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The Lorentzian scattering rigidity problem and rigidity of stationary metrics

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Abstract
We study scattering rigidity in Lorentzian geometry: recovery of a Lorentzian metric from the scattering relation $\mathcal{S}^\sharp$ known on a lateral boundary. We show that, under a non-conjugacy assumption, every defining function $r(x, y)$ of the submanifold of pairs of boundary points which can be connected by a lightlike geodesic plays the role of the boundary distance function in the Riemannian case in the following sense. Its linearization is the light ray transform of tensor fields of order two which are the perturbations of the metric; and each one of $\mathcal{S}^\sharp$ and r (up to an elliptic factor) determines the other uniquely. Next, we study scattering rigidity of stationary metrics in time-space cylinders and show that it can be reduced to boundary/lens rigidity of magnetic systems on the base; a problem studied previously. This implies several scattering rigidity results for stationary metrics.

Keywords Lens rigidity · Scattering rigidity · Inverse problem · Lorentizan · Stationary metric

1 Introduction

We study the scattering relation $\mathcal{S}^\sharp$ for Lorentzian manifolds with spacelike or timelike boundaries. The main question we are interested in is whether one can recover the metric g, up to some group of explicit gauge transformations, given $\mathcal{S}^\sharp$. Our convention is that $\mathcal{S}^\sharp$ acts on projections of lightlike *covectors* on the boundary; for its vector version we use the notation $\mathcal{S}$.

(A) Suppose we want to solve it by linearization near a background metric g. In fact, some of the strongest results in the Riemannian case are obtained this way. In the Riemannian case, a linearization of the boundary distance function ρ leads to the

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inversion of the geodesic X-ray transform of tensor fields

$$Xf(x,v)=\int \big\langle f,\dot\gamma_{x,v}(t)\otimes\dot\gamma_{x,v}(t)\big\rangle\,\mathrm{d}t, \tag{1.1}$$

where the two-tensor f is the perturbation of g. We want to show that $Xf=0$ implies that f is potential (see next section), which linearizes the diffeomorphism invariance of the scattering rigidity problem. Moreover, we want to prove a stability estimate allowing us to treat the nonlinearity, see, e.g., [44].

If we linearize $\mathcal{S}^\sharp$ instead, we get a not so simple looking formula. We still get a geodesic X-ray transform of symmetric tensor fields but of ∇f, with a weight, plus a zeroth order term, see [39, 46, 48]. The appearance of derivatives is not surprising in view of (2.2) below or by the fact that the generator of the geodesic flow contains first derivatives of the metric. The simple looking (1.1) is more attractive for analysis however, especially in the Euclidean case where one can use the Fourier transform, see, e.g., [33, 38]. On the other hand, at least for simple manifolds, $\mathcal{S}^\sharp$ and ρ determine each other but ρ has the advantage of being scalar, and as we said, with a simpler linearization.

There is no obvious extension of the boundary distance function *for this purpose* in the Lorentzian case even though distance/separation functions have been defined. In fact, the distance between two points that can be connected by a lightlike geodesic is zero, and if they cannot, one can define a separation function but that depends on more than the lightlike rays only.

We propose, for Lorentzian manifolds with spacelike or timelike boundaries, an equivalent to the boundary distance function which can serve as a generating function for the scattering relation and which linearization about a fixed metric leads to the light ray X-ray transform of tensor fields of order two

$$Lf(x,v)=\int \big\langle f,\dot\gamma_{x,v}(t)\otimes\dot\gamma_{x,v}(t)\big\rangle\,\mathrm{d}t, \tag{1.2}$$

which looks formally like (1.1) but the background metric is a Lorentzian one, and we integrate over lightlike geodesics only. This defining function can be taken, in fact, to be a defining function of the boundary pairs (x,y) which can be connected by a lightlike geodesic, and one way to construct it is through the energy functional. This legitimizes the interest in the Lorentzian version L of X.

(B) Is L invertible, up to some "natural" linear space which we expect to be its kernel? Eventually, we want them to be stably invertible, which is not true because timelike singularities are invisible for L [19, 20, 34, 38]. For some classes of f, that might still be true. We expect that $Lf=0$ with $f\in C_0^\infty$ at least, implies that f is a sum of a potential field and one conformal to g. This is the expectation based on the linearization of the (known so far) non-uniqueness of $\mathcal{S}^\sharp$. This kind of injectivity is proved in [13] when g is conformal to $\mathrm{d}t^2-h(x,\mathrm{d}x)$ with h Riemannian, under the assumption that for X_h related to h, we have injectivity modulo potential fields, even for tensor fields of orders $m>2$. This implies the same result for g Minkowski of course.

(C) We prove next scattering rigidity for one of those special cases where we expect it to hold: for stationary metrics, under some additional geometric conditions (after all, conditions are needed even in the Riemannian case). We use the observation made in [15] that once one projects the lightlike geodesics on the spatial base, one gets a magnetic system in space. Boundary rigidity for magnetic systems was studied in [10], see also [3, 56]. The equivalent to the boundary distance function there was taken to be the boundary action function $\mathbb{A}$, see also Appendix Appendix A. We show that $\mathbb{A}$ appears naturally when we reduce the Lorentzian scattering relation to knowledge of a defining function of pairs of boundary points which can be connected by a lightlike geodesic; and then project to the spatial base. Then all rigidity results in [10] apply and imply rigidity for classes of stationary metrics.

The boundary and the lens/scattering rigidity of Riemannian manifolds have rich history. It goes back to 1905 and 1907, when Herglotz [18] and Wiechert and Zoeppritz [55] resolved the conformal spherically symmetric case motivated by seismology. The conformal case for simple metrics was solved in [24] and [25]. Further results can be found in [4, 5, 7, 9, 16, 21, 30, 31, 39, 41, 46, 48]. The lens rigidity problem, more appropriate for non-simple geometries, is studied in [6, 8, 17, 22, 45].

Inversion of the geodesic X-ray transform on tensors on Riemannian manifolds has been well studied as well, see [11, 23, 33, 35, 36, 40, 42, 43, 47, 51] generalizing its version on functions.

There are no so many results for Lorentzian geometry. The author and Yang [49] showed that the scattering relation appears as the canonical relation of the associated Dirichlet-to-Neumann map, which is an FIO. A linearization of the scattering rigidity problem from a spacelike to a spacelike hypersurface was studied in [19] motivated by a problem in cosmology. It was shown that microlocally, a vanishing linearization implies that the perturbation is a sum of a conformal tensor plus a potential part. The recent paper [52] studies recovery of stationary metrics from the time separation function under an additional condition on the form ω, see section 4, but the data there uses information coming from not lightlike geodesics only. Of course, any result about lens rigidity of a Riemannian manifold (N, h) implies rigidity for the Lorentzian one $M = \mathbb{R}_t \times N_x$ with $g = -\mathrm{d}t^2 + h(x, \mathrm{d}x)$. Microlocal study of the light ray transform on functions was done in [20].

Light ray transform results in the Lorentzian setting exist as well: [12, 20, 34, 37, 53, 54] for functions, [32] for one-forms, and [13, 19] for higher order tensor fields. One major distinction is that one can recover spacelike singularities, probably say something about the lightlike ones, but the timelike ones are hidden. This can be interpreted as ability to see signals moving slower than light (or sound, etc.) but not ones moving faster than it. On the other hand, in some specific situations motivated by physics, signals moving faster than light should not exist (well, ignoring the discussion what a signal is, and ignoring the distinction between phase and group velocities at the moment). This has been used in [53] to show that if f solves a wave equation with speed one or less, one can recover all singularities, and in fact invert the light ray transform stably on a subclass of functions. The light ray transform on two-tensor fields however remains not well understood, and even its relation to the scattering relation was not clear. Clarifying the latter is one of our goals here.

Scattering rigidity of stationary metrics via timelike geodesics was studied in [28] by Sebastián Muñoz-Thon after the completion of this work. The dynamical system then projects to a magnetic-potential one studied by the same author in [26, 27]. The group of gauge transformations looks different then.

A few words about the conventions in this paper. All functions or maps are smooth. By $\langle \omega, v\rangle$ we denote the action of the covector ω on the vector v, sometimes denoted by $\omega(v)$ in the literature. For a covariant tensor f or order two, $\langle f, v\otimes w\rangle = f_{ij}v^i w^j$ in local coordinates, sometimes denoted as $f(v, w)$ in the literature. The notation $(u, v)_g$ is reserved for the scalar products of two vectors in the metric g. For a vector $v \in T_x M$ with a base point x on the boundary ∂M, v' stands for its orthogonal projection to $T_x(\partial M)$. Note that this makes sense as long as $T(\partial M)$ is either time or space like. In section 4, in local coordinates (t, x), we will write $v = (v_t, v_x)$ for a vector v, where v_t is the zeroth (time) component of v, while v_x is the n-vector consisting of the spatial components, not to be confused with partial derivatives.

