2n})$ for larger n . For $\mathcal{U}_q(\mathfrak{so}_6)$ and $\mathcal{U}_q(\mathfrak{so}_8)$, the matrices are small enough to perform computations symbolically, but computing determinants and inverses of large symbolic matrices is intractable. These difficulties are especially prevalent in the computation of the dual elements $e_{\mu\lambda}^*$ and $f_{\lambda\mu}^*$. As such, for $\mathcal{U}_q(\mathfrak{so}_{10})$, the process of obtaining the dual elements is done with a numerical value of $q = 10$ to speed up computation.

4.2. Proof of Theorem 3.2. Under the appropriate basis, the quantum Hamiltonian $\hat{H}$ decomposes as a direct sum of one 10×10 block, forty 2×2 blocks, and ten 1×1 blocks.

The 1×1 blocks all have entry zero, so taking four of these blocks corresponds to the four 1×1 blocks from the generator matrix $\mathcal{L}$ in (1).

The 2×2 blocks are

$$\begin{bmatrix} -q^{10} + 2q^8 - q^6 - \frac{1}{q^8} + \frac{2}{q^{10}} - \frac{1}{q^{12}} & \frac{(q^2-1)^2(q^{18}+1)}{q^{11}} \\ \frac{(q^2-1)^2(q^{18}+1)}{q^{11}} & -q^{12} + 2q^{10} - q^8 - \frac{1}{q^6} + \frac{2}{q^8} - \frac{1}{q^{10}} \end{bmatrix},$$

which have eigenvector $\begin{bmatrix} q \\ 1 \end{bmatrix}$, and conjugating by the corresponding diagonal matrix $\begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix}$ results in one of the 2×2 blocks $\mathcal{L}_2$ in the generator (1) multiplied by a constant factor of r^2 .

Let

$$\begin{aligned} B_1 &= -q^5 + 3q^3 - 3q + \frac{1}{q} - \frac{2}{q^5} + \frac{4}{q^7} - \frac{2}{q^9}, \\ B_2 &= -2q^3 + 4q - \frac{2}{q} + \frac{1}{q^5} - \frac{3}{q^7} + \frac{3}{q^9} - \frac{1}{q^{11}}, \\ B_3 &= q^{10} - 2q^8 + q^6 - 2q^2 + 4 - \frac{2}{q^2} + \frac{1}{q^6} - \frac{2}{q^8} + \frac{1}{q^{10}}. \end{aligned}$$

Then the 10×10 block has form $U^T + D + U$, where

$$U = \begin{bmatrix} 0 & B_1 & qB_1 & q^2B_1 & q^3B_1 & B_3 & q^6B_1 & q^5B_1 & q^4B_1 & q^3B_1 \\ 0 & 0 & q^2B_1 & q^3B_1 & q^4B_1 & B_2 & B_3 & q^6B_1 & q^5B_1 & q^4B_1 \\ 0 & 0 & 0 & q^4B_1 & q^5B_1 & qB_2 & B_2 & B_3 & q^6B_1 & q^5B_1 \\ 0 & 0 & 0 & 0 & q^6B_1 & q^2B_3 & qB_2 & B_2 & B_3 & q^6B_1 \\ 0 & 0 & 0 & 0 & 0 & q^3B_3 & q^2B_2 & qB_2 & B_2 & B_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & q^6B_2 & q^5B_2 & q^4B_2 & q^3B_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^4B_2 & q^3B_2 & q^2B_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^2B_2 & qB_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and D is the diagonal matrix with entries

