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RESEARCH

Bi-Sobolev boundary singularities

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Abstract
We provide sharp geometric descriptions of boundary singularities in the form of cusps that can be created by a deformation of a hyperelastic body which stores a given bi-Sobolev type energy. Guided by Hookes' Law, we investigate when the deformed configuration returns to its original shape by applying the inverse deformation with the same finite energy.

Keywords Cusp · Bi-Sobolev homeomorphisms · Quasiball

Mathematics Subject Classification Primary 30C65

1 Introduction

Throughout this description $\mathbb{X}$ and $\mathbb{Y}$ are bounded Euclidean domains in $\mathbb{R}^n$ of the same topological type, that is, there is a homeomorphism $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$. A far reaching addition of the Geometric Function Theory (GFT) [2,9,15,16,31] comes from mathematical models of hyperelasticity [1,3,6]. By the very assumptions of hyperelasticity, we enquire into homeomorphisms $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ of smallest *stored energy*

$$\mathsf{E}_{\mathbb{X}}[h] = \int_{\mathbb{X}} \mathbf{E}(x, h, Dh)\, \mathrm{d}x\,, \qquad \mathbf{E} : \mathbb{X} \times \mathbb{Y} \times \mathbb{R}^{n\times n} \to \mathbb{R}_+ \tag{1.1}$$

where the *stored energy function* $\mathbf{E}$ is determined by the mechanical and elastic properties of the materials occupying the configurations. The so-called *bi-p-harmonic energy* serves as a model example,

$$\mathsf{E}^p[h] \overset{\text{def}}{=\!=} \int_{\mathbb{X}} |Dh(x)|^p \,\mathrm{d}x + \int_{\mathbb{Y}} |Dh^{-1}(y)|^p \,\mathrm{d}y\,. \tag{1.2}$$

Hereafter $|\cdot|$ stands for the operator norm of matrices. When $p = n = 2$, the situation is reminiscent of the *Riemann Mapping Problem*. The conformal mappings between simply connected planar domains minimize the *bi-Dirichlet energy* subject to all homeomorphisms $h \in \mathscr{W}^{1,2}(\mathbb{X}, \mathbb{Y})$ sliding freely along the boundary [18]. Such a minimization is known as *frictionless* problem in the theory of Nonlinear Elasticity [3,4,6,7].

As a first step toward the existence of the energy-minimal deformations one must provide an affirmative answer to the following general question.

Question 1.1 Whether or not a pair of two domains $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^n$ of the same topological type admits a homeomorphism $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ of finite $\mathsf{E}_{\mathbb{X}}$-energy?

There is broad literature dealing with related problems in the GFT. Of wide interest are three problems; impose conditions on $\mathbb{X}$ and $\mathbb{Y}$ to ensure that

P1. There exists a bi-Lipschitz deformation $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$.
P2. There exists a quasiconformal mapping $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$.
P3. There exists a deformation $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ of finite bi-n-harmonic energy. In this case the energy-minimal mappings are called *bi-n-harmonics*.

Dedicated to Pekka Koskela on the occasion of his 60th birthday.

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1.1 Bi-Lipschitz singularities

Let $\mathbb{X}$ and $\mathbb{Y}$ be subsets of metric spaces $\mathbf{X}$ and $\mathbf{Y}$, respectively. A map $F : \mathbb{X} \overset{\text{onto}}{\longrightarrow} \mathbb{Y}$ is said to be *bi-Lipschitz* if for all $x_1, x_2 \in \mathbb{X}$ it holds that

$$L^{-1}\text{dist}_{\mathbf{X}}[x_1, x_2] \le \text{dist}_{\mathbf{Y}}[F(x_1), F(x_2)] \le L\, \text{dist}_{\mathbf{X}}[x_1, x_2] \tag{1.3}$$

where the *bi-Lipschitz constant* $L \ge 1$ is independent of the points x_1, x_2. In particular, $\mathbb{X}$ and $\mathbb{Y}$ are of the same topological type. We say that $\mathbb{X}$ and $\mathbb{Y}$ are bi-Lipschitz equivalent. When it is necessary to emphasize the constant L, we say that F is L-bi-Lipschitz. Certainly, the inverse map $F^{-1} : \mathbb{Y} \overset{\text{onto}}{\longrightarrow} \mathbb{X}$ is also L-bi-Lipschitz. A composition of L_1 and L_2 bi-Lipschitz maps is $L_1 L_2$-bi-Lipschitz. It should be noted that if $\mathbf{X}$ and $\mathbf{Y}$ are complete then F extends as a bi-Lipschitz map between the closures of $\mathbb{X}$ and $\mathbb{Y}$ still denoted by $F : \overline{\mathbb{X}} \overset{\text{onto}}{\longrightarrow} \overline{\mathbb{Y}}$. Here we are concerned with domains $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^n$ (open connected subsets). Thus $F : \mathbb{X} \overset{\text{onto}}{\longrightarrow} \mathbb{Y}$ gives rise to a bi-Lipschitz map between the boundaries, again denoted by $F : \partial\mathbb{X} \overset{\text{onto}}{\longrightarrow} \partial\mathbb{Y}$. In particular $\partial\mathbb{X}$ and $\partial\mathbb{Y}$ are of the same topological type.

Remark 1.2 It is possible for two domains $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^n$ to have a Lipschitz homeomorphism $f : \mathbb{X} \overset{\text{onto}}{\longrightarrow} \mathbb{Y}$ and a Lipschitz homeomorphism $g : \mathbb{Y} \overset{\text{onto}}{\longrightarrow} \mathbb{X}$, but not having any bi-Lipschitz map. One of the reasons is that $\partial\mathbb{X}$ and $\partial\mathbb{Y}$ can be of different topological type, see [19,20].

As noted above, a domain $\mathbb{X} \subset \mathbb{R}^n$ can be bi-Lipschitz equivalent to a smooth domain $\mathbb{Y} \subset \mathbb{R}^n$ only when every point $a \in \partial\mathbb{X}$ has a neighborhood (in $\partial\mathbb{X}$) that is bi-Lipschitz equivalent to a domain in $\mathbb{R}^{n-1}$. In this case we say that $\partial\mathbb{X}$ is locally *bi-Lipschitz flat*. However, the converse is far from obvious.

It is generally a highly nontrivial question whether a bi-Lipschitz singularity can be removed by a larger class of deformations naturally determined by the stored energy of the domains. We have in mind the classes which are invariant under bi-Lipschitz change of variables; such are Sobolev mappings. Let us take a quick look at the deformations of finite distortion, including quasiconformal mappings.

1.2 Mappings of finite distortion

A homeomorphism $h : \mathbb{X} \overset{\text{onto}}{\longrightarrow} \mathbb{Y}$ of the Sobolev class $\mathscr{W}^{1,1}_{\text{loc}}(\mathbb{X}, \mathbb{R}^n)$ between domains $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^n$ has *finite distortion* if for some measurable function $1 \le K(x) < \infty$ the following *distortion inequality* holds almost everywhere

$$|Dh(x)|^n \le K(x)\, J(x,h)\,, \quad \text{where} \quad J(x,h) = \det Dh(x)\,. \tag{1.4}$$

It is called K-*quasiconformal* if $K(x) \le K$. The smallest such $K(x)$, denoted by $K_h(x)$, is called the *(outer) distortion function* of h. The concept of mappings of finite distortion has emerged in GFT, going back as far as the paper [13]. It was carried on in a methodical way by starting in the papers [17,24,25], see the monographs [2,15,16].

