LINK SPLITTING DEFORMATION OF COLORED KHOVANOV–ROZANSKY HOMOLOGY

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Abstract. We introduce a multi-parameter deformation of the triply-graded Khovanov–Rozansky homology of links colored by one-column Young diagrams, generalizing the "y-ified" link homology of Gorsky–Hogancamp and work of Cautis–Lauda–Sussan. For each link component, the natural set of deformation parameters corresponds to interpolation coordinates on the Hilbert scheme of the plane. We extend our deformed link homology theory to braids by introducing a monoidal dg 2-category of curved complexes of type A singular Soergel bimodules. Using this framework, we promote to the curved setting the categorical colored skein relation from [HRW21] and also the notion of splitting map for the colored full twists on two strands. As applications, we compute the invariants of colored Hopf links in terms of ideals generated by Haiman determinants and use these results to establish general link splitting properties for our deformed, colored, triply-graded link homology. Informed by this, we formulate several conjectures that have implications for the relation between (colored) Khovanov–Rozansky homology and Hilbert schemes.

Contents

1. Introduction

The last two decades have seen the introduction of powerful homological invariants of knots, links, braids, and tangles, which are connected to classical quantum invariants through a decategorification relationship [Kho00, Rou04, KR08a, KR08b]. These invariants are best understood in the context of differential graded categories: each tangle diagram $\mathbf{D}$ is assigned a chain complex $\mathcal{Z}(\mathbf{D})$ over an additive category, Reidemeister moves between such diagrams are assigned specific chain maps that are invertible up to homotopy, and movies between diagrams that encode certain braid/tangle cobordisms are assigned (generally non-invertible) chain maps that are natural, up to homotopy [CMW09, Cap08, Bla10, EK10, ETW18]. In the case that $\mathbf{D}$ is a knot or link diagram, the homology $H_{\mathcal{Z}}(\mathbf{L})$ of $\mathcal{Z}(\mathbf{D})$ is therefore an invariant of the link $\mathbf{L}$ determined by $\mathbf{D}$. In fact, $\mathcal{Z}(\mathbf{D})$ can typically be equipped with additional structure (see below) and determines an invariant $\mathcal{Z}(\mathbf{L})$ of the corresponding link $\mathbf{L}$ up to quasi-isomorphism. Instances of this higher structure are the subject of this paper.

1.1. **Local operators and monodromy.** In many cases, a choice of point $\mathsf{p} \in \mathbf{D}$ equips $\mathcal{Z}(\mathbf{D})$ with an action of a graded-commutative algebra of *local operators* $A_{\mathcal{Z}}$. In prototypical examples, $A_{\mathcal{Z}} = \mathcal{Z}(\bigcirc)$ can be identified with the invariant of the unknot[1], and the action of $A_{\mathcal{Z}}$ at p is induced by the saddle cobordism $\bigcirc \sqcup \mathbf{D} \to \mathbf{D}$ that merges a small unknot with $\mathbf{D}$ near the point p. We now mention several specific instances of this setup. For the duration, we work over the rationals $\mathbb{Q}$ (see §1.7).

Example 1.1. Let $\mathcal{Z}(\mathbf{D}) = C_{\mathrm{KR}_N}(\mathbf{D})$ be the $\mathfrak{gl}_N$ Khovanov–Rozansky complex, whose homology is the $\mathfrak{gl}_N$ Khovanov–Rozansky homology $H_{\mathrm{KR}_N}(\mathbf{L})$ of the link $\mathbf{L}$ determined by $\mathbf{D}$ [KR08a]. A choice of $\mathsf{p} \in \mathbf{D}$ equips $C_{\mathrm{KR}_N}(\mathbf{D})$ with an action of the graded algebra $A_{\mathrm{KR}_N} := H^*(\mathbb{C}P^{N-1}) \cong \mathbb{Q}[x]/(x^N)$. The unknot invariant $C_{\mathrm{KR}_N}(\bigcirc)$ is the free A_{KR_N}-module generated by a single element, in degree $1-N$, so we may identify A_{KR_N} and $C_{\mathrm{KR}_N}(\bigcirc)$ up to shift. Under this identification, the action of $C_{\mathrm{KR}_N}(\bigcirc)$ at $\mathsf{p} \in \mathbf{D}$ is induced by the map

$$C_{\mathrm{KR}_N}(\bigcirc) \otimes C_{\mathrm{KR}_N}(\mathbf{D}) \xrightarrow{\cong} C_{\mathrm{KR}_N}(\bigcirc \sqcup \mathbf{D}) \to C_{\mathrm{KR}_N}(\mathbf{D})$$

where the first part is due to the monoidality of C_{KR_N} and the second part is induced by the saddle cobordism.

Remark 1.2. The algebra $A_{\mathcal{Z}}$ is sometimes referred to as the *sheet algebra* of the theory $\mathcal{Z}$. It suggest that elements of $A_{\mathcal{Z}}$ should be visualized as the identity cobordism of the trivial $(1,1)$-tangle, suitably decorated. See e.g. [MN08, Corollary 2.4], where this terminology appears to have originated.

Example 1.3. Let $\mathbf{D}$ be a closed braid diagram of a link $\mathbf{L}$, and let $\mathcal{Z}(\mathbf{D}) = C_{\mathrm{KR}}(\mathbf{D})$ be the triply-graded Khovanov–Rozansky complex, whose homology is the HOMFLYPT homology $\mathrm{HHH}(\mathbf{L}) := H_{\mathrm{KR}}(\mathbf{D})$ [KR08b]. There are two common choices for the sheet algebra in this case: the *underived sheet algebra* $\mathbb{Q}[x]$, or the *derived sheet algebra* $\mathbb{Q}[x] \otimes \wedge[\eta] \cong \mathrm{HH}^\bullet(\mathbb{Q}[x])$. Here the degrees of the variables, written multiplicatively following Convention 3.6 below, are given by $\mathrm{wt}(x) = \mathbf{q}^2$ and $\mathrm{wt}(\eta) = \mathbf{a}\mathbf{q}^{-2}$.

The action of this sheet algebra is best understood using Khovanov's formulation of $\mathrm{HHH}(\mathbf{L})$ using the Hochschild homology of Soergel bimodules [Kho07], given that this homology theory is not functorial with respect to general link cobordisms. We will refer to the $\mathbf{a}$-degree of the variable η in the following as the *Hochschild-degree*.

Example 1.4. In this paper, we are primarily interested in colored link homologies and, more specifically, the $\wedge$-colored extension of triply-graded Khovanov–Rozansky homology [WW17]. This homology theory defines invariants of framed oriented links $\mathbf{L}$ in which each component $\mathsf{c} \in \pi_0(\mathbf{L})$ is labeled by a non-negative integer $b(\mathsf{c})$, the *color*, each of which defines a sheet algebra $A_{\mathcal{Z},b(\mathsf{c})}$. For $b \in \mathbb{N}$, fix an alphabet $\mathbb{X}^b = \{x_1, \ldots, x_b\}$, then the b-colored sheet algebra is given by the ring of symmetric polynomials $\mathrm{Sym}(\mathbb{X}^b) := \mathbb{Q}[x_1, \ldots, x_b]^{\mathfrak{S}_b}$ and the derived sheet algebra by $\mathrm{HH}^\bullet(\mathrm{Sym}(\mathbb{X}^b))$.

In addition to the action of sheet algebras, $\mathcal{Z}(\mathbf{D})$ is typically also equipped with higher structures stemming from the fact that choices of different points $\mathsf{p}_1, \mathsf{p}_2 \in \mathbf{D}$ that lie in the same component[2] $\mathsf{c} \in \pi_0(\mathbf{L})$ should induce homotopic actions of the relevant sheet algebra $A_{\mathcal{Z}}$. To be precise, let $\gamma \subset \mathbf{D}$ be an oriented path from p_1 to p_2. For each $a \in A_{\mathcal{Z}}$, let $a(\mathsf{p}_i)$ denote the action of $a \in A_{\mathcal{Z}}$ at $\mathsf{p}_i \in \mathbf{D}$. Then, the path γ determines a homotopy $\Psi_\gamma(a)$, with

$$[\delta_{\mathbf{D}}, \Psi_\gamma(a)] = a(\mathsf{p}_2) - a(\mathsf{p}_1).$$

[1]More precisely, $\mathcal{Z}(\bigcirc)$ is typically a free $A_{\mathcal{Z}}$-module of rank 1, so $A_{\mathcal{Z}}$ and $\mathcal{Z}(\bigcirc)$ are isomorphic up to grading shift.

[2]Here, and in the following, we slightly abuse terminology and identify components $\mathsf{c} \in \pi_0(\mathbf{L})$ of the link determined by $\mathbf{D}$ with the corresponding equivalence class of points in the diagram $\mathbf{D}$. Thus, we can talk about components of $\mathbf{D}$.

Here $\delta_{\mathbf{D}}$ is the differential on the complex $\mathcal{Z}(\mathbf{D})$, so the super-commutator $[\delta_{\mathbf{D}}, -]$ is the differential on the dg algebra $\mathrm{End}(\mathcal{Z}(\mathbf{D}))$. Note, however, that $a(\mathsf{p}_1)$ and $a(\mathsf{p}_2)$ are homotopic in two *different* ways. We can choose two complementary paths $\gamma, \gamma' \colon \mathsf{p}_1 \to \mathsf{p}_2$ in $\mathbf{D}$ so that traversing γ followed by the reverse of γ' yields a loop. It follows that the difference

$$\Psi_a := \Psi_\gamma(a) - \Psi_{\gamma'}(a)$$

is a closed endomorphism of $\mathcal{Z}(\mathbf{D})$ of degree $\mathrm{wt}(\Psi_a) = \mathrm{wt}(a)\mathbf{t}^{-1}$, called the *monodromy* of a along $\mathbf{D}$. These monodromy endomorphisms, together with certain higher operations, (should) assemble to give an action of the Hochschild homology $\mathrm{HH}_\bullet(A_{\mathcal{Z}})$ (itself an algebra, since $A_{\mathcal{Z}}$ is graded-commutative) on $\mathcal{Z}(\mathbf{D})$. This should not be confused with the passage to a derived sheet algebra as in Example 1.3. Instead, the guiding principle is the following.

Principle 1.5. *Each component $\mathsf{c} \in \pi_0(\mathbf{L})$ determines an action of $\mathrm{HH}_\bullet(A_{\mathcal{Z}})$ on $\mathcal{Z}(\mathbf{L})$, well-defined up to quasi-isomorphism, in such a way that, for $a \in A_{\mathcal{Z}}$, the Kähler differential $d(a) \in \mathrm{HH}_1(A_{\mathcal{Z}})$ acts as the monodromy Ψ_a along c. The actions along various components (anti-)commute and assemble to give an action of $\bigotimes_{\mathsf{c}\in\pi_0(\mathbf{L})} \mathrm{HH}_\bullet(A_{\mathcal{Z}})$.*

Remark 1.6. In the above statement, we have invoked the well-known fact that for a commutative $\mathbb{Q}$-algebra A, $\mathrm{HH}_1(A)$ is isomorphic to the A-module of Kähler differentials on A, i.e. the quotient of the free A-module generated by symbols $d(a)$ with $a \in A$, modulo the relations $d(ab) = ad(b) + bd(a)$ and $d(a + b) = d(a) + d(b)$ for $a, b \in A$ and $d(s) = 0$ for $s \in \mathbb{Q}$.

Remark 1.7. In the setting of colored link homologies, the statement of Principle 1.5 should be modified accordingly to account for the fact that the sheet algebras depend on a choice of color.

We thus refer to $\mathrm{HH}_\bullet(A_{\mathcal{Z}})$ as the *monodromy algebra* of $A_{\mathcal{Z}}$. For the duration, we restrict to the setting of (colored) triply-graded Khovanov–Rozansky homology for concreteness, since this will be the invariant we study in this work.

Example 1.8. Continuing Example 1.3, for the (uncolored) triply-graded Khovanov–Rozansky homology with sheet algebra $\mathbb{Q}[x]$, the associated monodromy algebra is $\mathbb{Q}[x] \otimes \wedge[\xi]$, where $\mathrm{wt}(x) = \mathbf{q}^2$ and $\mathrm{wt}(\xi) = \mathbf{q}^2\mathbf{t}^{-1}$. Thus, one would expect an action of $\mathbb{Q}[x_1, \ldots, x_r] \otimes \wedge[\xi_1, \ldots, \xi_r]$ on $C_{\mathrm{KR}}(\mathbf{D})$ when $\mathbf{D}$ has r components. Such an action was constructed in [GH] in the form of "y-ified" Khovanov–Rozansky homology.

Example 1.9. For colored Khovanov–Rozansky homology, the b-colored sheet algebra $\mathbb{Q}[x_1, \ldots, x_b]^{\mathfrak{S}_b}$ can be identified with, for example, $\mathbb{Q}[p_1, \ldots, p_b]$ or $\mathbb{Q}[e_1, \ldots, e_b]$, where p_i and e_i are the power sum and elementary symmetric functions in the alphabet $\mathbb{X}^b = \{x_1, \ldots, x_b\}$. Correspondingly, the b-colored monodromy algebra can be identified with $\mathbb{Q}[p_1, \ldots, p_b] \otimes \wedge[\Xi_1, \ldots, \Xi_b]$ or $\mathbb{Q}[e_1, \ldots, e_b] \otimes \wedge[\Psi_1, \ldots, \Psi_b]$, where $\mathrm{wt}(p_i) = \mathrm{wt}(e_i) = \mathbf{q}^{2i}$ and $\Xi_i = d(p_i)$ and $\Psi_i = \delta(e_i)$ with $\mathrm{wt}(\Xi_i) = \mathrm{wt}(\Psi_i) = \mathbf{q}^{2i}\mathbf{t}^{-1}$. If one prefers a more basis independent description, we can identify the monodromy algebra with $\mathfrak{S}_b$-invariants in $\mathbb{Q}[x_1, \ldots, x_b] \otimes \wedge[d(x_1), \ldots, d(x_b)]$.

1.2. **From monodromy to deformation.** Now, we show how the action of the monodromy algebra $\mathrm{HH}_\bullet(A_{\mathcal{Z}})$ on $\mathcal{Z}(\mathbf{D})$ allows us to construct *link splitting deformations*[3] of $\mathcal{Z}(\mathbf{D})$. Again, we work explicitly with triply-graded Khovanov–Rozansky homology, first in the uncolored case (considered in [GH]) and then its extension to the colored case which is achieved in this paper.

Suppose that $\mathbf{D}$ is a braid closure diagram for an r-component oriented link $\mathbf{L}$ and let $C_{\mathrm{KR}}(\mathbf{D})$ be the Khovanov–Rozansky complex associated to $\mathbf{D}$, which carries an action of the monodromy algebra

[3]In this paper, the word *deformation* will always refer to such link splitting deformations, which are based on monodromy data. These are of an entirely different nature than the deformations of finite-rank type A link homologies based on deformations of underlying Frobenius algebras, that were studied by the second- and third-named authors in [RW16].

$\mathbb{Q}[x_1,\dots,x_r]\otimes\bigwedge[\xi_1,\dots,\xi_r]$ on $C_{\mathrm{KR}}(\mathbf{D})$. The deformed (or "$y$-ified") complex $\mathcal{Y}C_{\mathrm{KR}}(\mathbf{D})$ is constructed from this action using Koszul duality [BGS96] (see the earlier [BGG78] for the specific case of polynomial/exterior algebras). Explicitly, introduce formal parameters $y_1,\dots,y_r$ with $\mathrm{wt}(y_i)=\mathbf{q}^{-2}\mathbf{t}^2$ and form the complex

$$\mathcal{Y}C_{\mathrm{KR}}(\mathbf{D}) := C_{\mathrm{KR}}(\mathbf{D})\otimes\mathbb{Q}[y_1,\dots,y_r]\,, \quad \delta_{\mathrm{KR}}+\sum_{i=1}^{r} y_i\xi_i. \tag{1}$$

In [GH], it is shown that the homology of $\mathcal{Y}C_{\mathrm{KR}}(\mathbf{D})$ is a well-defined invariant of the oriented link $\mathbf{L}$, up to isomorphism.

The main goal of the present paper is to investigate the colored version of this invariant. For this, suppose that $\mathbf{L}$ is a (framed, oriented) *colored* link, i.e. each component $\mathsf{c}\in\pi_0(\mathbf{L})$ is assigned a color $b(\mathsf{c})\geq 0$. Recall from Example 1.9 that the monodromy algebra associated to a b-labeled component is $\mathbb{Q}[p_1,\dots,p_b]\otimes\bigwedge[\Xi_1,\dots,\Xi_b]\cong\mathrm{Sym}(\mathbb{X}^b)\otimes\bigwedge[\Xi_1,\dots,\Xi_b]$. The following is a consequence of Lemma 5.38; see Remark 5.39.

Proposition 1.10. *Let $\mathbf{D}$ be a diagram for a framed, oriented, colored link $\mathbf{L}$, presented as a braid closure. The colored Khovanov–Rozansky complex $C_{\mathrm{KR}}(\mathbf{D})$ admits an action of the algebra*

$$\bigotimes_{\mathsf{c}\in\pi_0(\mathbf{L})}\Big(\mathbb{Q}[p_1,\dots,p_{b(\mathsf{c})}]\otimes\textstyle\bigwedge[\Xi_1,\dots,\Xi_{b(\mathsf{c})}]\Big)\cong\big(\mathbb{Q}[p_{\mathsf{c},i}]\otimes\bigwedge[\Xi_{\mathsf{c},i}]\big)_{\mathsf{c}\in\pi_0(\mathbf{L}),1\leq i\leq b(\mathsf{c})}$$

in which $\Xi_{\mathsf{c},k}$ is the monodromy of $\frac{1}{k}p_k$ along c.

Using this monodromy action (and Koszul duality), we can build the following deformed complex:

$$\mathcal{Y}C_{\mathrm{KR}}(\mathbf{D}) := C_{\mathrm{KR}}(\mathbf{D})\otimes\mathbb{Q}[v_{\mathsf{c},k}]_{\mathsf{c}\in\pi_0(\mathbf{L}),1\leq k\leq b(\mathsf{c})}\,, \quad \delta_{\mathrm{KR}}+\sum_{\mathsf{c},k} v_{\mathsf{c},k}\Xi_{\mathsf{c},k}. \tag{2}$$

In Theorem 5.30, we establish the following.

Theorem 1.11. *The complex $\mathcal{Y}C_{\mathrm{KR}}(\mathbf{D})$ is a well-defined invariant of the framed, oriented, colored link $\mathbf{L}$, up to quasi-isomorphism of modules over $\mathbb{Q}[p_{\mathsf{c},i},v_{\mathsf{c},j}]_{\mathsf{c}\in\pi_0(\mathbf{L}),1\leq i,j\leq b(\mathsf{c})}$. Consequently, its homology $\mathcal{Y}H_{\mathrm{KR}}(\mathbf{L})$ is an invariant of $\mathbf{L}$ up to isomorphism of $\mathbb{Q}[p_{\mathsf{c},i},v_{\mathsf{c},j}]$-modules.*

Remark 1.12. In this paper, we actually take a different approach to the definition of the deformed complex from (2). Indeed, rather than first constructing the monodromy morphisms for a link diagram $\mathbf{D}$ and then using them to build $\mathcal{Y}C_{\mathrm{KR}}(\mathbf{D})$, we instead build the complex $\mathcal{Y}C_{\mathrm{KR}}(\mathbf{D})$ from local pieces that encode the homotopies associated with paths in $\mathbf{D}$ that traverse a single crossing. These take the form of certain *curved complexes*, that we discuss next. Nonetheless, the two approaches are equivalent; see Remark 5.39.

1.3. Curving Rickard complexes.

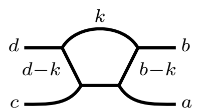
The undeformed colored Khovanov–Rozansky complex can be studied at the level of braids (without closing up to obtain a link) using the framework of singular Soergel bimodules. It will be quintessential to extend our deformed theory to the level of braids as well. Similar to the considerations in [GH], the notion of curved complexes of singular Soergel bimodules appears naturally.

Recall the monoidal 2-category[4] of *singular Soergel bimodules*, denoted SSBim. Objects of this category are sequences $\boldsymbol{a}=(a_1,\dots,a_m)$ of positive integers, and a 1-morphism from $\boldsymbol{b}=(b_1,\dots,b_{m'})$ to $\boldsymbol{a}=(a_1,\dots,a_m)$ is a certain kind of graded bimodule over $(R^{\boldsymbol{a}},R^{\boldsymbol{b}})$. Here, $R^{\boldsymbol{a}}$ denotes the ring of polynomials in $\sum_{i=1}^m a_i$ variables that are invariant with respect to the action of $\mathfrak{S}_{a_1}\times\cdots\times\mathfrak{S}_{a_m}$. Specifically, 1-morphisms SSBim are generated (with respect to direct sum and summands, grading shift, horizontal composition $\star$, and external tensor product $\boxtimes$) by induction and restriction bimodules

[4] Throughout, by 2-category we always mean a weak 2-category, also known as a bicategory.

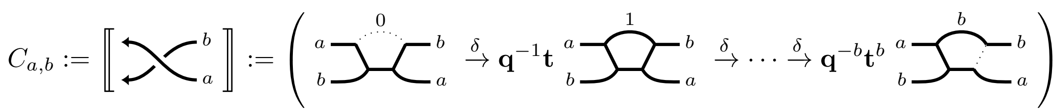
relating the rings $R^{a,b}$ and R^{a+b}. The 2-morphisms are maps of graded bimodules. See §3.4 for full details.

The 2-category SSBim contains the *singular Bott-Samelson bimodules*, those bimodules that are constructed from induction and restriction bimodules using only grading shift, $\star$, and $\boxtimes$. These bimodules can be depicted diagrammatically as certain trivalent graphs called *webs*, e.g.:



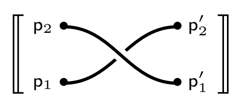
(Here, the unlabeled edge has label $a+b-k = c+d-k$). In diagrams such as these, a trivalent "merge" vertex (when read right-to-left) corresponds to a restriction bimodule ${}_{R^{a+b}}(R^{a,b})_{R^{a,b}}$ and a "split" vertex corresponds to an induction bimodule ${}_{R^{a,b}}(R^{a,b})_{R^{a+b}}$.

To each colored braid, one can associate a complex of singular Soergel bimodules using the notion of Rickard complexes. For instance, to the (a,b)-colored elementary crossing with $a \geq b$, one associates a complex of the form

(3) 

To each b-labeled edge appearing in such diagrams (either a colored braid diagram, a web depicting a singular Bott-Samelson, or a composition thereof) there is an action of the sheet algebra $\mathrm{Sym}(\mathbb{X}^b)$ from Example 1.4 on the associated (complex of) 1-morphism(s). We will denote the sheet algebra simply by $\mathrm{Sym}(\mathbb{X}_{\mathsf{p}})$, if we wish to emphasize the point where the sheet algebra is acting (thus $|\mathbb{X}_{\mathsf{p}}| = b(\mathsf{c})$).

It is well-known that the actions on the four endpoints of the Rickard complex are homotopic along the strands, see e.g. [RW16, Proposition 5.7]. For example, choosing points $\mathsf{p}_1, \mathsf{p}_2, \mathsf{p}'_1, \mathsf{p}'_2$ on $C_{a,b}$ as follows:

(4) 

we find that the actions of $f(\mathbb{X}_{\mathsf{p}_2})$ and $f(\mathbb{X}_{\mathsf{p}'_1})$ and the actions of $g(\mathbb{X}_{\mathsf{p}_1})$ and $g(\mathbb{X}_{\mathsf{p}'_2})$ on $C_{a,b}$ are homotopic, for all $f \in \mathrm{Sym}(\mathbb{X}^a)$ and $g \in \mathrm{Sym}(\mathbb{X}^b)$. Hence, there exists homotopies $\Psi^{\mathsf{o}}(f)$ and $\Psi^{\mathsf{u}}(g)$ so that

$$[\delta, \Psi^{\mathsf{o}}(f)] = f(\mathbb{X}_{\mathsf{p}_2}) - f(\mathbb{X}_{\mathsf{p}'_1}), \quad [\delta, \Psi^{\mathsf{u}}(g)] = g(\mathbb{X}_{\mathsf{p}_1}) - g(\mathbb{X}_{\mathsf{p}'_2}).$$

Following the recipe from §1.2, we should incorporate the homotopies for a collection of generators of $\mathrm{Sym}(\mathbb{X}^a)$ into our differential, as in (2). One choice of generating set is the collection of *elementary* symmetric functions $e_1, \ldots, e_a \in \mathrm{Sym}(\mathbb{X}^a)$. Considering the corresponding homotopies $\{\Psi^{\mathsf{o}}_k\}_{k=1}^a$ and $\{\Psi^{\mathsf{u}}_k\}_{k=1}^b$ (built in Lemma 4.20 below) and extending scalars, we thus can consider $C_{a,b}$, equipped with the "differential"

$$\delta^{\mathrm{tot}} := \delta + \sum_{k=1}^{a} \Psi^{\mathsf{o}}_k u^{\mathsf{o}}_k + \sum_{k=1}^{b} \Psi^{\mathsf{u}}_k u^{\mathsf{u}}_k \in \mathrm{End}(C_{a,b}) \otimes \mathbb{Q}[\mathbb{U}] \tag{5}$$

where $\mathbb{U} = \{u^{\mathsf{o}}_k\}_{1 \leq k \leq a} \cup \{u^{\mathsf{u}}_k\}_{1 \leq k \leq b}$. As we show, the homotopies $\Psi^{\mathsf{o}/\mathsf{u}}_i$ each square to zero and pairwise anti-commute, so we find that

$$(\delta^{\mathrm{tot}})^2 = \sum_{k=1}^{a} \big(e_k(\mathbb{X}_{\mathsf{p}_2}) - e_k(\mathbb{X}_{\mathsf{p}'_1})\big) u^{\mathsf{o}}_k + \sum_{k=1}^{b} \big(e_k(\mathbb{X}_{\mathsf{p}_1}) - e_k(\mathbb{X}_{\mathsf{p}'_2})\big) u^{\mathsf{u}}_k. \tag{6}$$

Hence, $\mathcal{Y}C_{a,b} := (C_{a,b}, \delta^{\mathrm{tot}})$ is not a chain complex in the traditional sense, but rather a *curved complex*. Recall that the latter is a generalization of the notion of chain complex, and consists of a pair $(X, \delta^{\mathrm{tot}})$ where the *curved differential* δ^{tot} squares to a nonzero element $F \in \mathrm{End}(X)$. We will refer to the curvature in (6), and its analogue for more general braids, as Δe-curvature.

More generally, using the operations $\star$ and $\boxtimes$ in SSBim, we can associate a Rickard complex $C(\beta_{\boldsymbol{b}})$ to any colored braid $\beta_{\boldsymbol{b}}$ that is built from complexes $C_{a,b}$ and $C^{\vee}_{a,b}$ assigned to colored positive and negative crossings, as in (3). In §4, we show that these complexes can be deformed in a similar manner to $C_{a,b}$.

Theorem 1.13. *The Rickard complex $C(\beta_{\boldsymbol{b}})$ associated to a colored braid $\beta_{\boldsymbol{b}}$ admits a deformation to a curved complex $\mathcal{Y}C(\beta_{\boldsymbol{b}})$ with Δe-curvature. Such a deformation is unique, up to homotopy equivalence.*

(See Theorem 4.24 and Lemma 4.15 for the precise statements.)

In fact, the construction of the *curved Rickard complex* $\mathcal{Y}C(\beta_{\boldsymbol{b}})$ closely parallels the construction of $C(\beta_{\boldsymbol{b}})$ from the elementary pieces $C_{a,b}$ and $C^{\vee}_{a,b}$. Indeed, with the curved complexes $\mathcal{Y}C_{a,b}$ and $\mathcal{Y}C^{\vee}_{a,b}$ associated to positive and negative crossings in hand, one need only construct well-defined composition operations to build the curved complexes associated to arbitrary braids. Hence, we establish the following.

Theorem 1.14. *There exists a monoidal dg 2-category $\mathcal{Y}(\mathrm{SSBim})$ wherein 1-morphisms are curved complexes of singular Soergel bimodules with Δe-curvature. Appropriate horizontal compositions $\star$ and external tensor products $\boxtimes$ of the curved complexes $\mathcal{Y}C_{a,b}$ and $\mathcal{Y}C^{\vee}_{a,b}$ associated with positive and negative crossings assign a 1-morphism in $\mathcal{Y}(\mathrm{SSBim})$ to any colored braid (word), which satisfies the braid relations up to canonical homotopy equivalence.*

In our description of $\mathcal{Y}C_{a,b}$ above (and subsequent statements about $\mathcal{Y}C(\beta_{\boldsymbol{b}})$), we chose the elementary symmetric functions as the generators of the sheet algebra $\mathrm{Sym}(\mathbb{X}^b)$ of a b-colored strand. Thus our homotopies Ψ_k encode Δe-curvature. At times, we will find it beneficial to work with curved complexes built from other homotopies, which similarly identify sheet algebra actions at the ends of braid strands. Specifically, let p and p' be points at the left and right ends of a b-colored braid strand, and let $N(\mathbb{X}_{\mathsf{p}}, \mathbb{X}_{\mathsf{p}'}) \lhd \mathrm{Sym}(\mathbb{X}_{\mathsf{p}}|\mathbb{X}_{\mathsf{p}'}) \cong \mathrm{Sym}(\mathbb{X}_{\mathsf{p}}) \otimes \mathrm{Sym}(\mathbb{X}_{\mathsf{p}'})$ denote the *diagonal ideal*, which is generated by all elements of the form $f(\mathbb{X}_{\mathsf{p}}) - f(\mathbb{X}_{\mathsf{p}'})$ with $f \in \mathrm{Sym}(\mathbb{X}^b)$. In addition to the set of generators

$$\mathcal{N}_e := \{e_k(\mathbb{X}_{\mathsf{p}}) - e_k(\mathbb{X}_{\mathsf{p}'}) \mid 1 \leq k \leq b\}$$

for $N(\mathbb{X}_{\mathsf{p}}, \mathbb{X}_{\mathsf{p}'})$, we can also work with the generating sets

$$\mathcal{N}_h := \{h_k(\mathbb{X}_{\mathsf{p}} - \mathbb{X}_{\mathsf{p}'}) \mid 1 \leq k \leq b\}, \quad \mathcal{N}_p := \{p_k(\mathbb{X}_{\mathsf{p}}) - p_k(\mathbb{X}_{\mathsf{p}'}) \mid 1 \leq k \leq b\}.$$

(See §2 for details on symmetric functions.) By change of variables, it is possible to pass from curved complexes of singular Soergel bimodules with Δe-curvature (i.e. curvature modeled on $\mathcal{N}_e$) to curved complexes with $h\Delta$-curvature and Δp-curvature, modeled on $\mathcal{N}_h$ and $\mathcal{N}_p$, respectively. See §4.5 and §5.5.

The choice of generators for $N(\mathbb{X}_{\mathsf{p}}, \mathbb{X}_{\mathsf{p}'})$ is conceptually immaterial, but each of the above leads to a notion of curved complex of singular Soergel bimodules that is useful in particular instances. For example, complexes with $h\Delta$-curvature appear most often "in the wild," and a straightforward change of variables leads to complexes with Δe-curvature that are well-behaved 2-categorically. Passing to Δp-curvature requires working over a field of characteristic zero, but such curvature is best adapted to establishing Markov invariance. In particular, Proposition 1.10 and Theorem 1.11 proceed by passing from curved Rickard complexes in $\mathcal{Y}(\mathrm{SSBim})$ to curved complexes with (appropriate) Δp-curvature, before taking braid closure to obtain (uncurved) complexes associated to the corresponding link diagram.

Remark 1.15 (y-variables vs. u-variables vs. v-variables)**.** In defining the curved differential δ^{tot} in (5), we used the variable name u to distinguish from the y variables appearing in the uncolored case (1). Similarly, we will use the variable names v and $\dot{v}$ in the setting of $h\Delta$- and Δp-curvature, respectively. In each of these settings, these deformation parameters play the role of (but are **not** literally equal to) a generating set of symmetric functions in the uncolored y-variables. The relation between the uncolored y deformation parameters and the v (and u) parameters is outlined below and discussed in detail in §4.6 and §4.7. It is best understood in the context of interpolation theory and is related to the geometry of the Hilbert scheme $\mathrm{Hilb}_n(\mathbb{C}^2)$.

1.4. **Relation to previous work.** A notion of curved Rickard complexes, and its application to link homology, has been studied by Cautis–Lauda–Sussan (CLS) in [CLS20]. Their construction starts at the level of a categorified quantum group $\mathcal{U}_Q(\mathfrak{sl}_m)$, where m corresponds to the number of braid strands. As such, their construction does not see one alphabet $\mathbb{X}_i$ per (left) braid boundary point $1 \leq i \leq m$, but only the formal difference alphabets $\mathbb{X}_i - \mathbb{X}_{i+1}$. Related to this, CLS consider only *one* deformation parameter u of weight $\mathbf{q}^{-2}\mathbf{t}^2$. On the other hand, our construction starts at the level of complexes of singular Soergel bimodules and considers a *family* of deformation parameters for each strand, whose size is given by the strand label. These parameters account for higher degree homotopies of Rickard complexes, as proposed in [CLS20, Section 1.3]. We expect that our constructions can be lifted from singular Soergel bimodules to a suitable version of the categorified quantum group $\mathcal{U}_Q(\mathfrak{gl}_m)$.

For a basic comparison of the constructions in [CLS20] to our construction, consider the curved Rickard complex associated to the positive crossing (3). Retaining the notation from (4), the curvature considered by CLS takes the form

$$\big(e_1(\mathbb{X}_{\mathsf{p}_2}) - e_1(\mathbb{X}_{\mathsf{p}'_1})\big)z^{\mathsf{o}}u - \big(e_1(\mathbb{X}_{\mathsf{p}_1}) - e_1(\mathbb{X}_{\mathsf{p}'_2})\big)z^{\mathsf{u}}u \tag{7}$$

This expression has only a single deformation parameter u, that is weighted by scalars z^{o} and z^{u}, and curved complexes with this curvature encode homotopic actions of $e_1(\mathbb{X}_{\mathsf{p}_2}) - e_1(\mathbb{X}_{\mathsf{p}_1})$ and $e_1(\mathbb{X}_{\mathsf{p}'_1}) - e_1(\mathbb{X}_{\mathsf{p}'_2})$. By contrast, our curvature (6) has a family of deformation parameters $\{u^{\mathsf{o}}_k\}_{k=1}^a$ associated to the over strand and a family $\{u^{\mathsf{u}}_k\}_{k=1}^b$ associated to the under strand. Curved complexes with this curvature encode the homotopic actions of *all* $e_k(\mathbb{X})$ along *each* strand. Note that by specializing $u^{\mathsf{o}}_1 = z^{\mathsf{o}}u$, $u^{\mathsf{u}}_1 = -z^{\mathsf{u}}u$ and $u^{\mathsf{o}}_k = 0 = u^{\mathsf{u}}_k$ for all $k > 0$ in (6), we recover (7). Hence, our link homologies encodes the CLS invariant as a special case, see §5.6 and 10.1.

Our multi-parameter curved Rickard complexes give rise to a variety of deformations of colored, triply-graded Khovanov–Rozansky link homology that appear closely related to the geometry of the Hilbert scheme $\mathrm{Hilb}_m(\mathbb{C}^2)$, and its isospectral analogue X_m. Indeed, recall that the aforementioned work of Gorsky–Hogancamp [GH] uses the y-ified (uncolored) triply-graded homology to establish a precise relation between $\mathcal{Y}H_{\mathrm{KR}}(\mathbf{L})$ and the *isospectral Hilbert scheme* X_m. There, the variables $\{x_1, \ldots, x_m, y_1, \ldots, y_m\}$ occurring in the curved Rickard complex assigned to an m-strand braid correspond to local coordinates on an open subset of X_m. In particular, in lowest Hochschild degree the y-ified Khovanov–Rozansky homology of an uncolored unknot is

$$\mathcal{Y}H^{\mathrm{low}}_{\mathrm{KR}}(\bigcirc) \cong \mathbb{Q}[x,y]$$

which (after extending scalars) is the coordinate ring of $\mathbb{C}^2 = \mathrm{Hilb}_1(\mathbb{C}^2)$. In the colored case, we have

$$\mathcal{Y}H^{\mathrm{low}}_{\mathrm{KR}}(\bigcirc_b) \cong \mathrm{Sym}(\mathbb{X}^b)[v_1, \ldots, v_b] = \mathbb{Q}[e_1(\mathbb{X}^b), \ldots, e_b(\mathbb{X}^b), v_1, \ldots, v_b] \tag{8}$$

after changing from the u to v variables, as in Remark 1.15. As explained in §4.7, these v-variables are *interpolation coordinates* for y-variables in terms of x-variables:

$$y_i = \sum_{r=1}^{b} x_i^{r-1} v_r \,. \tag{9}$$

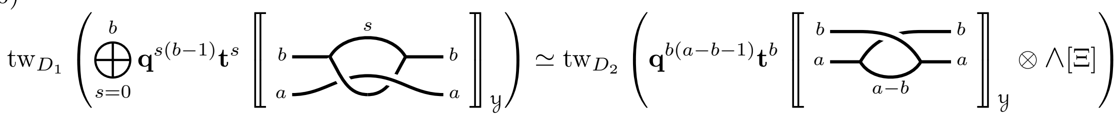
Comparing to [Hai01, Proposition 3.6.3], we see that the generators $\{e_1(\mathbb{X}^b), \dots, e_b(\mathbb{X}^b), v_1, \dots, v_b\}$ of $\mathcal{Y}H_{\mathrm{KR}}(\bigcirc_b)$ can be understood as coordinates on an open affine subset of $\mathrm{Hilb}_b(\mathbb{C}^2)$. We will comment further on relations between colored $\mathcal{Y}H_{\mathrm{KR}}(\mathbf{L})$ and Hilbert schemes in the following section.

1.5. **Curved colored skein relation, link splitting, and the Hopf link.** Important applications of deformed link homologies derive from their controlled behavior under unlinking, i.e. their *link splitting* properties, see e.g. Batson–Seed [BS15]. Closer to the present paper, the connection between (uncolored) $\mathcal{Y}H_{\mathrm{KR}}(\mathbf{L})$ and Hilbert schemes mentioned above relies on the *link splitting map* from the full twist braid to the identity braid. This map identifies the lowest Hochschild-degree summand of the homology of the (m, m)-torus link with an ideal $I_{1^m} \lhd \mathbb{Q}[x_1, \dots, x_m, y_1, \dots, y_m] =: \mathbb{Q}[\mathbb{X}, \mathbb{Y}]$, and X_m is precisely the blowup of $(\mathbb{C}^2)^m$ at $\mathbb{C} \otimes_{\mathbb{Q}} I_{1^m}$, by work of Haiman [Hai01, Proposition 3.4.2].

The (uncolored) link splitting map is determined by a deformation of the categorified HOMFLYPT skein relation. In order to understand link splitting behavior in the colored, triply-graded context, we develop a curved version of the colored skein relation that we introduced in the companion paper [HRW21]. The culmination of §6 is the following result. Herein, the brackets $[\![-]\!]_{\mathcal{Y}}$ denote curved lifts of the Rickard complexes associated to shown tangled webs, and tw_D denotes a twist in the differential by a curved Maurer–Cartan element D (see §4.1 for a review of this terminology).

Theorem 1.16 (Corollary 6.20)**.** *Let* $\Xi := \{\xi_1, \dots, \xi_b\}$ *be a set of exterior variables with* $\mathrm{wt}(\xi_i) = \mathbf{t}^{-1}\mathbf{q}^{2i}$*, then there exists a homotopy equivalence*

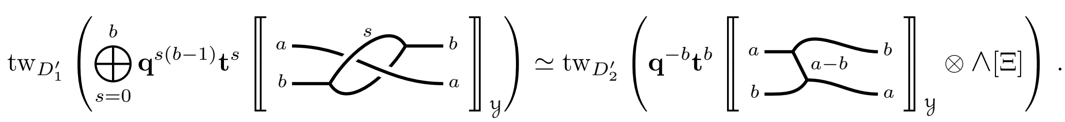
(10)



of curved twisted complexes. Further, the curved Maurer–Cartan element D_1 is one-sided with respect to the partial order by the index s, and the right-hand side is a certain curved Koszul complex.

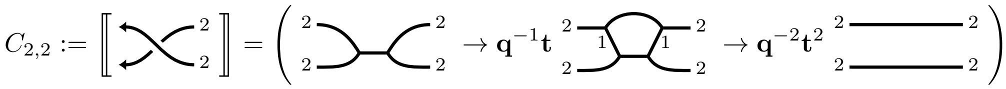
For obvious reasons, we will sometimes refer to the left-hand side of (10) as the complex of *threaded digons*, denoted by $\mathrm{TD}_b(a)$. As the notation suggests, it is useful to view this complex as a function of the threading a-colored strand. One consequence of Theorem 1.16 is that $\mathrm{TD}_b(a) \simeq 0$ when $a < b$ (in that case, the webs on the right-hand side correspond to the zero bimodule).

Remark 1.17. An equivalent formulation of the curved colored skein relation (10) that more closely resembles the usual HOMFLYPT skein relation (relating positive and negative crossings to their oriented resolution) is as follows:



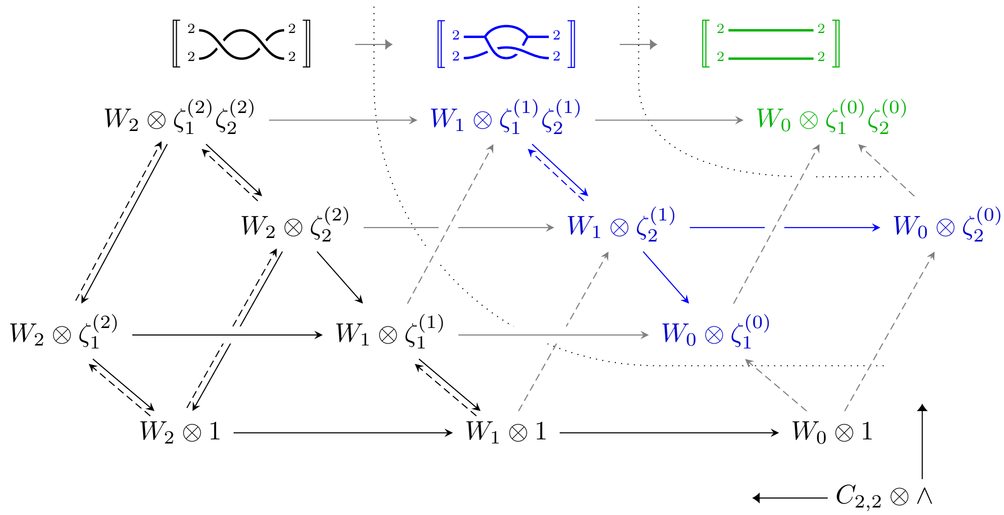
Note that the left-hand side involves complexes that interpolate between a positive and negative crossing. The twists here have similar properties to those in Theorem 1.16.

Example 1.18. The Rickard complex for a crossing between a pair of 2-colored strands has the form



We denote the webs appearing in this complex as W_2, W_1 and W_0 respectively. After basis change in the exterior algebras, the curved twisted complex on the right-hand side of (10) has the following schematic form:

(11)



The subquotients with respect to the filtration indicated by the dotted lines (and colored **black**, **blue**, and **green**) are homotopy equivalent to the indicated complexes that appear on the left-hand side of (10). (Compare to the corresponding figure in [HRW21, Section 1], where the dashed components of the differential do not appear.) Additional details can be found in Example 6.14.

In §7 we use the curved colored skein relation (10) to study splitting properties of the deformed, colored, triply-graded link homology. In particular, we obtain an explicit model for the *colored link splitting map* from the colored full twist on two strands to the identity braid in §7.2. In Theorem 7.14 we obtain a *simultaneous splitting* of the complex of threaded digons.

In §8, we begin the study of the colored full twist. Let us briefly recall the uncolored case, restricting to the lowest Hochschild degrees for brevity.

Theorem 1.19 ([GH]). *There is a canonical map (the splitting map) $\mathcal{Y}\,\mathrm{FT}_m \to \mathbf{1}_m$ relating the y-ified Rouquier complexes of the full-twist and trivial braids. This map induces an injective map*

$$\mathcal{Y}H_{\mathrm{KR}}(T(m,m))^{\mathrm{low}} \hookrightarrow \mathcal{Y}H_{\mathrm{KR}}(T(m,0))^{\mathrm{low}} = \mathbb{Q}[\mathbb{X},\mathbb{Y}]$$

of y-ified triply graded homologies, where $T(m,l)$ denotes the (m,l) torus link (in particular $T(m,0)$ is the m-component unlink and $T(m,m)$ is the closure of the full twist braid). Moreover, the image of the above map coincides with the ideal $I_{1^m} \lhd \mathbb{Q}[\mathbb{X},\mathbb{Y}]$ defined by

$$I_{1^m} := \mathbb{Q}[\mathbb{X},\mathbb{Y}]\cdot\{f(\mathbb{X},\mathbb{Y}) \mid f \in \mathbb{Q}[\mathbb{X},\mathbb{Y}] \text{ is antisymmetric for the diagonal action of } \mathfrak{S}_m\}\,.$$

A key component of this result is the *parity* property [EH19] enjoyed by $T(m,m)$ (see also [Mel17, HM19]).

Paralleling the uncolored case, we conjecture that the parity property holds in the colored setting as well; see Conjecture 8.5. Under this assumption, we show that, in lowest Hochschild degree, the deformed, colored homology of the $\boldsymbol{b}$-colored (m,m)-torus link embeds as an ideal $J_{\boldsymbol{b}}$ in the algebra

$$E_{\boldsymbol{b}} := \bigotimes_{i=1}^{m} \mathrm{Sym}(\mathbb{X}^{b_i})[v_{i,1},\ldots,v_{i,b_i}]\,.$$

In the uncolored case, the ideal J_{1^m} is precisely the ideal I_{1^m} mentioned above. Generalizing this, we pose the following (restated in the main text as Conjecture 8.15):

Conjecture 1.20. *Let $\boldsymbol{b} = (b_1, \dots, b_m)$ and $N = \sum_{i=1}^m b_i$. Introduce alphabets $\mathbb{X} = \{x_1, \dots, x_N\}$, $\mathbb{Y} = \{y_1, \dots, y_N\}$, and $\mathbb{V} = \{v_{i,r} \,|\, 1 \le i \le m,\ 1 \le r \le b_i\}$, and regard $\mathbb{Q}[\mathbb{X}, \mathbb{Y}]$ as a subalgebra of $\mathbb{Q}[\mathbb{X}, \mathbb{V}]$ via the appropriate analogue of (9). Let $T(m, m; \boldsymbol{b})$ denote the $\boldsymbol{b}$-colored (m, m)-torus link, then the colored splitting map identifies the lowest Hochschild degree summand of $\mathcal{Y}H_{\mathrm{KR}}\big(T(m, m; \boldsymbol{b})\big)$ with the ideal*

$$I_{\boldsymbol{b}} := E_{\boldsymbol{b}} \cdot \left\{ \frac{f(\mathbb{X}, \mathbb{Y})}{\Delta(\mathbb{X}^{b_1}) \cdots \Delta(\mathbb{X}^{b_m})} \mid f \in \mathbb{Q}[\mathbb{X}, \mathbb{Y}] \text{ is antisymmetric for } \mathfrak{S}_N \right\},$$

where $\Delta(\mathbb{X}^{b_i})$ denotes the Vandermonde determinant.

(To see that $I_{\boldsymbol{b}}$ is well-defined, we refer the reader to Lemma 8.13.)

The 1-strand case $\boldsymbol{b} = (b)$ of Conjecture 1.20 is trivial and the m-strand uncolored case $\boldsymbol{b} = 1^m$ was proven in [GH]. In §9, we prove this conjecture in the 2-strand case, i.e. for the (a, b)-colored Hopf link. First, in Proposition 9.4 we confirm that the colored Hopf link is indeed parity. We then embark on a complicated inductive journey, using the simultaneous colored skein splitting referenced above, which culminates in Theorem 9.33 with the verification of Conjecture 1.20 when $m = 2$. Along the way, we encounter specific *Haiman determinants* (reviewed in §2.4) and in Corollary 9.40 we give an explicit set of generators for $I_{a,b}$ using these elements.

Finally, in §10, we use Theorem 9.33 to extend our link splitting results from §7 to the case of arbitrary colored links. For certain specializations, this generalizes results obtained in [BS15, GH] to our setting, and recovers the colored link splitting result from [CLS20]. We also speculate on the interpretation of more-interesting specializations, making contact with Conjecture 1.25, which is stated in the following section.

1.6. **Further conjectures.** To conclude this (extended) introduction, we collect two particularly enticing conjectures.

1.6.1. *Homology of cables.* We propose a precise relation between deformed, colored, triply-graded homology and the deformed (uncolored) triply-graded homology of cables, focusing on the case of cabled knots, for ease of exposition. Recall the following result in the undeformed setting.

Theorem 1.21 ([GW19, Theorem 6.1 and Corollary 6.5])**.** *Let $\mathbf{K}$ be a framed, oriented knot, and let $\mathbf{K}^b$ denote the b-cable of $\mathbf{K}$ (a b-component link). Both the triply-graded Khovanov–Rozansky complex $C_{\mathrm{KR}}(\mathbf{K}^b)$ and the $\mathfrak{gl}_N$ Khovanov–Rozansky complex $C_{\mathrm{KR}_N}(\mathbf{K}^b)$ carry an action of the symmetric group $\mathfrak{S}_b$, up to homotopy, induced by braiding components of the cable. In the $\mathfrak{gl}_N$ case, the isotypic component corresponding to the trivial representation is equivalent to the b-colored $\mathfrak{gl}_N$ Khovanov–Rozansky complex $C_{\mathrm{KR}_N}(\mathbf{K}; b)$.*

It is natural to consider the extension of this result for deformed triply-graded link homology. Interestingly, the most naïve extension of this result is false.

Example 1.22. Let $\mathbf{U}$ be the 0-framed unknot, and $\mathbf{U}^b$ its b-cable. Then,

$$\mathcal{Y}C^{\mathrm{low}}_{\mathrm{KR}}(\mathbf{U}^b) = \mathbb{Q}[x_1, \dots, x_b, y_1, \dots, y_b]$$

with the standard $\mathfrak{S}_b$-action. The $\mathfrak{S}_b$-invariant part of $\mathcal{Y}C^{\mathrm{low}}_{\mathrm{KR}}(\mathbf{U}^b)$ does not equal the b-colored invariant

$$\mathcal{Y}C^{\mathrm{low}}_{\mathrm{KR}}(\mathbf{U}; b) = \mathbb{Q}[x_1, \dots, x_b]^{\mathfrak{S}_b} \otimes \mathbb{Q}[v_1, \dots, v_b]$$

from (8), since $\mathrm{wt}(v_k) = \mathbf{q}^{-2k}\mathbf{t}^2$ while $\mathrm{wt}(y_i) = \mathbf{q}^{-2}\mathbf{t}^2$ for all i.

Remark 1.23. Note that there is an injective algebra map

$$\mathcal{Y}C^{\mathrm{low}}_{\mathrm{KR}}(\mathbf{U}^b)^{\mathfrak{S}_b} = \mathbb{Q}[\mathbb{X}, \mathbb{Y}]^{\mathfrak{S}_b} \to \mathbb{Q}[\mathbb{X}, \mathbb{V}]^{\mathfrak{S}_b} = \mathcal{Y}C^{\mathrm{low}}_{\mathrm{KR}}(\mathbf{U}; b), \quad y_i \mapsto \sum_{k=1}^{b} x_i^{k-1} v_k. \tag{12}$$

By Lemma 4.33, the variables v_k can be expressed as certain ratios of the form $\frac{f(\mathbb{X},\mathbb{Y})}{\Delta(\mathbb{X})}$ in which $f(\mathbb{X},\mathbb{Y})$ is $\mathfrak{S}_b$-antisymmetric, and the map (12) extends to an isomorphism $\mathbb{Q}[\mathbb{X},\mathbb{Y},I_{1^b}/\Delta(\mathbb{X})]^{\mathfrak{S}_b} \to \mathcal{Y}C^{\mathrm{low}}_{\mathrm{KR}}(\mathbf{U};b)$, where $\mathbb{Q}[\mathbb{X},\mathbb{Y},I_{1^b}/\Delta(\mathbb{X})]$ denotes the subalgebra of $\mathbb{Q}[\mathbb{X},\mathbb{Y},\Delta(\mathbb{X})^{-1}]$ generated by $\mathbb{Q}[\mathbb{X},\mathbb{Y}]$ and elements $f\Delta(\mathbb{X})^{-1}$ with $f \in I_{1^b}$.

To generalize this pattern to general knots, we must reinterpret the process of adjoining ratios $f/\Delta(\mathbb{X})$ in which $f \in I_{1^b}$. Motivated by the relation between I_{1^b} and full twists (Theorem 1.19), we formulate the following.

Conjecture 1.24. *Let $\mathbf{K}$ be a framed, oriented knot, and let $\chi\mathbf{K}$ denote $\mathbf{K}$ with framing increased by one. For each integer $b \geq 0$ we consider a directed system of complexes*

$$\mathcal{Y}C_{\mathrm{KR}}(\mathbf{K}^b)^{triv} \to \mathcal{Y}C_{\mathrm{KR}}((\chi\mathbf{K})^b)^{sgn} \to \mathcal{Y}C_{\mathrm{KR}}((\chi^2\mathbf{K})^b)^{triv} \to \cdots \tag{13}$$

in which the maps are inherited from the "bottom eigenmap" for the curved Rickard complex $\mathcal{Y}C(\mathrm{FT}_b)$ associated to the b-strand full-twist FT_b. The (homotopy) colimit of this directed system is quasi-isomorphic to the b-colored curved Rickard complex $\mathcal{Y}C_{\mathrm{KR}}(\mathbf{K},b)$, as complexes of modules over

$$\mathcal{Y}C_{\mathrm{KR}}(\mathbf{U};b) \cong \operatorname{colim}\left(\mathcal{Y}C_{\mathrm{KR}}(\mathbf{U}^b)^{triv} \to \mathcal{Y}C_{\mathrm{KR}}((\chi\mathbf{U})^b)^{sgn} \to \cdots\right).$$

Let us comment on this conjecture. First, note that the b-cable of $\chi\mathbf{K}$ and the b-cable of $\mathbf{K}$ differ by the insertion of a full twist braid FT_b. Thus, the directed system (13) is inherited from a directed system

$$\mathbf{1}_b \to \mathrm{FT}_b \to \mathrm{FT}_b^2 \to \cdots$$

in the category $\mathcal{Y}\mathrm{SBim}_b$ defined in [GH]. There are many choices one can make for the connecting maps. Indeed, results in [GH] show that

$$\mathrm{Hom}_{\mathcal{Y}\mathrm{SBim}_b}(\mathrm{FT}_b^k, \mathrm{FT}_b^{k+1}) \simeq \mathrm{Hom}_{\mathcal{Y}\mathrm{SBim}_b}(\mathbf{1}_b, \mathrm{FT}_b) \simeq \langle \mathbb{Q}[\mathbb{X},\mathbb{Y}]^{\mathrm{sgn}} \rangle \subset \mathbb{Q}[\mathbb{X},\mathbb{Y}].$$

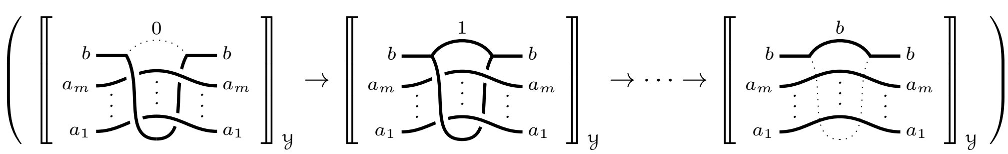
Among all such morphisms, the one corresponding to the Vandermonde determinant $\Delta(\mathbb{X})$ is distinguished as the generator of cohomological degree zero. The associated morphism $\mathbf{1}_b \to \mathrm{FT}_b$ in $\mathcal{Y}\mathrm{SBim}_b$ (or in the undeformed category of complexes or Soergel bimodules) is referred to as the "bottom eigenmap", adopting terminology from [EH17].

Conjecture 1.24 is supported by our Theorem 9.33, i.e. our verification of Conjecture 1.20 in the case of colored Hopf links, since taking colimits of directed systems of the form (13) is akin to inverting the Vandermonde $\Delta(\mathbb{X})$. We save explorations along these lines for future work.

1.6.2. *Threaded digons and the Hilbert scheme.* Finally, in a different direction, we propose that a generalization of the complex $\mathrm{TD}_b(a)$ of threaded digons from Theorem 1.16 constructs a family of complexes desired in the Gorsky–Neguţ–Rasmussen (GNR) conjecture [GNR21].

Conjecture 1.25. *Let $a_1,\ldots,a_m,b \geq 0$ and set $\boldsymbol{a} = (a_1,\ldots,a_m)$, then there exists a one-sided twisted complex of threaded digons:*

$\mathrm{TD}_b(\boldsymbol{a}) =$



(potentially with longer arrows pointing to the right) such that:

(i) $\mathrm{TD}_b(\boldsymbol{a}) \simeq 0$ *if* $b > a_1 + \cdots + a_m$*, and*

(ii) the partial trace $\mathrm{Tr}^b(\mathrm{TD}_b(1,\ldots,1) \otimes \mathbb{Q}[v_1,\ldots,v_b])$ *categorifies the* b^{th} *elementary symmetric function in the Jucys-Murphy braids on m strands (here, $\{v_1,\ldots,v_b\}$ are deformation parameters for the b-labeled strand). Further, this is the complex $\mathcal{E}_b \in \mathcal{Y}(\mathrm{SBim}_m)$ corresponding to the* b^{th} *exterior*

power of the tautological bundle on the flag Hilbert scheme $\mathrm{FHilb}_m(\mathbb{C}^2)$*, which has rank* m*, under the GNR conjecture.*

The $m = 1$ case of Conjecture 1.25 follows from Theorem 1.16. Indeed, we noted above that $\mathrm{TD}_b(a) \simeq 0$ when $a < b$, and $\mathrm{Tr}^1(\mathrm{TD}_1(1) \otimes \mathbb{Q}[v_1])$ can be explicitly identified with the identity braid on one strand using the right-hand side of (10). For item (ii), observe that the elementary symmetric function $e_b(-)$ has similar behavior to the (conjectured) behavior of $\mathrm{TD}_b(-)$: it vanishes when the number of variables is smaller than b. Less-heuristically, results of Morton [Mor02] show that the complex $\mathrm{Tr}^1(\mathrm{TD}_1(1, \ldots, 1) \otimes \mathbb{Q}[v_1])$ would categorify the sum (i.e. 1^{st} elementary symmetric function) of the Jucys-Murphy elements. The extension to general b is suggested by this, and Conjecture 1.24. Lastly, we mention that work of Elias [Eli] proposes a different approach to constructing $\mathcal{E}_1$.

1.7. **Coefficient conventions.** Throughout, we work over the field $\mathbb{Q}$ of rational numbers for simplicity. All of our results remain true over an arbitrary field of characteristic zero. We believe that all results should hold over the integers; however, the proof we present for Markov invariance of $\mathcal{Y}H_{\mathrm{KR}}(\mathbf{L})$ uses the power-sum symmetric functions $\frac{1}{k}p_k(\mathbb{X})$, thus requires working over a field of characteristic zero. Nonetheless, our 2-category $\mathcal{Y}(\mathrm{SSBim})$ is defined using Δe-curvature (equivalently, $h\Delta$-curvature), which allows for an integral version of this 2-category.

1.8. **Organization of the paper.** In Section 2 we recall background on symmetric functions, including the formalism of symmetric functions in the difference of two alphabets and Haiman determinants. Section 3 introduces categorical background and sets up conventions for gradings and homological algebra necessary to introduce curved Rickard complexes of singular Soergel bimodules, which happens in Section 4. This section also includes a thorough discussion of different choices of deformation parameters with their relative advantages and relations between them. Section 5 develops the deformed, colored triply-graded link homology. Proceeding towards a study of link splitting, in Section 6 we obtain a curved colored skein relation, which we use in Section 7 to construct splitting maps. Section 8 introduces conjectures relating the deformed homology of colored torus links to Hilbert schemes, which we prove in the case of two strands in Section 9 by computing the homology of the colored Hopf link using the curved colored skein relation. In the final Section 10 we study the link splitting properties of the deformed colored triply-graded link homology.

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2. Symmetric functions

In this section, we collect assorted background material on symmetric functions. The reader can safely skip §2.1, §2.2, and §2.3 and return whenever results or formulas from here are used in the later parts of the paper. We do, however, recommend a look at §2.4 for readers unfamiliar with Haiman determinants.

2.1. Symmetric functions.

Symmetric functions play an important role throughout this paper. A more detailed exposition appears in [HRW21, Section 2.1].

Alphabets are finite or countably infinite sets that we denote by blackboard letters, such as $\mathbb{X}$, $\mathbb{Y}$ etc. Given an alphabet $\mathbb{X}$, the $\mathbb{Q}$-algebra of symmetric functions on $\mathbb{X}$ will be denoted $\mathrm{Sym}(\mathbb{X})$. Symmetric functions on a finite alphabet will also be called symmetric polynomials. For pairwise disjoint alphabets $\mathbb{X}_1, \dots, \mathbb{X}_r$, we write

$$\mathrm{Sym}(\mathbb{X}_1|\cdots|\mathbb{X}_r) \cong \mathrm{Sym}(\mathbb{X}_1) \otimes \cdots \otimes \mathrm{Sym}(\mathbb{X}_r)$$

for the ring of functions on $\mathbb{X}_1 \cup \cdots \cup \mathbb{X}_r$ that are separately symmetric in each of the alphabets $\mathbb{X}_i$.

Definition 2.1. The *elementary* symmetric functions $e_j(\mathbb{X})$, *complete* symmetric functions $h_j(\mathbb{X})$, and *power sum* symmetric functions $p_j(\mathbb{X})$ are each defined via their generating functions as follows:

$$\begin{aligned} E(\mathbb{X},t) &:= \prod_{x\in\mathbb{X}}(1+xt) =: \sum_{j\geq 0} e_j(\mathbb{X})t^j \\ H(\mathbb{X},t) &:= \prod_{x\in\mathbb{X}}(1-xt)^{-1} =: \sum_{j\geq 0} h_j(\mathbb{X})t^j \\ P(\mathbb{X},t) &:= \sum_{x\in\mathbb{X}}\frac{xt}{1-xt} =: \sum_{j\geq 1} p_j(\mathbb{X})t^j \end{aligned}$$

By convention $e_0(\mathbb{X}) = h_0(\mathbb{X}) = 1$ and $p_0(\mathbb{X})$ is undefined. In the case of countably infinite alphabets, we sometimes drop the alphabet from the notation and write $E(t)$, $H(t)$, and $P(t)$, and e_j, h_j, and p_j, for the functions introduced above.

The elementary and complete symmetric functions are related by the identity

$$H(\mathbb{X},t)E(\mathbb{X},-t) = 1\,, \quad \text{i.e.} \quad \sum_{i+j=k}(-1)^j h_i(\mathbb{X})e_j(\mathbb{X}) = \delta_{k,0} \quad \forall k\geq 0\,, \tag{14}$$

and each are related to the power sum symmetric functions by the Newton identity:

$$\frac{t\frac{d}{dt}H(\mathbb{X},t)}{H(\mathbb{X},t)} = P(\mathbb{X},t)\,, \quad \text{i.e.} \quad H(\mathbb{X},t) = \exp\int P(\mathbb{X},t)\frac{dt}{t}\,.$$

We will work with the highly useful formalism of linear combinations of alphabets, see [HRW21, Definition 2.3]. In particular, for the generating functions in Definition 2.1, we have

$$\begin{aligned} H(a_1\mathbb{X}_1 + a_2\mathbb{X}_2, t) &= H(\mathbb{X}_1,t)^{a_1}H(\mathbb{X}_2,t)^{a_2} \\ E(a_1\mathbb{X}_1 + a_2\mathbb{X}_2, t) &= E(\mathbb{X}_1,t)^{a_1}E(\mathbb{X}_2,t)^{a_2} \\ P(a_1\mathbb{X}_1 + a_2\mathbb{X}_2, t) &= a_1P(\mathbb{X}_1,t) + a_2P(\mathbb{X}_2,t) \end{aligned}$$

for any $a_1, a_2 \in \mathbb{Q}$. A good illustration of how this formalism works is given by the following computation:

$$H(\mathbb{X}-\mathbb{X}',t) = H(\mathbb{X},t)H(\mathbb{X}',t)^{-1} = H(\mathbb{X},t)E(\mathbb{X}',-t) = E(\mathbb{X}',-t)E(\mathbb{X},-t)^{-1},$$

from which we obtain

$$h_k(\mathbb{X}-\mathbb{X}') = \sum_{i+j=k}(-1)^j h_i(\mathbb{X})e_j(\mathbb{X}'). \tag{15}$$

2.2. **Kernel of multiplication.** If A is a commutative algebra, we let $N \lhd A \otimes A$ be the kernel of the multiplication map $A \otimes A \to A$. Equivalently, N is the ideal inside $A \otimes A$ generated by differences $a \otimes 1 - 1 \otimes a$.

Remark 2.2. The Hochschild homology $\mathrm{HH}_\bullet(A)$ of a commutative algebra A is itself a graded-commutative algebra. We have $\mathrm{HH}_0(A) \cong A$ and $\mathrm{HH}_1(A) \cong N/N^2$. More generally, each $\mathrm{HH}_k(A)$ is a module over $\mathrm{HH}_0(A) \cong A$.

We will focus on the case $A = \mathrm{Sym}(\mathbb{X})$ for a finite alphabet $\mathbb{X}$ and identify $\mathrm{Sym}(\mathbb{X})^{\otimes 2} = \mathrm{Sym}(\mathbb{X}|\mathbb{X}')$. Let $N(\mathbb{X}, \mathbb{X}') \lhd \mathrm{Sym}(\mathbb{X}|\mathbb{X}')$ be the ideal generated by elements of the form $f(\mathbb{X}) - f(\mathbb{X}')$. In this section, we will record various relationships between generating sets of $N(\mathbb{X}, \mathbb{X}')$.

Proposition 2.3. *If $|\mathbb{X}| = |\mathbb{X}'| = a$, then $N(\mathbb{X}, \mathbb{X}') \lhd \mathrm{Sym}(\mathbb{X}|\mathbb{X}')$ is generated by any of the following:*

$$\{e_k(\mathbb{X}) - e_k(\mathbb{X}')\}_{1\le k\le a}\,, \quad \{h_k(\mathbb{X}) - h_k(\mathbb{X}')\}_{1\le k\le a}\,, \quad \{p_k(\mathbb{X}) - p_k(\mathbb{X}') = p_k(\mathbb{X} - \mathbb{X}')\}_{1\le k\le a}$$
$$\{h_k(\mathbb{X} - \mathbb{X}')\}_{1\le k\le a}\,, \quad \{e_k(\mathbb{X} - \mathbb{X}')\}_{1\le k\le a}\,.$$

Furthermore, the element $h_k(\mathbb{X} - \mathbb{X}')$ equals $\frac{1}{k} p_k(\mathbb{X} - \mathbb{X}')$ modulo $N(\mathbb{X}, \mathbb{X}')^2$.

Proof. Lemma 2.4 below shows that the families of elements $e_k(\mathbb{X}) - e_k(\mathbb{X}')$, $h_k(\mathbb{X} - \mathbb{X}')$, $h_k(\mathbb{X}) - h_k(\mathbb{X}')$, and $e_k(\mathbb{X} - \mathbb{X}')$ for $1 \le k \le a$ generate the same ideal. Since any symmetric function f can be written as a polynomial in the e_k for various k, it is straightforward to check that this ideal is $N(\mathbb{X}, \mathbb{X}')$. Indeed, one direction of containment is immediate, so it suffices to prove that any $f(\mathbb{X}) - f(\mathbb{X}')$ is in the ideal generated by the $e_k(\mathbb{X}) - e_k(\mathbb{X}')$. To see this, we may restrict to the case where f is a monomial in the e_k and then induct on the degree of the monomial. If $f = e_k$, the statement is trivial. Otherwise write $f = e_{k_0} f'$ for some k_0 and expand

$$f(\mathbb{X}) - f(\mathbb{X}') = (e_{k_0}(\mathbb{X}) - e_{k_0}(\mathbb{X}'))f'(\mathbb{X}) + e_{k_0}(\mathbb{X}')(f'(\mathbb{X}) - f'(\mathbb{X}'))$$

which is in the ideal generated by the $e_k(\mathbb{X}) - e_k(\mathbb{X}')$ by the induction hypothesis. Finally, observe that $p_k(\mathbb{X}) - p_k(\mathbb{X}') = p_k(\mathbb{X} - \mathbb{X}')$, so each of the above symmetric functions involves the virtual alphabet $\mathbb{X} - \mathbb{X}'$. The relations (16) and (19) then imply the second and third statement. □

Lemma 2.4. *Let $\mathbb{X}, \mathbb{X}'$ be alphabets. For $k \ge 1$, we have*

$$p_k(\mathbb{X} - \mathbb{X}') = \sum_{j=1}^{k} (-1)^{k-j} j h_j(\mathbb{X} - \mathbb{X}') e_{k-j}(\mathbb{X} - \mathbb{X}')\,, \tag{16}$$

$$h_k(\mathbb{X} - \mathbb{X}') = \sum_{j=1}^{k} (-1)^{j-1} h_{k-j}(\mathbb{X})(e_j(\mathbb{X}) - e_j(\mathbb{X}'))\,, \tag{17}$$

and their inverse rewriting formulas

$$e_k(\mathbb{X}) - e_k(\mathbb{X}') = \sum_{j=1}^{k} (-1)^{j-1} e_{k-j}(\mathbb{X}) h_j(\mathbb{X} - \mathbb{X}')\,, \tag{18}$$

$$h_k(\mathbb{X} - \mathbb{X}') = \frac{1}{k} \sum_{j=1}^{k} h_{k-j}(\mathbb{X} - \mathbb{X}') p_j(\mathbb{X} - \mathbb{X}')\,. \tag{19}$$

Moreover,

$$(-1)^k e_k(\mathbb{X} - \mathbb{X}') = h_k(\mathbb{X}' - \mathbb{X}) = -\sum_{j=1}^{k} h_{k-j}(\mathbb{X}' - \mathbb{X}) h_j(\mathbb{X} - \mathbb{X}')\,. \tag{20}$$

Note that we obtain another collection of identities by applying the algebra involution $p_k \mapsto -p_k$, $h_k \leftrightarrow (-1)^k e_k$, or by swapping the roles of $\mathbb{X}$ and $\mathbb{X}'$.

Proof. These equations are efficiently proved by the following manipulations of generating functions, i.e.

$$P(\mathbb{X}-\mathbb{X}',t) = \big(t\frac{d}{dt}H(\mathbb{X}-\mathbb{X}',t)\big)E(\mathbb{X}-\mathbb{X}',-t)$$
$$H(\mathbb{X},t)H(\mathbb{X}',t)^{-1}-1 = -H(\mathbb{X},t)\big(E(\mathbb{X},-t)-E(\mathbb{X}',-t)\big)$$

and

$$\big(E(\mathbb{X},-t)-E(\mathbb{X}',-t)\big) = -E(\mathbb{X},-t)(H(\mathbb{X},t)H(\mathbb{X}',t)^{-1}-1)$$
$$t\frac{d}{dt}H(\mathbb{X}-\mathbb{X}',t) = H(\mathbb{X}-\mathbb{X}',t)P(\mathbb{X}-\mathbb{X}',t)$$

establish (16) – (19). Equation (20) is proved by the computation

$$E(\mathbb{X}-\mathbb{X}',-t)-1 = H(\mathbb{X}'-\mathbb{X},t)-1 = -H(\mathbb{X}'-\mathbb{X},t)(H(\mathbb{X}-\mathbb{X}',t)-1)\,.$$ □

2.3. **Hook Schur functions and h-reduction.** Let $\mathbb{X}$ be an alphabet (or a formal linear combination of alphabets). If $\mathbb{Y}$ is an alphabet with $|\mathbb{Y}| \leq c$, then complete symmetric functions $h_N(\mathbb{X})$ can be expressed as $\mathrm{Sym}(\mathbb{X}+\mathbb{Y})$-linear combinations of complete symmetric functions $h_n(\mathbb{X})$ for $n \leq c$. We refer to this process as *h-reduction* and describe it explicitly in Lemma 2.9.

Example 2.5. Consider the following identity in the polynomial ring $\mathbb{Q}[x_1,\dots,x_a]$ (for each $1 \leq i \leq a$)

$$0 = \prod_{j=1}^{a}(x_i-x_j) = \sum_{k=0}^{a}(-1)^{a-k}e_{a-k}(\mathbb{X})x_i^k\,.$$

This allows us to write x_i^a as a $\mathbb{Q}[\mathbb{X}]^{\mathfrak{S}_a}$-linear combination of monomials x_i^k with $0 \leq k \leq a-1$.

$$x_i^a = \sum_{k=0}^{a-1}(-1)^{a-k-1}e_{a-k}(\mathbb{X})x_i^k\,.$$

The general case of h-reduction requires Schur functions associated to hook shapes.

Definition 2.6. For $i,j \geq 0$ we write $\mathfrak{s}_{(i|j)} := \mathfrak{s}_{(i+1,1^j)}$ for the hook Schur functions, which can be described as the family of symmetric functions satisfying:

$$\mathfrak{s}_{(i-1|0)} = h_i\,,\quad \mathfrak{s}_{(0|j-1)} = e_j\,,\quad h_ie_j = \mathfrak{s}_{(i|j-1)} + \mathfrak{s}_{(i-1|j)}\,.$$

By convention $\mathfrak{s}_{(i|j)} = 0$ if $i < 0$ or $j < 0$. We denote the two-parameter generating function of the hook Schur functions in an alphabet $\mathbb{X}$ by

$$S(t,u) := \sum_{i,j\geq 0}\mathfrak{s}_{(i|j)}t^iu^j\,.$$

Lemma 2.7. *The two-parameter generating function of the hook Schur functions satisfies*

$$S(t,u) = \frac{H(t)E(u)-1}{t+u}\,.$$

Proof. The rearrangement $H(t)E(u) = 1+(t+u)S(t,u)$ is a generating function restatement of the characterizing identity

$$h_ie_j = \mathfrak{s}_{(i|j-1)} + \mathfrak{s}_{(i-1|j)}\,,\quad (i,j) \in \mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}\smallsetminus\{(0,0)\}\,,$$

provided we interpret $\mathfrak{s}_{(i|-1)}$ and $\mathfrak{s}_{(-1|j)}$ as zero. □

In particular the algebra automorphism $\mathrm{Sym}(\mathbb{X}) \to \mathrm{Sym}(\mathbb{X})$ sending $h_k \leftrightarrow e_k$ also sends $S(t,u) \mapsto S(u,t)$, hence $\mathfrak{s}_{(i|j)} \mapsto \mathfrak{s}_{(j|i)}$. The following description of hook Schur functions is also useful.