2 The Defining Function *r* of Σ and its Linearization

2.1 The Riemannian Case

Let (M, g) be a compact Riemannian manifold of dimension n with boundary. The lens rigidity problem is to recover the metric (and the topology, if unknown) from the lens data $(\mathcal{S}, \ell)$ consisting of the images $(y, w) = \mathcal{S}(x, v)$ of all boundary points x and unit incoming directions v, where y is the outgoing point (assuming $\gamma_{x,v}$ non-trapping), and w is the outgoing direction of the geodesic $\gamma_{x,v}$ with $\ell(x, v)$ being its length. The map $\mathcal{S}$ alone is called the scattering relation.[1] This recovery is expected to be done up to an isometry fixing the boundary ∂M pointwise. If it holds, then (M, g) is called lens rigid. It is a well studied problem, as we pointed out in the introduction. The boundary rigidity problem has the same goal but the data is the geodesic distance $\rho(x, y)$ between boundary points. Under "simplicity" assumptions, we have $v = \exp_x^{-1}(y)/|\exp_x^{-1}(y)|$, which allows us to express ρ through ℓ as

$$\rho(x, y) = \ell\left(x, \exp_x^{-1}(y)/|\exp_x^{-1}(y)|\right). \tag{2.1}$$

A better way to think about the parameterizations of the geodesics leaving and arriving at ∂M is to project v and w on $T_x(\partial M)$ and $T_y(\partial M)$, respectively; let v' and w', respectively, be those projections. They determine v and w uniquely. Then we can view $\mathcal{S}$ and ℓ as maps from $(x, v') \in B(\partial M)$ (the tangent unit ball bundle) to (y, w') which belongs to the same bundle. Only then will $\mathcal{S}$ and ℓ be invariant under isometries as above.

The so redefined map $\mathcal{S}$ is then symplectic when lifted to $\mathcal{S}^\sharp$ on $B^*(\partial M)$. When $x_0 \in \partial M$ and $y_0 \in \partial M$ are not conjugate to each other along some geodesic, ρ

[1] In author's view, the lens and the scattering data notions should be swapped: lens data should refer to $\mathcal{S}$ only, with no reference to time of propagation, while (time-dependent) near field scattering data should include the latter. In time-space, time is already included, so calling $\mathcal{S}$ scattering data is justified.

is well defined near (x, y) when the distance is the geodesic length restricted to a neighborhood of that geodesic, and

$$\mathcal{S}^\sharp\left(x, -\mathrm{d}'_x\rho(x, y)\right) = \left(y, \mathrm{d}'_y\rho(x, y)\right), \quad (x, y) \in \partial M \times \partial M, \tag{2.2}$$

where d' stands for the tangential projection of d onto $T^*(\partial M)$. The same formula can be written for $\mathcal{S}$ by replacing d by grad, which requires knowledge of g on the boundary. Formula (2.2) is an observation by Michel [22], see also [35]. Therefore, $\mathcal{S}^\sharp$ which maps $\mathbb{R}^{2n-2}$ to itself locally, is actually determined by the derivatives of a (scalar function) mapping $\mathbb{R}^{2n-2}$ to $\mathbb{R}$ locally.

It is straightforward to see that under the non-conjugacy assumption, ρ determines $(\mathcal{S}^\sharp, \ell)$ locally, and vice versa. Indeed, knowing ρ, we can recover $\mathcal{S}^\sharp$ by (2.2), and then ℓ by (2.1). On the other hand, given $\mathcal{S}^\sharp$, for $(x, y) \in \partial M \times \partial M$ fixed, we can recover the projection ξ' of the incoming codirection by (2.2). Then we recover $\rho(x, y)$ by (2.1) knowing ℓ. We want to mention that the non-conjugacy assumption makes $\mathcal{S}^\sharp$ a "free" canonical transformation in the terminology of [2, chapter 47], and then guarantees the existence of a generating function, which happens to be $\rho^2/2$ here. When (M, g) is simple, in fact $\mathcal{S}^\sharp$ suffices to recover ρ. Indeed, by the arguments above, for a fixed $x \in \partial M$, we know $\mathrm{d}'_y\rho(x, y)$ for all y. We can integrate that along a curve on ∂M connecting x and y to recover $\rho(x, y)$.

One of the approaches to boundary/lens rigidity is to linearize near a fixed metric and try to invert stably the resulting linear transform first. A simple variational argument, see, e.g., [35], shows that the linearization of the boundary distance function leads to the quite nicely looking geodesic X-ray transform of symmetric tensor fields of order two:

$$\delta\rho(x, y) = X(\delta g)(x, \exp_x^{-1}(y)),$$

with Xf defined in (1.1), where f is a two-tensor field, and one often normalizes v to unit vectors. We want to invert it stably up a potential field $\mathrm{d}^s v$ with $v = 0$ on ∂M, where $(d^s v)_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$ is the symmetrized differential. Such potential fields linearize the non-uniqueness of the nonlinear problem due to isometries. The geodesic X-ray transform of symmetric tensor fields has been studied extensively, as we pointed out in the Introduction.

As we mentioned in the Introduction, if we linearize $\mathcal{S}^\sharp$ instead, we get a not so simple looking formula. We still get a geodesic X-ray transform of symmetric tensor fields but of ∇f, with a weight, plus a zeroth order term, see [39, 46, 48].

2.2 The Lorentzian Case

Assume that we have a Lorentzian manifold M of dimension $1+n$ now with a metric of signature $(-, +, \ldots, +)$. One example is $g = -\mathrm{d}t^2 + h_{\alpha\beta}(x)\mathrm{d}x^\alpha \mathrm{d}x^\beta$ with h a Riemannian metric on a compact manifold N with boundary, which leads to essentially the same problem as before, so it is not so interesting but is a good start to understand how the scattering rigidity problem would be formulated in that case. The setup above

leads naturally to light rays starting from the cylindrical boundary of $\mathbb{R}_t \times N_x$ and ending up there. Assuming a general Lorentzian metric and lateral boundary which is timelike (or spacelike), we get a scattering relation. One of the new features is that light rays do not have natural parameterization: their "speed squared" $(\dot\gamma, \dot\gamma)_g$ is zero, and rescaling the parameter along each one still yields a null geodesic. Moreover, given a family of such light rays, we may rescale by a factor changing from geodesic to geodesic, and obtain a different parameterization, no better or worse than the initial one. In any case, fixing some parameterization locally, a linearization of $\mathcal{S}^\sharp$ near a fixed light ray would produce a transform at least as unpleasant as in the Riemannian case. On the other hand, the much simpler form (1.2) of L has already being studied with the anticipation that it must have something to do with the linearization of the Lorentzian scattering rigidity problem.

The natural questions then are the following. Under a non-conjugacy assumption, can we define, on $(\mathbb{R} \times \partial N)^2$, an equivalent $r(x, y)$ of the boundary distance function so that

(i) (2.1) and (2.2) hold in some form,
(ii) in particular, r and $\mathcal{S}^\sharp$ determine each other,
(iii) the linearization of r near some g gives us Lf defined in (1.2).

The problem with the attempt to choose r to be a distance function as a direct analog of ρ is that given two points, they may not be connected by a (unique or not) lightlike geodesic. This is a property we really want in order to get (1.2), and relate it to $\mathcal{S}$. If they are, then their geodesic distance is zero. The Lorentzian distance, see, e.g., [1], does not seem to give a direct answer: it is zero on one side of the light cone (away from the chronological future), and singular at the light cone, where it vanishes.

The solution we propose is the following. Given a timelike lateral boundary ∂M, generalizing the cylinder $\mathbb{R} \times \partial N$ above, the set of points connected by a unique lightlike geodesic is a submanifold of $\partial M \times \partial M$ of codimension one locally under a non-conjugacy assumption. Fix any defining function $r(x, y)$ of it. Then r satisfies (i), (ii), (iii) above in an appropriate sense. Of course, r is defined up to an elliptic factor only. On the other hand, null directions can be parameterized up to a scaling factor, so those two peculiarities correlate well.

2.3 Main Result About the Defining Function

Our point of view here is local, so for this reason, we assume that we work in an ambient manifold with a complete Lorentzian metric g, and we have two "small" either timelike (will be referred to as the (T) case), or spacelike (the (S) case) surfaces U and V corresponding to initial and endpoints, respectively, see Fig. 1. One can assume one of those cases on one side, and the other one on the other one, as well. We fix $x_0 \in U$, $y_0 \in V$ so that they are connected by a lightlike geodesic $[0, 1] \ni t \to \gamma_0(t)$. Assume that

$$x_0 \text{ and } y_0 \text{ are not conjugate along } \gamma_0, \tag{NC}$$

and that γ_0 is transversal to U and V at their only points of intersection, x and y.