$$\left\{ \begin{aligned} & -q^{10} + 2q^8 - q^6 - q^4 + 3q^2 - 3 + \frac{1}{q^2} - \frac{2}{q^6} + \frac{3}{q^8} - \frac{1}{q^{12}}, \\ & -q^{10} + 2q^8 - 2q^6 + 3q^4 - 3q^2 + 1 - \frac{2}{q^4} + \frac{4}{q^6} - \frac{3}{q^8} + \frac{2}{q^{10}} - \frac{1}{q^{12}}, \\ & -q^{10} + q^8 + 2q^6 - 3q^4 + q^2 - \frac{2}{q^2} + \frac{4}{q^4} - \frac{2}{q^6} - \frac{1}{q^8} + \frac{2}{q^{10}} - \frac{1}{q^{12}}, \\ & -2q^{10} + 5q^8 - 4q^6 + q^4 - 2 + \frac{4}{q^2} - \frac{2}{q^4} - \frac{1}{q^8} + \frac{2}{q^{10}} - \frac{1}{q^{12}}, \\ & -q^{12} + 2q^{10} - q^8 - 2q^2 + 4 - \frac{2}{q^2} - \frac{1}{q^8} + \frac{2}{q^{10}} - \frac{1}{q^{12}}, \\ & -q^{12} + 3q^8 - 2q^6 + q^2 - 3 + \frac{3}{q^2} - \frac{1}{q^4} - \frac{1}{q^6} + \frac{2}{q^8} - \frac{1}{q^{10}}, \\ & -q^{12} + 2q^{10} - 3q^8 + 4q^6 - 2q^4 + 1 - \frac{3}{q^2} + \frac{3}{q^4} - \frac{2}{q^6} + \frac{2}{q^8} - \frac{1}{q^{10}}, \\ & -q^{12} + 2q^{10} - q^8 - 2q^6 + 4q^4 - 2q^2 + \frac{1}{q^2} - \frac{3}{q^4} + \frac{2}{q^6} + \frac{1}{q^8} - \frac{1}{q^{10}}, \\ & -q^{12} + 2q^{10} - q^8 - 2q^4 + 4q^2 - 2 + \frac{1}{q^4} - \frac{4}{q^6} + \frac{5}{q^8} - \frac{2}{q^{10}}, \\ & -q^{12} + 2q^{10} - q^8 - 2q^2 + 4 - \frac{2}{q^2} - \frac{1}{q^8} + \frac{2}{q^{10}} - \frac{1}{q^{12}} \end{aligned} \right\}.$$

The 10×10 block has rank 6 and has four linearly independent eigenvectors

$$g_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -q^2 \\ q \\ 0 \\ 0 \\ 0 \\ -1 \\ q \end{bmatrix}, \quad g_1 = \begin{bmatrix} 0 \\ 0 \\ -q^2 \\ q \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ q \end{bmatrix}, \quad g_2 = \begin{bmatrix} 0 \\ -q^2 \\ q \\ 0 \\ 0 \\ 0 \\ -1 \\ q \\ 0 \\ 0 \end{bmatrix}, \quad g_3 = \begin{bmatrix} -q^2 \\ q \\ 0 \\ 0 \\ 0 \\ -1 \\ q \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

These are the four choices of ground state vector corresponding to the Type D ASEP with parameters $(q, 5, 0)$, $(q, 5, 1)$, $(q, 5, 2)$, and $(q, 5, 3)$, respectively.

For each $\delta = 0, 1, 2, 3$, set $L_\delta = G_\delta^{-1} \hat{H} G_\delta$. Then,

$$L_0 = \begin{bmatrix} * & B_1 & qB_1 & \infty & \infty & B_3 & q^6 B_1 & q^5 B_1 & \infty & \infty \\ B_1 & * & q^2 B_1 & \infty & \infty & B_2 & B_3 & q^6 B_1 & \infty & \infty \\ qB_1 & q^2 B_1 & * & \infty & \infty & qB_2 & B_2 & B_3 & \infty & \infty \\ 0 & 0 & 0 & * & -q^5 B_1 & 0 & 0 & 0 & \frac{B_3}{q^2} & -q^5 B_1 \\ 0 & 0 & 0 & -q^7 B_1 & * & 0 & 0 & 0 & -\frac{B_2}{q} & B_3 \\ B_3 & B_2 & qB_2 & \infty & \infty & * & q^6 B_2 & q^5 B_2 & \infty & \infty \\ q^6 B_1 & B_3 & B_2 & \infty & \infty & q^6 B_2 & * & q^4 B_2 & \infty & \infty \\ q^5 B_1 & q^6 B_1 & B_3 & \infty & \infty & q^5 B_2 & q^4 B_2 & * & \infty & \infty \\ 0 & 0 & 0 & q^2 B_3 & -qB_2 & 0 & 0 & 0 & * & -qB_2 \\ 0 & 0 & 0 & -q^7 B_1 & B_3 & 0 & 0 & 0 & -\frac{B_2}{q} & * \end{bmatrix},$$