The inverse of a quasiconformal mapping is again quasiconformal [2,33]. In particular, quasiconformal mapping are *bi-n-harmonics* which are interesting in their own right [18,21,22]. At this point, we mention a useful formula that holds for all homeomorphisms $h \in \mathscr{W}^{1,n}_{\text{loc}}(\mathbb{X}, \mathbb{R}^n)$,

$$\int_{\mathbb{E}} J(x,h)\, \mathrm{d}x = |h(\mathbb{E})|, \quad \text{for every Borel set } \mathbb{E} \subset \mathbb{X}.$$

1.3 The (p, q)-energy

The idea behind our concept of bi-p-harmonic energy at (1.2) is relevant to Hooke's Law; for, it admits the following interpretation. When restoring the original shape of the deformed body the inverse map has also finite bi-p-harmonic energy.

For greater generality, we introduce the following class of deformations.

Definition 1.3 The term (p,q)-*bi-Sobolev homeomorphism* refers to an invertible mapping $h : \mathbb{X} \overset{\text{onto}}{\longrightarrow} \mathbb{Y}$ of Sobolev class $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{R}^n)$ whose inverse $f \overset{\text{def}}{=\!=} h^{-1}$ belongs to $\mathscr{W}^{1,q}(\mathbb{Y}, \mathbb{R}^n)$, where $1 \le p, q \le \infty$. In case $p, q < \infty$ we define the associated (p,q)-*energy* by the rule

$$\mathsf{E}^{p,q}_{\mathbb{X},\mathbb{Y}}[h] = \mathsf{E}^{q,p}_{\mathbb{Y},\mathbb{X}}[f] = \mathsf{E}^{p,q}_{\mathbb{X},\mathbb{Y}}[h, f] \overset{\text{def}}{=\!=} \int_{\mathbb{X}} |Dh(x)|^p\, \mathrm{d}x + \int_{\mathbb{Y}} |Df(y)|^q\, \mathrm{d}y < \infty \tag{1.5}$$

In case $p = q$, we call it *bi-p-harmonic energy*.

The following implications are straightforward from the very definitions of the mappings in question.

$$\boxed{\text{P1.} \implies \text{P2.} \implies \text{P3.}}$$

All the above-mentioned classes of deformations are invariant under bi-Lipschitz change of variables in both $\mathbb{X}$ and $\mathbb{Y}$, so the flatness and singularities can be recognized by way of bi-Lipshitz equivalence with the model singularities.

1.4 Model cusps

To set up a cusp, we begin with a function $u : [0, T] \overset{\text{onto}}{\longrightarrow} [0, M)$, which is continuous and strictly increasing from $u(0) = 0$ to $u(T) = M$. The inverse function $v \overset{\text{def}}{=\!=} u^{-1} : [0, M] \overset{\text{onto}}{\longrightarrow} [0, T]$ is also strictly increasing. Our standing assumption is that

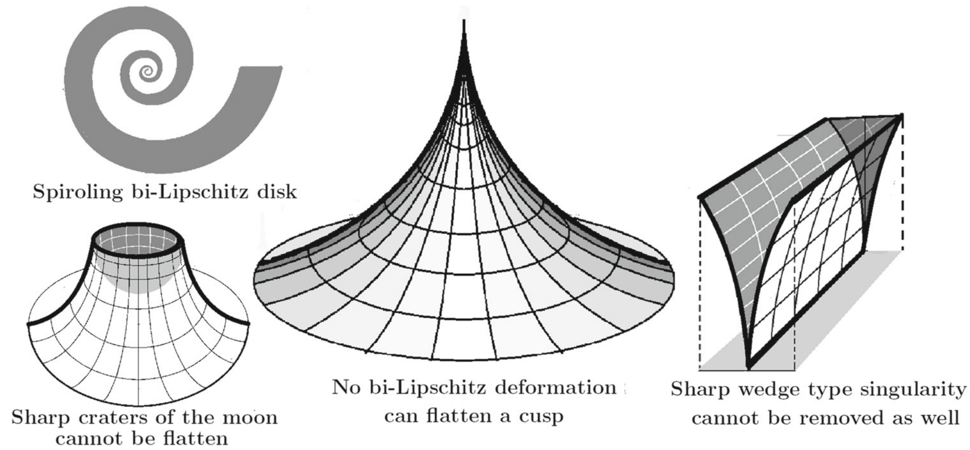

Fig. 1 Illustration of possible singularities

$$\lim_{t\searrow 0}\frac{u(t)}{t}=0\ ;\ \text{equivalently,}\ \lim_{s\searrow 0}\frac{v(s)}{s}=\infty\,. \tag{1.6}$$

Thus we have a Jordan arc in the (t,s) -plane

$$\prec\ \stackrel{\text{def}}{=\!=}\ \{(t,s)\ ;\ 0\le t\le T\ ,\ s=u(|t|)\,\} \tag{1.7}$$

which exhibits a cusp at $(0,0)$. Few of various model singularities may be constructed using this arc called *generatrix*, see Fig. 1.

- A *cone-like cusp*, briefly a cusp, is created by rotating $\prec$ around its axis of symmetry. This is an $(n-1)$-dimensional surface in $\mathbb{R}^n$,

$$\mathbf{S}_u\ \stackrel{\text{def}}{=\!=}\ \{(t,x)\in[0,T]\times\mathbb{R}^{n-1}\,;\quad |x|=u(t)\,\}\subset\mathbb{R}^+\times\mathbb{R}^{n-1}\,. \tag{1.8}$$

- When the axis of rotation is parallel and sufficiently far from the axis of symmetry, then a wedge-like cusp is formed with singularity along a circle.
- A straight line wedge-like cusp is just the Cartesian product of $\prec$ and an interval $\mathbf{I}\subset\mathbb{R}$ or, more generally, $\prec\times\mathbf{I}^{n-2}$.

1.5 Inward versus outward cusps

We shall further scrutinize the domains with power cone-like cusps as their boundaries. Throughout this context we assume that the exponent $\beta>1$ and consider $u(t)=t^\beta$. To simplify the notation we write $\mathbf{S}_\beta=\mathbf{S}_u\in\mathbb{R}^+\times\mathbb{R}^{n-1}$, that is,

$$\mathbf{S}_\beta=\{(t,x)\in[0,\infty)\times\mathbb{R}^{n-1}:|x|=t^\beta\}\,. \tag{1.9}$$

Definition 1.4 The *β-power inward cusp* in a ball is defined and denoted by

$$\mathbb{B}^{\prec}_{\beta}\ \stackrel{\text{def}}{=\!=}\ \mathbb{B}^n(0,1)\setminus\{(t,x)\in\mathbb{R}^+\times\mathbb{R}^{n-1}:|x|\le t^\beta\}. \tag{1.10}$$

Similarly, the *β-power outward cusp* is

$$\mathbb{B}^{\succ}_{\beta}\ \stackrel{\text{def}}{=\!=}\ \mathbb{B}^n((2,0),\sqrt{2})\cup\{(t,x)\in(0,1]\times\mathbb{R}^{n-1}:|x|<t^\beta\}\,. \tag{1.11}$$

The number $\beta>1$ measures the degree of sharpness of the cusp; the larger the value of β the sharper the cusp is (Fig. 2).