Lemma 2.8. *For $i, j \geq 0$ we have*

$$(-1)^j \mathfrak{s}_{(i|j)} = \sum_{k+l=j} (-1)^l h_{i+k+1} e_l = \sum_{k+l=i} (-1)^{l+j} h_k e_{j+l+1}\,.$$

Proof. We will prove the first identity; the second follows by symmetry (apply the involution $h_k \leftrightarrow e_k$). First, note that we have the generating function identity

$$\frac{H(t)-H(u)}{t-u} = \sum_{k\geq 0} \frac{t^k - u^k}{t-u} h_k = \sum_{i,j\geq 0} h_{i+j+1} t^i u^j.$$

Then we rewrite the hook Schur generating function as follows

$$S(t,-u) = \frac{H(t)E(-u)-1}{t-u} = \frac{H(t)-H(u)}{t-u} E(-u) = \sum_{i,j,k\geq 0} (-1)^k h_{i+j+1} e_k t^i u^{j+k},$$

which gives rise to the identity in the statement. □

Lemma 2.9 (h-reduction)**.** *If $\mathbb{Y}$ has cardinality $|\mathbb{Y}| \leq c$, then for any $\mathbb{X}$ and $r \geq 1$ we have*

$$h_{c+r}(\mathbb{X}) = \sum_{0\leq i\leq c} (-1)^{c-i} \mathfrak{s}_{(r-1|c-i)}(\mathbb{X}+\mathbb{Y}) h_i(\mathbb{X})\,. \tag{21}$$

Before proving this lemma, we note the following special cases.

Corollary 2.10 (Reducing monomials)**.** *For all $m \geq a \geq 1$ and $1 \leq i \leq a$, we have*

$$x_i^m = \sum_{1\leq j\leq a} (-1)^{a-j} \mathfrak{s}_{(m-a|a-j)}(x_1,\ldots,x_a) x_i^{j-1}.$$

Proof. Take $\mathbb{X} = \{x_i\}$, $\mathbb{Y} = \{x_1,\ldots,\widehat{x_i},\ldots,x_a\}$, and $c = a-1$ in Lemma 2.9. □

Corollary 2.11. *If $\mathbb{X}$ and $\mathbb{X}'$ are alphabets of cardinality c, then we have*

$$h_{c+r}(\mathbb{X}-\mathbb{X}') = \sum_{1\leq i\leq c} (-1)^{c-i} \mathfrak{s}_{(r-1|c-i)}(\mathbb{X}) h_i(\mathbb{X}-\mathbb{X}')\,.$$

Proof. Take $\mathbb{X} \mapsto \mathbb{X}-\mathbb{X}'$ and $\mathbb{Y} \mapsto \mathbb{X}'$ in Lemma 2.9. Note that the $i=0$ summand in (21) is zero in this case since the Young diagram for the hook $(r-1|c)$ has $c+1$ rows, which exceeds the cardinality of $\mathbb{X}$. □

Proof of Lemma 2.9. We begin by proving the $\mathbb{Y} = \varnothing$ case. Lemma 2.8 gives us the identity

$$(-1)^i \mathfrak{s}_{(r-1|i)}(\mathbb{X}) h_j(\mathbb{X}) = \sum_{k+l=i} (-1)^l h_{r+k}(\mathbb{X}) e_l(\mathbb{X}) h_j(\mathbb{X})\,.$$

Summing up over all indices $i, j \geq 0$ with $i+j=c$ yields

$$\sum_{i+j=c} (-1)^i \mathfrak{s}_{(r-1|i)}(\mathbb{X}) h_j(\mathbb{X}) = \sum_{j+k+l=c} (-1)^l h_{k+r}(\mathbb{X}) e_l(\mathbb{X}) h_j(\mathbb{X}) \overset{(14)}{=} h_{c+r}(\mathbb{X})\,,$$

which is the $|\mathbb{Y}| = 0$ case of the lemma.

Now, we deduce the general identity. First, we compute

$$h_{c+r}(\mathbb{X}) = h_{c+r}((\mathbb{X}+\mathbb{Y})-\mathbb{Y}) = \sum_{i=0}^{c} (-1)^i h_{c-i+r}(\mathbb{X}+\mathbb{Y}) e_i(\mathbb{Y})\,.$$

We have used the fact that $e_i(\mathbb{Y}) = 0$ for $i > c \geq |\mathbb{Y}|$. Next, we apply the $\mathbb{Y} = \varnothing$ case of the lemma to rewrite $h_{c-i+r}(\mathbb{X}+\mathbb{Y})$. The resulting identity is

$$h_{c+r}(\mathbb{X}) = \sum_{i+j+k=c} (-1)^{k+i} \mathfrak{s}_{(r-1|k)}(\mathbb{X}+\mathbb{Y}) h_j(\mathbb{X}+\mathbb{Y}) e_i(\mathbb{Y}) \,.$$

Finally, we fix k and sum over all indices i, j with $i + j = c - k$ to obtain

$$h_{c+r}(\mathbb{X}) = \sum_{0 \leq k \leq c} (-1)^k \mathfrak{s}_{(r-1|k)}(\mathbb{X}+\mathbb{Y}) h_{c-k}(\mathbb{X}) \,. \qquad \square$$

2.4. **Haiman determinants.** Our description of the deformed, colored homology of (m, m)-torus knots (conjectural, when $m > 2$) in §8 and §9 relies on certain (anti)symmetric polynomials constructed using determinants.

Let R be a commutative $\mathbb{Q}$-algebra and consider a tuple $(f_1, \ldots, f_N)$ of elements $f_i \in R$. Let $f_{i,j}$ denote the element of $R^{\otimes N}$ given by $f_{i,j} = 1 \otimes \cdots \otimes f_i \otimes \cdots \otimes 1$, where f_i occurs in the j-th position, and set

$$\operatorname{hdet}(f_1, \ldots, f_N) := \det(f_{i,j})_{1 \leq i,j \leq N} := \begin{vmatrix} f_{1,1} & \cdots & f_{1,N} \\ \vdots & \ddots & \vdots \\ f_{N,1} & \cdots & f_{N,N} \end{vmatrix} \in R^{\otimes N} \tag{22}$$

Equivalently, $\operatorname{hdet}(f_1, \ldots, f_N)$ is the anti-symmetrization of $f_1 \otimes \cdots \otimes f_N$ inside $R^{\otimes N}$.

Remark 2.12. We will consider this construction in the special cases $R = \mathbb{Q}[x]$ and $R = \mathbb{Q}[x, y]$. In such cases, we will identify $R^{\otimes N}$ with the polynomial ring $\mathbb{Q}[\mathbb{X}]$ or $\mathbb{Q}[\mathbb{X}, \mathbb{Y}]$, respectively, where $\mathbb{X} = \{x_1, \ldots, x_N\}$ and $\mathbb{Y} = \{y_1, \ldots, y_N\}$. When the $f_i \in R$ are monic monomials, we refer to the elements constructed in (22) as *Haiman determinants*, due to their appearance in [Hai01, Section 2.2].

It will be useful to introduce the following short-hand.

Definition 2.13. Let $\lambda_1 \geq \cdots \geq \lambda_N \geq 0$ be a weakly decreasing sequence of non-negative integers of length N, then we associate to it an N-tuple of monomials in x as follows:

$$\mathcal{M}_N(\lambda) := (x^{\lambda_1+N-1}, \ldots, x^{\lambda_{N-1}+1}, x^{\lambda_N}).$$

Convention 2.14. Given a positive integers $N \geq l$ and a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ with l parts, we will sometimes view λ as a weakly decreasing sequence of length N by appending to it $N - l$ zeros.

Example 2.15. We have $\mathcal{M}_N(\emptyset) = (x^{N-1}, \ldots, x, 1)$ and thus $\operatorname{hdet}(\mathcal{M}_N(\emptyset)) = \Delta(\mathbb{X})$, where $\Delta(\mathbb{X}) = \prod_{1 \leq i < j \leq N}(x_i - x_j)$ is the usual *Vandermonde determinant*. More generally, for a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ with $l \leq N$, Jacobi's bialternant formula implies

$$\operatorname{hdet}(\mathcal{M}_N(\lambda)) = \Delta(\mathbb{X}) \mathfrak{s}_\lambda(\mathbb{X}), \tag{23}$$

where $\mathfrak{s}_\lambda(\mathbb{X})$ denotes the Schur polynomial associated to λ in the alphabet $\mathbb{X}$ of cardinality N.

Definition 2.16. Let S be a finite set of monic monomials in $R = \mathbb{Q}[x, y]$ and set $N := |S|$. We identify $R^{\otimes N} = \mathbb{Q}[\mathbb{X}, \mathbb{Y}]$ where $|\mathbb{X}| = N = |\mathbb{Y}|$ and consider the associated *Haiman determinant*

$$\Delta_S(\mathbb{X}, \mathbb{Y}) := \operatorname{hdet}(S) \in \mathbb{Q}[\mathbb{X}, \mathbb{Y}].$$

Our convention is to order the monomials in S by writing $S = S^0 \cup S^1 y \cup \cdots \cup S^r y^r$ where each S^k is a tuple of monomials in x, written in decreasing order.

Example 2.17. For $S = \{x^2, x, 1, y\}$, the Haiman determinant is

$$\Delta_S(\mathbb{X},\mathbb{Y}) = \begin{vmatrix} x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1 & x_2 & x_3 & x_4 \\ 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix}.$$

3. Categorical background

In this section we review background on homological algebra and singular Soergel bimodules.

3.1. **Categories of coefficients.** We will assume familiarity with the notion of a category *enriched* in a symmetric monoidal category $\mathcal{R}$. In particular, if $\mathcal{B}$ is $\mathcal{R}$-enriched, then $\mathrm{Hom}_{\mathcal{B}}(X,Y)$ is an object of $\mathcal{R}$ for all $X, Y \in \mathcal{B}$, and the composition of morphisms in $\mathcal{B}$ is given by morphisms in $\mathcal{R}$

$$\mathrm{Hom}_{\mathcal{B}}(Y,Z) \otimes \mathrm{Hom}_{\mathcal{B}}(X,Y) \to \mathrm{Hom}_{\mathcal{B}}(X,Z)$$

where $\otimes$ is the monoidal structure in $\mathcal{R}$. We view $\mathcal{R}$ as the *category of coefficients* for $\mathcal{B}$. In this section, we discuss the various categories of coefficients that will appear in this paper.

We let $\mathcal{K}$ denote the category of finite-dimensional $\mathbb{Q}$-vector spaces and let $\overline{\mathcal{K}}$ denote the category of all $\mathbb{Q}$-vector spaces. A category is $\mathbb{Q}$-*linear* if it is enriched in $\overline{\mathcal{K}}$.

Let Γ be an abelian group and let $\mathcal{K}[\Gamma]$ denote the category of finite-dimensional Γ-graded $\mathbb{Q}$-vector spaces. An object of this category is a $\mathbb{Q}$-vector space of the form $\bigoplus_{\gamma\in\Gamma} M_\gamma$ where each M_γ is finite-dimensional, and $M_\gamma = 0$ for all but finitely many $\gamma \in \Gamma$. Morphism spaces in this category are the Γ-graded $\mathbb{Q}$-vector spaces

$$\mathrm{Hom}_{\mathcal{K}[\Gamma]}(M,N) = \bigoplus_{\gamma\in\Gamma} \mathrm{Hom}^{\gamma}_{\mathcal{K}[\Gamma]}(M,N)$$

where

$$\mathrm{Hom}^{\gamma}_{\mathcal{K}[\Gamma]}(M,N) = \prod_{\gamma'\in\Gamma} \mathrm{Hom}_{\mathcal{K}}(M_{\gamma'}, N_{\gamma'+\gamma}) .$$

Now, suppose Γ is equipped with a symmetric bilinear form $\langle -,- \rangle \colon \Gamma \times \Gamma \to \mathbb{Z}/2\mathbb{Z}$. This determines a monoidal structure on $\mathcal{K}[\Gamma]$, given on objects by

$$(M \otimes N)_\gamma = \bigoplus_{\gamma_1+\gamma_2=\gamma} M_{\gamma_1} \otimes N_{\gamma_2}$$

and morphisms by

$$(f \otimes g)(m \otimes n) = (-1)^{\langle \deg(g), \deg(m) \rangle} f(m) \otimes g(n).$$

The monoidal structure just defined is symmetric, with braiding given by

$$M \otimes N \to N \otimes M\,, \quad m \otimes n \mapsto (-1)^{\langle \deg(m), \deg(n) \rangle} n \otimes m\,.$$

At times, we will relax the finiteness conditions that define $\mathcal{K}[\Gamma]$, and let $\overline{\mathcal{K}}[\![\Gamma]\!]$ denote the category of Γ-graded $\mathbb{Q}$-vector spaces. This category is again symmetric monoidal, via the same formulae.

Now, if Γ is additionally equipped with an element $\mathsf{d} \in \Gamma$ satisfying $\langle \mathsf{d}, \mathsf{d} \rangle = 1 \in \mathbb{Z}/2\mathbb{Z}$, then we can consider the category $\mathcal{K}[\Gamma]_{\mathrm{dg}}$ of Γ-graded complexes with differentials of degree d. (Note that we suppress $\langle -,- \rangle$ and d from this notation, as they will be clear in context.) Objects in $\mathcal{K}[\Gamma]_{\mathrm{dg}}$ are pairs (X, δ) where $X \in \mathcal{K}[\Gamma]$ and $\delta \in \mathrm{End}^{\mathsf{d}}_{\mathcal{K}[\Gamma]}(X)$ satisfies $\delta^2 = 0$, and morphism spaces are the complexes

$$\mathrm{Hom}_{\mathcal{K}[\Gamma]_{\mathrm{dg}}}\Big((X,\delta_X),(Y,\delta_Y)\Big) = \mathrm{Hom}_{\mathcal{K}[\Gamma]}(X,Y)$$

with differential $f \mapsto \delta_Y \circ f - (-1)^{|f|} f \circ \delta_X$. This category is the prototypical differential Γ-graded category. It comes equipped with a tensor product

$$(X, \delta_X) \otimes (Y, \delta_Y) = \Big(X \otimes Y \,,\ \delta_X \otimes \mathrm{id}_Y + \mathrm{id}_X \otimes \delta_Y \Big) .$$

The category $\overline{\mathcal{K}}[\![\Gamma]\!]_{\mathrm{dg}}$ is defined similarly.

Definition 3.1. A $\mathbb{Q}$-linear category $\mathcal{B}$ is Γ-*graded* if it is enriched in $\overline{\mathcal{K}}[\![\Gamma]\!]$ and *differential* Γ-*graded* if it is enriched in $\overline{\mathcal{K}}[\![\Gamma]\!]_{\mathrm{dg}}$.

The most important Γ for our uses are introduced in the following examples.

Example 3.2. Let $\Gamma = \mathbb{Z}$ with trivial bilinear form $\langle i, j \rangle = 0$. We will identify the group algebra $\mathbb{Q}[\mathbb{Z}]$ with the algebra of Laurent polynomials $\mathbb{Q}[\mathbb{Z}] = \mathbb{Q}[\mathbf{q}^{\pm}]$, and similarly indicate the choice of coordinate $\mathbf{q}$ in $\mathbb{Z}$ by writing $\mathbb{Z} = \mathbb{Z}_{\mathbf{q}}$. Paralleling this notation, we denote

$$\mathcal{K}[\mathbf{q}^{\pm}] := \mathcal{K}[\mathbb{Z}_{\mathbf{q}}] \,, \quad \overline{\mathcal{K}}[\![\mathbf{q}^{\pm}]\!] := \overline{\mathcal{K}}[\![\mathbb{Z}_{\mathbf{q}}]\!].$$

In both cases, we also let $\mathbf{q}$ denote the grading shift functor, defined by $(\mathbf{q}M)_i = M_{i-1}$. These category appear when we consider graded rings, modules, and bimodules. Note that the dg categories $\mathcal{K}[\Gamma]_{\mathrm{dg}}$ and $\overline{\mathcal{K}}[\![\mathbb{Z}_{\mathbf{q}}]\!]_{\mathrm{dg}}$ are trivial for this choice of $\langle -, - \rangle$, since there is no element $\mathsf{d} \in \mathbb{Z}$ with $\langle \mathsf{d}, \mathsf{d} \rangle = 1$.

Example 3.3. Let $\Gamma = \mathbb{Z}_{\mathbf{t}}$ with $\langle j, j' \rangle = jj'$ and $\mathsf{d} = 1$, then $\mathcal{K}[\Gamma]_{\mathrm{dg}}$ is the usual category of bounded complexes of finite-dimensional $\mathbb{Q}$-vector spaces (with the cohomological convention for complexes, i.e. differentials have degree $+1$).

Example 3.4. Combining Examples 3.2 and 3.3, let $\Gamma = \mathbb{Z}_{\mathbf{q}} \times \mathbb{Z}_{\mathbf{t}}$ with $\langle (i, j), (i', j') \rangle = jj'$. On the level of group algebras, we have $\mathbb{Q}[\mathbb{Z}_{\mathbf{q}} \times \mathbb{Z}_{\mathbf{t}}] = \mathbb{Q}[\mathbf{q}^{\pm}, \mathbf{t}^{\pm}]$, and we denote the categories of graded $\mathbb{Q}$-vector spaces similarly:

$$\mathcal{K}[\mathbf{q}^{\pm}, \mathbf{t}^{\pm}] := \mathcal{K}[\mathbb{Z}_{\mathbf{q}} \times \mathbb{Z}_{\mathbf{t}}] \,, \quad \overline{\mathcal{K}}[\![\mathbf{q}^{\pm}, \mathbf{t}^{\pm}]\!] := \overline{\mathcal{K}}[\![\mathbb{Z}_{\mathbf{q}} \times \mathbb{Z}_{\mathbf{t}}]\!] \,.$$

As above, we regard monomials $\mathbf{q}^i \mathbf{t}^j$ as a grading shift functors, via $(\mathbf{q}^i \mathbf{t}^j M)_{k,l} = M_{k-i, l-j}$. Taking $\mathsf{d} = (0, 1)$ gives the dg categories

$$\mathcal{K}[\mathbf{q}^{\pm}, \mathbf{t}^{\pm}]_{\mathrm{dg}} := \mathcal{K}[\mathbb{Z}_{\mathbf{q}} \times \mathbb{Z}_{\mathbf{t}}]_{\mathrm{dg}} \,, \quad \overline{\mathcal{K}}[\![\mathbf{q}^{\pm}, \mathbf{t}^{\pm}]\!]_{\mathrm{dg}} := \overline{\mathcal{K}}[\![\mathbb{Z}_{\mathbf{q}} \times \mathbb{Z}_{\mathbf{t}}]\!]_{\mathrm{dg}}$$

which appear when we consider complexes of graded modules or bimodules over $\mathbb{Z}_{\mathbf{q}}$-graded rings.

Finally, triply-graded Khovanov–Rozansky homology takes values in the following symmetric monoidal category of triply-graded $\mathbb{Q}$-vector spaces.

Example 3.5. Let $\Gamma = \mathbb{Z}_{\mathbf{a}} \times \mathbb{Z}_{\mathbf{q}} \times \mathbb{Z}_{\mathbf{t}}$. As in Examples 3.3 and 3.4, the $\mathbf{t}$-grading is cohomological, so we take $\mathsf{d} = (0, 0, 1)$. The $\mathbf{a}$-grading also has a cohomological flavor, but is *independent* from $\mathbf{t}$. This is reflected in our choice of symmetric bilinear form:

$$\langle (i, j, k), (i', j', k') \rangle = ii' + kk' \,.$$

The resulting categories of triply-graded $\mathbb{Q}$-vector spaces will be denoted $\mathcal{K}[\mathbf{a}^{\pm}, \mathbf{q}^{\pm}, \mathbf{t}^{\pm}]$ and $\overline{\mathcal{K}}[\![\mathbf{a}^{\pm}, \mathbf{q}^{\pm}, \mathbf{t}^{\pm}]\!]$, and complexes therein by $\mathcal{K}[\mathbf{a}^{\pm}, \mathbf{q}^{\pm}, \mathbf{t}^{\pm}]_{\mathrm{dg}}$ and $\overline{\mathcal{K}}[\![\mathbf{a}^{\pm}, \mathbf{q}^{\pm}, \mathbf{t}^{\pm}]\!]_{\mathrm{dg}}$. These categories occur when we consider (co)homological functors, such as Hochschild (co)homology, applied to complexes of graded bimodules.

Convention 3.6. In (differential) Γ-graded categories for Γ as in Examples 3.2, 3.4, and 3.5, we will typically indicate the degree of a morphism multiplicatively, by indicating its *weight*. For example, $\mathrm{wt}(f) = \mathbf{a}^i \mathbf{q}^j \mathbf{t}^k$ means that the $\mathbb{Z}_{\mathbf{a}} \times \mathbb{Z}_{\mathbf{q}} \times \mathbb{Z}_{\mathbf{t}}$-degree of f is (i, j, k).

3.2. **Complexes and curved complexes.** Retaining notation from the previous section, the most important differential Γ-graded categories are categories of (curved) complexes. To begin, suppose $\mathcal{A}$ is a $\mathbb{Q}$-linear category. Let $\mathcal{A}[\mathbf{t}^{\pm}]$ denote the category whose objects are sequences $(X_i)_{i\in\mathbb{Z}}$ with $X_i \in \mathcal{A}$ and $X_i = 0$ for all but finitely many i, and morphisms

$$\mathrm{Hom}_{\mathcal{A}[\mathbf{t}^{\pm}]}(X,Y) := \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}^k_{\mathcal{A}[\mathbf{t}^{\pm}]}(X,Y)\,, \quad \mathrm{Hom}^k_{\mathcal{A}[\mathbf{t}^{\pm}]}(X,Y) := \prod_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(X_i, Y_{i+k})\,.$$

In other words $\mathrm{Hom}_{\mathcal{A}[\mathbf{t}^{\pm}]}(X,Y)$ is the $\mathbb{Z}$-graded $\mathbb{Q}$-vector space spanned by homogeneous *multimaps* $(f_i)_{i\in\mathbb{Z}}$ with $f_i \in \mathrm{Hom}_{\mathcal{A}}(X_i, Y_{i+k})$.

The dg category of bounded chain complexes $\mathcal{C}(\mathcal{A})$ can be built from $\mathcal{A}[\mathbf{t}^{\pm}]$ in a standard way. Objects of $\mathcal{C}(\mathcal{A})$ are complexes: pairs (X,δ) where $X \in \mathcal{A}[\mathbf{t}^{\pm}]$ and $\delta \in \mathrm{End}^1_{\mathcal{A}[\mathbf{t}^{\pm}]}(X)$ with $\delta^2 = 0$. The morphism spaces in $\mathcal{C}(\mathcal{A})$ are the complexes

$$\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(X,Y) = \mathrm{Hom}_{\mathcal{A}[\mathbf{t}^{\pm}]}(X,Y)\,, \quad d\colon f \mapsto \delta_Y \circ f - (-1)^{|f|} f \circ \delta_X\,.$$

Remark 3.7. If $\mathcal{A}$ is Γ-graded, then $\mathcal{A}[\mathbf{t}^{\pm}]$ is $\Gamma \times \mathbb{Z}_{\mathbf{t}}$ graded. Here, we equip the latter with the $\mathbb{Z}/2\mathbb{Z}$-valued symmetric bilinear form

$$\langle (\gamma, j), (\gamma', j') \rangle_{\Gamma\times\mathbb{Z}} = \langle \gamma, \gamma' \rangle + jj'$$

and let $\mathsf{d} = (0,1)$. In other words, when forming categories of complexes over Γ-graded categories, our differentials have degree $(0,1) \in \Gamma \times \mathbb{Z}_{\mathbf{t}}$. In this way, $\mathcal{C}(\mathcal{A})$ is a differential $\Gamma \times \mathbb{Z}_{\mathbf{t}}$-graded category.

We will define the category of curved complexes in a similar fashion. Informally, the basic idea is to replace the equation $\delta^2 = 0$ with the equation $\delta^2 = F$, where F is an element of the center of $\mathcal{A}[\mathbf{t}^{\pm}]$. (We will see below that this does not work *sensu stricto*, but that there is any easy fix.)

Definition 3.8. The *center* of a $\mathbb{Z}_{\mathbf{t}}$-graded category $\mathcal{B}$ is the $\mathbb{Z}$-graded algebra $\mathcal{Z}(\mathcal{B})$ of natural transformations from the identity functor of $\mathcal{B}$ to itself. Precisely, a degree k element of $\mathcal{Z}(\mathcal{B})$ is an assignment $X \mapsto F|_X \in \mathrm{End}^k_{\mathcal{B}}(X)$ satisfying *super-naturality*: $f \circ F|_X = (-1)^{k|f|} F|_Y \circ f$ for all $f \in \mathrm{Hom}_{\mathcal{B}}(X,Y)$.

Lemma 3.9. *We have* $\mathcal{Z}(\mathcal{A}[\mathbf{t}^{\pm}]) = \mathcal{Z}(\mathcal{A})$.

Proof. Each object $X \in \mathcal{A}[\mathbf{t}^{\pm}]$ comes equipped with a family of idempotent endomorphisms e_i which project on to the i-th object X_i. Any central element must commute with these e_i, hence must be degree zero. Now, a degree zero central element F acting on X is completely determined by how it acts on X_0, since such an F must commute with the degree k map relating X and its shift $\mathbf{t}^k X$. □

We now run into a slight snag: δ^2 has degree two, yet the only degree two element of $\mathcal{Z}(\mathcal{A})$ is zero. In order to remedy this, the two standard approaches are to collapse the grading from $\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$ or formally extend scalars from $\mathbb{Q}$ to an appropriate graded ring. We prefer to preserve the $\mathbb{Z}$-grading, hence extend scalars as follows.

Definition 3.10. If $\mathcal{B}$ is $\mathbb{Z}_{\mathbf{t}}$-graded category and R is a $\mathbb{Z}_{\mathbf{t}}$-graded ring, then we let $\mathcal{B} \otimes R$ denote the $\mathbb{Z}_{\mathbf{t}}$-graded category with the same objects as $\mathcal{B}$, and morphisms

$$\mathrm{Hom}_{\mathcal{B}\otimes R}(X,Y) := \mathrm{Hom}_{\mathcal{B}}(X,Y) \otimes R\,.$$

Remark 3.11. The center of $\mathcal{B} \otimes R$ is isomorphic to $\mathcal{Z}(\mathcal{B}) \otimes R$. In particular, the center of $\mathcal{A}[\mathbf{t}^{\pm}] \otimes R$ is isomorphic to $\mathcal{Z}(\mathcal{A}) \otimes R$.

Definition 3.12 (Curved complexes)**.** Let $\mathcal{A}$ be a $\mathbb{Q}$-linear category, R a $\mathbb{Z}_{\mathbf{t}}$-graded ring, and F a degree two element of $\mathcal{Z}(\mathcal{A}) \otimes R$. Let $\mathcal{C}_F(\mathcal{A}; R)$ denote the dg category whose objects are pairs (X,δ) with $X \in \mathcal{A}[\mathbf{t}^{\pm}]$ and $\delta \in \mathrm{End}^1_{\mathcal{A}[\mathbf{t}^{\pm}]\otimes R}(X)$ satisfying $\delta^2 = F|_X$. Morphism spaces in $\mathcal{C}_F(\mathcal{A}; R)$ are the complexes

$$\mathrm{Hom}_{\mathcal{C}_F(\mathcal{A})}(X,Y) := \mathrm{Hom}_{\mathcal{A}[\mathbf{t}^{\pm}]\otimes R}(X,Y)\,, \quad d\colon f \mapsto \delta_Y \circ f - (-1)^{|f|} f \circ \delta_X\,.$$

When the ring R is clear from context, we will omit it from the notation and denote the category $\mathcal{C}_F(\mathcal{A};R)$ simply by $\mathcal{C}_F(\mathcal{A})$.

Remark 3.13. If $\mathcal{A}$ is Γ-graded, then both $\mathcal{C}(\mathcal{A})=\mathcal{C}_0(\mathcal{A})$ and $\mathcal{C}_F(\mathcal{A})$ are dg $\Gamma\times\mathbb{Z}_{\mathbf{t}}$-graded categories.

3.3. **Frobenius extensions.** Frobenius extensions between rings of partially symmetric polynomial rings will play an important role in defining morphisms between singular Soergel bimodules.

Definition 3.14. A *Frobenius extension* is an inclusion of commutative rings $\iota\colon A\hookrightarrow B$ such that B is free and finitely generated as an A-module, together with a non-degenerate A-linear map $\partial\colon B\to A$, called the *trace*. Here, *non-degeneracy* asserts the existence of A-linear *dual bases* $\{x_\alpha\}$ and $\{x'_\alpha\}$ for B such that $\partial(x_\alpha x'_\beta)=\delta_{\alpha,\beta}$. For a *graded Frobenius extension* between graded rings, we require ι to be grading preserving and ∂ and the dual bases to be homogeneous.

Fix $N>0$, and let $R:=\mathbb{Q}[x_1,\dots,x_N]$ be the polynomial ring, $\mathbb{Z}_{\mathbf{q}}$-graded by declaring $\deg_{\mathbf{q}}(x_i)=2$. Given a parabolic subgroup $\mathfrak{S}_{\boldsymbol{a}}=\mathfrak{S}_{a_1}\times\cdots\times\mathfrak{S}_{a_m}$ of the symmetric group $\mathfrak{S}_N$, we let $R^{\boldsymbol{a}}\subseteq R$ denote the ring of polynomials invariant under the action of $\mathfrak{S}_{\boldsymbol{a}}$. Note that $R^{\boldsymbol{b}}\subset R^{\boldsymbol{a}}$ if and only if $\mathfrak{S}_{\boldsymbol{b}}\supset\mathfrak{S}_{\boldsymbol{a}}$.

Lemma 3.15. *If $\mathfrak{S}_{\boldsymbol{b}}\supset\mathfrak{S}_{\boldsymbol{a}}$, then $R^{\boldsymbol{b}}\hookrightarrow R^{\boldsymbol{a}}$ is a graded Frobenius extension of rank $|\mathfrak{S}_{\boldsymbol{b}}/\mathfrak{S}_{\boldsymbol{a}}|$. To describe the trace, let $w_{\boldsymbol{b}}$ denote the longest element of $\mathfrak{S}_{\boldsymbol{b}}$ and $w_{\boldsymbol{b}/\boldsymbol{a}}=s_{i_1}\cdots s_{i_k}$ a minimal length coset representative for $w_{\boldsymbol{b}}\mathfrak{S}_{\boldsymbol{a}}$. Then we have:*

$$\partial\colon R^{\boldsymbol{a}}\to R^{\boldsymbol{b}}\,,\quad f\mapsto\partial_{i_1}\cdots\partial_{i_k}f\,,\quad \textit{where}\quad \partial_i(g)=\frac{g-s_i(g)}{x_i-x_{i+1}}\,.$$

Here, the transposition s_i acts by swapping variables x_i and x_{i+1} in g.

Proof. See [Wil08, Theorem 3.1.1 and Corollary 2.14]. □

More specifically, the following two examples will be used throughout.

Example 3.16. Let $\mathbb{X}=\{x_1,\dots,x_N\}$ be an alphabet with $\deg_{\mathbf{q}}(x_i)=2$. Then $\mathrm{Sym}(\mathbb{X})\hookrightarrow\mathbb{Q}[\mathbb{X}]$ is a graded Frobenius extension of rank $N!$ with non-degenerate trace given by:

$$\mathbb{Q}[\mathbb{X}]\ni f\mapsto(\partial_1\cdots\partial_{N-1})\cdots(\partial_1\partial_2)\partial_1 f=\frac{\mathrm{Alt}(f)}{\Delta(\mathbb{X})}\in\mathrm{Sym}(\mathbb{X})\,.$$

Here, Alt denotes the antisymmetrizer $\sum_{\sigma\in\mathfrak{S}_a}(-1)^\sigma\sigma\in\mathbb{Q}[\mathfrak{S}_N]$ acting on $\mathbb{Q}[\mathbb{X}]$ by the permutation of variables and $\Delta(\mathbb{X})=\prod_{1\le i<j\le N}(x_i-x_j)$ denotes the Vandermonde determinant. A $\mathrm{Sym}(\mathbb{X})$-linear basis of $\mathbb{Q}[\mathbb{X}]$ is given by the monomials $x_1^{n_1}x_2^{n_2}\cdots x_{N-1}^{n_{N-1}}$ where $0\le n_i\le N-i$. We have

$$\partial(x_1^{n_1}x_2^{n_2}\cdots x_{N-1}^{n_{N-1}})=\begin{cases}1 & \text{if } n_i=N-i \text{ for } 1\le i\le N,\\ 0 & \text{otherwise.}\end{cases}$$

The basis dual to the monomial basis above has elements $\prod_{k=1}^{N-1}(-1)^{b_k}e_{b_k}(x_{N+1-k},\dots,x_N)$ where $b_k=k-n_{N-k}$ for $1\le k\le N-1$.

Example 3.17. Let $\mathbb{X}_1=\{x_1,\dots,x_a\}$ and $\mathbb{X}_2=\{x_{a+1},\dots,x_{a+b}\}$ be alphabets with $|\mathbb{X}_1|=a$ and $|\mathbb{X}_2|=b$. Then $\mathrm{Sym}(\mathbb{X}_1+\mathbb{X}_2)\hookrightarrow\mathrm{Sym}(\mathbb{X}_1|\mathbb{X}_2)$ is a graded Frobenius extension of rank ab with trace given by the *Sylvester operator*:

$$\partial_{a,b}\colon\ \mathrm{Sym}(\mathbb{X}_1|\mathbb{X}_2)\ni f\mapsto(\partial_b\cdots\partial_1)\cdots(\partial_{a+b-1}\cdots\partial_a)f\in\mathrm{Sym}(\mathbb{X}_1+\mathbb{X}_2)\,.$$

A $\mathrm{Sym}(\mathbb{X}_1+\mathbb{X}_2)$-linear basis of $\mathrm{Sym}(\mathbb{X}_1|\mathbb{X}_2)$ is given by the Schur functions $\mathfrak{s}_\lambda(\mathbb{X}_1)$ indexed by partitions λ with $\lambda_1\le b$ having at most a parts (i.e. the Young diagram for λ fits inside the $a\times b$ box). We denote the set of such partitions by $P(a,b)$. The dual basis is then given by the signed Schur functions $(-1)^{|\widehat{\lambda}|}\mathfrak{s}_{\widehat{\lambda}}(\mathbb{X}_2)$ where $\widehat{\lambda}\in P(b,a)$ denotes the dual complementary partition. The sum of products of pairs of basis and dual-basis elements appears when expanding the Schur function of the $a \times b$ box in the difference of alphabets:

$$\mathfrak{s}_{b^a}(\mathbb{X}_1 - \mathbb{X}_2) = \sum_{\lambda \in P(a,b)} (-1)^{|\widehat{\lambda}|} \mathfrak{s}_\lambda(\mathbb{X}_1) \mathfrak{s}_{\widehat{\lambda}}(\mathbb{X}_2)$$

3.4. **Singular Soergel bimodules and webs.** The $\mathbb{Q}$-linear monoidal 2-category of singular Soergel bimodules can be assembled from certain full subcategories of the categories of graded bimodules between rings of partially symmetric polynomials over $\mathbb{Q}$ in variables of degree two. These subcategories are generated by certain induction and restriction bimodules under bimodule composition $\star$, direct sum, taking direct summands, and grading shifts. We now unpack this description, using the notation established in §3.3.

Consider the 2-category Bim_N wherein

- Objects are tuples $\boldsymbol{a} = (a_1, \ldots, a_m)$ with $a_i \geq 1$ and $\sum_{i=1}^m a_i = N$.
- 1-morphisms $\boldsymbol{a} \to \boldsymbol{b}$ are graded $(R^{\boldsymbol{b}}, R^{\boldsymbol{a}})$-bimodules.
- 2-morphisms are homomorphisms of graded bimodules.

Horizontal composition is given by tensor product over the rings $R^{\boldsymbol{a}}$, and will be denoted by $\star$. The composition of 2-morphisms is the usual composition of bimodule homomorphisms. We will write $\mathbf{1}_{\boldsymbol{a}} := R^{\boldsymbol{a}}$ for the identity bimodule, saving the notation $R^{\boldsymbol{a}}$ for the rings themselves. We denote the number of entries in $\boldsymbol{a}$ by $\#(\boldsymbol{a})$, i.e. for $\boldsymbol{a} = (a_1, \ldots, a_m)$ we have $\#(\boldsymbol{a}) = m$. The collection $\mathrm{Bim} := \{\mathrm{Bim}_N\}_{N \geq 0}$ is a monoidal 2-category, with (external) tensor product

$$\boxtimes\colon \mathrm{Bim}_{N_1} \times \mathrm{Bim}_{N_2} \to \mathrm{Bim}_{N_1+N_2}$$

given on objects by concatenation of tuples: $(a_1, \ldots, a_{m_1}) \boxtimes (b_1, \ldots, b_{m_2}) := (a_1, \ldots, a_{m_1}, b_1, \ldots, b_{m_2})$. and on 1- and 2-morphisms by tensor product over $\mathbb{Q}$.

A *singular Bott-Samelson bimodule* is, by definition, any $(R^{\boldsymbol{a}_0}, R^{\boldsymbol{a}_r})$-bimodule of the form

$$B \;=\; R^{\boldsymbol{a}_0} \otimes_{R^{\boldsymbol{b}_1}} R^{\boldsymbol{a}_1} \otimes_{R^{\boldsymbol{b}_2}} \cdots \otimes_{R^{\boldsymbol{b}_r}} R^{\boldsymbol{a}_r}$$

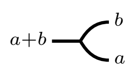
for some sequence of rings and subrings $R^{\boldsymbol{a}_0} \supset R^{\boldsymbol{b}_1} \subset \cdots \supset R^{\boldsymbol{b}_r} \subset R^{\boldsymbol{a}_r}$, or a grading shift thereof. In particular, we have the *merge* and *split bimodules* (terminology explained below) given by

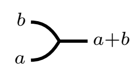
$$(24) \qquad {}_{\boldsymbol{b}}M_{\boldsymbol{a}} := \mathbf{q}^{\ell(\boldsymbol{a})-\ell(\boldsymbol{b})} R^{\boldsymbol{b}} \otimes_{R^{\boldsymbol{b}}} R^{\boldsymbol{a}}\,, \quad {}_{\boldsymbol{a}}S_{\boldsymbol{b}} := R^{\boldsymbol{a}} \otimes_{R^{\boldsymbol{b}}} R^{\boldsymbol{b}},$$

whenever $R^{\boldsymbol{b}} \subset R^{\boldsymbol{a}}$ (equivalently $\mathfrak{S}_{\boldsymbol{b}} \supset \mathfrak{S}_{\boldsymbol{a}}$). Here, $\mathbf{q}^k$ denotes a shift up in degree by k, and $\ell(\boldsymbol{a}) = \sum_i \binom{a_i}{2}$ denotes the length of the longest element in $\mathfrak{S}_{\boldsymbol{a}}$. Any singular Bott-Samelson bimodule is isomorphic to a shift of the horizontal composition of some sequence of merges and splits.

Definition 3.18. The 2-category SSBim_N of singular Soergel bimodules is the smallest full 2-subcategory of Bim_N that contains the singular Bott-Samelson bimodules and is closed under taking shifts, direct sums, and direct summands. The collection $\{\mathrm{SSBim}_N\}_{N \geq 0}$ assembles to form a full monoidal 2-subcategory of Bim that we denote SSBim. We will denote the 1-morphism category in SSBim from $\boldsymbol{a}$ to $\boldsymbol{b}$ by ${}_{\boldsymbol{b}}\mathrm{SSBim}_{\boldsymbol{a}}$.

If $\boldsymbol{a} = (a, b)$ and $\boldsymbol{a}' = (a + b)$, then we represent the corresponding merge and split bimodules by diagrams:

(25) ${}_{\boldsymbol{a}'}M_{\boldsymbol{a}} =$ 

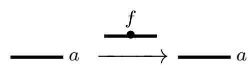
and ${}_{\boldsymbol{a}}S_{\boldsymbol{a}'} =$ 

.

All other singular Bott-Samelson bimodules can be obtained from these merge and split bimodules using grading shift, horizontal composition $\star$, and tensor product $\boxtimes$. Graphically, $\star$ corresponds to the horizontal glueing of diagrams along a common boundary, and $\boxtimes$ corresponds to vertical stacking of diagrams. We will refer to the graphs built from the diagrams in (25) via $\star$ and $\boxtimes$ as *webs*, which we always understand as mapping from the labels at their right endpoints to those at their left.

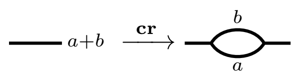
All maps between singular Bott-Samelson bimodules can be built using $\star$ and $\boxtimes$ from the following elemental maps[5] (which encode the Frobenius extension structures discussed in §3.3):

(1) *Decoration endomorphisms*

$R^{\boldsymbol{a}} =$  $= R^{\boldsymbol{a}}, \quad 1 \mapsto f$

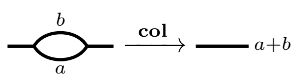
for $f \in R^{\boldsymbol{a}} = \mathrm{Sym}(\mathbb{X})$.

(2) *Digon creation* morphisms

$R^{a+b} =$  $= R^{(a,b)}, \quad 1 \mapsto 1$

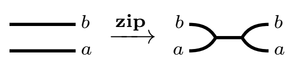
of weight $\mathbf{q}^{-ab}$.

(3) *Digon collapse* morphisms:

$R^{(a,b)} =$  $= R^{a+b}, \quad f \mapsto \partial_{a,b}(f)$

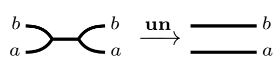
of weight $\mathbf{q}^{-ab}$. Here $\partial_{a,b}$ is the Sylvester operator from Example 3.17.

(4) *Zip* morphisms:

$R^{(a,b)} =$  $, \quad 1 \mapsto \mathfrak{s}_{b^a}(\mathbb{X}_1 - \mathbb{X}_2')$

of weight $\mathbf{q}^{ab}$. Here, the latter is viewed as an element in $\mathrm{Sym}(\mathbb{X}_1|\mathbb{X}_2) \otimes_{R^{a+b}} \mathrm{Sym}(\mathbb{X}_1'|\mathbb{X}_2')$.

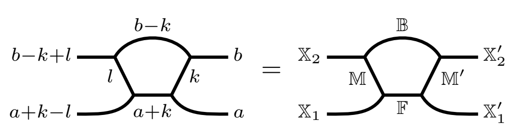
(5) *Un-zip* morphisms:

 $= R^{(a,b)}, \quad f \otimes g \mapsto fg$

of weight $\mathbf{q}^{ab}$.

In the cases (2)-(5), the degree/weight of the morphism is determined by the shift present in the definition of the merge bimodule in (24).

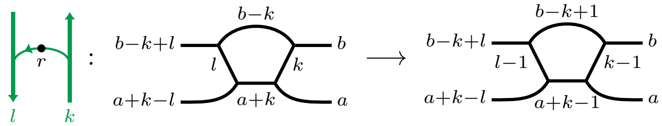
The following webs will play an important role in the following:

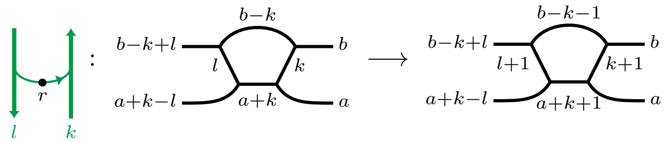
(26) $\mathsf{F}^{(l)}\mathsf{E}^{(k)}\mathbf{1}_{a,b} :=$ 

In the second diagram, we establish conventions for the alphabets associated with each edge; they have cardinalities as given by the corresponding labels in the first diagram. This will aid in specifying decoration endomorphisms of the corresponding bimodules. The notation $\mathsf{F}^{(l)}\mathsf{E}^{(k)}\mathbf{1}_{a,b}$ is borrowed from the theory of the categorified quantum group for $\mathfrak{sl}_2$, whose *extended graphical calculus* [KLMS12] can also be used to encode 2-morphisms between certain singular Bott–Samelson bimodules, see [HRW21, Proposition 2.18].

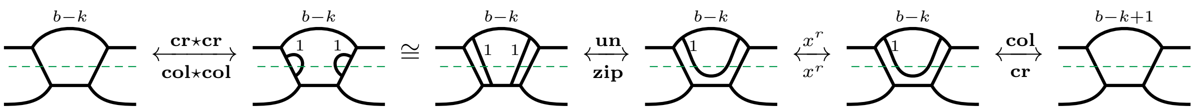
[5]This follows e.g. from the results in [Web17], which combine with [QR16] to show that the $n \to \infty$ limit of the $\mathfrak{gl}_n$ foam 2-category defined in the latter is equivalent to the 2-category of singular Bott-Samelson bimodules.

For example, we will use morphisms:

(27) $\chi_r^+ := (-1)^{b-k}$ 

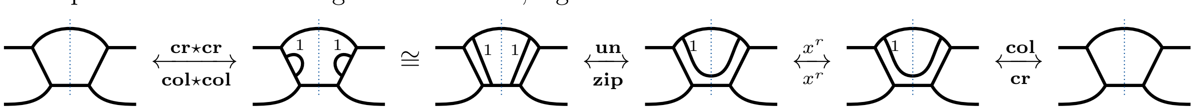
(28) $\chi_r^- := (-1)^{a+b+k+l-1}$ 

each of which has degree $a-b+k-l+1+2r$. Here, the signed[6] extended graphical calculus diagrams can be interpreted as encoding a *horizontal* (dashed) slice through a movie of webs that describes the morphism of singular Soergel bimodules:

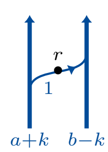
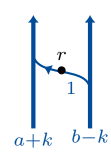


The morphism (27) is given by reading left-to-right with the top arrow labels and (28) is given by reading right-to-left with the bottom arrow labels. The unlabeled isomorphisms are composites of (co)associativity isomorphisms for merge (split) bimodules. The variable x is associated with the 1-labeled edge, and the extended graphical calculus diagram encodes the intersection with the dashed slice.

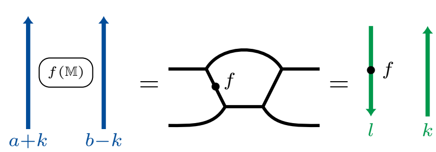
We will occasionally wish to encode such morphisms using *perpendicular graphical calculus*, which corresponds instead to taking a *vertical* slice, *e.g.*



and in this calculus the morphisms (27) and (28) are given by

(29) $\chi_r^+ :=$  and $\chi_r^- :=$  .

Note that some of the web edges are not visible in this calculus. We will denote decoration endomorphisms on such edges by drawing the endomorphism in an appropriate region, e.g. using the conventions in (26) we have



All of the relations used in the sequel between perpendicular graphical calculus diagrams can be deduced either from the corresponding relations in extended graphical calculus, or from relations in the $\mathfrak{gl}_n$ foam 2-category defined in [QR16]. As mentioned above, the latter is known to describe the

[6]The indicated signs are required to give a well-defined 2-functor from the categorified quantum group in [KLMS12] to the 2-category of singular Soergel bimodules. We will always depict the signs in green for signed diagrams that are sent to the "naïve" (unsigned) movie of webs in the image.

2-category of singular Bott-Samelson bimodules in the $n \to \infty$ limit (see e.g. [QRS18, Section 5.2], [Wed19, Proposition 3.4], or [HRW21, Appendix A]). See Figure 1 for a graphical depiction of such a foam, together with the slices giving (27) and (29).

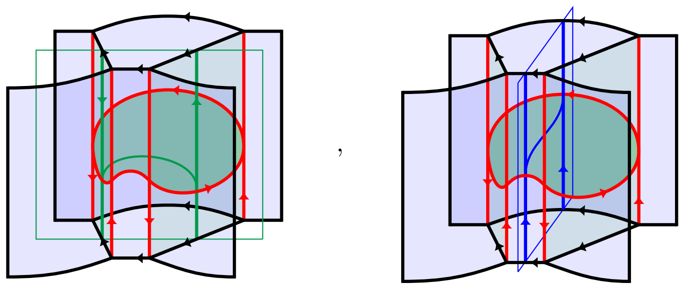


FIGURE 1. The foam corresponding to χ_0^+ and its slices that yield the corresponding extended and perpendicular graphical calculus diagrams, respectively.

There exist two $\star$- and $\circ$-contravariant duality functors on SSBim:

$$^{\vee}(-) \text{ and } (-)^{\vee} \colon {}_{\boldsymbol{a}}\mathrm{SSBim}_{\boldsymbol{b}} \to {}_{\boldsymbol{b}}\mathrm{SSBim}_{\boldsymbol{a}}$$

defined by

$$^{\vee}X := \mathrm{Hom}_{R^{\boldsymbol{a}}}(X, R^{\boldsymbol{a}})\,, \quad X^{\vee} := \mathrm{Hom}_{R^{\boldsymbol{b}}}(X, R^{\boldsymbol{b}})$$

which satisfy the adjunctions

$$\mathrm{Hom}_{\mathrm{SSBim}}(X \star Y, Z) \cong \mathrm{Hom}_{\mathrm{SSBim}}(Y, {}^{\vee}X \star Z)\,, \quad \mathrm{Hom}_{\mathrm{SSBim}}(X \star Y, Z) \cong \mathrm{Hom}_{\mathrm{SSBim}}(X, Z \star Y^{\vee})\,.$$

Since the bimodules ${}_{a+b}M_{a,b}$ and ${}_{a,b}S_{a+b}$ generate SSBim as a monoidal 2-category, this duality can be succinctly recorded as follows:

Proposition 3.19. *Let $a, b \geq 0$, then*

$${}_{a+b}M_{a,b}^{\vee} \cong \mathbf{q}^{ab}\,{}_{a,b}S_{a+b}\,, \quad {}^{\vee}_{a+b}M_{a,b} \cong \mathbf{q}^{-ab}\,{}_{a,b}S_{a+b}$$

and

$${}_{a,b}S_{a+b}^{\vee} \cong \mathbf{q}^{-ab}\,{}_{a+b}M_{a,b}\,, \quad {}^{\vee}_{a,b}S_{a+b} \cong \mathbf{q}^{ab}\,{}_{a+b}M_{a,b}\,.$$

Further, the relevant (co)unit morphisms are given by the digon creation/collapse and (un)zip morphisms. □

We will be interested in complexes of singular Soergel bimodules. The natural setting for their study is the dg 2-category of singular Soergel bimodules, which is obtained by taking the dg category of complexes is each Hom-category of SSBim.

Definition 3.20. Let $\mathcal{C}(\mathrm{SSBim})$ be the monoidal dg 2-category with the same objects as SSBim, and wherein the 1-morphism category $\boldsymbol{a} \to \boldsymbol{b}$ equals $\mathcal{C}({}_{\boldsymbol{b}}\mathrm{SSBim}_{\boldsymbol{a}})$.

In other words, 1-morphisms in $\mathcal{C}(\mathrm{SSBim})$ are complexes of singular Soergel bimodules and 2-morphism spaces in $\mathcal{C}(\mathrm{SSBim})$ are Hom-complexes of bimodule maps. Horizontal composition and external tensor product of 1-morphisms is defined as usual, e.g.

$$(X \star Y)^k = \bigoplus_{i+j=k} X^i \star Y^j\,, \quad \delta_{X \star Y} = \delta_X \star \mathrm{id}_Y + \mathrm{id}_X \star \delta_Y\,. \tag{30}$$

The components of the horizontal composition and external tensor product of 2-morphisms are defined using the Koszul sign rule. For example, if $f \in \mathrm{Hom}_{\mathcal{C}(\mathrm{SSBim})}(X, X')$ and $g \in \mathrm{Hom}_{\mathcal{C}(\mathrm{SSBim})}(Y, Y')$ are given, then $f \star g$ is defined component-wise by:

$$(f \star g)|_{X^i \star Y^j} = (-1)^{i|g|} f|_{X^i} \star g|_{Y^j} \,.$$

Note that the (graded) middle interchange law:

$$(f_1 \star g_1) \circ (f_2 \star g_2) = (-1)^{|g_1||f_2|}(f_1 \circ f_2) \star (g_1 \circ g_2) \tag{31}$$

holds in $\mathcal{C}(\mathrm{SSBim})$.

Convention 3.21. Since the 1-morphism categories of SSBim are $\mathbb{Z}_{\mathbf{q}}$-graded, the 1-morphism category $\mathcal{C}({}_{\boldsymbol{b}}\mathrm{SSBim}_{\boldsymbol{a}})$ is enriched in $\overline{\mathcal{K}}[\![\mathbf{q}^{\pm}, \mathbf{t}^{\pm}]\!]_{\mathrm{dg}}$. We will use the convention that $\deg(f) = (i, j)$ means f has $\mathbf{q}$-degree (or "Soergel degree") i and cohomological degree j. Further, the singly-indexed Hom-space $\mathrm{Hom}^k_{\mathcal{C}(\mathrm{SSBim})}(X, Y)$ always refers to cohomological degree, while the doubly-indexed $\mathrm{Hom}^{i,j}_{\mathcal{C}(\mathrm{SSBim})}(X, Y)$ consists of f with $\deg(f) = (i, j)$. For example, if X is a 1-morphism in $\mathcal{C}(\mathrm{SSBim})$, then its differential satisfies

$$\delta_X \in \mathrm{End}^{0,1}_{\mathcal{C}(\mathrm{SSBim})}(X) := \mathrm{Hom}^{0,1}_{\mathcal{C}(\mathrm{SSBim})}(X, X) \subseteq \mathrm{Hom}^{1}_{\mathcal{C}(\mathrm{SSBim})}(X, X) =: \mathrm{End}^{1}_{\mathcal{C}(\mathrm{SSBim})}(X) \,.$$

As in Convention 3.6, we will typically indicate these degrees multiplicatively by writing $\mathrm{wt}(f) = \mathbf{q}^i \mathbf{t}^j$, and will also use the variables $\mathbf{q}, \mathbf{t}$ to denote the corresponding shift functors. Thus, for example, $\mathrm{wt}(\delta_X) = \mathbf{q}^0 \mathbf{t}^1 = \mathbf{t}$.

3.5. **Colored braids and Rickard complexes.** We next recall the complexes of singular Soergel bimodules assigned to colored braids. In this paper, the set S of colors we will be $\mathbb{Z}_{\geq 1}$. Let Br_m denote the m-strand braid group, which acts on S^m by permuting coordinates (this action factors through the symmetric group $\mathfrak{S}_m$).

Definition 3.22. The *S-colored braid groupoid* $\mathfrak{Br}(S)$ is the category wherein

- objects are sequences $(a_1, \ldots, a_m)$ with $a_i \in S$, $m \geq 1$, and
- morphisms are given by

$$\mathrm{Hom}_{\mathfrak{Br}(S)}(\boldsymbol{a}, \boldsymbol{b}) = \left\{ \beta \in \mathrm{Br}_m \mid a_i = b_{\beta(i)} \text{ for } 1 \leq i \leq m \right\}$$

  with $\boldsymbol{a} = (a_1, \ldots, a_m)$ and $\boldsymbol{b} = (b_1, \ldots, b_m)$.

Morphisms in $\mathfrak{Br}(S)$ are called *S-colored braids* and elements in $\mathrm{Hom}_{\mathfrak{Br}}(\boldsymbol{a}, \boldsymbol{b})$ will be denoted by ${}_{\boldsymbol{b}}\beta_{\boldsymbol{a}}$, or occasionally by ${}_{\boldsymbol{b}}\beta$ or $\beta_{\boldsymbol{a}}$ since the domain/codomain determine one another. We will write $\mathrm{Br}_m(S)$ for the full subcategory of $\mathfrak{Br}(S)$ with objects having exactly m entries.

The colored braid groupoid is generated by the colored Artin generators

$$\beta_i \colon (a_1, \ldots, a_i, a_{i+1}, \ldots, a_m) \to (a_1, \ldots, a_{i+1}, a_i, \ldots, a_m)$$

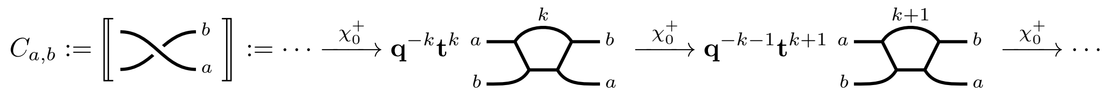
which, when composable, satisfy relations analogous to the usual (type A) braid relations. A *colored braid word* is a sequence of colored Artin generators and their inverses. We say that a colored braid word $(\underline{\beta})_{\boldsymbol{a}}$ *represents* the corresponding product of colored Artin generators in $\mathfrak{Br}(S)$.

We now use the colored Artin generators to associate complexes $C({}_{\boldsymbol{b}}\beta_{\boldsymbol{a}})$ in SSBim to $\mathbb{Z}_{\geq 1}$-colored braid words ${}_{\boldsymbol{b}}\beta_{\boldsymbol{a}}$. Here, it is convenient to abuse notation by writing:

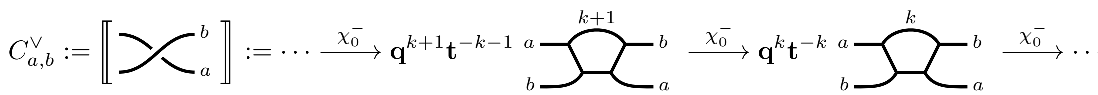
$$C({}_{\boldsymbol{b}}\beta_{\boldsymbol{a}}) = \mathbf{1}_{\boldsymbol{b}} C(\beta) \mathbf{1}_{\boldsymbol{a}} = \mathbf{1}_{\boldsymbol{b}} C(\beta) = C(\beta) \mathbf{1}_{\boldsymbol{a}}$$

(Note that $C(\beta)$ alone does not denote a well-defined complex.)

Definition 3.23. Let $a, b \geq 0$. The *2-strand Rickard complex* $C_{a,b}$ is the (bounded) complex



of singular Soergel bimodules. The rightmost non-zero term is either $\mathbf{q}^{-b}\mathbf{t}^{b}\mathsf{F}^{(a-b)}\mathbf{1}_{a,b}$ or $\mathbf{q}^{-a}\mathbf{t}^{a}\mathsf{E}^{(b-a)}\mathbf{1}_{a,b}$ depending on whether $a \geq b$ or $a \leq b$, respectively. Analogously, we also have:



As graded objects, we identify

$$C_{a,b} = \bigoplus_{k=0}^{\min(a,b)} \mathbf{q}^{-k}\mathbf{t}^{k} C^k_{a,b}\,, \quad C^\vee_{a,b} = \bigoplus_{k=0}^{\min(a,b)} \mathbf{q}^{k}\mathbf{t}^{-k} C^k_{a,b}$$

where $C^k_{a,b} := \mathsf{F}^{(a-k)}\mathsf{E}^{(b-k)}\mathbf{1}_{a,b}$. As the notation suggests, $(C_{a,b})^\vee = C^\vee_{b,a}$.

Definition 3.24. For the Artin generator β_i of the braid group Br_m and $\boldsymbol{a} = (a_1, \ldots, a_m)$, we set:

$$C(\beta_i)\mathbf{1}_{\boldsymbol{a}} := \mathbf{1}_{(a_1,\ldots,a_{i-1})} \boxtimes C_{a_i,a_{i+1}} \boxtimes \mathbf{1}_{(a_{i+2}\ldots,a_m)}$$
$$C(\beta_i^{-1})\mathbf{1}_{\boldsymbol{a}} := \mathbf{1}_{(a_1,\ldots,a_{i-1})} \boxtimes C^\vee_{a_i,a_{i+1}} \boxtimes \mathbf{1}_{(a_{i+2},\ldots,a_m)}\,.$$

This assignment extends to arbitrary colored braid words using horizontal composition. Given a braid word $\beta = \beta_{i_r}^{\varepsilon_r} \cdots \beta_{i_1}^{\varepsilon_1}$, we call

$$C(\beta)\mathbf{1}_{\boldsymbol{a}} = C(\beta_{i_r}^{\varepsilon_r} \cdots \beta_{i_1}^{\varepsilon_1})\mathbf{1}_{\boldsymbol{a}} := C(\beta_{i_r}^{\varepsilon_r}) \star \cdots \star C(\beta_{i_1}^{\varepsilon_1})\mathbf{1}_{\boldsymbol{a}} \tag{32}$$

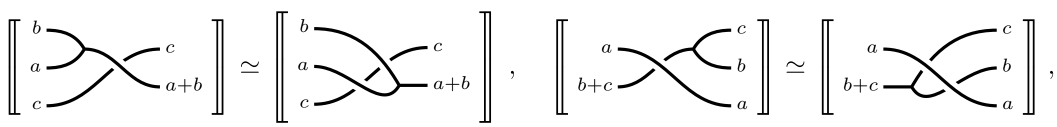
the *Rickard complex* assigned to the colored braid $\beta_{\boldsymbol{a}}$.

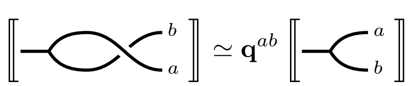
This terminology is justified by the following proposition.

Proposition 3.25 ([HRW21, Proposition 2.25])**.** *The complexes $C(\beta_{i_1}^{\varepsilon_1} \cdots \beta_{i_r}^{\varepsilon_r})\mathbf{1}_{\boldsymbol{a}}$ satisfy the (colored) braid relations, up to canonical homotopy equivalence.* □

Rickard complexes of colored braids extend to invariants of braided webs (using horizontal composition and external tensor product), since they satisfy the following *fork-slide* and *twist-zipper* relations.

Proposition 3.26 ([HRW21, Proposition 2.27])**.** *We have homotopy equivalences*

(33)


(34)


as well as reflections thereof. □

4. Curved Rickard complexes and interpolation coordinates

In this section, we introduce a dg 2-category of curved complexes of singular Soergel bimodules, and define *curved Rickard complexes* as certain special 1-morphisms.

4.1. **Perturbation theory for (curved) complexes.** In the following, we will need the notion of a twist. Suppose that $X \in \mathcal{A}[\mathbf{t}^{\pm}]$ comes equipped with two endomorphisms $\delta, \alpha \in \operatorname{End}_{\mathcal{A}[\mathbf{t}^{\pm}]}(X) \otimes R$ such that

$$\delta^2 = F_1 \quad \text{and} \quad (\delta+\alpha)^2 = F_1 + F_2$$

for $F_1, F_2 \in \mathcal{Z}(\mathcal{A}) \otimes R$ of (cohomological) degree two. (As a special case, we could have $F_1 = 0 = F_2$.) It follows that (X, δ) is an object of $\mathcal{C}_{F_1}(\mathcal{A})$, and the object $(X, \delta+\alpha) \in \mathcal{C}_{F_1+F_2}(\mathcal{A})$ is said to be a *twist* of (X, δ). We set

$$\operatorname{tw}_\alpha((X,\delta)) := (X, \delta+\alpha)$$

and will often simply write the former as $\operatorname{tw}_\alpha(X)$ when the differential δ on X is understood. Note that the element α satisfies the *Maurer–Cartan equation* with curvature:

$$[\delta, \alpha] + \alpha^2 = F_2 \,.$$

We will refer to α as a *(curved) Maurer–Cartan element*, or, by abuse of terminology, as a *twist*.

In various places, we will need to promote a homotopy equivalence $X \simeq Y$ to a homotopy equivalence between twists $\operatorname{tw}_\alpha(X) \simeq \operatorname{tw}_\beta(Y)$. This is the subject of homological perturbation theory. For our purposes, the following result suffices; see *e.g.* [Mar01, Hog] for a more-thorough discussion.

Proposition 4.1. *Let $X, Y \in \mathcal{C}_{F_1}(\mathcal{A})$ and let $f \in \operatorname{Hom}^0_{\mathcal{C}_{F_1}(\mathcal{A})}(X,Y)$ and $g \in \operatorname{Hom}^0_{\mathcal{C}_{F_1}(\mathcal{A})}(Y,X)$ determine a homotopy equivalence $X \simeq Y$ with associated homotopies $k_X \in \operatorname{End}^{-1}_{\mathcal{C}_{F_1}(\mathcal{A})}(X)$ and $k_Y \in \operatorname{End}^{-1}_{\mathcal{C}_{F_1}(\mathcal{A})}(Y)$ (satisfying $d(k_X) = \mathrm{id}_X - g \circ f$ and $d(k_Y) = \mathrm{id}_Y - f \circ g$). Suppose that $\alpha \in \operatorname{End}^1_{\mathcal{C}_{F_1}(\mathcal{A})}(X)$ is a Maurer–Cartan element with curvature F_2 such that $\mathrm{id}_X + \alpha \circ k_X \in \operatorname{End}^0_{\mathcal{C}(\mathcal{A})}(X)$ is invertible, then*

$$\widetilde{f} := f \circ (\mathrm{id}_X + \alpha \circ k_X)^{-1} \in \operatorname{Hom}^0_{\mathcal{C}_{F_1+F_2}(\mathcal{A})}(\operatorname{tw}_\alpha(X), \operatorname{tw}_\beta(Y))$$
$$\widetilde{g} := (\mathrm{id}_X + k_X \circ \alpha)^{-1} \circ g \in \operatorname{Hom}^0_{\mathcal{C}_{F_1+F_2}(\mathcal{A})}(\operatorname{tw}_\beta(Y), \operatorname{tw}_\alpha(X))$$

determine a homotopy equivalence $\operatorname{tw}_\alpha(X) \simeq \operatorname{tw}_\beta(Y)$ in $\mathcal{C}_{F_1+F_2}(\mathcal{A})$, where $\beta := f \circ \alpha \circ \widetilde{g}$. □

Remark 4.2. In all of our applications of Proposition 4.1, invertibility of $\mathrm{id}_X + \alpha \circ k_X$ will follow since $\alpha \circ k_X$ is nilpotent. For example, this holds when k_X acts summand-wise on a *finite one-sided twisted complex* $\operatorname{tw}_\alpha(X)$. Recall that the latter means that

$$(X, \delta) = \bigoplus_{i \in \mathbb{Z}} (X_i, \delta_i)$$

with $X_i = 0$ for all but finitely many $i \in \mathbb{Z}$, and the components $\alpha_{i,j} \colon X_j \to X_i$ of the twist α satisfy $\alpha_{i,j} = 0$ for $i \leq j$.

Further, invertibility of $\mathrm{id}_X + \alpha \circ k_X$ implies that $\mathrm{id}_X + k_X \circ \alpha$ is also invertible, so no further assumptions are necessary to define $\widetilde{g}$. Although the homotopy k_Y does not appear in the definition of $\widetilde{f}$ or $\widetilde{g}$, it would appear in the formula for the perturbed homotopy $\widetilde{k_Y} \in \operatorname{End}^{-1}_{\mathcal{C}_{F_1+F_2}(\mathcal{A})}(Y)$.

4.2. **A preliminary discussion on curvature.** Our curved complexes of singular Soergel bimodules will have curvatures modeled on *strand-wise curvature*[7] of the form

$$\sum_{k=1}^{a} h_k(\mathbb{X} - \mathbb{X}') v_k \,, \tag{35}$$

which we refer to as *$h\Delta$-curvature*. Here $\mathbb{X}, \mathbb{X}'$ are alphabets of cardinality a and $v_1, \dots, v_a$ are deformation parameters with $\operatorname{wt}(v_k) = \mathbf{q}^{-2k}\mathbf{t}^2$. However, in order to define horizontal composition in

[7] Informally, $\mathbb{X}$ and $\mathbb{X}'$ should be thought of as the alphabets on the left and right of an a-colored strand in a braid.

our 2-category of curved complexes of singular Soergel bimodules, it will be auspicious to work with a different curvature that is modeled on strand-wise Δe-*curvature*, which is of the form

$$\sum_{k=1}^{a}(e_k(\mathbb{X}) - e_k(\mathbb{X}'))u_k\,. \tag{36}$$

As it turns out, we can regard these curvatures as equivalent after an appropriate change of variables.

Definition 4.3. Let $\mathbb{X}$ be an alphabet of cardinality a and consider collections of deformation parameters $\mathbb{U} = \{u_1, \dots, u_a\}$ and $\mathbb{V} = \{v_1, \dots, v_a\}$ with $\mathrm{wt}(v_k) = \mathbf{q}^{-2k}\mathbf{t}^2 = \mathrm{wt}(u_k)$. Let us identify the algebras $\mathbb{Q}[\mathbb{X}, \mathbb{U}]$ and $\mathbb{Q}[\mathbb{X}, \mathbb{V}]$ by declaring

$$v_k = (-1)^{k-1}\sum_{k\le l\le a} e_{l-k}(\mathbb{X})u_l\,, \quad u_k = (-1)^{k-1}\sum_{k\le l\le a} h_{l-k}(\mathbb{X})v_l\,. \tag{37}$$

It is an easy exercise using (15) to verify that the formulae in (37) are mutually inverse.

Lemma 4.4. *Inside* $\mathbb{Q}[\mathbb{X}, \mathbb{U}] \otimes_{\mathbb{Q}[\mathbb{U}]} \mathbb{Q}[\mathbb{X}, \mathbb{U}]$*, we have*

$$1 \otimes v_k = \sum_{k\le l\le a} h_{l-k}(\mathbb{X} - \mathbb{X}') \cdot (v_l \otimes 1)$$

for all $1 \le k \le a$*, where* $\mathbb{X}'$ *is the alphabet* $1 \otimes \mathbb{X}$.

Proof. We compute

$$\begin{aligned} 1 \otimes v_k &= (-1)^{k-1}\sum_{k\le l\le a} 1 \otimes e_{l-k}(\mathbb{X})u_l = (-1)^{k-1}\sum_{k\le l\le a} u_l \otimes e_{l-k}(\mathbb{X}) \\ &= (-1)^{k-1}\sum_{k\le l\le m\le a} (-1)^{l-1}h_{m-l}(\mathbb{X})v_m \otimes e_{l-k}(\mathbb{X}) = \sum_{k\le m\le a} h_{m-k}(\mathbb{X} - \mathbb{X}') \cdot (v_m \otimes 1)\,. \end{aligned}$$ □

Corollary 4.5. *Under the identification of* $\mathbb{Q}[\mathbb{X}, \mathbb{X}', \mathbb{U}] \cong \mathbb{Q}[\mathbb{X}, \mathbb{X}', \mathbb{V}]$ *given via* (37)*, we have*

$$\sum_{k<l\le a}(e_{l-k}(\mathbb{X}) - e_{l-k}(\mathbb{X}'))u_l = \sum_{k<l\le a} h_{l-k}(\mathbb{X} - \mathbb{X}')(v_l \otimes 1)$$

for all $1 \le k \le a$.

Proof. Identify $\mathbb{Q}[\mathbb{X}, \mathbb{X}', \mathbb{U}] \cong \mathbb{Q}[\mathbb{X}, \mathbb{U}] \otimes_{\mathbb{Q}[\mathbb{U}]} \mathbb{Q}[\mathbb{X}, \mathbb{U}]$. We then compute

$$1 \otimes v_k - v_k \otimes 1 = \sum_{k\le l\le a}\Big(e_{l-k}(\mathbb{X})u_l \otimes 1 - 1 \otimes e_{l-k}(\mathbb{X})u_l\Big) = \sum_{k\le l\le a}(e_{l-k}(\mathbb{X}) - e_{l-k}(\mathbb{X}'))u_l\,.$$

Note that the $k = l$ term in this sum is zero. On the other hand, Lemma 4.4 gives

$$1 \otimes v_k - v_k \otimes 1 = \left(\sum_{k\le l\le a} h_{l-k}(\mathbb{X} - \mathbb{X}') \cdot (v_l \otimes 1)\right) - v_k \otimes 1 = \sum_{k<l\le a} h_{l-k}(\mathbb{X} - \mathbb{X}')(v_l \otimes 1)\,.$$ □

Note that Lemma 4.4 and Corollary 4.5 remain true (with the same proof) if we extend $\mathbb{U}$ and $\mathbb{V}$ to include deformation parameters u_0, v_0, with weights $\mathbf{q}^0\mathbf{t}^2$, which we assume are related via the obvious extension of (37) to the case $k = 0$. Thus we also have the following identity, which is the "$k = 0$ case" of Corollary 4.5. (Alternatively, this could be established by a straightforward computation.)

Corollary 4.6. *We have*

$$\sum_{1\le l\le a}(e_l(\mathbb{X}) - e_l(\mathbb{X}'))u_l = \sum_{1\le l\le a} h_l(\mathbb{X} - \mathbb{X}')(v_l \otimes 1)\,.$$

4.3. **Curved complexes over** SSBim**.** We now introduce the monoidal dg 2-category of curved complexes of singular Soergel bimodules. Informally, this 2-category is formed via the following procedure. First, we consider a 2-subcategory consisting of 1-morphisms $X\colon \boldsymbol{a} \to \boldsymbol{b}$ in $\mathcal{C}(\mathrm{SSBim})$ where $\boldsymbol{b}$ is obtained by permuting the indices of $\boldsymbol{a}$. In particular, all Rickard complexes give 1-morphisms in this 2-category. Next, in each Hom-category we adjoin $\#(\boldsymbol{a})$ alphabets $\{\mathbb{U}_i\}_{i=1}^{\#(\boldsymbol{a})}$ of formal variables via Definition 3.10. Finally, we pass to a certain category of curved complexes in each Hom-category, for a choice of curvature that *e.g.* encodes the homotopies that "slide" the action of symmetric polynomials on the left boundary along a strand of a Rickard complex to the right.

We now make this informal description precise.