$$L_1 = \begin{bmatrix} * & B_1 & \infty & \infty & q^3 B_1 & B_3 & q^6 B_1 & \infty & \infty & q^3 B_1 \\ B_1 & * & \infty & \infty & q^4 B_1 & B_2 & B_3 & \infty & \infty & q^4 B_1 \\ 0 & 0 & * & -q^3 B_1 & 0 & 0 & 0 & \frac{B_3}{q^2} & -q^5 B_1 & 0 \\ 0 & 0 & -q^5 B_1 & * & 0 & 0 & 0 & -\frac{B_2}{q} & B_3 & 0 \\ q^3 B_1 & q^4 B_1 & \infty & \infty & * & q^3 B_2 & q^2 B_2 & \infty & \infty & B_3 \\ B_3 & B_2 & \infty & \infty & q^3 B_2 & * & q^6 B_2 & \infty & \infty & q^3 B_2 \\ q^6 B_1 & B_3 & \infty & \infty & q^2 B_2 & q^6 B_2 & * & \infty & \infty & q^2 B_2 \\ 0 & 0 & q^2 B_3 & -q B_2 & 0 & 0 & 0 & * & -q^3 B_2 & 0 \\ 0 & 0 & -q^7 B_1 & B_3 & 0 & 0 & 0 & -q B_2 & * & 0 \\ q^3 B_1 & q^4 B_1 & \infty & \infty & B_3 & q^3 B_2 & q^2 B_2 & \infty & \infty & * \end{bmatrix},$$

$$L_2 = \begin{bmatrix} * & \infty & \infty & q^2 B_1 & q^3 B_1 & B_3 & \infty & \infty & q^4 B_1 & q^3 B_1 \\ 0 & * & -q B_1 & 0 & 0 & 0 & \frac{B_3}{q^2} & -q^5 B_1 & 0 & 0 \\ 0 & -q^3 B_1 & * & 0 & 0 & 0 & -\frac{B_2}{q} & B_3 & 0 & 0 \\ q^2 B_1 & \infty & \infty & * & q^6 B_1 & q^2 B_2 & \infty & \infty & B_3 & q^6 B_1 \\ q^3 B_1 & \infty & \infty & q^6 B_1 & * & q^3 B_2 & \infty & \infty & B_2 & B_3 \\ B_3 & \infty & \infty & q^2 B_2 & q^3 B_2 & * & \infty & \infty & q^4 B_2 & q^3 B_2 \\ 0 & q^2 B_3 & -q B_2 & 0 & 0 & 0 & * & -q^5 B_2 & 0 & 0 \\ 0 & -q^7 B_1 & B_3 & 0 & 0 & 0 & -q^3 B_2 & * & 0 & 0 \\ q^4 B_1 & \infty & \infty & B_3 & B_2 & q^4 B_2 & \infty & \infty & * & B_2 \\ q^3 B_1 & \infty & \infty & q^6 B_1 & B_3 & q^3 B_2 & \infty & \infty & B_2 & * \end{bmatrix},$$

