Toward Strategies for Characterizing NISQ Device Requirements in Linear Systems Control via Stability and Profitability Analysis

1st Shilpa Narasimhan
Dept. of Chemical Engineering
and Materials Science
Wayne State University
Detroit, MI, USA
shilpa.narasimhan@wayne.edu

1st Keshav Kasturi Rangan
Dept. of Chemical Engineering
and Materials Science
Wayne State University
Detroit, MI, USA
keshav@wayne.edu

2nd Helen Durand
Dept. of Chemical Engineering
and Materials Science
Wayne State University
Detroit, MI, USA
helen.durand@wayne.edu

***Abstract*—Classical computing systems find utility in performing engineering computations in both design and operation of chemical manufacturing processes. During operation, computing systems may be used within process control systems (PCSs) to operate processes by computing control actions. Control engineers may eventually need to determine the utility of other computing devices such as quantum devices (QCs) within PCSs. A QC that is not fault-tolerant may introduce errors in control computations due to the noise (e.g., quantum noise) and result in the process states (e.g., temperature, concentration) deviating from desired operating limits. Deviations of process states may result in one or both of two undesirable consequences that need to be addressed: (1) unsafe situations (e.g., due to loss of containment) and (2) reduction in the profitability of the process (e.g., due to low quality product). Thus, it is currently unclear to what extent errors in noisy intermediate-scale quantum (NISQ) devices would need to be eliminated before they could have potential utility for control action computation. In the current work, we investigate how control theory might aid in guiding answers to this question. We first characterize the stability of linear processes with proportional control implemented on a QC by treating quantum noise as a bounded disturbance exogenous to the process. Then, we analyze the extent to which an optimal control formulation for processes with noise and plant/model mismatch might apply to processes with unexpected control actions from a NISQ device. We demonstrate the results using a single-input/single-output system under a control law implemented using a quantum simulator with a depolarizing error noise model affecting the control action computations.**

***Index Terms*—Control applied on quantum devices, Systems engineering, Stability and profitability of chemical processes**

I. INTRODUCTION

Automation systems play a key role in manufacturing toward keeping processes running efficiently online. They automate operation through monitoring and adjusting the process behavior using online sensors and computing devices that interface with actuators. To enhance efficiency, various computing and networking advances have been incorporated into process control as they have been developed. As examples, control algorithms have been designed that consider the use of wireless networks in data transmission between different elements of the automation system [1], or that consider that the computing unit is capable of carrying out nonlinear optimization algorithms [2]. Thus, just as control engineers have evaluated the utility of other advances in computing, they will need to evaluate whether quantum devices have any control-relevant applications.

Given that classical computers are used to determine control actions today, it is reasonable to ask whether quantum devices could have any utility for that same task. This question is particularly relevant for control laws for which control engineers make approximations today (e.g., using model order reduction of process models [3]) due to a lack of sufficiently fast algorithms for problems such as incorporating large-scale process models within optimization-based control laws. One direction that needs to be undertaken in answering this question is to evaluate existing quantum algorithms and investigate new ones to see whether they could provide benefits for such problems. However, another problem specific to control that would need to be addressed is evaluating how any new algorithms need to be designed to safely interact with the physical systems that would be then impacted by the results of the quantum algorithm, without significantly reducing profits compared to using a classical device.

This second question remains relatively unexplored, and the answer to this question should help define directions for the first. Applications of quantum com-

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puting related to control have included discussions of potential algorithms in quantum computation that might be relevant toward control [4], studies exploring the use of quantum computing in optimal control calculations for quantum systems [5], using a quantum annealer for model predictive control [6], or in reinforcement learning [7]. However, control-theoretic safety/stability notions have not been the focus. Our group has performed several analyses of control algorithms implemented on quantum devices, primarily with simple control laws and quantum algorithms to chart a path toward understanding the interactions between nondeterminism in quantum algorithm outputs (due to either noise or probabilistic algorithm outputs) and the stability of the process on which the control actions are implemented [8], [9], [10]. However, these studies either did not provide control theories for the observations (in the case of noisy quantum devices) or did not consider noise. Also, none of our prior studies examined how the potentially unexpected control actions could impact profitability of operation.

In this work, we present safety and profitability of operation as key metrics in understanding the potential of NISQ devices to be used for computing control actions. We study the potential of control-theoretic approaches for creating safe and optimal control formulations for linear dynamic systems in providing guidance on the degree of error mitigation required by NISQ devices for them to be potentially considerable for control, and we discuss control and dynamic systems techniques as an idea for mitigating the effects of device noise when applying quantum computing in control action selection. In these analyses, we consider two theoretical premises (practical stability under bounded disturbances and optimal operation in the presence of disturbances under a linear quadratic Gaussian (LQG) control law). Through these studies, we identify areas where these theories succeed at providing insights into the potential of using noisy intermediate-scale quantum (NISQ) [11] devices for control law implementation, and where they describe the interaction of the NISQ devices with the control problem inadequately (which guides future directions for providing improved theories).

II. Preliminaries

A. Notation and Definitions

The set of real numbers is represented by $\mathbb{R}$, the set of integers is represented by $\mathbb{Z}$, and the set of complex numbers is represented by $\mathbb{C}$. I is the identity matrix. The eigenvalues of a matrix $A \in \mathbb{R}^{n\times n}$, are those values of λ for which $Av = \lambda v$ ($v \in R^n$ is the corresponding eigenvector). The spectral radius of $A \in \mathbb{R}^{n\times n}$ is defined as $\rho(A) := \max_{p=1}^{n}|\lambda_p|$. The Euclidean norm of $x = [x_1 \ \dots \ x_n]^T \in \mathbb{R}^n$ is $\|x\| = \left(x_1^2 + \dots + x_n^2\right)^{\frac{1}{2}}$. $\mathcal{N}(\mu, \sigma)$ denotes a Gaussian distribution with mean μ and standard deviation σ. Given $A \in \mathbb{C}^{n\times m}$, its Euclidean norm is defined as $\|A\| = \sqrt{\Sigma_{i=1}^{n}\Sigma_{j=1}^{m}\|a_{ij}\|^2}$, where a_{ij} are the elements of A. Given a constant $c \in \mathbb{R}$, $\|cA\| = |c|\|A\|$. Given matrices $A \in \mathbb{R}^{n\times o}$ and $B \in \mathbb{R}^{o\times m}$, the submultiplicative inequality of matrix norms states that $\|AB\| \le \|A\|\|B\|$. Similarly, given matrices $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{n\times m}$, the triangle inequality of norms states that $\|A + B\| \le \|A\| + \|B\|$.