Definition 4.7. Fix an integer $m \geq 0$ and let $\mathcal{Y}(\mathrm{SSBim}, m)$ be the 2-category wherein:

- objects are pairs $(\boldsymbol{a}, \sigma)$ where $\boldsymbol{a} = (a_1, \ldots, a_m)$ is an object in SSBim and $\sigma \in \mathfrak{S}_m$.
  For each such object $((a_1, \ldots, a_m), \sigma)$, we introduce an alphabet
  $$\mathbb{U}(\boldsymbol{a}, \sigma) = \{u_{i,r}(\boldsymbol{a}, \sigma) \mid 1 \leq i \leq m,\ 1 \leq r \leq a_i\}$$
  with $\mathrm{wt}(u_{i,r}(\boldsymbol{a}, \sigma)) = \mathbf{q}^{-2r}\mathbf{t}^2$. We also write $\mathbb{U}_i(\boldsymbol{a}, \sigma)$ for the subalphabet with fixed $i \in \{1, \ldots, m\}$. Further, we place an equivalence relation on objects by declaring $(\boldsymbol{a}, \sigma) \sim (\boldsymbol{b}, \tau)$ if $a_{\sigma(i)} = b_{\tau(i)}$ for all $1 \leq i \leq m$. If $(\boldsymbol{a}, \sigma) \sim (\boldsymbol{b}, \tau)$, then we will identify the associated $\mathbb{U}$-variables $u_{i,r}(\boldsymbol{a}, \sigma) = u_{i,r}(\boldsymbol{b}, \tau)$ and drop the dependence on $(\boldsymbol{a}, \sigma)$ from the notation.
- the Hom-category ${}_{\boldsymbol{b},\tau}\mathcal{Y}(\mathrm{SSBim})_{\boldsymbol{a},\sigma}$ from $(\boldsymbol{a}, \sigma)$ to $(\boldsymbol{b}, \tau)$ is empty unless $(\boldsymbol{a}, \sigma) \sim (\boldsymbol{b}, \tau)$, in which case we set
  $${}_{\boldsymbol{b},\tau}\mathcal{Y}(\mathrm{SSBim})_{\boldsymbol{a},\sigma} := \mathcal{C}_F\big({}_{\boldsymbol{b}}\mathrm{SSBim}_{\boldsymbol{a}}; \mathbb{Q}[\mathbb{U}]\big)$$
  where the curvature element is
  $$F = \sum_{i=1}^{m} \sum_{r=1}^{a_{\sigma(i)}} \big(e_r(\mathbb{X}_{\tau(i)}) - e_r(\mathbb{X}'_{\sigma(i)})\big)u_{i,r} \in \mathrm{Sym}(\mathbb{X}_1|\cdots|\mathbb{X}_m|\mathbb{X}'_1|\cdots|\mathbb{X}'_m) \otimes \mathbb{Q}[\mathbb{U}]\,. \tag{38}$$
  Here we identify the rings acting on the left and right (boundary) as $R^{\boldsymbol{b}} = \mathrm{Sym}(\mathbb{X}_1|\cdots, \mathbb{X}_m)$ and $R^{\boldsymbol{a}} = \mathrm{Sym}(\mathbb{X}'_1|\cdots|\mathbb{X}'_m)$, respectively. In such expressions, we will sometimes suppress the summation limits depending on colors $a_{\sigma(i)}$ by summing over all $r \geq 1$ and declaring $u_{i,r} := 0$ for $r > a_{\sigma(i)}$.

We let $\mathcal{Y}(\mathrm{SSBim}) := \bigsqcup_{m \geq 0} \mathcal{Y}(\mathrm{SSBim}, m)$. The horizontal composition in $\mathcal{Y}(\mathrm{SSBim})$ is defined by the usual rule

$$(X, \delta_X^{\mathrm{tot}}) \star (Y, \delta_Y^{\mathrm{tot}}) := (X \star Y\,,\ \delta_X^{\mathrm{tot}} \star \mathrm{id}_Y + \mathrm{id}_X \star \delta_Y^{\mathrm{tot}}) \tag{39}$$

where the horizontal composition of 2-morphisms is inherited from

$$\mathrm{Hom}(X, Y) \otimes \mathbb{Q}[\mathbb{U}] \otimes \mathrm{Hom}(X', Y') \otimes \mathbb{Q}[\mathbb{U}] \to \mathrm{Hom}(X \star X', Y \star Y') \otimes \mathbb{Q}[\mathbb{U}]$$
$$f \otimes g \otimes f' \otimes g' \mapsto (f \star f') \otimes (gg')\,.$$

Remark 4.8. The appearance of permutations in the objects of $\mathcal{Y}(\mathrm{SSBim})$ may appear surprising. Below, when we assign 1-morphisms in $\mathcal{Y}(\mathrm{SSBim})$ to braids, the permutations in the (co)domain objects will encode a numbering of the strands in the braid. E.g. if the domain object is $(\boldsymbol{a}, \sigma)$, then $\sigma(i) = j$ tells us that the strand meeting the j^{th} boundary point on the right of the braid is the i^{th} strand in this numbering.

We have written the differential on a 1-morphism $(X, \delta_X^{\mathrm{tot}})$ in $\mathcal{Y}(\mathrm{SSBim})$ using the superscript "tot" in order to emphasize the fact that δ_X^{tot} decomposes canonically (and uniquely) into a sum of terms:

$$\delta_X^{\mathrm{tot}} = \delta_X + \Delta_X$$

where δ_X lives in $\mathrm{End}_{\mathcal{C}(\mathrm{SSBim})}(X)$ and Δ_X lives in the ideal $\mathrm{End}_{\mathcal{C}(\mathrm{SSBim})}(X) \otimes \mathbb{Q}[\mathbb{U}]_{>0}$ generated by polynomials in $\mathbb{U}$ with zero constant term. Hence, (X, δ_X) defines a complex of singular Soergel bimodules, and $(X, \delta_X + \Delta_X) = \mathrm{tw}_{\Delta_X}((X, \delta_X))$. It will frequently be useful to decompose 2-morphisms in $\mathcal{Y}(\mathrm{SSBim})$ according to their $\mathbb{U}$-degree zero parts and their "strictly positive $\mathbb{U}$-degree" parts, according to the following definition.

Definition 4.9. A 2-morphism f in $\mathcal{Y}(\mathrm{SSBim})$ is $\mathbb{U}$-*irrelevant* if f is zero after setting all $\mathbb{U}$-variables equal to zero. In other words, f is $\mathbb{U}$-irrelevant if it is an element of $\mathrm{Hom}_{\mathcal{C}(\mathrm{SSBim})}(X, Y) \otimes \mathbb{Q}[\mathbb{U}]_{>0}$ for appropriate X, Y.

Convention 4.10. Henceforth, we will write 1-morphisms in $\mathcal{Y}(\mathrm{SSBim})$ in the form $\mathrm{tw}_{\Delta_X}(X)$ where X is a 1-morphism in $\mathcal{C}(\mathrm{SSBim})$ and Δ_X is a curved Maurer–Cartan element in $\mathrm{End}_{\mathcal{C}(\mathrm{SSBim})}(X) \otimes \mathbb{Q}[\mathbb{U}]_{>0}$ (that is to say, Δ is $\mathbb{U}$-irrelevant). In this language, the composition of 1-morphisms takes the form

$$\mathrm{tw}_{\Delta_X}(X) \star \mathrm{tw}_{\Delta_Y}(Y) := \mathrm{tw}_{\Delta_{X\star Y}}(X \star Y)\,, \quad \Delta_{X\star Y} := \Delta_X \star \mathrm{id}_Y + \mathrm{id}_X \star \Delta_Y. \tag{40}$$

(We will show in Lemma 4.12 that it is well-defined.) Further, we will sometime embellish this notation as $\mathrm{tw}_{\Delta_X}({}_{\boldsymbol{b},\tau}X_{\boldsymbol{a},\sigma})$ or $\mathrm{tw}_{\Delta_X}({}_{\boldsymbol{b}}X_{\boldsymbol{a}})$ when we wish to emphasize the data specifying the objects.

Remark 4.11. If $\mathrm{tw}_{\Delta_X}({}_{\boldsymbol{b},\tau}X_{\boldsymbol{a},\sigma})$ is a 1-morphism in $\mathcal{Y}(\mathrm{SSBim})$, then the linear part of Δ_X is a sum of terms of the form $\Psi_{i,r}u_{i,r}$, where $\Psi_{i,r} \in \mathrm{End}^{2r,-1}(X)$ satisfies

$$[\delta_X, \Psi_{i,r}] = e_r(\mathbb{X}_{\tau(i)}) - \; e_r(\mathbb{X}'_{\sigma(i)})\,.$$

If Δ_X is linear in the variables $u_{i,r}$, then $\mathrm{tw}_{\Delta_X}(X)$ is called a *strict 1-morphism* in $\mathcal{Y}(\mathrm{SSBim})$. This implies that the endomorphisms $\Psi_{i,r}$ necessarily square to zero and pairwise anti-commute, so

$$[\delta_X + \Delta_X, \Psi_{i,r}] = e_r(\mathbb{X}_{\tau(i)}) - \; e_r(\mathbb{X}'_{\sigma(i)})\,.$$

Hence, $e_r(\mathbb{X}_{\tau(i)}) - \; e_r(\mathbb{X}'_{\sigma(i)})$ is null-homotopic on $\mathrm{tw}_{\Delta_X}(X)$. This conclusion holds for non-strict morphisms as well, as can be seen by differentiating the equation $(\delta_X + \Delta_X)^2 = F$ with respect to the variable $u_{i,r}$.

Note that a typical 2-morphism $f \in \mathrm{Hom}_{\mathcal{Y}(\mathrm{SSBim})}(X, Y)$ is a formal sum

$$f = \sum_{\mathbf{i},\mathbf{r},\mathbf{k}} f_{\mathbf{i},\mathbf{r},\mathbf{k}} \otimes u_{\mathbf{i},\mathbf{r}}^{\mathbf{k}} \tag{41}$$

over finitely many triples $(\mathbf{i}, \mathbf{r}, \mathbf{k})$ where $u_{\mathbf{i},\mathbf{r}}^{\mathbf{k}} := u_{i_1,r_1}^{k_1} \cdots u_{i_\ell,r_\ell}^{k_\ell}$ and $f_{\mathbf{i},\mathbf{r},\mathbf{k}} \in \mathrm{Hom}_{\mathcal{C}(\mathrm{SSBim})}(X, Y)$. If f is homogeneous of weight $\mathbf{q}^{l_1}\mathbf{t}^{l_2}$, then we have

$$\mathrm{wt}(f_{\mathbf{i},\mathbf{r},\mathbf{k}}) = \mathbf{q}^{l_1 + 2\sum_j k_j r_j}\mathbf{t}^{l_2 - 2\sum_j k_j}, \tag{42}$$

since $u_{i,r}^k$ has weight $\mathbf{q}^{-2kr}\mathbf{t}^{2k}$. In particular, since the curved complexes $\mathrm{tw}_{\Delta_X}(X)$ in $\mathcal{Y}(\mathrm{SSBim})$ are bounded, this implies that the curved Maurer–Cartan element Δ_X is nilpotent.

We now elaborate on the 2-categorical structure on $\mathcal{Y}(\mathrm{SSBim})$. First, we check the following.

Lemma 4.12. *Horizontal composition is well-defined on* $\mathcal{Y}(\mathrm{SSBim})$.

Proof. It suffices to show that given composable 1-morphisms $\mathrm{tw}_{\Delta_X}({}_{\boldsymbol{a},\sigma}X_{\boldsymbol{a}',\sigma'})$ and $\mathrm{tw}_{\Delta_X}({}_{\boldsymbol{a}',\sigma'}X_{\boldsymbol{a}'',\sigma''})$, the twist

$$\Delta_{X\star Y} = \Delta_X \star \mathrm{id}_Y + \mathrm{id}_X \star \Delta_Y$$

satisfies the conditions in Definition 4.7. The only non-immediate check is to compute its curvature. Let $m = \#(\boldsymbol{a}) = \#(\boldsymbol{a}') = \#(\boldsymbol{a}'')$, and note that $a_{\sigma(i)} = a'_{\sigma'(i)} = a''_{\sigma''(i)}$ for all $1 \le i \le m$. We then have

$$(\delta_X + \Delta_X)^2 = \sum_{i=1}^{m} \sum_{r=1}^{a'_{\sigma'(i)}} \big(e_r(\mathbb{X}_{\sigma(i)}) - e_r(\mathbb{X}'_{\sigma'(i)})\big)u_{i,r}$$

and

$$(\delta_Y+\Delta_Y)^2=\sum_{i=1}^{m}\sum_{r=1}^{a''_{\sigma''(i)}}\big(e_r(\mathbb{X}'_{\sigma'(i)})-e_r(\mathbb{X}''_{\sigma''(i)})\big)u_{i,r}\,,$$

thus

$$\begin{aligned}\Big((\delta_X+\Delta_X)\star\mathrm{id}_Y+\mathrm{id}_X\star(\delta_Y+\Delta_Y)\Big)^2 &=(\delta_X+\Delta_X)^2\star\mathrm{id}_Y+\mathrm{id}_X\star(\delta_Y+\Delta_Y)^2\\ &\quad+[(\delta_X+\Delta_X)\star\mathrm{id}_Y,\mathrm{id}_X\star(\delta_Y+\Delta_Y)]\\ &=\sum_{i=1}^{m}\sum_{r=1}^{a''_{\sigma''(i)}}\big(e_r(\mathbb{X}_{\sigma(i)})-e_r(\mathbb{X}'_{\sigma'(i)})+e_r(\mathbb{X}'_{\sigma'(i)})-e_r(\mathbb{X}''_{\sigma''(i)})\big)u_{i,r}\\ &=\sum_{i=1}^{m}\sum_{r=1}^{a''_{\sigma''(i)}}\big(e_r(\mathbb{X}_{\sigma(i)})-e_r(\mathbb{X}''_{\sigma''(i)})\big)u_{i,r}\end{aligned}$$

as desired. Here, we use that $[(\delta_X+\Delta_X)\star\mathrm{id}_Y,\mathrm{id}_X\star(\delta_Y+\Delta_Y)]=0$ by (31). □

Next, we note that the external tensor product $\boxtimes$ can be extended from $\mathcal{C}(\mathrm{SSBim})$ to $\mathcal{Y}(\mathrm{SSBim})$ in a straightforward manner. On the level of objects, we have

$$(\boldsymbol{a}_1,\sigma_1)\boxtimes(\boldsymbol{a}_2,\sigma_2)=(\boldsymbol{a}_1\boxtimes\boldsymbol{a}_2,\sigma_1\boxtimes\sigma_2),$$

where $\sigma_1\boxtimes\sigma_2\in\mathfrak{S}_{\#(\boldsymbol{a}_1)+\#(\boldsymbol{a}_2)}$ is defined by

$$\sigma_1\boxtimes\sigma_2\colon i\mapsto\begin{cases}\sigma_1(i) & \text{if } 1\le i\le\#(\boldsymbol{a}_1)\\ \#(\boldsymbol{a}_1)+\sigma_2(i-\#(\boldsymbol{a}_1)) & \text{if } \#(\boldsymbol{a}_1)+1\le i\le\#(\boldsymbol{a}_1)+\#(\boldsymbol{a}_2)\,,\end{cases}$$

i.e. using the standard inclusion $\mathfrak{S}_{\#(\boldsymbol{a}_1)}\times\mathfrak{S}_{\#(\boldsymbol{a}_2)}\hookrightarrow\mathfrak{S}_{\#(\boldsymbol{a}_1)+\#(\boldsymbol{a}_2)}$.

On the level of 1-morphisms, if $\mathrm{tw}_{\Delta_i}(X_i)$ are 1-morphisms $(\boldsymbol{a}_i,\sigma_i)\to(\boldsymbol{b}_i,\tau_i)$ for $i=1,2$, then

$$\mathrm{tw}_{\Delta_1}(X_1)\boxtimes\mathrm{tw}_{\Delta_2}(X_2):=\mathrm{tw}_{\Delta_1\boxtimes\mathrm{id}_{X_2}+\mathrm{id}_{X_1}\boxtimes\Delta_2}(X_1\boxtimes X_2)\tag{43}$$

where, analogous to (30), we have $X_1\boxtimes X_2=\bigoplus_{i,j}X_1^i\boxtimes X_2^j$. The verification that the twist $\Delta_1\boxtimes\mathrm{id}_{X_2}+\mathrm{id}_{X_1}\boxtimes\Delta_2$ satisfies the curvature condition in Definition 4.7 is similar to the verification in Lemma 4.12, thus we omit it.

Finally, the external tensor product of 2-morphisms in $\mathcal{Y}(\mathrm{SSBim})$ is defined via the $\mathbb{Q}[\mathbb{U}]$-linear extension of the external tensor product in $\mathcal{C}(\mathrm{SSBim})$. Explicitly, for homogeneous $f\colon X_1\to Y_1$ and $g\colon X_2\to Y_2$ it is given by

$$(f\boxtimes g)|_{X_1^i\boxtimes X_2^j}=(-1)^{i|g|}f|_{X_1^i}\boxtimes g|_{X_2^j}.$$

Thus defined, $\boxtimes$ endows $\mathcal{Y}(\mathrm{SSBim})$ with the structure of a monoidal dg 2-category.

Remark 4.13. There is a monoidal dg 2-functor $\mathcal{Y}(\mathrm{SSBim})\to\mathcal{C}(\mathrm{SSBim})$ that forgets the permutations, sets all variables $u_{i,r}$ equal to zero, and sends $\mathrm{tw}_{\Delta_X}(X)\mapsto X$. We refer to $\mathrm{tw}_{\Delta_X}(X)$ as a *curved lift* or *curved deformation* of the complex X, and similarly for 2-morphisms in $\mathcal{Y}(\mathrm{SSBim})$.

Our next result allows us to upgrade homotopy equivalences between 1-morphisms in $\mathcal{C}(\mathrm{SSBim})$ to equivalences in $\mathcal{Y}(\mathrm{SSBim})$, provided we are given a curved lift of one of the 1-morphisms.

Proposition 4.14. *Suppose that $\mathrm{tw}_{\Delta_X}(X)$ is a 1-morphism in $\mathcal{Y}(\mathrm{SSBim})$, and $f\colon X\to Y$ is a homotopy equivalence in $\mathcal{C}(\mathrm{SSBim})$, then there exists a curved lift $\mathrm{tw}_{\Delta_Y}(Y)$ of Y and an induced homotopy equivalence $\widetilde{f}\colon\mathrm{tw}_{\Delta_X}(X)\to\mathrm{tw}_{\Delta_Y}(Y)$.*

Proof. We use Proposition 4.1, and its notation. The present result is an immediate consequence, applied to $X \in \mathcal{C}(\mathrm{SSBim})$ with $\alpha = \Delta_X$, once we have confirmed that $\Delta_X \circ k_X \in \mathrm{End}_{\mathcal{C}(\mathrm{SSBim})[\mathbb{U}]}(X)$ is nilpotent. To see the latter, note that since $\Delta_X = 0 \mod \langle u_{i,r} \rangle$, the same holds for $\Delta_X \circ k_X$. This implies that $(\Delta_X \circ k_X)^\ell = 0 \mod \langle u_{i,r} \rangle^\ell$. Writing $(\Delta_X \circ k_X)^\ell$ in components as in (41), this in turn implies that there exists $\ell \geq 0$ so that $(\Delta_X \circ k_X)^\ell = 0$ by equation (42), since X is bounded. □

Lemma 4.15. *Suppose that* $(\boldsymbol{a}, \sigma)$ *and* $(\boldsymbol{b}, \tau)$ *are objects in* $\mathcal{Y}(\mathrm{SSBim})$ *such that* $\#(\boldsymbol{a}) = \#(\boldsymbol{b})$ *and* $a_{\sigma(i)} = b_{\tau(i)}$ *for all* $1 \leq i \leq \#(\boldsymbol{a})$*. Let* $\mathbf{1}_{\boldsymbol{b}} \star L \star \mathbf{1}_{\boldsymbol{a}}$ *be an invertible* 1*-morphism in* $\mathcal{C}(\mathrm{SSBim})$ *such that* $f(\mathbb{X}_{\tau(i)}) \star \mathrm{id}_L \simeq \mathrm{id}_L \star f(\mathbb{X}'_{\sigma(i)})$ *for all* $1 \leq i \leq \#(\boldsymbol{a})$ *and all symmetric functions* f*, then there exists a curved lift*

$$\mathrm{tw}_{\Delta_L}({}_{\boldsymbol{b},\tau}L_{\boldsymbol{a},\sigma})$$

of L *in* $\mathcal{Y}(\mathrm{SSBim})$ *which is unique up to homotopy equivalence.*

Proof. The existence statement follows via obstruction-theoretic arguments analogous to those in [GH, Section 2.10]. We elaborate on the details, since such constructions are crucial for us.

Abbreviate by writing $E(L) := \mathrm{End}_{\mathcal{C}(\mathrm{SSBim})}(L) \otimes \mathbb{Q}[\mathbb{U}]$ and let $\mathfrak{m} = \mathbb{Q}[\mathbb{U}]_{>0}$ denote the irrelevant ideal, i.e. the maximal homogeneous ideal of $\mathbb{Q}[\mathbb{U}]$. Set $\mathfrak{m}^k(L) := \mathrm{End}_{\mathcal{C}(\mathrm{SSBim})}(L) \otimes \mathfrak{m}^k \subset E(L)$; by convention, $\mathfrak{m}^0(L) = E(L)$. We first claim that $E(L)$ is complete (in the graded sense) with respect to the filtration by ideals

$$E(L) \supset \mathfrak{m}(L) \supset \mathfrak{m}^2(L) \supset \cdots .$$

In other words, given a sequence of elements $f_k \in \mathfrak{m}^k(L)$, each of which is homogeneous of cohomological degree ℓ (not depending on k), we claim that the infinite sum $\sum_{k=0}^\infty f_k$ is a well-defined element of $E(L)$.

Indeed, each f_k may be written as a sum of terms $h \otimes g$ where $g \in \mathbb{Q}[\mathbb{U}]$ is a polynomial in which each monomial has total $\mathbb{U}$-degree $\geq k$ (hence cohomological degree $\geq 2k$) and where $f \in \mathrm{End}_{\mathcal{C}(\mathrm{SSBim})}(L)$, which necessarily has cohomological degree $\leq \ell - 2k$. By the definition of $\mathcal{C}(\mathrm{SSBim})$, L is a bounded complex in $\mathcal{C}(\mathrm{SSBim})$, so such terms are zero for $k \gg 0$. Thus $\sum_{i=0}^\infty f_i$ is in fact a finite sum, establishing the claim.

Now, establishing the existence of a curved lift of L amounts to constructing an element

$$\Delta = \sum_{k=0}^{\infty} \Delta_k \in E(L)$$

with $\Delta_k \in \mathfrak{m}^k(L)$ of cohomological degree 1 such that

$$d(\Delta) + \Delta^2 = F \tag{44}$$

where $d = [\delta_L, -]$ is the differential on the dg algebra $E(L)$ and F is the curvature element in (38). Without loss of generality, we assume that Δ_k is $\mathbb{U}$-homogeneous of degree exactly k, and then write equation (44) in terms of its $\mathbb{U}$-homogeneous components, obtaining the family of equations

$$\begin{aligned} d(\Delta_1) &= F \\ d(\Delta_2) &= -\Delta_1^2 \\ d(\Delta_3) &= -(\Delta_2\Delta_1 + \Delta_1\Delta_2) \\ &\;\;\vdots \\ d(\Delta_k) &= -\sum_{i=1}^{k-1} \Delta_i \Delta_{k-i} . \end{aligned} \tag{45}$$

We construct such Δ_k by induction. For the base case $k = 1$, the hypotheses on L guarantee that $F \simeq 0$ in $E(L)$, which gives the existence of Δ_1. Now, let $k > 1$ and assume that $\Delta_1, \ldots, \Delta_{k-1}$ that satisfy (45) have been constructed. Consider the element $O_k := -\sum_{i=1}^{k-1} \Delta_i \Delta_{k-i} \in \mathfrak{m}^k(L)$, which is the obstruction to defining Δ_k. A straightforward computation using (45) shows that O_k is closed. Additionally, we can write $O_k = \sum_g h_g \otimes g$ in which g runs over all monomials of total $\mathbb{U}$-degree k, and therefore each $h_g \in \mathrm{End}_{\mathcal{C}(\mathrm{SSBim})}(L)$ is a closed element of cohomological degree $1 - 2k$. Since L is invertible, we have a quasi-isomorphism of dg algebras

$$\mathrm{End}_{\mathcal{C}(\mathrm{SSBim})}(L) \overset{\mathrm{qis}}{\cong} \mathrm{End}_{\mathcal{C}(\mathrm{SSBim})}(\mathbf{1}_{\boldsymbol{a}}) \,.$$

In particular, neither of these algebras have cohomology in negative degrees, thus the obstruction O_k vanishes up to homotopy. This gives the existence of Δ_k and completes the proof of the existence of the curved lift $\mathrm{tw}_\Delta(L)$.

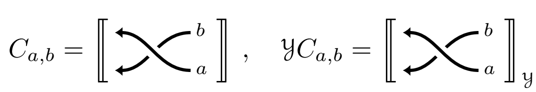
To show uniqueness of the lift up to homotopy equivalence, let $X = \mathrm{tw}_\Delta(L)$ and $Y = \mathrm{tw}_{\Delta'}(L)$ be two curved lifts of L and $X^\vee = \mathrm{tw}_{\Delta''}(L^\vee)$ a curved lift of an (up to homotopy) inverse of L. It follows that $X^\vee \star Y$ is a curved lift of a complex that is homotopy equivalent to $\mathbf{1}_{\boldsymbol{a}}$ in $\mathcal{C}(\mathrm{SSBim})$. Hence, Proposition 4.14 implies the existence of a homotopy equivalence

$$X^\vee \star Y \simeq \mathrm{tw}_{\Delta'''}(\mathbf{1}_{\boldsymbol{a}})$$

for some twist Δ'''. However, since $\mathrm{End}^k_{\mathcal{C}(\mathrm{SSBim})}(\mathbf{1}_{\boldsymbol{a}}) = 0$ for $k < 0$, (42) implies that $\Delta''' = 0$. Thus $X^\vee \star Y \simeq \mathbf{1}_{\boldsymbol{a},\sigma}$, which shows that Y is a two-sided inverse to $X^\vee$ up to homotopy. The same argument shows that X is a two-sided inverse to $X^\vee$ up to homotopy. Thus $X \simeq Y$ by uniqueness of two-sided inverses. □

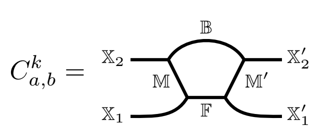
Remark 4.16. Using obstruction-theoretic arguments, it is possible to strengthen the uniqueness statement in Lemma 4.15 as follows. Suppose, that $X \in \mathcal{C}(\mathrm{SSBim})$ is invertible and that $\mathrm{tw}_\Delta(X)$ and $\mathrm{tw}_{\Delta'}(X)$ are two curved lifts of X in $\mathcal{Y}(\mathrm{SSBim})$. Then, in fact, there is a (closed) *isomorphism* $\varphi\colon \mathrm{tw}_\Delta(X) \to \mathrm{tw}_{\Delta'}(X)$ of curved complexes of singular Soergel bimodules such that $\varphi = \mathrm{id}_X + \varphi_{>0}$ with $\varphi_{>0} \in \mathrm{End}_{\mathcal{C}(\mathrm{SSBim})}(X) \otimes \mathbb{Q}[\mathbb{U}]_{>0}$.

4.4. **Curved Rickard complexes.** Our first goal is to define a lift of the two-strand Rickard complex $C_{a,b} \in \mathcal{C}(\mathrm{SSBim})$ from Definition 3.23 to a *curved Rickard complex* $\mathcal{Y}C_{a,b} \in \mathcal{Y}(\mathrm{SSBim})$. More precisely, writing $\boldsymbol{a} := (a,b)$, $\boldsymbol{a}' := (b,a)$, and t for the transposition in $\mathfrak{S}_2$, we define lifts of $C_{a,b}$ to 1-morphisms in ${}_{\boldsymbol{b},\mathsf{t}\circ\sigma}\mathcal{Y}(\mathrm{SSBim})_{\boldsymbol{a},\sigma}$ for both possible permutations $\sigma \in \mathfrak{S}_2$. Pictorially, we denote the 2-strand Rickard complex and its curved analogue by



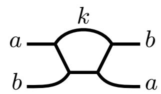
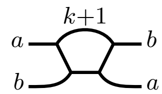
respectively.

As in Definition 3.23, we will abbreviate by writing $C^k_{a,b} := \mathsf{F}^{(a-k)}\mathsf{E}^{(b-k)}\mathbf{1}_{a,b}$. This is the singular Soergel bimodule corresponding to the web



where the labels on the edges are given by the sizes of the alphabets $|\mathbb{X}_1| = |\mathbb{X}'_2| = b$, $|\mathbb{X}_2| = |\mathbb{X}'_1| = a$, $|\mathbb{M}| = a - k$, and $|\mathbb{M}'| = b - k$. This implies $|\mathbb{B}| = k$ and $|\mathbb{F}| = a + b - k$. Recall the morphisms $\chi_r^\pm\colon C^k_{a,b} \to C^{k\pm 1}_{a,b}$ from equations (27) and (28).

Proposition 4.17. *Let $a, b \geq 0$, $\sigma \in \mathfrak{S}_2$, and define $\boldsymbol{a} := (a_1, a_2) := (a, b)$, $\boldsymbol{a}' := (b, a)$. The following diagram define a 1-morphism $\mathcal{Y}C_{a,b} \in \mathcal{Y}(\mathrm{SSBim})$ from $(\boldsymbol{a}, \sigma)$ to $(\boldsymbol{a}', \mathsf{t} \circ \sigma)$, which is a curved lift of the Rickard complex $C_{a,b}$:*

(46) $\cdots \underset{\Delta}{\overset{\delta}{\rightleftarrows}} \mathbf{q}^{-k}\mathbf{t}^{k}$  $\underset{\Delta}{\overset{\delta}{\rightleftarrows}} \mathbf{q}^{-k-1}\mathbf{t}^{k+1}$  $\underset{\Delta}{\overset{\delta}{\rightleftarrows}} \cdots$.

Here δ, Δ are given componentwise by $\delta = \chi_0^+$ and

$$\Delta|_{C^k_{a,b}} = \sum_{1 \leq r \leq m < \infty} (-1)^{r+k-1} \Big(e_{m-r}(\mathbb{X}_2) u_{\sigma(1),m} - e_{m-r}(\mathbb{X}_2') u_{\sigma(2),m} \Big) \chi^-_{r-1}$$

(following Definition 4.7, $u_{\sigma(i),m} = 0$ for $m > a_i$). An analogous construction defines a 1-morphism $\mathcal{Y}C^\vee_{a,b} \in \mathcal{Y}(\mathrm{SSBim})$ from $(\boldsymbol{a}, \sigma)$ to $(\boldsymbol{a}', \mathsf{t} \circ \sigma)$, which is a curved lift of the inverse Rickard complex $C^\vee_{a,b}$.

The proof of this proposition is found below, but first we make some observations. The rightward differential δ is the usual differential on the Rickard complex $C_{a,b}$. Meanwhile the leftward differential Δ is linear in the variables $u_{\sigma(1),m}, u_{\sigma(2),m}$. Thus, (46) defines a *strict* curved deformation of $C_{a,b}$, in the sense of Remarks 4.11 and 4.13.

Definition 4.18. For each integer $r \geq 1$, let Θ_r denote the degree $\mathbf{q}^{2r}\mathbf{t}^{-1}$ endomorphism of $C_{a,b}$ which is given component-wise on $C^k_{a,b}$ by the morphisms $(-1)^k \chi^-_{r-1}$, where χ^-_{r-1} is the morphism from (28).

Lemma 4.19. *The endomorphisms $\Theta_r \in \mathrm{End}_{\mathcal{C}(\mathrm{SSBim})}(C_{a,b})$ satisfy*

$$\Theta_i^2 = 0\,, \quad \Theta_i\Theta_j + \Theta_j\Theta_i = 0\,, \quad [\delta, \Theta_r] = h_r(\mathbb{X}_2 - \mathbb{X}_1')\,.$$

Proof. The first two relations follow from the definition of Θ_r and an easy computation in the $\dot{\mathcal{U}}(\mathfrak{gl}_2)$ thick calculus [KLMS12]. The proof of the third relation is a computation analogous to the one appearing in the proof of [RW16, Proposition 5.7]. We omit the details here (they appear in [HRW21, Lemma 2.30], where a more-general result is established). □

We now pass from the homotopies Θ_r, which are related to $h\Delta$-curvature (35), to homotopies related to the Δe-curvature (36).

Lemma 4.20. *For each $m \geq 1$, let $\Psi_m, \Psi_m' \in \mathrm{End}_{\mathcal{C}(\mathrm{SSBim})}(C_{a,b})$ be given by*

$$\Psi_m := \sum_{r=1}^{m} (-1)^{r-1} e_{m-r}(\mathbb{X}_2)\Theta_r\,, \quad \Psi_m' := \sum_{r=1}^{m} (-1)^{r-1} e_{m-r}(\mathbb{X}_2')\Theta_r\,.$$

These endomorphisms satisfy

$$\Psi_i^2 = 0 = {\Psi_i'}^2\,, \quad \Psi_i\Psi_j + \Psi_j\Psi_i = 0\,, \quad \Psi_i\Psi_j' + \Psi_j'\Psi_i = 0\,, \quad \Psi_i'\Psi_j' + \Psi_j'\Psi_i' = 0\,,$$
$$[\delta, \Psi_m] = e_m(\mathbb{X}_2) - e_m(\mathbb{X}_1')\,, \quad [\delta, \Psi_m'] = e_m(\mathbb{X}_2') - e_m(\mathbb{X}_1)\,.$$

Proof. The first four relations are immediate from Lemma 4.19. For the penultimate relation, we compute

$$[\delta, \Psi_m] = \sum_{r=1}^{m} (-1)^{r-1} e_{m-r}(\mathbb{X}_2)[\delta, \Theta_r] = \sum_{r=1}^{m} (-1)^{r-1} e_{m-r}(\mathbb{X}_2) h_r(\mathbb{X}_2 - \mathbb{X}_1') \overset{(18)}{=} e_m(\mathbb{X}_2) - e_m(\mathbb{X}_1')\,.$$

For the last relation, we first note that

$$e_m(\mathbb{X}_2') - e_m(\mathbb{X}_1) \overset{(18)}{=} \sum_{r=1}^{m} (-1)^{r-1} e_{m-r}(\mathbb{X}_2') h_r(\mathbb{X}_2' - \mathbb{X}_1) = \sum_{r=1}^{m} (-1)^{r-1} e_{m-r}(\mathbb{X}_2') h_r(\mathbb{X}_2 - \mathbb{X}_1')\,.$$

Here, we use that $\mathbb{X}_2' - \mathbb{X}_1 = \mathbb{X}_2 - \mathbb{X}_1'$ when acting on ${}_{b,a}\mathrm{SSBim}_{a,b}$. We then have

$$[\delta, \Psi'_m] = \sum_{r=1}^{m} (-1)^{r-1} e_{m-r}(\mathbb{X}_2')[\delta, \Theta_r] = \sum_{r=1}^{m} (-1)^{r-1} e_{m-r}(\mathbb{X}_2') h_r(\mathbb{X}_2 - \mathbb{X}_1') = e_m(\mathbb{X}_2') - e_m(\mathbb{X}_1)\,. \quad \square$$

Remark 4.21. The homotopies Ψ_k^{o} and Ψ_k^{u} appearing in §1.3 are Ψ_k and $-\Psi'_k$ from Lemma 4.20, respectively.

Lemma 4.20 implies that Ψ_m is closed for $m > a$, since $\mathbb{X}_2$ and $\mathbb{X}_1'$ have cardinality a. Similarly, Ψ'_m is closed for $m > b$. The following says that these closed endomorphisms are in fact zero. Analogously, it shows that Θ_m is "redundant" for $m > a$ or $m > b$.

Proposition 4.22. *For $r > 0$, we have*

$$\Psi_{a+r} = 0\,, \quad \Psi_{b+r} = 0$$

and

$$\Theta_{a+r} = \sum_{k=1}^{a} (-1)^{a-k} \mathfrak{s}_{(r-1|a-k)}(\mathbb{X}_2)\Theta_k\,, \quad \Theta_{b+r} = \sum_{k=1}^{b} (-1)^{b-k} \mathfrak{s}_{(r-1|b-k)}(\mathbb{X}_2')\Theta_k\,.$$

Proof. Let us distinguish the alphabets $\mathbb{B}^{(k)}$ and $\mathbb{B}^{(k+1)}$ (of cardinality k and $k+1$), living on webs $C_{a,b}^k$ and $C_{a,b}^{k+1}$. Note that we may regard $\mathrm{Hom}_{\mathrm{SSBim}}(C_{a,b}^{k+1}, C_{a,b}^k)$ as a module over $\mathrm{Sym}(\mathbb{X}_2|\mathbb{X}_2'|\mathbb{B}^{(k)}|\mathbb{B}^{(k+1)})$. We also have actions of symmetric functions in the alphabets

$$\mathbb{M}^{(k)} := \mathbb{X}_2 - \mathbb{B}^{(k)}\,, \quad \mathbb{M}'^{(k)} := \mathbb{X}_2' - \mathbb{B}^{(k)}\,, \quad \mathbb{M}^{(k+1)} = \mathbb{X}_2 - \mathbb{B}^{(k+1)}\,, \quad \mathbb{M}'^{(k+1)} := \mathbb{X}_2' - \mathbb{B}^{(k+1)}\,.$$

Let $M \subset \mathrm{Hom}_{\mathrm{SSBim}}(C_{a,b}^{k+1}, C_{a,b}^k)$ be the $\mathrm{Sym}(\mathbb{X}_2|\mathbb{X}_2'|\mathbb{B}^{(k)}|\mathbb{B}^{(k+1)})$-submodule generated by the elements $\{\chi_r^-\}_{r\geq 0}$ and let $\mathbb{D} = \{x\}$ denote the alphabet on the cup of χ_0^-. Since $\chi_{r-1}^- = x^{r-1}\cdot\chi_0^- = h_{r-1}(\mathbb{D})\cdot\chi_0^-$, we may regard M as a module over $\mathrm{Sym}(\mathbb{X}_2|\mathbb{X}_2'|\mathbb{B}^{(k)}|\mathbb{B}^{(k+1)}|\mathbb{D})$. It follows (e.g. from the "movie" following (28), or the corresponding foam) that

$$\mathbb{D} = \mathbb{M}^{(k)} - \mathbb{M}^{(k+1)} = \mathbb{B}^{(k+1)} - \mathbb{B}^{(k)} = \mathbb{M}'^{(k)} - \mathbb{M}'^{(k+1)}$$

when acting on χ_0^-. Observe that $\Theta_m|_{C_{a,b}^k} = (-1)^k h_{m-1}(\mathbb{D})\cdot\chi_0^-$, so

$$\begin{aligned}\Psi_m|_{C_{a,b}^k} = \sum_{r=1}^{m} (-1)^{r-1} e_{m-r}(\mathbb{X}_2)\Theta_r &= (-1)^k \sum_{r=1}^{m} (-1)^{r-1} e_{m-r}(\mathbb{X}_2) h_{r-1}(\mathbb{D})\cdot\chi_0^- \\ &= (-1)^k e_{m-1}(\mathbb{X}_2 - \mathbb{D})\cdot\chi_0^-\end{aligned}$$

Now, $\mathbb{X}_2 - \mathbb{D} = \mathbb{M}^{(k+1)} + \mathbb{B}^{(k)}$ has cardinality $a-1$, which implies that $\Psi_m = 0$ for $m > a$. A similar argument shows that $\Psi'_m|_{C_{a,b}^k} = (-1)^{a+b-k} e_{m-1}(\mathbb{X}_2' - \mathbb{D})\cdot\chi_0^-$ which is zero for $m > b$ since $\mathbb{X}_2' - \mathbb{D}$ has cardinality $b-1$. Finally, the identities for Θ_{a+r} and Θ_{b+r} follow from h-reduction, i.e. Lemma 2.9. For instance:

$$\begin{aligned}\Theta_{a+r}|_{C_{a,b}^k} = (-1)^k h_{a+r-1}(\mathbb{D})\cdot\chi_0^- &= \sum_{i=1}^{a} (-1)^{a-i} \mathfrak{s}_{(r-1|a-i)}(\mathbb{X}_2)(-1)^k h_{i-1}(\mathbb{D})\cdot\chi_0^- \\ &= \sum_{i=1}^{a} (-1)^{a-i} \mathfrak{s}_{(r-1|a-i)}(\mathbb{X}_2)\Theta_i|_{C_{a,b}^k}\end{aligned}$$

where we have used (21) with $\mathbb{X} = \mathbb{D}$, $\mathbb{Y} = \mathbb{X}_2 - \mathbb{D}$, and $c = a-1$ on the first line. The relation for Θ_{b+r} is proven similarly. $\square$

Remark 4.23. Proposition 4.22 shows that, in $\mathrm{End}_{\mathcal{C}(\mathrm{SSBim})}(C_{a,b})$, the elements Θ_r and Ψ_r can all be written as $\mathrm{Sym}(\mathbb{X}_2|\mathbb{X}_2')$-linear combinations of $\Theta_1, \ldots, \Theta_{\min\{a,b\}}$.

Proof of Proposition 4.17. The fact that $\delta \circ \delta = 0$ is well-known (the Rickard complex is a complex); see e.g. [WW17, Corollary 6.5]. We write $\Delta = \sum_{m\geq 1}(\Psi_m u_{\sigma(1),m} - \Psi'_m u_{\sigma(2),m})$, where we again set $u_{\sigma(i),m} = 0$ for $m > a_i$. Lemma 4.20 implies that $\Delta \circ \Delta = 0$ and that

$$[\delta, \Delta] = \sum_{m\geq 1} u_{\sigma(1),m}\Big(e_m(\mathbb{X}_2) - e_m(\mathbb{X}'_1)\Big) - \sum_{m\geq 1} u_{\sigma(2),m}\Big(e_m(\mathbb{X}'_2) - e_m(\mathbb{X}_1)\Big),$$

which is as desired. □

We now use Proposition 4.17 to assign a curved Rickard complex to any colored braid. The input for this construction is a $\mathbb{Z}_{\geq 1}$-colored braid word $\beta_{\boldsymbol{a}}$ (in the sense of §3.5) and a numbering of the strands[8] in the braid. If $\beta \in \mathrm{Br}_m$, then we require that $\beta_{\boldsymbol{a}}$ has strands numbered from 1 to m. Reading the sequence of strand numbers on the incoming and outgoing boundaries (respectively) defines two permutations $\sigma, \tau \in \mathfrak{S}_m$ such that $\tau = \beta \circ \sigma$. (Here, and in the following, we abuse notation and write β for the permutation induced by the braid β under the canonical homomorphism $\mathrm{Br}_k \to \mathfrak{S}_k$.) We will denote the data of a $\mathbb{Z}_{\geq 1}$-colored braid $\beta_{\boldsymbol{a}} = {}_{\boldsymbol{b}}\beta_{\boldsymbol{a}}$ with numbered strands by ${}_{\boldsymbol{b},\tau}\beta_{\boldsymbol{a},\sigma}$ or by $\beta_{\boldsymbol{a},\sigma}$ (since $\tau = \beta \circ \sigma$). Further, we will occasionally omit the numbering from our notation when it is not locally relevant.

Theorem 4.24. *For each braid $\beta \in \mathrm{Br}_m$, each $\boldsymbol{a} \in \mathbb{Z}^m_{\geq 1}$, and each $\sigma \in \mathfrak{S}_m$, the Rickard complex $C(\beta_{\boldsymbol{a}}) \in \mathcal{C}(\mathrm{SSBim})$ has a curved lift to a 1-morphism $\mathcal{Y}C(\beta_{\boldsymbol{a},\sigma})$ from $(\boldsymbol{a}, \sigma)$ to $(\beta(\boldsymbol{a}), \beta\circ\sigma)$ in $\mathcal{Y}(\mathrm{SSBim})$ that is unique up to homotopy equivalence. In particular, the assignment $\beta_{\boldsymbol{a},\sigma} \mapsto \mathcal{Y}C(\beta_{\boldsymbol{a},\sigma})$ satisfies the (colored) braid relations up to homotopy.*

Before the proof, we remark that the curvature equation for $\mathcal{Y}C(\beta_{\boldsymbol{a},\sigma})$ is

$$(\delta + \Delta)^2 = \sum_{i=1}^{m}\sum_{r=1}^{\infty}(e_r(\mathbb{X}_{\tau(i)}) - e_r(\mathbb{X}'_{\sigma(i)}))u_{i,r}$$

for $\tau = \beta \circ \sigma$.

Proof. Since $C(\beta_{\boldsymbol{a}}) \in \mathcal{C}(\mathrm{SSBim})$ is an invertible 1-morphism and the Rickard complexes assigned to colored braids satisfy the (colored) braid relations up to homotopy, this is an immediate consequence of Lemma 4.15. However, since we only sketched the existence portion of that proof, we give an explicit construction here.

First, we prove the theorem for the identity braid $\mathbf{1}_{\boldsymbol{a}}$ and any chosen $\sigma \in \mathfrak{S}_m$. In this case,

$$\mathrm{End}_{\mathcal{C}(\mathrm{SSBim})[\mathbb{U}]}(\mathbf{1}_{\boldsymbol{a}}) = \mathrm{Sym}(\mathbb{X}_1|\cdots|\mathbb{X}_m) \otimes \mathbb{Q}[\mathbb{U}]$$

is supported in even cohomological degrees, so any (degree 1) differential on $\mathbf{1}_{\boldsymbol{a}}$ must be zero for degree reasons. In particular, $\delta_{\mathbf{1}_{\boldsymbol{a}}} = 0$, so the curvature equation for $\mathbf{1}_{\boldsymbol{a}}$ becomes

$$(\Delta_{\mathbf{1}_{\boldsymbol{a}}})^2 = \sum_{i=1}^{m}\sum_{r=1}^{\infty}(e_r(\mathbb{X}_{\sigma(i)}) - e_r(\mathbb{X}'_{\sigma(i)}))u_{i,r} = 0\,.$$

Here, we use that $f(\mathbb{X}_{\sigma(i)}) - f(\mathbb{X}'_{\sigma(i)})$ acts by zero on the identity bimodule $\mathbf{1}_{\boldsymbol{a}}$ for any symmetric function f. This equation has the unique solution $\Delta_{\mathbf{1}_{\boldsymbol{a}}} = 0$, so the curved Rickard complex associated to the identity braid is just $\mathbf{1}_{\boldsymbol{a}}$ with $\delta^{\mathrm{tot}} = \delta + \Delta = 0$, regardless of our choice of σ.

We next give a constructive proof of the existence of the curved Rickard complex associated to a non-trivial braid (word). For the Artin generator β_i and the identity permutation $\mathrm{id} \in \mathfrak{S}_m$, we define

$$\mathcal{Y}C((\beta_i)_{\boldsymbol{a},\mathrm{id}}) := \mathbf{1}_{(a_1,\dots a_{i-1}),\mathrm{id}} \boxtimes \mathcal{Y}C_{(a_i,a_{i+1}),\mathrm{id}} \boxtimes \mathbf{1}_{(a_{i+2},\dots a_m),\mathrm{id}}$$

[8] Thus, one can think of such braids as being $\mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$-colored.

where the middle tensor factor is the 2-strand curved Rickard complex from Proposition 4.17. For any other $\sigma \in \mathfrak{S}_m$, we obtain $\mathcal{Y}C((\beta_i)_{\boldsymbol{a},\sigma})$ from $\mathcal{Y}C((\beta_i)_{\boldsymbol{a},\mathrm{id}})$ by the substitution $u_{i,r} \mapsto u_{\sigma^{-1}(i),r}$. Analogously, we define $\mathcal{Y}C((\beta_i^{-1})_{\boldsymbol{a},\sigma})$ using $\mathcal{Y}C^{\vee}_{(a_i,a_{i+1}),\mathrm{id}}$. For a general braid word $\beta = \beta_{i_r}^{\varepsilon_r} \cdots \beta_{i_1}^{\varepsilon_1}$, we define $\mathcal{Y}C(\beta_{\boldsymbol{a},\sigma})$ by taking the horizontal composition of the curved Rickard complexes assigned to its constituent Artin generators $\mathcal{Y}C((\beta_{i_j}^{\varepsilon_j})_{\tau_j(\boldsymbol{a}),\tau_j\circ\sigma})$, in analogy to (32). Here, τ_j is the permutation associated with the braid $\beta_{i_{j-1}}^{\varepsilon_{j-1}} \cdots \beta_{i_r}^{\varepsilon_r}$. Lemma 4.12 guarantees that $\mathcal{Y}C(\beta_{\boldsymbol{a},\sigma})$ is a well-defined 1-morphism in $\mathcal{Y}(\mathrm{SSBim})$. It is the unique curved lift of $C(\beta_{\boldsymbol{a}})$ associated with the permutation σ by Lemma 4.15, since Proposition 3.25 gives that $C(\beta_{\boldsymbol{a}}) \in \mathcal{C}(\mathrm{SSBim})$ is an invertible 1-morphism.

Finally, we show that the assignment $\beta_{\boldsymbol{a},\sigma} \mapsto \mathcal{Y}C(\beta_{\boldsymbol{a},\sigma})$ satisfies the braid relations, up to homotopy equivalence. Suppose that $\beta_{\boldsymbol{a},\sigma}$ and $\gamma_{\boldsymbol{a},\sigma}$ are two braid words representing the same colored braid. By Proposition 3.25, we have a homotopy equivalence $C(\beta_{\boldsymbol{a},\sigma}) \simeq C(\gamma_{\boldsymbol{a},\sigma})$. Proposition 4.14 provides a lift to a homotopy equivalence between $\mathcal{Y}C(\beta_{\boldsymbol{a},\sigma})$ and some curved lift of $C(\gamma_{\boldsymbol{a},\sigma})$. However, by Lemma 4.15, such lifts are unique up to homotopy equivalence, and so $\mathcal{Y}C(\beta_{\boldsymbol{a},\sigma}) \simeq \mathcal{Y}C(\gamma_{\boldsymbol{a},\sigma})$. □

We pause to record a useful observation, which will help to establish a well-defined module structure on our deformed colored link homology. By construction, the complex $\mathcal{Y}C(\beta_{\boldsymbol{a},\sigma})$ built in Theorem 4.24 is a strict 1-morphism, in the sense of Remark 4.11. As such, the linear part of the curved Maurer–Cartan element gives homotopies $\Psi_{i,r}$, now considered as elements of $\mathrm{End}_{\mathcal{Y}(\mathrm{SSBim})}\left(\mathcal{Y}C(\beta_{\boldsymbol{a},\sigma})\right)$, for $1 \le i \le m$ and $1 \le r \le a_{\sigma(i)}$ that square to zero, pairwise anti-commute, and satisfy

$$[\delta_{\mathcal{Y}C(\beta)} + \Delta_{\mathcal{Y}C(\beta)}, \Psi_{i,r}] = e_r(\mathbb{X}_{\tau(i)}) - e_r(\mathbb{X}'_{\sigma(i)})\,. \tag{47}$$

It follows that, for each $1 \le i \le m$, $\mathcal{Y}C(\beta_{\boldsymbol{a},\sigma})$ is a well-defined dg module over the (i^{th}) *two point* dg algebra

$$A^i_{L,R} := \mathrm{Sym}(\mathbb{X}_{\tau(i)}|\mathbb{X}'_{\sigma(i)}) \otimes \wedge(\Psi_{i,1}, \ldots, \Psi_{i,a_{\sigma(i)}})\,, \quad d(\Psi_{i,r}) = e_r(\mathbb{X}_{\tau(i)}) - e_r(\mathbb{X}'_{\sigma(i)})\,, \tag{48}$$

and further that these actions assemble to give a dg $(\bigotimes_i A^i_{L,R})$-module structure on $\mathcal{Y}C(\beta_{\boldsymbol{a},\sigma})$. Let $\mathbb{W}_i$ be an alphabet with $|\mathbb{W}_i| = |\mathbb{X}_{\tau(i)}| = |\mathbb{X}'_{\sigma(i)}|$. By restricting the action of (48) to the left and right alphabets, we obtain two (possibly) distinct actions of the algebra $\mathrm{Sym}(\mathbb{W}_i)$ using the identifications

$$\mathrm{Sym}(\mathbb{W}_i) \cong \mathrm{Sym}(\mathbb{X}_{\tau(i)})\,, \quad \mathrm{Sym}(\mathbb{W}_i) \cong \mathrm{Sym}(\mathbb{X}'_{\sigma(i)})\,.$$

Our next result shows that these actions on $\mathcal{Y}C(\beta_{\boldsymbol{a},\sigma})$ give quasi-isomorphic dg $\mathrm{Sym}(\mathbb{W}_i)$-modules. We state the result in slightly greater generality.

Proposition 4.25. *Let X be a dg $(\bigotimes_{i=1^m} A^i_{L,R})$-module (e.g. a strict 1-morphism in $\mathcal{Y}(\mathrm{SSBim})$), then the induced left and right actions of $\mathrm{Sym}(\mathbb{W}_i)$ give quasi-isomorphic dg $\mathrm{Sym}(\mathbb{W}_i)$-modules (thus quasi-isomorphic dg $(\bigotimes_i \mathrm{Sym}(\mathbb{W}_i))$-modules).*

Proof. It suffices to show that, given a dg module over

$$\mathbb{Q}[e^L_1, \ldots, e^L_a, e^R_1, \ldots, e^R_a] \otimes \wedge(\Psi_1, \ldots, \Psi_a)\,, \quad d(\Psi_i) = e^L_i - e^R_i\,,$$

the corresponding left and right $\mathbb{Q}[e_1, \ldots, e_a]$-modules are quasi-isomorphic. Correspondingly, this immediately reduces to the 1-variable case.

Thus, let C be a dg module over

$$A = \mathbb{Q}[x_L, x_R] \otimes \wedge(\xi)\,, \quad d(\xi) = x_L - x_R$$

and consider $\mathcal{M} := C \otimes_A \mathcal{T}$ where $\mathcal{T} := \mathbb{Q}[z_L, z_R, z_M, T] \otimes \wedge(\xi_{LR}, \xi_{RM}, \xi_{LM})$ is a dg A-algebra via the inclusion $A \hookrightarrow \mathcal{T}$ sending $x_L \mapsto z_L$, $x_R \mapsto z_R$, and $\xi \mapsto \xi_{LR}$. The remaining differentials on $\mathcal{T}$ are given by

$$d(\xi_{RM}) = z_R - z_M\,, \quad d(\xi_{LM}) = z_L - z_M\,, \quad d(T) = \xi_{RM} - \xi_{LM} + \xi_{LR}\,.$$

Now, view $\mathcal{M}$ as a dg $\mathbb{Q}[x]$-module, where x acts via z_M, and let C_i denote the dg $\mathbb{Q}[x]$-module C wherein x acts via x_i, for $i = L, R$. The deformation retract $\mathcal{M} \to C_L$ determined by

$$z_M \mapsto x_L\,, \quad \xi_{LM} \mapsto 0\,, \quad \xi_{RM} \mapsto -\xi\,, \quad T \mapsto 0$$

is $\mathbb{Q}[x]$-linear; however, its homotopy inverse is not. Thus it (only) gives a quasi-isomorphism of $\mathbb{Q}[x]$-modules $\mathcal{M} \overset{\text{qis}}{\cong} C_L$. Similarly, there is a deformation retract $\mathcal{M} \to C_R$ determined by

$$z_M \mapsto x_R\,, \quad \xi_{LM} \mapsto \xi\,, \quad \xi_{RM} \mapsto 0\,, \quad T \mapsto 0$$

that gives a quasi-isomorphism of $\mathbb{Q}[x]$-modules $\mathcal{M} \overset{\text{qis}}{\cong} C_R$. We thus have $C_L \overset{\text{qis}}{\cong} \mathcal{M} \overset{\text{qis}}{\cong} C_R$. □

4.5. **Alphabet soup I: from u's to v's.** As discussed in §4.2 (and implicitly used in Lemma 4.20), we will find it convenient to translate back-and-forth between Δe- and $h\Delta$-curvatures, which are modeled on $\sum (e_k(\mathbb{X}) - e_k(\mathbb{X}'))u_k$ and $\sum h_k(\mathbb{X} - \mathbb{X}')v_k$ respectively. Indeed, equation (39) and Lemma 4.12 show that complexes with Δe-curvature possess a straightforward horizontal composition, while Lemma 4.19 suggests that complexes with $h\Delta$-curvature appear more regularly "in the wild."

Definition 4.26. Let $\boldsymbol{a} = (a_1, \dots, a_m)$ and $\boldsymbol{b} = (b_1, \dots, b_m)$ be objects in SSBim, and let $\sigma, \tau \in \mathfrak{S}_m$ be such that $a_{\sigma(i)} = b_{\tau(i)}$. For each $1 \leq i \leq m$, introduce deformation parameters $\mathbb{V}_i = \{v_{i,r}\}_{r=1}^{a_{\sigma(i)}}$ with $\mathrm{wt}(v_{i,r}) = \mathbf{q}^{-2r}\mathbf{t}^2$. Let

$$_{\boldsymbol{b},\tau}\mathcal{V}(\mathrm{SSBim})_{\boldsymbol{a},\sigma} := \mathcal{C}_Z\big(_{\boldsymbol{b}}\mathrm{SSBim}_{\boldsymbol{a}}; \mathbb{Q}[\mathbb{V}_1, \dots, \mathbb{V}_m]\big)$$

where the curvature element is

$$Z = \sum_{i=1}^{m} \sum_{r=1}^{a_{\sigma(i)}} h_r(\mathbb{X}_{\tau(i)} - \mathbb{X}'_{\sigma(i)})v_{i,r}\,. \tag{49}$$

We will abbreviate by writing $\mathbb{V} = \mathbb{V}_1 \cup \cdots \cup \mathbb{V}_m$ when m is understood, thus $\mathbb{Q}[\mathbb{V}] = \mathbb{Q}[\mathbb{V}_1, \dots, \mathbb{V}_m]$. Similarly, we will write

$$\mathcal{V}(\mathrm{SSBim}) := \bigsqcup_{(\boldsymbol{b},\tau),(\boldsymbol{a},\sigma)} {_{\boldsymbol{b},\tau}\mathcal{V}(\mathrm{SSBim})_{\boldsymbol{a},\sigma}}$$

which we understand simply as a disjoint union of categories.

Proposition 4.27. *Retaining the setup from Definition 4.26, there is an isomorphism*

$$_{\boldsymbol{b},\tau}\mathcal{V}(\mathrm{SSBim})_{\boldsymbol{a},\sigma} \cong {_{\boldsymbol{b},\tau}\mathcal{Y}(\mathrm{SSBim})_{\boldsymbol{a},\sigma}}$$

of dg categories determined by the mutually inverse assignments

$$v_{i,k} \mapsto (-1)^{k-1} \sum_{l=k}^{a_{\sigma(i)}} e_{l-k}(\mathbb{X}_{\tau(i)})u_{i,l}\,, \quad u_{i,k} \mapsto (-1)^{k-1} \sum_{l=k}^{a_{\sigma(i)}} h_{l-k}(\mathbb{X}_{\tau(i)})v_{i,l}\,, \tag{50}$$

(cf. Definition 4.3).

Proof. The functor $_{\boldsymbol{b},\tau}\mathcal{V}(\mathrm{SSBim})_{\boldsymbol{a},\sigma} \to {_{\boldsymbol{b},\tau}\mathcal{Y}(\mathrm{SSBim})_{\boldsymbol{a},\sigma}}$ is defined on curved complexes $\mathrm{tw}_{\Delta_X}(X) = (X, \delta_X + \Delta_X)$ by sending X to itself, and defined on morphisms

$$\mathrm{Hom}_{\mathrm{SSBim}[\mathbf{t}^{\pm}]}(X,Y) \otimes \mathbb{Q}[\mathbb{V}] := \mathrm{Hom}_{\mathcal{V}(\mathrm{SSBim})}(X,Y) \to \mathrm{Hom}_{\mathcal{Y}(\mathrm{SSBim})}(X,Y) =: \mathrm{Hom}_{\mathrm{SSBim}[\mathbf{t}^{\pm}]}(X,Y) \otimes \mathbb{Q}[\mathbb{U}]$$

using the first substitution rule in (50). In particular, this determines the image of the curved Maurer–Cartan element Δ_X. This is well-defined since Corollary 4.6 implies that it takes curved complexes with $h\Delta$-curvature (49) to those with Δe-curvature (38). The functor in the other direction is defined analogously using the second substitution rule in (50). As for (37), a computation using (15) shows that the substitutions (50), and thus the associated functors, are mutually inverse. □

Remark 4.28. By pulling back structure from $\mathcal{Y}(\mathrm{SSBim})$, the categories ${}_{\boldsymbol{b},\tau}\mathcal{V}(\mathrm{SSBim})_{\boldsymbol{a},\sigma}$ assemble to give a monoidal dg 2-category. We will not record the precise formulae for the various operations, but do wish to point out a subtlety regarding the action of the deformation parameters $\mathbb{V}$. Since $\mathrm{End}_{\mathcal{V}(\mathrm{SSBim})}(\mathbf{1}_{\boldsymbol{b}}) \cong \mathrm{Sym}(\mathbb{X}_1|\cdots|\mathbb{X}_m)[\mathbb{V}]$ is a module over $\mathbb{Q}[\mathbb{V}]$, any curved complex ${}_{\boldsymbol{b}}X \in \mathcal{V}(\mathrm{SSBim})$ inherits an action of the latter by acting **on the left**. Specifically, given $g \in \mathbb{Q}[\mathbb{V}]$, we let $g \cdot \mathrm{id}_X$ denote the endomorphism

$$X \cong \mathbf{1}_{\boldsymbol{b}} \star X \xrightarrow{g \star \mathrm{id}_X} \mathbf{1}_{\boldsymbol{b}} \star X \cong X\,.$$

However, we could also consider the action of such g act on the right, via

$$X \cong X \star \mathbf{1}_{\boldsymbol{a}} \xrightarrow{\mathrm{id}_X \star g} X \star \mathbf{1}_{\boldsymbol{a}} \cong X\,.$$

These actions **do not** necessarily agree. Rather, they are related by an appropriate analogue of Lemma 4.4. Specifically, on ${}_{\boldsymbol{b},\tau}X_{\boldsymbol{a},\sigma}$ we have that

$$\mathrm{id}_X \star v_{i,r} = \sum_{r \le l \le a_{\sigma(i)}} h_{l-r}(\mathbb{X}_{\tau(i)} - \mathbb{X}'_{\sigma(i)}) \cdot (v_{i,l} \star \mathrm{id}_X)\,.$$

Moreover, given another 1-morphism ${}_{\boldsymbol{a},\sigma}Y_{\mathbf{c},\rho}$, we can consider the composite ${}_{\boldsymbol{b},\tau}X \star Y_{\mathbf{c},\rho}$ and use the above formula to change from the action of $v_{i,k} \star \mathrm{id}_X \star \mathrm{id}_Y$ to $\mathrm{id}_X \star v_{i,l} \star \mathrm{id}_Y$ and further to $\mathrm{id}_X \star \mathrm{id}_Y \star v_{i,r}$ or directly in one step. The consistency of these changes is a consequence of the formula

$$h_k(\mathbb{X}_{\tau(i)} - \mathbb{X}''_{\rho(i)}) = \sum_{r+s=k} h_r(\mathbb{X}_{\tau(i)} - \mathbb{X}'_{\sigma(i)}) h_s(\mathbb{X}'_{\sigma(i)} - \mathbb{X}''_{\rho(i)})\,.$$

In any case, the action of $h_{l-r}(\mathbb{X}_{\tau(i)} - \mathbb{X}'_{\sigma(i)})$ on any 1-morphism in $\mathcal{V}(\mathrm{SSBim})$ is null-homotopic (see e.g. Remark 4.11), thus the left and right actions of $\mathbb{V}$ are always homotopic.

Convention 4.29. Henceforth, we will not distinguish between the dg categories ${}_{\boldsymbol{b},\tau}\mathcal{V}(\mathrm{SSBim})_{\boldsymbol{a},\sigma}$ and ${}_{\boldsymbol{b},\tau}\mathcal{Y}(\mathrm{SSBim})_{\boldsymbol{a},\sigma}$, and in each instance will use the notation coinciding with the relevant deformation parameters $\mathbb{U}$ or $\mathbb{V}$.

In §6 – §9, we will be particularly interested in the $\sigma = \mathrm{id} = \tau$ case of Definition 4.26. To simplify notation, we will use the shorthand

(51)
$$\mathcal{V}_{\boldsymbol{a}} := {}_{\boldsymbol{a},\mathrm{id}}\mathcal{V}(\mathrm{SSBim})_{\boldsymbol{a},\mathrm{id}} = \mathcal{C}_Z\big({}_{\boldsymbol{a}}\mathrm{SSBim}_{\boldsymbol{a}}; \mathbb{Q}[\mathbb{V}_1, \ldots, \mathbb{V}_m]\big)$$

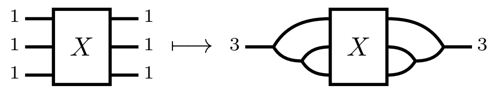
for the category of curved complexes of singular Soergel bimodules with curvature

$$Z = \sum_{i=1}^{m} \sum_{r=1}^{a_i} h_r(\mathbb{X}_i - \mathbb{X}'_i) v_{i,r}\,.$$

4.6. **Alphabet soup II: from y's to u's and v's.** In the special (uncolored) case of $\boldsymbol{a} = 1^m = \boldsymbol{b}$, the categories ${}_{1^m,\tau}\mathcal{Y}(\mathrm{SSBim})_{1^m,\sigma}$ recover (a version of) the category $\mathcal{Y}(\mathrm{SBim})$ of curved complexes of Soergel bimodules from [GH]. Setting $\mathbb{X} = \{x_1, \ldots, x_m\}$ and $\mathbb{X}' = \{x'_1, \ldots, x'_m\}$, this incarnation of $\mathcal{Y}(\mathrm{SBim})$ is the category of curved complexes of $(\mathbb{Q}[\mathbb{X}], \mathbb{Q}[\mathbb{X}'])$-bimodules (equivalently, $\mathbb{Q}[\mathbb{X}, \mathbb{X}']$-modules) with "thin" curvature

(52)
$$\sum_{i=1}^{m} (x_{\tau(i)} - x_{\sigma(i)}) y_i$$

for deformation parameters $\mathbb{Y} = \{y_1, \ldots, y_m\}$. These parameters can be understood as equaling either the u's from Definition 4.7 or the v's from Definition 4.26 (since $h_1(x_i - x'_i) = x_i - x'_i = e_1(x_i) - e_1(x'_i)$). Note that $\mathbb{Q}[\mathbb{X}, \mathbb{X}']$-modules can be viewed as $\mathrm{Sym}(\mathbb{X}|\mathbb{X}')$-modules, by restricting the left and right actions. In terms of SSBim, this corresponds to horizontally composing Soergel bimodules X by appropriate merge and split bimodules which, on each side of X, merge the boundaries to a single m-colored strand, e.g.



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Since colored links can be interpreted as the "(anti)symmetric part" of appropriate cables of links in a similar manner (recall Theorem 1.21 and Conjecture 1.24), we will find it fortuitous to relate the thin curvature in (52) to the $h\Delta$-curvature from (35) and the Δe-curvature from (36). Indeed, certain endomorphisms that appear naturally in this story (see §4.7 below) are crucial in our investigation of the colored link splitting map in §7.

To this end, let $a \geq 1$ and fix alphabets

$$\mathbb{X} = \{x_1, \ldots, x_a\}, \quad \mathbb{X}' = \{x'_1, \ldots, x'_a\}, \quad \mathbb{Y} = \{y_1, \ldots, y_a\}, \quad \mathbb{U} = \{u_1, \ldots, u_a\}, \quad \mathbb{V} = \{v_1, \ldots, v_a\}.$$

Let $\mathcal{A}$ denote the category of $\mathbb{Q}[\mathbb{X}, \mathbb{X}']$-modules and consider the following categories of curved complexes

$$\mathcal{VA} := \mathcal{C}_{\sum_{k=1}^a h_k(\mathbb{X}-\mathbb{X}')v_k}(\mathcal{A}, \mathbb{Q}[\mathbb{V}]), \quad \mathcal{YA} := \mathcal{C}_{\sum_{k=1}^a (e_k(\mathbb{X})-e_k(\mathbb{X}'))u_k}(\mathcal{A}, \mathbb{Q}[\mathbb{U}])$$
$$\mathsf{Y}\mathcal{A} := \mathcal{C}_{\sum_{i=1}^a (x_i - x'_i)y_i}(\mathcal{A}, \mathbb{Q}[\mathbb{Y}]).$$

Writing $h_k(\mathbb{X}-\mathbb{X}')$ and $e_k(\mathbb{X}) - e_k(\mathbb{X}')$ as $\mathbb{Q}[\mathbb{X}, \mathbb{X}']$-linear combinations of the elements $x_i - x'_i$ produces dg functors from $\mathsf{Y}\mathcal{A}$ to $\mathcal{VA}$ and $\mathcal{YA}$, respectively. Precisely, we have the following.

Proposition 4.30. *The dg categories $\mathcal{VA}$ and $\mathcal{YA}$ are isomorphic via the mutually inverse substitutions*

$$v_k \mapsto (-1)^{k-1} \sum_{k \leq l \leq a} e_{l-k}(\mathbb{X}) u_l, \quad u_k \mapsto (-1)^{k-1} \sum_{k \leq l \leq a} h_{l-k}(\mathbb{X}) v_l.$$

Moreover, the substitutions

$$y_i \mapsto \sum_{l=1}^a h_{l-1}\big(\{x_i, x_{i+1}, \ldots, x_a\} - \{x'_{i+1}, \ldots, x'_a\}\big) v_l \tag{53}$$

and

$$y_i \mapsto \sum_{l=1}^a e_{l-1}(x_1, \ldots, x_{i-1}, x'_{i+1}, \ldots, x'_a) u_l \tag{54}$$

determine dg functors $\mathsf{Y}\mathcal{A} \to \mathcal{VA}$ and $\mathsf{Y}\mathcal{A} \to \mathcal{YA}$ that are compatible with the isomorphism $\mathcal{VA} \cong \mathcal{YA}$.

Proof. The isomorphism between $\mathcal{VA}$ and $\mathcal{UA}$ follows from the discussion in §4.2, which is the 1-strand case of Proposition 4.27.

Similarly, the substitutions (53) and (54) define algebra homomorphisms $\mathbb{Q}[\mathbb{X}, \mathbb{X}', \mathbb{Y}] \to \mathbb{Q}[\mathbb{X}, \mathbb{X}', \mathbb{V}]$ and $\mathbb{Q}[\mathbb{X}, \mathbb{X}', \mathbb{Y}] \to \mathbb{Q}[\mathbb{X}, \mathbb{X}', \mathbb{U}]$ that determine dg functors which are the identity on objects and are given on morphism complexes by extension of scalars $\mathbb{Q}[\mathbb{X}, \mathbb{X}', \mathbb{V}] \otimes_{\mathbb{Q}[\mathbb{X}, \mathbb{X}', \mathbb{Y}]} (-)$ and $\mathbb{Q}[\mathbb{X}, \mathbb{X}', \mathbb{U}] \otimes_{\mathbb{Q}[\mathbb{X}, \mathbb{X}', \mathbb{Y}]} (-)$.

To confirm that these functors are indeed well-defined, it suffices to show that the algebra maps preserve curvature. We first confirm this for the functor $\mathsf{Y}\mathcal{A} \to \mathcal{YA}$. To begin, we explicitly write $e_k(\mathbb{X}) - e_k(\mathbb{X}')$ as an element of the ideal generated by $x_i - x'_i \in \mathbb{Q}[\mathbb{X}, \mathbb{X}']$. Consider the difference of monomials:

$$x_{i_1} \cdots x_{i_k} - x'_{i_1} \cdots x'_{i_k} = \sum_{j=1}^k x_{i_1} \cdots x_{i_{j-1}} (x_{i_j} - x'_{i_j}) x'_{i_{j+1}} \cdots x'_{i_k}.$$

Summing over all sequences with $1 \leq i_1 < \cdots < i_k \leq a$ gives the identity

$$e_k(\mathbb{X}) - e_k(\mathbb{X}') = \sum_{i=1}^a e_{k-1}(x_1, \ldots, x_{i-1}, x'_{i+1}, \ldots, x'_a) \cdot (x_i - x'_i).$$

Hence, we compute

$$\sum_{k=1}^{a}\big(e_k(\mathbb{X})-e_k(\mathbb{X}')\big)u_k = \sum_{k=1}^{a}\sum_{i=1}^{a} e_{k-1}(x_1,\dots,x_{i-1},x'_{i+1},\dots,x'_a)\cdot(x_i-x'_i)\cdot u_k$$
$$= \sum_{i=1}^{a}(x_i-x'_i)\left(\sum_{k=1}^{a} e_{k-1}(x_1,\dots,x_{i-1},x'_{i+1},\dots,x'_a)u_k\right),$$

which proves that the algebra map $\mathbb{Q}[\mathbb{X},\mathbb{X}',\mathbb{Y}]\to\mathbb{Q}[\mathbb{X},\mathbb{X}',\mathbb{U}]$ defined by (54) preserves curvature.

Since $\mathcal{YA}\cong\mathcal{VA}$, the proof is completed by showing that the algebra maps $\mathbb{Q}[\mathbb{X},\mathbb{X}',\mathbb{Y}]\to\mathbb{Q}[\mathbb{X},\mathbb{X}',\mathbb{U}]$ and $\mathbb{Q}[\mathbb{X},\mathbb{X}',\mathbb{Y}]\to\mathbb{Q}[\mathbb{X},\mathbb{X}',\mathbb{V}]$ are intertwined by the isomorphism $\mathbb{Q}[\mathbb{X},\mathbb{X}',\mathbb{U}]\cong\mathbb{Q}[\mathbb{X},\mathbb{X}',\mathbb{V}]$. Indeed:

$$\begin{aligned} y_i &\mapsto \sum_{k=1}^{a} e_{k-1}(x_1,\dots,x_{i-1},x'_{i+1},\dots,x'_a)u_k \\ &\mapsto \sum_{1\le k\le l\le a}(-1)^{k-1}e_{k-1}(x_1,\dots,x_{i-1},x'_{i+1},\dots,x'_a)h_{l-k}(x_1,\dots,x_a)v_l \\ &= \sum_{l=1}^{a} h_{l-1}\big(\{x_1,\dots,x_a\}-\{x_1,\dots,x_{i-1},x'_{i+1},\dots,x'_a\}\big)v_l \\ &= \sum_{l=1}^{a} h_{l-1}\big(\{x_i,\dots,x_a\}-\{x'_{i+1},\dots,x'_a\}\big)v_l\,. \end{aligned}$$ □

4.7. **Alphabet soup III: interpolation coordinates.** Retain the notation from §4.6 and consider the identity bimodule $\mathbf{1}_{(1,\dots,1)}\in\mathsf{Y}\mathcal{A}$. In $\mathrm{End}_{\mathsf{Y}\mathcal{A}}(\mathbf{1}_{(1,\dots,1)})$, we have that $x_i=x'_i$, thus we find that the dg functor $\mathsf{Y}\mathcal{A}\to\mathcal{VA}$ induces a map $\mathbb{Q}[\mathbb{X},\mathbb{Y}]\cong\mathrm{End}_{\mathsf{Y}\mathcal{A}}(\mathbf{1}_{(1,\dots,1)})\to\mathrm{End}_{\mathcal{VA}}(\mathbf{1}_{(1,\dots,1)})\cong\mathbb{Q}[\mathbb{X},\mathbb{V}]$ sending

$$y_i\mapsto\sum_{r=1}^{a}h_{r-1}(x_i)v_r\,.$$

This motivates the following.