and

$$L_3 = \begin{bmatrix} * & -\frac{B_1}{q} & 0 & 0 & 0 & \frac{B_3}{q^2} & -q^5 B_1 & 0 & 0 & 0 \\ -qB_1 & * & 0 & 0 & 0 & -\frac{B_2}{q} & B_3 & 0 & 0 & 0 \\ \infty & \infty & * & q^4 B_1 & q^5 B_1 & \infty & \infty & B_3 & q^6 B_1 & q^5 B_1 \\ \infty & \infty & q^4 B_1 & * & q^6 B_1 & \infty & \infty & B_2 & B_3 & q^6 B_1 \\ \infty & \infty & q^5 B_1 & q^6 B_1 & * & \infty & \infty & qB_2 & B_2 & B_3 \\ q^2 B_3 & -qB_2 & 0 & 0 & 0 & * & -q^7 B_2 & 0 & 0 & 0 \\ -q^7 B_1 & B_3 & 0 & 0 & 0 & -q^5 B_2 & * & 0 & 0 & 0 \\ \infty & \infty & B_3 & B_2 & qB_2 & \infty & \infty & * & q^2 B_2 & qB_2 \\ \infty & \infty & q^6 B_1 & B_3 & B_2 & \infty & \infty & q^2 B_2 & * & B_2 \\ \infty & \infty & q^5 B_1 & q^6 B_1 & B_3 & \infty & \infty & qB_2 & B_2 & * \end{bmatrix},$$

where the diagonal entries $*$ are just the entries of D . Remove any rows containing ∞ and any columns containing 0 from L_δ , and divide all entries by r^2 to obtain a 4×4 matrix $\tilde{L}_\delta$. These are

$$\tilde{L}_0 = \begin{bmatrix} \frac{-2q^{18}+q^{16}-2q^8-1}{q^{10}} & \frac{q^{10}-q^8+2}{q^2} & \frac{q^{16}-2q^8+1}{q^{10}} & \frac{q^{10}-q^8+2}{q^2} \\ q^{10}-q^8+2 & \frac{-q^{20}-2q^{10}-1}{q^{10}} & \frac{2q^{10}-q^2+1}{q^{10}} & \frac{q^{16}-2q^8+1}{q^8} \\ \frac{q^{16}-2q^8+1}{q^6} & \frac{2q^{10}-q^2+1}{q^8} & \frac{-q^{18}-2q^{10}+q^2-2}{q^8} & \frac{2q^{10}-q^2+1}{q^8} \\ q^{10}-q^8+2 & \frac{q^{16}-2q^8+1}{q^8} & \frac{2q^{10}-q^2+1}{q^{10}} & \frac{-q^{20}-2q^{10}-1}{q^{10}} \end{bmatrix},$$

$$\tilde{L}_1 = \begin{bmatrix} \frac{-q^{18}-q^{16}+q^{14}-2q^6-1}{q^{10}} & \frac{q^{10}-q^8+2}{q^4} & \frac{q^{16}-2q^8+1}{q^{10}} & \frac{q^{10}-q^8+2}{q^2} \\ \frac{q^{10}-q^8+2}{q^2} & \frac{-2q^{18}+q^{16}-2q^8-1}{q^{10}} & \frac{2q^{10}-q^2+1}{q^{10}} & \frac{q^{16}-2q^8+1}{q^8} \\ \frac{q^{16}-2q^8+1}{q^6} & \frac{2q^{10}-q^2+1}{q^8} & \frac{-q^{18}-2q^{12}+q^4-q^2-1}{q^8} & \frac{2q^{10}-q^2+1}{q^6} \\ q^{10}-q^8+2 & \frac{q^{16}-2q^8+1}{q^8} & \frac{2q^{10}-q^2+1}{q^8} & \frac{-q^{18}-2q^{10}+q^2-2}{q^8} \end{bmatrix},$$

$$\tilde{L}_2 = \begin{bmatrix} \frac{-q^{18}-q^{14}+q^{12}-2q^4-1}{q^{10}} & \frac{q^{10}-q^8+2}{q^6} & \frac{q^{16}-2q^8+1}{q^{10}} & \frac{q^{10}-q^8+2}{q^2} \\ \frac{q^{10}-q^8+2}{q^4} & \frac{-q^{18}-q^{16}+q^{14}-2q^6-1}{q^{10}} & \frac{2q^{10}-q^2+1}{q^{10}} & \frac{q^{16}-2q^8+1}{q^8} \\ \frac{q^{16}-2q^8+1}{q^6} & \frac{2q^{10}-q^2+1}{q^8} & \frac{-q^{18}-2q^{14}+q^6-q^4-1}{q^8} & \frac{2q^{10}-q^2+1}{q^4} \\ q^{10}-q^8+2 & \frac{q^{16}-2q^8+1}{q^8} & \frac{2q^{10}-q^2+1}{q^6} & \frac{-q^{18}-2q^{12}+q^4-q^2-1}{q^8} \end{bmatrix},$$