B. Control Theory for Discrete-Time Linear Dynamic Systems

We highlight control-theoretic principles for processes that can be modeled as discrete-time linear dynamic systems as follows:

$$x(t + 1) = Ax(t) \tag{1a}$$

$$y(t) = x(t) \tag{1b}$$

where $x \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^n$ is the state measurement (measured output) from the sensors, $A \in \mathbb{R}^{n\times n}$ is a matrix, and $t \in \mathbb{Z}$ is the time step. A typical goal in control is the following: given any initial state (state at time step $t = 0$), $x(0) \in \mathbb{R}^n$, it is desired to drive the process states to the desired setpoint which is the origin $x = 0$. To characterize when this occurs for the system in Eq. 1, we utilize precise definitions of "stability." The process in Eq. 1 is considered to be stable if the response of the process is bounded, i.e., $\|x(t)\| < M < \infty$ for all $t \in \mathbb{Z}$, where $M \in \mathbb{R}^+$ is a positive constant. The process in Eq. 1 is called asymptotically stable if it is stable and if the states of the process converge to the origin after extended period of operation, i.e., if $\lim_{t\to\infty}\|x(t)\| \to 0$. Because Eq. 1a implies that, given an initial state $x(0) \in \mathbb{R}$, the state at any time step $t > 0$ is $x(t) = A^{t-1}x(0)$, and by writing $x(0)$ as a linear combination of an orthonormal set of eigenvectors for A, the state $x(t)$ can be written as a linear combination of the basis vectors where the coefficients of the linear combination include the eigenvectors to the power $t - 1$. Thus, if the complex magnitudes of the eigenvalues are less than one (i.e., $\rho(A) < 1$), each of the coefficients of the linear combination will decay to zero as $t \to \infty$ so that the process will be asymptotically stable [12].

C. Quantum Computing Notation and Definitions

A quantum state $|\psi\rangle \in \mathbb{C}^n$ may be represented in vector form as: $[a_1 \ a_2 \ a_3 \ \dots \ a_n]^T$, where $a_i \in \mathbb{C}$, for $p = 1, 2, 3, \dots, n$, are coefficients such that $\Sigma_{p=1}^{n}|a_p|^2 = 1$ and the square of the complex magnitude of the p-th coefficient $|a_p|^2$ represents the probability that the system may be found in state i when measured. The computational basis states for qubits are $|0\rangle = [1 \ 0]^T$ and $|1\rangle = [0 \ 1]^T$. $A|x\rangle$ describes the evolution of the quantum state $|x\rangle$ under the operation as prescribed by the gate $A \in \mathbb{C}^{n\times n}$.

III. Toward NISQ Benchmarking Concepts Through Stability and Profitability Analysis of Linear Systems Under Proportional Control On Quantum Devices

If quantum computers are used to compute control actions, then it is important to understand how nondeterminism (e.g., due to noise in today's quantum devices) would impact safety of processes and profitability of operation compared to using a classical device. It would be preferable if full fault-tolerance was not required before we could consider implementing control algorithms on quantum devices; however, it is not currently known how to establish targets for quantum device error mitigation to make them suitable for various control actions. This section provides a first effort toward utilizing control theory to move toward filling this gap.

A. *Stability of Linear Systems Under Proportional Control On Quantum Devices*

In this section, we focus on the stability analysis of a linear system (with no inherent plant/model mismatch) for which the proportional control law is implemented using a quantum device. We demonstrate that practical stability can be achieved, but in practice this result provides useful guarantees only for inherently safe systems or systems where the control actions are restricted to being very small. To develop these results, we first describe the class of process systems considered when a classical computer is used for defining the control actions. We then discuss the change in the closed-loop dynamics when the quantum device with noise is used, followed by a description of the practical stability guarantees that result due to finite quantum register sizes. However, for larger quantum register sizes, this result may fail to bring useful stability results, so we suggest a method for post-processing inputs computed by the quantum device to reduce the size of the bounded set into which the closed-loop state would be driven. We discuss how setting a target size for that set may help with assessing the noise characteristics required by a NISQ device for it to be useful in control.

1) *Class of Systems Without Quantum Noise*

In this section, we describe the structure of the system to which control actions will be applied (which in subsequent sections will be considered to come from a quantum device, but in this section is developed in the absence of perturbations to the control law due to noise). Though mathematical models describing the dynamics of chemical processes are typically derived using a variety of physics-based and/or data-based approaches for specific systems [13], we derive the results of this section without reference to a specific process but instead for any system that can be modeled through the following class of discrete-time linear dynamic systems [14]:

$$x(t+1) = Ax(t) + Bu(t) \tag{2a}$$

$$y(t) = x(t) \tag{2b}$$

where $u(t) \in \mathbb{R}^m$ is the control input communicated by the controller and $B \in \mathbb{R}^{n\times m}$ is a matrix.

We define the following proportional control law which we would like to implement on the process:

$$u(t) = -K'y(t) = -K'x(t) \tag{3}$$

where the factor $K' \in \mathbb{R}^{m\times n}$ is the controller gain and is a controller design parameter. Combining Eq. 2a and Eq. 3, the dynamics of the closed-loop process (when the desired control law is implemented at every sampling time, i.e., a noisy quantum device is not used in the implementation of the control law) may be described as:

$$x(t+1) = (A - BK')x(t) \tag{4}$$

The control objective of the PCS is to cause the origin $x = 0$ to be asymptotically stable. Therefore, as discussed in Section II-B, it is necessary that $\rho(A-BK') < 1$ (i.e., K' is assumed to be picked such that this is true).