Definition 4.31. Set

$$y_i := \sum_{r=1}^{a} x_i^{r-1}v_r\in\mathbb{Q}[\mathbb{X},\mathbb{V}]\,. \tag{55}$$

In other words, y_i is defined to be the polynomial of degree $|\mathbb{X}|-1$ in x_i with coefficients v_r. We call these coefficients *interpolation coordinates* and highlight that they are independent of i.

Using (55), we can view $\mathbb{Q}[\mathbb{X},\mathbb{Y}]$ as a subalgebra of $\mathbb{Q}[\mathbb{X},\mathbb{V}]$. This inclusion is equivariant for the action of $\mathfrak{S}_a$ that (simultaneously) permutes the x_i (and y_i). Note, however, that (55) is a special case of (53), which is only compatible with Proposition 4.30 when $\mathbb{X}=\mathbb{X}'$. Nevertheless, symmetric functions in the alphabet $\mathbb{Y}$ give well-defined elements of $\mathrm{Sym}(\mathbb{X})[\mathbb{V}]\cong\mathrm{End}_{\mathcal{V}(\mathrm{SSBim})}(\mathbf{1}_{(a)})$, thus we can consider them as operators acting (on the left or right) on suitable 1-morphisms in $\mathcal{V}(\mathrm{SSBim})$. For the duration of the paper, the variables y_i will always be understood in this context. (See Remark 4.28 above, which addresses a subtle point concerning these actions.)

Our terminology in Definition 4.31 is chosen since we can express the v_i in terms of x_i, y_i by formulae familiar from interpolation theory. We now make this precise, and establish further identities involving the interpolation coordinates. We will make use of identities from §2.3 and the Haiman determinants from §2.4.

Example 4.32. When $a = 2$, the elements $y_i \in \mathbb{Q}[x_1, x_2, v_1, v_2]$ are given by $y_i = v_1 + x_i v_2$ and

$$v_2 = \frac{y_1 - y_2}{x_1 - x_2} = \frac{\operatorname{hdet}(y, 1)}{x_1 - x_2} = -\frac{\Delta_{\mathcal{M}_1}(\mathbb{X}, \mathbb{Y})}{\Delta(\mathbb{X})}, \quad v_1 = \frac{x_1 y_2 - x_2 y_1}{x_1 - x_2} = \frac{\operatorname{hdet}(x, y)}{x_1 - x_2} = \frac{\Delta_{\mathcal{M}_2}(\mathbb{X}, \mathbb{Y})}{\Delta(\mathbb{X})}$$

for $\mathcal{M}_1 = \{1, y\}$ and $\mathcal{M}_2 = \{x, y\}$.

Generalizing Example 4.32, the following gives the general rule to recover the interpolation coordinates from the x_i and y_i. (See also Lemma 7.7 below for another formulation.)

Lemma 4.33. *Let $\mathbb{X} = \{x_1, \dots, x_a\}$ and $\mathbb{Y} = \{y_1, \dots, y_a\}$, then we have*

$$v_{a-k+1} = \frac{\operatorname{hdet}(x^{a-1}, \dots, x^{a-k+1}, y, x^{a-k-1}, \dots x^0)}{\operatorname{hdet}(x^{a-1}, \dots, x^0)} = (-1)^{a-k} \frac{\Delta_{\mathcal{M}_k}(\mathbb{X}, \mathbb{Y})}{\Delta(\mathbb{X})}$$

where $\mathcal{M}_k = \{x^{a-1}, \dots, \widehat{x^{a-k}}, \dots, 1, y\}$.

Proof. The second equation follows from the definitions and reordering rows. For the first equation, consider the matrix defining the determinant $\operatorname{hdet}(x^{a-1}, \dots, x^{a-k+1}, y, x^{a-k-1}, \dots x^0)$ and rewrite its kth row as:

$$\begin{bmatrix} y_1 & \cdots & y_a \end{bmatrix} = \sum_{r=1}^{a} v_r \begin{bmatrix} x_1^{r-1} & \cdots & x_a^{r-1} \end{bmatrix}. \qquad \square$$

The only summand with a nonzero contribution is the one for $r = a - k + 1$, which yields v_{a-k+1} in the quotient.

In Definition 4.31 we have defined the variables y_i in terms of x_i and a family of interpolation coordinates v_r which depends on the cardinality of the alphabet $\mathbb{X}$. Sometimes, however, it is useful to take the opposite viewpoint and start with alphabets $\mathbb{X}$ and $\mathbb{Y}$ and the assumption that the variables y_i can be interpolated by polynomials in x_i (with coefficients then determined by Lemma 4.33). For example, in later parts of the paper, we will need to understand the behavior of the interpolation coordinates v_r under inclusions of the alphabets $\mathbb{X}$ and $\mathbb{Y}$.

Proposition 4.34. *Consider integers $1 \leq c \leq d$ and alphabets*

$$\mathbb{X}^{(c)} := \{x_1, \dots, x_c\} \subset \mathbb{X}^{(d)} := \{x_1, \dots, x_d\}, \quad \mathbb{Y}^{(c)} := \{y_1, \dots, y_c\} \subset \mathbb{Y}^{(d)} := \{y_1, \dots, y_d\}$$

with associated collections of interpolation coordinates $\mathbb{V}^{(c)} = \{v_1^{(c)}, \dots, v_c^{(c)}\}$ and $\mathbb{V}^{(d)} = \{v_1^{(d)}, \dots, v_d^{(d)}\}$. The standard inclusion $\mathbb{Q}[\mathbb{X}^{(c)}, \mathbb{Y}^{(c)}] \hookrightarrow \mathbb{Q}[\mathbb{X}^{(d)}, \mathbb{Y}^{(d)}]$ extends uniquely to an algebra map

$$\mathbb{Q}[\mathbb{X}^{(c)}, \mathbb{V}^{(c)}] \hookrightarrow \mathbb{Q}[\mathbb{X}^{(d)}, \mathbb{V}^{(d)}]$$

sending

$$v_k^{(c)} \mapsto v_k^{(d)} + (-1)^{c-k} \sum_{l=c+1}^{d} \mathfrak{s}_{(l-c-1|c-k)}(\mathbb{X}^{(c)}) v_l^{(d)}. \tag{56}$$

Proof. We first prove uniqueness. Suppose $\varphi, \psi \colon \mathbb{Q}[\mathbb{X}^{(c)}, \mathbb{V}^{(c)}] \hookrightarrow \mathbb{Q}[\mathbb{X}^{(d)}, \mathbb{V}^{(d)}]$ are two algebra maps extending the inclusion $\mathbb{Q}[\mathbb{X}^{(c)}, \mathbb{Y}^{(c)}] \hookrightarrow \mathbb{Q}[\mathbb{X}^{(d)}, \mathbb{Y}^{(d)}]$. Since $\Delta(\mathbb{X}^{(c)}) v_k^{(c)} \in \mathbb{Q}[\mathbb{X}^{(c)}, \mathbb{Y}^{(c)}]$, it follows that $\varphi(v_k^{(c)}) - \psi(v_k^{(c)})$ is an element of $\mathbb{Q}[\mathbb{X}^{(d)}, \mathbb{Y}^{(d)}]$ that is annihilated by $\Delta(\mathbb{X}^{(c)})$, hence zero.

Now, define $\varphi \colon \mathbb{Q}[\mathbb{X}^{(c)}, \mathbb{V}^{(c)}] \hookrightarrow \mathbb{Q}[\mathbb{X}^{(d)}, \mathbb{V}^{(d)}]$ to be the algebra map defined by $\varphi(x_i) = x_i$ and the substitution rule (56). It suffices to verify that $\varphi(y_i) = y_i$ for $1 \leq i \leq c$. Indeed:

$$\varphi(y_i) = \sum_{k=1}^{c} x_i^{k-1} \varphi(v_k^{(c)}) = \sum_{k=1}^{c} x_i^{k-1} \left(v_k^{(d)} + \sum_{l=c+1}^{d} (-1)^{c-k} \mathfrak{s}_{(l-c-1|c-k)}(\mathbb{X}^{(c)}) v_l^{(d)} \right)$$

$$= \sum_{k=1}^{c} x_i^{k-1} v_k^{(d)} + \sum_{l=c+1}^{d} v_l^{(d)} \sum_{k=1}^{c} (-1)^{c-k} \mathfrak{s}_{(l-c-1|c-k)}(\mathbb{X}^{(c)}) x_i^{k-1}$$

$$\overset{\text{Cor. 2.10}}{=} \sum_{k=1}^{c} x_i^{k-1} v_k^{(d)} + \sum_{l=c+1}^{d} x_i^{l-1} v_l^{(b)} = \sum_{k=1}^{d} x_i^{k-1} v_k^{(d)} = y_i. \qquad \square$$

There is an interesting generalization of Proposition 4.34 that we will use (in various forms) throughout this paper.

Lemma 4.35. *Let $\mathbb{X}^{(c)} = \{x_1, \ldots, x_c\}$ and $\mathbb{X}'^{(c)} = \{x'_1, \ldots, x'_c\}$. For each subset $S \subset \{1, \ldots, c\}$, consider the element of $\mathbb{Q}[\mathbb{X}^{(c)}, \mathbb{X}'^{(c)}, \mathbb{V}^{(c)}]$ defined by*

$$Z_S^{(c)} := \sum_{k=1}^{c} h_k(\mathbb{X}_S - \mathbb{X}'_S) v_k^{(c)},$$

where $\mathbb{X}_S \subset \mathbb{X}^{(c)}$ and $\mathbb{X}'_S \subset \mathbb{X}'^{(c)}$ denote the corresponding subalphabets. For $c \leq d$, the algebra map

$$\varphi\colon \mathbb{Q}[\mathbb{X}^{(c)}, \mathbb{X}'^{(c)}, \mathbb{V}^{(c)}] \hookrightarrow \mathbb{Q}[\mathbb{X}^{(d)}, \mathbb{X}'^{(d)}, \mathbb{V}^{(d)}] \tag{57}$$

determined by $x_i \mapsto x_i$, $x'_i \mapsto x'_i$, and the rule (56) sends $Z_S^{(c)} \mapsto Z_S^{(d)} := \sum_{k=1}^{d} h_k(\mathbb{X}_S - \mathbb{X}'_S) v_k^{(d)}$.

Proof. We compute:

$$\varphi(Z_S^{(c)}) = \sum_{k=1}^{c} h_k(\mathbb{X}_S - \mathbb{X}'_S)\varphi(v_k^{(c)}) = \sum_{k=1}^{c} h_k(\mathbb{X}_S - \mathbb{X}'_S)\left(v_k^{(d)} + \sum_{l=c+1}^{d} (-1)^{c-k} \mathfrak{s}_{(l-c-1|c-k)}(\mathbb{X}^{(c)}) v_l^{(d)}\right)$$

$$= \sum_{k=1}^{c} h_k(\mathbb{X}_S - \mathbb{X}'_S) v_k^{(d)} + \sum_{l=c+1}^{d} v_l^{(d)} \sum_{k=1}^{c} (-1)^{c-k} \mathfrak{s}_{(l-c-1|c-k)}(\mathbb{X}^{(c)}) h_k(\mathbb{X}_S - \mathbb{X}'_S)$$

$$= \sum_{k=1}^{c} h_k(\mathbb{X}_S - \mathbb{X}'_S) v_k^{(d)} + \sum_{l=c+1}^{d} h_l(\mathbb{X}_S - \mathbb{X}'_S) v_l^{(d)} = \sum_{k=1}^{d} h_k(\mathbb{X}_S - \mathbb{X}'_S) v_k^{(d)} = Z_S^{(d)}.$$

In passing to the last line, we have used Lemma 2.9 with $\mathbb{X} = \mathbb{X}_S - \mathbb{X}'_S$ and $\mathbb{Y} = (\mathbb{X}^{(c)} - \mathbb{X}_S) + \mathbb{X}'_S$ (which has cardinality c). $\square$

Remark 4.36. Lemma 4.35 establishes a crucial "stability" property for the expressions $Z_S^{(c)}$ under the inclusion (57). The fact that $y_i \mapsto y_i$ in Proposition 4.34 is essentially the special case of Lemma 4.35 corresponding to $S = \{i\}$ and $\mathbb{X}'^{(c)} = 0$ (hence $Z_S = x_i y_i$).

In this section, we have discussed the variables y_i associated with a single a-colored strand. Indeed, as mentioned above, any symmetric polynomial in the alphabet $\mathbb{Y}$ gives a well-defined element in $\mathrm{Sym}(\mathbb{X})[\mathbb{V}] = \mathrm{End}_{\mathcal{Y}(\mathrm{SSBim})}(\mathbf{1}_{(a)})$. Paralleling the passage from §4.2 to §4.5, it is possible to pass from the one-strand case to the general case, since curvature in $\mathcal{Y}(\mathrm{SSBim})$ is modeled on the strand-wise $h\Delta$- and Δe-curvatures. See §4.8 and §8.1 for aspects of the (pure) 2-strand and m-strand cases, respectively.

Remark 4.37. The reader familiar with the Hilbert scheme $\mathrm{Hilb}_a(\mathbb{C}^2)$ should note that interpolation coordinates arise naturally in its study. Compare the generators $\{e_1(\mathbb{X}), \ldots, e_a(\mathbb{X}), v_1, \ldots, v_a\}$ of $\mathrm{Sym}(\mathbb{X})[\mathbb{V}] = \mathrm{End}_{\mathcal{Y}(\mathrm{SSBim})}(\mathbf{1}_{(a)})$ to the coordinates on the open affine set $U_x \subset \mathrm{Hilb}_a(\mathbb{C}^2)$ in [Hai01, Proposition 3.6.3].

4.8. **Alphabet soup IV: the two-strand categories.** In this section we set up some framework and notation for working with the categories $\mathcal{V}_{a,b}$, i.e. the $m = 2$ case of (51). This section can be skipped on first reading, and referred to as needed.

First, we will use the abbreviation

$$\mathcal{C}_{a,b} := \mathcal{C}({}_{a,b}\mathrm{SSBim}_{a,b})$$

for the relevant dg category of (uncurved) complexes of singular Soergel bimodules. We fix alphabets as follows:

$$\begin{aligned} \mathbb{X} &= \{x_1, \ldots, x_{a+b}\}, \quad \mathbb{X}_1 = \{x_1, \ldots, x_a\}, \quad \mathbb{X}_2 = \{x_{a+1}, \ldots, x_{a+b}\} \\ \mathbb{X}' &= \{x'_1, \ldots, x'_{a+b}\}, \quad \mathbb{X}'_1 = \{x'_1, \ldots, x'_a\}, \quad \mathbb{X}'_2 = \{x'_{a+1}, \ldots, x'_{a+b}\} \\ \mathbb{Y} &= \{y_1, \ldots, y_{a+b}\}, \quad \mathbb{Y}_1 = \{y_1, \ldots, y_a\}, \quad \mathbb{Y}_2 = \{y_{a+1}, \ldots, y_{a+b}\}. \end{aligned}$$

We denote[9] the deformation parameters associated with the first and second entries of $\boldsymbol{a} = (a, b)$ by

$$\mathbb{V}_L^{(a)} := \{v_{L,1}^{(a)}, \ldots, v_{L,a}^{(a)}\}, \quad \mathbb{V}_R^{(b)} := \{v_{R,1}^{(b)}, \ldots, v_{R,b}^{(b)}\} \tag{58}$$

respectively, and let $\mathbb{V} := \mathbb{V}_L^{(a)} \cup \mathbb{V}_R^{(b)}$. We regard $\mathbb{Q}[\mathbb{X}, \mathbb{Y}]$ as a subalgebra of $\mathbb{Q}[\mathbb{X}, \mathbb{V}]$ by the identification

$$y_i = \begin{cases} \sum_{k=1}^{a} x_i^{k-1} v_{L,k}^{(a)} & \text{if } 1 \le i \le a \\ \sum_{k=1}^{b} x_i^{k-1} v_{R,k}^{(b)} & \text{if } a < i \le a+b \,, \end{cases} \tag{59}$$

which is the (pure) 2-strand analogue of (55).

We now observe that it is possible to specify objects in $\mathcal{V}_{a,b}$ using a reduced collection of deformation parameters.

Definition 4.38. Let $\overline{\mathcal{V}}_{a,b} := \mathcal{C}_{\bar{Z}}({}_{a,b}\mathrm{SSBim}_{a,b}, \mathbb{Q}[\bar{v}_1, \ldots, \bar{v}_b])$ be the category of curved complexes where $\mathrm{wt}(\bar{v}_r) = \mathbf{q}^{-2r}\mathbf{t}^2$ and with curvature element

$$\bar{Z} = \sum_{r=1}^{b} h_r(\mathbb{X}_2 - \mathbb{X}'_2)\bar{v}_r \,.$$

In other words, $\overline{\mathcal{V}}_{a,b}$ is the category of curved complexes over ${}_{a,b}\mathrm{SSBim}_{a,b}$ where we have only deformed "on the b-labeled strand."

We now aim to introduce functors $\overline{\mathcal{V}}_{a,b} \leftrightarrow \mathcal{V}_{a,b}$ which lift the identity on $\mathcal{C}_{a,b}$. The first functor $\mathcal{V}_{a,b} \to \overline{\mathcal{V}}_{a,b}$ is just the specialization $v_{L,i}^{(a)} = 0$ and $v_{R,j}^{(b)} = \bar{v}_j$ for all i, j. The functor in the other direction is more interesting. The starting point for its constructing is the observation that since

$$f(\mathbb{X}_1 + \mathbb{X}_2 - \mathbb{X}'_1 - \mathbb{X}'_2)|_X = 0$$

for any $X \in {}_{a,b}\mathrm{SSBim}_{a,b}$ and any (positive degree) symmetric function f, we have that

$$h_r(\mathbb{X}_1 - \mathbb{X}'_1) = h_r\big((\mathbb{X}_1 + \mathbb{X}_2 - \mathbb{X}'_1 - \mathbb{X}'_2) + (\mathbb{X}'_2 - \mathbb{X}_2)\big) = h_r(\mathbb{X}'_2 - \mathbb{X}_2) \,.$$

When $a \ge b$, an application of Lemma 4.35 shows that if we set

$$v_{L,j}^{(b)} := v_{L,j}^{(a)} + (-1)^{b-j} \sum_{i=1}^{a-b} \mathfrak{s}_{(i-1|b-j)}(\mathbb{X}'_2) v_{L,b+i}^{(a)} \tag{60}$$

[9] Our notation here indicates that $\mathbb{V}_L^{(a)}$ and $\mathbb{V}_R^{(b)}$ should be viewed as associated to the "left" a-colored and "right" b-colored strands of a 2-strand pure braid, when drawn vertically. They would be called $\mathbb{V}_1$ and $\mathbb{V}_2$ in the language of §4.5.

for $1 \leq j \leq b$, then

$$\sum_{k=1}^{a} h_k(\mathbb{X}_1 - \mathbb{X}'_1) v^{(a)}_{L,k} = \sum_{k=1}^{a} h_k(\mathbb{X}'_2 - \mathbb{X}_2) v^{(a)}_{L,k} = \sum_{k=1}^{b} h_k(\mathbb{X}'_2 - \mathbb{X}_2) v^{(b)}_{L,k}\,.$$

Further, this remains true (trivially) when $a < b$, provided we let $v^{(a)}_{L,j} = 0$ for $j > 0$, since in this case (60) gives

$$v^{(b)}_{L,j} := \begin{cases} v^{(a)}_{L,j} & \text{if } 1 \leq j \leq a \\ 0 & \text{if } a < j \leq b\,. \end{cases}$$

It follows that

$$\begin{aligned} Z &= \sum_{1\leq i\leq a} h_i(\mathbb{X}_1 - \mathbb{X}'_1) v^{(a)}_{L,i} + \sum_{1\leq j\leq b} h_j(\mathbb{X}_2 - \mathbb{X}'_2) v^{(b)}_{R,j} \\ &= \sum_{1\leq k\leq b} h_k(\mathbb{X}'_2 - \mathbb{X}_2) v^{(b)}_{L,k} + \sum_{1\leq j\leq b} h_j(\mathbb{X}_2 - \mathbb{X}'_2) v^{(b)}_{R,j} \\ &\overset{(20)}{=} - \sum_{1\leq j\leq k\leq b} h_j(\mathbb{X}_2 - \mathbb{X}'_2) h_{k-j}(\mathbb{X}'_2 - \mathbb{X}_2) v^{(b)}_{L,k} + \sum_{1\leq j\leq b} h_j(\mathbb{X}_2 - \mathbb{X}'_2) v^{(b)}_{R,j} \\ &= \sum_{1\leq j\leq b} h_j(\mathbb{X}_2 - \mathbb{X}'_2) \left(v^{(b)}_{R,j} - \sum_{j\leq k\leq b} h_{k-j}(\mathbb{X}'_2 - \mathbb{X}_2) v^{(b)}_{L,k} \right)\,. \end{aligned} \tag{61}$$

This suggests the following.

Proposition 4.39. *There is a functor $\overline{\mathcal{V}}_{a,b} \to \mathcal{V}_{a,b}$ which is the identity on objects and morphisms in $\mathcal{C}_{a,b}$, and which sends*

$$\bar{v}_j \mapsto v^{(b)}_{R,j} - \sum_{k=j}^{b} h_{k-j}(\mathbb{X}'_2 - \mathbb{X}_2) v^{(b)}_{L,k}\,. \tag{62}$$

Proof. It need only be checked that (62) is compatible with curvatures, i.e. that

$$\bar{Z} = \sum_{r=1}^{b} h_r(\mathbb{X}_2 - \mathbb{X}'_2)\bar{v}_r \mapsto \sum_{r=1}^{a} h_r(\mathbb{X}_1 - \mathbb{X}'_1) v^{(a)}_{L,r} + \sum_{r=1}^{b} h_r(\mathbb{X}_2 - \mathbb{X}'_2) v^{(b)}_{R,r} = Z\,. \tag{63}$$

This is true by the computation (61) preceding this proposition. □

Definition 4.40. Let $\pi\colon \mathcal{V}_{a,b} \to \mathcal{V}_{a,b}$ denote the functor which is the identity on objects and morphisms in $\mathcal{C}_{a,b}$, and which sends $v^{(a)}_{L,i} \mapsto 0$ and sends $v^{(b)}_{R,i}$ to the expression on the right-hand side of (62). We call π the *reduction functor*. We say that an object $X \in \mathcal{V}_{a,b}$ is *reduced* if $\pi(X) = X$, and that a morphism $f\colon X \to Y$ between reduced objects is reduced if $\pi(f) = f$.

Remark 4.41. It is clear that $\pi^2 = \pi$ is idempotent, and has image equal (or canonically isomorphic to) $\overline{\mathcal{V}}_{a,b}$. Thus, we may identify $\overline{\mathcal{V}}_{a,b}$ as a (non-full) subcategory of $\mathcal{V}_{a,b}$ consisting of reduced objects and morphisms in $\mathcal{V}_{a,b}$. The notion of reduction saves us some work when constructing curved lifts of complexes in $\mathcal{C}_{a,b}$. Indeed, it suffices to construct a curved lift in $\overline{\mathcal{V}}_{a,b}$, which then immediately gives a curved lift in $\mathcal{V}_{a,b}$.

We conclude this section by establishing further notation for working with the reduced categories $\overline{\mathcal{V}}_{a,b}$. At times, we will need to consider the relation between such categories as we allow b to vary. For each $\ell \geq 0$, introduce interpolation coordinates $\overline{\mathbb{V}}^{(\ell)} = \{\bar{v}_1^{(\ell)}, \ldots, \bar{v}_\ell^{(\ell)}\}$ and elements

$$\overline{y}_i := \sum_{k=1}^{\ell} x_i^{k-1} \bar{v}_k^{(\ell)} \tag{64}$$

for all $a+1 \leq i \leq a+\ell$. A priori, the definition of $\overline{y}_i$ depends on ℓ. However, if we let

$$\mathbb{X}_{[1,\ldots,c]} = \{x_1, \ldots, x_c\}\,, \quad \mathbb{X}'_{[1,\ldots,c]} = \{x'_1, \ldots, x'_c\} \tag{65}$$

then for each pair of integers $\ell \leq b$ we have an inclusion of algebras

$$\mathbb{Q}[\mathbb{X}_{[1,a+\ell]}, \mathbb{X}'_{[1,a+\ell]}, \overline{\mathbb{V}}^{(\ell)}] \hookrightarrow \mathbb{Q}[\mathbb{X}_{[1,a+b]}, \mathbb{X}'_{[1,a+b]}, \overline{\mathbb{V}}^{(b)}] = \mathbb{Q}[\mathbb{X}, \mathbb{X}', \overline{\mathbb{V}}^{(b)}] \tag{66}$$

sending $x_i \mapsto x_i$, $x'_i \mapsto x'_i$, and

$$\bar{v}_k^{(\ell)} \mapsto \bar{v}_k^{(b)} + (-1)^{\ell-k} \sum_{r=\ell+1}^{b} \mathfrak{s}_{(r-\ell-1|\ell-k)}(\mathbb{X}_{[a+1,a+\ell]}) \bar{v}_r^{(b)}\,. \tag{67}$$

Crucially, Lemma 4.34 implies that this sends $\overline{y}_i \mapsto \overline{y}_i$. Furthermore, Lemma 4.35 implies that the inclusion (66) is compatible with curvature, in the sense that

$$\sum_{i=1}^{\ell} h_i(\mathbb{X}_{[a+1,a+\ell]} - \mathbb{X}'_{[a+1,a+\ell]}) \bar{v}_i^{(\ell)} \mapsto \sum_{i=1}^{b} h_i(\mathbb{X}_{[a+1,a+b]} - \mathbb{X}'_{[a+1,a+b]}) \bar{v}_i^{(b)}\,. \tag{68}$$

5. Deformed colored link homology

In this section, we use curved complexes of singular Soergel bimodules to construct our deformed, colored, triply-graded link homology. We begin by recalling the construction of colored, triply-graded Khovanov–Rozansky homology, which is an invariant of framed, oriented, colored links taking values in the symmetric monoidal category $\overline{\mathcal{K}}[\![\mathbf{a}^{\pm}, \mathbf{q}^{\pm}, \mathbf{t}^{\pm}]\!]$ from §3.1. Khovanov–Rozansky homology is defined by applying the Hochschild homology functor to the Rickard complex of a braid representative of a link $\mathbf{L}$, and our deformation is defined by replacing the Rickard complex with its curved analogue from §4.4. In this section (and those following) we will typically denote our (co)domain objects in SSBim by $\boldsymbol{b}$ and $\mathbf{c}$, since our notation for the object $\boldsymbol{a}$ is easily confused with the Hochschild degree $\mathbf{a}$.

5.1. Hochschild (co)homology.

We begin by recalling the basics of Hochschild homology and cohomology. Suppose that R and S are $\mathbb{Z}_{\mathbf{q}}$-graded algebras and let ${}_R\mathcal{B}_S$ denote the category of $\mathbb{Z}_{\mathbf{q}}$-graded (R,S)-bimodules. When $R = S$, a graded (R,R)-bimodule $M \in {}_R\mathcal{B}_R$ may be viewed as a module over the enveloping algebra $R^e = R \otimes R^{\mathrm{op}}$, and the i^{th} *Hochschild homology and cohomology* of M are defined as:

$$\mathrm{HH}_i(M) := \mathrm{Tor}_i^{R^e}(R, M)\,, \quad \mathrm{HH}^i(M) := \mathrm{Ext}^i_{R^e}(R, M) \tag{69}$$

for $i \geq 0$, respectively. Note that both inherit a $\mathbb{Z}_{\mathbf{q}}$-grading from ${}_R\mathcal{B}_R$, thus are objects in $\overline{\mathcal{K}}[\![\mathbf{q}^{\pm}]\!]$. The *total Hochschild (co)homology functors* are defined by

$$\mathrm{HH}_\bullet(M) := \bigoplus_{i \geq 0} \mathbf{a}^{-i}\, \mathrm{HH}_i(M)\,, \quad \mathrm{HH}^\bullet(M) := \bigoplus_{i \geq 0} \mathbf{a}^{i}\, \mathrm{HH}^i(M) \tag{70}$$

which we therefore view as functors ${}_R\mathcal{B}_R \to \overline{\mathcal{K}}[\![\mathbf{a}^{\pm}, \mathbf{q}^{\pm}]\!]$ (see §3.1).

Remark 5.1. We view Hochschild homology as supported in negative $\mathbf{a}$-degree, while Hochschild cohomology is supported in positive $\mathbf{a}$-degrees.

The following is a standard fact about Hochschild homology; see e.g. [Rou17].

Proposition 5.2. *Let R and S be $\mathbb{Z}_{\mathbf{q}}$-graded algebras. Suppose that ${}_RM_S \in {}_R\mathcal{B}_S$ and ${}_SN_R \in {}_S\mathcal{B}_R$ are $\mathbb{Z}_{\mathbf{q}}$-graded bimodules that are projective as R-modules and S-modules, then there is an isomorphism of $\mathbb{Z}_{\mathbf{a}} \times \mathbb{Z}_{\mathbf{q}}$-graded $\mathbb{Q}$-vector spaces*

$$\mathrm{HH}_\bullet(M \otimes_S N) \cong \mathrm{HH}_\bullet(N \otimes_R M)$$

that is natural in M and N. □

The total Hochschild (co)homology functors can be extended from bimodules to complexes of bimodules term-wise. Explicitly, if $X \in \mathcal{C}({}_R\mathcal{B}_R)$ is a complex of graded (R,R)-bimodules, then we may write $X = \mathrm{tw}_\delta(\bigoplus_k \mathbf{t}^k X^k)$ for graded (R,R)-bimodules $X^k \in {}_R\mathcal{B}_R$. Define the complex

$$\mathrm{HH}_\bullet(X) := \mathrm{tw}_{\mathrm{HH}_\bullet(\delta)}\Big(\bigoplus_k \mathbf{t}^k\,\mathrm{HH}_\bullet(X^k)\Big) \tag{71}$$

which is an object in $\overline{\mathcal{K}}[\![\mathbf{a}^\pm, \mathbf{q}^\pm, \mathbf{t}^\pm]\!]_{\mathrm{dg}}$. In other words, $\mathrm{HH}_\bullet(X)$ equals the $\mathbb{Z}_{\mathbf{a}} \times \mathbb{Z}_{\mathbf{q}} \times \mathbb{Z}_{\mathbf{t}}$-graded $\mathbb{Q}$-vector space $\bigoplus_{i,k} \mathbf{a}^{-i}\mathbf{t}^k\,\mathrm{HH}_i(X^k)$, with appropriate differential. Proposition 5.2 extends to this setting as follows.

Proposition 5.3. *Let R and S be $\mathbb{Z}_{\mathbf{q}}$-graded algebras. Suppose that ${}_RX_S \in \mathcal{C}({}_R\mathcal{B}_S)$ and ${}_SY_R \in \mathcal{C}({}_S\mathcal{B}_R)$ are complexes of $\mathbb{Z}_{\mathbf{q}}$-graded bimodules, that are projective as R-modules or S-modules, then there is an isomorphism of $\mathbb{Z}_{\mathbf{a}} \times \mathbb{Z}_{\mathbf{q}} \times \mathbb{Z}_{\mathbf{t}}$-graded complexes*

$$\mathrm{HH}_\bullet(X \otimes_S Y) \cong \mathrm{HH}_\bullet(Y \otimes_R X),$$

that is natural in X, Y (in the dg sense).

Proof. Denote the natural isomorphism from Proposition 5.2 by

$$\tau_{M,N}\colon\ \mathrm{HH}_\bullet(M \otimes_S N) \to \mathrm{HH}_\bullet(N \otimes_R M)\,.$$

This induces an isomorphism on the level of $\mathbb{Z}_{\mathbf{a}} \times \mathbb{Z}_{\mathbf{q}} \times \mathbb{Z}_{\mathbf{t}}$-graded $\mathbb{Q}$-vector spaces:

$$\mathrm{HH}_\bullet(X \otimes_S Y) \cong \bigoplus_{k,l\in\mathbb{Z}} \mathbf{t}^{k+l}\,\mathrm{HH}_\bullet(X^k \otimes_S Y^l) \xrightarrow{\bigoplus_{k,l}(-1)^{kl}\tau_{X^k,Y^l}} \bigoplus_{k,l\in\mathbb{Z}} \mathbf{t}^{k+l}\,\mathrm{HH}_\bullet(Y^l \otimes_R X^k) \cong \mathrm{HH}_\bullet(Y \otimes_R X)$$

that we denote by $\tau_{X,Y}$.

We now observe that $\tau_{X,Y}$ is natural. Let $f \in \mathrm{Hom}_{\mathcal{C}({}_R\mathcal{B}_S)}(X_1, X_2)$ and $g \in \mathrm{Hom}_{\mathcal{C}({}_S\mathcal{B}_R)}(Y_1, Y_2)$ be morphisms of complexes of bimodules. Let $z \in \mathrm{HH}_\bullet(X_1^k \otimes_S Y_1^l) \subset \mathrm{HH}_\bullet(X_1 \otimes_S Y_1)$, then the Koszul sign rule tells us that

$$\mathrm{HH}_\bullet(f \otimes g)(z) = (-1)^{|g|k}\,\mathrm{HH}_\bullet(f|_{X_1^k} \otimes g|_{Y_1^l})(z)\,.$$

After applying τ_{X_2,Y_2}, we obtain

$$(-1)^{(k+|f|)(l+|g|)+|g|k}\,\mathrm{HH}_\bullet(g|_{Y_1^l} \otimes f|_{X_1^k})(\tau_{X_1^k,Y_1^l}(z))$$

using naturality of the isomorphism from Proposition 5.2. The sign here is equal to $(-1)^{|f||g|+kl+|f|l}$. On the other hand, we have

$$\left(\mathrm{HH}_\bullet(g \otimes f) \circ \tau_{X_1,Y_1}\right)(z) = (-1)^{|f|l+kl}\,\mathrm{HH}_\bullet(g|_{Y_1^l} \otimes f|_{X_1^k})(\tau_{X_1^k,Y_1^l}(z))\,,$$

thus

$$\tau_{X_2,Y_2} \circ \mathrm{HH}_\bullet(f \otimes g) = (-1)^{|f||g|}\,\mathrm{HH}_\bullet(g \otimes f) \circ \tau_{X_1,Y_1}$$

as desired.

Indeed, this is the appropriate (signed) version of naturality in this context. For example, it guarantees that the isomorphism $\tau_{X,Y} : \mathrm{HH}_\bullet(X \otimes_S Y) \cong \mathrm{HH}_\bullet(Y \otimes_R X)$ intertwines the differentials $\mathrm{HH}_\bullet(\delta_X \otimes \mathrm{id}_Y + \mathrm{id}_X \otimes \delta_Y)$ and $\mathrm{HH}_\bullet(\delta_Y \otimes \mathrm{id}_X + \mathrm{id}_Y \otimes \delta_X)$. □

Remark 5.4. The hypotheses of Propositions 5.2 and 5.3 hold for (complexes of) singular Soergel bimodules. Indeed, any $X \in {}_{\boldsymbol{b}}\mathrm{SSBim}_{\mathbf{c}}$ is free as either a left $R^{\boldsymbol{b}}$-module or right $R^{\mathbf{c}}$-module.

From this point forward, we focus on the case when R is a polynomial ring, since this is the relevant setting for singular Soergel bimodules. In this case, the Koszul resolution provides a direct means for computing Hochschild (co)homology and gives an explicit relation between these two invariants. Let[10] $R = \mathbb{Q}[z_1, \ldots, z_N]$ with $\deg_{\mathbf{q}}(z_i) = d_i$ for $1 \leq i \leq N$. The Koszul resolution of R is the complex of graded (R, R)-bimodules given by

$$\mathbf{K} := \bigotimes_{i=1}^{N} \left(\mathbf{a}^{-1}\mathbf{q}^{d_i} R \otimes R \xrightarrow{z_i \otimes 1 - 1 \otimes z_i} R \otimes R \right) . \tag{72}$$

It follows from (69) that $\mathrm{HH}_\bullet(M)$ is the homology of $\mathbf{K} \otimes_{R \otimes R} M$, while $\mathrm{HH}^\bullet(M)$ is the homology of $\mathrm{Hom}_{R \otimes R}(\mathbf{K}, M)$. This implies that $\mathrm{HH}_i(M) \cong \mathbf{q}^{d_1 + \cdots + d_N} \, \mathrm{HH}^{N-i}(M)$, hence

$$\mathrm{HH}_\bullet(M) \cong \mathbf{a}^{-N}\mathbf{q}^{d_1 + \cdots + d_N} \, \mathrm{HH}^\bullet(M) \tag{73}$$

by (70). Moreover,

$$\mathrm{HH}_\bullet(R) \cong \wedge[\eta_1, \ldots, \eta_N] \otimes R \, , \quad \mathrm{wt}(\eta_i) = \mathbf{a}^{-1}\mathbf{q}^{d_i}$$

and

$$\mathrm{HH}^\bullet(R) \cong \wedge[\eta_1^*, \ldots, \eta_N^*] \otimes R \, , \quad \mathrm{wt}(\eta_i^*) = \mathbf{a}\mathbf{q}^{-d_i} \, .$$

The latter acts on the former by identifying η_i^* with the derivation sending $\eta_i \mapsto 1$ and $\eta_j \mapsto 0$ for $j \neq i$.

5.2. **Partial traces.** We now recall the partial Hochschild (co)homology functors from [RT21], which generalize the (uncolored) partial trace functors first introduced in [Hog18]. These functors refine the Hochschild (co)homology functors from §5.1, and allow them to be applied to the complex $C(\beta_{\boldsymbol{b}})$ "one strand at a time." As such, they are useful in proving the invariance of colored, triply-graded link homology (and its deformation defined in §5.4 below) under the second Markov move. We will refer to these functors collectively as the *colored partial trace* functors.

Recall that SSBim is a full 2-subcategory of the 2-category Bim from §3.4, and that in these 2-categories the Hom-categories are enriched in the symmetric monoidal category $\overline{\mathcal{K}}[\![\mathbf{q}^\pm]\!]$ of $\mathbb{Z}_{\mathbf{q}}$-graded $\mathbb{Q}$-vector spaces. We now consider the bounded derived category of Bim, denoted $\mathcal{D}(\mathrm{Bim})$, which is enriched in the category $\overline{\mathcal{K}}[\![\mathbf{a}^\pm, \mathbf{q}^\pm]\!]$ of $\mathbb{Z}_{\mathbf{a}} \times \mathbb{Z}_{\mathbf{q}}$-graded $\mathbb{Q}$-vector spaces. This 2-category $\mathcal{D}(\mathrm{Bim})$ is the natural setting for Hochschild (co)homology of singular Soergel bimodules, via the inclusion

$$\mathrm{SSBim} \hookrightarrow \mathrm{Bim} \hookrightarrow \mathcal{D}(\mathrm{Bim}) \, . \tag{74}$$

We emphasize to the reader that the cohomological grading in $\mathcal{D}(\mathrm{Bim})$ is the $\mathbf{a}$-degree, which is independent from the cohomological grading used in the dg category of (curved) complexes $\mathcal{C}(\mathrm{SSBim})$, which is the $\mathbf{t}$-degree.

Definition 5.5. Let $\mathcal{D}(\mathrm{Bim})$ be the monoidal 2-category wherein:

- objects of $\mathcal{D}(\mathrm{Bim})$ are the same as in Bim.
- the Hom-category ${}_{\boldsymbol{b}}\mathcal{D}(\mathrm{Bim})_{\mathbf{c}}$ from $\mathbf{c}$ to $\boldsymbol{b}$ is the bounded derived category $D^b({}_{\boldsymbol{b}}\mathrm{Bim}_{\mathbf{c}})$ of graded $(R^{\boldsymbol{b}}, R^{\mathbf{c}})$-bimodules (equivalently, this is the bounded derived category of graded $R^{\boldsymbol{b}} \otimes R^{\mathbf{c}}$-modules). Horizontal composition of 1-morphisms is given by derived tensor product over the intermediate rings $R^{\boldsymbol{b}}$.
- The external tensor product $\boxtimes$ is defined as in §3.4.

[10]Here, we call our variables $\{z_i\}$, since in general they will not be the variables $\mathbb{X} = \{x_i\}$, but rather symmetric functions in subalphabets of the alphabet $\mathbb{X}$.

Since singular Soergel bimodules in ${}_{\boldsymbol{b}}\mathrm{SSBim}_{\mathbf{c}}$ are free as either left $R^{\boldsymbol{b}}$-modules or right $R^{\mathbf{c}}$-modules, horizontal composition of such bimodules is exact. Thus, the inclusion in (74) is indeed a 2-functor (i.e. derived tensor product over the rings $R^{\boldsymbol{b}}$ equals the usual tensor product for such bimodules). Since $\boxtimes$ is given in both settings by tensor product over $\mathbb{Q}$, (74) is a monoidal 2-functor.

For $X, Y \in {}_{\boldsymbol{b}}\mathrm{SSBim}_{\mathbf{c}}$, we have

$$\mathrm{Hom}_{\mathcal{D}(\mathrm{Bim})}(X,Y) = \mathrm{Ext}_{R^{\boldsymbol{b}}\otimes R^{\mathbf{c}}}(X,Y) \supset \mathrm{Hom}_{\mathrm{SSBim}}(X,Y)\,.$$

In particular, for $X \in {}_{\boldsymbol{b}}\mathrm{SSBim}_{\boldsymbol{b}}$ this gives

$$\mathrm{Hom}_{\mathcal{D}(\mathrm{Bim})}(R^{\boldsymbol{b}}, X) = \mathrm{HH}^{\bullet}(X)\,.$$

Definition 5.6 (Partial trace)**.** Let $X \in {}_{\boldsymbol{b}}\mathcal{D}(\mathrm{Bim})_{\mathbf{c}}$ be a 1-morphism between objects of the form $\boldsymbol{b} = \boldsymbol{b}' \boxtimes c$ and $\mathbf{c} = \mathbf{c}' \boxtimes c$, then the *partial Hochschild homology* of X is:

$$\mathrm{Tr}_c(X) := \mathrm{tw}_\delta\left(X \otimes \wedge[\eta_1,\dots,\eta_c]\right), \quad \mathrm{wt}(\eta_i) = \mathbf{a}^{-1}\mathbf{q}^{2i}$$

with twist $\delta = \sum_{1\le i\le c}(e_i(\mathbb{X}) - e_i(\mathbb{X}')) \otimes \eta_i^*$. Here, $\mathbb{X}$ and $\mathbb{X}'$ denote the alphabets corresponding to the last (c-labeled) boundary points. The complex $\mathrm{Tr}_c(X)$ is regarded as an object in the 1-morphism category ${}_{\boldsymbol{b}'}\mathcal{D}(\mathrm{Bim})_{\mathbf{c}'}$; in particular, its cohomological degree is the $\mathbf{a}$-degree. Paralleling the relation in (73), we also define the *partial Hochschild cohomology* for such X to be:

$$\mathrm{Tr}^c(X) := \mathbf{a}^c\mathbf{q}^{-c(c+1)}\,\mathrm{Tr}_c(X)$$

Remark 5.7 (Relation to Hochschild (co)homology)**.** Given a 1-morphism $X \in {}_{\boldsymbol{b}}\mathcal{D}(\mathrm{Bim})_{\boldsymbol{b}}$ with $\boldsymbol{b} = (b_1,\dots,b_m)$, it is immediate from (72) and Definition 5.6 that:

$$(\mathrm{Tr}_{b_1}\circ\cdots\circ\mathrm{Tr}_{b_m})(X) \cong \mathrm{HH}_{\bullet}(X)\,, \quad (\mathrm{Tr}^{b_1}\circ\cdots\circ\mathrm{Tr}^{b_m})(X) \cong \mathrm{HH}^{\bullet}(X)\,.$$

We now recall various properties of the colored partial trace functors. Further details are provided in [RT21, Section 4.C] and [Hog18, Sections 3.2 and 3.3]; hence, our treatment is concise.

We first record an adjunction that will be used to compute various Hom-spaces.

Proposition 5.8. *Let $\boldsymbol{b} = \boldsymbol{b}' \boxtimes c$ and $\mathbf{c} = \mathbf{c}' \boxtimes c$ be objects in* Bim*, then partial Hochschild cohomology gives a functor*

$$\mathrm{Tr}^c\colon {}_{\boldsymbol{b}}\mathcal{D}(\mathrm{Bim})_{\mathbf{c}} \to {}_{\boldsymbol{b}'}\mathcal{D}(\mathrm{Bim})_{\mathbf{c}'}$$

that is right adjoint to the functor ${}_{\boldsymbol{b}'}\mathcal{D}(\mathrm{Bim})_{\mathbf{c}'} \to {}_{\boldsymbol{b}}\mathcal{D}(\mathrm{Bim})_{\mathbf{c}}$ sending $Y \mapsto Y \boxtimes \mathbf{1}_c$. □

Using the Koszul resolution of $\mathrm{Sym}(\mathbb{X})$ as a bimodule over itself, it is easy to see the following.

Proposition 5.9. *Let $X \in \mathcal{D}(\mathrm{Bim})$ and $c \ge 1$, then*

$$\mathrm{Tr}_c(X \boxtimes \mathbf{1}_c) \cong X \otimes \mathrm{HH}_{\bullet}(\mathrm{Sym}(\mathbb{X})) \cong X \otimes \mathbb{Q}[e_1(\mathbb{X}),\dots,e_c(\mathbb{X})] \otimes \wedge[\eta_1,\dots,\eta_c]$$

where $|\mathbb{X}| = c$, $\mathrm{wt}(e_i(\mathbb{X})) = \mathbf{q}^{2i}$, and $\mathrm{wt}(\eta_i) = \mathbf{a}^{-1}\mathbf{q}^{2i}$. □

The next result establishes a certain "bilinearity" of the colored partial trace.

Proposition 5.10. *Let $\boldsymbol{b} = \boldsymbol{b}' \boxtimes c$ and $\mathbf{c} = \mathbf{c}' \boxtimes c$ be objects in* Bim*, and let $Y_1, Y_2 \in {}_{\boldsymbol{b}'}\mathcal{D}(\mathrm{Bim})_{\mathbf{c}'}$ and $X \in {}_{\boldsymbol{b}}\mathcal{D}(\mathrm{Bim})_{\mathbf{c}}$ be 1-morphisms. There is an isomorphism*

$$\mathrm{Tr}^c\left((Y_1 \boxtimes \mathbf{1}_c) \star X \star (Y_2 \boxtimes \mathbf{1}_c)\right) \cong Y_1 \star \mathrm{Tr}^c(X) \star Y_2 \tag{75}$$

in ${}_{\boldsymbol{b}'}\mathcal{D}(\mathrm{Bim})_{\mathbf{c}'}$ that is natural in X, Y_1, and Y_2. In particular, if $Z_1 \boxtimes Z_2 \in {}_{\boldsymbol{b}}\mathcal{D}(\mathrm{Bim})_{\mathbf{c}}$, then

$$\mathrm{Tr}^c(Z_1 \boxtimes Z_2) \cong Z_1 \boxtimes \mathrm{Tr}^c(Z_2)$$

naturally in Z_1 and Z_2. □

Our next result describes the behavior of Rickard complexes under the second Markov move. In order to state it, we first need the following.

Definition 5.11. Let $\mathcal{C}(\mathcal{D}(\mathrm{Bim}))$ denote the monoidal 2-category with the same objects as $\mathcal{D}(\mathrm{Bim})$ (and Bim), and 1-morphism categories

$$_{\boldsymbol{b}}\mathcal{C}(\mathcal{D}(\mathrm{Bim}))_{\mathbf{c}} := \mathcal{C}({}_{\boldsymbol{b}}\mathcal{D}(\mathrm{Bim})_{\mathbf{c}})\,.$$

Note that $\mathcal{C}(\mathcal{D}(\mathrm{Bim}))$ is enriched in $\overline{\mathcal{K}}[\![\mathbf{a}^{\pm}, \mathbf{q}^{\pm}, \mathbf{t}^{\pm}]\!]_{\mathrm{dg}}$, and that the functors Tr_c and Tr^c induce functors $_{\boldsymbol{b}}\mathcal{C}(\mathcal{D}(\mathrm{Bim}))_{\mathbf{c}} \to {}_{\boldsymbol{b}'}\mathcal{C}(\mathcal{D}(\mathrm{Bim}))_{\mathbf{c}'}$ by applying Tr_c and Tr^c to complexes term-wise as in (71). Further, the inclusion (74) induces an inclusion $\mathcal{C}(\mathrm{SSBim}) \hookrightarrow \mathcal{C}(\mathcal{D}(\mathrm{Bim}))$.

Lemma 5.12. *Let $b \geq 1$, then we have homotopy equivalences*

$$\mathrm{Tr}_b(\mathbf{1_c} \boxtimes C_{b,b}) \simeq \mathbf{a}^{-b}\mathbf{q}^{b^2}\mathbf{t}^b \mathbf{1_c} \boxtimes \mathbf{1}_b \quad \textit{and} \quad \mathrm{Tr}_b(\mathbf{1_c} \boxtimes C^{\vee}_{b,b}) \simeq \mathbf{q}^{-b^2}\mathbf{1_c} \boxtimes \mathbf{1}_b$$

inside $_{\mathbf{c}\boxtimes b}\mathcal{C}(\mathcal{D}(\mathrm{Bim}))_{\mathbf{c}\boxtimes b}$.

Proof. This appears in [RT21, Lemma 4.12] in terms of partial Hochschild cohomology, namely:

$$\mathrm{Tr}^b(\mathbf{1_c} \boxtimes C_{b,b}) \simeq \mathbf{q}^{-b}\mathbf{t}^b \mathbf{1_c} \boxtimes \mathbf{1}_b \quad \text{and} \quad \mathrm{Tr}^b(\mathbf{1_c} \boxtimes C^{\vee}_{b,b}) \simeq \mathbf{a}^b\mathbf{q}^{-2b^2-b}\mathbf{1_c} \boxtimes \mathbf{1}_k. \tag{76}$$

After shifting the grading by $\mathbf{a}^b\mathbf{q}^{-b(b+1)}$ according to Remark 5.7, we recover the shown expressions. ☐

Proposition 5.13. *For any 1-morphism X in $_{\mathbf{c}\boxtimes b}\mathcal{C}(\mathrm{SSBim})_{\mathbf{c}\boxtimes b}$, we have equivalences*

$$\mathrm{HH}_\bullet\left((X \boxtimes \mathbf{1}_b) \star (\mathbf{1_c} \boxtimes C_{b,b})\right) \simeq \mathbf{a}^{-b}\mathbf{q}^{b^2}\mathbf{t}^b\,\mathrm{HH}_\bullet(X)\,, \quad \mathrm{HH}_\bullet\left((X \boxtimes \mathbf{1}_b) \star (\mathbf{1_c} \boxtimes C^{\vee}_{b,b})\right) \simeq \mathbf{q}^{-b^2}\,\mathrm{HH}_\bullet(X)$$

that are natural in X.

Proof. This is an immediate consequence of Remark 5.7, Proposition 5.10, and Lemma 5.12. ☐

5.3. **Colored, triply-graded homology.** In [Kho07], Khovanov showed that triply-graded Khovanov–Rozansky homology [KR08b] can be reformulated using the Hochschild homology of Soergel bimodules. We now recall the extension of this result to colored, triply-graded link homology via singular Soergel bimodules, as described in [MSV11, Wed19, Cau17]. See also [WW17] for a geometric analogue of this construction. We say that a $\mathbb{Z}_{\geq 1}$-colored braid $_{\boldsymbol{b}}\beta_{\mathbf{c}}$ is *balanced* if it is an endomorphism in $\mathfrak{Br}(\mathbb{Z}_{\geq 1})$, i.e. if $\mathbf{c} = \boldsymbol{b}$. In this case, the (standard) braid closure $\widehat{\beta_{\boldsymbol{b}}}$ represents a colored link. The classical Alexander theorem implies that every colored link in S^3 arises in this way, and Markov's theorem classifies the redundancy in presenting links by braid closures.

Following the references above, we will define the colored, triply-graded link homology as the homology of a properly normalized version of the Hochschild homology of Rickard complexes. The result will be an invariant of colored, framed, oriented links. First, we need some setup.

Definition 5.14. If $\omega \in \mathfrak{S}_m$ is a permutation then we put an equivalence relation on $\{1, \ldots, m\}$ by declaring $i \sim j$ if $i = \omega^k(j)$ for some $k > 0$. The set of equivalence classes $\Omega(\omega) := \{1, \ldots, m\}/\sim$ is called the set of *cycles* of ω. We denote by $[i]$ the cycle containing $i \in \{1, \ldots, m\}$. If $\boldsymbol{b} = (b_1, \ldots, b_m)$ satisfies $\omega(\boldsymbol{b}) = \boldsymbol{b}$, then b_i depends only on the cycle containing i, so we may write $b_{[i]} := b_i$.

Note that if ω is the permutation represented by a braid β, then $\Omega(\omega)$ is in canonical bijection with the set of link components of $\widehat{\beta}$. We now introduce some numerical invariants of our colored braid $\beta_{\boldsymbol{b}}$.

Definition 5.15. Suppose β is an m-strand braid and ω is the permutation represented by β, and let $\boldsymbol{b} = (b_1, \ldots, b_m)$ be a set of colors. The *colored writhe* $\varepsilon(\beta_{\boldsymbol{b}}) \in \mathbb{Z}$ of a colored braid $\beta_{\boldsymbol{b}} \in \mathfrak{Br}(\mathbb{Z}_{\geq 1})$ is defined by

$$\varepsilon(\beta_{\boldsymbol{b}}) := \sum_{\mathsf{x}} s(\mathsf{x})a(\mathsf{x})b(\mathsf{x})$$

where x ranges over all crossings in the braid β, $\{a(\mathsf{x}), b(\mathsf{x})\}$ is the multiset of colors meeting x, and $s(\mathsf{x}) \in \{\pm 1\}$ is the sign of x. Furthermore, when $\beta_{\boldsymbol{b}}$ is balanced we set:

$$n(\beta_{\boldsymbol{b}}) := \sum_{[i]\in\Omega(\omega)} b_{[i]}\,, \quad N(\beta_{\boldsymbol{b}}) := \sum_{1\le i\le m} b_i\,, \quad Q(\beta_{\boldsymbol{b}}) := \sum_{1\le i\le m} b_i(b_i+1)\,.$$

These quantities give the sum of the colors of the components of the closure of $\beta_{\boldsymbol{b}}$, the colors on the strands of $\beta_{\boldsymbol{b}}$, and the sum of the $\mathbf{q}$-degrees of all the generators of the partially symmetric polynomial ring $R^{\boldsymbol{b}}$.

Definition 5.16. Let $\mathbf{L}$ be a colored link, presented as the closure of a balanced, colored braid $\beta_{\boldsymbol{b}}$. The *colored, triply-graded Khovanov–Rozansky complex* of $\beta_{\boldsymbol{b}}$ is

$$C_{\mathrm{KR}}(\beta_{\boldsymbol{b}}) := (\mathbf{a}\mathbf{t}^{-1})^{\frac{1}{2}(\varepsilon(\beta_{\boldsymbol{b}})+N(\beta_{\boldsymbol{b}})-n(\beta_{\boldsymbol{b}}))}\mathbf{q}^{-\varepsilon(\beta_{\boldsymbol{b}})}\,\mathrm{HH}_{\bullet}(C(\beta_{\boldsymbol{b}}))$$

This is an object[11] of $\overline{\mathcal{K}}[\![\mathbf{a}^{\pm},\mathbf{q}^{\pm},\mathbf{t}^{\pm}]\!]_{\mathrm{dg}}$ (see §3.1). The *colored, triply-graded Khovanov–Rozansky homology* of $\beta_{\boldsymbol{b}}$ is defined by $H_{\mathrm{KR}}(\beta_{\boldsymbol{b}}) := H(C_{\mathrm{KR}}(\beta_{\boldsymbol{b}}))$, which is an object of $\overline{\mathcal{K}}[\![\mathbf{a}^{\pm},\mathbf{q}^{\pm},\mathbf{t}^{\pm}]\!]$.

Remark 5.17. It can be seen that $\varepsilon(\beta_{\boldsymbol{b}})+N(\beta_{\boldsymbol{b}})-n(\beta_{\boldsymbol{b}})$ is even, so the shift $(\mathbf{a}\mathbf{t}^{-1})^{\frac{1}{2}(\varepsilon(\beta_{\boldsymbol{b}})+N(\beta_{\boldsymbol{b}})-n(\beta_{\boldsymbol{b}}))}$ is well-defined. We leave verification of this as an exercise.

Remark 5.18. One can also express $C_{\mathrm{KR}}(\beta_{\boldsymbol{b}})$ in terms of Hochschild cohomology:

$$C_{\mathrm{KR}}(\beta_{\boldsymbol{b}}) = \mathbf{a}^{\frac{1}{2}(\varepsilon(\beta_{\boldsymbol{b}})-N(\beta_{\boldsymbol{b}})-n(\beta_{\boldsymbol{b}}))}\mathbf{q}^{Q(\beta_{\boldsymbol{b}})-\varepsilon(\beta_{\boldsymbol{b}})}\mathbf{t}^{\frac{1}{2}(-\varepsilon(\beta_{\boldsymbol{b}})-N(\beta_{\boldsymbol{b}})+n(\beta_{\boldsymbol{b}}))}\,\mathrm{HH}^{\bullet}(C(\beta_{\boldsymbol{b}}))$$

via (73).

Theorem 5.19. *A change of braid representative $\beta_{\boldsymbol{b}}$ for a framed, oriented, colored link $\mathbf{L}$ induces a homotopy equivalence between the corresponding complexes $C_{\mathrm{KR}}(\beta_{\boldsymbol{b}})$. Consequently, $H_{\mathrm{KR}}(\mathbf{L}) := H_{\mathrm{KR}}(\beta_{\boldsymbol{b}}) \in \overline{\mathcal{K}}[\![\mathbf{a}^{\pm},\mathbf{q}^{\pm},\mathbf{t}^{\pm}]\!]$ is a well-defined invariant of the framed, oriented, colored link $\mathbf{L}$, up to isomorphism. Furthermore, both $C_{\mathrm{KR}}(\beta_{\boldsymbol{b}})$ and $H_{\mathrm{KR}}(\mathbf{L})$ are monoidal under split (disjoint) union $\sqcup$:*

$$C_{\mathrm{KR}}(\beta_{\boldsymbol{b}}\sqcup\beta'_{\boldsymbol{b}'}) \cong C_{\mathrm{KR}}(\beta_{\boldsymbol{b}})\otimes_{\mathbb{Q}} C_{\mathrm{KR}}(\beta'_{\boldsymbol{b}'})\,, \quad H_{\mathrm{KR}}(\mathbf{L}\sqcup\mathbf{L}') \cong H_{\mathrm{KR}}(\mathbf{L})\otimes_{\mathbb{Q}} H_{\mathrm{KR}}(\mathbf{L}')\,.$$

Changing framing by ± 1 on a b-labeled component shifts $H_{\mathrm{KR}}(\mathbf{L})$ by $(\mathbf{a}\mathbf{t}^{-1})^{\pm\frac{1}{2}b(b-1)}$.

A choice of a point p on a b-labeled component equips $C_{\mathrm{KR}}(\beta_{\boldsymbol{b}})$ and $H_{\mathrm{KR}}(\mathbf{L})$ with an action of the symmetric polynomial ring $\mathrm{Sym}(\mathbb{X}_{\mathsf{p}})$ with $|\mathbb{X}_{\mathsf{p}}| = b$. Different choices of points on a component give quasi-isomorphic $\mathrm{Sym}(\mathbb{X}_{\mathsf{p}})$-module structures on $C_{\mathrm{KR}}(\beta_{\boldsymbol{b}})$ and equal actions on $H_{\mathrm{KR}}(\mathbf{L})$.

Proof. Following the classical Markov theorem, the invariance statement is a consequence of Proposition 3.25, conjugacy invariance of Hochschild homology (Proposition 5.3), and the behavior of Hochschild homology of Rickard complexes under the second Markov move (Lemma 5.12). See also [WW17, Theorem 1.1] or [Cau17, Theorem 4.1]. The monoidality follows since split links can be represented by split braids, the fact that the Hochschild homology of Rickard complexes is monoidal under the external tensor product $\boxtimes$, and the observation that the numerical invariants $\varepsilon(-)$, $N(-)$, and $n(-)$ are additive under $\boxtimes$. The framing behavior follows from Lemma 5.12. The module structure on $C_{\mathrm{KR}}(\beta_{\boldsymbol{b}})$ is inherited from the module structure of singular Soergel bimodules. Homotopies relating the module structures specified by points on a single strand on two sides of a crossing were studied in §4.4, and the last statement therefore follows from Proposition 4.25. □

Example 5.20. Let $\mathbf{U}(b)$ be the 0-framed b-colored unknot, presented as the closure of a single b-labeled strand. In this case, no differentials or grading shifts enter into Definition 5.16. To compute the unknot invariant, let $\mathbb{X}$ be the size b alphabet associated to the b-labeled strand. Then, we have:

$$H_{\mathrm{KR}}(\mathbf{U}(b)) = C_{\mathrm{KR}}(\mathbf{U}(b)) = \mathrm{HH}_{\bullet}(\mathrm{Sym}(\mathbb{X})) \cong \mathbb{Q}[e_1(\mathbb{X}),\dots,e_b(\mathbb{X})]\otimes\textstyle\bigwedge[\eta_1,\dots,\eta_b]$$

[11] In fact, the homogeneous components of $C_{\mathrm{KR}}(\beta_{\boldsymbol{b}})$ are finite-dimensional, the $\mathbf{a}$- and $\mathbf{t}$-grading are bounded and the $\mathbf{q}$-grading is bounded from below. The same holds for $H_{\mathrm{KR}}(\beta_{\boldsymbol{b}})$.

where $\mathrm{wt}(e_i(\mathbb{X})) = \mathbf{q}^{2i}$ and $\mathrm{wt}(\eta_i) = \mathbf{a}^{-1}\mathbf{q}^{2i}$. The Poincaré series of this homology is:

$$\left(\frac{1+\mathbf{a}^{-1}\mathbf{q}^{2}}{1-\mathbf{q}^{2}}\right)\cdots\left(\frac{1+\mathbf{a}^{-1}\mathbf{q}^{2b}}{1-\mathbf{q}^{2b}}\right).$$

As is typical in link homology theory, what is usually called "the unknot invariant" actually serves two distinct roles: first literally as the invariant of the unknot, and second as the algebra that acts on the invariant of any link upon specifying a chosen point. The latter is the *derived sheet algebra* from Example 1.4, so-named because it computes endomorphisms of a single strand in $\mathcal{D}(\mathrm{Bim})$. In the colored, triply-graded Khovanov–Rozansky theory, the derived sheet algebra for a b-labeled strand is given by the dg algebra $\mathrm{HH}^\bullet(\mathrm{Sym}(\mathbb{X}))$, whose Poincaré series is:

$$\left(\frac{1+\mathbf{a}\mathbf{q}^{-2}}{1-\mathbf{q}^{2}}\right)\cdots\left(\frac{1+\mathbf{a}\mathbf{q}^{-2b}}{1-\mathbf{q}^{2b}}\right).$$

The action of the derived sheet algebra on the unknot invariant coincides with the classical action of Hochschild cohomology on Hochschild homology. By (73), the underlying triply-graded vector spaces of the unknot invariant and the derived sheet algebra only differ by a grading shift.

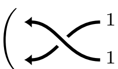
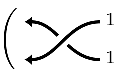
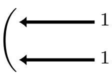
Remark 5.21. We expect that Theorem 5.19 can be strengthened as follows. If the labels on the components of $\mathbf{L}$ are $b_1, \ldots, b_r$, then we expect that $C_{\mathrm{KR}}(\mathbf{L})$ is well-defined up to quasi-isomorphism as a dg-module over the dg algebra given as the tensor product of the *derived* sheet algebras of all b_i. We will not need this stronger statement in the present paper.

Remark 5.22. It is sometimes desirable to use a renormalized version of Definition 5.16 that favors Hochschild cohomology over Hochschild homology, and in which the unknot invariant is identified with the derived sheet algebra. For this one sets:

$$\begin{aligned} C'_{\mathrm{KR}}(\mathbf{L}) &= \mathbf{a}^{\frac{1}{2}(\varepsilon(\beta_{\boldsymbol{b}})+N(\beta_{\boldsymbol{b}})+n(\beta_{\boldsymbol{b}}))}\mathbf{t}^{\frac{1}{2}(-\varepsilon(\beta_{\boldsymbol{b}})-N(\beta_{\boldsymbol{b}})+n(\beta_{\boldsymbol{b}}))}\mathbf{q}^{-\varepsilon(\beta_{\boldsymbol{b}})-n_2(\beta_{\boldsymbol{b}})}\,\mathrm{HH}_\bullet(C(\beta_{\boldsymbol{b}})) \\ &= \mathbf{a}^{\frac{1}{2}(\varepsilon(\beta_{\boldsymbol{b}})-N(\beta_{\boldsymbol{b}})+n(\beta_{\boldsymbol{b}}))}\mathbf{t}^{\frac{1}{2}(-\varepsilon(\beta_{\boldsymbol{b}})-N(\beta_{\boldsymbol{b}})+n(\beta_{\boldsymbol{b}}))}\mathbf{q}^{-\varepsilon(\beta_{\boldsymbol{b}})-n_2(\beta_{\boldsymbol{b}})+Q(\beta_{\boldsymbol{b}})}\,\mathrm{HH}^\bullet(C(\beta_{\boldsymbol{b}})), \end{aligned}$$

where $n_2(\beta_{\boldsymbol{b}}) := \sum_{[i]\in\Omega(\omega)} b_{[i]}(b_{[i]}+1)$.

Remark 5.23. The invariant $H_{\mathrm{KR}}(\mathbf{L})$ decategorifies to a version of the HOMFLYPT invariant $P(\mathbf{L})$ of links colored by one-column Young diagrams by specializing the three-variable Poincaré series at $\mathbf{t} = -1$. More specifically, the *uncolored* part of the invariant (i.e. where all colors are single box Young diagrams) is determined by the unknot invariant that we read off Example 5.20 and the following skein relation:

$\mathbf{q}P\Big($  $\Big) + \mathbf{a}\mathbf{q}^{-1}P\Big($  $\Big) = (-\mathbf{a})^r(\mathbf{q}-\mathbf{q}^{-1})P\Big($  $\Big)$

where $r = 0$ if all shown strands on the left-hand side belong to the same link component, and $r = 1$ otherwise.

5.4. **The deformed, colored, triply-graded homology.** We now define our deformed, colored, triply-graded homology. This proceeds in parallel to §5.3 by replacing the Rickard complex $C(\beta_{\boldsymbol{b}})$ with an appropriate curved analogue constructed from $\mathcal{Y}C(\beta_{\boldsymbol{b},\omega^{-1}})$ and the set $\Omega(\omega)$ of cycles of β. We will use the $\mathbb{V}$-variables description of the Hom-categories in $\mathcal{Y}(\mathrm{SSBim})$, as it is more convenient for our present considerations.

Suppose we are given a 1-morphism $\mathrm{tw}_\Delta(X) \in {}_{\boldsymbol{b},\mathrm{id}}\mathcal{Y}(\mathrm{SSBim})_{\boldsymbol{b},\omega^{-1}}$, where $\boldsymbol{b} = (b_1, \ldots, b_m)$ and $\omega \in \mathfrak{S}_m$. If $\omega = \mathrm{id}$, then the $h\Delta$-curvature on $\mathrm{tw}_\Delta(X)$ (given by (49)) takes the form $\sum_{i,k} h_k(\mathbb{X}_i - \mathbb{X}'_i)v_{i,k}$, which becomes zero after applying $\mathrm{HH}_\bullet$. Thus,

$$\mathrm{tw}_{\mathrm{HH}_\bullet(\Delta)}\Big(\mathrm{HH}_\bullet(X)\otimes\mathbb{Q}[\mathbb{V}]\Big) \tag{77}$$

is a well-defined complex (with zero curvature), and we obtain our sought after deformation of (5.16) by replacing $\mathrm{HH}_\bullet$ with (77). If $\omega \neq \mathrm{id}$, then we will need to "prepare" $\mathrm{tw}_\Delta(X)$ before taking Hochschild homology.

Hence, we introduce $\mathbb{X}$-alphabets parametrized by elements of $\Omega(\omega)$ (i.e. the cycles of ω) by setting

$$\mathbb{X}_{[i]} := \sum_{j\in[i]} \mathbb{X}_j \, , \quad \mathbb{X}'_{[i]} := \sum_{j\in[i]} \mathbb{X}'_j,$$

and we introduce deformation parameters indexed by $\Omega(\omega)$, denoted

$$\mathbb{V}_{[i]} := \{v_{[i],1},\dots,v_{[i],b_i}\}\, , \quad \mathbb{V}^{\Omega(\omega)} := \bigcup_{[i]\in\Omega(\omega)} \mathbb{V}_{[i]}. \tag{78}$$

We now change variables from $\mathbb{V}$ to $\mathbb{V}^{\Omega(\omega)}$ so that the resulting complex has curvature

$$\sum_{[i]\in\Omega(\omega)} \sum_{k=1}^{b_i} h_k(\mathbb{X}_{[i]} - \mathbb{X}'_{[i]}) v_{[i],k}. \tag{79}$$

Note that if $\mathrm{tw}_\Delta(X) = \mathcal{Y}C(\beta_{\boldsymbol{b}})$ is the curved complex associated to a braid, then (79) has one alphabet $\mathbb{V}_{[i]} = \{v_{[i],1},\dots,v_{[i],b_i}\}$ of deformation parameters for each link component in $\widehat{\beta_{\boldsymbol{b}}}$. We refer to curvature of the form (79) as *bundled curvature*, since the deformation parameters on various braid strands are bundled according to the corresponding link components.

The following gives a method for obtaining bundled curvature.

Definition 5.24. Define a surjective algebra map $\mathbb{Q}[\mathbb{X},\mathbb{X}',\mathbb{V}] \to \mathbb{Q}[\mathbb{X},\mathbb{X}',\mathbb{V}^{\Omega(\omega)}]$ by declaring

$$v_{i,k} \mapsto \sum_{k\leq l\leq b_i} h_{l-k}\left(\sum_{\substack{j<i \\ j\sim i}} (\mathbb{X}_j - \mathbb{X}'_{\omega^{-1}(j)}) \right) v_{[i],l}\, . \tag{80}$$

Lemma 5.25. *Suppose* $\mathrm{tw}_\Delta(X) \in {}_{\boldsymbol{b},1}\mathcal{Y}(\mathrm{SSBim})_{\boldsymbol{b},\omega^{-1}}$. *The substitution* (80) *yields a complex*

$$\mathrm{tw}_{\bar{\Delta}}\left(X \otimes_{\mathbb{Q}[\mathbb{X},\mathbb{X}',\mathbb{V}]} \mathbb{Q}[\mathbb{X},\mathbb{X}',\mathbb{V}^{\Omega(\omega)}]\right)$$

with curvature (79).

Proof. For simplicity, we consider the case when ω has only a single cycle $[1]=[2]=\dots=[m]$, so that $\mathbb{X}_{[1]} = \mathbb{X}_1 + \dots + \mathbb{X}_m$ and similarly for $\mathbb{X}'_{[1]}$. Observe that

$$\begin{aligned} h_l(\mathbb{X}_{[1]} - \mathbb{X}'_{[1]}) &= h_l\Big((\mathbb{X}_1 - \mathbb{X}'_{\omega^{-1}(1)}) + \dots + (\mathbb{X}_m - \mathbb{X}'_{\omega^{-1}(m)}) \Big) \\ &= \sum_{i=1}^{m}\sum_{k=1}^{l} h_k(\mathbb{X}_i - \mathbb{X}'_{\omega^{-1}(i)}) h_{l-k}(\mathbb{X}_1 + \dots + \mathbb{X}_{i-1} - \mathbb{X}'_{\omega^{-1}(1)} - \dots - \mathbb{X}'_{\omega^{-1}(i-1)})\, , \end{aligned}$$

where the second line is obtained by iterating identities of the form

$$h_l(\mathbb{Z}_1 + \mathbb{Z}_2) = h_l(\mathbb{Z}_1) + \sum_{k=1}^{l} h_{l-k}(\mathbb{Z}_1) h_k(\mathbb{Z}_2).$$

It thus follows that

$$\sum_{l=1}^{b} h_l(\mathbb{X}_1 + \dots + \mathbb{X}_m - \mathbb{X}'_1 - \dots - \mathbb{X}'_m) v_{[1],l} = \sum_{i=1}^{m}\sum_{k=1}^{l} h_k(\mathbb{X}_i - \mathbb{X}'_{\omega^{-1}(i)}) v_{i,k}$$

under the substitution $v_{i,k} = \sum_{l=k}^{b} h_{l-k}(\mathbb{X}_1 + \cdots + \mathbb{X}_{i-1} - \mathbb{X}'_{\omega^{-1}(1)} - \cdots - \mathbb{X}'_{\omega^{-1}(i-1)})v_{[1],l}$. This completes the proof when ω has one cycle. The proof for general ω is accomplished by applying the above computation to each cycle of ω. It differs only in more-tedious bookkeeping, so we omit the details. □

There may be other changes of variables that obtain the curvature (79) from the curvature $\sum_{i,k} h_k(\mathbb{X}_i - \mathbb{X}'_{\omega^{-1}(i)})v_{i,k}$. The following says that, for curved Rickard complexes $\mathcal{Y}C(\beta_{\boldsymbol{b}})$, any two choices are equivalent.

Lemma 5.26. *Let $\beta_{\boldsymbol{b}} \in \mathrm{Br}_m(\mathbb{Z}_{\geq 1})$ be balanced and let $\omega \in \mathfrak{S}_m$ be the permutation represented by β. If $\bar{\Delta}, \bar{\Delta}' \in \mathrm{End}_{\mathcal{C}(\mathrm{SSBim})}(C(\beta_{\boldsymbol{b}})) \otimes \mathbb{Q}[\mathbb{V}^{\Omega(\omega)}]$ are two Maurer–Cartan elements with the same curvature (79), then $\mathrm{tw}_{\bar{\Delta}}(C(\beta_{\boldsymbol{b}})) \simeq \mathrm{tw}_{\bar{\Delta}'}(C(\beta_{\boldsymbol{b}}))$.*

Proof. Similar to Lemma 4.15. □

Note that the curvature (79) vanishes upon identifying $\mathbb{X}_{[i]}$ and $\mathbb{X}'_{[i]}$. Since Hochschild homology factors through the quotient $\mathbb{X}_i = \mathbb{X}'_i$ (which implies $\mathbb{X}_{[i]} = \mathbb{X}'_{[i]}$), we now arrive at the definition of our link invariant.