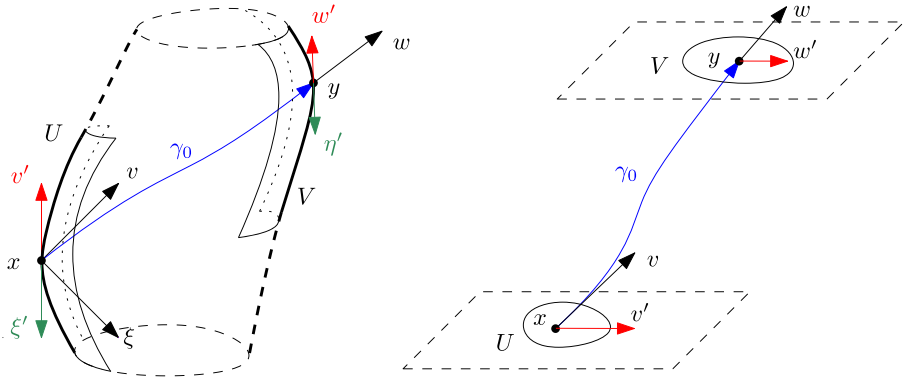

Fig. 1 The scattering relation $\mathcal{S}(x, v') = (y, w')$ on the tangent bundle for U, V either timelike (left) or spacelike (right), and its version $\mathcal{S}^\sharp(x, \xi') = (y, \eta')$ on the cotangent bundle (left only)

In the (T) case, we fix a time orientation on U that we call future pointing, see also [49]. Assume that γ_0 is future pointing at x_0, and we choose a time orientation on V so that γ_0 is future pointing at y as well. We also fix orientation on U and V in classical sense, calling the sides containing γ_0 interior, and the other ones exterior. In the (S) case, the classical orientation relates to time orientation: the interior side of U is the future one; the interior side of V is the past. Set $v_0 := \dot\gamma(0)$, $w_0 := \dot\gamma(1)$. Let v_0', w_0' be their orthogonal projections on TU and TV, respectively; see [29, chapter 2]. They must be timelike/spacelike depending on which case we have (T) or (S). Let $\mathcal{U}$, $\mathcal{V}$ be small timelike/spacelike conic neighborhoods of (x_0, v_0') in TU, and of (y_0, w_0') in TV, respectively. We define the *scattering relations* $\mathcal{S}$, $\mathcal{S}^\sharp$ below, see also Fig. 1.

Definition 2.1 The scattering relation $\mathcal{S} : \mathcal{U} \to \mathcal{V}$ is defined by $\mathcal{S}(x, v') = (y, w')$ as follows. Let v be the lightlike vector at x with orthogonal projection v' on T_xU, pointing to the interior; then $y \in V$ is the point where the geodesic $\gamma_{x,v}$ issued from (x, v) meets V, and w' is the orthogonal projection on T_yV of its direction there. Identifying vectors with covectors by the metric g restricted to TU, we define $\mathcal{S}$ on the cotangent bundle as well, we call it $\mathcal{S}^\sharp$.

Clearly, $\mathcal{S}$ and $\mathcal{S}^\sharp$ are positively homogeneous of order one in the fiber variable. We have $\mathcal{S}(x, a(x, v')v') = (y, a(x, v')w')$ for every $a > 0$. We may normalize v' in some way to reduce the number of variables. For example, we may require $(v', v')_{g'} = \mp 1$ (recall that v' is spacelike/timelike depending of whether U is timelike/spacelike), where g' is the induced metric. If x^0 is a local time variable in the former case, we may require $v^0 = 1$. More generally, we may restrict v to some hypersurface so that each radial ray intersects it transversally. We call each such restriction of $\mathcal{S}$ a *reduced representation* of $\mathcal{S}$, similarly for $\mathcal{S}^\sharp$. Each of them is just a representation of $\mathcal{S}$ or $\mathcal{S}^\sharp$ modulo rescaling by positive factors, depending on (x, v'). Knowing the reduced version recovers the full one in a trivial way.

Remark 2.1 The non-conjugacy and the transversality assumptions imply that $\exp_x : \exp_x^{-1}(V) \to V$ is a diffeomorphism for every $x \in U$ as long as U and V are small

enough. Assuming that, we can also assume that $\mathcal{U} = \{\exp_x^{-1}(V);\ x \in U\}$; then we have a global diffeomorphism for $(x, v) \in \mathcal{U}$ above. Finally, we can project v to T_xU to get a diffeomorphic map $T_xU \ni v' \to y \in V$ for every $x \in U$. We are going to consider geodesics (lightlike or not) issued from $(x, v) \in \mathcal{U}$ only.

Definition 2.2 (a) The set $\Sigma \subset U \times V$ consists of pairs (x, y) so that x and y are connected by a unique lightlike geodesic (locally).

(b) The smooth function $r : U \times V \to \mathbb{R}$ is called a defining function of Σ, if (i) $r = 0, \mathrm{d}r \neq 0$ on Σ, and (ii) $r(x, y) < 0$ if and only if the locally unique geodesic connecting x and y is timelike.

Condition (ii) is just a sign convention for r; negativity means that y is in the chronological future of x.

Theorem 2.1 *With the assumptions above, and in particular, with the non-conjugate assumption (NC), we have*

(a) Σ *is a codimension one submanifold of* $U \times V$ *when* U *and* V *are small enough. Let* $r : U \times V \to \mathbb{R}$ *be any defining function of* Σ. *Then*

(b) $\left\{(x, -\mathrm{d}'_x r, y, \mathrm{d}'_y r);\ r(x, y) = 0\right\}$ *coincides locally with the graph of some reduced representation of* $\mathcal{S}^\sharp$.

(c) If g_τ *is an one-parameter family of Lorentzian metrics smoothly depending on* τ *near* $\tau = 0$, *so that* $g_0 = g$, *and* r_τ *are associated defining functions smoothly depending on* τ, *we have*

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Big|_{\tau=0} r_\tau(x, y) = \kappa(x, y) \int_0^1 \left\langle f, \dot\gamma_{[x,y]}(t) \otimes \dot\gamma_{[x,y]}(t) \right\rangle \mathrm{d}t \tag{2.3}$$

on $\Sigma = \{r(x, y) = 0\}$, *where* $f = \mathrm{d}g_\tau/\mathrm{d}\tau|_{\tau=0}$, $[0, 1] \ni t \to \gamma_{[x,y]}(t)$ *is the locally unique lightlike geodesic in the metric* g *connecting* x *and* y, *and* κ *is a smooth non-vanishing function.*

Before proceeding with the proof, we make a few observations.

Remark 2.2 The non-uniqueness of the defining function r due to the freedom to multiply by any elliptic factor $\kappa(x, y)$ implies the following. If $r_\kappa = \kappa r$ is another defining function, then in (a), $\mathcal{S}^\sharp(x, \xi') = (y, \eta')$ with $\xi' = -\mathrm{d}'_x r$, $\eta' = \mathrm{d}'_y r$ is replaced by $\mathcal{S}^\sharp(x, \kappa\xi') = (y, \kappa\eta')$, which is just another reduced representation of $\mathcal{S}^\sharp$. Each one of them determines $\mathcal{S}^\sharp$ by homogeneity. Replacing r_τ by $\kappa_\tau r_\tau$ in (2.3) multiplies the right-hand side by κ_0.

Remark 2.3 The non-trivial elliptic factor κ in (2.3) is inevitable since the light ray transform on the right is determined up to rescaling of the parameter t, and the defining function is determined by such a factor as well, as we already emphasized.

Remark 2.4 Accounting for the homogeneity of $\mathcal{S}^\sharp$ with respect to the fiber variable, Theorem 2.1(b) implies that the graph of $\mathcal{S}^\sharp$ (locally) coincides with the twisted conormal bundle $N^*\Sigma' \setminus 0$. In particular, $\mathcal{S}^\sharp$ is a symplectic local diffeomorphism. The latter already follows from [49] since the canonical relation of the associated DN map related to the wave equation $\Box_g u = 0$, at least in the (T) case, is the graph of $\mathcal{S}^\sharp$. Under the non-conjugacy condition (NC), that FIO has a Schwartz kernel conormal to Σ, which can be shown using the parametrix.

Corollary 2.1 *The scattering relation $\mathcal{S}^\sharp$ determines r uniquely up to an elliptic factor. On the other hand, each defining function r determines $\mathcal{S}^\sharp$ uniquely.*

Example 1 The Riemannian case: Let $g = -\mathrm{d}t^2 + h(x, \mathrm{d}x)$ on $M = \mathbb{R} \times N$, where (N, h) is a compact Riemannian manifold with convex boundary. Multiplying g by a positive conformal factor $\lambda(t, x) > 0$ leaves $\mathcal{S}^\sharp$ unchanged, see section 3.1, so this example covers the static case, in fact.