2) *Class of Systems With Quantum Noise*

In the prior section, we developed the closed-loop dynamic model in the case that the desired (and stabilizing) control action is implemented on the process. In this section, we update the closed-loop dynamics to include the effects of noise from quantum devices on the process. This requires the development of a strategy for representing the noise. One way to do this would be to represent the result of the quantum computation as the desired result plus an additional bias that could take a range of values. To develop theoretical results which characterize the impacts of noise specifically on the process dynamics (and to avoid obscuring the noise effects via also considering nondeterminism due to quantum algorithms), we consider that the computation of the desired control action can be carried out by a theoretically deterministic algorithm (e.g., QFT-based addition [15]). with this assumption, and denoting u as the result of Eq. 3, we model the effect of the noise on the process through its creation of a modified input u':

$$u'(t) = u(t) + \delta(t) = -K'x(t) + \delta(t) \tag{5}$$

where $\delta(t) \in \mathbb{R}^m$ represents the error introduced because of quantum noise, and $u(t)$ is the correct value of the control input. Combining Eq. 20 and Eq. 5, the dynamics of the closed-loop process with control implemented on the quantum device may be represented as:

$$x(t+1) = (A - BK')x(t) + B\delta(t) \tag{6a}$$

$$y(t) = x(t) \tag{6b}$$

Fig. 1 illustrates the block diagram of a PCS using a quantum device to compute the control input per Eq. 5.

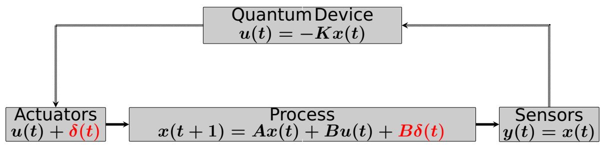


Fig. 1: Block diagram of a PCS implemented using a quantum device.

3) Theoretical Results and Implications

If $\delta(t)$ in Eq. 6a can be arbitrarily large in magnitude, it can cause $x(t+1)$ to move farther from the origin, despite that K' was chosen to cause $x(t+1)$ to move toward $x = 0$ in the case that $\delta \equiv 0$. However, if we assume that δ can take only finite and bounded values (e.g., the qubit register size is fixed such that the range of values which might be output on a noisy device is limited to those which can be represented on that register), then practical stability can be achieved (i.e., the closed-loop state will remain bounded within a compact set whose size depends on the magnitudes of the values of δ within the compact set $\Delta \subset \mathbb{R}^m$ of all values that it can take). The following proposition, a well-known result in control theory (e.g., [12]) for linear systems with bounded disturbances, formalizes this result. We repeat the proof here to tie it directly to quantum computing and to analyze some of the implications for attempting to mitigate the impacts of error in NISQ devices as revealed by analyzing the proof.

Proposition 1: Consider the closed-loop process in Eq. 6 with a noisy quantum device with a finite register size used to compute control inputs per Eq. 5. Let the controller gain K' be selected to render the nominal (noise-free) process asymptotically stable with $\rho(A - BK') < 1$ and let all possible values of δ be bounded within a compact set $\Delta \subset \mathbb{R}^m$. Given $x(0) \in \mathbb{R}^n$, $\|x(t)\| < M$ as $t \to \infty$, where $M \in \mathbb{R}^+$ is a positive constant.

Proof 1: The proof of this theorem follows [12]. Consider the process in Eq. 6 with $\rho(A - BK') < 1$, where $x(0) \in \mathbb{R}^n$. The evolution of the process states is driven by the realization of the quantum noise acting on the process at each time step, so that the norm of the realization of the process state at a time step $t = n$ is:

$$\begin{aligned}\|x(n)\| = \|&(A-BK')^n x(0) + \\ &(A-BK')^{n-1}B\delta(0) + (A-BK')^{n-2}B\delta(1) \\ &+ \ldots + B\delta(n-1)\|\end{aligned} \tag{7}$$

Using the triangle inequality and the submultiplicative properties of norms, Eq. 7 can be simplified to:

$$\begin{aligned}\|x(n)\| \le\ & \|(A-BK')^n\|\|x(0)\| \\ &+ \|(A-BK')^{n-1}\|\|B\|\|\delta(0)\| \\ &+ \|(A-BK')^{n-2}\|\|B\|\|\delta(1)\| + \ldots \\ &+ \|B\|\|\delta(n-1)\|\end{aligned} \tag{8}$$

The norm of the m^{th} term containing δ in Eq. 8, where $1 \le m \le n$, is $\|(A-BK')^{n-m}\|\|\delta(m-1)\|\|B\|$. By writing $(A-BK')^k$ in its Jordan canonical form PJP^{-1} for an invertible P and block diagonal J, and using the submultiplicative property of norms, $\|(A-BK')^{n-m}\| \le \|P\|\|P^{-1}\|\|J^{n-m}\|$. Considering p distinct eigenvalues with m_i repetitions of the i-th eigenvalue, we obtain the following expression for $\|J^n\|$:

$$\|J^n\| = \sqrt{\sum_{i=1}^{p}\sum_{k=1}^{m_i}\sum_{j=0}^{k-1}\left(\frac{n!}{(n-j)!j!}\right)^2 \|\lambda_i\|^{2(n-j)}} \tag{9}$$

Each term within the square root of $\|J^n\|$ is the n-th element of a convergent sequence. The sums of convergent sequences are convergent, meaning that the whole of the term under the square root in Eq. 9 is a convergent sequence, and its square root also converges. Since convergent sequences are bounded, this implies that for every value of n, the right-hand side of Eq. 9 can be bounded by $\bar{M}_n$. Then, $\|(A-BK')^{n-m}\| \le \|P\|\|P^{-1}\|\bar{M}_{n-m} := C_{n-m}$, so that the m^{th} term containing δ in Eq. 8 can be simplified to: $C_{n-m}\|\delta(m-1)\|\|B\|$. Since the quantum noise is bounded within a compact set (Δ) at all times (i.e., $\delta(m-1) \in \Delta$ for all $m > 1$), we have:

$$\|\delta(m-1)\| \le R_\delta := \max_{\delta' \in \Delta}\|\delta'\| \text{ for all } m > 1 \tag{10}$$

Thus, the m^{th} term containing δ of Eq 8 can be simplified to: $C_{n-m}R_\delta\|B\|$. Substituting this in Eq. 8:

$$\begin{aligned}\|x(n)\| \le\ & C_n\|x(0)\| \\ &+ \underbrace{(C_{n-1} + C_{n-2} + \ldots + C_0)}_{C_{\delta,n}} R_\delta\|B\|\end{aligned} \tag{11}$$

Since the sequence of values of C_n converges to zero and $\|x(0)\| > 0$ is finite, the first term in Eq. 11 converges to 0 as $n \to \infty$, and is a finite and decreasing value for all other n. Furthermore, $C_{\delta,n}$ is a convergent sequence. Calling its upper bound $\bar{C}$ indicates that when n is finite, $\|x(n)\|$ is bounded by a finite sequence of bounded terms and thus is also bounded (we will call that upper bound M) in Eq. 11 and as $n \to \infty$, $\|x(n)\|$ converges to $\bar{C}R_\delta\|B\| < M$. This completes the proof.