As noted above there exists a Lipschitz homeomorphism $f:\mathbb{B}^{\prec}_{\beta}\overset{\text{onto}}{\longrightarrow}\mathbb{B}$ and a Lipschitz homeomorphism $h:\mathbb{B}\overset{\text{onto}}{\longrightarrow}\mathbb{B}^{\prec}_{\beta}$. Such Lipschitz homeomorphisms even exist in the limit case when $\beta=\infty$; that is, in the case of the unit ball with a cut along the straight line segment $\mathbf{I}\stackrel{\text{def}}{=\!=}\{(t,0,\dots,0):0\le t\le 1\}$, see [20]. However, there is no bi-Lipschitz homeomorphism from the unit ball $\mathbb{B}$ onto any cusp domain $\mathbb{B}^{\prec}_{\beta}$ or $\mathbb{B}^{\succ}_{\beta}$ with exponent $\beta>1$.

A bounded domain $\mathbb{Y}\subset\mathbb{R}^n$ which is a quasiconformal image of the unit ball $\mathbb{B}\subset\mathbb{R}^n$ is called *quaisiball*. When $n=2$ the Riemann Mapping Theorem characterizes quasiballs as simply connected domains. It is, however, a highly nontrivial question when a domain $\mathbb{Y}\subset\mathbb{R}^n$ is a quasiball

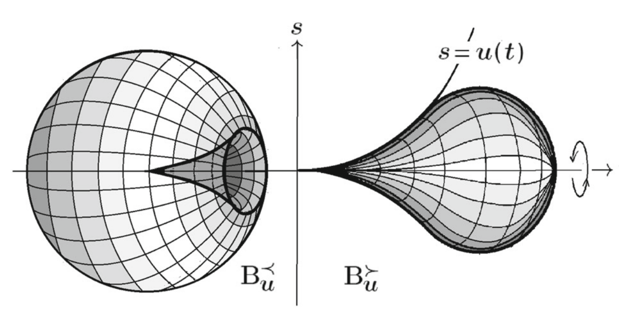

Fig. 2 Inward and outward cusp domains

if $n \geq 3$. Among geometric obstructions are the inward cusps [10], see also [9]. A ball with outward cusp, however, is always a quasiball. Precisely, Gehring and Väisälä [10] proved that any outward cusp domain is a quasiball, whenever the cusp function u is Lipschitz. In particular, such outward cups domains are bi-n-harmonic energy equivalent with the unit ball $\mathbb{B} \subset \mathbb{R}^n$. It turns out that any power-type cups domain is equivalent with the unit ball $\mathbb{B}$ through finite bi-n-harmonic energy. Actually, we proved in [22] that there exists a homeomorphism from the unit ball $\mathbb{B} \subset \mathbb{R}^n$, $n \geq 3$ onto an inward cusp domain $\mathbb{B}_u^{\prec}$ whose generatrix arc is given by

$$u(t) = \frac{e}{\exp\left(\frac{1}{t}\right)^{\alpha}} \quad \text{for } 0 \leq t \leq 1\,, \quad \text{where } 0 < \alpha < n\,. \tag{1.12}$$

We summarize the above observations about power-type cusps domains in the following table. For the definitions of equivalencies P1., P2., and P3. we refer to the lines just after Question 1.1.

Equivalency	Inward cusp domain	Outward cusp domain
P1.	No	No
P2.	No	Yes
P3.	Yes	Yes

Planar cusp domains are standard examples of Jordan domains which are not quasidisks. A planar domain is a *quasidisk* if is the image of an open disk under a quasiconformal self mapping of $\mathbb{C}$. A question whether or not a planar domain with singular boundary is a generalized quasidisk under a global mapping of finite distortion is studied thoroughly in [14,15,23,26–28]. In any case any planar cusp domain, inward or outward, is conformally equivalent with the unit disk. So we see that it is bi-(2)-harmonic flat; synonymously, *bi-Dirichlet flat*. Thus the question we are concerned with is whether or not a planar cusp domain is a bi-p-harmonic flat for $p > 2$.

1.6 Inward cusp in planar domains

In spite of being bi-Lipschitz singular every inward cusp domain is bi-p-harmonic flat for all $p \geq 2$. Precisely,

Theorem 1.5 *Let $\mathbb{B} \subset \mathbb{R}^2$ be the unit disk and $\mathbb{B}_\beta^{\prec} \subset \mathbb{R}^2$ an inward cusp domain. Then there exits a Lipschitz homeomorphism $h : \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}_\beta^{\prec}$ whose inverse $f = h^{-1} \in \mathscr{W}^{1,\infty}(\mathbb{B}_\beta^{\prec}, \mathbb{C})$, but it can never be Lipschitz continuous.*

The celebrated Brennan conjecture merits mentioning here.

Conjecture 1.6 [5] Every conformal map of a simply connected domain $\Omega \subset \mathbb{C}$ onto the unit disk belongs to $\mathscr{W}^{1,p}(\Omega)$, whenever $\frac{4}{3} < p < 4$.

1.7 Inward cusp domains and $n \geq 3$

Recall that $\mathbb{B} \subset \mathbb{R}^n$ and $\mathbb{B}_\beta^{\prec}$ are not quasiconformally equivalent for $\beta > 1$ when $n \geq 3$. However, there is a homeomorphism $h : \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}_\beta^{\prec}$ with finite total bi-n-harmonic energy. Such a homeomorphism extends as a homeomorphism up to the closure of $\mathbb{B}$ when $n \geq 3$. In particular, if there exists a homeomorphism $h : \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}_\beta^{\prec}$ with finite bi-p-harmonic energy, $p > n \geq 3$, then both the boundary homeomorphism $h : \partial\mathbb{B} \xrightarrow{\text{onto}} \partial\mathbb{B}_\beta^{\prec}$ and its inverse are Hölder continuous with exponent $\alpha = 1 - n/p$. The corresponding $(n-1)$-dimensional cusp surface $\mathbf{S}_\beta$ (1.9) is α-bi-Hölder equivalent with a smooth $(n-1)$-dimensional surface if $\beta < \frac{1}{\alpha}$. This seemingly natural approach does not lead to the best possible result in terms of the sharpness exponent. This can be seen from our next result which also substantially relaxes the condition on Sobolev regularity of the inverse deformation. For simplicity of the writing, we introduce the *critical exponent*

$$\beta_\circ = \beta_\circ(n,p,q) \overset{\text{def}}{=\!=} \begin{cases} \frac{pq+p-n}{(p-n)(q+1-n)} & \text{if } n < p < \infty \text{ and } n-1 < q < \infty \\ \frac{q+1}{q+1-n} & \text{if } p = \infty \text{ and } n-1 < q < \infty \\ \frac{p}{p-n} & \text{if } n < p < \infty \text{ and } q = \infty. \end{cases} \tag{1.13}$$