Take $r(t, x, s, y) = t - s + \rho(x, y)$. Then $r = 0$ if and only if (t, x) and (s, y) are connected by the lightlike geodesic $\Gamma : [t, s] \ni \sigma \mapsto (\sigma, \gamma(\sigma))$, which we will parameterize by a unit length parameter, where γ is the unique (locally, under the non-conjugacy assumption) unit speed geodesic in the x-space, connecting x and y. We have $\mathrm{d}'_{t,x} r = (1, \mathrm{d}'_x \rho)$, $\mathrm{d}'_{s,y} r = (-1, \mathrm{d}'_y \rho)$, where d' stands for the tangential gradient. Converting them to vectors using the associated Lorentzian metric, we get $-\operatorname{grad}'_{t,x} r = (1, -\operatorname{grad}'_x \rho)$, $\operatorname{grad}'_{s,y} r = (1, \operatorname{grad}'_y \rho)$. Those are exactly the fiber components of the so reduced (normalized) Lorentzian scattering relation; compare with (2.2), see also Theorem 4.1 below.

One can also take $r_1 = -(t-s)^2 + \rho^2(x, y)$. In a neighborhood of a fixed pair on Σ, r_1 equals r times a non-vanishing factor, and is therefore another defining function. The points of any such pair (t, x), (s, y) are connected by a unique (locally) geodesic, not necessarily lightlike. Indeed, one can take γ as above but now $\dot\gamma$ is not unit anymore; we require $|\dot\gamma|_h = \rho(x, y)/(s - t)$. Then $r_1/2$ is the defining function obtained from the energy discussed below.

Proof of Theorem 2.1 To prove (a), notice that for $x \in U$ fixed, $\exp_x(\cdot)$ maps the light cone to a smooth hypersurface in M near x_0 by the non-conjugacy condition. By the transversality assumption, the intersection with V is a codimension one submanifold of V. We can include x in this argument by adding $n - 1$ more dimensions, and complete the proof of (a).

We proceed with the proof of (b). We construct first a defining function using the notion of energy of a curve. For a smooth curve $[a, b] \ni t \to c(t)$, one defines the (non positive or negative definite) energy functional

$$E(c) = \frac{1}{2} \int_a^b (\dot c(t), \dot c(t))_g \, \mathrm{d}t.$$

For a smooth variation $H : [-\varepsilon, \varepsilon] \times [a, b] \to M$ of c (so that $H(\tau, t)|_{\tau=0} = c(t)$), we have the following variational formula

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Big|_{\tau=0} E = (W, \dot c)_g \Big|_a^b - \int_a^b (W, D_t \dot c)_g \, \mathrm{d}t, \tag{2.4}$$

where $W = \partial_s H(0,\cdot)$ is the variation field of H, and D_t is the t-covariant derivative. The proof is the same as in the Riemannian case since it depends on the calculus of the covariant derivatives and it is independent of the signature of the metric. In particular, each geodesic, lightlike or not, is a critical point of E under proper variations (those fixing the endpoints).

Let g_τ be the family of Lorentzian metrics as in the theorem. Let $\gamma_\tau(t)$ be the unique geodesic in the metric g_τ connecting x and y. We parameterize it by $t \in [0,1]$. The associated energy in the metric g_τ is given by

$$E_{g_\tau}(\gamma_\tau) = \frac{1}{2}\int_0^1 (\dot\gamma_\tau(t), \dot\gamma_\tau(t))_{g_\tau}\, \mathrm{d}t.$$

If we fix τ in g_τ on the right (only), the τ-derivative would vanish since each geodesic with fixed endpoints is a critical point of the energy functional. Thus we get

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Big|_{\tau=0} E_{g_\tau}(\gamma_\tau) = \frac{1}{2}\int_0^1 \big\langle f, \dot\gamma_0\otimes\dot\gamma_0\big\rangle\, \mathrm{d}t, \quad f := \frac{\mathrm{d}}{\mathrm{d}\tau}\Big|_{\tau=0} g_\tau. \tag{2.5}$$

With this in mind, we define the following function

$$r(x,y) = E(\gamma_{[x,y]}), \quad (x,y)\in U\times V, \tag{2.6}$$

where $[0,1]\ni t\to\gamma_{[x,y]}(t)$ is the unique (locally) geodesic connecting x and y. We claim that r is a defining function of Σ. Indeed, the integrand in (2.5) with $\gamma_0 = \gamma_{[x,y]}$ there, is constant along $\gamma_{[x,y]}$; it is zero if and only if $\gamma_{[x,y]}$ is lightlike and then $(x,y)\in\Sigma$ by definition. The integrand is negative, if and only if γ_0 is timelike, and then $r<0$, as required. We will show that $\mathrm{d}'_{x,y} r \neq 0$ on Σ as a byproduct of the analysis below.

Let W in (2.4) correspond to variations of $\gamma_{[x,y]}$, see also (2.6), with $x\in U$ fixed and y varying near y_0 in V. Then $(\mathrm{d}/\mathrm{d}\tau) r(x,y(\tau)) = (\partial y/\partial\tau, \dot\gamma(1))_g$ at $\tau = 0$ by (2.4). On the other hand, we have $(\mathrm{d}/\mathrm{d}\tau) r(x,y(\tau)) = \langle \mathrm{d}_y r, \partial y/\partial\tau\rangle$ at $\tau = 0$. Converting $\mathrm{d}_y r$ to a vector by the metric, we see that $\mathrm{grad}_y\, r(x,y)$ and $\dot\gamma(1)$ have the same projection on $T_y V$. Now, we can fix y and vary x to get a similar conclusion but with a negative sign coming from (2.4).

Next, we get $\mathrm{d}'_y r\neq 0$, also $\mathrm{d}'_x r\neq 0$, because $\dot\gamma'(1)\neq 0$ since that projection is either timelike in the (T) case, or spacelike in the (S) case. Any other defining function would be of the type αr, and the factor $\alpha(x,y)$ just multiplies both $-\mathrm{d}_x r$ and $\mathrm{d}_y r$ on $\alpha r = r = 0$. This completes the proof of (b).

Finally, (2.3) with $\kappa = 1/2$ follows from (2.5) when $r = r_\tau$ is as (2.6) for each τ. A different defining function of Σ_τ would be of the form $r_1 = \kappa_\tau r_\tau$ with some elliptic κ_τ, and its τ derivative at $\tau = 0$ just gains the factor κ_0 (multiplied by $1/2$). This proves (c). □

3 Gauge Invariance of the Scattering Relation and of the Light Ray Transform

In this section, we collect some facts about the two maps which are known in general, see, e.g., [19] but not necessarily clearly formulated elsewhere.

There are several obvious groups of transformations which leave the scattering relation invariant, and their linearizations are in the kernel of L.

3.1 Invariance Under Diffeomorphisms

Let ψ be a local diffeomorphism from a neighborhood of γ_0 to its image in M. Assume $\psi|_U = \mathrm{Id}$, $\psi|_V = \mathrm{Id}$ in the setup in section 2. Then $\mathcal{S}_g = \mathcal{S}_{\psi^* g}$ and $\mathcal{S}_g^\sharp = \mathcal{S}_{\psi^* g}^\sharp$ in a trivial way.

The linear counterpart of this is the following. Assume we have a smooth one-parameter family of diffeomorphisms ψ_τ fixing U and V near $\tau = 0$ with $\psi_0 = \mathrm{Id}$. Then for $g_\tau := \psi_\tau^* g$ we have $(\mathrm{d}/\mathrm{d}t) g_\tau|_{\tau=0} = 2\mathrm{d}^s v$, where $v = (\mathrm{d}/\mathrm{d}t)\psi_\tau|_{\tau=0}$. Note that $v = 0$ on U and on V. Moreover, all v's like that are possible linearizations of one-parameter groups of diffeomorphisms fixing U and V. Therefore, we get that $L(\mathrm{d}^s v) = 0$ for all such v's. This can be verified independently by applying the Fundamental Theorem of Calculus to the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}(v, \dot\gamma(t))_g = \langle \mathrm{d}^s v, \dot\gamma \otimes \dot\gamma \rangle,$$

where on the right, v is identified with its covector version by lowering the indices. This is a well known fact in tensor tomography on Riemannian manifolds, at least.

3.2 Invariance Under Conformal Changes

Let $\tilde g = c(x) g$, where $c(x) > 0$ is a smooth function. Then lightlike/spacelike/timelike vectors or covectors in the metric g are such in the metric $\tilde g$ as well. Moreover, the lightlike geodesics in the metric g remain lightlike in the metric $\tilde g$ as well as curves but with a changed parameterization preserving the direction (in general, they do not solve the geodesic equation).