Though the result above is well-known in control theory, we have presented it to enable discussion of what this proof indicates about error mitigation (in the sense of improving the stability results) for NISQ devices. Thus, we will now make several observations based on different aspects of the closed-loop behavior based on Proposition 1.

Observation 1. Proposition 1 demonstrates that the closed-loop state of Eq. 6 with a quantum device computing control inputs per Eq. 5 will remain bounded, with an upper bound given by Eq. 11 for every n and therefore dependent on R_δ. Specifically, despite that the device noise may prevent the closed-loop state from converging to 0 (the desired setpoint), practical stability is ensured (which means that the process states remain finite). Based on the definition of R_δ in Eq. 10, the magnitude of R_δ depends on the size of the set Δ in which δ lies. Since $u(t)$ must always be within the set of values that are possible for a quantum device to compute, δ is restricted by the quantum register size. However, for larger quantum registers, this value may become very large. Thus, the boundedness of the closed-loop state may not translate to safety (i.e., the bound on the state may be so large that there are states which satisfy $\|x(n)\| \leq M$ but which correspond to some problematic operating condition from a manufacturing viewpoint such as excessively high temperatures which might create a hazardous operating condition). Thus, it is necessary to consider how the result of Proposition 1 can be made more practical to provide meaningful guidelines regarding the use of NISQ devices for manufacturing systems control.

One idea would be to restrict the value of R_δ (e.g., using only small quantum devices) to attempt to ensure that no unsafe states satisfy $\|x(n)\| < M$ with M dependent on R_δ from Eq. 11. This could be considered an approach toward inherently safe design with the NISQ device in the sense that it would not permit the NISQ device to open the possibility that unsafe conditions could be reached. However, this could also translate into R_δ being small such that a very limited number of control actions can be computed by the quantum register, which may reduce the ability to use the actuators to their full potential. Thus, to ensure practical stability and safety through the specific result of Proposition 1, one way to interpret the result is that if we would like to continuously apply $u = -K'x$, we may need the noise profile on the NISQ device (due to the device construction and algorithm used) to keep $\delta(n)$ bounded within smaller sets than would be suggested via the full register size.

Observation 2. As mentioned in Observation 1, the specific result guaranteed by Proposition 1 may not be useful from a manufacturing perspective in cases where larger quantum registers are considered. Thus, we wish to find ways to adjust the result to make it more useful for guaranteeing safety of a manufacturing system and thereby guiding NISQ device requirements (combined with algorithm properties/gate depth which also affect the degree of noise observed in a result). In [10], our group proposed that one way to achieve safety during operation would be to define a safe operating region (a set of states denoted Ω_ρ) and a sufficiently conservative subset of this region of operation (a sets of states denoted Ω_{ρ_e}) such that if the state is initialized within Ω_{ρ_e}, then between two time steps, it cannot leave Ω_ρ. Then, when it is outside Ω_{ρ_e}, a control law is implemented classically (such that $\delta = 0$ for all times over which the classical controller is implemented) to drive the closed-loop state back into Ω_{ρ_e} so that the quantum computer can be used to compute control actions again. The underlying assumption was that the quantum device was computing a control law that was more computationally-intensive than the classical back-up controller such that there was a motivation to using the quantum computer for some of the control action computations. We can try to understand what the extension of that concept might look like for the linear system of Eq. 6 with control inputs computed using the quantum computer per Eq. 5, to see whether this method of attempting to circumvent the issues with state boundedness in a potentially large set in Proposition 1 due to the consistent application of the quantum computer (i.e., no back-up control law) provides insights into requirements for a NISQ device and algorithm implemented on such a device.

The requirement for the design of Ω_{ρ_e} above could be written as follows:

$$\Omega_{\rho_e} := \{x(n) \in R^n \ : \ x(n+1) \in \Omega_\rho, \ \forall \ \delta(n) \in \Delta\} \tag{12}$$

Making this explicit for the closed-loop dynamics of Eq. 6 under Eq. 5, the requirement becomes that $x(n) \in \Omega_{\rho_e}$ if $x(n+1) = (A - BK)x(n) + B\delta(n)$ is in Ω_ρ, for all $\delta(n) \in \Delta$. This begins to become a requirement that could be tested. Specifically, one could define a region Ω_{ρ_e} and a set Δ based on the register size and check what the worst-case value of $x(n+1)$ is given that register size. Then, Ω_ρ needs to be large enough to contain all such $x(n+1)$. The region Ω_{ρ_e} may need to be very small if $\delta(n)$ can be large; furthermore, depending on the set of safe states for the system, with large $\delta(n)$, there may not exist any Ω_ρ and Ω_{ρ_e} combination which is suitable. One way to interpret that would be that until the values of $\delta(n)$ produced by a NISQ device/algorithm combination are reduced (i.e., there is a bound on the error that they produce in the computations) to a level that enables Ω_ρ and Ω_{ρ_e} to be characterized, that would imply that the back-up controller-based strategy for mitigating the effects of noise in a NISQ device is not viable.

Another approach to analyzing this idea, however, would be to loosen the requirement that the state needs to remain bounded within Ω_ρ at all times under the NISQ-computed control actions to instead requiring that it needs to do so a certain percentage of the time. This idea would utilize a risk management-based approach, where a sufficiently small risk that the closed-loop state may leave Ω_ρ is tolerated because there is no process without risk. If we were to frame this concept, this might include, for example, that given $x(n) \in \Omega_{\rho_e}$, $x(n+1)$ needs to be within Ω_ρ with a certain probability. As a first step toward understanding how this impacts device requirements, one might try simulating the closed-loop state under various device noise profiles to see which translate to this requirement being met. That could help to then showcase what types of noise profiles could lead to acceptable behavior.