and

$$\tilde{L}_3 = \begin{bmatrix} \frac{-q^{18}-q^{12}+q^{10}-2q^2-1}{q^{10}} & \frac{q^{10}-q^8+2}{q^8} & \frac{q^{16}-2q^8+1}{q^{10}} & \frac{q^{10}-q^8+2}{q^2} \\ \frac{q^{10}-q^8+2}{q^6} & \frac{-q^{18}-q^{14}+q^{12}-2q^4-1}{q^{10}} & \frac{2q^{10}-q^2+1}{q^{10}} & \frac{q^{16}-2q^8+1}{q^8} \\ \frac{q^{16}-2q^8+1}{q^6} & \frac{2q^{10}-q^2+1}{q^8} & \frac{-q^{18}-2q^{16}+q^8-q^6-1}{q^8} & \frac{2q^{10}-q^2+1}{q^2} \\ q^{10}-q^8+2 & \frac{q^{16}-2q^8+1}{q^8} & \frac{2q^{10}-q^2+1}{q^4} & \frac{-q^{18}-2q^{14}+q^6-q^4-1}{q^8} \end{bmatrix}.$$

These matrices are written with respect to the ordered bases $(v_4 \otimes v_9, v_5 \otimes v_{10}, v_9 \otimes v_4, v_{10} \otimes v_5), (v_3 \otimes v_8, v_4 \otimes v_9, v_8 \otimes v_3, v_9 \otimes v_4), (v_2 \otimes v_7, v_3 \otimes v_8, v_2 \otimes v_7, v_8 \otimes v_3), (v_1 \otimes v_6, v_2 \otimes v_7, v_6 \otimes v_1, v_2 \otimes v_7)$, respectively.

Observe that $\tilde{L}_\delta$ is exactly the 4×4 block $\mathcal{L}_1$ in the generator (1) with parameter $(q, 5, \delta)$, as desired.

4.3. Proof of Theorem 3.3. This theorem generalizes Theorem 3.1 from [1] for all n . The inductive step remains the same as from that paper, so it remains to verify the base case of $L = 2$ for general n .

Let $\mathcal{L}$ be the 16×16 generator matrix for the two-site model with parameters $(q, n, 0)$ from (1). Let $\mathcal{D}$ be the 16×16 matrix whose rows and columns are

indexed by the states of Type D ASEP (in the same order as for $\mathcal{L}$) and whose (η, ξ) entry is $D_{\alpha_1, \alpha_2}^L(\eta, \xi)$. Setting $D_i^j = 1 - \frac{q^j}{\alpha_i}$, we have