The easiest way to prove this is to pass to the Hamiltonian formalism. An alternative proof is given in [19, Lemma 6.1]. With $H = \frac{1}{2} g^{ij}(x)\xi_i \xi_j$, the Hamiltonian curves $(x(s), \xi(s))$ at the level $H = 0$, transformed into curves on the tangent bundle coincide with the lightlike geodesics. For $\tilde H$, associated with $\tilde g$, we have the system

$$\dot{\tilde x}^i = c^{-1} g^{ij} \tilde\xi_j, \quad \dot{\tilde\xi}_i = -\frac{1}{2} c^{-1} \partial_{x^i} g^{kl} \tilde\xi_k \tilde\xi_l - \frac{1}{2} (\partial_{x^i} c^{-1}) g^{ij}(x) \tilde\xi_i \tilde\xi_j.$$

In addition, we have initial conditions $(x(0), \xi(0)) = z \in \mathcal{U}$; we will denote then the solutions by $(x(s, z), \xi(s, z))$. On $\{\tilde H = 0\} = \{H = 0\}$, the last summand vanishes; therefore we are left with

$$\dot{\tilde{x}}^i = c^{-1} g^{ij} \tilde{\xi}_j, \quad \dot{\tilde{\xi}}_i = -\frac{1}{2} c^{-1} \partial_{x^i} g^{kl} \tilde{\xi}_k \tilde{\xi}_l. \tag{3.1}$$

Assume initial conditions at $s = 0$. Let α solve

$$\dot{\alpha}(s, z) = c^{-1}(x(s, z)), \quad \alpha(0, z) = 0,$$

where $z \in \mathcal{U}$ is the initial condition, $x(s, z)$ is the x-component of the solution of (3.1) with $c = 1$ (those quantities have no tildes over them). Then $(\mathrm{d}/\mathrm{d}s)x(\alpha(s, z)) = c^{-1}\dot{x}$, and similarly for $\xi(\alpha(s, z))$, with initial condition z at $s = 0$. Comparing this to (3.1), we conclude $(\tilde{x}, \tilde{\xi}) = (x, \xi) \circ \alpha(s, \cdot)$. Note that this conclusion presumes the same initial conditions at $s = \tilde{s} = 0$.

We show next that $\mathcal{S}^\sharp = \tilde{\mathcal{S}}^\sharp$. This follows from the fact that for (x, ξ) so that $(x, \xi^\sharp)$ is as in Definition 2.1, the exit points and codirections related to g and $\hat{g}$ are independent of the re-parameterization but they happen for possibly different values of s: s_0 and $\tilde{s}_0$ so that $\alpha(\tilde{s}_0) = s_0$.

On the other hand, one can write $\mathcal{S} = g^{-1} \circ \mathcal{S}^\sharp \circ g$; therefore, $\tilde{\mathcal{S}} = c^{-1} g^{-1} \circ \tilde{\mathcal{S}}^\sharp \circ g c = c^{-1} g^{-1} \circ \mathcal{S}^\sharp \circ g c = c^{-1} \circ \tilde{\mathcal{S}} \circ c$, where g denotes the operator $(x, v) \mapsto (x, gv)$, similarly for the other multiplication operators there. Therefore, $\mathcal{S}$ is *not* invariant under general conformal changes; unless the conformal factor is constant on the boundary; then it is.

To linearize this, assume $g_\tau = c_\tau g$. Then the linearization of g_τ at $\tau = 0$ is $c(x)g$ with $c = (\mathrm{d}c/\mathrm{d}\tau)c_\tau|_{\tau=0}$. Thus $L(cg) = 0$. This is obvious by itself since the integrand in (1.2) vanishes pointwise when $f = cg$.

3.3 Lens Rigidity and Light Ray Transform Injectivity Formulations

With the above in mind, we can formulate the scattering rigidity problem as follows. Show that $\mathcal{S}^\sharp_{g_1} = \mathcal{S}^\sharp_{g_2}$ implies $g_2 = c\psi^* g_1$ with $c > 0$, and a diffeomorphism ψ fixing U and V. For $\mathcal{S}$, we need to add the additional requirement $c = \text{const.}$ on ∂M. We are vague on purpose here about the assumptions and the region we expect to prove that equality since we have various cases even in the Riemannian case.

The injectivity of L under the gauge can be formulated like this: show that under some assumptions, $Lf = 0$ implies $f = \mathrm{d}^s v + \lambda g$, where $v = 0$ on U and on V, and λ is a scalar function.

Both problems are open, and in section 4, we will consider the special case of stationary metrics for the scattering rigidity problem.

4 Rigidity of Stationary Metrics

We consider stationary metrics in this section. We refer to [13, 52] for a justification of the interest in such metrics.

4.1 Stationary Spacetime Geometry

4.1.1 Stationary Metrics

In $\mathbb{R}^{1+n}$, consider metrics of the form

$$g = -\lambda(x)\mathrm{d}t^2 + 2\tilde{\omega}_j(x)\mathrm{d}t\,\mathrm{d}x^j + \tilde{h}_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j \tag{4.1}$$

with $\lambda > 0$, $\tilde{\omega} = \tilde{\omega}_j \mathrm{d}x^j$ an 1-form in $\mathbb{R}^n$, and $\tilde{h}$ a symmetric tensor on $\mathbb{R}^n$; all time-independent. In matrix form,

$$g = \begin{pmatrix} -\lambda & \tilde{\omega}_1 & \dots & \tilde{\omega}_n \\ \tilde{\omega}_1 & \tilde{h}_{11} & \dots & \tilde{h}_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\omega}_n & \tilde{h}_{n1} & \dots & \tilde{h}_{nn} \end{pmatrix}.$$

Since we want g to be Lorentzian, it is convenient to complete the square, and after useful rescaling of ω and h by λ, write g in the form

$$g = \lambda(x)\left(-\left(\mathrm{d}t + \omega_j(x)\mathrm{d}x^j\right)^2 + h_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j\right) \tag{4.2}$$

with $h_{ij} = \lambda^{-1}\tilde{h}_{ij} + \lambda^{-2}\tilde{\omega}_i\tilde{\omega}_j$, assumed positive definitive, and $\omega = -\lambda^{-1}\tilde{\omega}$. Occasionally, we will use the notation $g_{\lambda,\omega,h}$ for a metric of the kind (4.2). The metric h would be positive definitive, if $\tilde{h}$ is positive definite as well. Given a Lorentzian manifold, one can derive that form of the metric globally as well, where ω and λ are invariantly defined, from abstract assumptions of global hyperbolicity, and the existence of a complete timelike Killing field, see, e.g, [13, 50]. We are not going to go into details of that and just will assume that our Lorentzian manifold is $\mathbb{R}_t \times N_x$, with (N, h) Riemannian, and that the Lorentzian metric there is given by (4.2) locally, which is actually a global definition assuming ω a well-defined one-form on N.

The dot product $(\cdot,\cdot)_g$ can be derived from (4.1) by polarization. For $v = (v_t, v_x)$, $w = (w_t, w_x)$, we get

$$(v, w)_g = \lambda\big(-(v_t + \langle\omega, v_x\rangle)(w_t + \langle\omega, w_x\rangle) + \langle h, v_x \otimes w_x\rangle\big). \tag{4.3}$$

4.1.2 Invariance of the Scattering Relation and Gauge Equivalence

There are two obvious groups of diffeomorphisms keeping g in the form (4.2), and keeping the scattering relation $\mathcal{S}^\sharp$ the same. First, for any diffeomorphism $\psi : N \to N$ fixing ∂N pointwise, setting $\Psi := \mathrm{Id} \otimes \psi$, we have that $\Psi^* g$ is of the same form with h, ω and λ replaced by $\psi^* h$, $\psi^*\omega$ and $\psi^*\lambda$. Next, it is easy to see that adding an exact form to ω provides an isometric metric. Indeed, let $\phi(x)$ be a smooth function. Then with $\tilde{\omega} := \omega + \mathrm{d}\phi$, we have

$$\mathrm{d}t + \omega_j(x)\mathrm{d}x^j = \mathrm{d}(t - \phi(x)) + \tilde{\omega}_j(x)\mathrm{d}x^j,$$

therefore, keeping x the same and doing the change

$$t' = t - \phi(x) \tag{4.4}$$

yields a metric like (4.2) but with $\tilde{w}$ instead. As we see below, at least locally, only $\mathrm{d}\omega$ matters for the projections of the lightlike geodesics on M, consistent with the observation we just made. Therefore, for the diffeomorphism $\Phi(t, x) = (t - \phi(x), x)$, we get $\Phi^* g$ of the same form with h and λ the same, and ω replaced by $\omega + \mathrm{d}\phi$. Since we want this diffeomorphism to fix $\mathbb{R} \times \partial N$ pointwise, we will require $\phi = 0$ on ∂N. This observation also shows that while there is a well defined time direction $\partial/\partial t$ preserving the geometry, roughly speaking (a Killing vector field), there is no natural time variable t, say defined up to translations and time reversal since we have time shifts depending on x preserving the form of the metric as well.

Finally, a conformal factor $\mu > 0$ keeps the scattering relation $\mathcal{S}^\sharp$ intact, as well. This suggests the following.