Observation 3. The third observation that we make concerns the closed-loop behavior of the process. In particular, the time evolution of the state as suggested by the equation within the norm on the right-hand side of Eq. 8 occurs under the assumption that $u = K'x(t)$ (in the absence of device noise). However, we could imagine attempting to create different closed-loop dynamics through updating $u(t)$ for several sampling periods. Specifically, consider $x(1)$, with $u(0) = -K'x(0) + \delta(0)$, as follows:

$$x(1) = (A - BK')x(0) + B\delta(0) \tag{13}$$

When $u(1)$ is being computed, $\delta(0)$ has already been realized. Thus, in principle, one could design $u(1)$ to attempt to cancel its effect. Specifically, consider the prediction $x(2) = Ax(1) + Bu(1) = A[(A - BK')x(0) + B\delta(0)] + Bu(t)$ (for the case that $\delta(1)$ is predicted to be zero). If the goal of the controller is to drive $x(n)$ to 0, we could attempt to choose $u(1)$ such that, if the next realization of the quantum device noise is $\delta(1) = 0$, then $x(2) = 0$. The control law that would achieve this is $u(t) = -B^{-1}[A(A - BK')x(0) + AB\delta(0)]$ (assuming an invertible B). Since the actual control action that is then applied to the system is $u(t) = -B^{-1}[A(A - BK')x(0) + AB\delta(0)] + B\delta(1)$ (due to the device noise), the value of $x(2)$ becomes:

$$x(2) = B\delta(1) \tag{14}$$

We could then perform the same concept to develop $u(2)$ in an attempt to drive $x(3)$ to zero, given $\delta(1)$. Specifically, since $x(3) = Ax(2) + Bu(2) = AB\delta(1) + Bu(2)$, we can attempt to design $u(2)$ such that it causes $x(3)$ to go to zero if $\delta(2) = 0$. This would lead to $u(2) = -B^{-1}AB\delta(1)$ (again assuming an invertible B). With this, the actual closed-loop dynamics would be $x(3) = B\delta(2)$, which has the same form as Eq. 14. Thus, we can apply the procedure recursively (i.e., use $u(i) = -B^{-1}AB\delta(i-1)$ for $i \geq 2$), and continue to get that $x(i) = B\delta(i-1)$, $i \geq 2$. The significance of this is that this control strategy causes the value of $x(i)$ to be an explicit function of only B and $\delta(i)$, which can make it more straightforward to examine the consequences of a given noise profile on the state trajectory. For example, if one would like to specify that $\|x(i)\| \leq \nu$ with a certain probability constitutes acceptable behavior for the closed-loop system, then since $\|x(i)\| \leq \|B\|\|\delta(i-1)\| \leq \|B\|\|\delta(i-1)\|$, $i \geq 2$, this implies that meeting the requirement corresponds to requiring that $\|\delta(i)\| \leq \frac{\nu}{\|B\|}$ the required percentage of the time that the result must hold. This then becomes an explicit requirement on the noise profile, which can be useful from the perspective of understanding whether a given NISQ device/algorithm combination would "meet the mark." We note that the ease with which the benchmarking was performed here comes from choosing a specific control law. However, we could consider using this same type of principle to obtain similar results from Eq. 11. Specifically, replacing R_δ with $\|\delta(n-1)\|$ (based on Eq. 10), then the requirement that $\|x(n)\| \leq \nu$ could translate to a requirement that $C_n\|x(0)\| + C_{\delta,n}\|B\|\|\delta(n-1)\| \leq \nu$, which can be solved for $\|\delta(n-1)\|$ to specify the conditions on the noise required to meet the control objective under this control scheme. We note, however, that the specific requirement on $\|\delta(i)\|$ may be different with the control scheme of Proposition 1 and with the control scheme of Observation 3, highlighting that the specific control strategy can impact what device/algorithm requirements are needed to hit a specific control objective metric with the quantum computer.

Observation 4. As a fourth observation, we note that one of the challenges with using Proposition 1 is the magnitude of R_δ. One idea for attempting to create a more practical control situation would be to use a classical post-processing strategy on a control action computed by a quantum device. The post-processing strategy may receive the control input from the quantum device before it is implemented on the process and "correct" the value computed by the quantum device so that the control action applied does not exceed safety bounds. For example, the post-processing classical filter may check if at each time step, $\|u(t) + K'x(t)\| < \bar{\nu}$, where $\bar{\nu} > 0$ is a pre-specified tolerance on the control input applied at the time step t. If the control input computed by the quantum device exceeds the tolerance, then the classical computer may override the quantum computer and communicate a control input of $u(t) = K'x(t)$ to the actuators. In this case, $\|\delta(n)\|$ is restricted by $\bar{\nu}$ (rather than R_δ) in the proof of Proposition 1, which means that the bound in Eq. 11 can be arbitrarily specified by specifying the value of $\bar{\nu}$ used in the post-processing filter. This would be unmotivated if $u(t)$ was truly intended to be $-K'x$ (since the post-processing algorithm is then also classically computing that value). Nevertheless, the strategy might be studied for cases where u is computed via other control laws to see whether it aids with mitigating the effects of noise on closed-loop stability.

These observations demonstrate the potential benefits of control theory for guiding the design of control laws and post-processing strategies that are appropriate for NISQ devices, both in terms of their ability to serve the needs of control and to attempt to both mitigate device errors and reflect what characteristics are needed from a quantum device/algorithm combination in terms of noise profiles.

B. Profitability of Linear Systems Under Proportional Control On Quantum Devices

The results related to stability in the prior section inspire asking whether another important metric for process operation, profitability/optimality, can also be handled similarly (i.e., if control theory can guide the formulation of control algorithms and post-processing strategies that can reach performance goals in the midst of NISQ era noise while providing guidelines for what the noise profiles would need to be to hit such performance goals for control action computation using quantum devices). As a first step toward investigating this, we consider a control law known as the linear quadratic Gaussian (LQG) formulation. This is a control law which is optimal with respect to a specific objective function/metric in the presence of noise with a certain shape. We discuss the formulation of this strategy and demonstrate that despite its utility for systems with measurement noise and uncertainty, it causes a component of the control law to become unimportant when used with the proportional part of the controller implemented using a quantum device, suggesting a fundamental difference in noise-handling for systems affected by measurement noise and plant/model mismatch compared to quantum noise. This section is structured as follows: first, we discuss the class of systems to which LQG is typically applied. Then, we make analogy between the closed-loop dynamics of such systems and of a linear system under a controller implemented with the aid of a noisy quantum device. We then discuss differences between the LQG formulation when implemented for the system with measurement noise and plant/model mismatch versus with the quantum noise, despite the analogies noted between them. We close with a simulation that demonstrates that the design of control strategies for noise-handling may be different between the traditional control-theoretic case with disturbances and plant/model mismatch and the case of quantum noise.