Definition 5.27. Let $\mathbf{L}$ be a colored link which is presented as the closure of a balanced colored m-strand braid $\beta_{\boldsymbol{b}}$ and let $\omega \in \mathfrak{S}_m$ be the permutation represented by β. Let

$$\bar{\Delta} \in \mathrm{End}_{\mathcal{C}(\mathrm{SSBim})}(C(\beta_{\boldsymbol{b}})) \otimes \mathbb{Q}[\mathbb{V}^{\Omega(\omega)}]$$

be the curved Maurer–Cartan element with curvature (79) from Lemma 5.25. Let

$$\mathcal{Y}\mathrm{HH}_\bullet(\beta_{\boldsymbol{b}}) := \mathrm{tw}_{\mathrm{HH}_\bullet(\bar{\Delta})}\Big(\mathrm{HH}_\bullet(C(\beta)) \otimes \mathbb{Q}[\mathbb{V}^{\Omega(\omega)}]\Big) \tag{81}$$

then the *deformed, colored, triply-graded link homology* $\mathcal{Y}H_{\mathrm{KR}}(\mathbf{L})$ is the homology of the chain complex

$$\mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}}) := (\mathbf{a}\mathbf{t}^{-1})^{\frac{1}{2}(\varepsilon(\beta_{\boldsymbol{b}})+N(\beta_{\boldsymbol{b}})-n(\beta_{\boldsymbol{b}}))}\mathbf{q}^{-\varepsilon(\beta_{\boldsymbol{b}})}\mathcal{Y}\mathrm{HH}_\bullet(\beta_{\boldsymbol{b}})\,. \tag{82}$$

In other words,

$$\mathcal{Y}H_{\mathrm{KR}}(\mathbf{L}) := H(\mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}}))\,.$$

Remark 5.28. Although we have used the specific Maurer–Cartan element from Lemma 5.25 to define $\mathcal{Y}H_{\mathrm{KR}}(\mathbf{L})$, Lemma 5.26 shows that we could have used any Maurer–Cartan element with bundled curvature (79).

Remark 5.29. We will sometimes abuse notation by writing $\mathcal{Y}C_{\mathrm{KR}}(\mathbf{L})$ instead of $\mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}})$. This is justified by Theorem 5.30 below. As noted above, the set of cycles $\Omega(\omega)$ of the permutation ω determined by β can be identified with the set $\pi_0(\mathbf{L})$ of components of the link $\mathbf{L}$. We will thus also denote $\mathbb{V}^{\pi_0(\mathbf{L})} := \mathbb{V}^{\Omega(\omega)}$ in this context, and further write $\mathbb{V}^{\mathsf{c}} := \mathbb{V}^{[i]}$ and $v_{\mathsf{c},r} := v_{[i],r}$ when $\mathsf{c} \in \pi_0(\mathbf{L})$ corresponds to the cycle $[i] \in \Omega(\omega)$.

As defined, $\mathcal{Y}H_{\mathrm{KR}}(\mathbf{L})$ is an object of $\overline{\mathcal{K}}[\![\mathbf{a}^{\pm}, \mathbf{q}^{\pm}, \mathbf{t}^{\pm}]\!]$, and (as with $H_{\mathrm{KR}}(\mathbf{L})$) the homogeneous components are finite-dimensional, the $\mathbf{a}$- and $\mathbf{t}$-grading are bounded, and the $\mathbf{q}$-grading is bounded from below. In fact, the module structure on $\mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}})$ allows us to endow $\mathcal{Y}H_{\mathrm{KR}}(\mathbf{L})$ with additional structure, that we now describe.

Let $\mathbf{L}$ be the closure of a colored braid $\beta_{\boldsymbol{b}}$. For each component $\mathsf{c} \in \pi_0(\mathbf{L})$ of color $b(\mathsf{c})$, introduce an alphabet $\mathbb{X}_{\mathsf{c}}$ of cardinality $b(\mathsf{c})$ and set

$$A_{\mathbf{L}} := \bigotimes_{\mathsf{c} \in \pi_0(\mathbf{L})} \mathrm{Sym}(\mathbb{X}_{\mathsf{c}}) \otimes \mathbb{Q}[\mathbb{V}^{\mathsf{c}}]\,.$$

Given a point $\mathsf{p} \in \beta_{\boldsymbol{b}}$ (away from a crossing), the 2-categorical structure of $\mathcal{Y}(\mathrm{SSBim})$ endows $\mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}})$ with the structure of a dg module over $\mathrm{Sym}(\mathbb{X}_{\mathsf{c}})$, where c is the component of $\mathbf{L}$ containing p. If we choose one such point for each component of $\mathbf{L}$, this endows $\mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}})$ with a dg $A_{\mathbf{L}}$-module structure. We call such a choice of points a *pointing* of $\mathbf{L}$.

We can now state precisely in what sense $\mathcal{Y}H_{\mathrm{KR}}(\mathbf{L})$ is a colored link invariant.

Theorem 5.30. *Choose a pointing of $\widehat{\beta_{\boldsymbol{b}}}$, then the renormalized complex $\mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}})$ from (82) depends only on the framed, oriented, colored link $\mathbf{L} := \widehat{\beta_{\boldsymbol{b}}}$, up to quasi-isomorphism of $A_{\mathbf{L}}$-modules. Consequently, $\mathcal{Y}H_{\mathrm{KR}}(\mathbf{L})$ is a well-defined $A_{\mathbf{L}}$-module, up to isomorphism.*

The proof of Theorem 5.30 (i.e. the Markov invariance of $\mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}})$) is established in the following section. There, we show that $\mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}})$ can equivalently be described using curved Rickard complexes with strand-wise curvature $\sum_{k=1}^{a}\frac{1}{k}(p_k(\mathbb{X})-p_k(\mathbb{X}'))\dot{v}_k$; this curvature is the most-straightforwardly adapted to the Markov moves.

Before doing so, however, we first establish some easy consequences of Definition 5.27.

Example 5.31. Let $\mathbf{U}(b)$ be the 0-framed b-colored unknot. Since $\mathbf{U}(b)$ can be presented as the closure of a single b-labeled strand, no differentials and grading shifts enter into Definition 5.27. To compute the unknot invariant, let $\mathbb{X}$ and $\mathbb{V}$ be the size b alphabets associated to the b-labeled strand. Then, we have:

$$\mathcal{Y}H_{\mathrm{KR}}(\mathbf{U}(b)) = \mathcal{Y}C_{\mathrm{KR}}(\mathbf{U}(b)) = \mathrm{HH}_\bullet(\mathrm{Sym}(\mathbb{X}))\otimes\mathbb{Q}[\mathbb{V}] \cong \mathbb{Q}[e_1(\mathbb{X}),\dots,e_b(\mathbb{X})]\otimes\textstyle\bigwedge[\eta_1,\dots,\eta_b]\otimes\mathbb{Q}[v_1,\dots,v_b]$$

where $\mathrm{wt}(e_i(\mathbb{X}))=\mathbf{q}^{2i}$, $\mathrm{wt}(\eta_i)=\mathbf{a}^{-1}\mathbf{q}^{2i}$, $\mathrm{wt}(v_i)=\mathbf{q}^{-2i}\mathbf{t}^2$. The Poincaré series of the unknot homology is thus:

$$\frac{(1+\mathbf{a}^{-1}\mathbf{q}^2)}{(1-\mathbf{q}^2)(1-\mathbf{q}^{-2}\mathbf{t}^2)}\cdots\frac{(1+\mathbf{a}^{-1}\mathbf{q}^{2b})}{(1-\mathbf{q}^{2b})(1-\mathbf{q}^{-2b}\mathbf{t}^2)}$$

while the corresponding derived sheet algebra is $\mathrm{HH}^\bullet(\mathrm{Sym}(\mathbb{X}))\otimes\mathbb{Q}[\mathbb{V}]$ with Poincaré series:

$$\frac{(1+\mathbf{a}\mathbf{q}^{-2})}{(1-\mathbf{q}^2)(1-\mathbf{q}^{-2}\mathbf{t}^2)}\cdots\frac{(1+\mathbf{a}\mathbf{q}^{-2b})}{(1-\mathbf{q}^{2b})(1-\mathbf{q}^{-2b}\mathbf{t}^2)}\,.$$

Remark 5.32. As for the undeformed invariant, we expect that Theorem 5.30 can be strengthened to exhibit $\mathcal{Y}C_{\mathrm{KR}}(\mathbf{L})$ as a dg-module over the tensor product of derived sheet algebras, well defined up to quasi-isomorphism; see Remark 5.21.

Note that $\mathcal{Y}H_{\mathrm{KR}}(\mathbf{U}(b))$ is a free module over $\mathbb{Q}[v_1,\dots,v_b]$. In fact, this behavior persists for all b-colored *knots*.

Proposition 5.33. *Consider a framed oriented b-colored knot $\mathbf{K}$. Then we have a $\mathrm{Sym}(\mathbb{X}^b)\otimes\mathbb{Q}[v_1,\dots,v_b]$-linear homotopy equivalence*

$$\mathcal{Y}C_{\mathrm{KR}}(\mathbf{K}) \simeq C_{\mathrm{KR}}(\mathbf{K})\otimes\mathbb{Q}[v_1,\dots,v_b]$$

and therefore an isomorphism

$$\mathcal{Y}H_{\mathrm{KR}}(\mathbf{K}) \cong H_{\mathrm{KR}}(\mathbf{K})\otimes\mathbb{Q}[v_1,\dots,v_b]$$

of triply-graded $\mathrm{Sym}(\mathbb{X}^b)\otimes\mathbb{Q}[v_1,\dots,v_b]$-modules.

Proof. Let β_{b^m} be a braid representative of $\mathbf{K}$. Since $\mathbf{K}$ is a knot, the associated permutation $\omega\in\mathfrak{S}_m$ is an m-cycle, so $v_{[1],k}=v_{[2],k}=\cdots=v_{[m],k}$ for all $1\le k\le b$ and the bundled curvature element (79) equals

$$\sum_{k=1}^{m} h_k(\mathbb{X}_1+\cdots+\mathbb{X}_m-(\mathbb{X}'_1+\cdots+\mathbb{X}'_m))v_{[1],k}=0\,.$$

The uniqueness of curved lifts with bundled curvature (i.e. Lemma 5.26) implies that we may take $\bar{\Delta}=0$ when forming $\mathcal{Y}C_{\mathrm{KR}}(\mathbf{K})$. This implies the first statement (after identifying $v_k=v_{[1],k}$), and taking homology gives the second. □

5.5. **Alphabet soup V: power sums.** In our considerations thus far, we have worked with strand-wise curvature modeled on $h\Delta$-curvature (or, equivalently, Δe-curvature). In order to most easily establish invariance of the complex $\mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}})$ under the Markov moves, we find it beneficial to also consider strand-wise curvature modeled on Δp-curvature:

$$\sum_{k=1}^{a} \frac{1}{k}(p_k(\mathbb{X}) - p_k(\mathbb{X}'))\dot{v}_k \, .$$

The following lemma will allow us to translate between such curvature and those previously considered.

Lemma 5.34. *We have* $\sum_{k=1}^{a} \frac{1}{k}(p_k(\mathbb{X})-p_k(\mathbb{X}'))\dot{v}_k = \sum_{k=1}^{a} h_k(\mathbb{X}-\mathbb{X}')v_k$ *under the following mutually inverse substitutions*

$$\dot{v}_k = \sum_{l=k}^{a} \frac{k}{l} h_{l-k}(\mathbb{X}-\mathbb{X}')v_l \, , \quad v_k = \sum_{l=k}^{a} \frac{k}{l}(-1)^{l-k} e_{l-k}(\mathbb{X}-\mathbb{X}')\dot{v}_l \, . \tag{83}$$

Proof. This is a straightforward application of Newton's identity relating the power sum and complete symmetric functions, as manifest in (16) and (19). For example, after the stated change of variables, we have

$$\sum_{1\le k\le a} \frac{1}{k} p_k(\mathbb{X}-\mathbb{X}')\dot{v}_k = \sum_{1\le k\le l\le a} \frac{1}{k} p_k(\mathbb{X}-\mathbb{X}')\frac{k}{l} h_{l-k}(\mathbb{X}-\mathbb{X}')v_l \overset{(19)}{=} \sum_{1\le l\le a} h_l(\mathbb{X}-\mathbb{X}')v_l \, .$$

Since $p_k(\mathbb{X}-\mathbb{X}') = p_k(\mathbb{X}) - p_k(\mathbb{X}')$ we obtain the identity in the statement. □

Remark 5.35. The above substitutions send $\dot{v}_k \leftrightarrow v_k$ modulo $N(\mathbb{X},\mathbb{X}')$. In practice, the alphabets $\mathbb{X}$ and $\mathbb{X}'$ will be associated to the left and right endpoints of a strand of a braid. The above substitution is not compatible with horizontal composition of braids and will only be applied immediately before closing the braid, during which we identify $\mathbb{X}$ and $\mathbb{X}'$.

For each $\boldsymbol{b} = (b_1, \ldots, b_m)$ of SSBim, introduce alphabets of deformation parameters $\dot{\mathbb{V}}_1, \ldots, \dot{\mathbb{V}}_m$ where $\dot{\mathbb{V}}_i = \{\dot{v}_{i,1}, \ldots, \dot{v}_{i,b_i}\}$ and $\mathrm{wt}(\dot{v}_{i,r}) = \mathbf{q}^{-2r}\mathbf{t}^2$. Set $\dot{\mathbb{V}}_{\boldsymbol{b}} := \dot{\mathbb{V}}_1 \cup \cdots \cup \dot{\mathbb{V}}_m$. Given $\boldsymbol{b}$ and a permutation $\omega \in \mathfrak{S}_m$, it is convenient to consider two copies of the alphabets $\dot{\mathbb{V}}_{\boldsymbol{b}}$, which will act on two sides of a $\boldsymbol{b}$-colored braid with underlying permutation ω. As a bookkeeping tool, we introduce:

$$(S_\omega)_{\boldsymbol{b}} := \Big(\mathbb{Q}[\dot{\mathbb{V}}_{\omega(\boldsymbol{b})}] \otimes \mathbb{Q}[\dot{\mathbb{V}}_{\boldsymbol{b}}]\Big) \Big/ \langle 1 \otimes \dot{v}_{i,k} - \dot{v}_{\omega(i),k} \otimes 1 \mid 1 \le i \le m, 1 \le k \le b_i \rangle \, . \tag{84}$$

We regard $(S_\omega)_{\boldsymbol{b}}$ as both a $(\mathbb{Q}[\dot{\mathbb{V}}_{\omega(\boldsymbol{b})}], \mathbb{Q}[\dot{\mathbb{V}}_{\boldsymbol{b}}])$-bimodule and a commutative algebra with multiplication

$$(f_1 \otimes g_1) \cdot (f_2 \otimes g_2) = f_1 f_2 \otimes g_1 g_2 \, .$$

Paralleling our notation for the alphabets $\mathbb{X}_i$ and $\mathbb{X}'_i$, we will write $\dot{v}_{i,k} := \dot{v}_{i,k} \otimes 1$ and $\dot{v}'_{i,k} := 1 \otimes \dot{v}_{i,k}$. We will sometimes denote $(S_\omega)_{\boldsymbol{b}}$ by either ${}_{\omega(\boldsymbol{b})}(S_\omega)_{\boldsymbol{b}}$ or ${}_{\omega(\boldsymbol{b})}(S_\omega)$. If we let $\star = \otimes_{\mathbb{Q}[\dot{\mathbb{V}}]}$ (in this context), then

$$(S_{\omega_1})_{\omega_2(\boldsymbol{b})} \star (S_{\omega_2})_{\boldsymbol{b}} \cong (S_{\omega_1\omega_2})_{\boldsymbol{b}} \, .$$

Finally, if S is any $\mathbb{Q}$-algebra and B is an (S,S)-bimodule, then we will let $[B]$ denote the S-*coinvariants*, i.e.

$$[B] := B \big/ \big(\mathbb{Q} \cdot \{sb - bs \mid b \in B, s \in S\}\big)$$

Note that $[B] = \mathrm{HH}_0(B)$, but we wish to not confuse the reader with this occurrence of Hochschild homology and the functor $\mathrm{HH}_\bullet$, which (in this paper) we apply exclusively to singular Soergel bimodules. Observe that

$$[(S_\omega)_{\boldsymbol{b}}] \cong \mathbb{Q}[\dot{\mathbb{V}}_1, \ldots, \dot{\mathbb{V}}_m]/(\dot{v}_{i,k} \sim \dot{v}_{\omega(i),k}) \cong \mathbb{Q}[\dot{v}_{[i],k}]_{[i]\in\Omega(\omega), k\in\{1,\ldots,b_i\}} =: \mathbb{Q}[\dot{\mathbb{V}}^{\Omega(\omega)}] \, . \tag{85}$$

Lemma 5.36. *Let $\beta_{\boldsymbol{b}}$ be a balanced, colored braid, and let $\omega \in \mathfrak{S}_m$ be the permutation represented by β. There exists a Maurer–Cartan element*[12] $\Delta \in \mathrm{End}_{\mathcal{C}(\mathrm{SSBim})}\left(C(\beta_{\boldsymbol{b}})\right) \otimes \left({}_{\boldsymbol{b}}(S_\omega)_{\omega^{-1}(\boldsymbol{b})}\right)$ *with curvature*

$$\sum_{i=1}^{m}\sum_{k=1}^{b_i}\frac{1}{k}\Big((p_k(\mathbb{X}_i)\otimes \dot{v}_{i,k}) - (p_k(\mathbb{X}'_i)\otimes \dot{v}'_{i,k})\Big). \tag{86}$$

This twist is unique up to homotopy equivalence in the sense of Lemma 5.26.

Proof. By (84), the curvature element (86) can also be written as

$$\sum_{i=1}^{m}\sum_{k=1}^{b_i}\frac{1}{k}\Big(p_k(\mathbb{X}_i) - p_k(\mathbb{X}'_{\omega^{-1}(i)})\Big)\otimes \dot{v}_{i,k}.$$

Thus a Maurer–Cartan element with curvature (86) can be constructed from the Maurer–Cartan element on the curved complex $\mathcal{Y}C(\beta_{\boldsymbol{b},\omega^{-1}})$, which has curvature $\sum_{i=1}^{m}\sum_{k=1}^{b_i} h_k(\mathbb{X}_i - \mathbb{X}'_{\omega^{-1}(i)})v_{i,k}$, using Lemma 5.34. □

We next show how to obtain bundled Δp-curvature, establishing the analogue of Lemma 5.25 in this context. In the following two results, the Maurer–Cartan element Δ is understood to be the one constructed in the proof of Lemma 5.36.

Lemma 5.37. *Let $\beta_{\boldsymbol{b}}$ be a balanced, colored braid, and let $\omega \in \mathfrak{S}_m$ be the permutation represented by β. There exists a Maurer–Cartan element* $[\Delta] \in \mathrm{End}_{\mathcal{C}(\mathrm{SSBim})}\left(C(\beta_{\boldsymbol{b}})\right) \otimes [{}_{\boldsymbol{b}}(S_\omega)_{\omega^{-1}(\boldsymbol{b})}]$ *with curvature*

$$\sum_{[i]\in\Omega(\omega)}\sum_{k=1}^{b_i}\frac{1}{k}\Big(p_k(\mathbb{X}_{[i]}) - p_k(\mathbb{X}'_{[i]})\Big)\otimes \dot{v}_{[i],k}\,. \tag{87}$$

Proof. The quotient map ${}_{\boldsymbol{b}}(S_\omega)_{\omega^{-1}(\boldsymbol{b})} \twoheadrightarrow [{}_{\boldsymbol{b}}(S_\omega)_{\omega^{-1}(\boldsymbol{b})}]$ gives us an algebra map

$$\mathrm{End}_{\mathcal{C}(\mathrm{SSBim})}\left(C(\beta_{\boldsymbol{b}})\right) \otimes \left({}_{\boldsymbol{b}}(S_\omega)_{\omega^{-1}(\boldsymbol{b})}\right) \to \mathrm{End}_{\mathcal{C}(\mathrm{SSBim})}\left(C(\beta_{\boldsymbol{b}})\right) \otimes [{}_{\boldsymbol{b}}(S_\omega)_{\omega^{-1}(\boldsymbol{b})}]\,.$$

Taking $[\Delta]$ to be the image of Δ under this algebra map produces a Maurer–Cartan element with curvature (87). □

The twist in Lemma 5.37 is unique, in the sense of Lemma 5.26. Using this $[\Delta]$, we can recover the complex $\mathcal{Y}\mathrm{HH}_\bullet(\beta_{\boldsymbol{b}})$ from (81), up to homotopy equivalence.

Lemma 5.38. *Let $\beta_{\boldsymbol{b}}$ be a balanced, colored braid, and let $\omega \in \mathfrak{S}_m$ be the permutation represented by β. After identifying $\dot{v}_{[i],k} = v_{[i],k}$, we have*

$$\mathrm{tw}_{\mathrm{HH}_\bullet([\Delta])}\Big(\mathrm{HH}_\bullet\left(C(\beta_{\boldsymbol{b}})\right)\otimes[{}_{\boldsymbol{b}}(S_\omega)_{\omega^{-1}(\boldsymbol{b})}]\Big) \simeq \mathcal{Y}\mathrm{HH}_\bullet(\beta_{\boldsymbol{b}}) \tag{88}$$

where $\mathrm{HH}_\bullet([\Delta])$ *denotes the image of* $[\Delta]$ *from Lemma 5.37 under the algebra map*

$$\mathrm{End}_{\mathcal{C}(\mathrm{SSBim})}\left(C(\beta_{\boldsymbol{b}})\right) \otimes [{}_{\boldsymbol{b}}(S_\omega)_{\omega^{-1}(\boldsymbol{b})}] \to \mathrm{End}_{\mathbb{Q}}\left(\mathrm{HH}_\bullet(C(\beta_{\boldsymbol{b}}))\right)\otimes[{}_{\boldsymbol{b}}(S_\omega)_{\omega^{-1}(\boldsymbol{b})}]\,.$$

Proof. Note that a substitution as in Lemma 5.34 will convert the curvature (87) into (79). Further, since we work with bundled curvature and Hochschild homology identifies the alphabets $\mathbb{X}_{[i]}$ and $\mathbb{X}'_{[i]}$, Remark 5.35 shows that the relevant substitution simply sets $\dot{v}_{[i],k} = v_{[i],k}$. Our uniqueness statement (Lemma 5.26) then establishes the lemma. □

[12] Since $\beta_{\boldsymbol{b}}$ is balanced, $\omega^{-1}(\boldsymbol{b}) = \boldsymbol{b}$, but we wish to emphasize the formal similarity to the complex $\mathcal{Y}C(\beta_{\boldsymbol{b},\omega^{-1}})$.

Remark 5.39. The Maurer–Cartan element Δ from Lemma 5.36 is strict, in the sense of Remark 4.11, thus its linear part determines null-homotopies $\widetilde{\Xi}_{[i],k}$ for the action of $\frac{1}{k}\Big(p_k(\mathbb{X}_{[i]}) - p_k(\mathbb{X}'_{[i]})\Big)$. Applying $\mathrm{HH}_\bullet$ then produces the monodromy maps $\Xi_{\mathbf{c},k}$ from Proposition 1.10. This pairs with the discussion preceding Theorem 5.30 to establish Proposition 1.10.

Using this "power sum model" for $\mathcal{Y}\mathrm{HH}_\bullet(\beta_{\boldsymbol{b}})$ established in Lemma 5.38, we now prove Markov invariance of $\mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}})$.

Proof of Theorem 5.30. It suffices to show that $\mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}})$ is invariant under the Markov moves, up to quasi-isomorphism of $A_{\mathbf{L}}$-modules. We will establish the Markov moves in turn, and then observe that all maps used are $A_{\mathbf{L}}$-module quasi-isomorphisms.

Markov I: Let $\beta_{\boldsymbol{b}} = \beta'_{\mathbf{c}}\beta''_{\boldsymbol{b}}$ be balanced, where β' and β'' are m-strand braids with corresponding permutations ω' and ω''. Hence, $\boldsymbol{b} = \omega'(\mathbf{c})$ and $\mathbf{c} = \omega''(\boldsymbol{b})$, so ${}_{\mathbf{c}}\beta''\beta'_{\mathbf{c}}$ is a balanced, colored braid as well. Let $\Delta' \in \mathrm{End}_{\mathcal{C}(\mathrm{SSBim})}\big(C({}_{\boldsymbol{b}}\beta'_{\mathbf{c}})\big) \otimes \big({}_{\boldsymbol{b}}(S_{\omega'})_{\mathbf{c}}\big)$ and $\Delta'' \in \mathrm{End}_{\mathcal{C}(\mathrm{SSBim})}\big(C({}_{\mathbf{c}}\beta''_{\boldsymbol{b}})\big) \otimes \big({}_{\mathbf{c}}(S_{\omega''})_{\boldsymbol{b}}\big)$ be Maurer–Cartan elements as in Lemma 5.36.

A straightforward computation shows that the Maurer–Cartan elements

$$\Delta' \star \mathrm{id} + \mathrm{id} \star \Delta'' \in \mathrm{End}_{\mathcal{C}(\mathrm{SSBim})}\big(C(\beta'_{\mathbf{c}}) \star C(\beta''_{\boldsymbol{b}})\big) \otimes \big({}_{\boldsymbol{b}}(S_{\omega'})_{\mathbf{c}} \star {}_{\mathbf{c}}(S_{\omega''})_{\boldsymbol{b}}\big) \tag{89}$$

and

$$\Delta'' \star \mathrm{id} + \mathrm{id} \star \Delta' \in \mathrm{End}_{\mathcal{C}(\mathrm{SSBim})}\big(C(\beta''_{\boldsymbol{b}}) \star C(\beta'_{\mathbf{c}})\big) \otimes \big({}_{\mathbf{c}}(S_{\omega''})_{\boldsymbol{b}} \star {}_{\boldsymbol{b}}(S_{\omega'})_{\mathbf{c}}\big) \tag{90}$$

have curvature as in Lemma 5.36. Now, using Proposition 5.3, we have an isomorphism of dg algebras

$$\mathrm{End}_{\mathbb{Q}}\Big(\mathrm{HH}_\bullet\big(C(\beta'_{\mathbf{c}})\star C(\beta''_{\boldsymbol{b}})\big)\Big)\otimes\big[\big({}_{\boldsymbol{b}}(S_{\omega'})\star(S_{\omega''})_{\boldsymbol{b}}\big)\big] \cong \mathrm{End}_{\mathbb{Q}}\Big(\mathrm{HH}_\bullet\big(C(\beta''_{\boldsymbol{b}})\star C(\beta'_{\mathbf{c}})\big)\Big)\otimes\big[\big({}_{\mathbf{c}}(S_{\omega''})\star(S_{\omega'})_{\mathbf{c}}\big)\big]$$

which exchanges the Maurer–Cartan elements induced by (89) and (90). Since $C(\beta'\beta''_{\boldsymbol{b}}) = C(\beta'_{\mathbf{c}})\star C(\beta''_{\boldsymbol{b}})$ and $C(\beta''\beta'_{\mathbf{c}}) = C(\beta''_{\boldsymbol{b}}) \star C(\beta'_{\mathbf{c}})$, this implies that

$$\begin{aligned} &\mathrm{tw}_{\mathrm{HH}_\bullet([\Delta'\star\mathrm{id}+\mathrm{id}\star\Delta''])}\Big(\mathrm{HH}_\bullet\big(C(\beta'\beta''_{\boldsymbol{b}})\big)\otimes[{}_{\boldsymbol{b}}(S_{\omega'\omega''})_{\boldsymbol{b}}]\Big) \\ &\qquad\cong \mathrm{tw}_{\mathrm{HH}_\bullet([\Delta''\star\mathrm{id}+\mathrm{id}\star\Delta'])}\Big(\mathrm{HH}_\bullet\big(C(\beta''\beta'_{\mathbf{c}})\big)\otimes[{}_{\mathbf{c}}(S_{\omega''\omega'})_{\mathbf{c}}]\Big). \end{aligned} \tag{91}$$

Thus, by Lemma 5.38, $\mathcal{Y}\mathrm{HH}_\bullet(\beta'\beta''_{\boldsymbol{b}}) \simeq \mathcal{Y}\mathrm{HH}_\bullet(\beta''\beta'_{\mathbf{c}})$. Since the numerical invariants $\varepsilon(-)$, $n(-)$, and $N(-)$ from Definition 5.15 agree for agree for $\beta'\beta''_{\boldsymbol{b}}$ and $\beta''\beta'_{\mathbf{c}}$, this establishes invariance of $\mathcal{Y}C_{\mathrm{KR}}(-)$ under the first Markov move.

Markov II: Let $\boldsymbol{b} = \mathbf{c} \boxtimes b$ and suppose that $\beta_{\boldsymbol{b}}$ is a balanced, colored m-strand braid that can be written as ${}_{\boldsymbol{b}}\beta = {}_{\boldsymbol{b}}(\beta' \boxtimes \mathbf{1}_b)\beta_{m-1}$. (Recall that β_{m-1} denotes the $(m-1)^{st}$ Artin generator.) Let $\omega \in \mathfrak{S}_m$ be the permutation represented by β and let $\omega' \in \mathfrak{S}_{m-1}$ be the permutation represented by β'. Observe that $\beta'_{\mathbf{c}}$ is necessarily also balanced.

For $1 \le i \le m-1$, the cycles $[i]_{\omega'}$ and $[i]_\omega$ are related by

$$\begin{cases} [i]_{\omega'} = [i]_\omega \smallsetminus \{m\} & \text{if } [i]_\omega = [m]_\omega \\ [i]_{\omega'} = [i]_\omega & \text{otherwise} \end{cases}$$

Thus, there is a canonical bijection between the cycles of ω and ω' given by sending $[i]_{\omega'} \mapsto [i]_\omega$ for $1 \le i \le m-1$. (Note that $[m]_\omega = [m-1]_\omega$ by our hypotheses on the braid β.) Henceforth we will identify the algebras

$$[{}_{\boldsymbol{b}}(S_\omega)_{\boldsymbol{b}}] = [{}_{\mathbf{c}}(S_{\omega'})_{\mathbf{c}}]$$

and we will use the notation $\dot{v}_{[i],k}$ without specifying whether $[i]$ is regarded as a cycle of ω or ω'.

Introduce alphabets $\mathbb{X}_i, \mathbb{X}'_i, \mathbb{X}''_i$ which act by left-, middle-, and right-multiplication (respectively) on

$$C({}_{\boldsymbol{b}}\beta) = \mathbf{1}_{\boldsymbol{b}} \star C(\beta' \boxtimes \mathbf{1}_b) \star \mathbf{1}_{\boldsymbol{b}} \star C(\beta_{m-1}) \star \mathbf{1}_{\boldsymbol{b}}\,.$$

In particular we have $\mathbb{X}'_i = \mathbb{X}''_i$ for $i = 1, \dots, m-2$ and $\mathbb{X}_m = \mathbb{X}'_m$ when acting on $C({}_{\boldsymbol{b}}\beta)$. The bundled curvature (87) equals

$$\sum_{i=1}^{m}\sum_{k=1}^{b_i}\frac{1}{k}\Big(p_k(\mathbb{X}_i) - p_k(\mathbb{X}''_i)\Big)\dot{v}_{[i],k} = \sum_{i=1}^{m}\sum_{k=1}^{b_i}\frac{1}{k}\Big(p_k(\mathbb{X}_i) - p_k(\mathbb{X}'_i)\Big)\dot{v}_{[i],k} + \sum_{i=1}^{m}\sum_{k=1}^{b_i}\frac{1}{k}\Big(p_k(\mathbb{X}'_i) - p_k(\mathbb{X}''_i)\Big)\dot{v}_{[i],k}\,.$$

Since $\mathbb{X}_m = \mathbb{X}'_m$, the $i = m$ term of the first summation on the right is zero for all k. Furthermore, since $\mathbb{X}'_i = \mathbb{X}''_i$ for $i = 1, \dots, m-2$, we have

$$\sum_{i=1}^{m}\sum_{k=1}^{b_i}\frac{1}{k}\Big(p_k(\mathbb{X}'_i) - p_k(\mathbb{X}''_i)\Big)\dot{v}_{[i],k} = \sum_{k=1}^{b_i}\frac{1}{k}\Big(p_k(\mathbb{X}_{m-1}) + p_k(\mathbb{X}_m) - p_k(\mathbb{X}'_{m-1}) - p_k(\mathbb{X}'_m)\Big)\dot{v}_{[m]} = 0\,.$$

Here, we have also used the fact that $\mathbb{X}'_{m-1} + \mathbb{X}'_m = \mathbb{X}''_{m-1} + \mathbb{X}''_m$ when acting on $C({}_{\boldsymbol{b}}\beta_{m-1})$. Therefore, the curvature element (87) in the present setting reduces to

$$\sum_{i=1}^{m}\sum_{k=1}^{b_i}\frac{1}{k}\Big(p_k(\mathbb{X}_i) - p_k(\mathbb{X}''_i)\Big)\dot{v}_{[i],k} = \sum_{i=1}^{m-1}\sum_{k=1}^{b_i}\frac{1}{k}\Big(p_k(\mathbb{X}_i) - p_k(\mathbb{X}'_i)\Big)\dot{v}_{[i],k},$$

which coincides with the bundled curvature (87) for β'.

Thus, if $[\Delta'] \in \mathrm{End}_{\mathcal{C}(\mathrm{SSBim})}\big(C(\beta'_{\mathbf{c}})\big) \otimes [{}_{\mathbf{c}}(S_{\omega'})_{\mathbf{c}}]$ satisfies the conditions of Lemma 5.36 for $\beta'_{\mathbf{c}}$, then

$$([\Delta'] \boxtimes \mathrm{id}) \star \mathrm{id} \in \mathrm{End}_{\mathcal{C}(\mathrm{SSBim})}\big(C(\beta_{\boldsymbol{b}})\big) \otimes [{}_{\boldsymbol{b}}(S_\omega)_{\boldsymbol{b}}]$$

satisfies the conditions of Lemma 5.36 for $\beta_{\boldsymbol{b}}$. Lemma 5.38 now gives that

$$\mathrm{tw}_{\mathrm{HH}_\bullet(([\Delta']\boxtimes\mathrm{id})\star\mathrm{id})}\Big(\mathrm{HH}_\bullet\big(C({}_{\boldsymbol{b}}(\beta' \boxtimes \mathbf{1}_b)\beta_{m-1})\big)\Big) \simeq \mathcal{Y}\mathrm{HH}_\bullet(\beta_{\boldsymbol{b}})\,.$$

Recall that Proposition 5.13 gives a homotopy equivalence

$$\mathrm{HH}_\bullet\Big(C({}_{\boldsymbol{b}}(\beta' \boxtimes \mathbf{1}_b)\beta_{m-1})\Big) \simeq \mathbf{a}^{-b}\mathbf{q}^{b^2}\mathbf{t}^b\,\mathrm{HH}_\bullet(\beta'_{\mathbf{c}})$$

of undeformed Rickard complexes, which intertwines the actions of $\mathrm{End}_{\mathcal{C}(\mathrm{SSBim})}(C(\beta'_{\mathbf{c}}))$ on both sides. This implies that the induced map

$$\mathrm{HH}_\bullet\Big(C({}_{\boldsymbol{b}}(\beta' \boxtimes \mathbf{1}_b)\beta_{m-1})\Big) \otimes [{}_{\boldsymbol{b}}(S_\omega)_{\boldsymbol{b}}] \to \mathbf{a}^{-b}\mathbf{q}^{b^2}\mathbf{t}^b\,\mathrm{HH}_\bullet(\beta'_{\mathbf{c}}) \otimes [{}_{\mathbf{c}}(S_{\omega'})_{\mathbf{c}}]$$

intertwines the actions of Maurer–Cartan elements $\mathrm{HH}_\bullet(([\Delta'] \boxtimes \mathrm{id}) \star \mathrm{id})$ and $\mathrm{HH}_\bullet([\Delta'])$. Hence, Lemma 5.38 gives that

$$\mathcal{Y}\mathrm{HH}_\bullet({}_{\boldsymbol{b}}(\beta' \boxtimes \mathbf{1}_b)\beta_{m-1}) \simeq \mathbf{a}^{-b}\mathbf{q}^{b^2}\mathbf{t}^b\,\mathcal{Y}\mathrm{HH}_\bullet(\beta'_{\mathbf{c}})\,, \tag{92}$$

and a similar argument gives that

$$\mathcal{Y}\mathrm{HH}_\bullet({}_{\boldsymbol{b}}(\beta' \boxtimes \mathbf{1}_b)\beta_{m-1}^{-1}) \simeq \mathbf{q}^{-b^2}\mathcal{Y}\mathrm{HH}_\bullet(\beta'_{\mathbf{c}})\,.$$

Comparing the shifts in (82), this establishes the requisite behavior of $\mathcal{Y}C_{\mathrm{KR}}(-)$ under the second Markov move.

Module structure: First, note that all homotopy equivalences used above are $\mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}]$-linear, so it suffices to show that they are quasi-isomorphisms of $\bigotimes_{\mathsf{c}\in\pi_0(\mathbf{L})}\mathrm{Sym}(\mathbb{X}_{\mathsf{c}})$-modules. This follows from Proposition 4.25. Indeed, therein it is shown that, up to quasi-isomorphism, we can assume that the $\mathrm{Sym}(\mathbb{X}_{\mathsf{c}})$ action is given at any point p on the corresponding component. In particular, all homotopy equivalences following from our uniqueness results are quasi-isomorphisms, since we can assume

$\mathrm{Sym}(\mathbb{X}_{\mathsf{c}})$ is acting via alphabets on the left, and these homotopy equivalences are equivalences in categories of curved complexes of bimodules. This similarly shows that the maps establishing Markov II invariant are quasi-isomorphisms: we can assume that the $\mathrm{Sym}(\mathbb{X}_{\mathsf{c}})$ action is given on the left, and does not act via $\mathbb{X}_m$. It remains to show that (91) is a quasi-isomorphism. For this, we can use Proposition 4.25 to assume that the $\mathrm{Sym}(\mathbb{X}_{\mathsf{c}})$ actions occur in the "middle" of the left-hand side (i.e. via the action of $\mathrm{End}_{\mathrm{SSBim}}(\mathbf{1}_{\mathsf{c}})$ on $C(\beta'_{\mathsf{c}}) \star \mathbf{1}_{\mathsf{c}} \star C(\beta''_{\boldsymbol{b}})$), and on the left on the right-hand side. This map is a $\mathrm{Sym}(\mathbb{X}_{\mathsf{c}})$-linear isomorphism for these actions, thus a quasi-isomorphism. □

Remark 5.40. Theorem 5.30 describes the link invariant $\mathcal{Y}H_{\mathrm{KR}}(\mathbf{L})$ as a module over the algebra $A_{\mathbf{L}} = \bigotimes_{\mathsf{c}\in\pi_0(\mathbf{L})} \mathrm{Sym}(\mathbb{X}_{\mathsf{c}}) \otimes \mathbb{Q}[\mathbb{V}^{\mathsf{c}}]$, i.e. the deformation parameters acting here are $\{\dot{v}_{[i],k}\}_{[i]\in\Omega(\omega),k\in\{1,\dots,b_i\}}$. However, it is straightforward to describe the action on $\mathcal{Y}H_{\mathrm{KR}}(\mathbf{L})$ of the various other deformation parameters. By definition (i.e. by (85)), the parameters $\dot{v}_{[i],k}$ act as the parameters $\dot{v}_{i,k}$. Further, it follows from (47) that elements of $N(\mathbb{X}_i, \mathbb{X}'_{\omega^{-1}(i)})$ acts null-homotopically on $\mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}})$, and thus by zero on $\mathcal{Y}H_{\mathrm{KR}}(\mathbf{L})$. Remark 5.35 then implies that the parameters $\dot{v}_{i,k}$ and $v_{i,k}$ act identically on $\mathcal{Y}H_{\mathrm{KR}}(\mathbf{L})$, and the assignment (80) implies that the parameters $v_{i,k}$ and $v_{[i],k}$ acts identically as well. Lastly, the assignment (50) implies that the parameters $u_{i,k}$ act on $\mathcal{Y}H_{\mathrm{KR}}(\mathbf{L})$ by $(-1)^{k-1}\sum_{l=k}^{b_i} h_{l-k}(\mathbb{X}_{[i]})\dot{v}_{[i],l}$.

5.6. Coefficients and spectral sequences.

We now collect some straightforward results on homology with coefficients and spectral sequences that we need for our link splitting results in §8 – 10 below. First, we will use the following common generalization of Definitions 5.27 and 5.16.

Definition 5.41. Let $\beta_{\boldsymbol{b}}$ be a balanced, colored braid. We consider two types of homology with coefficients. If M is a $\mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}]$-module, we define

$$\mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}}, M) := \mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}}) \otimes_{\mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}]} M\,. \tag{93}$$

If instead M' is an $A_{\mathbf{L}}$-module, we define

$$\mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}}, M') := \mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}}) \overset{L}{\otimes}_{A_{\mathbf{L}}} M'\,. \tag{94}$$

If either case, if $\mathbf{L}$ is the colored link obtained as the closure of $\beta_{\boldsymbol{b}}$, then the *deformed, colored, triply-graded Khovanov–Rozansky homology* of $\mathbf{L}$ *with coefficients* in M is defined by

$$\mathcal{Y}H_{\mathrm{KR}}(\mathbf{L}, M) := H(\mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}}, M))\,.$$

Remark 5.42. Since $\mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}})$ is free when regarded as a module over $\mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}]$, the tensor product in (93) coincides with the derived tensor product $\overset{L}{\otimes}_{\mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}]} M$. So, (93) can (and often will) be thought of as a special case of (94), with $M' = A_{\mathbf{L}} \otimes_{\mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}]} M$. Moreover, we expect that $\mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}})$ is free as an $A_{\mathbf{L}}$-module, so that the derived tensor product in (94) may be replaced with the ordinary tensor product. This amounts to showing that Hochschild homology of any singular Soergel bimodule $\mathbf{1}_{\boldsymbol{b}}B\mathbf{1}_{\boldsymbol{b}}$ is free as a module over the appropriate symmetric polynomial ring $\mathrm{End}_{\mathrm{SSBim}}(\mathbf{1}_{\boldsymbol{b}})$. (Note that this holds in the uncolored case.)

Remark 5.43. As in Remark 5.29, we will sometimes write $\mathcal{Y}C_{\mathrm{KR}}(\mathbf{L}, M)$ instead of $\mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}}, M)$. Strictly speaking, this complex depends on a choice of braid representative of $\mathbf{L}$, but the resulting complex depends only on the framed, oriented, colored link $\mathbf{L}$ up to quasi-isomorphism of $A_{\mathbf{L}}$-modules. Thus, the homology with coefficients $\mathcal{Y}H_{\mathrm{KR}}(\mathbf{L}, M)$ is a well-defined module over $A_{\mathbf{L}}/\mathrm{Ann}(M)$, up to isomorphism (here $\mathrm{Ann}(M) \subset A_{\mathbf{L}}$ denotes the annihilator of M).

Remark 5.44. We do not require that M be a doubly graded module over $A_{\mathbf{L}}$, so tensoring with M may involve a collapse of gradings.

Example 5.45. We consider the following examples:

(1) For $M = \mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}]$, we have $\mathcal{Y}C_{\mathrm{KR}}(\mathbf{L}, M) = \mathcal{Y}C_{\mathrm{KR}}(\mathbf{L})$.
(2) For the trivial module $M = \mathbb{Q}$ (on which all variables $v_{[i],k}$ act by zero), we have $\mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}}, M) = C_{\mathrm{KR}}(\beta_{\boldsymbol{b}})$.
(3) Fix a scalar $z_{\mathsf{c}} \in \mathbb{Q}$ for each link component $\mathsf{c} \in \pi_0(\mathbf{L})$ and let u be a formal variable of weight $\mathrm{wt}(u) = \mathbf{q}^{-2}\mathbf{t}^2$. Consider the $\mathbb{Z}_{\mathbf{q}} \times \mathbb{Z}_{\mathbf{t}}$-graded $\mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}]$-module $M_{\underline{z}} := \mathbb{Q}[u]$ on which $v_{\mathsf{c},1} \in \mathbb{V}^{\pi_0(\mathbf{L})}$ acts as multiplication by $z_{\mathsf{c}}u$, and $v_{\mathsf{c},r}$ acts by zero when $r > 1$. In this case, $\mathcal{Y}H_{\mathrm{KR}}(\mathbf{L}, M_{\underline{z}})$ recovers the deformed triply-graded homology of Cautis–Lauda–Sussan [CLS20, Theorem 6.3], which satisfies splitting properties between components labeled by distinct scalars z_{c} after inverting u.

Remark 5.46. The bigraded Khovanov–Rozansky $\mathfrak{gl}_N$ link homologies (as well as certain deformations thereof) can be computed from the Rickard complexes of colored braids by applying a functor that is trace-like up to homotopy, and which induces homotopy equivalences for the second Markov move similar as in Lemma 5.12; see [Wed19, Theorem 3.21] or [QR18, Section 6]. This implies the existence of link splitting deformations of bigraded colored Khovanov–Rozansky $\mathfrak{gl}_N$ link homologies. Specifically, for coefficients in $M_{\underline{z}}$ as in Example 5.45 (3), one obtains bigraded colored homologies that satisfy link splitting properties and agree with the invariants from [CLS20, Theorem 5.4] (modulo conventions).

Finally, we collect various spectral sequences associated with $\mathcal{Y}C_{\mathrm{KR}}(\mathbf{L})$. Let $\mathbf{L}$ be a colored link. For each $\mathsf{c} \in \pi_0(\mathbf{L})$, introduce an alphabet of (odd) variables $\Xi_{\mathsf{c}} = \{\xi_{\mathsf{c},r}\}_{r=1}^{b(\mathsf{c})}$ with $\mathrm{wt}(\xi_{\mathsf{c},r}) = \mathbf{q}^{2r}\mathbf{t}^{-1}$ and where $b(\mathsf{c})$ denotes the color of the component c. Set $\Xi^{\pi_0(\mathbf{L})} := \bigcup_{\mathsf{c}\in\pi_0(\mathbf{L})} \Xi_{\mathsf{c}}$.

Proposition 5.47. *We have*

$$\mathcal{Y}C_{\mathrm{KR}}(\mathbf{L}) \cong \mathrm{tw}_D(C_{\mathrm{KR}}(\mathbf{L}) \otimes \mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}])$$

where the twist D is $\mathbb{Q}[\Xi^{\pi_0(\mathbf{L})}]$-linear and $\mathbb{V}^{\pi_0(\mathbf{L})}$-irrelevant. We also have

$$C_{\mathrm{KR}}(\mathbf{L}) \simeq \mathrm{tw}_{D'}(\mathcal{Y}C_{\mathrm{KR}}(\mathbf{L}) \otimes \wedge[\Xi^{\pi_0(\mathbf{L})}])$$

where the twist D' equals $\sum_{\mathsf{c}\in\pi_0(\mathbf{L})} \sum_{r=1}^{b(\mathsf{c})} v_{\mathsf{c},r}\xi^*_{\mathsf{c},r}$.

Proof. The first statement is true by construction.The second statement follows from the first since the additional polynomial variables can be cancelled against extra exterior variables via the twist D', as in standard Koszul duality (relating complexes of modules over polynomial and exterior algebras). □

The twist D in Proposition 5.47 strictly increases $\mathbb{V}$-degree, and D' strictly decreases Ξ-degree. Taking the spectral sequence associated to these filtered complexes thus yields the following.

Corollary 5.48. *There are spectral sequences*

$$C_{\mathrm{KR}}(\mathbf{L}) \otimes \mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}] \implies \mathcal{Y}C_{\mathrm{KR}}(\mathbf{L}), \quad \mathcal{Y}C_{\mathrm{KR}}(\mathbf{L}) \otimes \wedge[\Xi^{\pi_0(\mathbf{L})}] \implies C_{\mathrm{KR}}(\mathbf{L}). \qquad \square$$

Remark 5.49. The filtration by Ξ-degree on the complex $\mathrm{tw}_{D'}(\mathcal{Y}C_{\mathrm{KR}}(\mathbf{L}) \otimes \wedge[\Xi^{\pi_0(\mathbf{L})}])$ has finitely many steps, so the associated spectral sequence converges after finitely many steps. On the other hand, the filtration by $\mathbb{V}$-degree on $\mathrm{tw}_\delta(C_{\mathrm{KR}}(\mathbf{L}) \otimes \mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}])$ is infinite. Nonetheless, the associated spectral sequence is a first quadrant spectral sequence (bounded below in both cohomological degree and $\mathbb{V}$-degree), hence is reasonably well-behaved.

The relation between $C_{\mathrm{KR}}(\mathbf{L})$ and $\mathcal{Y}C_{\mathrm{KR}}(\mathbf{L})$ can be reformulated in terms of homological perturbation theory as follows.

Lemma 5.50. *There is a homotopy equivalence*

$$\mathcal{Y}C_{\mathrm{KR}}(\mathbf{L}) \simeq \mathrm{tw}_{D''}(H_{\mathrm{KR}}(\mathbf{L}) \otimes \mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}])$$

of differential $\mathbb{Z}_{\mathbf{a}} \times \mathbb{Z}_{\mathbf{q}} \times \mathbb{Z}_{\mathbf{t}}$*-graded* $\mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}]$*-modules, for some* $\mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}]$*-linear twist* D''*. Here,* $H_{\mathrm{KR}}(\mathbf{L}) \otimes \mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}]$ *is viewed as a complex with zero differential.*

Proof. Since we are working with field coefficients, we can choose a homotopy equivalence in $\overline{\mathcal{K}}[\![\mathbf{a}^{\pm}, \mathbf{q}^{\pm}, \mathbf{t}^{\pm}]\!]_{\mathrm{dg}}$ relating $C_{\mathrm{KR}}(\mathbf{L})$ and its homology:

$$C_{\mathrm{KR}}(\mathbf{L}) \simeq H_{\mathrm{KR}}(\mathbf{L}).$$

Homological perturbation (Proposition 4.1) now gives us a homotopy equivalence

$$\mathcal{Y}C_{\mathrm{KR}}(\mathbf{L}) \cong \mathrm{tw}_D(C_{\mathrm{KR}}(\mathbf{L}) \otimes \mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}]) \simeq \mathrm{tw}_{D''}(H_{\mathrm{KR}}(\mathbf{L}) \otimes \mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}])$$

for some twist D''. Here, we use Remark 4.2 (and the analogue of (42) to see the necessary nilpotence). □

Lemma 5.50 is particularly useful in the following context.

Definition 5.51. A colored link $\mathbf{L}$ is *parity* if its (undeformed) triply-graded link homology $H_{\mathrm{KR}}(\mathbf{L})$ is supported in purely even (or purely odd) cohomological degrees.

Theorem 5.52. *Suppose that* $\mathbf{L}$ *is parity, then*

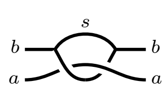
$$\mathcal{Y}C_{\mathrm{KR}}(\mathbf{L}) \simeq H_{\mathrm{KR}}(\mathbf{L}) \otimes \mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}]$$

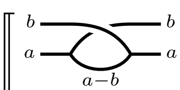
as differential $\mathbb{Z}_{\mathbf{a}} \times \mathbb{Z}_{\mathbf{q}} \times \mathbb{Z}_{\mathbf{t}}$*-graded* $\mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}]$*-modules. In particular, parity implies that* $\mathcal{Y}H_{\mathrm{KR}}(\mathbf{L})$ *is a free* $\mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}]$*-module.*

Proof. The twist D'' from Lemma 5.50 is necessarily zero for degree reasons: $\deg_{\mathbf{t}}(D'') = 1$ and $H_{\mathrm{KR}}(\mathbf{L}) \otimes \mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}]$ is supported in exclusively even (or exclusively odd) cohomological degrees. □

6. The curved colored skein relation

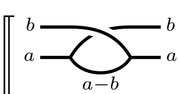
In [HRW21], we proved the *colored skein relation* for singular Soergel bimodules, which takes the form of a homotopy equivalence:

(95) $$\mathrm{tw}_{D_1}\left(\bigoplus_{s=0}^{b} \mathbf{q}^{s(b-1)}\mathbf{t}^s \left[\!\left[\right.\right.$$ 

$$\left.\left.\right]\!\right]\right) \simeq \mathbf{q}^{b(a-b-1)}\mathbf{t}^b K\left(\left[\!\left[\right.\right.$$ 

$$\left.\left.\right]\!\right]\right),$$

where $K(-)$ denotes the Koszul complex associated to the action of $h_i(\mathbb{X}_2 - \mathbb{X}_2')$ for $1 \leq i \leq b$. This is a homotopy equivalence of filtered complexes, where the filtration on the left-hand side is given by the index of summation s, and the filtration on the right-hand side requires some preparation to describe. We will use the shorthand $\mathbf{q}^{s(b-1)}\mathbf{t}^s\mathrm{MCCS}^s_{a,b}$ for the summands on the left-hand side, because, reading left-to-right, the associated diagram is the horizontal composition of "Merge-Crossing-Crossing-Split." We denote the term on the right-hand side (without the shift) by $\mathrm{KMCS}_{a,b}$, since it is a Koszul complex built on the complex

(96) $$\mathrm{MCS}_{a,b} := \left[\!\left[\right.\right.$$ 

$$\left.\left.\right]\!\right]$$

whose notation is again suggestive, namely to be read left-to-right as "Merge-Crossing-Split."

The purpose of this section is to lift the colored skein relation, which is a homotopy equivalence in $\mathcal{C}_{a,b}$, to the curved setting. We begin in §6.1 with a quick discussion of (curved) Koszul complexes, and then in §6.2 use this language to describe the homotopy equivalence (103). In §6.3 and 6.4, we curve both sides of the skein relation and promote (103) to a homotopy equivalence in $\mathcal{V}_{a,b}$.

6.1. **Koszul complexes with curvature.** We will use the following construction of (curved) Koszul complexes. To this end, recall the reduced 2-strand categories $\overline{\mathcal{V}}_{a,b}$ from Definiton 4.38.

Definition 6.1. Fix integers $a, b \geq 0$ and let $\xi_1, \ldots, \xi_b$ be formal odd variables with $\mathrm{wt}(\xi_i) = \mathbf{q}^{2i}\mathbf{t}^{-1}$. For $X \in \mathcal{C}_{a,b}$, let $K(X) \in \mathcal{C}_{a,b}$ denote the twisted complex

$$K(X) := \mathrm{tw}_\delta(X \otimes \textstyle\bigwedge[\xi_1, \ldots, \xi_b]), \quad \delta = \sum_{1 \leq i \leq b} h_i(\mathbb{X}_2 - \mathbb{X}'_2) \otimes \xi_i^*.$$

Let $\mathcal{V}K(X) \in \overline{\mathcal{V}}_{a,b} \subset \mathcal{V}_{a,b}$ denote the curved twisted complex

$$\mathcal{V}K(X) := \mathrm{tw}_\Delta(K(X)) = \mathrm{tw}_{\delta+\Delta}(X \otimes \textstyle\bigwedge[\xi_1, \ldots, \xi_b]), \quad \Delta = \sum_{1 \leq i \leq b} \bar{v}_i \otimes \xi_i.$$

Remark 6.2. We can write $K(X)$ and $\mathcal{V}K(X)$ as one-sided twisted complexes constructed from $K(X^k)$ and $\mathcal{V}K(X^k)$, where $X = (\bigoplus_k X^k, \delta_X)$. More precisely:

$$K(X) = \left(\cdots \xrightarrow{K(\delta_X)} \mathbf{t}^k K(X^k) \xrightarrow{K(\delta_X)} \mathbf{t}^{k+1} K(X^{k+1}) \xrightarrow{K(\delta_X)} \cdots \right),$$

and similarly for $\mathcal{V}K(X)$.

Proposition 6.3. *K and $\mathcal{V}K(-)$ extend to dg functors $\mathcal{C}_{a,b} \to \mathcal{C}_{a,b}$ and $\mathcal{C}_{a,b} \to \overline{\mathcal{V}}_{a,b} \subset \mathcal{V}_{a,b}$, respectively.*

Proof. This follows since we may describe

$$K(X) \cong X \otimes_{\mathrm{Sym}(\mathbb{X}_2|\mathbb{X}'_2)} \mathrm{tw}_\delta \left(\mathrm{Sym}(\mathbb{X}_2|\mathbb{X}'_2) \otimes \textstyle\bigwedge[\xi_1, \ldots, \xi_b] \right)$$

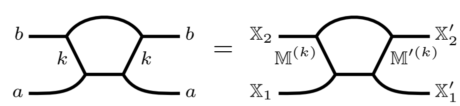
and

$$\mathcal{V}K(X) \cong X \otimes_{\mathrm{Sym}(\mathbb{X}_2|\mathbb{X}'_2)} \mathrm{tw}_{\delta+\Delta} \left(\mathrm{Sym}(\mathbb{X}_2|\mathbb{X}'_2) \otimes \textstyle\bigwedge[\xi_1, \ldots, \xi_b] \right). \qquad \square$$

The formation of curved Koszul complexes such as $\mathcal{V}K(X)$ is one of the most basic methods for constructing objects of $\overline{\mathcal{V}}_{a,b}$.

6.2. **The uncurved skein relation.** We now recall the uncurved version of the skein relation, which was established in the companion paper [HRW21].

Consider the singular Soergel bimodules $W_k \in {}_{a,b}\mathrm{SSBim}_{a,b}$ indicated by the following webs (with relevant alphabets specified in the second diagram):

(97) $W_k :=$ 

We identify the alphabets with subalphabets of $\mathbb{X} = \{x_1, \ldots, x_{a+b}\}$ and $\mathbb{X}' = \{x'_1, \ldots, x'_{a+b}\}$ as follows:

$$\begin{aligned} \mathbb{X}_1 &= \{x_1, \ldots, x_a\}, \quad \mathbb{X}_2 = \{x_{a+1}, \ldots, x_{a+b}\}, \quad \mathbb{M}^{(k)} = \{x_{a+1}, \ldots, x_{a+k}\} \\ \mathbb{X}'_1 &= \{x'_1, \ldots, x'_a\}, \quad \mathbb{X}'_2 = \{x'_{a+1}, \ldots, x'_{a+b}\}, , \quad \mathbb{M}'^{(k)} = \{x'_{a+1}, \ldots, x'_{a+k}\}. \end{aligned} \tag{98}$$

Definition 6.4. Recall the complex $\mathrm{MCS}_{a,b}$ from (96). By convention, $\mathrm{MCS}_{a,b} = 0$ if $a < b$. Also define

$$\mathsf{MCS}_{a,b} := \left(W_b \xrightarrow{\delta^H} \mathbf{q}^{-(a-b+1)}\mathbf{t}W_{b-1} \xrightarrow{\delta^H} \cdots \xrightarrow{\delta^H} \mathbf{q}^{-b(a-b+1)}\mathbf{t}^b W_0 \right), \quad \delta^H\big|_{W_k} := \chi_0^+|_{W_k}$$

Denote the corresponding Koszul complexes by $\mathrm{KMCS}_{a,b} := K(\mathrm{MCS}_{a,b})$ and $\mathsf{KMCS}_{a,b} := K(\mathsf{MCS}_{a,b})$.

Proposition 6.5 ([HRW21, Equation (26)])**.** *We have*

$$\mathsf{MCS}_{a,b} \simeq \mathrm{MCS}_{a,b} \quad \textit{and} \quad K(\mathsf{MCS}_{a,b}) \simeq K(\mathrm{MCS}_{a,b}). \qquad \square$$

The relevant mnemonic is that $\mathsf{MCS}_{a,b}$ is the result of certain Gaussian eliminations on $\mathrm{MCS}_{a,b}$, i.e. a "slimmer" version thereof. We can write $\mathsf{KMCS}_{a,b}$ as a one-sided twisted complex as in Remark 6.2:

$$K(\mathsf{MCS}_{a,b}) = \Big(K(W_b) \xrightarrow{\delta^H} \mathbf{q}^{a-b+1}\mathbf{t}K(W_{b-1}) \xrightarrow{\delta^H} \cdots \xrightarrow{\delta^H} \mathbf{q}^{b(a-b+1)}\mathbf{t}^b K(W_0)\Big), \tag{99}$$

where $\delta^H = K(\chi_0^+)\colon K(W_k) \to K(W_{k-1})$. Next, we change basis within each $K(W_k)$ by declaring

$$\zeta_j^{(k)} := \sum_{i=1}^{j} (-1)^{i-1} e_{j-i}(\mathbb{M}^{(k)}) \otimes \xi_i\,, \quad \xi_i = \sum_{j=1}^{i} (-1)^{j-1} h_{i-j}(\mathbb{M}^{(k)}) \otimes \zeta_j^{(k)}. \tag{100}$$

The effect of this change of basis is captured by the following.

Proposition 6.6 ([HRW21, Proposition 3.10])**.** *We have*

$$K(W_k) \cong \mathrm{tw}_\delta(W_k \otimes \textstyle\bigwedge[\zeta_1^{(k)}, \dots, \zeta_b^{(k)}])\,, \quad \delta = \displaystyle\sum_{i=1}^{k} (e_{i_j}(\mathbb{M}^{(k)}) - e_{i_j}(\mathbb{M}'^{(k)})) \otimes (\zeta_i^{(k)})^*.$$

With respect to this isomorphism, the differential $\delta^H : K(W_k) \to K(W_{k-1})$ has a nonzero component

$$W_k \otimes \zeta_{i_1}^{(k)} \cdots \zeta_{i_r}^{(k)} \xrightarrow{\delta^H} W_{k-1} \otimes \zeta_{j_1}^{(k-1)} \cdots \zeta_{j_r}^{(k-1)}$$

if and only if $i_p - j_p \in \{0,1\}$ for all $1 \le p \le r$, in which case it equals χ_m^+ (see (27)) where $m = \sum_{p=1}^{r}(i_p - j_p)$. □

A priori, the complex $\mathbf{q}^{b(a-b-1)}\mathbf{t}^b\mathsf{KMCS}_{a,b}$ (which is homotopy equivalent to the right-hand side of the skein relation) is graded both by cohomological degree in $\mathsf{MCS}_{a,b}$ and the exterior algebra degree in $\bigwedge[\xi_1, \dots, \xi_b]$. According to [HRW21], the key to understanding the colored skein relation is a refinement of the exterior grading into two independent gradings. This is accomplished with the following.

Definition 6.7. Let

$$P_{k,l,s} := \mathbf{q}^{k(a-b+1)-2b}\mathbf{t}^{2b-k}\; W_k \otimes \textstyle\bigwedge^l[\zeta_1^{(k)}, \dots, \zeta_k^{(k)}] \otimes \bigwedge^s[\zeta_{k+1}^{(k)}, \dots, \zeta_b^{(k)}],$$

with indices constrained by $0 \le s \le b$ and $0 \le l \le k \le b-s$.

Proposition 6.8 ([HRW21, Proposition 3.12])**.** *We have*

$$\mathbf{q}^{b(a-b-1)}\mathbf{t}^b\mathsf{KMCS}_{a,b} \;\cong\; \mathrm{tw}_{\delta^v+\delta^h+\delta^c}\left(\bigoplus_{0\le l\le k\le b-s} P_{k,l,s}\right), \tag{101}$$

where δ^v, δ^h, δ^c are pairwise anti-commuting differentials given as follows:

- *the vertical differential $\delta^v : P_{k,l,s} \to P_{k,l-1,s}$ is the direct sum of Koszul differentials, up to the sign $(-1)^k$; its component*

$$W_k \otimes \zeta_{i_1}^{(k)} \cdots \zeta_{i_r}^{(k)} \xrightarrow{\delta^v} W_k \otimes \zeta_{i_1}^{(k)} \cdots \widehat{\zeta_{i_j}^{(k)}} \cdots \zeta_{i_r}^{(k)}$$

  is $(-1)^{-k+j-1}(e_{i_j}(\mathbb{M}^{(k)}) - e_{i_j}(\mathbb{M}'^{(k)}))$ if $1 \le i_j \le k$ (and all other components are zero).
- *the horizontal differential δ^h and the connecting differential δ^c are uniquely characterized by $\delta^h + \delta^c = \delta^H$ from Proposition 6.6, together with*

$$\delta^h(P_{k,l,s}) \subset P_{k-1,l,s}\,, \quad \delta^c(P_{k,l,s}) \subset P_{k-1,l-1,s+1}.$$

□

In other words, δ^h is the part of δ^H which preserves s-degree and δ^c is the part of δ^H which increases s-degree by 1. Since the differentials δ^v and δ^h preserve s-degree, we may reorganize the direct sum (101) as follows:

$$\mathbf{q}^{b(a-b-1)}\mathbf{t}^b\mathsf{KMCS}_{a,b} \cong \mathrm{tw}_{\delta^c}\left(\bigoplus_{0\le s\le b}\mathrm{tw}_{\delta^v+\delta^h}\left(\bigoplus_{0\le l\le k\le b-s} P_{k,l,s}\right)\right).$$

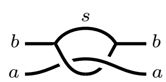
The skein relation essentially amounts to a topological interpretation of the s^{th} summand above.

Proposition 6.9 ([HRW21, Theorem 3.4])**.** *The complexes*

$$\mathsf{MCCS}^s_{a,b} := \mathbf{q}^{-s(b-1)}\mathbf{t}^{-s}\,\mathrm{tw}_{\delta^v+\delta^h}\left(\bigoplus_{0\le l\le k\le b-s} P_{k,l,s}\right). \tag{102}$$

satisfy

$$\mathsf{MCCS}^s_{a,b} \simeq \left[\!\left[\quad \right]\!\right]$$

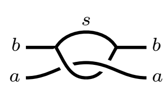


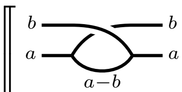
by [HRW21, Corollary 3.29]. *Consequently, the isomorphism*

$$\mathrm{tw}_{\delta^c}\left(\bigoplus_{s=0}^{b}\mathbf{q}^{s(b-1)}\mathbf{t}^s\mathsf{MCCS}^s_{a,b}\right) \cong \mathbf{q}^{b(a-b-1)}\mathbf{t}^b\mathsf{KMCS}_{a,b}$$

yields a homotopy equivalence

$$\mathrm{tw}_{D_1}\left(\bigoplus_{s=0}^{b}\mathbf{q}^{s(b-1)}\mathbf{t}^s\left[\!\left[\quad \right]\!\right]\right) \simeq \mathbf{q}^{b(a-b-1)}\mathbf{t}^b K\left(\left[\!\left[\quad \right]\!\right]\right) \tag{103}$$





by Propositions 6.5 and 4.1. □

6.3. **Curving the right-hand side.** We now aim to promote Proposition 6.9 to the curved setting. To begin, we immediately note that the complex

$$\mathcal{V}\mathsf{KMCS}_{a,b} := \mathcal{V}K(\mathsf{MCS}) \simeq \mathcal{V}K(\mathrm{MCS}) \tag{104}$$

provides a curved lift of $\mathsf{KMCS}_{a,b}$. We will write $\mathcal{V}\mathsf{KMCS}_{a,b}$ explicitly as a one-sided twisted complex

$$\mathcal{V}\mathsf{KMCS}_{a,b} = \left(\mathcal{V}K(W_b) \xrightarrow{\delta^H} \mathbf{q}^{-(a-b+1)}\mathbf{t}\mathcal{V}K(W_{b-1}) \xrightarrow{\delta^H} \cdots \xrightarrow{\delta^H} \mathbf{q}^{-b(a-b+1)}\mathbf{t}^b\mathcal{V}K(W_0)\right) \tag{105}$$

where we slightly abuse notation in writing δ^H for $\mathcal{V}K(\delta^H)$. Note that each of the objects $\mathcal{V}K(W_k)$ is itself a curved complex, whose differentials we draw as arrows pointing downward and upward in illustrations such as (11) and Example 6.14 below.

Our goal is to write down the appropriate curved version of Proposition 6.8. First, consider the curved Koszul complex $\mathcal{V}K(W_k) := \mathrm{tw}_{\delta+\Delta}(W_k \otimes \wedge[\xi_1,\dots,\xi_b])$ from Definition 6.1. Recall that $\delta = \sum_{i=1}^b h_i(\mathbb{X}_2 - \mathbb{X}_2') \otimes \xi_i^*$ and $\Delta = \sum_{i=1}^b \bar{v}_i \otimes \xi_i$, so $(\delta+\Delta)^2 = \sum_i h_i(\mathbb{X}_2 - \mathbb{X}_2')\bar{v}_i$.

Lemma 6.10. *In terms of the ζ-basis we have*

$$\mathcal{V}K(W_k) \cong \mathrm{tw}_{\delta+\Delta}(W_k \otimes \wedge[\zeta_1^{(k)},\dots,\zeta_b^{(k)}]),$$

where

$$\delta = \sum_{1\le i\le b}(e_i(\mathbb{M}) - e_i(\mathbb{M}')) \otimes (\zeta^{(k)})^*, \quad \Delta = \sum_{1\le j\le l\le b}(-1)^{j-1}h_{l-j}(\mathbb{M})\bar{v}_l \otimes \zeta_j^{(k)}.$$

Proof. The formula for the uncurved differential δ is given above in Proposition 6.6; the formula for Δ is immediate from (100). □

Proposition 6.11. *We have*

$$\mathbf{q}^{b(a-b-1)}\mathbf{t}^b\mathcal{V}\mathsf{KMCS}_{a,b} \;\cong\; \mathrm{tw}_{\delta^v+\Delta^v+\delta^h+\delta^c+\Delta^c}\left(\bigoplus_{0\le l\le k\le b-s} P_{k,l,s}\right), \tag{106}$$

where δ^v, δ^h, δ^c are as in Proposition 6.8, and

$$\begin{aligned} &\Delta^v\colon P_{k,l,s}\to P_{k,l+1,s}\,, \quad \Delta^v = \sum_{\substack{1\le j\le l\le b\\ j\le k}} (-1)^{j-1}h_{l-j}(\mathbb{M}^{(k)})\otimes \zeta_j^{(k)}\bar{v}_l\,,\\ &\Delta^c\colon P_{k,l,s}\to P_{k,l,s+1}\,, \quad \Delta^c = \sum_{\substack{1\le k\le l\le b\\ j> k}} (-1)^{j-1}h_{l-j}(\mathbb{M}^{(k)})\otimes \zeta_j^{(k)}\bar{v}_l\,. \end{aligned} \tag{107}$$

Moreover, the endomorphisms $\delta^v, \Delta^v, \delta^h, \delta^c, \Delta^c$ satisfy the following relations:

$$(\delta^h+\delta^v+\Delta^v)^2 = \sum_i h_i(\mathbb{X}_2-\mathbb{X}_2')\bar{v}_i\,, \quad (d^c+\Delta^c)^2=0\,, \quad [\delta^h+\delta^v+\Delta^v, d^c+\Delta^c]=0$$

Proof. The first statement holds by construction, and everything else follows by taking components in $(\delta+\Delta)^2 = \sum_i h_i(\mathbb{X}_2-\mathbb{X}_2')\bar{v}_i$. □

Since Δ^v preserves s, we may define curved lifts of $\mathsf{MCCS}^s_{a,b}$ as follows.

Definition 6.12. Let

$$\mathcal{V}\mathsf{MCCS}^s_{a,b} := \mathbf{q}^{-s(b-1)}\mathbf{t}^{-s}\;\mathrm{tw}_{\delta^v+\delta^h+\Delta^v}\left(\bigoplus_{0\le l\le k\le b-s} P_{k,l,s}\right). \tag{108}$$

This is a well-defined 1-morphism in $\overline{\mathcal{V}}_{a,b}$ by Proposition 6.11.

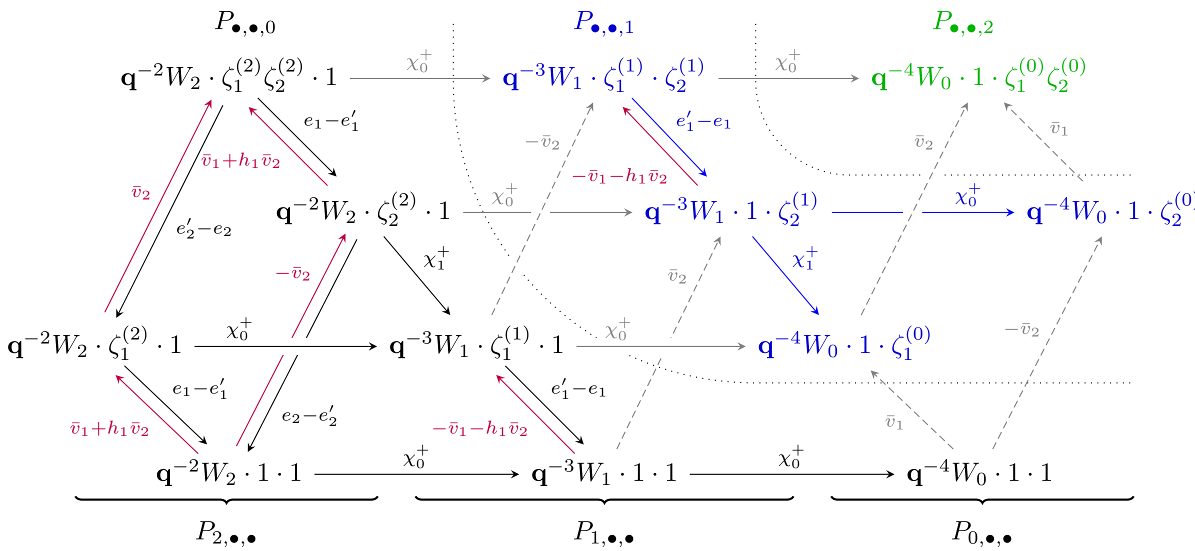
The following holds by construction.

Proposition 6.13. *For all integers $a, b \ge 0$ we have*

$$\mathrm{tw}_{\delta^c+\Delta^c}\left(\bigoplus_{s=0}^{b}\mathbf{q}^{s(b-1)}\mathbf{t}^s\mathcal{V}\mathsf{MCCS}^s_{a,b}\right)\cong \mathbf{q}^{b(a-b-1)}\mathbf{t}^b\;\mathcal{V}\mathsf{KMCS}_{a,b} \tag{109}$$

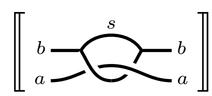
in which the Maurer–Cartan element $\delta^c+\Delta^c$ increases the index s by one. □

Example 6.14. We illustrate the complex $\mathcal{V}\mathsf{KMCS}_{2,2}$, as well as the subquotients $P_{\bullet,\bullet,s} = \mathbf{q}^s\mathbf{t}^s\,\mathsf{MCCS}^s_{2,2}$ for $0 \le s \le 2$. We use the symbol $\cdot$ instead of $\otimes$ to declutter the diagram.