$$\mathcal{D} = \begin{bmatrix} D_1^2 D_2^2 & D_2^2 & 1 & D_1^2 & D_1^2 & 1 & D_2^2 & 1 & D_1^2 D_2^2 & D_1^2 & D_1^2 D_2^2 & D_2^2 & 1 & D_1^2 & D_2^2 & D_1^2 D_2^2 \\ D_2^2 & D_1^4 D_2^2 & D_1^4 & 1 & 1 & D_1^4 & D_2^2 & 1 & D_1^2 D_2^2 & D_1^2 & D_2^2 & D_1^4 D_2^2 & 1 & D_1^2 & D_2^2 & D_1^2 D_2^2 \\ 1 & D_1^4 & D_1^4 D_2^4 & D_2^4 & 1 & D_1^4 & 1 & D_2^4 & D_1^2 & D_1^2 D_2^4 & D_2^2 & D_1^4 D_2^2 & 1 & D_1^2 & D_2^2 & D_1^2 D_2^2 \\ D_1^2 & 1 & D_2^2 & D_1^2 D_2^2 & D_1^2 & 1 & 1 & D_2^2 & D_1^2 & D_1^2 D_2^2 & D_1^2 D_2^2 & D_2^2 & 1 & D_1^2 & D_2^2 & D_1^2 D_2^2 \\ D_1^2 & 1 & 1 & D_1^2 & D_1^2 & 1 & 1 & 1 & D_1^2 & D_1^2 & D_1^2 & 1 & 1 & D_1^2 & 1 & D_1^2 \\ 1 & D_1^4 & D_1^4 & 1 & 1 & D_1^4 & 1 & 1 & D_1^2 & D_1^2 & 1 & D_1^4 & 1 & D_1^2 & 1 & D_1^2 \\ D_2^2 & D_2^2 & 1 & 1 & 1 & 1 & D_2^2 & 1 & D_2^2 & 1 & D_2^2 & D_2^2 & 1 & 1 & D_2^2 & D_2^2 \\ 1 & 1 & D_2^4 & D_2^4 & 1 & 1 & 1 & D_2^4 & 1 & D_2^4 & D_2^2 & D_2^2 & 1 & 1 & D_2^2 & D_2^2 \\ D_1^4 D_2^2 & D_1^4 D_2^2 & D_1^4 & D_1^4 & D_1^4 & D_1^4 & D_2^2 & 1 & D_1^2 D_1^4 D_2^2 & D_1^2 D_1^4 & D_1^4 D_2^2 & D_1^4 D_2^2 & 1 & D_1^2 D_1^4 & D_2^2 & D_1^2 D_1^4 D_2^2 \\ D_1^4 & D_1^4 & D_1^4 D_2^4 & D_1^4 D_2^4 & D_1^4 & D_1^4 & 1 & D_2^4 & D_1^2 D_1^4 & D_1^2 D_1^4 D_2^4 & D_1^4 D_2^2 & D_1^4 D_2^2 & 1 & D_1^2 D_1^4 & D_2^2 & D_1^2 D_1^4 D_2^2 \\ D_1^2 D_2^4 & D_2^4 & D_2^2 & D_1^2 D_2^4 & D_1^2 & 1 & D_2^4 & D_2^4 & D_1^2 D_2^4 & D_1^2 D_2^4 & D_1^2 D_2^4 D_2^4 & D_2^2 D_2^4 & 1 & D_1^2 & D_2^2 D_2^4 & D_1^2 D_2^4 D_2^4 \\ D_2^4 & D_1^4 D_2^4 & D_1^4 D_2^4 & D_2^4 & 1 & D_1^4 & D_2^4 & D_2^4 & D_1^2 D_2^4 & D_1^2 D_2^4 & D_2^2 D_2^4 & D_1^4 D_2^2 D_2^4 & 1 & D_1^2 & D_2^2 D_2^4 & D_1^2 D_2^2 D_2^4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ D_1^4 & D_1^4 & D_1^4 & D_1^4 & D_1^4 & D_1^4 & 1 & 1 & D_1^2 D_1^4 & D_1^2 D_1^4 & D_1^4 & D_1^4 & 1 & D_1^2 D_1^4 & 1 & D_1^2 D_1^4 \\ D_2^4 & D_2^4 & D_2^4 & D_2^4 & 1 & 1 & D_2^2 & D_2^4 & D_2^4 & D_2^4 & D_2^2 D_2^4 & D_2^2 D_2^4 & 1 & 1 & D_2^2 D_2^4 & D_2^2 D_2^4 \\ D_1^4 D_2^2 & D_1^4 D_2^2 & D_1^4 D_2^2 & D_1^4 D_2^2 & D_1^4 & D_1^4 & D_2^2 & D_2^2 & D_1^2 D_1^4 D_2^2 & D_1^2 D_1^4 D_2^2 & D_1^4 D_2^2 D_2^4 & D_1^4 D_2^2 D_2^4 & 1 & D_1^2 D_1^4 & D_2^2 D_2^4 & D_1^2 D_1^4 D_2^2 D_2^4 \end{bmatrix}.$$

One can verify the duality relation $\mathcal{L}\mathcal{D} = \mathcal{D}\mathcal{L}^T$ for the base case of $L = 2$, completing the proof.

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