1) Class of Systems Without Quantum Noise

In this section, we seek to analyze whether for optimal operation of a linear system, we can perform a task similar to that in the Section III-A. Specifically, we investigate whether we can take a well-known control-theoretic result and make an analogy between its application to the traditional process systems (impacted by measurement noise and plant/model mismatch) and those corrupted by a noisy quantum device to understand the extent to which the traditional control-theoretic results provide insights on how control-theoretic principles might be used in both understanding the effects of quantum device noise on closed-loop dynamics and determining what such studies imply for the requirements on NISQ era devices/algorithms for control applications. In this section, we thus describe the traditional process systems to which LQG is applied, so that we can proceed in the next section to make an analogy between these dynamics and those of a linear system with control action computations corrupted by quantum device noise.

The LQG-based control strategy [16] is designed for processes that are subject to process disturbances and measurement noise per the following equation:

$$x(t+1) = Ax(t) + Bu(t) + w(t) \tag{15a}$$

$$y(t) = x(t) + v(t) \tag{15b}$$

where $w \in \mathbb{R}^n$ is the process disturbance, and $v \in \mathbb{R}^n$ is the measurement noise vector. w and v are assumed to be independent and both are white Gaussian noise with zero mean. LQG is a controller given by $u(t) = -K\hat{x}(t)$, where $\hat{x}(t) \in R^n$ is called the "state estimate" and is determined by the following equation:

$$\hat{x}(t+1) = A\hat{x}(t) + Bu(t) + L(y(t) - \hat{y}(t)) \tag{16a}$$

$$\hat{y}(t) = \hat{x}(t) \tag{16b}$$

where $\hat{y} \in \mathbb{R}^n$ is the estimate of the measured output generated by the state estimator, and $L \in \mathbb{R}^{n\times n}$ is the estimator gain. Thus, the closed-loop dynamics of the process of Eq. 15 under LQG become:

$$x(t+1) = Ax(t) - BK\hat{x}(t) + w(t) \tag{17}$$

$$\hat{x}(t+1) = A\hat{x}(t) - BK\hat{x}(t) + Lx(t) + Lv(t) - L\hat{x}(t) \tag{18}$$

We note that in general, $x(t+1)$ and $\hat{x}(t+1)$ are not equivalent for all t, even if $x(0) = \hat{x}(0)$, as long as $w(t)$ or $v(t)$ are not both zero. To see this, we can analyze the error dynamics of the system of Eqs. 17-18, where the error is given by $e(t) := x - \hat{x}$ as follows:

$$e(t+1) = (A - L)e(t) + w(t) - Lv(t) \tag{19}$$

This equation is structurally similar to Eq. 6, but with BK' replaced by L and $B\delta(t)$ replaced by $w(t) - Lv(t)$. If $\rho(A - L) < 1$, then by the same steps as in Eq. 11, $\|e(n)\|$ is bounded by a term that decays to zero as $t \to \infty$ and another term that depends on the magnitude of the disturbances. If $w(t)$ and $v(t)$ are unbounded, then we would not guarantee practical stability because the bounds on the disturbance in Eq. 11 could not be established; however, if we consider a finite value of n and consider that the disturbance realizations until that point were bounded, we can place an upper bound on $\|e(n)\|$ that depends on the maximum value of $\|w(t) - Lv(t)\|$ observed until $t = n$ and thereby conclude that in general we do not expect $e(n) = 0$ when $w(t)$ and $v(t)$ are not both zero for all t.

The LQG control law is optimal with respect to a certain objective function. Thus, if we are able to make analogies between the process in this section and one in which errors come from a quantum device (instead of plant/model mismatch and measurement noise), there could be potential for making statements regarding profitability of operation with respect to that same objective function. This motivates exploring potential analogies in the subsequent section.

2) Class of Systems With Quantum Noise

We now make the analogy between the dynamics of the system of Eq. 20 when $u(t)$ is computed by a quantum device with noise and Eq. 15. Specifically, if $u(t) = \bar{u}(t) + \delta(t)$, where $\bar{u}(t)$ is the control action corresponding to the intended control law and $\delta(t)$ represents the deviation from this intended control action due to the device noise, we obtain the following dynamics:

$$x(t+1) = Ax(t) + B\bar{u}(t) + B\delta(t) \tag{20a}$$

$$y(t) = x(t) \tag{20b}$$

Comparing Eqs. 20a-20b with Eqs. 15a-15b, we see that Eqs. 20a-20b are equivalent to Eqs. 15a-15b in that case that $v(t) \equiv 0$ in Eq. 15b and that $B\delta(t)$ is white Gaussian noise with zero mean. Though the specific noise profile corresponding to a device/algorithm may not result in practice in $B\delta(t)$ being white Gaussian noise with zero mean, the analogy between these equations inspires us to examine the closed-loop dynamics to see whether the process of Eqs. 20a-20b may obtain any type of optimality result in analogy to LQG for the system of Eq. 15.

Because LQG involves both a proportional control computation and a computation of a state estimate, we must designate which parts of the LQG will be computed using the quantum device. In the study that follows, we consider a control architecture that uses both a classical computer and a quantum device. While the quantum device computes the control inputs, the classical computer generates the estimates of states. Under the proposed architecture, at each time step, the measured output communicated by the sensors is received by the classical computer that generates estimates of process states. The quantum device receives the estimates of state from the classical computer and computes the control inputs to be applied on the process. The control input is communicated to the actuators which implement the control action on the process, and to the classical computer for use within the state estimate computation.