Black and blue horizontal arrows correspond to components of δ^h. All other black and blue arrows indicate non-zero components of δ^v. The curved twist Δ^v is indicated by red arrows. Finally the connecting differential δ^c and its curved correction Δ^c are marked using grey horizontal arrows and grey dashed arrows, respectively.

6.4. **The curved colored skein relation.** The curved colored skein relation will follow by showing that the complex $\mathcal{V}\mathsf{MCCS}^s_{a,b}$ is homotopy equivalent to a curved lifts of the complex

$$\mathrm{MCCS}^s_{a,b} = \left[\!\left[\right.\right.$$  $$\left.\left. \right]\!\right] .$$

We begin by defining these curved lifts. Since $\mathrm{MCCS}^s_{a,b}$ is not invertible for $s \neq 0, b$, we cannot simply invoke Lemma 4.15 to define the curved lift. Instead, we will bootstrap from the $s = 0$ case.

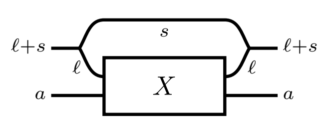
Convention 6.15. Since

$$\mathrm{MCCS}^0_{a,b} = \left[\!\left[\right.\right.$$  $$\left.\left. \right]\!\right]$$

is the Rickard complex assigned to the (a, b)-colored (2-strand) *full twist* braid, we will denote this complex by $\mathrm{FT}_{a,b} := \mathrm{MCCS}^0_{a,b}$. Similarly, we let $\mathsf{FT}_{a,b} := \mathsf{MCCS}^0_{a,b}$.

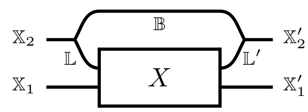
We will make use of the following functors in studying $\mathrm{MCCS}^s_{a,b}$ when $s > 0$.

Definition 6.16. Let $I^{(s)} : \mathcal{C}_{a,\ell} \to \mathcal{C}_{a,\ell+s}$ denote the functor defined by

$$I^{(s)}(X) :=$$  $$.$$

More precisely, $I^{(s)} \colon \mathcal{C}_{a,\ell} \to \mathcal{C}_{a,\ell+s}$ is defined as the composition of first applying $(-) \boxtimes \mathbf{1}_s$ and then horizontal pre- and post-composing with $\mathbf{1}_a \boxtimes {}_{(\ell,s)}S_{(\ell+s)}$ and $\mathbf{1}_a \boxtimes {}_{(\ell+s)}M_{(\ell,s)}$, respectively. Here, ${}_{(\ell,s)}S_{(\ell+s)}$ and ${}_{(\ell+s)}M_{(\ell,s)}$ denote the split and merge bimodules from (25), and $|\mathbb{L}| = \ell = |\mathbb{L}'|$.

Convention 6.17. When considering endomorphisms of complexes of the form $I^{(s)}(X)$, we will often use the alphabet naming convention indicated below:

$I^{(s)}(X) :=$ 

.

In other words, the above diagram indicates how $\mathrm{Sym}(\mathbb{X}_1|\mathbb{X}_1'|\mathbb{X}_2|\mathbb{X}_2'|\mathbb{L}|\mathbb{L}'|\mathbb{B})$ acts on the functor $I^{(s)}$ by natural transformations.

Set $b = \ell + s$. We now extend $I^{(s)}$ to categories of curved complexes $I^{(s)} \colon \overline{\mathcal{V}}_{a,\ell} \to \overline{\mathcal{V}}_{a,b}$. Note that since ${}_{(\ell,s)}S_{(\ell+s)}$ and ${}_{(\ell+s)}M_{(\ell,s)}$ are not 1-morphisms in $\mathcal{Y}(\mathrm{SSBim})$, we cannot simply invoke the 2-categorical operations from §4.3.

Definition 6.18. Let $I^{(s)} \colon \overline{\mathcal{V}}_{a,\ell} \to \overline{\mathcal{V}}_{a,b}$ be the functor defined on objects by

$$I^{(s)}\big(\operatorname{tw}_\Delta(X)\big) = \operatorname{tw}_{I^{(s)}(\Delta)}(I^{(s)}(X))$$

and on morphisms by the map

$$\mathrm{Hom}_{\mathcal{C}_{a,\ell}}(X,Y) \otimes \mathbb{Q}[\overline{\mathbb{V}}^{(\ell)}] \to \mathrm{Hom}_{\mathcal{C}_{a,b}}\left(I^{(s)}(X), I^{(s)}(Y)\right) \otimes \mathbb{Q}[\overline{\mathbb{V}}^{(b)}]$$

induced from the map

$$I^{(s)} \colon \mathrm{Hom}_{\mathcal{C}_{a,\ell}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}_{a,b}}\left(I^{(s)}(X), I^{(s)}(Y)\right)$$

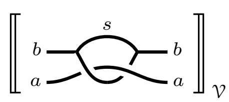
and the assignment (67) (with $\mathbb{L} = \mathbb{X}_{[a+1,\ldots,a+\ell]}$ and $\mathbb{L}' = \mathbb{X}'_{[a+1,\ldots,a+\ell]}$).

Note that $I^{(s)}$ preserves the curvature elements by (68), together with the observation that

$$h_i(\mathbb{L} - \mathbb{L}') = h_i(\mathbb{X}_2 - \mathbb{X}_2').$$

Thus, this functor is indeed well-defined.

Let $\mathcal{V}\mathrm{FT}_{a,b-s} \in \overline{\mathcal{V}}_{a,b-s} \subset \mathcal{V}_{a,b-s}$ denote a chosen curved lift of the full twist Rickard complex $\mathrm{FT}_{a,b-s}$; this exists since the latter is invertible in $\mathcal{C}_{a,b-s}$. Applying the functor $I^{(s)}$ then defines the following curved lift of $\mathrm{MCCS}^s_{a,b}$:



$:= I^{(s)}(\mathcal{V}\mathrm{FT}_{a,b-s})\,.$

Proposition 6.19. *We have* $\mathcal{V}\mathsf{MCCS}^s_{a,b} \simeq I^{(s)}(\mathcal{V}\mathrm{FT}_{a,b-s})$ *for all* $0 \le s \le b$.

Proof. Recall that [HRW21, Theorem 3.24] proves $\mathsf{MCCS}^0_{a,b-s} \simeq \mathrm{FT}_{a,b-s}$. Proposition 4.14 guarantees that there exists an induced homotopy equivalence between the curved lift $\mathcal{V}\mathsf{MCCS}^0_{a,b-s}$ of $\mathsf{MCCS}^0_{a,b-s}$ and some curved lift of $\mathrm{FT}_{a,b-s}$. However, since $\mathrm{FT}_{a,b-s}$ is invertible in $\mathcal{C}_{a,b-s}$, Lemma 4.15 implies such lifts are unique up to homotopy equivalence, and we get $\mathcal{V}\mathsf{MCCS}^0_{a,b-s} \simeq \mathcal{V}\mathrm{FT}_{a,b-s}$. Applying the functor $I^{(s)}$ then gives

$$I^{(s)}(\mathcal{V}\mathsf{MCCS}^0_{a,b-s}) \simeq I^{(s)}(\mathcal{V}\mathrm{FT}_{a,b-s})\,.$$

To complete the proof, we will show that

$$\mathcal{V}\mathsf{MCCS}^s_{a,b} \cong I^{(s)}(\mathcal{V}\mathsf{MCCS}^0_{a,b-s})\,. \tag{110}$$

To distinguish the webs W_k living in the categories ${}_{a,b-s}\mathrm{SSBim}_{a,b-s}$ and those in ${}_{a,b}\mathrm{SSBim}_{a,b}$, we will use the notation $W_k^{(b-s)}$ and $W_k^{(b)}$, mimicking our notation for the v-variables. First, note that

$I^{(s)}(\mathcal{V}\mathsf{MCCS}^0_{a,b-s})$ is a direct sum (with shifts) of terms of the form $I^{(s)}(W_k^{(b-s)}) \otimes \wedge[\zeta_1^{(k)}, \dots, \zeta_k^{(k)}]$. [HRW21, Lemma 3.28] establishes an isomorphism

$$I^{(s)}(W_k^{(b-s)}) \cong W_k^{(b)} \otimes \wedge^s[\zeta_{k+1}^{(k)}, \dots, \zeta_b^{(k)}]\,.$$

Crucially for our present considerations, this isomorphism is $\mathrm{Sym}(\mathbb{M}^{(k)})$-linear.

To prove (110), we only need to verify the commutativity of squares of the form

$$\begin{array}{ccc} I^{(s)}(W_k^{(b-s)} \otimes \wedge[\zeta_1^{(k)}, \dots, \zeta_k^{(k)}]) & \xrightarrow{I^{(s)}(\Delta^v)} & I^{(s)}(W_k^{(b-s)} \otimes \wedge[\zeta_1^{(k)}, \dots, \zeta_k^{(k)}]) \\ \uparrow \cong & & \uparrow \cong \\ W_k^{(b)} \otimes \wedge[\zeta_1^{(k)}, \dots, \zeta_k^{(k)}] \otimes \wedge^s[\zeta_{k+1}^{(k)}, \dots, \zeta_b^{(k)}] & \xrightarrow{\Delta^v} & W_k^{(b)} \otimes \wedge[\zeta_1^{(k)}, \dots, \zeta_k^{(k)}] \otimes \wedge^s[\zeta_{k+1}^{(k)}, \dots, \zeta_b^{(k)}] \end{array}$$

since the commutativity of analogous squares for δ^v and δ^h has already been established in the proof of [HRW21, Proposition 3.27].

Equation (107) shows that the action of Δ^v on $W_k^{(b-s)} \otimes \wedge[\zeta_1^{(k)}, \dots, \zeta_k^{(k)}]$ is given by

$$\Delta^v\Big|_{W_k^{(b-s)} \otimes \wedge[\zeta_1^{(k)}, \dots, \zeta_k^{(k)}]} = \sum_{j=1}^{k} (-1)^{j-1} \zeta_j^{(k)} \sum_{i=j}^{b-s} h_{i-j}(\mathbb{M}^{(k)}) \bar{v}_i^{(b-s)}\,.$$

(Here, and in the following, we slightly abuse notation in the ordering of our tensor factors; since both $h_i(\mathbb{M}^{(k)})$ and all v-variables have even cohomological degree, this does not cause any hidden sign issues.) Now, we apply $I^{(s)}$ to this, obtaining

$$\begin{aligned} I^{(s)}\left(\Delta^v\Big|_{W_k^{(b-s)} \otimes \wedge[\zeta_1^{(k)}, \dots, \zeta_k^{(k)}]}\right) &= \sum_{j=1}^{k} (-1)^{j-1} \zeta_j^{(k)} \sum_{i=j}^{b-s} h_{i-j}(\mathbb{M}^{(k)}) I^{(s)}(\bar{v}_i^{(b-s)}) \\ &= \sum_{j=1}^{k} (-1)^{j-1} \zeta_j^{(k)} \sum_{i=j}^{b-s} h_{i-j}(\mathbb{M}^{(k)}) \left(\bar{v}_i^{(b)} + (-1)^{b-s-i} \sum_{m=b-s+1}^{b} \mathfrak{s}_{(m-b+s-1|b-s-i)}(\mathbb{L}) \bar{v}_m^{(b)}\right). \end{aligned} \tag{111}$$

On the other hand,

$$\Delta^v\Big|_{W_k^{(b)} \otimes \wedge[\zeta_1^{(k)}, \dots, \zeta_k^{(k)}]} = \sum_{j=1}^{k} (-1)^{j-1} \zeta_j^{(k)} \sum_{i=j}^{b} h_{i-j}(\mathbb{M}^{(k)}) \bar{v}_i^{(b)} \tag{112}$$

and it suffices to show that (112) equals (111). By comparing coefficients of $\zeta_j^{(k)}$, this follows from

$$\sum_{i=j}^{b} h_{i-j}(\mathbb{M}^{(k)}) \bar{v}_i^{(b)} = \sum_{i=j}^{b-s} h_{i-j}(\mathbb{M}^{(k)}) \bar{v}_i^{(b)} + \sum_{i=j}^{b-s} (-1)^{b-s-i} h_{i-j}(\mathbb{M}^{(k)}) \sum_{m=b-s+1}^{b} \mathfrak{s}_{(m-b+s-1|b-s-i)}(\mathbb{L}) \bar{v}_m^{(b)}$$

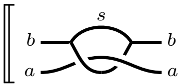
for $1 \le j \le k$, or, after reordering terms:

$$\sum_{m=b-s+1}^{b} h_{m-j}(\mathbb{M}^{(k)}) \bar{v}_m^{(b)} = \sum_{m=b-s+1}^{b} \sum_{i=j}^{b-s} (-1)^{b-s-i} h_{i-j}(\mathbb{M}^{(k)}) \mathfrak{s}_{(m-b+s-1|b-s-i)}(\mathbb{L}) \bar{v}_m^{(b)}$$

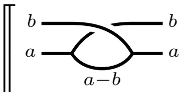
The latter is an application of Lemma 2.9 with $\mathbb{X} = \mathbb{M}^{(k)}$, $\mathbb{Y} = \mathbb{L} - \mathbb{M}^{(k)}$, $r = m - b + s$, and $c = b - s - j$. $\square$

Corollary 6.20 (Curved skein relation)**.** *We have a homotopy equivalence in $\mathcal{V}_{a,b}$ of the form*

$$\mathrm{tw}_{(\delta')^c+(\Delta')^c}\left(\bigoplus_{s=0}^{b}\mathbf{q}^{s(b-1)}\mathbf{t}^{s}\left[\!\!\left[\ \right.\right.$$



$$\left.\left.\ \right]\!\!\right]_{\mathcal{V}}\right)\simeq \mathbf{q}^{b(a-b-1)}\mathbf{t}^{b}\cdot \mathcal{V}K\left(\left[\!\!\left[\ \right.\right.\right.$$



$$\left.\left.\left.\ \right]\!\!\right]\right)$$

where the twist on the left-hand side strictly increases the parameter s and $(\Delta')^c$ is $\mathbb{V}$-irrelevant.

Proof. The right-hand side is a shift of $\mathcal{V}K(\mathrm{MCS}) \simeq \mathcal{V}\mathsf{KMCS}_{a,b}$, as observed in (104). Proposition 6.13 shows this is homotopy equivalent to $\mathrm{tw}_{\delta^c+\Delta^c}\left(\bigoplus_{s=0}^{b}\mathbf{q}^{s(b-1)}\mathbf{t}^{s}\mathcal{V}\mathsf{MCCS}^{s}_{a,b}\right)$ where the twist $\delta^c+\Delta^c$ raises s-degree by one. In Proposition 6.19 we have seen that $\mathcal{V}\mathsf{MCCS}^{s}_{a,b} \simeq I^{(s)}(\mathcal{V}\mathrm{FT}_{a,b-s})$. Homological perturbation (i.e. Proposition 4.1) now extends the direct sum of these equivalences to a homotopy equivalence:

$$\mathrm{tw}_{\delta^c+\Delta^c}\left(\bigoplus_{s=0}^{b}\mathbf{q}^{s(b-1)}\mathbf{t}^{s}\mathcal{V}\mathsf{MCCS}^{s}_{a,b}\right)\simeq \mathrm{tw}_{(\delta')^c+(\Delta')^c}\left(\bigoplus_{s=0}^{b}\mathbf{q}^{s(b-1)}\mathbf{t}^{s}I^{(s)}(\mathcal{V}\mathrm{FT}_{a,b-s})\right)$$

for some twist $(\delta')^c + (\Delta')^c$ that necessarily increases s-degree by the formula for the induced twist in Proposition 4.1. □

7. The splitting map

Recall from Convention 6.15 that $\mathsf{FT}_{a,b} = \mathsf{MCCS}^0_{a,b}$, which is a simplified model for the Rickard complex $\mathrm{FT}_{a,b} = \mathrm{MCCS}^0_{a,b}$ of the (a,b)-colored full twist braid. Let $\mathcal{V}\mathsf{FT}_{a,b} := \mathcal{V}\mathsf{MCCS}^0_{a,b}$ be its curved lift from Definition 6.12. In this section, we study a canonical closed morphism $\Sigma_{a,b} \in \mathrm{Hom}^0_{\mathcal{V}_{a,b}}(\mathcal{V}\mathsf{FT}_{a,b}, \mathbf{1}_{a,b})$ that is the colored analogue of the "link splitting" map from [GH]. We will use the notation from §4.8 throughout.

7.1. **Characterization of the splitting map.** To begin, we show that, up to homotopy and scalar multiple, there is a unique closed degree-zero map $\mathcal{V}\mathsf{FT}_{a,b} \to \mathbf{1}_{a,b}$. This morphism admits an abstract description as a perturbation of the following.

Definition 7.1. Let the *undeformed splitting map* $\mathcal{S}_{a,b} \in \mathrm{Hom}_{\mathcal{C}_{a,b}}(\mathsf{FT}_{a,b}, \mathbf{1}_{a,b})$ be the closed, degree-zero morphism given as the composite of the projection $\mathsf{FT}_{a,b} \to \mathbf{q}^{ab}W_b$ with the unzip morphism $\mathbf{un}\colon \mathbf{q}^{ab}W_b \to \mathbf{1}_{a,b}$.

We now have the following.

Proposition 7.2. *Precomposition with $\mathcal{S}_{a,b}$ gives an $\mathrm{End}_{\mathcal{C}_{a,b}}(\mathbf{1}_{a,b})$-linear homotopy equivalence*

$$\mathrm{End}_{\mathcal{C}_{a,b}}(\mathbf{1}_{a,b}) \to \mathrm{Hom}_{\mathcal{C}_{a,b}}(\mathsf{FT}_{a,b}, \mathbf{1}_{a,b}).$$

Moreover, there exists a closed, degree-zero lift $\Sigma_{a,b} \in \mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathcal{V}\mathsf{FT}_{a,b}, \mathbf{1}_{a,b})$ of $\mathcal{S}_{a,b}$ such that precomposition with $\Sigma_{a,b}$ gives an $\mathrm{End}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b})$-linear homotopy equivalence

$$\mathrm{End}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}) \to \mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathcal{V}\mathsf{FT}_{a,b}, \mathbf{1}_{a,b}).$$

Proof. Note that $\mathsf{FT}_{a,b}$ equals the finite one-sided twisted complex $\mathrm{tw}_{\delta^v}\left(\bigoplus_{l=0}^{b} R_l\right)$, where $R_l := \left(\bigoplus_{k=l}^{b} P_{k,l,0}, \delta^h\right)$ and δ^v decreases the l-degree by one. It follows that

$$\mathrm{Hom}_{\mathcal{C}_{a,b}}(\mathsf{FT}_{a,b}, \mathbf{1}_{a,b}) = \mathrm{tw}_{(\delta^v)^*}\left(\bigoplus_{l=0}^{b}\mathrm{Hom}_{\mathcal{C}_{a,b}}(R_l, \mathbf{1}_{a,b})\right)$$

is likewise a one-sided twisted complex ($(\delta^v)^*$ now increases l-degree by one). Considering the inclusion

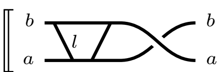
$$\iota\colon \mathrm{Hom}_{\mathcal{C}_{a,b}}(R_b, \mathbf{1}_{a,b}) \hookrightarrow \mathrm{Hom}_{\mathcal{C}_{a,b}}(\mathsf{FT}_{a,b}, \mathbf{1}_{a,b})\,, \tag{113}$$

we see that

$$\mathrm{cone}(\iota) \simeq \mathrm{tw}_{(\delta^v)^*}\left(\bigoplus_{l=0}^{b-1} \mathrm{Hom}_{\mathcal{C}_{a,b}}(R_l, \mathbf{1}_{a,b})\right).$$

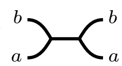
Now, [HRW21, Proposition 3.20] gives that

$$R_l \simeq \left[\!\left[\right]\!\right]$$



so Corollary 9.3 (proved below) implies that $\mathrm{Hom}_{\mathcal{C}_{a,b}}(R_l, \mathbf{1}_{a,b}) \simeq 0$ when $0 \leq l \leq b-1$. Proposition 4.1 allows for these equivalences to be applied to $\mathrm{cone}(\iota)$ term-wise (see Remark 4.2), thus $\mathrm{cone}(\iota) \simeq 0$. As a consequence, (113) is a homotopy equivalence; observe that it is $\mathrm{End}_{\mathcal{C}_{a,b}}(\mathbf{1}_{a,b})$-linear. Now,

$$R_b = \mathbf{q}^{ab}$$



so Corollary A.3 implies that

$$\mathrm{End}_{\mathcal{C}_{a,b}}(\mathbf{1}_{a,b}) \to \mathrm{Hom}_{\mathcal{C}_{a,b}}(R_b, \mathbf{1}_{a,b}), \quad \varphi \mapsto \varphi \circ \mathbf{un}$$

is an $\mathrm{End}_{\mathcal{C}_{a,b}}(\mathbf{1}_{a,b})$-linear isomorphism. The first result now follows from Definition 7.1.

Next, we consider the $\mathbb{V}$-deformation. Let

$$k \circlearrowright \mathrm{Hom}_{\mathcal{C}_{a,b}}(\mathsf{FT}_{a,b}, \mathbf{1}_{a,b})[\mathbb{V}] \underset{g}{\overset{f}{\rightleftarrows}} \mathrm{End}_{\mathcal{C}_{a,b}}(\mathbf{1}_{a,b})[\mathbb{V}] \circlearrowleft \overline{k}$$

be the data giving the homotopy equivalence just constructed, with scalars extended to $\mathbb{Q}[\mathbb{V}]$. Note that all of the indicated maps are therefore $\mathrm{End}_{\mathcal{C}_{a,b}}(\mathbf{1}_{a,b})[\mathbb{V}]$-linear, and we have that $g(1) = \mathcal{S}_{a,b}$. Next, note that $\mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathcal{V}\mathsf{FT}_{a,b}, \mathbf{1}_{a,b}) = \mathrm{tw}_\alpha\left(\mathrm{Hom}_{\mathcal{C}_{a,b}}(\mathsf{FT}_{a,b}, \mathbf{1}_{a,b})[\mathbb{V}]\right)$ for

$$\alpha \in \mathrm{End}_{E_{a,b}}\left(\mathrm{Hom}_{\mathcal{C}_{a,b}}(\mathsf{FT}_{a,b}, \mathbf{1}_{a,b})\right) \otimes_{\mathbb{Q}} \mathbb{Q}[\mathbb{V}]_{>0} \tag{114}$$

with $\mathrm{wt}(\alpha) = \mathbf{q}^0\mathbf{t}^1$. Here, we use the shorthand

$$E_{a,b} := \mathrm{End}_{\mathcal{C}_{a,b}}(\mathbf{1}_{a,b})[\mathbb{V}] = \mathrm{End}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}). \tag{115}$$

Since $\mathsf{FT}_{a,b}$ is bounded, this implies that $k \circ \alpha$ is nilpotent. Proposition 4.1 then implies that $\widetilde{g} = (1 + k \circ \alpha)^{-1} \circ g$ is a homotopy equivalence from $\mathrm{tw}_{\alpha'}\left(\mathrm{End}_{\mathcal{C}_{a,b}}(\mathbf{1}_{a,b})[\mathbb{V}]\right)$ to $\mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathcal{V}\mathsf{FT}_{a,b}, \mathbf{1}_{a,b})$ for some twist α', which must be zero since $\mathbf{1}_{a,b}$ is supported in a single cohomological degree. Thus,

$$\widetilde{g} \colon \mathrm{End}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}) \to \mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathcal{V}\mathsf{FT}_{a,b}, \mathbf{1}_{a,b})$$

is a homotopy equivalence. Set $\Sigma_{a,b} = \widetilde{g}(1)$. Equation (114) implies that $\widetilde{g}(1) = g(1) \mod \mathbb{Q}[\mathbb{V}]_{>0}$, so $\Sigma_{a,b}$ is indeed a curved lift of $g(1) = \mathcal{S}_{a,b}$. Finally, since g, k, and α are all $E_{a,b}$-linear, the same is true for $\widetilde{g}$, thus for $\varphi \in \mathrm{End}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b})$ we compute that

$$\widetilde{g}(\varphi) = \varphi \circ \widetilde{g}(1) = \varphi \circ \Sigma_{a,b}$$

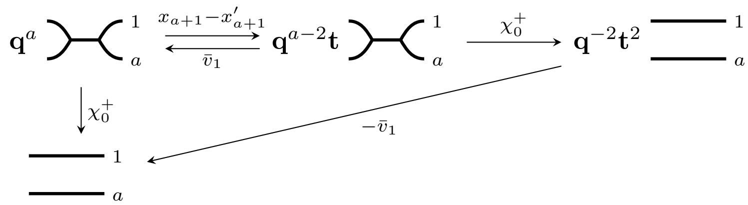
as desired. □

Remark 7.3. Note that the morphism $\Sigma_{a,b}$ "constructed" in the proof of Proposition 7.2 is not explicitly specified, since we have not specified the constituent morphisms (this will be done in Definition 7.8 below). However, any two closed, degree-zero lifts of $\mathcal{S}_{a,b}$ are necessarily homotopic, so $\Sigma_{a,b}$ is uniquely determined up to homotopy. We call this map the (deformed) *splitting map*. Indeed, the difference between any two such curved lifts of $\mathcal{S}_{a,b}$ is a closed, degree-zero element in the $\mathbb{V}$-irrelevant submodule $\mathrm{Hom}_{\mathcal{C}_{a,b}}(\mathcal{V}\mathsf{FT}_{a,b}, \mathbf{1}_{a,b}) \otimes \mathbb{Q}[\mathbb{V}]_{>0} \subset \mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathcal{V}\mathsf{FT}_{a,b}, \mathbf{1}_{a,b})$. However,

$\mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathcal{V}\mathsf{FT}_{a,b}, \mathbf{1}_{a,b}) \simeq \mathrm{End}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b})$ and the only such element in the latter is zero (which is null-homotopic).

Example 7.4. For $a \geq b = 1$, and using the notation introduced in §4.8, the curvature element in $\overline{\mathcal{V}}_{a,1}$ is $(x_{a+1} - x'_{a+1})\bar{v}_1$. A curved lift $\Sigma_{a,1} \colon \mathcal{V}\mathsf{FT}_{a,1} \to \mathbf{1}_{a,1}$ of the splitting map is then given by:

(116)

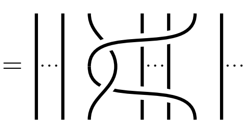


.

Note that $\Sigma_{a,1} = \chi_0^+ \mod \bar{v}_1$.

Remark 7.5. Since $\mathcal{V}\mathsf{FT}_{a,b} \simeq \mathcal{V}\mathrm{FT}_{a,b}$, we also obtain a degree-zero closed splitting map $\Sigma_{a,b} \colon \mathcal{V}\mathrm{FT}_{a,b} \to \mathbf{1}_{a,b}$, that is unique up to homotopy. Using this, we can construct a *deformed splitting map* for the colored positive full twist braid on any number of strands. Indeed, let $\mathrm{FT}_{\boldsymbol{b}}$ denote the Rickard complex associated to the full twist braid with strands colored $b_1, \ldots, b_m$. The full twist braid can be written as a composition of (pure) braids of the form:

$$A_{i,j} = \beta_{j-1} \cdots \beta_{i+1}\beta_i^2\beta_{i+1}^{-1} \cdots \beta_{j-1}^{-1} =$$



(here, in the interest of space, we break our conventions and write the braid vertically). Thus, the 2-strand splitting maps $\Sigma_{a,b} \colon \mathcal{V}\mathrm{FT}_{a,b} \to \mathbf{1}_{a,b}$ assemble (possibly non-uniquely) to define a version of the desired splitting map $\Sigma_{\boldsymbol{b}} \colon \mathcal{V}\mathrm{FT}_{\boldsymbol{b}} \to \mathbf{1}_{\boldsymbol{b}}$. Here, $\mathcal{V}\mathrm{FT}_{\boldsymbol{b}}$ is the unique curved lift of $\mathrm{FT}_{\boldsymbol{b}}$ that is guaranteed to exist by Lemma 4.15. This more-general type of splitting map plays a role in §8 below, and is further generalized in §10.

7.2. **Explicit description of the full twist splitting map.** In this section, we explicitly describe the splitting map $\Sigma_{a,b} \colon \mathcal{V}\mathsf{FT}_{a,b} \to \mathbf{1}_{a,b}$ as a morphism in $\overline{\mathcal{V}}_{a,b}$. We will adopt the convention for alphabets labeling the web W_k as in (98) and the convention for deformation parameters in $\overline{\mathcal{V}}_{a,b}$ as in §4.8. In particular, we consider deformation parameters $\overline{\mathbb{V}}^{(k)} = \{\bar{v}_1^{(k)}, \ldots, \bar{v}_k^{(k)}\}$ for each $k \geq 0$, as in the discussion following Remark 4.41. By convention we abbreviate $\bar{v}_i := \bar{v}_i^{(b)}$ and $\overline{\mathbb{V}} := \overline{\mathbb{V}}^{(b)}$. The alphabets $\overline{\mathbb{V}}^{(k)}$ and $\overline{\mathbb{V}}$ are therefore related by the substitution rule (67), with $\mathbb{M}^{(k)} = \mathbb{X}_{[a+1,a+k]}$, and we identify each $\mathbb{Q}[\mathbb{X}_{[1,a+k]}, \mathbb{X}'_{[1,a+k]}, \overline{\mathbb{V}}^{(k)}]$ as a subalgebra of $\mathbb{Q}[\mathbb{X}, \mathbb{X}', \overline{\mathbb{V}}]$ accordingly. For $a+1 \leq i \leq a+k$, we have the elements

$$\bar{y}_i = \sum_{l=1}^{k} x_i^{l-1}\bar{v}_l^{(k)} \in \mathbb{Q}[\mathbb{X}_{[1,a+k]}, \mathbb{X}'_{[1,a+k]}, \overline{\mathbb{V}}^{(k)}] \subset \mathbb{Q}[\mathbb{X}, \mathbb{X}', \overline{\mathbb{V}}]$$

which, as elements of $\mathbb{Q}[\mathbb{X}, \mathbb{X}', \overline{\mathbb{V}}]$, do not depend on k so long as $1 \leq k \leq b$.

Remark 7.6. The algebra $\mathbb{Q}[\mathbb{X}, \mathbb{X}', \overline{\mathbb{V}}]^{\mathfrak{S}_a \times \mathfrak{S}_k \times \mathfrak{S}_{b-k}}$ acts on the web W_k, thought of as an object of $\mathcal{C}_{a,b} \otimes \mathbb{Q}[\overline{\mathbb{V}}]$. In particular, $\bar{v}_i^{(k)}$ and appropriately partially symmetric expressions in the $\bar{y}_i$ may be regarded as an endomorphism of W_k.

We begin with a reformulation of Lemma 4.33.

Lemma 7.7. *For $1 \leq r \leq b$, we have*

$$\bar{v}_r^{(k)} = (-1)^{r-1} \partial_{a+1} \cdots \partial_{a+k-1} \big(e_{k-r}(\mathbb{X}_{[a+1,a+k-1]}) \cdot \overline{y}_{a+k} \big) \tag{117}$$

Here the operators ∂_i are considered $\mathbb{Q}[\overline{\mathbb{V}}^{(k)}]$-linear.

Proof. Example 3.17 shows that $\big\{ e_{k-i}(\mathbb{X}_{[a+1,a+k-1]}) \big\}_{i=1}^{k}$ and $\big\{ (-1)^{i-1} x_{a+k}^{i-1} \big\}_{i=1}^{k}$ are dual bases with respect to the Sylvester operator

$$\partial_{a+1} \cdots \partial_{a+k-1} \colon \operatorname{Sym}(\mathbb{X}_{[a+1,a+k-1]} | \{x_{a+k}\}) \to \operatorname{Sym}(\mathbb{X}_{[a+1,a+k]}) \,.$$

We thus compute

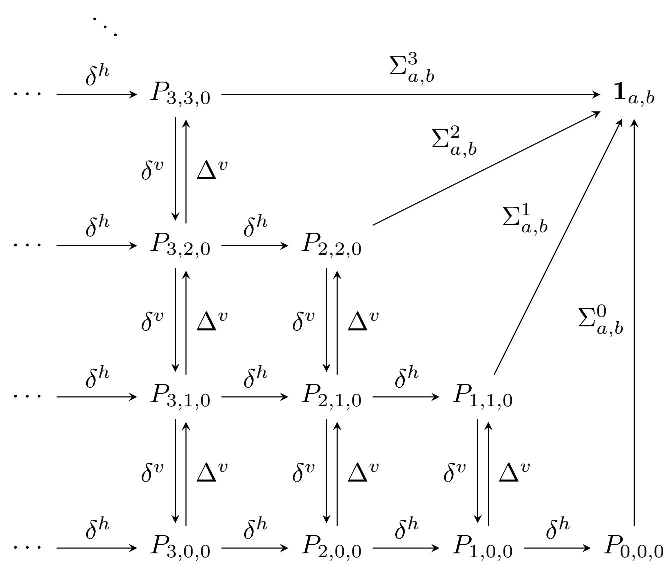
$$\begin{aligned} \partial_{a+1} \cdots \partial_{a+k-1} \big(e_{k-r}(\mathbb{X}_{[a+1,a+k-1]}) \cdot \overline{y}_{a+k} \big) &= \sum_{l=1}^{k} \partial_{a+1} \cdots \partial_{a+k-1} \big(e_{k-r}(\mathbb{X}_{[a+1,a+k-1]}) \cdot x_{a+k}^{l-1} \bar{v}_l^{(k)} \big) \\ &= \sum_{l=1}^{k} \partial_{a+1} \cdots \partial_{a+k-1} \big(e_{k-r}(\mathbb{X}_{[a+1,a+k-1]}) \cdot x_{a+k}^{l-1} \big) \bar{v}_l^{(k)} \\ &= (-1)^{r-1} \bar{v}_r^{(k)} \,. \end{aligned}$$

□

We now give an explicit model for the deformed splitting map from Proposition 7.2.

Definition 7.8. Let $\Sigma_{a,b} \in \operatorname{Hom}_{\mathcal{V}_{a,b}}(\mathcal{V}\mathsf{FT}_{a,b}, \mathbf{1}_{a,b})$ have non-zero components as indicated by the following diagram:

(118)



The map $\Sigma^k_{a,b}$ is given as $(-1)^{\binom{b}{2}}$ times the composition

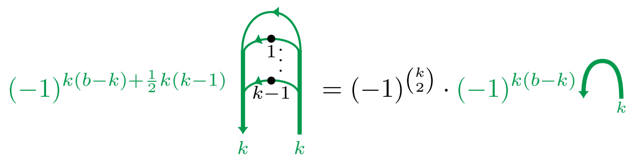
$$P_{k,k,0} \xrightarrow{\approx} W_k \xrightarrow{\overline{y}_{a+k+1} \cdots \overline{y}_{a+b}} W_k \xrightarrow{\mathcal{X}^+} \mathbf{1}_{a,b}.$$

Here, the first map (denoted $\approx$) is a "slanted identity" $P_{k,k,0} = \mathbf{q}^{k(a-b+k)-2(b-k)} \mathbf{t}^{2(b-k)} W_k \xrightarrow{\mathrm{id}} W_k$ (which therefore has weight $\mathbf{q}^{-k(a-b+k)+2(b-k)} \mathbf{t}^{-2(b-k)}$). The second is multiplication by $\overline{y}_{a+k+1} \cdots \overline{y}_{a+b}$, which is a well-defined element in $\operatorname{Sym}(\mathbb{B}) \otimes \mathbb{Q}[\overline{\mathbb{V}}]$. The final morphism is

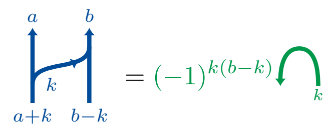
$$\mathcal{X}^+ := \chi_0^+ \cdots \chi_{k-1}^+,$$

which has weight $\mathbf{q}^{k(a-b+k)}\mathbf{t}^0$.

Remark 7.9. The morphism $\mathcal{X}^+ \colon W_k \to \mathbf{1}_{a,b}$ is given in extended graphical calculus as

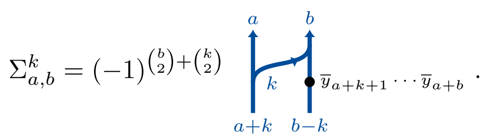


In the perpendicular graphical calculus from §3.4, we have



thus the components of $\Sigma_{a,b}$ from Definition 7.8 are succinctly described as

(119)



Theorem 7.10. *The map $\Sigma_{a,b}$ from Definition 7.8 is closed, has degree zero, and is a curved lift of $\mathcal{S}_{a,b}$. As such, it is an explicit model for the deformed splitting map $\Sigma_{a,b} \colon \mathcal{V}\mathsf{FT}_{a,b} \to \mathbf{1}_{a,b}$ from Proposition 7.2.*

Proof. Since $\mathrm{wt}(\overline{y}_{a+k+1} \cdots \overline{y}_{a+b}) = \mathbf{q}^{-2(b-k)}\mathbf{t}^{2(b-k)}$, $\Sigma_{a,b}$ has degree zero. The component $\Sigma^b_{a,b} \colon P_{b,b,0} = \mathbf{q}^{ab} W_b \to \mathbf{1}_{a,b}$ is $(-1)^{\binom{b}{2}}\mathcal{X}^+ = \mathbf{un}$, thus agrees with $\mathcal{S}_{a,b}$. Since the sum of the remaining components is an element of $\mathrm{Hom}_{\mathcal{C}_{a,b}}(\mathsf{FT}_{a,b}, \mathbf{1}_{a,b}) \otimes \mathbb{Q}[\mathbb{V}]_{>0}$, it remains to check that the prescribed map is a chain map, i.e. that $\Sigma^k_{a,b} \circ \Delta^v = -\Sigma^{k-1}_{a,b} \circ \delta^h$ as maps $P_{k,k-1,0} \to \mathbf{1}_{a,b}$.

This verification is most-easily given in terms of the variables $\{\xi_i\}_{i=1}^b$, rather than the variables $\{\zeta^{(k)}_i\}_{i=1}^b$ that are adapted to the "columns" $\mathcal{V}K(W_k)$. Recall from (100) that the change of variables between these sets of variables is lower-triangular, so we have identifications

$$P_{k,k,0} \approx W_k \otimes \xi_1 \cdots \xi_k \quad \text{and} \quad P_{k,k-1,0} \approx W_k \otimes \mathbb{Q}\{\xi_1 \cdots \widehat{\xi_i} \cdots \xi_k\}_{i=1}^k .$$

To begin, we compute

$$\begin{aligned}
\Delta^v|_{P_{k,\bullet,0}} &= \sum_{j=1}^k (-1)^{j-1} \zeta^{(k)}_j \sum_{l=j}^b h_{l-j}(\mathbb{M}^{(k)}) \bar{v}_l \\
&\overset{(100)}{=} \sum_{j=1}^k (-1)^{j-1} \sum_{i=1}^j (-1)^{i-1} e_{j-i}\big(\mathbb{M}^{(k)}\big) \xi_i \sum_{l=j}^b h_{l-j}(\mathbb{M}^{(k)}) \bar{v}_l \\
&= \sum_{i=1}^k \left(\sum_{l=1}^b \left(\sum_{j=1}^k (-1)^{j-i} e_{j-i}(\mathbb{M}^{(k)}) h_{l-j}(\mathbb{M}^{(k)}) \right) \bar{v}_l \right) \xi_i .
\end{aligned}$$

Next, we have that

$$\sum_{j=1}^k (-1)^{j-i} e_{j-i}(\mathbb{M}^{(k)}) h_{l-j}(\mathbb{M}^{(k)}) = \begin{cases} 1 & \text{if } l = i (\leq k) \\ (-1)^{k-i} \mathfrak{s}_{(l-k-1|k-i)}(\mathbb{M}^{(k)}) & \text{else} \end{cases}$$

using Lemma 2.8. In the latter case, note that this is zero unless $l > k$. Thus, (67) shows that in the variables $\bar{v}_i^{(k)}$ the map Δ^v takes the simplified form

$$\Delta^v|_{P_{k,\bullet,0}} = \sum_{i=1}^{k} \bar{v}_i^{(k)} \xi_i \,.$$

Next, we compute $\delta^h|_{P_{k,k-1,0}}$ in the variables $\{\xi_i\}_{i=1}^k$. This is simply given by the composition

$$P_{k,k-1,0} \hookrightarrow \mathcal{V}K(W_k) \xrightarrow{\delta^H} \mathcal{V}K(W_{k-1}) \twoheadrightarrow P_{k-1,k-1,0}$$

where δ^H is induced by χ_0. In particular, δ^H is determined in Koszul degree 1 by sending $W_k \otimes \xi_i \xrightarrow{\chi_0} W_{k-1} \otimes \xi_i$. Lower-triangularity of the change of basis from $\{\xi_i\}_{i=1}^{k-1}$ to $\{\zeta^{(k-1)}\}_{i=1}^{k-1}$ implies that δ^h still acts by χ_0 on terms indexed by (products of) $\xi_1, \ldots, \xi_{k-1}$. However, for ξ_k, we compute

$$\xi_k = \sum_{j=1}^{k-1} (-1)^{j-1} h_{k-j}(\mathbb{M}^{(k-1)}) \zeta_j^{(k-1)} + (-1)^{k-1} \zeta_k^{(k-1)} \,.$$

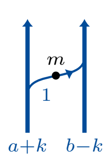
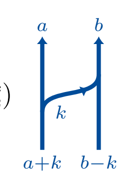
The value of $\delta^h(\xi_k)$ is obtained by projecting onto the span of $\{\zeta_j^{(k-1)}\}_{j=1}^{k-1}$, thus

$$\begin{aligned}
\delta^h|_{P_{k,\bullet,0}} \colon \xi_k \overset{\chi_0}{\longmapsto} & \sum_{j=1}^{k-1} (-1)^{j-1} h_{k-j}(\mathbb{M}^{(k-1)}) \zeta_j^{(k-1)} \\
&= \sum_{j=1}^{k-1} (-1)^{j-1} h_{k-j}(\mathbb{M}^{(k-1)}) \sum_{i=1}^{j} (-1)^{i-1} e_{j-i}(\mathbb{M}^{(k-1)}) \xi_i \\
&= \sum_{i=1}^{k-1} \left(\sum_{j=i}^{k-1} (-1)^{j-i} h_{k-j}(\mathbb{M}^{(k-1)}) e_{j-i}(\mathbb{M}^{(k-1)}) \right) \xi_i \\
&= \sum_{i=1}^{k-1} (-1)^{k-i+1} e_{k-i}(\mathbb{M}^{(k-1)}) \xi_i \,.
\end{aligned}$$

Recalling that $P_{k,l,0} = \mathbf{q}^{k(a-b+1)-2b} \mathbf{t}^{2b-k} \; W_k \otimes \wedge^l[\zeta_1^{(k)}, \ldots, \zeta_k^{(k)}]$, it therefore remains to verify that the diagram

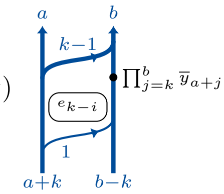
$$\begin{array}{ccc}
\mathbf{t}^{2b-k} W_k \otimes \xi_1 \cdots \xi_i \cdots \xi_k & \xrightarrow{\chi_0^+ \cdots \chi_{k-1}^+ \cdot \overline{y}_{a+k+1} \cdots \overline{y}_{a+b}} & \mathbf{1}_{a,b} \\
{\scriptstyle (-1)^{i-1-k} \bar{v}_i^{(k)}} \uparrow & & \uparrow {\scriptstyle \chi_0^+ \cdots \chi_{k-2}^+ \cdot \overline{y}_{a+k} \cdots \overline{y}_{a+b}} \\
\mathbf{t}^{2b-k} W_k \otimes \xi_1 \cdots \widehat{\xi_i} \cdots \xi_k & \xrightarrow{e_{k-i}(\mathbb{M}^{(k-1)}) \cdot \chi_0^+} & \mathbf{t}^{2b-k+1} W_{k-1} \otimes \xi_1 \cdots \xi_i \cdots \xi_{k-1}
\end{array} ,$$

anti-commutes for $1 \leq i \leq k$ (here we have omitted the $\mathbf{q}$-degree shifts on the bimodules, and the overall factor of $(-1)^{\binom{b}{2}}$ on the components of $\Sigma_{a,b}$). To do so, we use the perpendicular graphical calculus. Recall from (29) and Remark 7.9 that

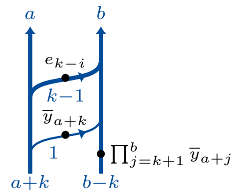
$\chi_m^+ =$  $: W_k \to W_{k-1} \,, \quad \mathcal{X}^+ = \chi_0^+ \cdots \chi_{k-1}^+ = (-1)^{\binom{k}{2}}$  $: W_k \to W_0 \,.$

We then compute

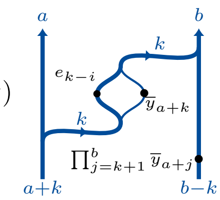
$$\chi_0^+ \cdots \chi_{k-2}^+ \cdot \overline{y}_{a+k} \cdots \overline{y}_{a+b} \cdot e_{k-i}(\mathbb{M}^{(k-1)}) \cdot \chi_0 = (-1)^{\binom{k-1}{2}}$$



$$= (-1)^{\binom{k-1}{2}}$$

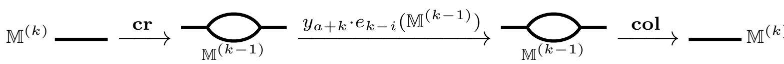


$$= (-1)^{\binom{k-1}{2}}$$



.

Now, we may simplify this latter diagram since the middle portion is locally the composition

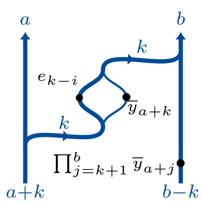


i.e. it is multiplication by the element

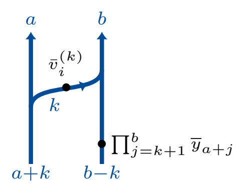
$$\partial_{a+1} \cdots \partial_{a+k-1}\big(\overline{y}_{a+k} \cdot e_{k-i}(\mathbb{M}^{(k-1)})\big) \overset{(117)}{=} (-1)^{i-1}\overline{v}_i^{(k)}.$$

This implies that

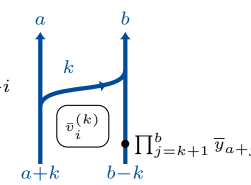
$$(-1)^{\binom{k-1}{2}}$$



$$= (-1)^{\binom{k-1}{2}+i-1}$$



$$= (-1)^{\binom{k}{2}-k+i}$$



and the result follows. (Here, we used that $\binom{k-1}{2} = \binom{k}{2} - k + 1$.) □

We record an immediate consequence, which will be used below.

Corollary 7.11. *Let* $\overline{\psi}_{a,b} \colon \mathbf{1}_{a,b} \to \mathcal{V}\mathsf{FT}_{a,b}$ *be the morphism of weight* $\mathbf{q}^{-2b}\mathbf{t}^{2b}$ *given by the composition*

$$\mathbf{1}_{a,b} \xrightarrow{\approx} \mathbf{q}^{-2b}\mathbf{t}^{2b}\mathbf{1}_{a,b} = P_{0,0,0} \hookrightarrow \mathcal{V}\mathsf{FT}_{a,b}\,,$$

then $\Sigma_{a,b} \circ \overline{\psi}_{a,b} = \overline{y}_{a+1} \cdots \overline{y}_{a+b} \cdot \mathrm{id}_{\mathbf{1}_{a,b}}$ *and* $\overline{\psi}_{a,b} \circ \Sigma_{a,b} \sim \overline{y}_{a+1} \cdots \overline{y}_{a+b} \cdot \mathrm{id}_{\mathcal{V}\mathsf{FT}_{a,b}}$.

Proof. The first identity follows directly from the explicit description of $\Sigma_{a,b}$ in Theorem 7.10.

Next, $\mathcal{V}\mathsf{FT}_{a,b}$ is invertible since it is (homotopy equivalent to) a curved Rickard complex, so we have

$$\mathrm{End}_{\overline{\mathcal{V}}_{a,b}}(\mathcal{V}\mathsf{FT}_{a,b}) \simeq \mathrm{End}_{\overline{\mathcal{V}}_{a,b}}(\mathbf{1}_{a,b}) = \mathrm{Sym}(\mathbb{X}_1|\mathbb{X}_2)[\overline{\mathbb{V}}]\,.$$

Thus, $\overline{\psi}_{a,b} \circ \Sigma_{a,b}$ is homotopic to some multiple of $\mathrm{id}_{\mathcal{V}\mathsf{FT}_{a,b}}$, i.e. $\overline{\psi}_{a,b} \circ \Sigma_{a,b} \sim c \cdot \mathrm{id}_{\mathcal{V}\mathsf{FT}_{a,b}}$ for some $c \in \mathrm{Sym}(\mathbb{X}_1|\mathbb{X}_2)[\overline{\mathbb{V}}]$. We then compute

$$\overline{y}_{a+1} \cdots \overline{y}_{a+b} \cdot \Sigma_{a,b} \simeq (\Sigma_{a,b} \circ \overline{\psi}_{a,b}) \circ \Sigma_{a,b} = \Sigma_{a,b} \circ (\overline{\psi}_{a,b} \circ \Sigma_{a,b}) \simeq c \cdot \Sigma_{a,b}\,.$$

Proposition 7.2 implies that $\mathrm{Hom}_{\overline{\mathcal{V}}_{a,b}}(\mathcal{V}\mathsf{FT}_{a,b}, \mathbf{1}_{a,b})$ is a torsion-free $\mathrm{Sym}(\mathbb{X}_1|\mathbb{X}_2)[\overline{y}_{a+1} \cdots \overline{y}_{a+b}]$-module, thus $c = \overline{y}_{a+1} \cdots \overline{y}_{a+b}$. □

7.3. **Splitting the skein relation.** We now consider the curved splitting map $\Sigma_{a,b}\colon \mathcal{V}\mathsf{FT}_{a,b} \to \mathbf{1}_{a,b}$ in the context of the skein relation. Recall that in the proof of Proposition 6.19 (see (110)) we have established the isomorphism

$$\mathcal{V}\mathsf{MCCS}^s_{a,b} \cong I^{(s)}(\mathcal{V}\mathsf{FT}_{a,b-s})$$

where $I^{(s)}$ is the functor from Definition 6.18. We use this isomorphism to identify these complexes, which in turn allows us to identify the left-hand side of equation (109) with the complex:

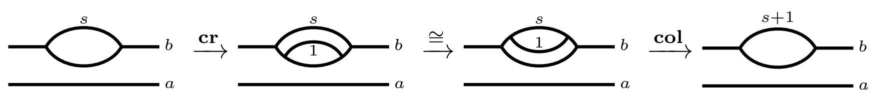
$$\mathcal{V}\mathsf{TD}_b(a) := \mathrm{tw}_{\delta^c+\Delta^c}\left(\bigoplus_{s=0}^{b} \mathbf{q}^{s(b-1)}\mathbf{t}^s I^{(s)}(\mathcal{V}\mathsf{FT}_{a,b-s})\right). \tag{120}$$

In light of the topological interpretation of $I^{(s)}(\mathcal{V}\mathsf{FT}_{a,b-s}) \cong \mathcal{V}\mathsf{MCCS}^s_{a,b}$ afforded by Proposition 6.19, we will refer to this as the curved complex of *threaded digons*. The curved splitting map suggests the consideration of the analogous (curved) complex wherein each instance of $\mathcal{V}\mathsf{FT}_{a,b-s}$ is replaced by the corresponding identity bimodule $\mathbf{1}_{a,b-s}$.

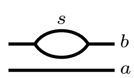
Definition 7.12. The complex of *(unthreaded) digons* is the complex

$$\mathbf{1}_a \boxtimes \mathcal{V}\mathsf{TD}_b(0) = \mathrm{tw}_d\left(\bigoplus_{s=0}^{b} \mathbf{q}^{s(b-1)}\mathbf{t}^s I^{(s)}(\mathbf{1}_{a,b-s})\right)$$

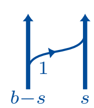
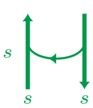
with differential $d = \bigoplus_{s=0}^{b-1} d_s$ given by

d_s : 

We introduce the following notation:

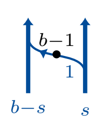
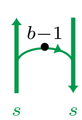
$\mathrm{Dig}^s_{a,b} := I^{(s)}(\mathbf{1}_{a,b-s}) =$  $= \mathbf{1}_a \boxtimes \left(\mathsf{E}^{(s)}\mathsf{F}^{(s)}\mathbf{1}_{b,0}\right)$

for the digon webs appearing in $\mathbf{1}_a \boxtimes \mathcal{V}\mathsf{TD}_b(0)$. In this notation, the components of the differential d take the form $d_s = \mathrm{id}_{\mathbf{1}_a} \boxtimes d'_s$, where d'_s admits the following descriptions in perpendicular and extended graphical calculus:

$d'_s :=$  $= (-1)^s$  $(b,0)$.

We now observe that the complex $\mathbf{1}_a \boxtimes \mathcal{V}\mathsf{TD}_b(0)$ is homotopically trivial.

Lemma 7.13. *The complex $\mathbf{1}_a \boxtimes \mathcal{V}\mathsf{TD}_b(0)$ is contractible, with null homotopy given by $k = \bigoplus_{s=1}^{b} k_s$ with $k_s = \mathbf{1}_a \boxtimes k'_s$ where*

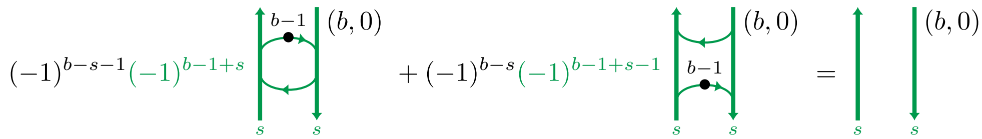
$k'_s := (-1)^{b-s}$  $= (-1)^{b-s} \cdot (-1)^{b-1}$  $(b,0)$

Moreover, the components satisfy $k_{s-1} \circ k_s = 0$.

Proof. We illustrate the chain complex and the homotopy as:

$$\left(\mathrm{Dig}^0_{a,b} \underset{k_1}{\overset{d_0}{\rightleftarrows}} \mathbf{q}^{b-1}\mathbf{t}^1\mathrm{Dig}^1_{a,b} \underset{k_2}{\overset{d_1}{\rightleftarrows}} \cdots \underset{k_b}{\overset{d_{b-1}}{\rightleftarrows}} \mathbf{q}^{b(b-1)}\mathbf{t}^b\mathrm{Dig}^b_{a,b} \right)$$

It suffices to show that $k'_{s+1} \circ d'_s + d'_{s-1} \circ k'_s = \mathrm{id}_{\mathsf{E}^{(s)}F^{(s)}\mathbf{1}_{b,0}}$. In extended graphical calculus, this is

(121) 

which is a specialization of [KLMS12, Lemma 4.6.4]. The relation $k'_{s-1} \circ k'_s = 0$ similarly follows from a computation in extended graphical calculus. ☐

Note that $\mathrm{Dig}^s_{a,b}$ can be viewed as an object in $\overline{\mathcal{V}}_{a,b}$, since the relevant curvature acts on it by zero. Applying the functor $I^{(s)}$ to the curved splitting maps $\Sigma_{a,b-s}\colon \mathcal{V}\mathsf{FT}_{a,b-s} \to \mathbf{1}_{a,b-s}$ gives a family of maps

$$\mathcal{V}\mathsf{MCCS}^s_{a,b} \xrightarrow{I^{(s)}(\Sigma_{a,b-s})} \mathrm{Dig}^s_{a,b}$$

The main goal of this section is to assemble these into a chain map $\mathcal{V}\mathsf{TD}_b(a) \to \mathbf{1}_a \boxtimes \mathcal{V}\mathsf{TD}_b(0)$ as follows.

Theorem 7.14 (Skein relation splitter)**.** *There is a degree-zero closed morphism*

$$\Phi\colon \mathcal{V}\mathsf{TD}_b(a) \to \mathbf{1}_a \boxtimes \mathcal{V}\mathsf{TD}_b(0)$$

whose component $\Phi^{r,s}\colon \mathbf{q}^{s(b-1)}\mathbf{t}^s\mathcal{V}\mathsf{MCCS}^s_{a,b} \to \mathbf{q}^{r(b-1)}\mathbf{t}^r\mathrm{Dig}^r_{a,b}$ *is homotopic to the splitting map* $I^{(s)}(\Sigma_{a,b-s})$ *if $s = r$, and zero otherwise.*

The proof of this theorem will occupy the remainder of this section. We begin with a number of preparatory lemmata. First, the following implies that Theorem 7.14 holds in the undeformed setting.

Lemma 7.15. *The square*

(122)
$$\begin{array}{ccc} I^{(s)}(\mathsf{FT}_{a,b-s}) & \xrightarrow{\delta^c} & I^{(s+1)}(\mathsf{FT}_{a,b-s-1}) \\ \big\downarrow{\scriptstyle I^{(s)}(\mathcal{S}_{a,b-s})} & & \big\downarrow{\scriptstyle I^{(s+1)}(\mathcal{S}_{a,b-s-1})} \\ I^{(s)}(\mathbf{1}_{a,b-s}) & \xrightarrow{d} & I^{(s+1)}(\mathbf{1}_{a,b-s-1}) \end{array}$$

commutes in $\mathcal{C}_{a,b}$. By slight abuse of notation, δ^c here denotes the morphism

$$I^{(s)}(\mathsf{FT}_{a,b-s}) \cong \mathsf{MCCS}^s_{a,b} \xrightarrow{\delta^c} \mathsf{MCCS}^{s+1}_{a,b} \cong I^{(s+1)}(\mathsf{FT}_{a,b-s-1})\,.$$

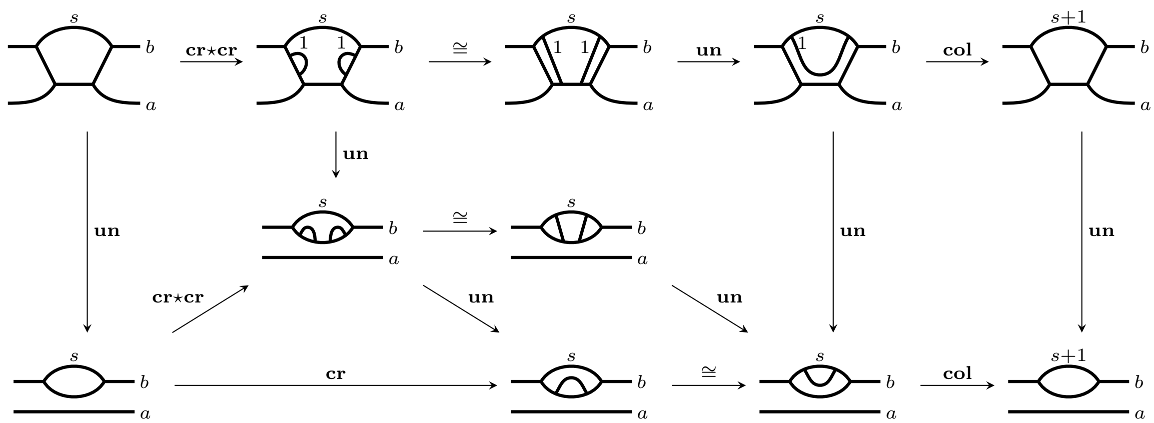
Proof. Since $\mathcal{S}_{a,b-s}$ is supported on the minimal $\mathbf{t}$-degree summand, the same is true for $I^{(s)}(\mathcal{S}_{a,b-s})$. In $\mathsf{MCCS}^s_{a,b} \cong I^{(s)}(\mathsf{FT}_{a,b-s})$, that summand is $\mathbf{q}^{a(b-s)-b(b+1)}\mathbf{t}^b W_{b-s}\zeta_1^{(b-s)} \cdots \zeta_{b-s}^{(b-s)} \cong \mathbf{q}^{a(b-s)} W_{b-s}$ (see e.g. Definition 6.7 and equation (108)). By definition (see Proposition 6.8) the connecting differential restricted to such summands is given by

$$\mathbf{q}^{a(b-s)-b(b+1)}\mathbf{t}^b W_{b-s}\zeta_1^{(b-s)} \cdots \zeta_{b-s}^{(b-s)} \xrightarrow{\chi_0^+} \mathbf{q}^{a(b-s)-b(b+1)}\mathbf{t}^b W_{b-s}\zeta_1^{(b-s-1)} \cdots \zeta_{b-s-1}^{(b-s)}\,.$$

Definition 7.1 then implies that it thus suffices to verify that the square

$$\begin{array}{ccc} \mathbf{q}^{a(b-s)}W_{b-s} & \xrightarrow{\chi_0^+} & \mathbf{q}^{a(b-s-1)}W_{b-s-1} \\ \big\downarrow{\scriptstyle \mathbf{un}} & & \big\downarrow{\scriptstyle \mathbf{un}} \\ \mathrm{Dig}^s_{a,b} & \xrightarrow{d} & \mathrm{Dig}^{s+1}_{a,b} \end{array}$$

commutes. This follows from the definitions of χ_0^+ and d. Indeed, (ignoring the shifts) the above square can be expanded to the diagram



All of the squares are readily seen to commute. The Frobenius extension structure discussed in §3.3 implies that the triangle commutes, and commutativity of the hexagon can be verified using an explicit computation (or is a consequence of [QR16, Equation (3.8)] and foam isotopy). □

Adjoining the squares (122) for $0 \leq s \leq b$ gives a chain map

$$\mathsf{KMCS}_{a,b} = \mathrm{tw}_{\delta^c}\left(\bigoplus_{s=0}^{b} \mathbf{q}^{s(b-1)}\mathbf{t}^s \mathsf{MCCS}^s_{a,b}\right) \longrightarrow \mathbf{1}_a \boxtimes \mathcal{V}\mathsf{TD}_b(0)$$

which is the uncurved analogue of Theorem 7.14. It remains to deform this map, and our argument will proceed in two steps. First, we will show that the a deformation of the square (122) commutes up to homotopy. Second, we will "straighten" these maps to give the desired closed morphism.

To aid in the former, we next establish a particular instance of an adjunction involving the functors $I^{(s)}$. Set

$$\overline{E}_{a,(b-s,s)} := \mathrm{End}_{\overline{\mathcal{V}}_{a,b}}(\mathrm{Dig}^s_{a,b}) = \mathrm{Sym}(\mathbb{X}_1|\mathbb{L}|\mathbb{B}) \otimes \mathbb{Q}[\bar{v}_1^{(b)}, \ldots, \bar{v}_b^{(b)}]$$

Here, we follow Convention 6.17 and denote the alphabet on the s-labeled edge in $\mathrm{Dig}^s_{a,b}$ by $\mathbb{B}$ and the alphabet on the $(b-s)$-labeled edge by $\mathbb{L}$. We view $\overline{E}_{a,(b-s,s)}$ as a module over

$$\overline{E}_{a,b-s} := \mathrm{End}_{\overline{\mathcal{V}}_{a,b-s}}(\mathbf{1}_{a,b-s}) = \mathrm{Sym}(\mathbb{X}_1|\mathbb{L}) \otimes \mathbb{Q}[\bar{v}_1^{(b-s)}, \ldots, \bar{v}_{b-s}^{(b-s)}]$$

via the inclusion $\mathrm{Sym}(\mathbb{X}_1|\mathbb{L}) \hookrightarrow \mathrm{Sym}(\mathbb{X}_1|\mathbb{L}|\mathbb{B})$ and the map

$$\mathbb{Q}[\bar{v}_1^{(b-s)}, \ldots, \bar{v}_{b-s}^{(b-s)}] \to \mathrm{Sym}(\mathbb{L}) \otimes \mathbb{Q}[\bar{v}_1^{(b)}, \ldots, \bar{v}_b^{(b)}]$$

given in (67).

Lemma 7.16. *Let X be a curved complex in $\overline{\mathcal{V}}_{a,b-s}$, then there is an isomorphism*

$$\mathbf{q}^{-s(b-s)}\,\mathrm{Hom}_{\overline{\mathcal{V}}_{a,b-s}}(X, \mathbf{1}_{a,b-s}) \otimes_{\overline{E}_{a,b-s}} \overline{E}_{a,(b-s,s)} \xrightarrow{\cong} \mathrm{Hom}_{\overline{\mathcal{V}}_{a,b}}(I^{(s)}(X), \mathbf{1}_{a,b})$$

of dg $\overline{E}_{a,(b-s,s)}$-modules that is natural in X.

Proof. First, we consider the undeformed setting. Let X be a 1-morphism in $\mathcal{C}_{a,b}$, then we have that

$$I^{(s)}(X) = (\mathbf{1}_a \boxtimes {}_bM_{b-s,s}) \star (X \boxtimes \mathbf{1}_s) \star (\mathbf{1}_a \boxtimes {}_{b-s,s}S_b).$$

Using Proposition 3.19 and Corollary A.2, we compute

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}_{a,b}}(I^{(s)}(X),\mathbf{1}_{a,b}) &= \mathrm{Hom}_{\mathcal{C}_{a,b}}\big((\mathbf{1}_a \boxtimes {}_bM_{b-s,s})\star(X\boxtimes\mathbf{1}_s)\star(\mathbf{1}_a\boxtimes {}_{b-s,s}S_b),\mathbf{1}_{a,b}\big)\\ &\cong \mathbf{q}^{-2s(b-s)}\,\mathrm{Hom}_{\mathcal{C}_{a,b}}\big(X\boxtimes\mathbf{1}_s,\mathbf{1}_a\boxtimes({}_{b-s,s}S_b\star {}_bM_{b-s,s})\big)\\ &\cong \mathbf{q}^{-s(b-s)}\,\mathrm{Hom}_{\mathcal{C}_{a,b}}\big(X,\mathbf{1}_{a,b-s}\big)\otimes\mathrm{End}_{\mathcal{C}_{a,b}}(\mathbf{1}_s)\,.\end{aligned} \tag{123}$$

In our present notation, we have that $\mathrm{End}_{\mathcal{C}_{a,b}}(\mathbf{1}_s)=\mathrm{Sym}(\mathbb{B})$. Set $\overline{E}_{a,b-s}(\mathbb{B}):=\overline{E}_{a,b-s}\otimes\mathrm{Sym}(\mathbb{B})$, then Definition 6.18 then gives that

$$\begin{aligned}\mathrm{Hom}_{\overline{\mathcal{V}}_{a,b}}(I^{(s)}(X),\mathbf{1}_{a,b}) &= \mathrm{Hom}_{\mathcal{C}_{a,b}}(I^{(s)}(X),\mathbf{1}_{a,b})\otimes\mathbb{Q}[\bar v_1^{(b)},\dots,\bar v_b^{(b)}]\\ &\cong \mathrm{Hom}_{\mathcal{C}_{a,b}}(I^{(s)}(X),\mathbf{1}_{a,b})\otimes_{\mathrm{Sym}(\mathbb{X}_1|\mathbb{L}|\mathbb{B})}\overline{E}_{a,(b-s,s)}\\ &\cong \mathrm{Hom}_{\mathcal{C}_{a,b}}(I^{(s)}(X),\mathbf{1}_{a,b})\otimes_{\mathrm{Sym}(\mathbb{X}_1|\mathbb{L}|\mathbb{B})}\overline{E}_{a,b-s}(\mathbb{B})\otimes_{\overline{E}_{a,b-s}(\mathbb{B})}\overline{E}_{a,(b-s,s)}\,.\end{aligned}$$

Using the isomorphism in (123), we obtain

$$\begin{aligned}\mathrm{Hom}_{\overline{\mathcal{V}}_{a,b}}(I^{(s)}(X),\mathbf{1}_{a,b}) &\cong \mathbf{q}^{-s(b-s)}\big(\mathrm{Hom}_{\mathcal{C}_{a,b-s}}(X,\mathbf{1}_{a,b-s})\otimes\mathrm{Sym}(\mathbb{B})\big)\otimes_{\mathrm{Sym}(\mathbb{X}_1|\mathbb{L}|\mathbb{B})}\overline{E}_{a,b-s}(\mathbb{B})\otimes_{\overline{E}_{a,b-s}(\mathbb{B})}\overline{E}_{a,(b-s,s)}\\ &\cong \mathbf{q}^{-s(b-s)}\,\mathrm{Hom}_{\mathcal{C}_{a,b-s}}(X,\mathbf{1}_{a,b-s})\otimes_{\mathrm{Sym}(\mathbb{X}_1|\mathbb{L})}\overline{E}_{a,b-s}\otimes_{\overline{E}_{a,b-s}}\overline{E}_{a,(b-s,s)}\,.\end{aligned}$$

Since

$$\mathrm{Hom}_{\overline{\mathcal{V}}_{a,b-s}}(X,\mathbf{1}_{a,b-s})\cong\mathrm{Hom}_{\mathcal{C}_{a,b-s}}(X,\mathbf{1}_{a,b-s})\otimes_{\mathrm{Sym}(\mathbb{X}_1,\mathbb{L})}\overline{E}_{a,b-s}$$

this implies our result. Indeed, it remains to see that the isomorphism is an isomorphism of complexes, and this follows from Definition 6.18, which implies that the differential on both complexes is induced from that on X. □

Corollary 7.17. *There is an $\overline{E}_{a,b}$-linear homotopy equivalence*

$$\mathrm{Hom}_{\overline{\mathcal{V}}_{a,b}}(I^{(s)}(\mathbf{1}_{a,b-s}),\mathbf{1}_{a,b})\xrightarrow{\simeq}\mathrm{Hom}_{\overline{\mathcal{V}}_{a,b}}(I^{(s)}(\mathcal{V}\mathsf{FT}_{a,b-s}),\mathbf{1}_{a,b})$$

defined by sending $f\mapsto f\circ I^{(s)}(\Sigma_{a,b-s})$.

Proof. The preceding lemma gives the commutative diagram

$$\begin{array}{ccc}\mathbf{q}^{-s(b-s)}\,\mathrm{Hom}_{\overline{\mathcal{V}}_{a,b-s}}(\mathbf{1}_{a,b-s},\mathbf{1}_{a,b-s})\otimes_{\overline{E}_{a,b-s}}\overline{E}_{a,(b-s,s)} & \xrightarrow{\cong} & \mathrm{Hom}_{\overline{\mathcal{V}}_{a,b}}(I^{(s)}(\mathbf{1}_{a,b-s}),\mathbf{1}_{a,b})\\ \big\downarrow{\scriptstyle(-)\circ\Sigma_{a,b-s}} & & \big\downarrow{\scriptstyle(-)\circ I^{(s)}(\Sigma_{a,b-s})}\\ \mathbf{q}^{-s(b-s)}\,\mathrm{Hom}_{\overline{\mathcal{V}}_{a,b-s}}(\mathcal{V}\mathsf{FT}_{a,b-s},\mathbf{1}_{a,b-s})\otimes_{\overline{E}_{a,b-s}}\overline{E}_{a,(b-s,s)} & \xrightarrow{\cong} & \mathrm{Hom}_{\overline{\mathcal{V}}_{a,b}}(I^{(s)}(\mathcal{V}\mathsf{FT}_{a,b-s}),\mathbf{1}_{a,b})\,.\end{array}$$

Proposition 7.2 gives that the left vertical map is a homotopy equivalence, thus the right vertical map is as well. □

We now have a curved analogue of Lemma 7.15.

Lemma 7.18. *The square*

$$\begin{array}{ccc}I^{(s)}(\mathcal{V}\mathsf{FT}_{a,b-s}) & \xrightarrow{\delta^c+\Delta^c} & I^{(s+1)}(\mathcal{V}\mathsf{FT}_{a,b-s-1})\\ \big\downarrow{\scriptstyle I^{(s)}(\Sigma_{a,b-s})} & & \big\downarrow{\scriptstyle I^{(s+1)}(\Sigma_{a,b-s-1})}\\ I^{(s)}(\mathbf{1}_{a,b-s}) & \xrightarrow{d} & I^{(s+1)}(\mathbf{1}_{a,b-s-1})\end{array} \tag{124}$$

in $\overline{\mathcal{V}}_{a,b}$ commutes up to homotopy.

Proof. First, observe that (124) specializes to (122), thus by Lemma 7.15 we have that the chain maps $d \circ I^{(s)}(\Sigma_{a,b-s})$ and $I^{(s+1)}(\Sigma_{a,b-s}) \circ (\delta^c + \Delta^c)$ are curved lifts in $\mathrm{Hom}_{\overline{\mathcal{V}}_{a,b}}\big(I^{(s)}(\mathcal{V}\mathsf{FT}_{a,b-s}), I^{(s+1)}(\mathbf{1}_{a,b-s-1})\big)$ of the same morphism in $\mathrm{Hom}_{\mathcal{C}_{a,b}}\big(I^{(s)}(\mathsf{FT}_{a,b-s}), I^{(s+1)}(\mathbf{1}_{a,b-s-1})\big)$. Hence, the morphism

$$d \circ I^{(s)}(\Sigma_{a,b-s}) - I^{(s+1)}(\Sigma_{a,b-s-1}) \circ (\delta^c + \Delta^c)$$

is a closed, degree-zero element of the $\mathbb{V}$-irrelevant submodule of $\mathrm{Hom}_{\overline{\mathcal{V}}_{a,b}}\big(I^{(s)}(\mathcal{V}\mathsf{FT}_{a,b-s}), I^{(s+1)}(\mathbf{1}_{a,b-s-1})\big)$.

Now, since

$$I^{(s+1)}(\mathbf{1}_{a,b-s-1}) = \mathrm{Dig}^{s+1}_{a,b} \cong \bigoplus_{\left[\begin{smallmatrix} b \\ s+1 \end{smallmatrix}\right]} \mathbf{1}_{a,b}\,,$$

Corollary 7.17 gives an $\overline{E}_{a,b}$-linear homotopy equivalence

$$\mathrm{Hom}_{\overline{\mathcal{V}}_{a,b}}\big(I^{(s)}(\mathcal{V}\mathsf{FT}_{a,b-s}), I^{(s+1)}(\mathbf{1}_{a,b-s-1})\big) \simeq \mathrm{Hom}_{\overline{\mathcal{V}}_{a,b}}\big(I^{(s)}(\mathbf{1}_{a,b-s}), I^{(s+1)}(\mathbf{1}_{a,b-s-1})\big)\,.$$

As in Remark 7.3, any degree-zero element in the latter that is $\mathbb{V}$-irrelevant is zero, hence null-homotopic. Thus

$$d \circ I^{(s)}(\Sigma_{a,b-s}) - I^{(s+1)}(\Sigma_{a,b-s-1}) \circ (\delta^c + \Delta^c) \sim 0$$

in $\mathrm{Hom}_{\overline{\mathcal{V}}_{a,b}}\big(I^{(s)}(\mathcal{V}\mathsf{FT}_{a,b-s}), I^{(s+1)}(\mathbf{1}_{a,b-s-1})\big)$, as desired. □

Using this, we can now give the following.