Theorem 1.7 *Let $n \geq 3$ and $\mathbb{B}_\beta^{\prec} \subset \mathbb{R}^n$ an inward cusp domain with degree $\beta \geq 1$. Suppose that $n < p \leq \infty$ and $n-1 < q \leq \infty$ with $\min\{p,q\} < \infty$. Then there exists a homeomorphism $h : \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}_\beta^{\prec}$ of the Sobolev class $\mathscr{W}^{1,p}(\mathbb{B},\mathbb{R}^n)$ whose inverse mapping $f \overset{\text{def}}{=\!=} h^{-1}$ belongs to $\mathscr{W}^{1,q}(\mathbb{B}_\beta^{\prec},\mathbb{R}^n)$ if and only if $\beta < \beta_\circ$.*

Neither the Sobolev regularity condition on $h \in \mathscr{W}^{1,p}(\mathbb{B},\mathbb{R}^n)$ with $p > n$, nor the requirement that $h^{-1} \in \mathscr{W}^{1,q}(\mathbb{B}_\beta^{\prec},\mathbb{R}^n)$, $q > n-1$, can be lessen for inward cusp domains.

Theorem 1.8 *Let $\mathbb{B}_\beta^{\prec}$ be any inward cusp domain in $\mathbb{R}^n$, $n \geq 3$. Then there exist the following deformations:*

- *a homeomorphism $h_1 : \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}_\beta^{\prec}$ of the class $\mathscr{W}^{1,n}(\mathbb{B},\mathbb{R}^n)$ whose inverse $f_1 : \mathbb{B}_\beta^{\prec} \xrightarrow{\text{onto}} \mathbb{B}$ is Lipschitz regular, and*
- *a Lipschitz homeomorphism $h_2 : \mathbb{B} \xrightarrow{\text{onto}} \mathbb{B}_\beta^{\prec}$ whose inverse mapping $f_2 : \mathbb{B}_\beta^{\prec} \xrightarrow{\text{onto}} \mathbb{B}$ lies in the Sobolev class $\mathscr{W}^{1,n-1}(\mathbb{B}_\beta^{\prec},\mathbb{R}^n)$.*

1.8 Outward cusp domains

It turns out that much less energy is needed to create and flatten a planar outward cusp domain than that for inward cusp domain with the same exponent of sharpness. However, such a difference does not exist in the case of higher dimensional cusp domains. This is slightly surprising when one compares the bi-p-harmonic energy with the bi-n-harmonic energy case. Recall that a cusp domain generated by u can be deformed to the unit ball $\mathbb{B} \subset \mathbb{R}^n$, $n \geq 3$ by finite bi-n-harmonic energy whenever:

- (in case of outward cusp) the function u is Lipschitz continuous
- (in case of inward cusp) if and only if u satisfies condition (1.12).

The next result shows that a full variant of Theorem 1.7 is valid for outward cusp domains including planar ones.

Theorem 1.9 *Let $n \geq 2$ and $\mathbb{B}^{\succ}_{\beta} \subset \mathbb{R}^n$ an outward cusp domain with degree $\beta \geq 1$. Suppose that $n < p \leq \infty$ and $n-1 < q \leq \infty$ with $\min\{p,q\} < \infty$. Then there exists a homeomorphism $h : \mathbb{B} \xrightarrow{onto} \mathbb{B}^{\succ}_{\beta}$ of the Sobolev class $\mathscr{W}^{1,p}(\mathbb{B}, \mathbb{R}^n)$ whose inverse mapping $f \stackrel{def}{=} h^{-1}$ belongs to $\mathscr{W}^{1,q}(\mathbb{B}^{\succ}_{\beta}, \mathbb{R}^n)$ if and only if $\beta < \beta_\circ$.*

Again, as in the inward cups case, one cannot lessen the Sobolev regularity assumptions on the deformations that could flatten the outward β-cusps.

Theorem 1.10 *Let $n \geq 2$ and $\mathbb{B}^{\succ}_{\beta}$ be any outward cusp domain in $\mathbb{R}^n$. Then there exist*

- *a homeomorphism $h_1 : \mathbb{B} \xrightarrow{onto} \mathbb{B}^{\succ}_{\beta}$ of the class $\mathscr{W}^{1,n}(\mathbb{B}, \mathbb{R}^n)$ whose inverse $f_1 : \mathbb{B}^{\succ}_{\beta} \xrightarrow{onto} \mathbb{B}$ is Lipschitz regular;*
- *a Lipschitz homeomorphism $h_2 : \mathbb{B} \xrightarrow{onto} \mathbb{B}^{\succ}_{\beta}$ whose inverse mapping $f_2 : \mathbb{B}^{\succ}_{\beta} \xrightarrow{onto} \mathbb{B}$ lies in the Sobolev class $\mathscr{W}^{1,n-1}(\mathbb{B}^{\succ}_{\beta}, \mathbb{R}^n)$.*

The interested reader finds more about the studied questions and their connections to the theory of composition operators for Sobolev spaces, see [11,12,29,32].

2 Proof of Theorem 1.5

Proof Theorem 1.5 follows once we construct a Sobolev homeomorphism $h : \mathbb{B} \xrightarrow{onto} \mathbb{B}^{\prec}_{\beta}$ whose deformation gradient Dh is bounded and $|Df| \in \mathscr{L}^{\infty}(\mathbb{B}^{\prec}_{\beta})$, where $f = h^{-1}$. Indeed, the class of Sobolev functions $\mathscr{W}^{1,\infty}(\Omega)$ coincides with the class of Lipschitz functions if Ω is a quasiconvex domain in $\mathbb{R}^n$. Recall that a domain is *quasiconvex* if there exists a constant C such that any two points $a, b \in \Omega$ can be joined by a curve γ of length at most $C|a-b|$. In particular, convex domains have this property with $C = 1$. A planar inward cusp domain $\mathbb{B}^{\prec}_{\beta}$, $\beta > 1$, is not a quasiconvex domain.