Definition 4.1 The metrics $g_{\lambda,\omega,h}$ and $g_{\hat{\lambda}}, \hat{\omega}, \hat{h}$ are called gauge equivalent, if there exists a diffeomorphism $\psi : N \to N$ fixing ∂N pointwise, and a function ϕ vanishing on ∂N, so that

$$\hat{\omega} = \psi^*(\omega + \mathrm{d}\phi), \quad \hat{h} = \psi^* h. \tag{4.5}$$

The transformations (4.5) determined by (ψ, ϕ) form a group with generators the two elementary transformations above. Application of (ψ_1, ϕ_1), and then (ψ_2, ϕ_2) is of the same kind with $\psi = \psi_2^* \circ \psi_1^*$, $\phi = \phi_2 + \psi_1^* \phi_1$.

Since we are interested in scattering rigidity (related to lightlike geodesics), by the results of section 3, we can replace g by $\lambda^{-1} g$ having the same scattering relation. Thus without loss of generality we can assume $\lambda = 1$. Then

$$g = -\Big(\mathrm{d}t + \omega_j(x)\mathrm{d}x^j\Big)^2 + h_{ij}(x)\mathrm{d}x^i \mathrm{d}x^j, \tag{4.6}$$

and lightlike vectors $v = (v_t, v_x)$ are characterized by

$$-(v_t + \langle \omega, v_x \rangle)^2 + |v_x|_h^2 = 0. \tag{4.7}$$

It is convenient to normalize the parameterization along the lightlike rays $(t(\sigma), x(\sigma))$ by requiring $|\dot{x}|_h = 1$.

4.1.3 Orthogonal Projection on the Boundary

There is a natural projection $\pi : M = \mathbb{R} \times N \to N$, invariant under changes (4.4), and N can be considered as the manifolds of orbits generated by the Killing vector field $\partial/\partial t$, see, e.g., [50]. It generates a projection $\mathrm{d}\pi$ between the tangent bundles. It is useful to understand orthogonal projections to $\mathbb{R} \times \partial N$ next. Let $(x, v) \in TN$ be such that $x \in \partial N$, and v is pointing to the exterior of $\mathbb{R} \times N$. Eventually, we

will apply this to (y, w) and to $(x, -v)$ in the notation of the scattering relation. Let ν_x be the exterior unit normal to ∂N in the metric h. It is straightforward to show, using (4.3), that $\nu := (-\langle \omega, \nu_x \rangle, \nu_x)$ is normal to $\mathbb{R} \times \partial N$, spacelike in particular, exterior, and unit in the sense $(\nu, \nu)_g = 1$. The projection v' under question is given by $v' = v - (v, \nu)_g \nu$ with ν as above. We have $(v, \nu)_g = (v_x, \nu_x)_h$, therefore, in local coordinates,

$$v' = \left(v_t + \langle \omega, \nu_x \rangle (v_x, \nu_x)_h, v'_x\right), \tag{4.8}$$

where v'_x is just the orthogonal projection of v_x on $T(\partial N)$.

We will further decompose v' in the following way. We write

$$v' = [v'_t, v'_x], \tag{4.9}$$

where $v'_t = -(v', \partial/\partial t)_{g'}$ is the (scalar) orthogonal projection of v' to $\partial/\partial t$ in the induced metric g', and $v'_x = (\mathrm{d}\pi)v'$. It is easy to see that v'_x is as before, which explains the same notation but the emphasis now is that it has an invariant meaning. For v'_t, we get

$$v'_t = v_t + \langle \omega, \nu_x \rangle (v_x, \nu_x)_h + \langle \omega', v'_x \rangle = v_t + \langle \omega, v_x \rangle,$$

where ω' is ω restricted to ∂M. We want to emphasize that (4.8) depends on (h, ω) but (4.9) does not.

Finally, if v is lightlike and future pointing, then $v'_t = |v_x|$ by (4.7), and we can assume that this equals one without a loss of generality.

It is useful to introduce boundary normal coordinates as in the proposition below.

Proposition 4.1 *Let $(t_0, x_0) \in \mathbb{R} \times \partial M$. Then there exist local coordinates (t, x) in $\mathbb{R} \times M$ near (t_0, x_0) in which g in (4.6) takes the form*

$$g = -(\mathrm{d}t + \omega_\alpha(x)\,\mathrm{d}x^\alpha)^2 + h_{\alpha\beta}(x)\mathrm{d}x^\alpha \mathrm{d}x^\beta + (\mathrm{d}x^n)^2$$

with some ω; and $\mathbb{R} \times \partial N$ is given by $\mathbb{R} \times \{x^n = 0\}$. Summation over Greek indices is taken from 1 to $n - 1$.

Proof We put h in boundary normal coordinates first; the construction is well known. Then we seek ϕ so that $(\omega - \mathrm{d}\phi)_n = 0$. The latter is equivalent to $\partial_{x^n}\phi = \omega_n$, which we solve with the initial condition $\phi = 0$ for $x^n = 0$. Then we set $t' = t - \phi(x)$. In the coordinates (t', x), we get the desired form. □

In those coordinates, $\nu = (0, \nu_x)$, $\nu_x = (0, \ldots, 0, 1)$, $v' = (v_t, v'_x)$, $v'_t = v_t + \langle \omega', v'_x \rangle$.

4.2 Reduction to a Magnetic System

We review some results in [15], see also [14]. The geodesics of g as in (4.2) there, projected to N under π, see section 4.1.3 and Fig. 2 below, are characterized as the

integral curves of a certain magnetic system. Note that if we include a non-constant conformal factor, as in (4.1) or (4.2), and we consider not null geodesics only, then the system has an additional electric potential, see also [3].

For the computations below, recall that if α is an one-form, then $d\alpha$ is a two-form satisfying $\langle \mathrm{d}\alpha, X \otimes Y\rangle = \langle \nabla_X \alpha, Y\rangle - \langle \nabla_Y \alpha, X\rangle$, where ∇ is the covariant differential, and this is true independently of the background metric.

It follows from (2.4) that a smooth curve connecting two points is geodesic if and only if it is a critical point of the energy functional without being a minimum or a maximum. Let $W = (T, X)$ be a variation of a geodesic $[0, 1] \ni \sigma \mapsto \gamma(\sigma) = (t(\sigma), x(\sigma))$ fixing the endpoints. The energy form takes the form

$$E = \int_0^1 \Big(-\big(\dot t + \langle \omega, \dot x\rangle\big)^2 + |\dot x|_h^2 \Big)\, \mathrm{d}\sigma.$$

Taking a variation of this, we get

$$0 = \int_0^1 \big(-2\big(\dot t + \langle \omega, \dot x\rangle\big)\big(\dot T + \langle \nabla_X \omega, \dot x\rangle + \langle \omega, \dot X\rangle\big) + 2(\dot x, \dot X)_h\big)\, \mathrm{d}\sigma.$$

We have $\langle \omega, \dot X\rangle = (\mathrm{d}/\mathrm{d}\sigma)\langle \omega, X\rangle - \langle D_\sigma \omega, X\rangle$ (where $D_\sigma = \nabla_{\dot x}$), thus

$$\begin{aligned}\langle \nabla_X \omega, \dot x\rangle + \langle \omega, \dot X\rangle &= \langle \nabla_X \omega, \dot x\rangle + \frac{\mathrm{d}}{\mathrm{d}\sigma}\langle \omega, X\rangle - \langle D_\sigma \omega, X\rangle \\ &= \frac{\mathrm{d}}{\mathrm{d}\sigma}\langle \omega, X\rangle + \langle -\mathrm{d}\omega, \dot x \otimes X\rangle.\end{aligned}$$

After some integration by parts we obtain

$$0 = \int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}\sigma}\big(\dot t + \langle \omega, \dot x\rangle\big)(T + \langle \omega, X\rangle) + \big(\dot t + \langle \omega, \dot x\rangle\big)\, \langle \mathrm{d}\omega, \dot x \otimes X\rangle - (D_\sigma \dot x, X)_h \right) \mathrm{d}\sigma.$$

Since this is true for every perturbation (T, X), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\sigma}\big(\dot t + \langle \omega, \dot x\rangle\big) &= 0, \\ D_\sigma \dot x^j &= \big(\dot t + \langle \omega, \dot x\rangle\big)\, (\mathrm{d}\omega)_i{}^j \dot x^i.\end{aligned}$$

Therefore, $\dot t + \langle \omega, \dot x\rangle = k = \text{const}$. This can also be interpreted as $(\partial/\partial t, \dot\gamma)_g = \text{const}$. for every geodesic, i.e., the energy of a particle is constant for all stationary observers. We get

$$\dot t + \langle \omega, \dot x\rangle = k, \tag{4.10}$$

$$D_\sigma \dot x = kY\dot x, \tag{4.11}$$

where $Y : TN \to TN$ is given by $(Yu)^i = (\mathrm{d}\omega)_i{}^j u^i$, in other words, $\langle Yu, v\rangle_h = \langle \mathrm{d}\omega, u \otimes v\rangle$ for every two vector fields u and v.