Proof of Theorem 7.14. We will prove that there exist (strictly) commutative squares of the form

$$\begin{array}{ccc} I^{(s)}(\mathcal{V}\mathsf{FT}_{a,b-s}) & \xrightarrow{\delta^c+\Delta^c} & I^{(s+1)}(\mathcal{V}\mathsf{FT}_{a,b-s-1}) \\ \downarrow{\scriptstyle g_s} & & \downarrow{\scriptstyle g_{s+1}} \\ I^{(s)}(\mathbf{1}_{a,b-s}) & \xrightarrow{d} & I^{(s+1)}(\mathbf{1}_{a,b-s-1}) \end{array} \tag{125}$$

wherein $g_s \sim I^{(s)}(\Sigma_{a,b-s})$. Commutativity of these diagrams then implies that the g_s assemble to give the requisite closed morphism. Set

$$D_s := \delta^c + \Delta^c \in \mathrm{Hom}_{\overline{\mathcal{V}}_{a,b}}\big(I^{(s)}(\mathcal{V}\mathsf{FT}_{a,b-s}), I^{(s+1)}(\mathcal{V}\mathsf{FT}_{a,b-s})\big)\,,$$

and let $k_s \in \mathrm{Hom}_{\overline{\mathcal{V}}_{a,b}}\big(I^{(s)}(\mathbf{1}_{a,b-s}), I^{(s-1)}(\mathbf{1}_{a,b-s+1})\big)$ be the homotopy from Lemma 7.13. Define

$$g_s := d_{s-1} \circ k_s \circ I^{(s)}(\Sigma_{a,b-s}) + k_{s+1} \circ I^{(s+1)}(\Sigma_{a,b-s-1}) \circ D_s\,,$$

then we compute that

$$d_s \circ g_s = d_s \circ k_{s+1} \circ I^{(s+1)}(\Sigma_{a,b-s-1}) \circ D_s = g_{s+1} \circ D_s$$

so the square (125) indeed strictly commutes. Finally, Lemma 7.18 gives that $d_s \circ I^{(s)}(\Sigma_{a,b-s}) \sim I^{(s+1)}(\Sigma_{a,b-s-1}) \circ D_s$, so

$$\begin{aligned} g_s &= d_{s-1} \circ k_s \circ I^{(s)}(\Sigma_{a,b-s}) + k_{s+1} \circ I^{(s+1)}(\Sigma_{a,b-s-1}) \circ D_s \\ &\sim d_{s-1} \circ k_s \circ I^{(s)}(\Sigma_{a,b-s}) + k_{s+1} \circ d_s \circ I^{(s)}(\Sigma_{a,b-s}) = I^{(s)}(\Sigma_{a,b-s}) \end{aligned}$$

as desired. □

8. Colored full twists and Hilbert schemes

As conjectured in [GNR21] and shown in [GH], the relation between Soergel bimodules and Hilbert schemes is mediated by the (positive) full twist braid. In this section, we speculate on the extension to the colored setting, culminating in Conjecture 8.15. In the following §9, we establish this conjecture in the 2-strand case.

8.1. **The full twist ideals.** Let $\boldsymbol{b} = (b_1, \ldots, b_m)$ be an object in SSBim. Extending the notation from §4.8, we let $\mathcal{C}_{\boldsymbol{b}} := \mathcal{C}({}_{\boldsymbol{b}}\mathrm{SSBim}_{\boldsymbol{b}})$, and denote the corresponding alphabets of cardinality b_i acting on the left and right by $\mathbb{X}_i$ and $\mathbb{X}'_i$, respectively (for $1 \leq i \leq m$). Following the notation in (51), we denote the corresponding dg category of curved complexes by $\mathcal{V}_{\boldsymbol{b}}$. Recall that the deformation parameters therein are denoted by $\mathbb{V}_i = \{v_{i,1}, \ldots, v_{i,b_i}\}$ and the curvature element is

$$(\delta_X + \Delta)^2 = \sum_{i=1}^{m} \sum_{r=1}^{b_i} h_r(\mathbb{X}_i - \mathbb{X}'_i) v_{i,r} \,.$$

Let $\mathrm{FT}_{\boldsymbol{b}} \in \mathcal{C}_{\boldsymbol{b}}$ denote the Rickard complex associated to the $\boldsymbol{b}$-colored positive full twist braid, and let $\mathcal{V}\mathrm{FT}_{\boldsymbol{b}}$ denote its (unique, up to homotopy) lift to $\mathcal{V}_{\boldsymbol{b}}$. As in Remark 7.5, we may construct a splitting map $\Sigma_{\boldsymbol{b}} \colon \mathcal{V}\mathrm{FT}_{\boldsymbol{b}} \to \mathbf{1}_{\boldsymbol{b}}$, which a priori depends on the presentation of the full twist braid as a product of generators.

Conjecture 8.1. *Precomposing with $\Sigma_{\boldsymbol{b}}$ defines a homotopy equivalence of* Hom*-complexes* $\mathrm{End}_{\mathcal{V}_{\boldsymbol{b}}}(\mathbf{1}_{\boldsymbol{b}}) \to \mathrm{Hom}_{\mathcal{V}_{\boldsymbol{b}}}(\mathcal{V}\mathrm{FT}_{\boldsymbol{b}}, \mathbf{1}_{\boldsymbol{b}})$.

One consequence of this conjecture would be that the splitting map $\Sigma_{\boldsymbol{b}}$ is unique up to homotopy and nonzero scalar. One may fix the scalar by prescribing the restriction of $\Sigma_{\boldsymbol{b}}$ to the term in cohomological degree zero. For the remainder of §8, we will assume Conjecture 8.1 and refer to $\Sigma_{\boldsymbol{b}}$ as *the* splitting map.

We now introduce the main objects of interest.

Definition 8.2. Extending (115), we let

$$E_{\boldsymbol{b}} := \mathrm{End}_{\mathcal{V}_{\boldsymbol{b}}}(\mathbf{1}_{\boldsymbol{b}}) = \mathrm{Sym}(\mathbb{X}_1 | \cdots | \mathbb{X}_m)[\mathbb{V}_1, \ldots, \mathbb{V}_m]$$

and set

$$M_{\boldsymbol{b}} := \mathrm{Hom}_{\mathcal{V}_{\boldsymbol{b}}}(\mathbf{1}_{\boldsymbol{b}}, \mathcal{V}\mathrm{FT}_{\boldsymbol{b}}) \,,$$

which we regard as a (differential) bigraded $E_{\boldsymbol{b}}$-module.

Note that $M_{\boldsymbol{b}}$ is a bigraded complex whose homology $H(M_{\boldsymbol{b}})$ is the lowest Hochschild-degree summand of the deformed homology of the $\boldsymbol{b}$-colored (m, m)-torus link $T(m, m; \boldsymbol{b})$, up to a $\mathbf{q}$-shift. We can also regard $E_{\boldsymbol{b}}$ as a bigraded complex with zero differential.

The splitting map $\Sigma_{\boldsymbol{b}} \colon \mathcal{V}\mathrm{FT}_{\boldsymbol{b}} \to \mathbf{1}_{\boldsymbol{b}}$ from Remark 7.5 induces an $E_{\boldsymbol{b}}$-linear morphism

$$M_{\boldsymbol{b}} \xrightarrow{\Sigma_{\boldsymbol{b}} \circ -} E_{\boldsymbol{b}}, \tag{126}$$

which is a chain map, since $\Sigma_{\boldsymbol{b}}$ is degree-zero and closed.

Definition 8.3. Let $H(\Sigma_{\boldsymbol{b}}) \colon H(M_{\boldsymbol{b}}) \to E_{\boldsymbol{b}}$ denote the map induced on homology by the chain map (126). We set $J_{\boldsymbol{b}} := \mathrm{im}(H(\Sigma_{\boldsymbol{b}}))$ and call $J_{\boldsymbol{b}} \lhd E_{\boldsymbol{b}}$ the *$\boldsymbol{b}$-colored full twist ideal*. (By construction, it is an ideal in $E_{\boldsymbol{b}}$.)

Remark 8.4. Recall that the (two strand) splitting map $\Sigma_{a,b} \colon \mathcal{V}\mathrm{FT}_{a,b} \to \mathbf{1}_{a,b}$ is only canonical only up to homotopy. Since the general splitting map is built from $\Sigma_{a,b}$'s, replacing the latter with a homotopic map would in turn replace $\Sigma_{\boldsymbol{b}}$ with a homotopic map. However, since $E_{\boldsymbol{b}}$ has zero differential, the full twist ideal $J_{\boldsymbol{b}}$ remains well-defined, regardless of our specific model for $\Sigma_{a,b}$.

It is an important problem to compute $H(M_{\boldsymbol{b}})$ and $J_{\boldsymbol{b}}$ explicitly. Indeed, in the case that $\boldsymbol{b} = 1^m := (1, \ldots, 1)$, they were shown to be isomorphic, and the latter ideal was described explicitly by E. Gorsky and the first-named author [GH]. In turn, this computation was used to provide an explicit connection between triply-graded Khovanov–Rozansky homology and the geometry of the Hilbert scheme of points in $\mathbb{C}^2$. As such, we anticipate that an explicit presentation of $J_{\boldsymbol{b}}$ for general $\boldsymbol{b}$ will provide an analogous relation between *colored* Khovanov–Rozansky homology and the geometry of the Hilbert scheme.

The explicit description of $H(M_{1^m}) \cong J_{1^m}$ in [GH] relies on work of Elias and the first-named author [EH19]. Therein, they show that the *uncolored* (m,m)-torus link $T(m,m)$ is *parity* (Definition 5.51). Based on this, and Conjecture 1.24, we propose the following.

Conjecture 8.5. *The $\boldsymbol{b}$-colored (m,m)-torus link $T(m,m;\boldsymbol{b})$ is parity. More generally, all colored positive torus links are parity.*

As Theorem 5.52 shows, the following is an immediate consequence of parity.

Proposition 8.6. *If Conjecture 8.5 holds, then there is an isomorphism of bigraded $\mathbb{Q}[\mathbb{V}]$-modules:*

$$\mathcal{Y}H_{\mathrm{KR}}(T(m,m;\boldsymbol{b})) \cong H_{\mathrm{KR}}(T(m,m;\boldsymbol{b})) \otimes \mathbb{Q}[\mathbb{V}]\,. \qquad \square$$

We next record an important consequence of the expected "flatness" statement in Proposition 8.6.

Corollary 8.7. *Let $T(m,m;\boldsymbol{b})$ be the $\boldsymbol{b}$-colored (m,m)-torus link, and let $U(\boldsymbol{b})$ be the $\boldsymbol{b}$-colored unlink. If Conjecture 8.5 holds, then the map $\mathcal{Y}H_{\mathrm{KR}}(\Sigma_{\boldsymbol{b}})\colon \mathcal{Y}H_{\mathrm{KR}}\big(T(m,m;\boldsymbol{b})\big) \to \mathcal{Y}H_{\mathrm{KR}}\big(U(\boldsymbol{b})\big)$ induced by the splitting map $\Sigma_{\boldsymbol{b}}\colon \mathcal{V}\mathrm{FT}_{\boldsymbol{b}} \to \mathbf{1}_{\boldsymbol{b}}$ is injective. In particular, the map $H(\Sigma_{\boldsymbol{b}})\colon H(M_{\boldsymbol{b}}) \to E_{\boldsymbol{b}}$ is injective and thus an isomorphism onto its image $J_{\boldsymbol{b}} \subset E_{\boldsymbol{b}}$.*

Proof. Consider the $m=2$ case (i.e. $\boldsymbol{b}=(a,b)$). Corollary 7.11 shows that there exists a morphism $\overline{\psi}_{a,b} \in \mathrm{Hom}_{\overline{\mathcal{V}}_{a,b}}(\mathbf{1}_{a,b}, \mathcal{V}\mathrm{FT}_{a,b})$ that satisfies

$$\overline{\psi}_{a,b} \circ \Sigma_{a,b} \sim \overline{y}_{a+1} \cdots \overline{y}_{a+b} \cdot \mathrm{id}_{\mathcal{V}\mathrm{FT}_{a,b}}$$

(in our current notation, the $\overline{y}$ variables correspond to the alphabet $\mathbb{V}_2$ corresponding to the b-labeled strand). It follows that the induced map on homology satisfies

$$\mathcal{Y}H_{\mathrm{KR}}(\overline{\psi}_{a,b}) \circ \mathcal{Y}H_{\mathrm{KR}}(\Sigma_{a,b}) = \overline{y}_{a+1} \cdots \overline{y}_{a+b} \cdot \mathrm{id}_{\mathcal{Y}H_{\mathrm{KR}}(T(2,2;(a,b)))}\,.$$

Assuming Conjecture 8.5, Proposition 8.6 implies that $H(M_{(a,b)})$ is $\mathbb{V}$-torsion free. Since

$$\overline{y}_j = \sum_{k=1}^{b} x_j^{k-1} \bar{v}_k$$

we have that $\overline{y}_{a+1} \cdots \overline{y}_{a+b}$ is the sum of the monomial $(\bar{v}_1)^b$ and terms involving the alphabet $\mathbb{X}_2$ that are lower order with respect to the alphabet $\overline{\mathbb{V}}$ (if we impose the monomial order $\bar{v}_1 > \cdots > \bar{v}_b$). Since[13] $\mathbb{Q}[\mathbb{V}] = \mathbb{Q}[\mathbb{V}_L^{(a)}, \mathbb{V}_R^{(b)}] = \mathbb{Q}[\mathbb{V}_L^{(a)}, \overline{\mathbb{V}}]$, this implies that multiplication by $\overline{y}_{a+1} \cdots \overline{y}_{a+b}$ is injective. Hence, $\mathcal{Y}H_{\mathrm{KR}}(\Sigma_{a,b})$ is injective as well.

The general case follows analogously, using the description of the full twist braid as the horizontal composition of two-strand full twists from Remark 7.5. See also Theorem 10.10 below for a more general result. $\square$

Assuming the validity of Conjecture 8.5, the computation of the deformed colored homology of the colored torus link $T(m,m;\boldsymbol{b})$ in lowest Hochschild-degree amounts to finding a presentation of the ideal $J_{\boldsymbol{b}} \lhd E_{\boldsymbol{b}}$. We now aim to formulate a precise conjectural description of $J_{\boldsymbol{b}}$. To help motivate this, we now recall the description in the uncolored case from [GH]. We begin by establishing some conventions.

Convention 8.8. As above, we write 1^N for the sequence $(1,\dots,1)$ of length N. If $\boldsymbol{b}=1^N$ then each alphabet $\mathbb{X}_i$ consists of a single variable x_i and we have a single deformation parameter for each i that we denote by $y_i := v_{i,1}$. The latter thus satisfy $\mathrm{wt}(y_i) = \mathbf{q}^{-2}\mathbf{t}^2$. In this case, define the *total alphabets* by

$$\mathbb{X} = \bigsqcup_{i=1}^{m} \mathbb{X}_i = \{x_1,\dots,x_N\}\,, \quad \mathbb{Y} = \bigsqcup_{i=1}^{m} \mathbb{V}_i = \{y_1,\dots,y_N\}$$

[13] In the 2-strand notation established in §4.8, we denote $V_1 = \mathbb{V}_L^{(a)}$ and $\mathbb{V}_2 = \mathbb{V}_R^{(b)}$.

so $E_{1^N} = \mathbb{Q}[\mathbb{X}, \mathbb{Y}]$.

Convention 8.9. Below, we will consider various algebras E and ideals $I \lhd E$ that are generated by collections of elements $S \subset E$. Since we will consider different algebras that contain the same subsets S, we will denote such ideals by $I = E \cdot S$ in order to make clear the algebra in which each ideal lives.

The following combines the torus link computations of [EH19] (see also [Mel17, HM19]) with [GH, Theorem 6.16 and Corollary 6.17]. (We take the liberty of renaming J_N from [GH] as J_{1^N}, in order to avoid clashes with our notation for the colored case.)

Theorem 8.10. *The parity conjecture (Conjecture 8.5) holds when $\boldsymbol{b} = 1^N$. Further, the full twist ideal J_{1^N} is equal to the ideal $I_{1^N} \lhd \mathbb{Q}[\mathbb{X}, \mathbb{Y}]$ generated by polynomials in $\mathbb{Q}[\mathbb{X}, \mathbb{Y}]$ that are antisymmetric with respect to the diagonal $\mathfrak{S}_N$-action.*

Inspired by this theorem, and also our results in §9 below, we now formulate a conjecture concerning the ideals $J_{\boldsymbol{b}}$ for general $\boldsymbol{b} = (b_1, \ldots, b_m)$ that extends Theorem 8.10.

Convention 8.11. We extend Convention 8.8 to the case of general $\boldsymbol{b} = (b_1, \ldots, b_m)$ by defining the *total alphabets*

$$\mathbb{X} := \bigsqcup_{i=1}^{m} \mathbb{X}_i\,, \quad \mathbb{V} = \bigsqcup_{i=1}^{m} \mathbb{V}_i\,.$$

We write $\mathbb{X} = \{x_1, \ldots, x_N\}$ where $N := b_1 + \cdots + b_m$ and thus identify the alphabets $\mathbb{X}_i$ (of cardinality b_i) with the subsets

$$\mathbb{X}_i = \{x_j \mid b_1 + \cdots + b_{i-1} < j \le b_1 + \cdots + b_i\} \subset \mathbb{X}\,.$$

It follows that

$$E_{\boldsymbol{b}} = \mathrm{Sym}(\mathbb{X}_1|\cdots|\mathbb{X}_m)[\mathbb{V}_1, \ldots, \mathbb{V}_m] = \mathbb{Q}[\mathbb{X}, \mathbb{V}]^{\mathfrak{S}_{b_1} \times \cdots \times \mathfrak{S}_{b_m}}\,,$$

provided we let $\mathfrak{S}_{b_1} \times \cdots \times \mathfrak{S}_{b_m}$ act on the alphabet $\mathbb{V}$ trivially.

For the duration of this section, let $\boldsymbol{b} = (b_1, \ldots, b_m)$ and set $N = b_1 + \cdots + b_m$. Given an index $1 \le j \le N$, we write $\varpi(j) = i$ if $b_1 + \cdots + b_{i-1} + 1 \le j \le b_1 + \cdots + b_{i-1} + b_i$.

Definition 8.12. Introduce alphabets

$$\mathbb{Y}_i := \{y_{b_1+\cdots+b_{i-1}+1}, \ldots, y_{b_1+\cdots+b_{i-1}+b_i}\} \subset \mathbb{Y} := \{y_1, \ldots, y_N\}\,,$$

and regard $\mathbb{Q}[\mathbb{X}_i, \mathbb{Y}_i]$ as a subalgebra of $\mathbb{Q}[\mathbb{X}_i, \mathbb{V}_i]$ via the assignment

$$y_j \mapsto \sum_{r=1}^{b_{\varpi(j)}} x_j^{r-1} v_{\varpi(j),r} \tag{127}$$

that is analogous to (55).

Let $\mathfrak{S}_N$ act on $\mathbb{Q}[\mathbb{X}, \mathbb{Y}]$ by simultaneously permuting the variables x_i, y_i.

Lemma 8.13. *Let $\rho\colon \mathbb{Q}[\mathbb{X}, \mathbb{Y}] \to \mathbb{Q}[\mathbb{X}, \mathbb{V}]$ denote the subalgebra inclusion given by the assignment (127). If $f(\mathbb{X}, \mathbb{Y})$ is antisymmetric with respect to the $\mathfrak{S}_N$ action on $\mathbb{Q}[\mathbb{X}, \mathbb{Y}]$, then $\rho(f)(\mathbb{X}, \mathbb{V})$ is divisible by the product of Vandermonde determinants $\Delta(\mathbb{X}_1) \cdots \Delta(\mathbb{X}_m)$.*

Proof. The inclusion $\mathbb{Q}[\mathbb{X}, \mathbb{Y}] \hookrightarrow \mathbb{Q}[\mathbb{X}, \mathbb{V}]$ given by the assignment (127) is $\mathfrak{S}_{b_1} \times \cdots \times \mathfrak{S}_{b_m}$-equivariant, provided we let this group act trivially on the alphabet $\mathbb{V}$. Assume $f \in \mathbb{Q}[\mathbb{X}, \mathbb{Y}]$ is antisymmetric with respect to $\mathfrak{S}_N$. Then f is antisymmetric with respect to $\mathfrak{S}_{b_1} \times \cdots \times \mathfrak{S}_{b_m}$, and the same is true of $\rho(f)$. Thus, each $\mathbb{V}$-coefficient of $\rho(f)$ is an antisymmetric polynomial in the alphabets $\mathbb{X}_i$, hence is divisible by each $\Delta(\mathbb{X}_i)$. The $\Delta(\mathbb{X}_i)$ are relatively prime, as they live in distinct tensor factors of $\mathbb{Q}[\mathbb{X}, \mathbb{V}] = \bigotimes_i \mathbb{Q}[\mathbb{X}_i, \mathbb{V}_i]$. It follows that $\rho(f)$ is divisible by $\Delta(\mathbb{X}_1) \cdots \Delta(\mathbb{X}_m)$. ☐

Definition 8.14. Define the ideal

$$I_{\boldsymbol{b}} := E_{\boldsymbol{b}} \cdot \left\{ \frac{f(\mathbb{X},\mathbb{Y})}{\Delta(\mathbb{X}_1)\cdots\Delta(\mathbb{X}_m)} \mid f \in \mathbb{Q}[\mathbb{X},\mathbb{Y}] \text{ is antisymmetric for } \mathfrak{S}_N \right\} \tag{128}$$

(following Convention 8.9).

Recall from §2.4 that any set S of monic monomials in $\mathbb{Q}[x,y]$ with $|S| = N$ determines a Haiman determinant $\Delta_S(\mathbb{X},\mathbb{Y}) \in \mathbb{Q}[\mathbb{X},\mathbb{Y}]$. The latter are antisymmetric for the diagonal $\mathfrak{S}_N$ action on $\mathbb{Q}[\mathbb{X},\mathbb{Y}]$, thus determine elements

$$\frac{\Delta_S(\mathbb{X},\mathbb{Y})}{\Delta(\mathbb{X}_1)\cdots\Delta(\mathbb{X}_m)} \in I_{\boldsymbol{b}}\,. \tag{129}$$

In §9 below, we show that when $m = 2$ (i.e. when $\boldsymbol{b} = (a,b)$), certain elements of the form (129) generate the ideal $J_{\boldsymbol{b}}$. Motivated by this, we propose the following as a $\boldsymbol{b}$-colored analogue of Theorem 8.10.

Conjecture 8.15. *The $\boldsymbol{b}$-colored full twist ideal $J_{\boldsymbol{b}} \lhd E_{\boldsymbol{b}}$ agrees with the ideal $I_{\boldsymbol{b}}$ from Definition 8.14.*

When $\boldsymbol{b} = 1^N$, this is simply a restatement of Theorem 8.10. We will prove the $\boldsymbol{b} = (a,b)$ case of Conjecture 8.15 below in Theorem 9.33 by pairing explicit computations of certain Haiman determinants (from §2.4) with an elaborate inductive argument. In this case, Corollary 9.40 identifies a specific set of 2^b Haiman determinants that generate $I_{a,b}$.

One may study the colored homology of the full twist by comparison with the 2-strand case, as was done in the uncolored setting in [GH]. To make this precise, let $A_{i,j}$ be the generator of the pure braid group described in Remark 7.5, which we will regard as a colored braid via $\boldsymbol{b}$. Let $\mathcal{V}A_{i,j}$ denote the curved Rickard complex for $A_{i,j}$. Consider the ideal in $E_{\boldsymbol{b}}$ generated by $J_{b_i,b_j} \subset \mathrm{Sym}(\mathbb{X}_i|\mathbb{X}_j)[\mathbb{V}_i,\mathbb{V}_j] \subset E_{\boldsymbol{b}}$, which we will denote $E_{\boldsymbol{b}}J_{b_i,b_j}$. It is straightforward to check that $E_{\boldsymbol{b}}J_{b_i,b_j}$ is the ideal associated to $A_{i,j}$; in other words $E_{\boldsymbol{b}}J_{b_i,b_j}$ is the image of the map

$$H(\mathrm{Hom}_{\mathcal{V}_{\boldsymbol{b}}}(\mathcal{V}A_{i,j},\mathbf{1}_{\boldsymbol{b}})) \to E_{\boldsymbol{b}}$$

induced by the splitting map $\mathcal{V}A_{i,j} \to \mathbf{1}_{\boldsymbol{b}}$. Canonicity of splitting maps implies that $\Sigma_{\boldsymbol{b}}\colon \mathcal{V}\mathrm{FT}_{\boldsymbol{b}} \to \mathbf{1}_{\boldsymbol{b}}$ factors as the composition

$$\mathcal{V}\mathrm{FT}_{\boldsymbol{b}} \to \mathcal{V}A_{i,j} \to \mathbf{1}_{\boldsymbol{b}}\,,$$

hence $J_{\boldsymbol{b}} \subset E_{\boldsymbol{b}}J_{b_i,b_j}$ for all $1 \le i < j \le m$.

Conjecture 8.16. *We have $J_{\boldsymbol{b}} = \bigcap_{i<j} E_{\boldsymbol{b}}J_{b_i,b_j}$.*

In the uncolored case $\boldsymbol{b} = (1,\ldots,1)$ this was proven in [GH, Theorem 6.16]. Combining Conjecture 8.15 and Conjecture 8.16 gives the following, purely algebraic, conjecture.

Conjecture 8.17. *We have an equality of ideals in E_b:*

$$I_{\boldsymbol{b}} = \bigcap_{1\le i<j\le m} E_{\boldsymbol{b}} \cdot \left\{ \frac{f(\mathbb{X},\mathbb{Y})}{\Delta(\mathbb{X}_i)\Delta(\mathbb{X}_j)} \mid f \in \mathbb{Q}[\mathbb{X},\mathbb{Y}] \text{ is antisymmetric for } \mathfrak{S}_{b_i+b_j} \right\},$$

where $\mathfrak{S}_{b_i+b_j}$ acts by simultaneously permuting variables within the alphabets $\mathbb{X}_i \cup \mathbb{X}_j$ and $\mathbb{Y}_i \cup \mathbb{Y}_j$.

In the uncolored case this was proven by Haiman; see Lemma 8.18 below.

8.2. **Full twists and Hilbert schemes.** Pioneering work of Haiman [Hai01] shows that the ideal

$$I_{1^N} = \mathbb{Q}[\mathbb{X},\mathbb{Y}]\cdot\{f(\mathbb{X},\mathbb{Y}) \mid f\in\mathbb{Q}[\mathbb{X},\mathbb{Y}] \text{ is antisymmetric for } \mathfrak{S}_N\}$$

appearing in Theorem 8.10 plays a crucial role in the study of the Hilbert scheme $\mathrm{Hilb}_N(\mathbb{C}^2)$ of N points in $\mathbb{C}^2$. Recall that the *isospectral Hilbert scheme* X_N is defined to be the reduced fiber product

$$\begin{array}{ccc} X_N & \longrightarrow & (\mathbb{C}^2)^N \\ \downarrow & & \downarrow \\ \mathrm{Hilb}_N(\mathbb{C}^2) & \longrightarrow & S^N(\mathbb{C}^2) \end{array}$$

and that in [Hai01, Proposition 3.4.2] it is shown that X_N is isomorphic to the blowup of $(\mathbb{C}^2)^N$ at the ideal $\mathbb{C}\otimes_{\mathbb{Q}} I_{1^N} \lhd \mathbb{C}[\mathbb{X},\mathbb{Y}]$. Haiman goes on to to show that X_N is normal, Cohen-Macaulay, and Gorenstein. The following seemingly (but not) straightforward result plays a role in these considerations, and will be used below.

Lemma 8.18 ([Hai01, Corollary 3.8.3])**.** $I_{1^n} = \bigcap_{1\le i<j\le N} \mathbb{Q}[\mathbb{X},\mathbb{Y}]\cdot\{x_i-x_j, y_i-y_j\}$.

Inspired by the connections between uncolored, triply-graded link homology and the isospectral Hilbert scheme X_N, we propose the following as a "colored" analogue of the latter.

Definition 8.19. For $\boldsymbol{b}=(b_1,\dots,b_m)$, let $X_{\boldsymbol{b}}$ be the blowup of $\mathrm{Spec}(E_{\boldsymbol{b}})$ at the ideal $I_{\boldsymbol{b}} \lhd E_{\boldsymbol{b}}$. In other words, set

$$\mathcal{A}_{\boldsymbol{b}} := \bigoplus_d I_{\boldsymbol{b}}^d z^d \subset E_{\boldsymbol{b}}[z]$$

and let $X_{\boldsymbol{b}} := \mathrm{Proj}(\mathcal{A}_{\boldsymbol{b}})$.

We expect that the relation between Soergel bimodules and the (isospectral) Hilbert scheme in [GNR21, OR, GH] has an analogue for singular Soergel bimodules, with $X_{\boldsymbol{b}}$ playing the role of "colored" isospectral Hilbert scheme. In particular, let $\mathcal{B}_{\boldsymbol{b}}$ be the colored "full-twist dg algebra",

$$\mathcal{B}_{\boldsymbol{b}} := \bigoplus_d \mathrm{Hom}_{\mathcal{V}_{\boldsymbol{b}}}(\mathbf{1}_{\boldsymbol{b}}, \mathcal{V}\mathrm{FT}_{\boldsymbol{b}}^d)$$

Post-composing with the splitting maps $\mathcal{V}\mathrm{FT}_{\boldsymbol{b}}^d \to \mathbf{1}_{\boldsymbol{b}}$ gives us a map of dg algebras $\mathcal{B}_{\boldsymbol{b}} \to \mathcal{A}_{\boldsymbol{b}}$.

Conjecture 8.20. *The map of dg algebras $\mathcal{B}_{\boldsymbol{b}} \to A_{\boldsymbol{b}}$ is a quasi-isomorphism.*

We save investigations along these lines for future work.

9. Homology of the colored Hopf link

In this section we prove Conjecture 8.5 and Conjecture 8.15 for the (positive) Hopf link, using the explicit description of the splitting map from §7.2, as well as the curved skein relation from §6. After the general setup from §8 we now return to the notational conventions for the 2-strand case that were established in §4.8.

We start by showing that the colored Hopf link is parity, which implies that the link splitting map to the corresponding colored unlink homology is injective. In §9.2 we compute generators for the Hopf link homology in lowest **a**-degree and in §9.3 we show that their images under the splitting map are (reduced) Haiman determinants. This implies the inclusion $J_{a,b}\subset I_{a,b}$ between the ideals from Definitions 8.3 and 8.14, in the 2-strand case. We will refer to the ideal $J_{a,b}$ as the *Hopf link ideal*, since it is isomorphic to the (lowest Hochschild summand of the) deformed colored homology of the (a,b)-colored Hopf link. In Sections 9.5 and 9.6, we establish the opposite inclusion, thus proving Conjecture

8.15 when $\boldsymbol{b} = (a,b)$. The argument proceeds via induction using a family of ideals $J_{a,(b-s,s)}$ that are related to the curved colored skein relation.

Since the positive (a,b)-colored Hopf link is isotopic to the (b,a)-colored Hopf link, without loss of generality we assume that $a \geq b$ for the duration.

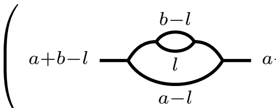
9.1. **Parity of the colored Hopf link.** Here, we compute the (undeformed) Khovanov–Rozansky homology of the (a,b)-colored Hopf link in lowest **a**-degree and show that it is supported in cohomological degrees congruent to ab modulo 2. In other words, we show that the colored Hopf link is *parity*. The argument is analogous to the one given in [Wed16, Example 4.14]. By Definition 3.23, we have

$$C_{b,a} \star C_{a,b} = \mathrm{tw}_{\delta \star \mathrm{id}} \left(\bigoplus_{l=0}^{b} \mathbf{q}^{-l} \mathbf{t}^{l} \mathsf{F}^{(b-l)} \mathsf{E}^{(a-l)} \star C_{a,b} \right).$$

We begin with the following computation.

Lemma 9.1. *There is a homotopy equivalence*

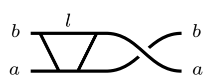
$$\mathrm{HH}_\bullet(\mathsf{F}^{(b-l)}\mathsf{E}^{(a-l)} \star C_{a,b}) \simeq \mathbf{a}^{-l}\mathbf{q}^{ab}\mathbf{t}^{l}\, \mathrm{HH}_\bullet \left(\quad \right)$$



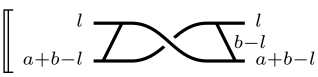
of complexes of bigraded vector spaces.

Proof. We compute

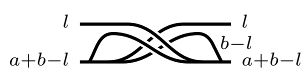
$$\mathrm{HH}_\bullet(\mathsf{F}^{(b-l)}\mathsf{E}^{(a-l)} \star C_{a,b}) = \mathrm{HH}_\bullet \left(\left[\!\left[\quad \right]\!\right] \right)$$



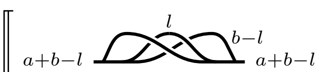
$$\cong \mathrm{HH}_\bullet \left(\left[\!\left[\quad \right]\!\right] \right)$$



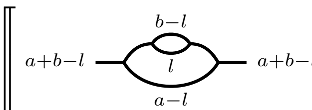
$$\simeq \mathrm{HH}_\bullet \left(\left[\!\left[\quad \right]\!\right] \right)$$



$$\simeq \mathbf{a}^{-l}\mathbf{q}^{l^2}\mathbf{t}^{l}\, \mathrm{HH}_\bullet \left(\left[\!\left[\quad \right]\!\right] \right)$$



$$\simeq \mathbf{a}^{-l}\mathbf{q}^{l^2+b(a-l)+l(b-l)}\mathbf{t}^{l}\, \mathrm{HH}_\bullet \left(\left[\!\left[\quad \right]\!\right] \right)$$



where we have used Propositions 3.26 and 5.3 and Lemma 5.12. □

We also record the following useful result, which has a completely analogous proof.

Proposition 9.2. *The complex of bigraded vector spaces* $\mathrm{HH}^\bullet(C_{b,a}^\vee \star \mathsf{F}^{(a-b+l)}\mathsf{E}^{(l)})$ *has homology supported in strictly positive* **a**-*degrees when* $0 \leq l \leq b-1$.

Corollary 9.3. *For* $0 \leq l \leq b-1$, *there is a homotopy equivalence of dg* $\mathrm{End}_{\mathcal{C}_{a,b}}(\mathbf{1}_{a,b})$-*modules* $\mathrm{Hom}_{\mathcal{C}_{a,b}}(\mathbf{1}_{a,b}\mathsf{F}^{(l)}\mathsf{E}^{(a-b+l)} \star C_{a,b}, \mathbf{1}_{a,b}) \simeq 0$.

Proof. The complex $\mathrm{Hom}_{\mathcal{C}_{a,b}}(\mathbf{1}_{a,b}\mathsf{F}^{(l)}\mathsf{E}^{(a-b+l)} \star C_{a,b}, \mathbf{1}_{a,b})$ inherits an action of $\mathrm{End}_{\mathcal{C}_{a,b}}(\mathbf{1}_{a,b})$ via its central action on the left in $\mathcal{C}_{a,b}$. We then have:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}_{a,b}}(\mathbf{1}_{a,b}\mathsf{F}^{(l)}\mathsf{E}^{(a-b+l)} \star C_{a,b}, \mathbf{1}_{a,b}) &\cong \mathrm{Hom}_{\mathcal{C}_{a,b}}(\mathbf{1}_{a,b}\mathsf{F}^{(l)}\mathsf{E}^{(a-b+l)}, C_{b,a}^\vee) \\
&\cong \mathrm{Hom}_{\mathcal{C}_{a,b}}(\mathbf{1}_{a,b}, C_{b,a}^\vee \star \mathsf{F}^{(a-b+l)}\mathsf{E}^{(l)}) \\
&= \mathrm{HH}^0(C_{b,a}^\vee \star \mathsf{F}^{(a-b+l)}\mathsf{E}^{(l)}) \simeq 0
\end{aligned}$$

where we have used duality, Proposition 3.19, and Proposition 9.2. □

The following is now an immediate consequence of Lemma 9.1 together with Proposition 4.1.

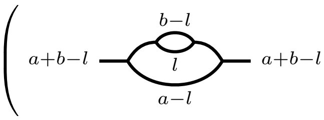
Proposition 9.4. *The (a,b)-colored Hopf link is parity.*

Proof. Up to a global shift of magnitude $\mathbf{a}^{ab}\mathbf{q}^{-2ab}\mathbf{t}^{-ab}$ (see Definition 5.16), the complex $C_{\mathrm{KR}}(\widehat{\mathrm{FT}_{a,b}})$ associated to the closure of the (a,b)-colored full twist is given by

$$\mathrm{HH}_\bullet(C_{b,a} \star C_{a,b}) = \mathrm{tw}_\alpha \left(\bigoplus_{l=0}^{b} \mathbf{q}^{-l}\mathbf{t}^{l}\, \mathrm{HH}_\bullet(\mathsf{F}^{(b-l)}\mathsf{E}^{(a-l)} \star C_{a,b}) \right)$$

for some twist α which strictly increases the index l. Here, the right-hand side is a finite one-sided twisted complex (see Remark 4.2), hence Proposition 4.1 (the homological perturbation lemma) allows us to apply the homotopy equivalences from Lemma 9.1 term-wise. We thus obtain

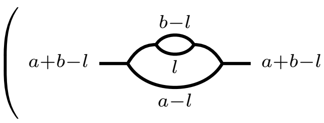
$$\mathrm{HH}_\bullet(C_{b,a} \star C_{a,b}) \simeq \mathrm{tw}_\beta \left(\bigoplus_{l=0}^{b} \mathbf{a}^{-l}\mathbf{q}^{ab-l}\mathbf{t}^{2l}\, \mathrm{HH}_\bullet \left(\right) \right)$$



for some Maurer–Cartan element β. Since the Maurer–Cartan element β has $\mathbf{t}$-degree one, it must be zero since it is acting on a complex which is supported in even cohomological degrees. □

For the reader's convenience, we include also the Hochschild cohomology version of the above (in particular note that the shift $\mathbf{a}^{-l}$ disappears):

$$\mathrm{HH}^\bullet(C_{b,a} \star C_{a,b}) \;\simeq\; \bigoplus_{l\geq 0} \mathbf{q}^{2(a-l)(b-l)-2l}\mathbf{t}^{2l} \left(\mathbf{q}^{ab-l^2}\, \mathrm{HH}^\bullet \left(\right) \right)$$



Corollary 9.5. *Let $T(2,2;a,b)$ be the (a,b)-colored Hopf link, and let $U(a,b)$ be the (a,b)-colored unlink. The map $\mathcal{Y}H_{\mathrm{KR}}(\Sigma_{a,b})\colon \mathcal{Y}H_{\mathrm{KR}}(T(2,2;a,b) \to \mathcal{Y}H_{\mathrm{KR}}(U(a,b))$ is injective. In particular, the map $H(\Sigma_{a,b})\colon H(M_{a,b}) \to E_{a,b}$ is injective.*

Proof. This is an immediate consequence of Corollary 8.7. □

Below, we will compute the image of the map $H(\Sigma_{a,b})\colon H(M_{a,b}) \to E_{a,b}$. Recall from §8.1 that

$$E_{a,b} = \mathrm{End}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b})\,, \quad M_{a,b} = \mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}, \mathcal{V}\mathsf{FT}_{a,b})\,,$$

and that the homology of the latter is isomorphic to the lowest Hochschild degree summand of $\mathcal{Y}H_{\mathrm{KR}}(T(2,2;a,b)$ (up to shift).

9.2. **Corner maps.** It will be useful to have an explicit set of generators for the homology of $M_{a,b} = \mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}, \mathcal{V}\mathsf{FT}_{a,b})$. Recall from Definition 6.12 that

$$\mathcal{V}\mathsf{FT}_{a,b} = \mathrm{tw}_{\delta^h+\delta^v+\Delta^v} \left(\bigoplus_{b\geq k\geq r\geq 0} P_{k,r,0} \right)$$

where $P_{k,r,0} := \mathbf{q}^{k(a-b+1)-2b}\mathbf{t}^{2b-k} W_k \otimes \wedge^r[\xi_1,\ldots,\xi_k]$. Motivated (visually) by (118), we will refer to the summands $P_{k,k,0} \subset \mathcal{V}\mathsf{FT}_{a,b}$ as *corners*. Reindexing by taking $k = b-l$, we have that

$$\begin{aligned} P_{b-l,b-l,0} &= \mathbf{q}^{(b-l)(a-b+1)-2b}\mathbf{t}^{b+l} W_k \otimes \xi_1\cdots\xi_{b-l} \\ &\cong \mathbf{q}^{(b-l)(a-b+1)-2b}\mathbf{t}^{b+l}\mathbf{q}^{(b-l)(b-l+1)}\mathbf{t}^{-b+l} W_{b-l} = \mathbf{q}^{(b-l)(a-l)-2l}\mathbf{t}^{2l} W_{b-l}. \end{aligned}$$

Definition 9.6. For $0 \le l \le b$ let $\mathrm{CM}_l\colon \mathrm{Hom}_{\mathrm{SSBim}}(\mathbf{1}_{a,b}, W_{b-l}) \to \mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}, \mathcal{V}\mathsf{FT}_{a,b})$ be the map sending $\varphi\colon \mathbf{1}_{a,b} \to W_{b-l}$ to the composite

$$\mathbf{1}_{a,b} \xrightarrow{\varphi} W_{b-l} \xrightarrow{\approx} \mathbf{q}^{(b-l)(a-l)-2l}\mathbf{t}^{2l} W_{b-l} \xrightarrow{\cong} P_{b-l,b-l,0} \hookrightarrow \mathcal{V}\mathsf{FT}_{a,b}\,,$$

where (as before) the map denoted $\approx$ is a slanted identity of W_{b-l}.

We can regard CM_l as a degree-zero map

$$\mathbf{q}^{(b-l)(a-l)-2l}\mathbf{t}^{2l}\,\mathrm{Hom}_{\mathrm{SSBim}}(\mathbf{1}_{a,b}, W_{b-l}) \to \mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}, \mathcal{V}\mathsf{FT}_{a,b})\,.$$

Lemma 9.7. *If we regard* $\mathrm{Hom}_{\mathrm{SSBim}}(\mathbf{1}_{a,b}, W_{b-l})$ *as a complex with zero differential, then the morphism* CM_l *from Definition 9.6 is closed. Furthermore, the induced map of complexes*

$$\bigoplus_{l=0}^{b} \mathbf{q}^{(b-l)(a-l)-2l}\mathbf{t}^{2l}\,\mathrm{Hom}_{\mathrm{SSBim}}(\mathbf{1}_{a,b}, W_{b-l}) \otimes \mathbb{Q}[\mathbb{V}] \xrightarrow{\bigoplus_l \mathrm{CM}_l} \mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}, \mathcal{V}\mathsf{FT}_{a,b})$$

is surjective in homology.

Proof. The differential of CM_l sends $\varphi \in \mathrm{Hom}_{\mathrm{SSBim}}(\mathbf{1}_{a,b}, W_{b-l})$ to $(\delta^h + \delta^v + \Delta^v)\circ\varphi$; here we identify $P_{b-l,b-l,0}$ with W_{b-l} (up to a shift). Each of the differentials Δ^v and δ^h restrict to zero at the corners, thus to see that CM_l is closed it suffices to show that $\delta^v \circ \varphi = 0$ for all $\varphi \in \mathrm{Hom}(\mathbf{1}_{a,b}, W_{b-l})$. This follows since δ^v is the Koszul differential associated to the action of $h_i(\mathbb{X}_2 - \mathbb{X}_2')$ on W_{b-l} for $1 \le i \le b-l$, and the central elements $h_i(\mathbb{X}_2 - \mathbb{X}_2')$ act by zero on $\mathrm{Hom}_{\mathrm{SSBim}}(\mathbf{1}_{a,b}, W_{b-l})$. This proves the first statement.

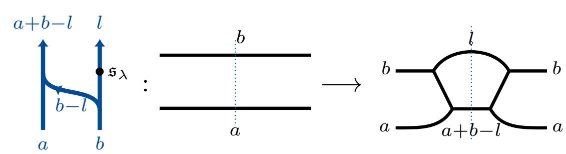
For the second statement, Corollary 8.7 implies that post-composing with the splitting map

$$\Sigma_{a,b}\colon \mathcal{V}\mathrm{FT}_{a,b} \to \mathbf{1}_{a,b}$$

gives a map $M_{a,b} = \mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}, \mathcal{V}\mathrm{FT}_{a,b}) \to \mathrm{End}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}) = E_{a,b}$ which is injective in homology. Thus $H(\Sigma_{a,b})\colon H(M_{a,b}) \to E_{a,b}$ is an isomorphism onto its image $J_{a,b}$. By Definition 7.8, the splitting map $\Sigma_{a,b}$ is supported on the corners, i.e. $\Sigma|_{P_{k,r,0}} = 0$ unless $k = r$. Thus, $J_{a,b}$ is spanned by elements of the form $\Sigma_{a,b} \circ \mathrm{CM}_l(\varphi)$, and the result follows from injectivity of $H(\Sigma_{a,b})$. □

Fix $0 \le l \le b$. We now explicitly describe the Hom-spaces $\mathrm{Hom}_{\mathrm{SSBim}}(\mathbf{1}_{a,b}, W_{b-l})$.

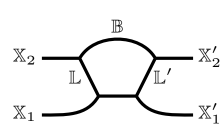
Definition 9.8. Consider the following elements of $\mathrm{Hom}_{\mathrm{SSBim}}(\mathbf{1}_{a,b}, W_{b-l})$,

$\varphi_{a,b,l}(\lambda) :=$ 

where $\lambda \in P(l, b-l)$ is a partition in the $l \times b-l$ rectangle.

Here, we have written $\varphi_{a,b,l}(\lambda)$ using the perpendicular graphical calculus from §3.4. In particular, taking $\lambda = \emptyset$ gives us the canonical map $\varphi_{a,b,l}(\emptyset)\colon \mathbf{1}_{a,b} \to W_{b-l}$ of weight $\mathbf{q}^{(a-l)(b-l)}$ given by **cr** (digon creation on the b-labeled strand) followed by **zip**.

We now show that these maps span $\mathrm{Hom}_{\mathrm{SSBim}}(\mathbf{1}_{a,b}, W_{b-l})$. Recall the alphabet labeling conventions from Convention 6.17, and observe that

$W_{b-l} = I^{(l)}({}_{a,b-l}S_{a+b-l}M_{a,b-l}) =$ 

This gives an action of $\mathrm{Sym}(\mathbb{X}_1|\mathbb{L}|\mathbb{B})$ on W_{b-l}, and hence on $\mathrm{Hom}_{\mathrm{SSBim}}(\mathbf{1}_{a,b}, W_{b-l})$ by post-composition. In our current situation, $|\mathbb{B}| = l$ and $|\mathbb{L}| = b-l$.

For use here and below, we record the following.

Lemma 9.9. *Let X be a complex in $\mathcal{C}_{a,b-l}$, then there is an isomorphism*

$$\mathrm{Hom}_{\mathcal{C}_{a,b}}(\mathbf{1}_{a,b}, I^{(l)}(X)) \cong \mathbf{q}^{-l(b-l)} \mathrm{Hom}_{\mathcal{C}_{a,b-l}}(\mathbf{1}_{a,b-l}, X) \otimes \mathrm{Sym}(\mathbb{B})$$

of dg $\mathrm{Sym}(\mathbb{X}_1|\mathbb{L}|\mathbb{B})$*-modules that is natural in X.*

Proof. Proposition 3.19 gives that

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}_{a,b}}(\mathbf{1}_{a,b}, I^{(l)}(X)) &= \mathrm{Hom}_{\mathcal{C}_{a,b}}\left(\mathbf{1}_{a,b}, (\mathbf{1}_a \boxtimes {}_bM_{b-l,l}) \star (X \boxtimes \mathbf{1}_l) \star (\mathbf{1}_a \boxtimes {}_{b-l,l}S_b)\right) \\ &\cong \mathbf{q}^{-2l(b-l)} \mathrm{Hom}_{\mathcal{C}_{a,b}}\left(\mathbf{1}_{a,b-l,l}, (\mathbf{1}_a \boxtimes ({}_{b-l,l}S_b \star {}_bM_{b-l,l})) \star (X \boxtimes \mathbf{1}_l)\right).\end{aligned}$$

Corollary A.2 then gives

$$\mathrm{Hom}_{\mathcal{C}_{a,b}}(\mathbf{1}_{a,b}, I^{(l)}(X)) \cong \mathbf{q}^{-l(b-l)} \mathrm{Hom}_{\mathcal{C}_{a,b-l}}(\mathbf{1}_{a,b-l}, X) \otimes \mathrm{Sym}(\mathbb{B}).$$

The result follows since each isomorphism is $\mathrm{Sym}(\mathbb{X}_1|\mathbb{L}|\mathbb{B})$-linear and natural in X. □

Corollary 9.10. *There is an isomorphism*

$$\mathbf{q}^{(a-l)(b-l)} \mathrm{Sym}(\mathbb{X}_1|\mathbb{L}|\mathbb{B}) \cong \mathrm{Hom}_{\mathrm{SSBim}}(\mathbf{1}_{a,b}, W_{b-l})$$

of $\mathrm{Sym}(\mathbb{X}_1|\mathbb{L}|\mathbb{B})$*-modules sending* $1 \to \varphi_{a,b,l}(\emptyset)$*. Consequently,* $\mathrm{Hom}_{\mathrm{SSBim}}(\mathbf{1}_{a,b}, W_{b-l})$ *is a free* $\mathrm{Sym}(\mathbb{X}_1|\mathbb{X}_2)$*-module with basis given by the morphisms $\varphi_{a,b,l}(\lambda)$ from Definition 9.8.*

Proof. Lemma 9.9 and Corollary A.3 give that

$$\begin{aligned}\mathrm{Hom}_{\mathrm{SSBim}}(\mathbf{1}_{a,b}, W_{b-l}) &\cong \mathbf{q}^{-l(b-l)} \mathrm{Hom}_{\mathcal{C}_{a,b-l}}(\mathbf{1}_{a,b-l}, {}_{a,b-l}S_{a+b-l}M_{a,b-l}) \otimes \mathrm{Sym}(\mathbb{B}) \\ &\cong \mathbf{q}^{(a-l)(b-l)} \mathrm{Sym}(\mathbb{X}_1|\mathbb{L}) \otimes \mathrm{Sym}(\mathbb{B})\end{aligned}$$

and show that under this isomorphism $\varphi_{a,b,l}(\emptyset)$ is sent to 1. The second statement then follows from Example 3.17. □

Lemma 9.7 and Corollary 9.10 immediately imply the following.

Proposition 9.11. *As a module over $E_{a,b}$, the homology of $M_{a,b} = \mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}, \mathcal{V}\mathsf{FT}_{a,b})$ is spanned by the classes of the elements $\mathrm{CM}(\varphi_{a,b,l}(\lambda))$, where $\mathrm{CM} := \bigoplus_L \mathrm{CM}_l$ is the inclusion from Lemma 9.7.* □

Recall the reduction functor $\pi\colon \mathcal{V}_{a,b} \to \mathcal{V}_{a,b}$ from Definition 4.40. This induces an algebra endomorphism of $E_{a,b} = \mathrm{End}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b})$ that we denote by the same symbol. By Definition 4.40, it maps

$$v_{L,i}^{(a)} \mapsto 0, \quad v_{R,i}^{(b)} \mapsto v_{R,i}^{(b)} - v_{L,i}^{(b)}$$

since the actions of $\mathbb{X}_2$ and $\mathbb{X}_2'$ appearing in (62) are identified on $E_{a,b}$. We will also use π to refer to the $\mathbb{Q}[\mathbb{X}_1, \mathbb{X}_2]$-linear endomorphism of $\mathbb{Q}[\mathbb{X}_1, \mathbb{X}_2, \mathbb{V}_L^{(a)}, \mathbb{V}_R^{(b)}]$ determined by the same formulas. This map sends:

$$y_i \mapsto \begin{cases} 0 & 1 \le i \le a \\ \overline{y}_i & a < i \le a+b \end{cases}$$

when we express y_i as in (59) and $\overline{y}_i$ as in (64).

Lemma 9.12. *We have $J_{a,b} = \langle \pi(J_{a,b}) \rangle$ as ideals in $E_{a,b}$.*

Proof. We claim that $J_{a,b}$ admits a set of generators, which is preserved by π. Indeed, as defined, $\mathcal{V}\mathsf{FT}_{a,b}$ and the splitting map $\Sigma_{a,b}\colon \mathcal{V}\mathsf{FT}_{a,b} \to \mathbf{1}_{a,b}$ are both reduced, hence fixed by π. Moreover, the elements $\mathrm{CM}(\varphi_{a,b,l}(\lambda))$ from Proposition 9.11 are fixed by π since the deformation parameters $\mathbb{V}$ do not enter into their definition. It follows that $\Sigma_{a,b} \circ \mathrm{CM}(\varphi_{a,b,l}(\lambda)) \in E_{a,b}$ is fixed by π. By Proposition 9.11, these elements generate $J_{a,b}$. □

9.3. **Haiman determinants and the Hopf link.** We now compute the generators $\Sigma_{a,b}\circ\mathrm{CM}(\varphi_{a,b,l}(\lambda)) \in J_{a,b}$ explicitly. We will see that these elements are special cases of the Haiman determinants from §2.4.

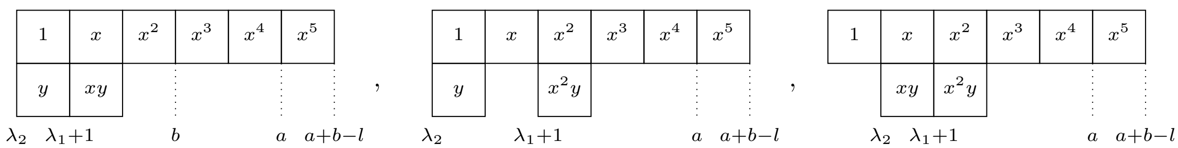
Definition 9.13. For $a \geq b \geq l \geq 0$ and a partition $\lambda \in P(l, b-l)$, the associated *key shape* $\mathrm{Key}_l(\lambda)$ is defined to be the set of monic monomials in $\mathbb{Q}[x, y]$:

$$\mathrm{Key}_l(\lambda) := \{x^{a+b-l-1}, \dots, x, 1\} \cup \{x^{\lambda_1+l-1}y, \dots, x^{\lambda_l}y\}$$

ordered as indicated.

Convention 9.14. We will identify finite sets S of monic monomials in $\mathbb{Q}[x, y]$ with finite subsets of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, illustrated by collections of boxes living in the 4^{th} quadrant. In general, we will call such collections of boxes *shapes*.

For example, for $a = 5$, $b = 3$ and $l = 2$ there are three key shapes:



which are associated with the partitions $\emptyset$, (1) and $(1, 1)$ inside the 2×1 box, respectively.

In a similar way, every partition λ specifies a finite set of monic monomials via the coordinates of the boxes in the Young diagram for λ. We caution the reader that this set of monomials naïvely associated to a partition λ is unrelated to both the key shape $\mathrm{Key}_l(\lambda)$ and the set of monomials $\mathcal{M}_N(\lambda)$ from Definition 2.13.

Example 9.15. The Haiman determinant associated to a key shape takes the form

$$\Delta_{\mathrm{Key}_l(\lambda)} = \begin{vmatrix} x_1^{a+b-l-1} & \cdots & x_a^{a+b-l-1} & x_{a+1}^{a+b-l-1} & \cdots & x_{a+b}^{a+b-l-1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & 1 & \cdots & 1 \\ x_1^{\lambda_1+l-1}y_1 & \cdots & x_a^{\lambda_1+l-1}y_a & x_{a+1}^{\lambda_1+l-1}y_{a+1} & \cdots & x_{a+b}^{\lambda_1+l-1}y_{a+b} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_1^{\lambda_l}y_1 & \cdots & x_a^{\lambda_l}y_a & x_{a+1}^{\lambda_l}y_{a+1} & \cdots & x_{a+b}^{\lambda_l}y_{a+b} \end{vmatrix} \tag{130}$$

Lemma 9.16 (Key lemma)**.** *For $a \geq b \geq l \geq 0$ and a partition $\lambda \in P(l, b-l)$, we have the following identity in $E_{a,b}$:*

$$\Sigma_{a,b} \circ \mathrm{CM}(\varphi_{a,b,k}(\lambda)) = \pm \frac{\pi(\Delta_{\mathrm{Key}_l(\lambda)}(\mathbb{X}, \mathbb{Y}))}{\Delta(\mathbb{X}_1)\Delta(\mathbb{X}_2)} \cdot \mathrm{id}\,. \tag{131}$$

Here, $\mathbb{X} = \mathbb{X}_1 \cup \mathbb{X}_2 = \{x_1, \dots, x_{a+b}\}$, $\mathbb{Y} = \{y_1, \dots, y_{a+b}\}$, and π is the reduction functor from Definition 4.40 (as discussed before Lemma 9.12 above).

Proof. We will use Definition 2.13 to help abbreviate parts of the proof. Specifically, for each $c \geq 1$ and each weakly decreasing sequence $\beta = (\beta_1 \geq \cdots \beta_c \geq 0)$ of length c, let $\mathcal{M}_c(\beta) = \{x^{\beta_1+c-1}, \dots, x^{\beta_{c-1}-1}, x^{\beta_c}\}$ be the associated list of monomials. We note the following properties of $\mathcal{M}_c(\beta)$:

(1) $\mathcal{M}_c(\emptyset) = \{x^{c-1}, \dots, x, 1\}$.
(2) $\mathrm{hdet}\, \mathcal{M}_c(\beta) = \mathfrak{s}_\beta(\mathbb{X})\Delta(\mathbb{X})$, where $|\mathbb{X}| = c$.
(3) If $\beta \in P(c, d)$, then the dual complementary partition $\widehat{\beta} \in P(d, c)$ satisfies

$$\mathcal{M}_d(\widehat{\beta}) = \{x^{c+d-1}, \dots, x, 1\} \smallsetminus \mathcal{M}_c(\beta)\,.$$

Now, consider the determinant $\Delta_{\mathrm{Key}_l(\lambda)}$, depicted in (130), regarded as an element of $\mathbb{Q}[\mathbb{X},\mathbb{V}]$ via (127). If we apply the algebra endomorphism π, the result is:

$$\pi(\Delta_{\mathrm{Key}_l(\lambda)}) = \begin{vmatrix} x_1^{a+b-l-1} & \cdots & x_a^{a+b-l-1} & x_{a+1}^{a+b-l-1} & \cdots & x_{a+b}^{a+b-l-1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & 1 & \cdots & 1 \\ 0 & \cdots & 0 & x_{a+1}^{\lambda_1+l-1}\overline{y}_{a+1} & \cdots & x_{a+b}^{\lambda_1+l-1}\overline{y}_{a+b} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & x_{a+1}^{\lambda_l}\overline{y}_{a+1} & \cdots & x_{a+b}^{\lambda_l}\overline{y}_{a+b} \end{vmatrix} \tag{132}$$

This determinant will be computed by (multiple column) Laplace expansion in the first a columns. Let M denote the matrix appearing on the right-hand side of (132), and let $C = \{1, \ldots, a\}$ be the indexing set for the selected columns. The result of the expansion is of the form:

$$|M| = \sum_{\substack{R\subset\{1,\ldots,a+b\} \\ |R|=a}} (-1)^{\epsilon_{R,C}} |M_{R,C}||M_{R^c,C^c}|$$

where the sum ranges over size a subsets of rows, denoted R. The summands are the product of the $a \times a$-minor determined by the pair R, C and the complementary $b \times b$-minor, with a sign determined by the permutation shuffling C to R.

It is clear that $|M_{R,C}| = 0$ unless $R \subset \{1, \ldots, a+b-l\}$. In other words, R corresponds to a selection of a of the first $a+b-l$ rows. Equivalently, the choice of R corresponds to the choice of a monomials from $\{x^{a+b-l-1}, \ldots, x, 1\}$. Such choices are parametrized exactly by partitions $\beta \in P(a, b-l)$, via the assignment $\beta \mapsto \mathcal{M}_a(\beta) \subset \{x^{a+b-l-1}, \ldots, x, 1\}$. Thus:

$$|M_{R,C}| = \begin{vmatrix} x_1^{\beta_1+a-1} & \cdots & x_a^{\beta_1+a-1} \\ \vdots & & \vdots \\ x_1^{\beta_a} & \cdots & x_a^{\beta_a} \end{vmatrix} = \mathrm{hdet}(\mathcal{M}_a(\beta))\,.$$

The complementary choice of monomials $\{x^{a+b-l-1}, \ldots, x, 1\} \smallsetminus \mathcal{M}_a(\beta)$ coincides with $\mathcal{M}_{b-l}(\widehat{\beta})$ for the dual complementary partition $\widehat{\beta} \in P(b-l, a)$, so:

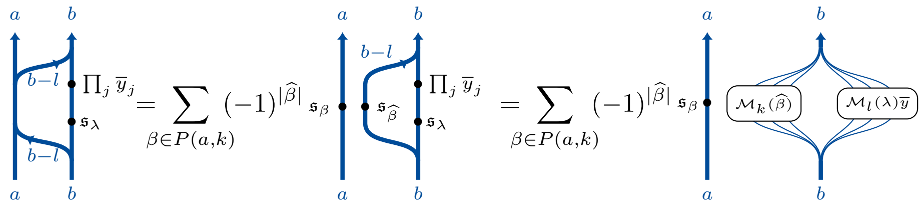
$$|M_{R^c,C^c}| = \begin{vmatrix} x_{a+1}^{\widehat{\beta}_1+b-l-1} & \cdots & x_{a+b}^{\widehat{\beta}_1+b-l-1} \\ \vdots & & \vdots \\ x_{a+1}^{\widehat{\beta}_{b-l}} & \cdots & x_{a+b}^{\widehat{\beta}_{b-l}} \\ x_{a+1}^{\lambda_1+l-1}\overline{y}_{a+1} & \cdots & x_{a+b}^{\lambda_1+l-1}\overline{y}_{a+b} \\ \vdots & \ddots & \vdots \\ x_{a+1}^{\lambda_l}\overline{y}_{a+1} & \cdots & x_{a+b}^{\lambda_l}\overline{y}_{a+b} \end{vmatrix} = \mathrm{hdet}(\mathcal{M}_{b-l}(\widehat{\beta}) \cup \mathcal{M}_l(\lambda)\overline{y})\,.$$

Thus, Laplace expansion yields the following identity:

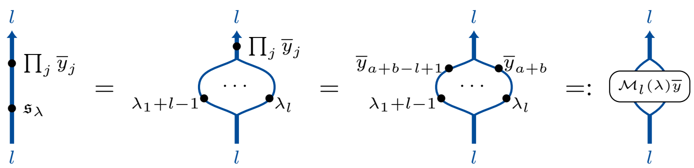
$$|M| = \sum_{\beta\in P_{a,b-l}} (-1)^{|\widehat{\beta}|}\, \mathrm{hdet}\left(\mathcal{M}_a(\beta)\right) \mathrm{hdet}\left(\mathcal{M}_{b-l}(\widehat{\beta}) \cup \mathcal{M}_l(\lambda)\overline{y}\right),$$

where the sign is obtained from shuffling $\{x^{a+b-l-1}, \ldots, x, 1\}$ into $\mathcal{M}_a(\beta) \cup \mathcal{M}_{b-l}(\widehat{\beta})$.

We now compute the left-hand side of (131) using perpendicular graphical calculus. Up to the $\pm$ sign coming from (119), this equals

(133)


where $\prod_j \overline{y}_j = \prod_{j=a+b-l+1}^{a+b} \overline{y}_j$. Here, $\mathcal{M}_l(\lambda)\overline{y} := \{x^{\lambda_1+l-1}\overline{y}, \dots, x^{\lambda_l}\overline{y}\}$, and we use the equality



In the middle step, we can slide $\prod_j \overline{y}_j$ through the top vertex (which acts via the Demazure operator associated with the longest element from Example 3.16) since it is $\mathfrak{S}_l$-symmetric. Now, we can express the right-hand side of (133) as:

$$(133) = \sum_{\beta\in P(a,k)} (-1)^{|\widehat{\beta}|} \left(\frac{\operatorname{hdet}(\mathcal{M}_a(\beta))}{\Delta(\mathbb{X}_1)}\right)\left(\frac{\pi(\operatorname{hdet}(\mathcal{M}_{b-l}(\widehat{\beta}) \cup \mathcal{M}_l(\lambda)y))}{\Delta(\mathbb{X}_2)}\right) = \frac{|M|}{\Delta(\mathbb{X}_1)\Delta(\mathbb{X}_2)}. \tag{134}$$

and the result follows. $\square$

Example 9.17. Let $a \geq b$ and consider the determinant associated to the "maximal" key shape $\operatorname{Key}_b(\emptyset)$. We compute

$$\pi(\Delta_{\operatorname{Key}_b(\emptyset)}(\mathbb{X},\mathbb{Y})) = \begin{vmatrix} x_1^{a-1} & \dots & x_a^{a-1} & x_{a+1}^{a-1} & \dots & x_{a+b}^{a-1} \\ \vdots & & \vdots & \vdots & & \vdots \\ 1 & \dots & 1 & 1 & \dots & 1 \\ 0 & \dots & 0 & x_{a+1}^{b-1}\overline{y}_{a+1} & \dots & x_{a+b}^{b-1}\overline{y}_{a+b} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & \overline{y}_{a+1} & \dots & \overline{y}_{a+b} \end{vmatrix} = \overline{y}_{a+1}\dots\overline{y}_{a+b}\Delta(\mathbb{X}_1)\Delta(\mathbb{X}_2)$$

by expanding the determinant in the first a columns. The product $\overline{y}_{a+1}\dots\overline{y}_{a+b}$ is familiar from Corollary 7.11 (which, in fact, had already shown that $\overline{y}_{a+1}\dots\overline{y}_{a+b} \in J_{a,b}$).

9.4. **Reduced vs. unreduced.** Recall that Conjecture 8.15 proposes an explicit algebraic description of the full twist ideal. In the present (two-strand) case, it posits that the Hopf link ideal $J_{a,b}$ is equal to the ideal

$$I_{a,b} = E_{a,b} \cdot \left\{ \frac{f(\mathbb{X},\mathbb{Y})}{\Delta(\mathbb{X}_1)\Delta(\mathbb{X}_2)} \mid f \in \mathbb{Q}[\mathbb{X},\mathbb{Y}] \text{ is antisymmetric for } \mathfrak{S}_{a+b} \right\}. \tag{135}$$

Here, $\mathbb{X} = \mathbb{X}_1 \cup \mathbb{X}_2 = \{x_1, \dots, x_{a+b}\}$ and $\mathbb{Y} = \{y_1, \dots, y_{a+b}\}$. In order to establish a relation between the full twist ideal $J_{a,b}$ and the ideal $I_{a,b}$, we now relate (certain) reduced and unreduced Haiman determinants.

To begin, we recall the method of computing determinants via the *Schur complement*.

Proposition 9.18. *Let A, B, C, and D be $n \times n$, $n \times m$, $m \times n$, and $m \times m$ matrices (respectively) with coefficients in a commutative ring, and consider the $(n+m) \times (n+m)$ matrix*

$$M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$$

If A is invertible, then $\det(M) = \det(A)\det(D - CA^{-1}B)$.

Proof. Gaussian elimination. □

This immediately implies the following.

Corollary 9.19. *Let* $\mathrm{Key}_l(\lambda)$ *be a key shape and let*

$$A = \begin{pmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_a \\ \vdots & & \vdots \\ x_1^{a-1} & \cdots & x_a^{a-1} \end{pmatrix}, \quad B = \begin{pmatrix} 1 & \cdots & 1 \\ x_{a+1} & \cdots & x_{a+b} \\ \vdots & & \vdots \\ x_{a+1}^{a-1} & \cdots & x_{a+b}^{a-1} \end{pmatrix}$$

$$C_l(\lambda) := \begin{pmatrix} x_1^a & \cdots & x_a^a \\ \vdots & & \vdots \\ x_1^{a+b-l-1} & \cdots & x_a^{a+b-l-1} \\ x_1^{\lambda_l} y_1 & \cdots & x_a^{\lambda_l} y_a \\ \vdots & & \vdots \\ x_1^{\lambda_1+l-1} y_1 & \cdots & x_a^{\lambda_1+l-1} y_a \end{pmatrix}, \quad D_l(\lambda) := \begin{pmatrix} x_{a+1}^a & \cdots & x_{a+b}^a \\ \vdots & & \vdots \\ x_{a+1}^{a+b-l-1} & \cdots & x_{a+b}^{a+b-l-1} \\ x_{a+1}^{\lambda_l} y_{a+1} & \cdots & x_{a+b}^{\lambda_l} y_{a+b} \\ \vdots & & \vdots \\ x_{a+1}^{\lambda_1+l-1} y_{a+1} & \cdots & x_{a+b}^{\lambda_1+l-1} y_{a+b} \end{pmatrix} \tag{136}$$

then $\Delta_{\mathrm{Key}_l(\lambda)}(\mathbb{X}, \mathbb{Y}) = \pm\Delta(\mathbb{X}_1)\det(D_l(\lambda) - C_l(\lambda)A^{-1}B)$.

Note that the $\pm$ sign occurs here since our ordering of monomials differs from the conventions for Haiman determinants (established in Definition 2.16) that is used in Definition 9.13. We will continue to use this ordering for the remainder of this section, since, for our current considerations, Proposition 9.18 is best-adapted to this ordering. Corollary 9.19 motivates the study of the matrix $D_l(\lambda) - C_l(\lambda)A^{-1}B$. Our next result computes the entries of this matrix.

Lemma 9.20. *Let z be an indeterminate and let $r, s \geq 0$. If A is the (Vandermonde) matrix from (136), then*

$$m^{r,s}_{\mathbb{X}_1,\mathbb{V}_L}(z) := \begin{pmatrix} x_1^r y_1^s & \cdots & x_a^r y_a^s \end{pmatrix} \cdot A^{-1} \cdot \begin{pmatrix} 1 \\ z \\ \vdots \\ z^{a-1} \end{pmatrix}$$

is a polynomial in $E_{a,0}[z]$ that satisfies

$$m^{r,s}_{\mathbb{X}_1,\mathbb{V}_L}(x_i) = x_i^r y_i^s \tag{137}$$

for all $x_i \in \mathbb{X}_1$ (thus is the unique such polynomial).

Proof. Note that $E_{a,0} = \mathrm{Sym}(\mathbb{X}_1)[\mathbb{V}_L]$. By equation (127) and Corollary 2.10, there exists a polynomial $m(z) \in E_{a,0}[z]$ of degree $\leq a-1$ that satisfies $m(x_i) = x_i^r y_i^s$ for all $x_i \in \mathbb{X}_1$. Now, by definition, $m^{r,s}_{\mathbb{X}_1,\mathbb{V}_L}(z)$ is a polynomial with coefficients in the field of fractions of $\mathbb{Q}[\mathbb{X}_1, \mathbb{Y}_1]$ of degree $\leq a-1$ satisfying (137). Since there is a unique such polynomial and $E_{a,0}$ is a subring of this field of fractions, we have $m^{r,s}_{\mathbb{X}_1,\mathbb{V}_L}(z) = m(z)$. Thus $m^{r,s}_{\mathbb{X}_1,\mathbb{V}_L}(z) \in E_{a,0}[z]$. □

We now relate the entries of $D_l(\lambda) - C_l(\lambda)A^{-1}B$ in the unreduced and reduced setting.

Lemma 9.21. *Let $x_j \in \mathbb{X}_2$ and let $r \geq 0$, then*

$$x_j^r y_j - m^{r,1}_{\mathbb{X}_1,\mathbb{V}_L}(x_j) = x_j^r \overline{y}_j + \sum_{k=a-r+1}^{a} \big(x_j^{r+k-1} - m^{r+k-1,0}_{\mathbb{X}_1,\mathbb{V}_L}(x_j)\big) v^{(a)}_{L,k}\,.$$

(By convention, the summation on the right-hand side is zero when $r = 0$.)