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(t,x) : t \in \mathbb{R}, x \in \mathbb{R}\}$$

and replace the unit disk $\mathbb{B} \subset \mathbb{R}^2$ by a bi-Lipschitz equivalent domain, $\mathbb{X} = \mathbb{X}_- \cup \mathbb{X}_+$, where

$$\mathbb{X}_- = \{(t,x) : -1 < t \leq 0 \text{ and } |x| < 1\}$$

and

$$\mathbb{X}_+ = \{(t,x) : 0 < t < 1 \text{ and } t < |x| < 1\}.$$

We replace the inward cusp domain $\mathbb{B}^{\prec}_{\beta}$ by the following bi-Lipschitz equivalent domain $\mathbb{Y} = \mathbb{Y}_- \cup \mathbb{Y}_+$, where

$$\mathbb{Y}_- = \{(s,y) : -1 < s \leq 0 \text{ and } |y| < 1\}$$

and

$$\mathbb{Y}_+ = \{(s,y) : 0 < s < 1 \text{ and } s^\beta < |y| < 1\}.$$

We define a homeomorphism $h : \mathbb{X} \xrightarrow{onto} \mathbb{Y}$ by setting

$$h(t,x) = \begin{cases} (t,x) & \text{in } \mathbb{X}_- \\ \left(t,\ \frac{1-t^\beta}{1-t} x + \frac{t^\beta - t}{1-t} \frac{x}{|x|}\right) & \text{in } \mathbb{X}_+. \end{cases}$$

Then for $(t,x) \in \mathbb{X}_+$ we have

$$Dh(t,x) = \begin{pmatrix} 1 & 0 \\ \frac{g(t)}{(1-t)^2} \cdot \left[x - \frac{x}{|x|}\right] & 1 \end{pmatrix}$$

where $g(t) = \beta t^{\beta-1}(t-1) + 1 - t^\beta \geq 0$. Therefore,

$$|Dh(t,x)| \leq 2\beta \qquad \text{where } (t,x) \in \mathbb{X}.$$

The inverse mapping $f = h^{-1} : \mathbb{Y} \xrightarrow{onto} \mathbb{X}$ takes the form

$$f(s,y) = \begin{cases} (s,y) & \text{in } \mathbb{Y}_- \\ \left(s,\ \frac{1-s}{1-s^\beta} y + \frac{s-s^\beta}{1-s^\beta} \frac{y}{|y|}\right) & \text{in } \mathbb{Y}_+ \end{cases}$$

and

$$Df(s,x) = \begin{pmatrix} 1 & 0 \\ \frac{g(s)}{(1-s^\beta)^2} \cdot \left[\frac{y}{|y|} - y\right] & \frac{1-s}{1-s^\beta} \end{pmatrix} \quad \text{for } (s,y) \in \mathbb{Y}_+$$

where again $g(s) = \beta s^{\beta-1}(s-1) + 1 - s^\beta \geq 0$. Therefore, $|Df(s, y)| < 2$ for every point $(s, y) \in \mathbb{Y}$. □

3 Nonexistence part of Theorems 1.7 and 1.9

To simplify our writing the notation $\mathbb{B}_\beta$ stands for either an outward cusp domain $\mathbb{B}_\beta^{\succ} \subset \mathbb{R}^n$ when $n \geq 2$ or an inward cusp domain $\mathbb{B}_\beta^{\prec} \subset \mathbb{R}^n$ when $n \geq 3$. Suppose that $n < p \leq \infty$ and $n-1 < q \leq \infty$ with $\min\{p, q\} < \infty$. Our goal in this section is to prove that there is no homeomorphism $h : \mathbb{B} \overset{\text{onto}}{\longrightarrow} \mathbb{B}_\beta$ such that $h \in \mathscr{W}^{1,p}(\mathbb{B}, \mathbb{R}^n)$ and $f = h^{-1} \in \mathscr{W}^{1,q}(\mathbb{B}_\beta, \mathbb{R}^n)$ if $\beta \geq \beta_\circ$ where the critical power $\beta_\circ$ is given by the formula (1.13). We will emphasize when the assumption $n \geq 3$ for inward cusp domains kicks in. We suppose to the contrary that there exists a homeomorphism $h : \mathbb{B} \overset{\text{onto}}{\longrightarrow} \mathbb{B}_\beta$ such that $h \in \mathscr{W}^{1,p}(\mathbb{B}, \mathbb{R}^n)$ and $f = h^{-1} \in \mathscr{W}^{1,q}(\mathbb{B}_\beta, \mathbb{R}^n)$. Under these standing assumptions in this section, the claim is $\beta < \beta_\circ$.

Case 1. when $p, q < \infty$. First any Sobolev mapping in $\mathscr{W}^{1,p}(\mathbb{B}, \mathbb{R}^n)$ extends as an α-Hölder continuous mapping with $\alpha = 1 - \frac{n}{p}$ up to the closure of $\mathbb{B}$ when $p > n$. In particular our homeomorphism $h : \mathbb{B} \overset{\text{onto}}{\longrightarrow} \mathbb{B}_\beta$ extends as a continuous mapping $h : \overline{\mathbb{B}} \to \overline{\mathbb{B}_\beta}$. We will still denote such an extension by h. Second, since $h(\overline{\mathbb{B}})$ is a compact subset of $\overline{\mathbb{B}_\beta}$, it follows that h takes $\overline{\mathbb{B}}$ onto $\overline{\mathbb{B}_\beta}$. Third, it is a topological fact [8] that such a continuous extension is a monotone mapping $h : \overline{\mathbb{B}} \overset{\text{onto}}{\longrightarrow} \overline{\mathbb{B}_\beta}$. By the definition, monotonicity, the concept of Morrey [30], simply means that for a continuous $h : \overline{\mathbb{X}} \to \overline{\mathbb{Y}}$ the preimage $h^{-1}(y_\circ)$ of a point $y_\circ \in \overline{\mathbb{Y}}$ is a connected set in $\overline{\mathbb{X}}$. We just obtained the following result:

Lemma 3.1 *The boundary mapping* $h : \partial\mathbb{B} \overset{\text{onto}}{\longrightarrow} \partial\mathbb{B}_\beta$ *is monotone and* $h \in \mathscr{C}^\alpha(\partial\mathbb{B})$ *with* $\alpha = 1 - \frac{n}{p}$.

To obtain the asserted bound $\beta < \beta_\circ$ we will combine the Hölder continuity of h on $\partial\mathbb{B}$ with a diameter estimate of $h^{-1}(C_t)$, where

$$C_t \overset{\text{def}}{=\!=} \{x \in \partial\mathbb{B}_\beta : |x| = t\} \qquad \text{for } 0 < t < 1.$$

Here we relay on our work in [22]. In [22] we studied homeomorphisms $h : \mathbb{B} \overset{\text{onto}}{\longrightarrow} \mathbb{B}_\beta^{\prec}$ under the weaker Sobolev regularity assumptions $h \in \mathscr{W}^{1,n}(\mathbb{B}, \mathbb{R}^n)$ and $f = h^{-1} \in \mathscr{W}^{1,q}(\mathbb{B}_\beta^{\prec}, \mathbb{R}^n)$. Even thought the results in [22] are stated explicitly only in the case of inward cusp targets the ones we will be referring to are valid for outward cusp domains as well. The proofs of these results remain unchanged. There is no need to repeat them here. We follow the notation in [22] including for $0 < t < 1$ we define

$$S_t \overset{\text{def}}{=\!=} \{x \in \mathbb{B}_\beta : |x| = t\},$$

$$S'_t \overset{\text{def}}{=\!=} h^{-1}(S_t) \quad \text{and} \quad C'_t \overset{\text{def}}{=\!=} \overline{S'_t} \cap \partial\mathbb{B}.$$

Furthermore the cusp $\partial\mathbb{B}_\beta$ has its vertex at $o = (0, \dots, 0) \in \partial\mathbb{B}_\beta$ and without loss of generality we may assume that $h(o') = o$, where $o' = (1, 0, \dots, 0) \in \partial\mathbb{B}$.