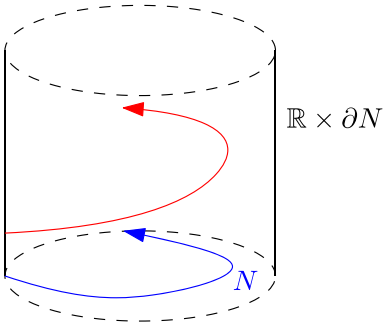

Fig. 2 A lightlike geodesic and its projection $x(\sigma)$ on N

Now we are in the framework of [10], see also Appendix A. Equation (4.11) is Newton's law of magnetic geodesics with Lorentzian force kY. Since the operator Y is anti-symmetric, we have $|\dot{x}|_h = m = \text{const.}$, i.e., σ is proportional to the arc-length parameter on the base N, which in general is not (proportional to) the time t. The constants k and m are σ-independent but they may change from geodesic to geodesic, in principle. For $\gamma = (t(\sigma), x(\sigma))$, we have $(\dot{\gamma}, \dot{\gamma})_g = -k^2 + m^2$. If γ is lightlike and future pointing, we can take $k = m = 1$ after replacing σ by $k\sigma$. When γ is not lightlike, we can have $k = 0$. Then $x(\sigma)$ is a geodesic in (N, h).

Next, integrating (4.10), we get

$$t(\sigma) + \int_{\sigma_0}^{\sigma} \langle \omega, \dot{x} \rangle - k\sigma = \text{const.} \tag{4.12}$$

In the case $k = 1$ which we really need below, this shows that t is actually an action variable, see (A.1).

We have thus proved part (a) of Theorem 4.1 below. In preparation to formulate part (b), (c), denote by $\mathcal{S}_{\text{mag}} : BN \to BN$ the scattering relation related to the magnetic system (4.11) with $k = 1$ there, which fixes unit speed along the magnetic geodesics. Let $\ell_{\text{mag}} : BN \to \mathbb{R}_+$ be the travel time, which is also the length of the magnetic geodesic inside N in the metric h. The boundary action function $\mathbb{A}$ is defined in Appendix A.

Recall that we assumed $\lambda = 1$ after Definition 4.1. Since we will use the theorem below for $\mathcal{S}^\sharp$, which is independent of λ, this is enough for our purposes. The theorem is easier to formulate for $\mathcal{S}$ however.

Theorem 4.1 *Let (N, h) be a compact Riemannian manifold with boundary, let ω be an one-form on M, and assume that N is magnetically simple for the magnetic system (h, ω). Let $M = \mathbb{R} \times N$ be equipped with the stationary Lorentzian metric*

$$g = -(\mathrm{d}t + \langle w, \mathrm{d}x \rangle)^2 + h(x, \mathrm{d}x).$$

Then we have the following.

(a) The lightlike geodesics $(t(\sigma), x(\sigma))$ solve (4.10), (4.11) with k independent of σ reflecting the freedom of affine changes of σ. Future propagation corresponds to $k > 0$. For the magnetic geodesics $x(\sigma)$ we have $|\dot{x}|_h = |k|$.

(b) With the notation in (4.9), if

$$\mathcal{S}\left(t, x, [v_t', v_x']\right) = \left(s, y, [w_t', w_x']\right), \tag{4.13}$$

with v' normalized so that $v_t' = 1$, then $w_t' = 1$ as well, and

$$\mathcal{S}_{\rm mag}(x, v_x') = (y, w_x'), \quad \ell_{\rm mag}(x, v_x') = s - t + \int_{\gamma_{x,v'}} \omega, \tag{4.14}$$

and in particular,

$$\mathbb{A}(x, y) = s - t \tag{4.15}$$

where $v \in T_x M$ is the incoming lightlike vector with projection v', and y is the exist point on ∂M of the magnetic geodesic issued from (x, v). (c) Each one of the three quantities determines the other two: $\mathcal{S}$, $\mathcal{S}_{\rm mag}$, and $\mathbb{A}$.

We note that simplicity is only needed to define $\mathbb{A}(x, y)$, and for a part of (c).

4.3 Reduction of the Scattering Relation to a Magnetic One

End of the proof of Theorem 4.1 By (4.10), (4.11), for the geodesic $\gamma_{x,v'} = (x(\sigma), t(\sigma))$ issued from (x, v) with projection v', normalized as in the theorem, we have $k = 1 = |\dot{x}|_h$ in (4.10), (4.11). Projecting on N, we get the first identity in (4.14). For the second one, we refer to (4.12): with $k = 1$ there and an initial condition $t(\sigma) = t$ (the t on the right is the initial moment in (4.13) and the one on the left is the zeroth component of $\gamma_{x,v'}$), we get that $s - t$ is equal to the action along the ray, which proves the rest of the (b) statement.

The proof of (c) is similar to that in the Riemannian case. That $\mathcal{S}_{\rm mag}$ and $\mathbb{A}$ determine each other is proved in [10], and the proof is similar to the arguments in section 2.1. Knowing $\mathcal{S}$, we recover $\mathcal{S}_{\rm mag}$ by part (b). Finally, knowing one of $\mathcal{S}_{\rm mag}$ and $\mathbb{A}$, we recover the other as well, and then we recover $\mathcal{S}$ by (4.14) and (4.15). □

4.4 Rigidity Results

We formulate some rigidity results as a consequence of the equivalence between the rigidity problems for stationary Lorentzian metrics and magnetic systems established in Theorem 4.1; and from the magnetic rigidity results in [10], see the appendix. We call (M, g) simple, if the projected magnetic system is simple, see the appendix. In the next statements, g is of the form (4.2), i.e., $g = g_{\lambda,\omega,h}$. Given another such metric $\hat{g}$, all associated quantities are decorated with a hat over them.

Theorem 4.2 *Let (M, g) and $(M, \hat{g})$ be simple and stationary. Then $\mathcal{S}^\sharp = \hat{\mathcal{S}}^\sharp$ implies that g and $\hat{g}$ are gauge equivalent if and only if $\mathbb{A} = \hat{\mathbb{A}}$ implies that (h, ω) and $(\hat{h}, \hat{\omega})$ are magnetically gauge equivalent. In particular, the simple magnetic system (N, h, ω)*

is boundary rigid if and only if the stationary Lorentzian manifold $(M, g_{\lambda,\omega,h})$ *is boundary rigid in its class.*

Corollary 4.1 *(Rigidity in* $1+2$ *dimensions) Simple stationary manifolds* (M, g) *of dimension* $\dim M = 1+2$ *are lens rigid.*

Corollary 4.2 *(Rigidity in a given conformal class) Let* (M, g) *and* $(M, \hat{g})$ *be two simple stationary Lorentzian systems so that* $\hat{h} = \mu(x)h$ *with some* $\mu > 0$. *If* $\mathcal{S}^\sharp = \hat{\mathcal{S}}^\sharp$, *then* $\mu = 1$ *and* $\hat{\omega} = \omega + \mathrm{d}\phi(x)$ *with some* ϕ *vanishing on* ∂N.

For the definition of the class $\mathcal{G}^k$ used in next theorem, we refer to Definition A.1.

Corollary 4.3 *(Generic local rigidity) There exists* $k \geq k_0$ *so that for every* $(h_0, \omega_0) \in \mathcal{G}^k$, *there exists* $\varepsilon > 0$ *such that for every two simple stationary metrics* $g = g_{\lambda,\omega,h}$ *and* $\hat{g} = g_{\hat{\lambda},\hat{\omega},\hat{h}}$ *for each of which* (h, α), $(\hat{h}, \hat{\alpha})$ *is an* ε *close to* (h_0, α_0) *in* $C^k(N)$, *we have the following:*

$$\mathcal{S}^\sharp = \hat{\mathcal{S}}^\sharp$$

implies that $\hat{g}$ *and* g *are gauge equivalent.*

Remark 4.1 The theorems state no conclusions about λ because the latter does not affect $S^\sharp$. On the other hand, reformulating them in terms of $\mathcal{S}$ and $\hat{\mathcal{S}}$ (on the tangent bundle) would require adding the requirement $\lambda = \text{const.}$ on ∂N to the notion of gauge equivalence. Then we can replace λ by that constant, and by homogeneity, we can replace it by $\lambda = 1$; which leads to the same results.

4.5 The Defining Function of Σ for Stationary Metrics

Although we did not need to resort to a defining function r in the stationary case, it would be interesting to see how the general theory developed in section 2 applies here. We will use the energy, see (2.6) as a definition of r. For (x, y) close to some $(x_0, y_0) \in \Sigma$, we have $|r(x, y)| \ll 1$, on the other hand, $r = -k^2 + m^2$, where k and m are as in (4.10), (4.11), and the remarks after it, related to the geodesic $\gamma_{[x,y]}(\sigma)$ parameterized by $\sigma \in [0, 1]$. Since $|\dot{x}|_h = m$, we have $m = \ell_{x,y}$. Integrating (4.10) along γ, we get

$$k = (s - t) + \int_{\gamma_{[x,y]}} \omega.$$

Therefore,

$$r(t, x, s, y) = \frac{1}{2}\ell_{x,y}^2 - \frac{1}{2}\Big((s - t) + \int_{\gamma_{[x,y]}} \omega\Big)^2.$$

Up to a positive (near $r = 0$) factor, we can replace r by the defining function

$$r_1(t, x, s, y) = -(s - t) + \ell_{x,y} - \int_{\gamma_{[x,y]}} \omega = t - s + \mathbb{A}(x, y).$$

This is a direct generalization of what we got in Example 1 for $g = -\mathrm{d}t^2 + h(x, \mathrm{d}x)$.