Using this framework, if we again consider the estimate to be generated from Eq. 16, but with the control action given by $u(t) = -K\hat{x} + \delta(t)$ due to the quantum device noise, we obtain the following closed-loop dynamics:

$$x(t+1) = Ax(t) - BK\hat{x}(t) + B\delta(t) \tag{21}$$

$$\hat{x}(t+1) = A\hat{x}(t) - BK\hat{x}(t) + Lx(t) + B\delta(t) - L\hat{x}(t) \tag{22}$$

Notably, these dynamics are different from those in Eqs. 17-18 in an important way: the error dynamics in this case are always zero if $x(0) = \hat{x}(0)$ (which is reasonable to expect given that full state feedback is assumed to be available with uncorrupted measurements according to Eq. 20b. To see this, consider the state evolution in the presence of quantum noise at $t = 1$:

$$x(1) = Ax(0) + B\bar{u}(0) + B\delta(0) \tag{23}$$

The state estimate is then given by:

$$\hat{x}(1) = A\hat{x}(0) + B\bar{u}(0) + B\delta(0) + L(y(0) - \hat{y}(0)) \tag{24}$$

Since perfect full state feedback is available, $y(0) = \hat{y}(0)$. Thus, the final term in Eq. 24 is zero. Since also $x(0) = \hat{x}(0)$, at time $t = 1$, the state and the state estimate values are exactly the same. This means that again $y(1) = x(1)$ and $\hat{y}(1) = \hat{x}(1)$ are equal, so that applying this result recursively, $\hat{x}(t)$ and $x(t)$ are equivalent for all t, such that $e(t) = 0$, $\forall\ t \geq 0$, when $\hat{x}(0) = x(0)$. We note that this is a fundamentally different result from Eq. 19. Furthermore, this is dependent only on the general structure of the dynamics of $\hat{x}(t)$ being given by Eq. 16 (i.e., in Eqs. 23-24, we did not specify what control law $\bar{u}(t)$ follows). This indicates a significant structural difference between linear systems impacted by measurement noise and disturbances versus those impacted by the quantum noise in the controller alone; in particular, it suggests that state estimators of the form in Eq. 16 have no ability to play a role in strategies for mitigating quantum noise impact when $x(0) = \hat{x}(0)$.

To make the significance of this more clear, for the process with plant/model mismatch and measurement noise in Eqs. 15, the role of the state estimator term is to create control actions $-K\hat{x}(t)$ that are different from those which would have been computed using $-Kx(t)$ (since in general there is an error between $\hat{x}(t)$ and $x(t)$ from Eq. 19 when $w(t) \not\equiv 0$ and $v(t) \not\equiv 0$. This means that in that case, the estimator can be used to create control actions that can be considered to have an "awareness" of, for example, the measurement noise (since it explicitly appears in Eq. 18), and also the disturbances (since from Eq. 19, those also create a deviation of $x(t+1)$ from $\hat{x}(t+1)$ and $x(t)$, which is impacted by $w(t)$, appears in the right-hand side of Eq. 18). However, this filter structure has no ability to make the controller "aware" of any realizations of the quantum noise since regardless of the control law implemented, it causes $e(t+1) = 0$, $\forall t$.

This result implies several points regarding NISQ device usage for control, and attempting to maintain profitability of operation with a noisy computation device. On one hand, it shows that for the system of Eq. 20 with $x(0) = 0$ and $\hat{x}(0) = 0$, a filter with the form in Eq. 16 cannot be used to attempt to modify control actions on-line to attempt to maintain profitability despite the noise. However, since the LQG solves a specific optimal control problem, it could be beneficial to seek to solve an optimal control problem that explicitly considers $u(t) = \bar{u}(t) + \delta(t)$ in the objective function definition.

3) Insights from Simulation Studies with Simulated Noise

In the prior section, we presented a case where measurement noise and plant/model mismatch create a fundamentally different behavior in the design of a control law than quantum noise does. This is significant because it helps to clarify that despite the utility of classical control-theoretic principles for systems subject to disturbances in developing concepts for NISQ devices to be used in process control as discussed in Section III-A, there may be cases where the techniques used for handling disturbances in traditional control do not create the same effects for processes impacted by noise from quantum devices. This helps to clarify that the problem of analyzing how and when NISQ devices can be used in control is not a fully solved problem by simply applying all results of disturbance or noise-handling from traditional control techniques without further analysis. To emphasize this, this section presents a numerical example that showcases similarities between the results of applying LQG with and without a filter to a linear system when the proportional component of the control law is computed using a quantum simulator with the depolarizing error noise model.

The specific process under consideration is as follows:

$$x(t+1) = x(t) + u(t) \tag{25a}$$

$$y(t) = x(t) \tag{25b}$$

The process in Eq. 25 fits the model for the process considered in Eq. 2a-2b with $A = 1$ and $B = 1$. The operational objective is to design a control law that drives the state to the origin. We analyze the behavior of the process under two strategies for control implemented on a quantum device: (1) LQG with $\hat{x}(t) = x(t)$ (i.e., no noise filtering) and (2) LQG with noise filtering over a classical computer. Through these simulations, we demonstrate the point from the prior section that the use of the state estimator does not create an appreciable difference between these strategies. We consider 100 different runs of the process of Eq. 25a under both control laws, each for 500 time steps. In each case, the process state is initialized at the setpoint of the process $x(0) = 0$. To model actuator limitations, we restrict the allowable control inputs to $-1 \leq u(t) \leq 1$ for all $t \geq 0$, meaning that if $u(t)$ at some time step is less than -1, we set $u(t) = -1$, and if $u(t)$ computed is greater than 1, we set $u(t) = 1$. All classical processing is performed using a 64-bit processor. The quantum computer is represented using a 32-qubit quantum simulator qasm_simulator provided by provided by IBM's open source software development kit, Qiskit (Version 0.46.0) [17]. We use the depolarizing error function within IBM's quantum simulator [18] by setting the error parameter (α) to 0.005. The control law $u(t) = -Kx(t)$ is implemented using a quantum Fourier transform (QFT)-based multiplication algorithm that is deterministic in the absence of noise.

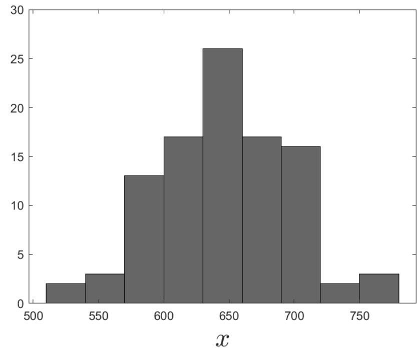
The specific algorithm that we implement leverages the fact that multiplication may be thought of as repeated addition and is discussed in [9]. Quantum addition and multiplication algorithms can be carried out using a QFT [15], [19]. The sequence of gates used by the QFT operator to achieve the desired transformation include Hadamard (H) and controlled Z-rotation (Z_k) gates. We give a condensed description of the use of the quantum simulator in the computation of the control actions here.