Lemma 2.11 in [22] tells us that

$$h(C'_t) = C_t \tag{3.1}$$

and Lemma 2.13 in [22] implies that C'_t divides $\partial\mathbb{B}$ into two disjoint components. We denote the component which contains o' by U'_t. According to Lemma 2.13 in [22], we also have

$$\partial U'_t = C'_t\,. \tag{3.2}$$

The key to the hunted estimate for $h^{-1}(C_t)$ is the following variant of the Sobolev embedding on spheres.

Lemma 3.2 *Suppose that a homeomorphism* $h : \mathbb{B} \overset{\text{onto}}{\longrightarrow} \mathbb{B}_\beta$ *belongs to* $\mathscr{W}^{1,p}(\mathbb{B}, \mathbb{R}^n)$, $p > n-1$ *and* $f = h^{-1} \in \mathscr{W}^{1,q}(\mathbb{B}_\beta, \mathbb{R}^n)$ *for* $q \in (n-1, \infty)$. *Then for almost every* $0 < t < 1$ *and every* $x'_t, y'_t \in C'_t$ *we have*

$$|x'_t - y'_t| \leq C|x_t - y_t|^{1-\frac{n-1}{q}} \left(\int_{S_t} |Df(x)|^q \, dx \right)^{\frac{1}{q}}. \tag{3.3}$$

Here $x_t = h(x'_t)$ *and* $y_t = h(y'_t)$ *and* C *is a positive constant independent of* t, x_t, *and* y_t.

Proof of Lemma 3.2 When $n = 2$, the estimate (3.3) is only valid for outward cups domains not for inward cups domains. Let $0 < t < 1$ be such that

$$\int_{S_t} |Df(x)|^q dx < \infty.$$

This happens for almost every t by Fubini's theorem. Let $x'_t, y'_t \in C'_t$. By (3.1) there are two sequences $\{x'_{t,i}\}_{i=1}^\infty$ and $\{y'_{t,i}\}_{i=1}^\infty$ in S'_t such that

$$\lim_{i\to\infty} x'_{t,i} = x'_t, \quad \lim_{i\to\infty} y'_{t,i} = y'_t$$

and

$$\lim_{i\to\infty} x_{t,i} = x_t \in C_t, , \quad \lim_{i\to\infty} y_{t,i} = y_t \in C_t, .$$

Here

$$x_{t,i} = h(x'_{t,i}),\ y_{t,i} = h(y'_{t,i}),\ x_t = h(x'_t) \text{ and } y_t = h(y'_t).$$

By the classical Sobolev embedding on sphere, we have

$$|x'_{t,i} - y'_{t,i}| \leq C|x_{t,i} - y_{t,i}|^{1-\frac{n-1}{p}} \left(\int_{S_t} |Df(x)|^p dx \right)^{\frac{1}{p}}.$$

Note that in the case of inward planar cusp domain we would get $|t - (x_{t,i} - y_{t,i})|$ in the place of $|x_{t,i} - y_{t,i}|$. Anyway, in our situations after passing to the limit, we obtain

$$|x'_t - y'_t| \le C|x_t - y_t|^{1-\frac{n-1}{p}} \left(\int_{S_t} |Df(x)|^p dx \right)^{\frac{1}{p}} .$$

□

Now, we choose a decreasing sequence $\{t_i\}$, which converges to 0 and satisfies (3.3). Furthermore we require that the sequence $\{t_i\}$ enjoys the property

$$\int_{S_{t_i}} |Df(x)|^q \,\mathrm{d}x < \frac{1}{t_i} . \tag{3.4}$$

This is possible according to Fubini's theorem we have

$$\int_0^1 \int_{S_t} |Df(x)|^q \,\mathrm{d}x\mathrm{d}t < \infty, \quad \text{hence}$$

$$\liminf_{t\to 0} t \int_{S_t} |Df(x)|^q \mathrm{d}x = 0.$$

Combining (3.1) with Lemma 3.2 we have

$$\operatorname{diam} C'_{t_i} \le C \cdot t_i^{\beta\left(1-\frac{n-1}{q}\right)} \left(\int_{S_{t_i}} |Df(x)|^q \mathrm{d}x \right)^{\frac{1}{q}} \tag{3.5}$$

which together with (3.4) gives

$$\operatorname{diam} C'_{t_i} \le C \cdot t_i^{\frac{\beta(q+1-n)-1}{q}} \tag{3.6}$$

In particular since $q > n-1$, this shows that $\operatorname{diam}(C'_{t_i}) \to 0$ as $i \to \infty$ and, therefore, U'_{t_i} lies on the half sphere of $\partial\mathbb{B}$. We now appeal to the geometric fact if $x, a \in U'_{t_i}$, then $|x - a| \le \operatorname{diam} \partial U'_{t_i}$. Now, for large enough i, by (3.2) we fix $x'_{t_i} \in C'_{t_i}$ and then

$$|x'_{t_i} - o'| \le \operatorname{diam} C'_{t_i} . \tag{3.7}$$

Therefore, by (3.6) and (3.7) we obtain our basic estimate for f.

Lemma 3.3 *There is $x'_{t_i} \in C'_{t_i}$ such that*

$$|x'_{t_i} - o'| \le C \cdot t_i^{\gamma} \qquad \text{where } \gamma \stackrel{def}{=} \beta \left(1 - \frac{n-1}{q}\right) - \frac{1}{q}.$$

Finally, combining Lemma 3.1 with Lemma 3.3 we obtain

$$t_i \le |h(x'_{t_i}) - o| \le C|x'_{t_i} - o'|^{\alpha} \le C t_i^{\alpha\gamma} . \tag{3.8}$$

Now $\alpha\gamma - 1 > 0$ if $\beta > \frac{pq+p-n}{(p-n)(q+1-n)} = \beta_\circ$ as claimed.

Case 2. $p = \infty$ **and** $q < \infty$. Now Lemma 3.1 holds for $\alpha = 1$ and in (3.8) we have

$$t_i \le C t_i^{\gamma} .$$

Now we have $\gamma > 1$ provided $\beta > \frac{q+1}{q-n+1} = \beta_\circ$ as claimed.

Case 3. $p < \infty$ **and** $q = \infty$. Now applying Hölder's inequality to (3.5), Lemma 3.3 holds for $\gamma = \beta$ and in (3.8) we have

$$t_i \le C t_i^{\alpha\beta}.$$

Since $\alpha = 1 - \frac{n}{p}$ we have $\alpha\beta > 1$ provided $\beta > \frac{p}{p-n} = \beta_\circ$ as claimed.

4 Existence part of Theorems 1.7 and 1.9

In this section we construct homeomorphisms asserted in Theorem 1.7 and Theorem 1.9, proving the existence part of such results. It is enough to construct such mappings only in the case when $p < \infty$ and $q < \infty$. Indeed, this is due to the openness of the asserted condition $\beta < \beta_\circ$ and the fact that

- the function $q \to \beta_\circ(n, p, q)$ is increasing and $\lim_{q\to\infty} \beta_\circ(n, p, q) = \beta_\circ(n, p, \infty)$, and
- the function $p \to \beta_\circ(n, p, q)$ is increasing and $\lim_{p\to\infty} \beta_\circ(n, p, q) = \beta_\circ(n, \infty, q)$.