A linearization of r_1, by [10, Lemma 3.1], is given by

$$\frac{1}{2}\int \langle \delta h, \dot\gamma \oplus \dot\gamma\rangle - \int_\gamma \delta\omega \tag{4.16}$$

with γ parameterized by an arc-length parameter, i.e., $|\dot\gamma|_h = 1$. We will compare it with the linearization (2.3) in Theorem 2.1(c). By (2.3), we should get

$$\int_0^1 \left\langle f, \dot\gamma_{[x,y]}(\sigma)\otimes\dot\gamma_{[x,y]}(\sigma)\right\rangle \mathrm{d}\sigma \quad \text{on } \Sigma = \{r(x, y) = 0\}, \tag{4.17}$$

where $f = \delta g$ with g as in (4.6). Therefore, dropping the subscript $[x, y]$, we get

$$\langle f, \dot\gamma(\sigma)\otimes\dot\gamma(\sigma)\rangle = -2(\dot\gamma_t + \langle\omega, \dot\gamma_x\rangle)\langle\delta\omega, \dot\gamma_x\rangle + \langle\delta h, \dot\gamma_x\otimes\dot\gamma_x\rangle, \tag{4.18}$$

where the (new) subscripts t and x denote the time and the spatial components of γ, respectively. By (4.10), (4.11) or (4.7), we have $\dot\gamma_t + \langle\omega, \dot\gamma_x\rangle = |\dot\gamma_x|$ for γ future pointing. In (4.18), $\gamma(\sigma)$ is parameterized by $\sigma \in [0, 1]$, thus $|\dot\gamma_{[x,y]}| = \ell_{x,y}$. The rescaling $\tilde\sigma = \ell_{x,y}\sigma$ makes $\tilde\sigma$ an arc-length parameter. Doing this in (4.18), we see that it equals $2\ell_{x,y}^2$ times (4.16). Therefore, the linearization (4.16) for the magnetic rigidity problem obtained in [10] coincides with the linearization (4.17), predicted by Theorem 2.1(c).

Appendix A Some Facts About Magnetic Systems

We recall some notions and results in [10]. On a Riemannian manifold (N, h), we are given a closed two-form Ω, which in our case would be $\Omega = \mathrm{d}\omega$. We define $Y : TN \to TN$ by $\langle\Omega, u\otimes v\rangle = (Yu, v)_h$. Then we consider the Newton-like equation

$$D_s\dot\gamma = Y\dot\gamma.$$

The solution curves $\gamma(\sigma)$ are called magnetic geodesics. An easy calculation yields $|\dot\gamma| = \text{const.}$ along each geodesic. Choosing different values of that constant generates different curves when $\Omega \neq 0$; and we fix $|\dot\gamma| = 1$. Time is not reversible along γ unless $\Omega = 0$.

We call N simple with respect to h and Ω of the magnetic exponential map at every point x is a diffeomorphism to N from its preimage, and if ∂N is strictly convex with respect to the magnetic flow in either direction. Then $\Omega = \mathrm{d}\omega$ with some one-form ω. We view (N, h, ω) as a magnetic system.

For every pair $(x, y) \in N \times N$, one defines the action $\mathbb{A}(x, y)$ by

$$\mathbb{A}(x, y) = \ell_{x,y} - \int_{\gamma_{[x,y]}} \omega, \tag{A.1}$$

where $\gamma_{[x,y]}$ is the unique (by simplicity) unit speed magnetic geodesic from x to y, and $\ell_{x,y}$ is the travel time. The action minimizes a certain time-free action functional. Restricted to $\partial N \times \partial N$, $\mathbb{A}$ is called the boundary action function. It plays the role, and generalizes of the boundary distance function when $\omega = 0$. Two magnetic systems (N, h, ω) and $(N, \hat{h}, \hat{\omega})$ are called gauge equivalent if there exists a diffeomorphism ψ on N fixing ∂N pointwise, and a function ϕ vanishing on ∂N, so that $\hat{h} = \psi^* h$, and $\hat{\omega} = \psi^* \omega + \mathrm{d}\phi$. Gauge equivalent magnetic systems have the same boundary action functions.

One defines the scattering relation $\mathcal{S}_{\mathrm{mag}}$, and the travel time ℓ_{mag} in the same way as we did in the Riemannian case. With the notation $\mathcal{S}_{\mathrm{mag}}(x, v') = (y, w')$, we have the following generalization of (2.2):

$$v' = -\mathrm{d}'_x \mathbb{A}(x, y) + \omega'(x), \quad w' = \mathrm{d}'_y \mathbb{A}(x, y) + \omega'(y),$$

where, as before, primes denote tangential projections, and in particular, ω' is ω restricted to $T(\partial N)$.

A.1 Linearization

A linearization of $\mathbb{A}(x, y)$ is the following X-ray transform

$$I[f, \beta](\gamma) = \int \langle f, \dot{\gamma} \otimes \dot{\gamma} \rangle + \int \beta,$$

see [10, Lemma 3.1], where f is a symmetric two-cotensor field, and β is an one-form, which play roles of perturbations of the background metric h (multiplied by $1/2$), and form $-\omega$. Lift Y to the cotangent bundle by $(Y\omega)_i = -Y_i{}^j \omega_j$, which corresponds to the isomorphism between TN and T^*N.

Based on the obvious gauge invariance of the nonlinear problem, after linearization, we expect $I[f, \beta] = 0$ to imply $h = \mathrm{d}^s v$ with $v = 0$ on ∂N, and $\beta = \mathrm{d}\phi - Yv$ with ϕ a function vanishing on ∂N. We called this property s-injectivity. We proved s-injectivity for simple magnetic systems in the following cases

(i) with an explicit bound of the curvature, following the energy method going back to Mukhometov, Romanov, Pestov and Sharafutdinov;
(ii) in a given conformal class,
(iii) for analytic ones using analytic microlocal analysis,
(iv) locally, near generic ones using the analytic result.

To describe the latter, we need the following definition.

Definition A.1 We define $\mathcal{G}^k$ to be the set of all simple C^k pairs (h, ω) on N with an s-injective magnetic ray transform $I_{h,\omega}$.

We can define a C^k topology on N (independent of a metric h which we eventually impose) by taking and fixing a finite atlas of local maps.

Theorem A.1 *([10, Theorem 4.11]) For some $k_0 > 0$, for every $k \geq k_0$, the set $\mathcal{G}^k$ is open and dense in the set of all C^k pairs (h, β) and contains all real analytic simple pairs.*

The magnetic ray transform is elliptic on the complement of the potential pairs, which allows for a stability estimate, which in turn allows to apply this to the nonlinear problem.

A.2 Rigidity Results

We sketch the rigidity results about simple magnetic system obtained in [10]. We proved boundary determination of the whole jet of h and ω, up to the gauge, first. Next, we showed the following results.

(i) Two-dimensional (simple) magnetic systems are boundary rigid. This was derived generalizing the Riemannian result by Pestov and Uhlmann [31], without a linearization.
(ii) If $\hat{g} = \mu g$ with $\mu > 0$ a function, then equality of the lens data implies $\mu = 1$ and $\hat{\omega}$ is gauge equivalent to ω.
(iii) Real analytic simple magnetic systems with the same lens data are gauge equivalent. This follows from a boundary determination of the jets of h and ω, and then by analytic continuation.
(iv) Generic local rigidity near simple magnetic systems with s-injective linearizations, following [41].

Recovery of a conformal factor and $\mathrm{d}\omega$ from local data near a strictly convex boundary point, and a global result under a foliation condition was proved in [56]. This requires knowledge of the action $\mathbb{A}$ *and* ℓ.

We formulate (iv) in the following.

Theorem A.2 *([10, Theorem 6.5]) There exists $k \geq k_0$ so that for every $(h_0, \omega_0) \in \mathcal{G}^k$, there exists $\varepsilon > 0$ such that for every two magnetic systems (h, ω), $(\hat{h}, \hat{\omega})$, each of which is an ε-close to (h_0, ω_0) in $C^k(N)$, we have the following:*

$$\hat{\mathbb{A}} = \mathbb{A} \quad \text{on } \partial N \times \partial N$$

implies that $(\hat{h}, \hat{\omega})$ and (h, ω) are gauge equivalent.

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