At every time t, either $\hat{x}$ or x is obtained. In the description of the algorithm that follows, we will consider that x is obtained (but it can be readily replaced with $\hat{x}$ when the LQG with state estimation is used). To work with binary representations of numbers, scale x and K by 10, round the result to the nearest integer, and convert it to its binary equivalent. Either x or K (whichever requires a larger number of bits to represent its binary equivalent) is then designated as "a", and the other as "b". Leading "0"s are appended to set b to be the same length as a. Shifted partial sums are then used to develop the multiplication result, where:

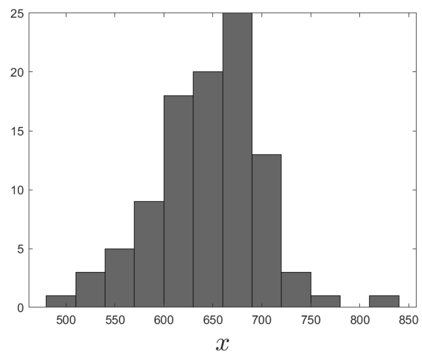
$$\begin{aligned} a \times b = & a_0(2^0 \times b) + a_1(2^1 \times b) + a_2(2^2 \times b) + \ldots + \\ & a_{n-2}(2^{n-2} \times b) + a_{n-1}(2^{n-1} \times b) \end{aligned} \tag{26}$$

where a_i represents a digit of a in binary. Specifically, each bit in the binary representation of b is scaled up or "shifted" by a factor of 2^t where $t = 0, 1, \ldots, n-1$. The length of each partial sum term on the right had side of Eq. 26, $a_i(2^i b)$, is fixed to $2n$. This will result in a set of terms which if multiplied by the corresponding bit of a and added will give the product of a and b. To reduce the number of partial sum terms to be computed, if $a_i = 0$, such that $i = 0, 1, \ldots, n-2, n-1$, then the partial sum term $a_i(2^i b)$ is set to 0 and represented as a $2n$-bit long set of zeros.

To compare the LQG with and without the state estimator, we first consider the case with no estimator. We set $K = 0.6180$ to minimize the quadratic cost $J = \sum_{i=0}^{\infty}(x(i)Qx(i)^T + u(i)Ru(i)^T)$, with the weighting matrices chosen as $Q = 5$ and $R = 5$. The trajectories of the state observed over all simulations remain bounded (as expected the realizations of the noise are sufficiently small according to the proof of Proposition 1), the states are continuously perturbed due to the presence of quantum noise around the equilibrium



(a)



(b)

Fig. 2: Distribution of the quadratic cost over 100 simulations of the process under control implemented on a quantum simulator with: (a) no noise filtering, and (b) noise filtering via an estimator implemented on a classical computer.

value. The values of the process states observed over all simulations had a mean of -0.07 and a standard deviation of 0.29. Over each of the 100 simulations, we evaluate the value of the quadratic cost function $J^e = \sum_{i=0}^{500}(x(i)Qx(i)^T + u(i)Ru(i)^T)$ and observed that J^e values had a mean of 646.67 and a standard deviation of 48.75.

We now consider implementing an LQG-based control law with the proposed control architecture under which control actions are computed over a quantum device using state estimates from a classical computer with a Kalman filter. Similar to the first case, we select $K = 0.6180$ to minimize the quadratic cost J, with $Q = 5$ and $R = 5$. To choose the Gaussian distributions for modeling quantum noise for the design of L, we compared the realizations of process states over two case studies with each consisting of 100 simulations considering the process for 500 time steps. Under the first case study, a linear control law with $K = 0.6$ is implemented over a classical computer modeling the process subject to additive process disturbances and measurement noise, whereas, under the second case study,

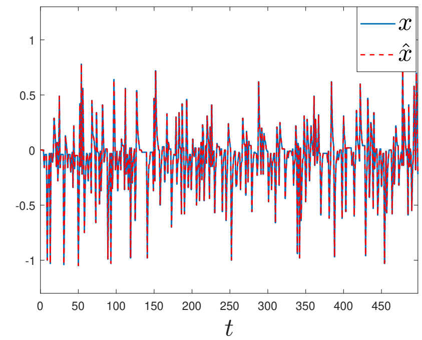


Fig. 3: Trajectories of state and estimated state over a simulation of process under control implemented on a quantum simulator with noise filtering over a classical computer.

we implemented the same control law over the quantum simulator modeling noise with a depolarizing error parameter $\alpha = 0.005$. Within the classical simulation case study, we varied the values of standard deviations of the disturbances and noise to choose the distribution which fit all realizations of process states generated over the case study considering the quantum simulator. The best fit Gaussian distributions were $\mathcal{N}(0, 0.666)$ and $\mathcal{N}(0, 0.0333)$ for the process disturbances and measurement noise, respectively. We utilize the covariances of the equivalent process disturbance and measurement noise to compute an observer gain of $L = 0.9975$. Over the simulations, we round the gain value to $L = 0.9$ (instead of $L = 1$) to avoid instability.

The trajectories of x and $\hat{x}$, as shown in Fig. 2b indicate that again the state is continuously perturbed but bounded, and the values that of process states over all simulations had a mean of -0.07 and 0.29, remaining at the same values as for the controller without a filter. The quadratic cost J^e computed over all simulations for the LQG with the filter had a mean of 645.54 and a standard deviation of 52.59, which is comparable to the performance of the control law without a filter (despite that different realizations of the noise were used for the two sets of simulations). Fig. 2a and Fig. 2b illustrate the distribution of the quadratic costs J^e over the first simulation set with no filter, and the second simulation set considering the LQG-based design strategy, respectively, which differ due to the different realizations of the noise. The estimation error for the LQG observed over all 100 simulations remained at zero as predicted from the discussion of the prior section. This is illustrated in Fig. 3 for one simulation of the process, where the state and the state estimate values are exactly the same for all time. Thus, as discussed in the prior section, for these conditions, a state estimator has no ability to mitigate the impact of the noise on a process from control implemented on a quantum device.

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