4.1 Theorem 1.7

Let $n \ge 3, n < p < \infty$ and $n-1 < q < \infty$. Fix $1 < \beta < \beta_o$ and the corresponding inward cusp domain $\mathbb{B}^{\prec}_{\beta}$. We write

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} = \{(t, x) : t \in \mathbb{R}, \ x \in \mathbb{R}^{n-1}\}$$

and replace $\mathbb{B} \subset \mathbb{R}^n$ by a bi-Lipschitz equivalent domain, $\mathbb{X} = \mathbb{X}_- \cup \mathbb{X}_+$, where

$$\mathbb{X}_- = \{(t, x) : -1 < t \le 0 \text{ and } |x| < 1\}$$

and

$$\mathbb{X}_+ = \{(t, x) : 0 < t < 1 \text{ and } t < |x| < 1\}.$$

We replace the inward cusp domain $\mathbb{B}^{\prec}_{\beta}$ by the following bi-Lipschitz equivalent domain $\mathbb{Y} = \mathbb{Y}_- \cup \mathbb{Y}_+$, where

$$\mathbb{Y}_- = \{(s, y) : -1 < s \le 0 \text{ and } |y| < 1\}$$

and

$$\mathbb{Y}_+ = \{(s, y) : 0 < s < 1 \text{ and } s^{\beta} < |y| < 1\}.$$

Since $1 < \beta < \beta_o = \frac{pq+p-n}{(p-n)(q+1-n)}$, we have $\frac{\beta(q+1-n)-1}{q} < \frac{p}{p-n}$ and

$$(1, \beta) \cap \left(\frac{\beta(q+1-n)-1}{q}, \frac{p}{p-n} \right) \neq \emptyset.$$

Fix $\alpha \in (1, \beta) \cap \left(\frac{\beta(q+1-n)-1}{q}, \frac{p}{p-n} \right)$, we define $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ by

$$h(t, x) = \begin{cases} (t, x) & \text{in } \mathbb{X}_- \\ \left(t^{\frac{1}{\alpha}}, \frac{1-t^{\frac{\beta}{\alpha}}}{1-t} x + \frac{t^{\frac{\beta}{\alpha}}-t}{1-t} \frac{x}{|x|} \right) & \text{in } \mathbb{X}_+. \end{cases}$$

Then the inverse mapping $f = h^{-1} : \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ takes the form

$$f(s, y) = \begin{cases} (s, y) & \text{in } \mathbb{Y}_- \\ \left(s^{\alpha}, \frac{1-s^{\alpha}}{1-s^{\beta}} y + \frac{s^{\alpha}-s^{\beta}}{1-s^{\beta}} \frac{y}{|y|} \right) & \text{in } \mathbb{Y}_+. \end{cases}$$

A simple computation shows that

$$|Dh(t, x)| \leq \begin{cases} 1 & \text{in } \mathbb{X}_- \\ Ct^{\frac{1}{\alpha}-1} & \text{in } \mathbb{X}_+ \end{cases}$$

and

$$|Df(s, y)| \leq \begin{cases} 1 & \text{in } \mathbb{Y}_- \\ Cs^{\alpha-\beta} & \text{in } \mathbb{Y}_+. \end{cases}$$

Hence we have

$$\begin{aligned} \int_{\mathbb{X}} |Dh|^p &\leq |\mathbb{X}_-| + C \int_0^1 |x|^{n-2} \int_0^{|x|} t^{\frac{p}{\alpha}-p} \, dt \, d|x| \\ &\leq |\mathbb{X}_-| + C \int_0^1 |x|^{n+\frac{p}{\alpha}-p-1} \, d|x| < \infty. \end{aligned}$$

Here we used the fact that $\alpha < \frac{p}{p-n}$. We also have

$$\begin{aligned} \int_{\mathbb{Y}} |Df|^q &\leq |\mathbb{Y}_-| + C \int_0^1 |y|^{n-2} \int_0^{|y|^{\frac{1}{\beta}}} s^{\alpha q-\beta q} \, ds \, d|y| \\ &\leq |\mathbb{Y}_-| + C \int_0^1 |y|^{n-2+\frac{\alpha q-\beta q+1}{\beta}} \, d|y| < \infty. \end{aligned}$$

because $\frac{\beta(q+1-n)-1}{q} < \alpha$.

4.2 Theorem 1.9

Let $n \geq 2$, $p > n$ and $q > n-1$. Fix $1 < \beta < \beta_o$ and the corresponding outward cusp domain $\mathbb{B}_\beta^{\succ} \subset \mathbb{R}^n$. We write

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} = \{(t, x) : t \in \mathbb{R}, \; x \in \mathbb{R}^{n-1}\}$$

and replace $\mathbb{B} \subset \mathbb{R}^n$ by a bi-Lipschitz equivalent domain,

$$\mathbb{X} = \{(t, x) : 0 < t < 1 \text{ and } |x| < t\}.$$

We replace the cusp domain $\mathbb{B}_\beta^{\succ}$ by the following bi-Lipschitz equivalent domain

$$\mathbb{Y} = \{(s, y) : 0 < s \leq 1 \text{ and } |y| < s^{\beta}\}.$$

Since $1 < \beta < \beta_\circ = \frac{pq+p-n}{(p-n)(q+1-n)}$, we have $\frac{p-n}{p} < \frac{q}{\beta(q+1-n)-1}$ and

$$\left(\frac{1}{\beta}, 1 \right) \cap \left(\frac{p-n}{p}, \frac{q}{\beta(q+1-n)-1} \right) \neq \emptyset.$$

Fix $\alpha \in (\frac{1}{\beta}, 1) \cap \left(\frac{p-n}{p}, \frac{q}{\beta(q+1-n)-1} \right)$, we define $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ by

$$h(t, x) = (t^{\alpha}, \; t^{\alpha\beta-1} x) \quad \text{in } \mathbb{X}.$$

Then the inverse mapping $f = h^{-1} : \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ takes the form

$$f(s, y) = (s^{\frac{1}{\alpha}}, \; s^{\frac{1}{\alpha}-\beta} y) \quad \text{in } \mathbb{Y}.$$

A straightforward computations shows that

$$|Dh(t, x)| \leq Ct^{\alpha-1} \quad \text{in } \mathbb{X}.$$

and

$$|Df(s, y)| \leq Cs^{\frac{1}{\alpha}-\beta} \quad \text{in } \mathbb{Y}.$$

First since $\frac{q}{\beta(q+1-n)-1} > \alpha$ we have

$$\begin{aligned} \int_{\mathbb{X}} |Dh|^p &\leq C \int_0^1 t^{\alpha p-p} \int_0^t |x|^{n-2} \, d|x| \, dt \\ &\leq C \int_0^1 t^{\alpha p-p+n-1} \, dt < \infty. \end{aligned}$$

Second using the fact that $\frac{q}{\beta(q+1-n)-1} > \alpha$ we obtain

$$\begin{aligned} \int_{\mathbb{Y}} |Df|^q &\leq C \int_0^1 s^{\frac{q}{\alpha}-\beta q} \int_0^{s^{\beta}} |y|^{n-2} \, d|y| \, ds \\ &\leq C \int_0^1 s^{\beta(n-1-q)+\frac{q}{\alpha}} \, ds < \infty. \end{aligned}$$

We finished the construction.

5 Proof of Theorem 1.8

Let $n \geq 3$. Fix an inward cusp domain $\mathbb{B}^{\prec}_{\beta} \subset \mathbb{R}^n$ with $\beta > 1$. As in the proof of Theorem 1.7, we simplify our writing and replace $\mathbb{B} \subset \mathbb{R}^n$ by a bi-Lipschitz equivalent domain, $\mathbb{X} = \mathbb{X}_- \cup \mathbb{X}_+$, where

$$\mathbb{X}_- = \{(t, x) \in \mathbb{R}^n : -1 < t \leq 0 \text{ and } |x| < 1\}$$

and

$$\mathbb{X}_+ = \{(t, x) \in \mathbb{R}^n : 0 < t < 1 \text{ and } t < |x| < 1\}.$$

We replace the inward cusp domain $\mathbb{B}^{\prec}_{\beta}$ by the following bi-Lipschitz equivalent domain $\mathbb{Y} = \mathbb{Y}_- \cup \mathbb{Y}_+$, where

$$\mathbb{Y}_- = \{(s, y) \in \mathbb{R}^n : -1 < s \leq 0 \text{ and } |y| < 1\}$$

and

$$\mathbb{Y}_+ = \{(s, y) \in \mathbb{R}^n : 0 < s < 1 \text{ and } s^{\beta} < |y| < 1\}.$$

We define the first searched homeomorphism $h_1 : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ by

$$h_1(t, x) = \begin{cases} (t, x) & \text{in } \mathbb{X}_- \\ \left(|x|^{\frac{1}{\beta}-1} t,\ x\right) & \text{in } \mathbb{X}_+. \end{cases}$$

Then the inverse mapping $f_1 = h_1^{-1} : \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ takes the form

$$f_1(s, y) = \begin{cases} (s, y) & \text{in } \mathbb{Y}_- \\ \left(|y|^{1-\frac{1}{\beta}} s,\ y\right) & \text{in } \mathbb{Y}_+ \end{cases}$$

and is a Lipschitz regular mapping on $\mathbb{Y}$. A simply computation gives us that

$$|Dh_1(t, x)| \leq \begin{cases} 1 & \text{in } \mathbb{X}_- \\ C|x|^{\frac{1}{\beta}-1} & \text{in } \mathbb{X}_+. \end{cases}$$

Therefore,

$$\int_{\mathbb{X}} |Dh_1|^n \leq |\mathbb{X}_-| + C \int_0^1 |x|^{\frac{n}{\beta}-1}\, \mathrm{d}|x| < \infty$$

We define the second asserted homeomorphism $h_2 : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ by

$$h_2(t, x) = \begin{cases} (t, x) & \text{in } \mathbb{X}_- \\ \left(t,\ \frac{1-t^{\beta}}{1-t} x + \frac{t^{\beta}-t}{1-t} \frac{x}{|x|}\right) & \text{in } \mathbb{X}_+. \end{cases}$$

Now, the mapping h_2 is Lipschitz regular. Then the inverse mapping $f_2 = h_2^{-1} : \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ takes the form

$$f_2(s, y) = \begin{cases} (s, y) & \text{in } \mathbb{Y}_- \\ \left(s,\ \frac{1-s}{1-s^{\beta}} y + \frac{s-s^{\beta}}{1-s^{\beta}} \frac{y}{|y|}\right) & \text{in } \mathbb{Y}_+ \end{cases}$$

and

$$|Df_2(s, y)| \leq \begin{cases} 1 & \text{in } \mathbb{Y}_- \\ C\frac{s}{|y|} & \text{in } \mathbb{Y}_+. \end{cases}$$

Therefore,

$$\int_{\mathbb{Y}} |Df_2|^{n-1} \leq |\mathbb{Y}_-| + C \int_0^1 s^{n-1} \int_{s^{\beta}}^1 \frac{1}{|y|}\, \mathrm{d}|y|\, \mathrm{d}s < \infty$$

which completes the proof of Theorem 1.8.

6 Proof of Theorem 1.10

Let $n \geq 2$. Fix an outward cusp domain $\mathbb{B}^{\succ}_{\beta} \subset \mathbb{R}^n$ for some $1 < \beta < \infty$. As in the proof of Theorem 1.9 we replace the unit ball $\mathbb{B} \subset \mathbb{R}^n$ by a bi-Lipschitz equivalent domain,

$$\mathbb{X} = \{(t, x) \in \mathbb{R}^n : 0 < t < 1 \text{ and } |x| < t\}.$$

We replace the outward cusp domain $\mathbb{B}^{\succ}_{\beta} \subset \mathbb{R}^n$ by the following bi-Lipschitz equivalent domain:

$$\mathbb{Y} = \{(s, y) \in \mathbb{R}^n : 0 < s < 1 \text{ and } |y| < s^{\beta}\}.$$

We will construct two bi-Sobolev homeomorphisms $h_1, h_2 : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$. First, we define the homeomorphism $h_1 : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ by

$$h_1(t, x) = (t^{\frac{1}{\beta}},\ x) \quad \text{in } \mathbb{X}.$$

Then the inverse mapping $f_1 = h_1^{-1} : \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ takes the form

$$f_1(s, y) = (s^{\beta},\ y) \quad \text{in } \mathbb{Y}$$

and it is Lipschitz regular. On the other hand, we have

$$|Dh_1(t, x)| \leq C t^{\frac{1}{\beta}-1} \quad \text{in } \mathbb{X}$$

and therefore

$$\int_{\mathbb{X}} |Dh_1|^n \leq C \int_0^1 t^{\frac{n}{\beta}-n} \int_0^t |x|^{n-2}\, \mathrm{d}|x|\, \mathrm{d}t \leq C \int_0^1 t^{\frac{n}{\beta}-1}\, \mathrm{d}t < \infty.$$

Second, we define the homeomorphism $h_2 : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ by

$$h_2(t, x) = (t,\ t^{\beta-1}x) \quad \text{in } \mathbb{X}.$$

Then the mapping h_2 is a Lipschitz regular and its inverse $f_2 = h_2^{-1} : \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ takes the form

$$f_2(s, y) = (s,\ s^{1-\beta}y) \quad \text{in } \mathbb{Y}.$$

Therefore,

$$|Df_2(s, y)| \le Cs^{1-\beta} \quad \text{in } \mathbb{Y}.$$

and we have

$$\begin{aligned}\int_{\mathbb{Y}} |Df_2|^{n-1} &\le C \int_0^1 s^{(n-1)(1-\beta)} \int_0^{s^\beta} |y|^{n-2}\, \mathrm{d}|y|\, \mathrm{d}s \\ &\le C \int_0^1 s^{n-1}\, \mathrm{d}s < \infty.\end{aligned}$$

This finishes the proof of Theorem 1.10.

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