Proof. To begin, note that Corollary 2.10 implies that

$$m^{c,0}_{\mathbb{X}_1,\mathbb{V}_L}(z) = \sum_{t=1}^{a} (-1)^{a-t} \mathfrak{s}_{(c-a|a-t)}(\mathbb{X}_1) z^{t-1} \tag{138}$$

when $c \geq a$. Next, suppose that $x_i \in \mathbb{X}_1$, then

$$\begin{aligned} x_i^r y_i &\overset{(127)}{=} x_i^r \sum_{k=1}^{a} x_i^{k-1} v^{(a)}_{L,k} = \sum_{k=1}^{a} x_i^{r+k-1} v^{(a)}_{L,k} \\ &\overset{(138)}{=} \sum_{k=1}^{a-r} x_i^{r+k-1} v^{(a)}_{L,k} + \sum_{k=a-r+1}^{a} \left(\sum_{t=1}^{a} (-1)^{a-t} \mathfrak{s}_{(r+k-1-a|a-t)}(\mathbb{X}_1) x_i^{t-1} \right) v^{(a)}_{L,k}\,. \end{aligned}$$

This implies that

$$m^{r,1}_{\mathbb{X}_1,\mathbb{V}_L}(z) = \sum_{k=1}^{a-r} z^{r+k-1} v^{(a)}_{L,k} + \sum_{k=a-r+1}^{a} \left(\sum_{t=1}^{a} (-1)^{a-t} \mathfrak{s}_{(r+k-1-a|a-t)}(\mathbb{X}_1) z^{t-1} \right) v^{(a)}_{L,k}\,.$$

On the other hand, recall from (62) that we have

$$\begin{aligned} \overline{y}_j &= \sum_{k=1}^{b} x_j^{k-1} \overline{v}_k = \sum_{k=1}^{b} x_j^{k-1} (v^{(b)}_{R,k} - v^{(b)}_{L,k}) = y_j - \sum_{k=1}^{b} x_j^{k-1} v^{(b)}_{L,k} \\ &= y_j - \sum_{k=1}^{b} x_j^{k-1} \left(v^{(a)}_{L,k} + (-1)^{b-k} \sum_{i=1}^{a-b} \mathfrak{s}_{(i-1|b-k)}(\mathbb{X}_2) v^{(a)}_{L,b+i} \right) \end{aligned}$$

(since $\mathbb{X}_2 = \mathbb{X}_2'$ in $E_{a,b}$). We thus compute that

$$\begin{aligned} x_j^r \overline{y}_j - x_j^r y_j + m^{r,1}_{\mathbb{X}_1,\mathbb{V}_L}(x_j) &= \sum_{k=1}^{a-r} x_j^{r+k-1} v^{(a)}_{L,k} - \sum_{k=1}^{b} x_j^{r+k-1} v^{(a)}_{L,k} \\ &+ \sum_{\substack{a-r+1 \leq k \leq a \\ 1 \leq t \leq a}} (-1)^{a-t} \mathfrak{s}_{(r+k-1-a|a-t)}(\mathbb{X}_1) x_j^{t-1} v^{(a)}_{L,k} - \sum_{\substack{1 \leq k \leq b \\ 1 \leq i \leq a-b}} (-1)^{b-k} \mathfrak{s}_{(i-1|b-k)}(\mathbb{X}_2) x_j^{r+k-1} v^{(a)}_{L,b+i}\,. \end{aligned} \tag{139}$$

There are now two cases. First, suppose that $b \geq a-r$, then

$$
\begin{aligned}
(139) = & \sum_{k=a-r+1}^{b} \left(\left(\sum_{t=1}^{a} (-1)^{a-t} \mathfrak{s}_{(r+k-1-a|a-t)}(\mathbb{X}_1) x_j^{t-1} \right) - x_j^{r+k-1} \right) v_{L,k}^{(a)} \\
& + \sum_{i=1}^{a-b} \left(\sum_{t=1}^{a} (-1)^{a-t} \mathfrak{s}_{(r+b+i-1-a|a-t)}(\mathbb{X}_1) x_j^{t-1} - \sum_{k=1}^{b} (-1)^{b-k} \mathfrak{s}_{(i-1|b-k)}(\mathbb{X}_2) x_j^{r+k-1} \right) v_{L,b+i}^{(a)} \\
\overset{(138)}{=} & \sum_{k=a-r+1}^{b} \left(m_{\mathbb{X}_1,\mathbb{V}_L}^{r+k-1,0}(x_j) - x_j^{r+k-1} \right) v_{L,k}^{(a)} \\
& + \sum_{i=1}^{a-b} \left(m_{\mathbb{X}_1,\mathbb{V}_L}^{r+b+i-1,0}(x_j) - x_j^{r} \sum_{k=1}^{b} (-1)^{b-k} \mathfrak{s}_{(i+b-1-b|b-k)}(\mathbb{X}_2) x_j^{k-1} \right) v_{L,b+i}^{(a)} \\
= & \sum_{k=a-r+1}^{b} \left(m_{\mathbb{X}_1,\mathbb{V}_L}^{r+k-1,0}(x_j) - x_j^{r+k-1} \right) v_{L,k}^{(a)} + \sum_{i=1}^{a-b} \left(m_{\mathbb{X}_1,\mathbb{V}_L}^{r+b+i-1,0}(x_j) - x_j^{r} \cdot x_j^{i+b-1} \right) v_{L,b+i}^{(a)} \\
= & \sum_{k=a-r+1}^{a} \left(m_{\mathbb{X}_1,\mathbb{V}_L}^{r+k-1,0}(x_j) - x_j^{r+k-1} \right) v_{L,k}^{(a)}
\end{aligned}
$$

as desired. When $b \leq a-r$, the computation is similar:

$$
\begin{aligned}
(139) = & \sum_{k=b+1}^{a-r} \left(x_j^{r+k-1} - \sum_{t=1}^{b} (-1)^{b-t} \mathfrak{s}_{(k-1-b|b-t)}(\mathbb{X}_2) x_j^{r+t-1} \right) v_{L,k}^{(a)} \\
& + \sum_{k=a-r+1}^{a} \left(\sum_{t=1}^{a} (-1)^{a-t} \mathfrak{s}_{(r+k-1-a|a-t)}(\mathbb{X}_1) x_j^{t-1} - \sum_{t=1}^{b} (-1)^{b-t} \mathfrak{s}_{(k-1-b|b-t)}(\mathbb{X}_2) x_j^{r+t-1} \right) v_{L,k}^{(a)} \\
\overset{(138)}{=} & \sum_{k=b+1}^{a-r} x_j^{r} \left(x_j^{k-1} - \sum_{t=1}^{b} (-1)^{b-t} \mathfrak{s}_{(k-1-b|b-t)}(\mathbb{X}_2) x_j^{t-1} \right) v_{L,k}^{(a)} \\
& + \sum_{k=a-r+1}^{a} \left(m_{\mathbb{X}_1,\mathbb{V}_L}^{r+k-1,0}(x_j) - x_j^{r} \sum_{t=1}^{b} (-1)^{b-t} \mathfrak{s}_{(k-1-b|b-t)}(\mathbb{X}_2) x_j^{t-1} \right) v_{L,k}^{(a)} \\
= & \, 0 + \sum_{k=a-r+1}^{a} \left(m_{\mathbb{X}_1,\mathbb{V}_L}^{r+k-1,0}(x_j) - x_j^{r} \cdot x_j^{k-1} \right) v_{L,k}^{(a)} = \sum_{k=a-r+1}^{a} \left(m_{\mathbb{X}_1,\mathbb{V}_L}^{r+k-1,0}(x_j) - x_j^{r+k-1} \right) v_{L,k}^{(a)} . \qquad \square
\end{aligned}
$$

Lemmata 9.20 and 9.21 immediately relate the (unreduced) Haiman determinants $\Delta_{\mathrm{Key}_l(\lambda)}(\mathbb{X},\mathbb{Y})$ to certain reduced Haiman determinants $\pi(\Delta_S(\mathbb{X},\mathbb{Y}))$. Namely, recall from Convention 9.14 that we identify sets of monic monomials S with finite subsets of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ which are illustrated as collections of boxes that we call shapes.

Definition 9.22. for $0 \leq l \leq b$, let S_l be the collection of all subsets of monomials S of the form

$$\{1, x, \dots, x^{a-1}\} \cup \{x^{s_1}, \dots, x^{s_{b-l}}\} \cup \{x^{r_1} y, \dots, x^{r_l} y\}$$

with $a \leq s_1 < \cdots < s_{b-l} \leq a+b$ and $0 \leq r_1 < \cdots < r_l \leq b$.

In other words, S_l consists of shapes that only have boxes in the first two rows, with exactly l boxes in the second row confined to the first b positions, and with $a+b-l$ boxes in the first row confined to the first $a+b$ positions and necessarily occupying the first a boxes in that row. In particular, note that $\mathrm{Key}_l(\lambda) \in \mathsf{S}_l$.

Proposition 9.23. *Let* $S \in \mathsf{S}_l$*, then*

$$\Delta_S(\mathbb{X},\mathbb{Y}) - \pi(\Delta_S(\mathbb{X},\mathbb{Y})) = \sum_{R\in \mathsf{S}_{\leq l-1}} c_{S,R}\cdot \pi(\Delta_R(\mathbb{X},\mathbb{Y}))$$

for $c_{S,R} \in \mathbb{Q}[\mathbb{V}_L]$ *and* $\mathsf{S}_{\leq l-1} = \bigcup_{k=0}^{l-1} \mathsf{S}_k$*.*

Proof. Proposition 9.18 gives an analogue of Corollary 9.19 for the Haiman determinant $\Delta_S(\mathbb{X},\mathbb{Y})$. Namely, it gives that

$$\Delta_S(\mathbb{X},\mathbb{Y}) = \det(A)\cdot \det(M_S)$$

for the $b \times b$ matrix

$$M_S = \begin{pmatrix} x_{a+1}^{s_1} & \cdots & x_{a+b}^{s_1} \\ \vdots & & \vdots \\ x_{a+1}^{s_{b-l}} & \cdots & x_{a+b}^{s_{b-l}} \\ x_{a+1}^{r_1}y_{a+1} & \cdots & x_{a+b}^{r_1}y_{a+b} \\ \vdots & & \vdots \\ x_{a+1}^{r_l}y_{a+1} & \cdots & x_{a+b}^{r_l}y_{a+b} \end{pmatrix} - \begin{pmatrix} x_1^{s_1} & \cdots & x_a^{s_1} \\ \vdots & & \vdots \\ x_1^{s_{b-l}} & \cdots & x_a^{s_{b-l}} \\ x_1^{r_1}y_1 & \cdots & x_a^{r_1}y_a \\ \vdots & & \vdots \\ x_1^{r_l}y_1 & \cdots & x_a^{r_l}y_a \end{pmatrix} A^{-1}B$$

with A and B as in (136). Lemma 9.20 then implies that

$$M_S = \begin{pmatrix} x_{a+1}^{s_1} - m_{\mathbb{X}_1,\mathbb{V}_L}^{s_1,0}(x_{a+1}) & \cdots & x_{a+b}^{s_1} - m_{\mathbb{X}_1,\mathbb{V}_L}^{s_1,0}(x_{a+b}) \\ \vdots & & \vdots \\ x_{a+1}^{s_{b-l}} - m_{\mathbb{X}_1,\mathbb{V}_L}^{s_{b-1},0}(x_{a+1}) & \cdots & x_{a+b}^{s_{b-l}} - m_{\mathbb{X}_1,\mathbb{V}_L}^{s_{b-1},0}(x_{a+1}) \\ x_{a+1}^{r_1}y_{a+1} - m_{\mathbb{X}_1,\mathbb{V}_L}^{r_1,1}(x_{a+1}) & \cdots & x_{a+b}^{r_1}y_{a+b} - m_{\mathbb{X}_1,\mathbb{V}_L}^{r_1,1}(x_{a+b}) \\ \vdots & & \vdots \\ x_{a+1}^{r_l}y_{a+1} - m_{\mathbb{X}_1,\mathbb{V}_L}^{r_l,1}(x_{a+1}) & \cdots & x_{a+b}^{r_l}y_{a+b} - m_{\mathbb{X}_1,\mathbb{V}_L}^{r_l,1}(x_{a+b}). \end{pmatrix} \tag{140}$$

The result now follows by applying Lemma 9.21 to the last l rows of (140). Indeed, this expresses each row of M_S as the sum of the corresponding row in $\pi(M_S)$ and a $\mathbb{Q}[\mathbb{V}_L]$-linear combination of rows corresponding to monomials x^s for $a \leq s \leq a+b-1$. Thus, we have that

$$\det(M_S) = \det(\pi(M_S)) + \sum_{R\in \mathsf{S}_{\leq l-1}} c_{S,R}\cdot \det(\pi(M_R))$$

for some $c_{R,S} \in \mathbb{Q}[\mathbb{V}_L]$ and the result follows. (Informally, when we use Lemma 9.21 and linearly expand $\det(M_S)$ along the relevant rows, in each term a box in the second row either: becomes reduced, or moves to row one in a position between the a^{th} and the $(a+b-1)^{st}$ with a coefficient in $\mathbb{Q}[\mathbb{V}_L]$. Note that there is no distinction between reduced versus unreduced monomials in the first row.) □

Corollary 9.24. *For any key shape* $\mathrm{Key}_l(\lambda)$*, we have that*

$$\pi(\Delta_{\mathrm{Key}_l(\lambda)}(\mathbb{X},\mathbb{Y})) = \sum_{R\in \mathsf{S}_{\leq l}} c_{l,\lambda,R}\cdot \Delta_R(\mathbb{X},\mathbb{Y})$$

for $c_{l,\lambda,R} \in \mathbb{Q}[\mathbb{V}_L]$ *and* $\mathsf{S}_{\leq l} = \bigcup_{k=0}^{l} \mathsf{S}_k$*.*

Proof. Proposition 9.23 shows that there is a unitriangular matrix with coefficients in $\mathbb{Q}[\mathbb{V}_L]$ relating $\{\Delta_S(\mathbb{X},\mathbb{Y}) \mid S \in \mathsf{S}_{\leq l}\}$ and $\{\pi(\Delta_S(\mathbb{X},\mathbb{Y})) \mid S \in \mathsf{S}_{\leq l}\}$. The result then follows since $\pi(\Delta_{\mathrm{Key}_l(\lambda)}(\mathbb{X},\mathbb{Y})) \in \{\pi(\Delta_S(\mathbb{X},\mathbb{Y})) \mid S \in \mathsf{S}_{\leq l}\}$. □

Combining this with Lemma 9.16 and the results from §9.2 gives the following.

Proposition 9.25. $J_{a,b} \subset I_{a,b}$

Proof. Proposition 9.11 and Lemmata 9.12 and 9.16 show that $J_{a,b}$ is generated by the elements $\frac{\pi(\Delta_{\mathrm{Key}_l(\lambda)}(\mathbb{X},\mathbb{Y}))}{\Delta(\mathbb{X}_1)\Delta(\mathbb{X}_2)}$ with $0 \le l \le b$ and $\lambda \in P(l, b-l)$. Corollary 9.24 expresses each such generator as

$$\sum_{R \in \mathsf{S}_{\le l}} c_{l,\lambda,R} \cdot \frac{\Delta_R(\mathbb{X},\mathbb{Y})}{\Delta(\mathbb{X}_1)\Delta(\mathbb{X}_2)}$$

with $c_{l,\lambda,R} \in \mathbb{Q}[\mathbb{V}_L] \subset E_{a,b}$. Since every $\Delta_R(\mathbb{X},\mathbb{Y})$ is $\mathfrak{S}_{a+b}$-antisymmetric, we see that $\frac{\pi(\Delta_{\mathrm{Key}_l(\lambda)}(\mathbb{X},\mathbb{Y}))}{\Delta(\mathbb{X}_1)\Delta(\mathbb{X}_2)} \in I_{a,b}$. □

9.5. **Families of ideals.** Our proof of the opposite inclusion $I_{a,b} \subset J_{a,b}$ for $b > 1$ requires an inductive argument that makes use of the curved skein relation via Theorem 7.14. We begin by introducing the relevant rings and ideals. First, we generalize the Hopf link ideal $J_{a,b}$ to a family of ideals that arise from the splitting map

$$I^{(s)}(\Sigma)\colon I^{(s)}(\mathcal{V}\mathsf{FT}_{a,b-s}) \to I^{(s)}(\mathbf{1}_{a,b-s})$$

for threaded digons. For the duration of this section, we will use Convention 6.17 for notation relevant to the functor $I^{(s)}$. In particular, we will consider $|\mathbb{B}| = s$ and $|\mathbb{L}| = \ell$ (often we have $\ell = b - s$).

Definition 9.26. Let

$$E_{a,(\ell,s)} := \mathrm{Sym}(\mathbb{X}_1|\mathbb{L}|\mathbb{B})[\mathbb{V}_L^{(a)}, \mathbb{V}_R^{(\ell+s)}]$$

where $\mathbb{V}_L^{(a)}$ and $\mathbb{V}_R^{(\ell+s)}$ are alphabets as in (58). Further, we identify $E_{a,(\ell,0)} = E_{a,\ell}$ (from Definition 8.2, with $\mathbb{L}$ in place of $\mathbb{X}_2$) and regard this as a subalgebra of $E_{a,(\ell,s)}$ via the identification:

$$v_{R,i}^{(\ell)} = v_{R,i}^{(\ell+s)} + (-1)^{\ell-i} \sum_{j=\ell+1}^{\ell+s} \mathfrak{s}_{(j-\ell-1|\ell-i)}(\mathbb{L}) v_{R,j}^{(\ell+s)}$$

that is analogous to (67).

The relevance of this algebra stems from the following result.

Lemma 9.27. *Let X be a curved complex in $\mathcal{V}_{a,\ell}$, then there is an isomorphism*

$$\mathrm{Hom}_{\mathcal{V}_{a,\ell+s}}(\mathbf{1}_{a,\ell+s}, I^{(s)}(X)) \cong \mathbf{q}^{-\ell s}\, \mathrm{Hom}_{\mathcal{V}_{a,\ell}}(\mathbf{1}_{a,\ell}, X) \otimes_{E_{a,(\ell,0)}} E_{a,(\ell,s)}$$

of dg $E_{a,(\ell,s)}$-modules that is natural in X.

Proof. The endomorphism algebra $\mathrm{End}_{\mathcal{V}_{a,b}}(I^{(s)}(X))$ is a $\mathrm{Sym}(\mathbb{X}_1|\mathbb{X}_1'|\mathbb{L}|\mathbb{L}'|\mathbb{B})[\mathbb{V}_L^{(a)}, \mathbb{V}_R^{(\ell+s)}]$-module. (As a reminder, we denote alphabets as in Convention 6.17.) The induced action on $\mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}, I^{(s)}(X))$ factors through the quotient in which we identify $\mathbb{X}_1 = \mathbb{X}_1'$ and $\mathbb{X}_2 = \mathbb{X}_2'$, and thus also $\mathbb{L} = \mathbb{L}'$. Hence, we see that $\mathrm{Hom}_{\mathcal{V}_{a,\ell+s}}(\mathbf{1}_{a,\ell+s}, I^{(s)}(X))$ is indeed a dg $E_{a,(\ell,s)}$-module. (The differential is induced from the differential on X, which commutes with this action.)

We now establish the isomorphism. Lemma 9.9 gives that

$$\mathrm{Hom}_{\mathcal{C}_{a,\ell+s}}(\mathbf{1}_{a,\ell+s}, I^{(s)}(X)) \cong \mathbf{q}^{-\ell s}\, \mathrm{Hom}_{\mathcal{C}_{a,\ell}}(\mathbf{1}_{a,\ell}, X) \otimes \mathrm{Sym}(\mathbb{B})\,.$$

and in the deformed setting, this gives

$$\begin{aligned}\mathrm{Hom}_{\mathcal{V}_{a,\ell+s}}(\mathbf{1}_{a,\ell+s}, I^{(s)}(X)) &= \mathrm{Hom}_{\mathcal{C}_{a,\ell+s}}(\mathbf{1}_{a,\ell+s}, I^{(s)}(X))[\mathbb{V}_L^{(a)}, \mathbb{V}_R^{(\ell+s)}] \\ &\cong \mathbf{q}^{-\ell s}\, \mathrm{Hom}_{\mathcal{C}_{a,\ell}}(\mathbf{1}_{a,\ell}, X) \otimes \mathrm{Sym}(\mathbb{B})[\mathbb{V}_L^{(a)}, \mathbb{V}_R^{(\ell+s)}] \\ &\cong \mathbf{q}^{-\ell s}\, \mathrm{Hom}_{\mathcal{V}_{a,\ell}}(\mathbf{1}_{a,\ell}, X) \otimes_{E_{a,(\ell,0)}} E_{a,(\ell,s)}\,.\end{aligned}$$

The result follows since Definitions 6.18 and 9.26 show that this is an isomorphism of complexes. □

Recall the notation $\mathrm{Dig}_{a,b}^s = I^{(s)}(\mathbf{1}_{a,b-s})$ from §7.3. Our last result has the following important consequence.

Corollary 9.28. *Let $a \geq b \geq 0$, then $\mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}, \mathrm{Dig}^s_{a,b}) \cong \mathbf{q}^{-s(b-s)} E_{a,(b-s,s)}$ and the map*

$$H(I^{(s)}(\Sigma))\colon H\big(\mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}, I^{(s)}(\mathcal{V}\mathsf{FT}_{a,b-s}))\big) \to E_{a,(b-s,s)}$$

on homology induced by the splitting map is injective.

Proof. Applying Lemma 9.27 to the (co)domain of the splitting map $\Sigma_{a,b-s} : \mathcal{V}\mathsf{FT}_{a,b-s} \to \mathbf{1}_{a,b-s}$ gives the commutative diagram:

(141)

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}, I^{(s)}(\mathcal{V}\mathsf{FT}_{a,b-s})) & \xrightarrow{\cong} & \mathbf{q}^{-s(b-s)} \mathrm{Hom}_{\mathcal{V}_{a,b-s}}(\mathbf{1}_{a,b-s}, \mathcal{V}\mathsf{FT}_{a,b-s}) \otimes_{E_{a,(b-s,0)}} E_{a,(b-s,s)} \\ \big\downarrow {\scriptstyle \mathrm{Hom}(\mathbf{1}_{a,b}, I^{(s)}(\Sigma_{a,b-s}))} & & \big\downarrow {\scriptstyle \mathrm{Hom}(\mathbf{1}_{a,b}, \Sigma_{a,b-s})} \\ \mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}, I^{(s)}(\mathbf{1}_{a,b-s})) & \xrightarrow{\cong} & \mathbf{q}^{-s(b-s)} \mathrm{Hom}_{\mathcal{V}_{a,b-s}}(\mathbf{1}_{a,b-s}, \mathbf{1}_{a,b-s}) \otimes_{E_{a,(b-s,0)}} E_{a,(b-s,s)} \end{array}$$

The first assertion follows since the bottom-right corner of this diagram is $\mathbf{q}^{-s(b-s)} E_{a,(b-s,s)}$. For the second, we note that Corollary 8.7 and Proposition 9.4 imply that the right vertical map induces an injective map in homology. Commutativity of the diagram implies that the same is true for the left vertical map. □

We next introduce notation that extends Definitions 8.2 and 8.3.

Definition 9.29. For $a \geq b \geq 0$, let $M_{a,(b-s,s)} := \mathbf{q}^{s(b-s)} \mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}, I^{(s)}(\mathcal{V}\mathsf{FT}_{a,b-s}))$ and set

$$J_{a,(b-s,s)} := \mathrm{im}\big(H(M_{a,(b-s,s)}) \xrightarrow{H(I^{(s)}(\Sigma))} E_{a,(b-s,s)}\big) \lhd E_{a,(b-s,s)} \,.$$

In the next section, we will achieve our goal of showing that $J_{a,b} = I_{a,b}$ by using an inductive argument involving the collection $\{J_{a,(b-s,s)}\}_{s=0}^{b}$. To this end, we now observe that $J_{a,(\ell,0)} = J_{a,\ell}$ completely determines $J_{a,(\ell,s)}$.

Lemma 9.30. *For $\ell + s \leq a$, we have $J_{a,(\ell,s)} = E_{a,(\ell,s)} \cdot J_{a,(\ell,0)}$.*

Proof. Lemma 9.27 gives that

$$M_{a,(\ell,s)} \cong M_{a,(\ell,0)} \otimes_{E_{a,(\ell,0)}} E_{a,(\ell,s)} \,.$$

Taking homology and applying Corollary 9.28 gives

$$J_{a,(\ell,s)} \cong J_{a,(\ell,0)} \otimes_{E_{a,(\ell,0)}} E_{a,(\ell,s)}$$

which is a restatement of the desired result. □

Motivated by this, we introduce the following family of ideals that generalize the ideal $I_{a,b} \lhd E_{a,b}$.

Definition 9.31. Let $\mathbb{X}^{(a+\ell)} = \mathbb{X}_1 \cup \mathbb{L} = \{x_1, \ldots, x_{a+\ell}\}$ and $\mathbb{Y}^{(a+\ell)} = \{y_1, \ldots, y_{a+\ell}\}$ and set

$$I_{a,(\ell,s)} := E_{a,(\ell,s)} \cdot \left\{ \frac{f(\mathbb{X}^{(a+\ell)}, \mathbb{Y}^{(a+\ell)})}{\Delta(\mathbb{X}_1)\Delta(\mathbb{L})} \mid f \in \mathbb{Q}[\mathbb{X}^{(a+\ell)}, \mathbb{Y}^{(a+\ell)}] \text{ is antisymmetric for } \mathfrak{S}_{a+\ell} \right\}.$$

Analogous to Lemma 9.30, the following holds (essentially by definition):

Lemma 9.32. *For $\ell + s \leq a$, we have $I_{a,(\ell,s)} = E_{a,(\ell,s)} \cdot I_{a,(\ell,0)}$.*

Proof. The ideals $I_{a,(\ell,0)} = I_{a,\ell}$ and $I_{a,(\ell,s)}$ both have generators of the form $\frac{f(\mathbb{X}^{(a+\ell)}, \mathbb{Y}^{(a+\ell)})}{\Delta(\mathbb{X}_1)\Delta(\mathbb{L})}$, which give elements of $E_{a,\ell}$ and $E_{a,(\ell,s)}$ by expanding $\{y_i\}_{i=1}^{a+\ell}$ in the alphabets $\mathbb{V}_L^{(a)} \cup \mathbb{V}_R^{(\ell)}$ and $\mathbb{V}_L^{(a)} \cup \mathbb{V}_R^{(\ell+s)}$. The result now follows from Definition 9.26 and Proposition 4.34, which shows that the triangle:

$$\begin{array}{ccc}
\left\{ \frac{f(\mathbb{X}^{(a+\ell)},\mathbb{Y}^{(a+\ell)})}{\Delta(\mathbb{X}_1)\Delta(\mathbb{L})} \mid f \in \mathbb{Q}[\mathbb{X}^{(a+\ell)},\mathbb{Y}^{(a+\ell)}] \text{ is antisymmetric for } \mathfrak{S}_{a+\ell} \right\} & \hookrightarrow & E_{a,(\ell,0)} \\
& \searrow & \downarrow \\
& & E_{a,(\ell,s)}
\end{array}$$

commutes. □

9.6. **The inductive argument.** In this section, we prove the following result, which establishes Conjecture 8.15 in the 2-strand case.

Theorem 9.33. *Let $a \geq b \geq 0$, then $J_{a,b} = I_{a,b}$.*

More generally, Lemmata 9.30 and 9.32 then immediately give:

Corollary 9.34. *Let $a \geq b \geq s \geq 0$, then $J_{a,(b-s,s)} = I_{a,(b-s,s)}$.* □

Proposition 9.25 implies that Theorem 9.33 will follow by showing that $I_{a,b} \subset J_{a,b}$. To begin, we establish the $b = 1$ case.

Lemma 9.35. *$I_{a,1} \subset J_{a,1}$ (and thus $J_{a,1} = I_{a,1}$).*

Proof. Recall that generators for $I_{a,1}$ take the form $\frac{f(\mathbb{X},\mathbb{Y})}{\Delta(\mathbb{X}_1)}$ where $f(\mathbb{X},\mathbb{Y})$ is anti-symmetric for the diagonal action of $\mathfrak{S}_{a+1}$ on $\mathbb{Q}[\mathbb{X},\mathbb{Y}]$. Here, $\mathbb{X} = \{x_1,\ldots,x_{a+1}\}$, $\mathbb{X}_1 = \mathbb{X} \smallsetminus \{x_{a+1}\}$, and $\mathbb{Y} = \{y_1,\ldots,y_{a+1}\}$. Fix such a generator and note that $f(\mathbb{X},\mathbb{Y}) \in I_{a+1}$ (see §8.2). Lemma 8.18 gives have that

$$I_{a+1} = \bigcap_{1\leq i<j\leq a+1} \mathbb{Q}[\mathbb{X},\mathbb{Y}] \cdot \{x_i - x_j, y_i - y_j\} \subset \bigcap_{1\leq i<j\leq a+1} \mathbb{Q}[\mathbb{X},\mathbb{V}] \cdot \{x_i - x_j, y_i - y_j\}\,.$$

Now, in $\mathbb{Q}[\mathbb{X},\mathbb{V}]$, for $1 \leq i < j \leq a$ we have

$$y_i - y_j = \sum_{r=1}^{a} \left(x_i^{r-1} - x_j^{r-1}\right) v_{L,r}^{(a)} \in \mathbb{Q}[\mathbb{X},\mathbb{V}] \cdot \{x_i - x_j\}$$

and for $1 \leq i \leq a$ we have

$$y_i - y_{a+1} = \sum_{r=1}^{a} \left(x_i^{r-1} - x_{a+1}^{r-1}\right) v_{L,r}^{(a)} + \left(\sum_{r=1}^{a} x_{a+1}^{r-1} v_{L,r}^{(a)}\right) - y_{a+1} = \left(\sum_{r=1}^{a} \left(x_i^{r-1} - x_{a+1}^{r-1}\right) v_{L,r}^{(a)}\right) - \overline{y}_{a+1}\,.$$

Hence,

$$\bigcap_{1\leq i<j\leq a+1} \mathbb{Q}[\mathbb{X},\mathbb{V}] \cdot \{x_i - x_j, y_i - y_j\} = \left(\bigcap_{1\leq i<j\leq a} \mathbb{Q}[\mathbb{X},\mathbb{V}] \cdot \{x_i - x_j\}\right) \cap \left(\bigcap_{1\leq i\leq a} \mathbb{Q}[\mathbb{X},\mathbb{V}] \cdot \{x_i - x_{a+1}, \overline{y}_{a+1}\}\right).$$

Now, we have that

$$\bigcap_{1\leq i<j\leq a} \mathbb{Q}[\mathbb{X},\mathbb{V}] \cdot \{x_i - x_j\} = \mathbb{Q}[\mathbb{X},\mathbb{V}] \cdot \Delta(\mathbb{X}_1)$$

and

$$\begin{aligned}
\bigcap_{1\leq i\leq a} \mathbb{Q}[\mathbb{X},\mathbb{V}] \cdot \{x_i - x_{a+1}, \overline{y}_{a+1}\} &= \mathbb{Q}[\mathbb{X},\mathbb{V}] \cdot \overline{y}_{a+1} + \bigcap_{1\leq i\leq a} \mathbb{Q}[\mathbb{X},\mathbb{V}] \cdot \{x_i - x_{a+1}\} \\
&= \mathbb{Q}[\mathbb{X},\mathbb{V}] \cdot \{e_a(\mathbb{X}_1 - \{x_{a+1}\}), \overline{y}_{a+1}\}
\end{aligned}$$

Thus, we have that

$$\bigcap_{1\leq i<j\leq a+1} \mathbb{Q}[\mathbb{X},\mathbb{V}] \cdot \{x_i - x_j, y_i - y_j\} = \big(\mathbb{Q}[\mathbb{X},\mathbb{V}] \cdot \Delta(\mathbb{X}_1)\big) \cap \big(\mathbb{Q}[\mathbb{X},\mathbb{V}] \cdot \{e_a(\mathbb{X}_1 - \{x_{a+1}\}), \overline{y}_{a+1}\}\big)$$

$$= \Delta(\mathbb{X}_1)\mathbb{Q}[\mathbb{X},\mathbb{V}] \cdot \{e_a(\mathbb{X}_1 - \{x_{a+1}\}), \overline{y}_{a+1}\}$$

and so

$$\frac{f(\mathbb{X},\mathbb{Y})}{\Delta(\mathbb{X}_1)} \in \mathbb{Q}[\mathbb{X},\mathbb{V}] \cdot \{e_a(\mathbb{X}_1 - \{x_{a+1}\}), \overline{y}_{a+1}\}\,.$$

Further, since $\frac{f(\mathbb{X},\mathbb{Y})}{\Delta(\mathbb{X}_1)} \in \mathbb{Q}[\mathbb{X},\mathbb{V}]$ is invariant under the $\mathfrak{S}_a$-action, this implies that in fact

$$\frac{f(\mathbb{X},\mathbb{Y})}{\Delta(\mathbb{X}_1)} \in \mathrm{Sym}(\mathbb{X}_1|\{x_{a+1}\})[\mathbb{V}] \cdot \{e_a(\mathbb{X}_1 - \{x_{a+1}\}), \overline{y}_{a+1}\} = \pi(J_{a,1}) \subset J_{a,1}$$

and thus $I_{a,1} \subset J_{a,1}$, as desired. □

Our proof of Theorem 9.33 for $b \geq 2$ relies on Theorem 7.14, which has the following consequence.

Lemma 9.36. *There is a (contractible) complex of the form*

$$E_{a,(b-\bullet,\bullet)} := \left(E_{a,(b-0,0)} \xrightarrow{d_0} \cdots \xrightarrow{d_{s-1}} \mathbf{q}^{s(s-1)}\mathbf{t}^s E_{a,(b-s,s)} \xrightarrow{d_s} \cdots \xrightarrow{d_{b-1}} \mathbf{q}^{b(b-1)}\mathbf{t}^b E_{a,(0,b)} \right)$$

with differentials induced from those in Definition 7.12. Further, $d_s(J_{a,(b-s,s)}) \subset J_{a,(b-s-1,s+1)}$, thus the ideals $J_{a,(b-s,s)} \lhd E_{a,(b-s,s)}$ form a subcomplex

$$J_{a,(b-\bullet,\bullet)} := \left(J_{a,(b-0,0)} \xrightarrow{d_0} \cdots \xrightarrow{d_{s-1}} \mathbf{q}^{s(s-1)}\mathbf{t}^s J_{a,(b-s,s)} \xrightarrow{d_s} \cdots \xrightarrow{d_{b-1}} \mathbf{q}^{b(b-1)}\mathbf{t}^b J_{a,(0,b)} \right)$$

of $E_{a,(b-\bullet,\bullet)}$.

We will occasionally abbreviate the notation of these complexes to $J_\bullet \subset E_\bullet$ when there is no confusion as to the values of a and b.

Proof. The complex $E_\bullet$ is obtained by applying the functor $\mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}, -)$ to the complex

$$\mathbf{1}_a \boxtimes \mathcal{V}\mathsf{TD}_b(0) = \left(\mathrm{Dig}^0_{a,b} \xrightarrow{d_0} \cdots \xrightarrow{d_{s-1}} \mathbf{q}^{s(b-1)}\mathbf{t}^s \mathrm{Dig}^s_{a,b} \xrightarrow{d_s} \cdots \xrightarrow{d_{b-1}} \mathbf{q}^{b(b-1)}\mathbf{t}^b \mathrm{Dig}^b_{a,b} \right)$$

from Definition 7.12 and using Lemma 9.27, which gives that

$$\mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}, \mathbf{q}^{s(b-1)}\mathbf{t}^s \mathrm{Dig}^s_{a,b}) \cong \mathbf{q}^{s(s-1)} E_{a,(b-s,s)}\,.$$

The component of the differential $\mathbf{q}^{s(s-1)}\mathbf{t}^s E_{a,(b-s,s)} \xrightarrow{d_s} \mathbf{q}^{(s+1)s}\mathbf{t}^s E_{a,(b-s-1,s+1)}$ is explicitly given by:

$$\begin{aligned}
\mathbf{q}^{s(s-1)} E_{a,(b-s,s)} &= \mathbf{q}^{s(s-1)}\, \mathrm{Sym}(\mathbb{X}_1|\mathbb{L}|\mathbb{B})[\mathbb{V}^{(a)}_L, \mathbb{V}^{(b)}_R] \\
&\hookrightarrow \mathbf{q}^{s(s-1)}\, \mathrm{Sym}(\mathbb{X}_1|\mathbb{L} \smallsetminus \{x_{a+b-s}\}|\{x_{a+b-s}\}|\mathbb{B})[\mathbb{V}^{(a)}_L, \mathbb{V}^{(b)}_R] \\
&\xrightarrow{\partial_{1,s}} \mathbf{q}^{(s+1)s}\, \mathrm{Sym}(\mathbb{X}_1|\mathbb{L} \smallsetminus \{x_{a+b-s}\}|\{x_{a+b-s}\} \cup \mathbb{B})[\mathbb{V}^{(a)}_L, \mathbb{V}^{(b)}_R] = \mathbf{q}^{(s+1)s} E_{a,(b-s-1,s+1)}
\end{aligned}$$

where here $\partial_{1,s}$ is (the tensor product of identity morphisms with) the Sylvester operator

$$\partial_{1,s}\colon \mathrm{Sym}(\{x_{a+b-s}\}|\mathbb{B}) \to \mathrm{Sym}(\{x_{a+b-s}\} \cup \mathbb{B})$$

from Example 3.17, which has degree $-2|\mathbb{B}| = -2s$. (The complex $E_\bullet$ is contractible by Lemma 7.13.) Finally, the differential on $E_{a,(b-\bullet,\bullet)}$ restricts to a map $J_{a,(b-s,s)} \to J_{a,(b-s-1,s+1)}$ by Theorem 7.14. □

We next investigate the homology of the subcomplex $J_{a,(b-\bullet,\bullet)}$, since we are interested in identifying the quotient complex $J_{a,(b,0)}$ of the latter. We now work towards the proof of the following:

Proposition 9.37. *The sequence*

$$0 \longrightarrow J_{a,(b-0,0)} \xrightarrow{d_0} \cdots \xrightarrow{d_{s-1}} \mathbf{q}^{s(s-1)}\mathbf{t}^s J_{a,(b-s,s)} \xrightarrow{d_s} \cdots \xrightarrow{d_{b-1}} \mathbf{q}^{b(b-1)}\mathbf{t}^b J_{a,(0,b)}$$

is exact.

The proof will follow almost immediately from the following two lemmata.

Lemma 9.38. *The complex $J_{a,(b-\bullet,\bullet)}$ is equal to the image of the complex $\mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}, \mathcal{V}\mathsf{KMCS}_{a,b})$ under the chain map to $E_{a,(b-\bullet,\bullet)}$ induced by the map*

$$\mathcal{V}\mathsf{KMCS}_{a,b} \cong \mathcal{V}\mathsf{TD}_b(a) \xrightarrow{\Phi} \mathbf{1}_a \boxtimes \mathcal{V}\mathsf{TD}_b(0)$$

from Theorem 7.14.

Proof. By definition, $J_{a,(b-s,s)}$ is the image of the homology

$$H(M_{a,(b-s,s)}) = H\big(\mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}, \mathbf{q}^{s(b-s)} I^{(s)}(\mathcal{V}\mathsf{FT}_{a,b-s}))\big)$$

under the map induced by $I^{(s)}(\Sigma_{a,b-s})$. By Theorem 7.14, this is the same as the image under the map induced on homology by the component

$$\Phi^{s,s} \colon \mathbf{q}^{s(b-s)} I^{(s)}(\mathcal{V}\mathsf{FT}_{a,b-s}) \to \mathbf{q}^{s(b-s)}\mathrm{Dig}^s_{a,b}$$

of Φ. (Recall from (120) that $I^{(s)}(\mathcal{V}\mathsf{FT}_{a,b-s})$ is the summand of $\mathcal{V}\mathsf{TD}_b(a)$ in $\mathbf{t}$-degree s.) Since $E_{a,(b-s,s)}$ has trivial differential, Theorem 7.14 implies that it suffices to show that every element in the image of the map

$$\Phi^{s,s} \colon \mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}, \mathbf{q}^{s(b-s)} I^{(s)}(\mathcal{V}\mathsf{FT}_{a,b-s})) \to \mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}, \mathbf{q}^{s(b-s)}\mathrm{Dig}^s_{a,b})$$

is the image of a cycle in $\mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}, \mathbf{q}^{s(b-s)} I^{(s)}(\mathcal{V}\mathsf{FT}_{a,b-s}))$.

To this end, recall from Definition 7.8 that the deformed splitting map $\Sigma_{a,b} \colon \mathcal{V}\mathsf{FT}_{a,b} \to \mathbf{1}_{a,b}$ is supported on the "corners" $\bigoplus_{l=0}^{b} P_{l,l,0} \subset \mathcal{V}\mathsf{FT}_{a,b}$. Lemma 9.27 and (141) imply that the same is true for $I^{(s)}(\mathcal{V}\mathsf{FT}_{a,b-s})$, i.e. $I^{(s)}(\Sigma_{a,b-s})$ is supported on the corners

$$P_{\bullet,\bullet,s} := \bigoplus_{l=0}^{b-s} P_{l,l,s} \subset I^{(s)}(\mathcal{V}\mathsf{FT}_{a,b-s})\,.$$

In fact, this implies the same statement with $I^{(s)}(\Sigma_{a,b-s})$ replaced by $\Phi^{s,s}$. Indeed, the proof of Theorem 7.14 gives the explicit formula

$$\Phi^{s,s} = d_{s-1} \circ k_s \circ I^{(s)}(\Sigma_{a,b-s}) + k_{s+1} \circ I^{(s+1)}(\Sigma_{a,b-s-1}) \circ (\delta^c + \Delta^c)$$

with d_s and k_s as in Definition 7.12 and Lemma 7.13. Since $I^{(s)}(\Sigma_{a,b-s})$ is supported on the corners, the same is true for the first summand. For the second summand, note that Proposition 6.8 and equation (107) imply that the connecting differential

$$\delta^c + \Delta^c \colon I^{(s)}(\mathcal{V}\mathsf{FT}_{a,b-s}) \to I^{(s+1)}(\mathcal{V}\mathsf{FT}_{a,b-s-1})$$

never sends "non-corners" (i.e. summands $P_{k,l,s}$ with $l < k$) to corners. Since $I^{(s+1)}(\Sigma_{a,b-s-1})$ is supported on the corners, the second summand is as well.

Finally, it follows from the description of the vertical component δ^v of the differential on $I^{(s)}(\mathcal{V}\mathsf{FT}_{a,b-s}) \cong \mathcal{V}\mathsf{MCCS}^s_{a,b}$ given by Proposition 6.8 that

$$\mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}, P_{\bullet,\bullet,s}) \subset \mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}, I^{(s)}(\mathcal{V}\mathsf{FT}_{a,b-s}))$$

is a subcomplex with zero differential. Thus every element in the support of $\Phi^{s,s}$ is a cycle, so its image is spanned by cycles. □

Lemma 9.39. *The inclusion $J_{a,(0,b)} \hookrightarrow J_{a,(b-\bullet,\bullet)}$ is surjective in homology.*

Proof. Lemma 9.38 gives a chain map

$$\Phi\colon \mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}, \mathcal{V}\mathsf{KMCS}_{a,b}) \to J_{a,(b-\bullet,\bullet)} \,. \tag{142}$$

Observe that the complex

$$\mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}, \mathcal{V}\mathsf{KMCS}_{a,b}) \cong \mathrm{tw}_{\delta^c+\Delta^c}\left(\bigoplus_{s=0}^{b} \mathrm{Hom}_{\mathcal{V}_{a,b}}\left(\mathbf{1}_{a,b}, \mathbf{q}^{s(b-1)}\mathbf{t}^s I^{(s)}(\mathcal{V}\mathsf{FT}_{a,b-s})\right)\right)$$

is filtered by s-degree, and we can likewise view $J_{a,(b-\bullet,\bullet)} = \mathrm{tw}_d\left(\bigoplus_{s=0}^{b} J_{a,(b-s,s)}\right)$ as s-filtered. Theorem 7.14 then implies that Φ is a filtered chain map. Further, by Corollary 9.28 and Definition 9.29, the chain map (142) induces an isomorphism in homology for the associated graded complexes. Since the s-filtration is bounded, a straightforward argument using the long exact sequence associated to a short exact sequence of chain complexes and the five lemma implies[14] that Φ induces an isomorphism

$$H\big(\mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}, \mathcal{V}\mathsf{KMCS}_{a,b})\big) \cong H\big(J_{a,(b-\bullet,\bullet)}\big) \,. \tag{143}$$

We are therefore interested in the complex:

$$\mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}, \mathcal{V}\mathsf{KMCS}_{a,b}) \cong \mathrm{tw}_{\sum \bar{v}_i\xi_i}\left(\mathrm{Hom}_{\mathcal{C}_{a,b}}(\mathbf{1}_{a,b}, \mathsf{MCS}_{a,b}) \otimes \mathbb{Q}[\mathbb{V}_L^{(a)}, \mathbb{V}_R^{(b)}] \otimes \textstyle\bigwedge[\xi_1, \dots, \xi_b]\right) \tag{144}$$

Observe that the inclusion of the summand

$$\mathrm{Hom}_{\mathcal{C}_{a,b}}(\mathbf{1}_{a,b}, \mathsf{MCS}_{a,b}) \otimes \mathbb{Q}[\mathbb{V}_L^{(a)}, \mathbb{V}_R^{(b)}] \otimes \xi_1 \cdots \xi_b \hookrightarrow \text{right-hand side of (144)} \tag{145}$$

is a chain map, since the components of the differential that leave the summand

$$\mathsf{MCS}_{a,b} \otimes \mathbb{Q}[\mathbb{V}_L^{(a)}, \mathbb{V}_R^{(b)}] \otimes \xi_1 \cdots \xi_b \subset \mathsf{KMCS}_{a,b}$$

become zero upon applying $\mathrm{Hom}_{\mathcal{C}_{a,b}}(\mathbf{1}_{a,b}, -)$. Further, (145) is surjective in homology, because the twist in the right-hand side of (144) is the Koszul differential associated to the action of the regular sequence $\bar{v}_1, \dots, \bar{v}_b$ on $\mathrm{Hom}_{\mathcal{C}_{a,b}}(\mathbf{1}_{a,b}, \mathsf{MCS}_{a,b}) \otimes \mathbb{Q}[\mathbb{V}_L^{(a)}, \mathbb{V}_R^{(b)}]$.

Now, let $\iota \in \mathrm{Hom}_{\mathcal{C}_{a,b}}(\mathbf{1}_{a,b}, \mathsf{MCS}_{a,b})$ be the inclusion of $\mathbf{1}_{a,b}$ into $\mathsf{MCS}_{a,b}$ as the right-most chain group, i.e. ι spans the chain group $\mathrm{Hom}^b_{\mathcal{C}_{a,b}}(\mathbf{1}_{a,b}, \mathsf{MCS}_{a,b})$. It follows from Proposition 6.5 and equations (75) and (76) that the homology of $\mathrm{Hom}_{\mathcal{C}_{a,b}}(\mathbf{1}_{a,b}, \mathsf{MCS}_{a,b})$ is supported in $\mathbf{t}$-degree b, hence is spanned by the class of ι. Thus, the composition

$$\mathrm{End}_{\mathcal{C}_{a,b}}(\mathbf{1}_{a,b}) \otimes \mathbb{Q}[\mathbb{V}_L^{(a)}, \mathbb{V}_R^{(b)}] \to \mathrm{Hom}_{\mathcal{C}_{a,b}}(\mathbf{1}_{a,b}, \mathsf{MCS}_{a,b}) \otimes \mathbb{Q}[\mathbb{V}_L^{(a)}, \mathbb{V}_R^{(b)}] \hookrightarrow \mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}, \mathcal{V}\mathsf{KMCS}_{a,b})$$

sending $1 \mapsto \iota \otimes \xi_1 \cdots \xi_b$ is surjective in homology. Upon inspection, we see that this is simply the inclusion

$$\mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}, I^{(b)}(\mathbf{1}_{a,0})) \hookrightarrow \mathrm{Hom}_{\mathcal{V}_{a,b}}(\mathbf{1}_{a,b}, \mathcal{V}\mathsf{KMCS}_{a,b}) \,.$$

It then follows, via (143), that the inclusion of $J_{a,(0,b)}$ into $J_{a,(b-\bullet,\bullet)}$ is surjective in homology. □

Proof (of Proposition 9.37). The homology of the complex

$$0 \longrightarrow J_{a,(b-0,0)} \xrightarrow{d_0} \cdots \xrightarrow{d_{s-1}} \mathbf{q}^{s(s-1)}\mathbf{t}^s J_{a,(b-s,s)} \xrightarrow{d_s} \cdots \xrightarrow{d_{b-1}} \mathbf{q}^{b(b-1)}\mathbf{t}^b J_{a,(0,b)} \longrightarrow 0$$

is supported on the far right. □

We now establish Theorem 9.33 and Corollary 9.34.

[14] Alternatively, this follows from the standard fact that if a morphism of spectral sequences is an isomorphism on a certain page, then it is an isomorphism on all subsequent pages.

Proof of Theorem 9.33. We show that $I_{a,\ell} = J_{a,\ell}$ for $1 \leq \ell < a$ implies that $I_{a,\ell+1} = J_{a,\ell+1}$. The result then follows inductively from the base case[15] $I_{a,1} = J_{a,1}$ that was established above in Lemma 9.35.

To begin, we first claim that the inclusion $E_{a,(b,0)} \hookrightarrow E_{a,(b-1,1)}$ restricts to an inclusion $I_{a,(b,0)} \hookrightarrow I_{a,(b-1,1)}$, i.e. it sends elements in the former to the latter. Indeed, recall that the ideal $I_{a,(b,0)}$ is generated over $E_{a,(b,0)}$ by expressions

$$\frac{f(\mathbb{X},\mathbb{Y})}{\Delta(\mathbb{X}_1)\Delta(\mathbb{X}_2)}$$

where $f \in \mathbb{Q}[\mathbb{X},\mathbb{Y}]$ is anti-symmetric for the diagonal action of $\mathfrak{S}_{a+b}$. Similarly, Definition 9.31 implies that $I_{a,(b-1,1)}$ is generated over $E_{a,(b-1,1)}$ by expressions

$$\frac{g(\mathbb{X} \smallsetminus \{x_{a+b}\}, \mathbb{Y} \smallsetminus \{y_{a+b}\})}{\Delta(\mathbb{X}_1)\Delta(\mathbb{X}_2 \smallsetminus \{x_{a+b}\})},$$

where $g \in \mathbb{Q}[\mathbb{X},\mathbb{Y}]$ is anti-symmetric for the diagonal action of $\mathfrak{S}_{a+b-1} \times \mathfrak{S}_1$. Thus, for $\mathfrak{S}_{a+b}$-anti-symmetric $f \in \mathbb{Q}[\mathbb{X},\mathbb{Y}]$, it suffices to show that

(146)
$$f(\mathbb{X},\mathbb{Y}) \in E_{a,(b-1,1)}\text{- span}\left\{\frac{\Delta(\mathbb{X}_2)}{\Delta(\mathbb{X}_2 \smallsetminus \{x_{a+b}\})} \cdot g(\mathbb{X},\mathbb{Y}) \mid g \in \mathbb{Q}[\mathbb{X},\mathbb{Y}] \text{ antisymmetric for } \mathfrak{S}_{a+b-1} \times \mathfrak{S}_1.\right\}$$

Note that $\Delta(\mathbb{X}_2)\Delta(\mathbb{X}_2 \smallsetminus \{x_{a+b}\})^{-1} = \prod_{j=a+1}^{a+b-1}(x_j - x_{a+b})$.

By the $E_{a,(b,0)}$-linearity of the inclusion $E_{a,(b,0)} \hookrightarrow E_{a,(b-1,1)}$, it suffices to check (146) in the case when f is an antisymmetrized monomial, i.e. when $f(\mathbb{X},\mathbb{Y}) = \Delta_S(\mathbb{X},\mathbb{Y}) = \operatorname{hdet}(S)$ for some collection

$$S = \{m_i(x,y)\}_{i=1}^{a+b} \subset \mathbb{Q}[x,y]$$

of monic monomials. In this case, we now see that (146) follows from Laplace expansion in the last column ($j = a+b$), after performing certain column operations on $f(\mathbb{X},\mathbb{Y}) = |m_i(x_j,y_j)|_{i,j=1}^{a+b}$.

To describe these column operations we use the identity

$$\sum_{j=a+1}^{a+b} (-1)^{a+b-j} \frac{\Delta(\mathbb{X}_2 \smallsetminus \{x_j\})}{\Delta(\mathbb{X}_2)} \cdot m_i(x_j,y_j) = \partial_{a+1}\cdots\partial_{a+b-1}(m_i(x_{a+b},y_{a+b})) \in E_{(a,b),0} \subset E_{(a,b-1),1}$$

which is straightforward to verify by induction in b. (The containment in $E_{(a,b),0}$ holds since the Sylvester operator $\partial_{a+1}\cdots\partial_{a+b-1}$ maps $\operatorname{Sym}(\mathbb{X}_2 \smallsetminus \{x_{a+b}\}|x_{a+b})[\mathbb{V}_R^{(b)}] \to \operatorname{Sym}(\mathbb{X}_2)[\mathbb{V}_R^{(b)}]$). The desired column operations change the entries of the last column as follows:

(147)
$$\begin{aligned} m_i(x_{a+b},y_{a+b}) \mapsto & m_i(x_{a+b},y_{a+b}) + \sum_{j=a+1}^{a+b-1} (-1)^{a+b-j} \frac{\Delta(\mathbb{X}_2 \smallsetminus \{x_j\})}{\Delta(\mathbb{X}_2 \smallsetminus \{x_{a+b}\})} \cdot m_i(x_j,y_j) \\ &= \frac{\Delta(\mathbb{X}_2)}{\Delta(\mathbb{X}_2 \smallsetminus \{x_{a+b}\})} \partial_{a+1}\cdots\partial_{a+b-1}(m_i(x_{a+b},y_{a+b})). \end{aligned}$$

Laplace expansion in the last column then gives

$$|m_i(x_j,y_j)|_{i,j=1}^{a+b} = \sum_{k=1}^{a+b} (-1)^{a+b+k} \partial_{a+1}\cdots\partial_{a+b-1}(m_k(x_{a+b},y_{a+b})) \frac{\Delta(\mathbb{X}_2)}{\Delta(\mathbb{X}_2 \smallsetminus \{x_{a+b}\})} |m_i(x_j,y_j)|_{i\neq k, j\neq a+b},$$

which verifies (146). (Note that we have to slightly extend scalars by inverting the expression $\Delta(\mathbb{X}_2 \smallsetminus \{x_{a+b}\})$ to perform (147); the value of the determinant remains unaffected regardless.)

[15] Our use of Proposition 9.37 later in the proof will make clear that we cannot simply induct up from the obvious equality $I_{a,0} = J_{a,0}$.

Suppose now that $2 \leq \ell \leq b$ and that we have shown that $I_{a,m} = J_{a,m}$ for $1 \leq m < \ell$. In particular, Lemmata 9.30 and 9.32 imply that $I_{a,(\ell-s,s)} = J_{a,(\ell-s,s)}$ for $1 \leq s \leq \ell - 2$. We thus consider the following commutative diagram:

$$\begin{array}{ccccccc}
 & & I_{a,(\ell,0)} & \xhookrightarrow{d_0} & I_{a,(\ell-1,1)} & \xrightarrow{d_1} & I_{a,(\ell-2,2)} \\
 & & \uparrow & & \| & & \| \\
0 & \longrightarrow & J_{a,(\ell,0)} & \xhookrightarrow{d_0} & J_{a,(\ell-1,1)} & \xrightarrow{d_1} & J_{a,(\ell-2,2)}
\end{array}$$

where the vertical inclusion is given by Proposition 9.25. Since $d_1 \circ d_0 = 0$ (by Lemma 9.36), we have that $I_{a,(\ell,0)} \subset \ker\left(I_{a,(\ell-1,1)} \xrightarrow{d_1} I_{a,(\ell-2,2)}\right)$. However,

$$\ker\left(I_{a,(\ell-1,1)} \xrightarrow{d_1} I_{a,(\ell-2,2)}\right) = \ker\left(J_{a,(\ell-1,1)} \xrightarrow{d_1} J_{a,(\ell-2,2)}\right)$$

and Proposition 9.37 implies that the latter equals $J_{a,(\ell,0)}$. Thus $I_{a,\ell} = I_{a,(\ell,0)} = J_{a,(\ell,0)} = J_{a,\ell}$, which establishes the result by induction. □

We record a consequence of Theorem 9.33, which gives a generating set for the ideal $I_{a,b} \lhd E_{a,b}$ of cardinality 2^b. Note that its statement (unlike its proof) appears to have nothing to do with link homology!

Corollary 9.40. $I_{a,b} = E_{a,b} \cdot \left\{ \frac{\Delta_{\mathrm{Key}_l(\lambda)}(\mathbb{X},\mathbb{Y})}{\Delta(\mathbb{X}_1)\Delta(\mathbb{X}_2)} \mid 0 \leq l \leq b\,,\ \lambda \in P(l, b-l) \right\}$.

Proof. Theorem 9.33 allows us to upgrade the statement of Proposition 9.23 to the following:

$$\Delta_{\mathrm{Key}_l(\lambda)}(\mathbb{X},\mathbb{Y}) - \pi(\Delta_{\mathrm{Key}_l(\lambda)}(\mathbb{X},\mathbb{Y})) = \sum_{\substack{m \leq l-1 \\ \mu \in P(m,b-m)}} c_{\lambda,\mu} \cdot \pi(\Delta_{\mathrm{Key}_m(\mu)}(\mathbb{X},\mathbb{Y})) \tag{148}$$

for $c_{\lambda,\mu} \in E_{a,b}$. Indeed, for $R \in \mathsf{S}_{l-1}$ (recall Definition 9.22), we have that $\frac{\Delta_R(\mathbb{X},\mathbb{Y})}{\Delta(\mathbb{X}_1)\Delta(\mathbb{X}_2)} \in I_{a,b} = J_{a,b}$, thus $\frac{\pi(\Delta_R(\mathbb{X},\mathbb{Y}))}{\Delta(\mathbb{X}_1)\Delta(\mathbb{X}_2)} \in \pi(J_{a,b}) \subset J_{a,b}$. By Proposition 9.11 and Lemma 9.16, this implies that

$$\frac{\pi(\Delta_R(\mathbb{X},\mathbb{Y}))}{\Delta(\mathbb{X}_1)\Delta(\mathbb{X}_2)} = \sum_{\substack{m \leq l-1 \\ \mu \in P(m,b-m)}} c_{R,\mu} \cdot \frac{\pi(\Delta_{\mathrm{Key}_m(\mu)}(\mathbb{X},\mathbb{Y}))}{\Delta(\mathbb{X}_1)\Delta(\mathbb{X}_2)}$$

where the bound $m \leq l-1$ follows by comparing y-degree. This gives (148).

Now, (148) implies that there is a unitriangular matrix with coefficients in $E_{a,b}$ relating

$$\{\Delta_{\mathrm{Key}_l(\lambda)}(\mathbb{X},\mathbb{Y}) \mid 0 \leq l \leq b\,,\ \lambda \in P(l,b-l)\}$$

and

$$\{\pi(\Delta_{\mathrm{Key}_l(\lambda)}(\mathbb{X},\mathbb{Y})) \mid 0 \leq l \leq b\,,\ \lambda \in P(l,b-l)\}\,.$$

Thus,

$$\begin{aligned}
I_{a,b} = J_{a,b} = E_{a,b} \cdot &\left\{ \frac{\pi(\Delta_{\mathrm{Key}_l(\lambda)}(\mathbb{X},\mathbb{Y}))}{\Delta(\mathbb{X}_1)\Delta(\mathbb{X}_2)} \mid 0 \leq l \leq b\,,\ \lambda \in P(l,b-l) \right\} \\
= E_{a,b} \cdot &\left\{ \frac{\Delta_{\mathrm{Key}_l(\lambda)}(\mathbb{X},\mathbb{Y})}{\Delta(\mathbb{X}_1)\Delta(\mathbb{X}_2)} \mid 0 \leq l \leq b\,,\ \lambda \in P(l,b-l) \right\}
\end{aligned}$$

where in the first line we again use Proposition 9.11 and Lemma 9.16. □

10. Link splitting and effective thickness

The explicit description of the 2-strand full twist ideal allows for the extension of the link splitting results from [BS15, GH] to the setting of colored, triply-graded link homology. In particular, in §10.1 we establish an analogue of [GH, Corollary 4.4] under the condition that certain elements are invertible. We call these elements *transparifers*, and their invertibility is the colored analogue of the condition from [GH] that the differences of deformation parameters y on distinct uncolored link components are invertible. However, in the colored setting, Definitions 5.24 and 5.27 endow each link component with an alphabet of deformation parameters of cardinality equal to the color of the link component. Consequently, there are numerous other specializations of interest that do not invert the transparifer. We save a complete investigation for future work, but briefly discuss a heuristic for these deformations in §10.2. This makes contact with Conjecture 1.25

10.1. **Crossing change morphisms and general splitting maps.** Much of the machinery of [GH, Section 4] applies mutatis mutandis, using the following analogue of Corollary 7.11.

Lemma 10.1. *Let $a \geq b \geq 0$. There exists a closed morphism $\psi_{a,b}\colon \mathbf{1}_{a,b} \to \mathcal{V}\mathsf{FT}_{a,b}$ of weight $\mathbf{q}^{-2b}\mathbf{t}^{2b}$ such that*

$$\Sigma_{a,b} \circ \psi_{a,b} = \frac{\Delta_{\mathrm{Key}_b(\emptyset)}(\mathbb{X},\mathbb{Y})}{\Delta(\mathbb{X}_1)\Delta(\mathbb{X}_2)} \cdot \mathrm{id}_{\mathbf{1}_{a,b}} \quad \textit{and} \quad \psi_{a,b} \circ \Sigma_{a,b} \sim \frac{\Delta_{\mathrm{Key}_b(\emptyset)}(\mathbb{X},\mathbb{Y})}{\Delta(\mathbb{X}_1)\Delta(\mathbb{X}_2)} \cdot \mathrm{id}_{\mathcal{V}\mathsf{FT}_{a,b}}\,.$$

Proof. Since $\Delta_{\mathrm{Key}_b(\emptyset)}(\mathbb{X},\mathbb{Y})\Delta(\mathbb{X}_1)^{-1}\Delta(\mathbb{X}_2)^{-1} \in I_{a,b} = J_{a,b} = \mathrm{im}(H(\Sigma_{a,b}))$ we can find closed $\psi_{a,b} \in M_{a,b}$ satisfying the first equation. The second relation follows as in the proof of Corollary 7.11. □

As a reminder, $\mathbb{X} = \mathbb{X}_1 \cup \mathbb{X}_2 = \{x_1, \ldots, x_{a+b}\}$ and $\mathbb{Y} = \{y_1, \ldots, y_{a+b}\}$. Further, recall from Remark 4.28 that expressions of the form $g \cdot \mathrm{id}_X$ with $g \in \mathrm{End}_{\mathcal{Y}(\mathrm{SSBim})}(\mathbf{1}_{a,b})$ are shorthand for the morphism

$$X \cong \mathbf{1}_{a,b} \star X \xrightarrow{g\star \mathrm{id}} \mathbf{1}_{a,b} \star X \cong X\,.$$

The morphism $\psi_{a,b}$ from Lemma 10.1 is the unreduced analogue of $\overline{\psi}_{a,b}$ from Corollary 7.11. We work with it here, since it treats the deformation parameters on the a- and b-labeled strands more "democratically" than $\overline{\psi}_{a,b}$.

Remark 10.2. As Corollary 7.11 and Lemma 10.1 show, the "sections" of the 2-strand full twist splitting map are not unique in the colored case. The maps $\psi_{a,b}$ and $\overline{\psi}_{a,b}$ are two valid choices.

Given alphabets $\mathbb{X}_i, \mathbb{X}_j$ and associated alphabets $\mathbb{Y}_i$, $\mathbb{Y}_j$, set $a = \max(|\mathbb{X}_i|, |\mathbb{X}_j|)$ and $b = \min(|\mathbb{X}_i|, |\mathbb{X}_j|)$. We will abbreviate

$$D_{i,j} := \frac{\Delta_{\mathrm{Key}_b(\emptyset)}(\mathbb{X}_i + \mathbb{X}_j, \mathbb{Y}_i + \mathbb{Y}_j)}{\Delta(\mathbb{X}_i)\Delta(\mathbb{X}_j)} \tag{149}$$

and call this the *transparifer* of the morphism $\psi_{a,b}$. Whenever the transparifer is invertible, the morphism $\psi_{a,b}$ from Lemma 10.1 becomes a homotopy equivalence between the identity bimodule and the 2-strand full twist. Thus, strands in the latter become *transparent* to each other.

Example 10.3. Under the substitution $v_{i,r} \mapsto 0$ and $v_{j,r} \mapsto 0$ for $r > 1$, the transparifer reduces to $\pm(v_{j,1} - v_{i,1})^b$. If we further specialize $v_{j,1} \mapsto z_j u$ for some scalars z_j and invertible u as in Example 5.45 (3), then the transparifer becomes invertible whenever $z_i \neq z_j$.

Lemma 10.4. *Consider the i^{th} colored Artin generator $(\beta_i)_{\boldsymbol{b}}$ in $\mathrm{Br}_m(\mathbb{Z}_{\geq 1})$ and let $\omega \in \mathfrak{S}_m$. Set $b := \min(b_i, b_{i+1})$, then there exists a pair of degree zero morphisms:*

$$\psi^+ \colon \mathcal{Y}C((\beta_i)_{\boldsymbol{b},\omega}) \to \mathcal{Y}C((\beta_i^{-1})_{\boldsymbol{b},\omega})\,, \quad \psi^- \colon \mathcal{Y}C((\beta_i^{-1})_{\boldsymbol{b},\omega}) \to \mathbf{q}^{2b}\mathbf{t}^{-2b}\mathcal{Y}C((\beta_i)_{\boldsymbol{b},\omega})$$

such that $\psi^+ \circ \psi^- \sim D_{\omega(i),\omega(i+1)} \cdot \mathrm{id}_{\mathcal{Y}C(\beta_i^{-1})}$, and $\psi^- \circ \psi^+ \sim D_{\omega(i),\omega(i+1)} \cdot \mathrm{id}_{\mathcal{Y}C(\beta_i)}$.

Proof. This follows from Lemma 10.1 by horizontal composition (on the right) with the curved Rickard complex for the inverse Artin generator. □

More generally, we have the following.

Proposition 10.5. *Let $\beta^+_{\boldsymbol{b}}$ and $\beta^-_{\boldsymbol{b}}$ be colored braids in $\mathrm{Br}_m(\mathbb{Z}_{\geq 1})$ that differ in a single crossing, i.e.:*

$$\beta^{\pm}_{\boldsymbol{b}} := \beta' \beta_i^{\pm} \beta''_{\boldsymbol{b}}$$

for some $\beta', \beta'' \in \mathrm{Br}_m(\mathbb{Z}_{\geq 1})$. Let $b := \min(\beta''(\boldsymbol{b})_i, \beta''(\boldsymbol{b})_{i+1})$ (i.e. the minimal color involved in the crossing in the Artin generator $\beta_i^{\pm}$) and let $\omega \in \mathfrak{S}_m$. Then, there exists a pair of degree zero morphisms

$$\psi^+ \colon \mathcal{Y}C(\beta^+_{\boldsymbol{b},\omega}) \to \mathcal{Y}C(\beta^-_{\boldsymbol{b},\omega}) \,, \quad \psi^- \colon \mathcal{Y}C(\beta^-_{\boldsymbol{b},\omega}) \to \mathbf{q}^{2b}\mathbf{t}^{-2b}\mathcal{Y}C(\beta^+_{\boldsymbol{b},\omega})$$

such that

$$\psi^+ \circ \psi^- \sim \mathrm{id}_{\mathcal{Y}C(\beta')} \star \big(D_{(\beta''\omega)(i),(\beta''\omega)(i+1)} \cdot \mathrm{id}_{\mathcal{Y}C(\beta_i^{-1})}\big) \star \mathrm{id}_{\mathcal{Y}C(\beta''_{\boldsymbol{b},\omega})}$$

and

$$\psi^- \circ \psi^+ \sim \mathrm{id}_{\mathcal{Y}C(\beta')} \star \big(D_{(\beta''\omega)(i),(\beta''\omega)(i+1)} \cdot \mathrm{id}_{\mathcal{Y}C(\beta_i)}\big) \star \mathrm{id}_{\mathcal{Y}C(\beta''_{\boldsymbol{b},\omega})} \,.$$

Proof. This is an immediate consequence of Lemma 10.4 using horizontal composition in $\mathcal{Y}(\mathrm{SSBim})$. □

Note that if the crossing in Proposition 10.5 occurs between strands that correspond to the same component of the closure of a balanced, colored braid, then the relevant transparifer acts null-homotopically on the complex $\mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}})$ from Definition 5.27 (since the alphabets $\mathbb{X}_i$ and $\mathbb{X}_j$ become identified, as do $\mathbb{Y}_i$ and $\mathbb{Y}_j$). Thus, in the setting of link homology, Proposition 10.5 is most interesting when the crossing is between strands in distinct components of the closure.

Now, given a link $\mathbf{L}$ presented as the closure $\widehat{\beta}$ of a braid β, recall that it is possible to unlink a component of $\mathbf{L}$ from the remaining components via a sequence of crossing changes. More precisely, if $\mathbf{L}$ has components $\{\mathbf{L}_i\}_{i=1}^{\ell}$, then there is a sequence of crossing changes that take the braid β to a braid β' where

$$\widehat{\beta} = \mathbf{L} = \mathbf{L}_1 \cup \cdots \cup \mathbf{L}_\ell$$

and

$$\widehat{\beta'} = (\mathbf{L}_1 \cup \cdots \cup \mathbf{L}_{i-1} \cup \mathbf{L}_{i+1} \cup \cdots \cup \mathbf{L}_\ell) \sqcup \mathbf{L}_i \,.$$

In the latter display, the square cup $\sqcup$ denotes the *split union*, which is given by placing two links in disjoint 3-balls in S^3. Passing to the complex $\mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}})$ from Definition 5.27 (which is the relevant complex when considering braid closures), now immediately gives the following.

Proposition 10.6. *Let $\mathbf{L}$ and $\mathbf{L}'$ be colored links presented as the closures of colored braids $\beta_{\boldsymbol{b}}$ and $\beta'_{\boldsymbol{b}}$, and suppose that there is a sequence of crossing changes taking $\beta_{\boldsymbol{b}}$ to $\beta'_{\boldsymbol{b}}$. Let $p_{i,j}$ and $n_{i,j}$ denote the number of positive-to-negative and negative-to-positive crossing changes between the components $\mathbf{L}_i$ and $\mathbf{L}_j$, then there exist closed morphisms*

$$\psi_{\mathbf{L}\to\mathbf{L}'} \colon \mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}}) \to \mathcal{Y}C_{\mathrm{KR}}(\beta'_{\boldsymbol{b}}) \,, \quad \psi_{\mathbf{L}'\to\mathbf{L}} \colon \mathcal{Y}C_{\mathrm{KR}}(\beta'_{\boldsymbol{b}}) \to \mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}})$$

so that

$$\psi_{\mathbf{L}'\to\mathbf{L}} \circ \psi_{\mathbf{L}\to\mathbf{L}'} \sim \prod_{i,j} D_{i,j}^{p_{i,j}+n_{i,j}} \cdot \mathrm{id}_{\mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}})}$$

and

$$\psi_{\mathbf{L}\to\mathbf{L}'} \circ \psi_{\mathbf{L}'\to\mathbf{L}} \sim \prod_{i,j} D_{i,j}^{p_{i,j}+n_{i,j}} \cdot \mathrm{id}_{\mathcal{Y}C_{\mathrm{KR}}(\beta'_{\boldsymbol{b}})} \,.$$

(Here, by slight abuse of notation, $D_{i,j}$ denotes the transparifer evaluated at the relevant quadrupel of alphabets associated with the i^{th} and j^{th} components of $\mathbf{L}$.)

Proof. This is essentially an immediate consequence of Proposition 10.5. The only subtle point is that we must identify the actions of transparifers "in the middle" of curved Rickard complexes with those acting on the left, up to homotopy. This follows since the actions of all the $\mathbb{X}$ alphabets along a strand are homotopic, thus the same holds for the associated $\mathbb{Y}$ alphabets. □

Remark 10.7. With notation as in Proposition 10.6, set

$$p = \sum_{i,j} p_{i,j} \min(\mathbf{L}_i, \mathbf{L}_j)\,, \quad n = \sum_{i,j} n_{i,j} \min(\mathbf{L}_i, \mathbf{L}_j)$$

where $\min(\mathbf{L}_i, \mathbf{L}_j)$ denotes the smaller of the colors of these components, then

$$\mathrm{wt}(\psi_{\mathbf{L}\to\mathbf{L}'}) = \mathbf{q}^{-2n}\mathbf{t}^{2n}\,, \quad \mathrm{wt}(\psi_{\mathbf{L}'\to\mathbf{L}}) = \mathbf{q}^{-2p}\mathbf{t}^{2p}\,.$$

Combining Definition 5.41 with Proposition 10.6 then gives the following.

Corollary 10.8. *Retain notation as in Proposition 10.6. Suppose that M is a coefficient module as in Definition 5.41 such that each occurring $D_{i,j}$ acts invertibly on M, then $\psi_{\mathbf{L}\to\mathbf{L}'}$ induces a homotopy equivalence $\mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}}, M) \simeq \mathcal{Y}C_{\mathrm{KR}}(\beta'_{\boldsymbol{b}}, M)$ and thus an isomorphism $\mathcal{Y}H_{\mathrm{KR}}(\mathbf{L}, M) \cong \mathcal{Y}H_{\mathrm{KR}}(\mathbf{L}', M)$.* □

In particular, the hypotheses of Corollary 10.8 hold under the specialization from Example 10.3. In this special case, we recover [CLS20, Theorem 6.3].

Next, given a colored link $\mathbf{L}$, let $\mathrm{split}(\mathbf{L})$ denote the colored link obtained as the split union of its components. In other words, if $\mathbf{L}$ has components $\{\mathbf{L}_i\}_{i=1}^{\ell}$, then

$$\mathbf{L} = \mathbf{L}_1 \cup \cdots \cup \mathbf{L}_\ell$$

while

$$\mathrm{split}(\mathbf{L}) = \mathbf{L}_1 \sqcup \cdots \sqcup \mathbf{L}_\ell\,.$$

By the discussion preceding Proposition 10.6, there exists a sequence of crossing changes taking a braid presentation for $\mathbf{L}$ to one for $\mathrm{split}(\mathbf{L})$.

Definition 10.9. Let $\mathbf{L}$ be a colored link that is presented as the closure of a colored braid $\beta_{\boldsymbol{b}}$. Given a sequence of crossing changes between distinct components that transforms $\beta_{\boldsymbol{b}}$ into a braid $\mathrm{split}(\beta_{\boldsymbol{b}})$ whose closure is $\mathrm{split}(\mathbf{L})$, the closed morphism

$$\psi_{\mathbf{L}\to\mathrm{split}(\mathbf{L})} \colon \mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}}) \to \mathcal{Y}C_{\mathrm{KR}}(\mathrm{split}(\beta_{\boldsymbol{b}}))$$

associated to this sequence of crossing changes via Proposition 10.6 will be called a *splitting map* for $\mathbf{L}$.

Note that splitting maps and, more generally, the morphisms from Proposition 10.6 may depend on the sequence of crossing changes (i.e. not just on the (co)domain complexes).

Theorem 10.10. *Suppose that $\mathcal{Y}H_{\mathrm{KR}}(\mathbf{L})$ is free over $\mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}]$, e.g. for parity reasons as in Theorem 5.52, then any splitting map induces an injective map on homology*

$$\psi_{\mathbf{L}\to\mathrm{split}(\mathbf{L})} \colon \mathcal{Y}H_{\mathrm{KR}}(\mathbf{L}) \to \mathcal{Y}H_{\mathrm{KR}}(\mathrm{split}(\mathbf{L})).$$

Proof. By Proposition 10.6, composing $\psi_{\mathbf{L}\to\mathrm{split}(\mathbf{L})}$ with the reverse map $\psi_{\mathrm{split}(\mathbf{L})\to\mathbf{L}}$ produces an endomorphism of $\mathcal{Y}H_{\mathrm{KR}}(\mathbf{L})$ that acts by multiplication with a product of transparifers. Denote this product of transparifers by T, and note that it is a homogeneous polynomial in $\mathrm{Sym}(\mathbb{X}^{\pi_0(\mathbf{L})})[\mathbb{V}^{\pi_0(\mathbf{L})}]$. Moreover, Example 10.3 shows that

$$T = T_0 + T_{>0}$$

where $T_{>0}$ is an element of the $\mathbb{X}$-irrelevant ideal (specifically, T_0 is a polynomial in the subalphabet $\{v_{c,1}\}_{c\in\pi_0(\mathbf{L})} \subset \mathbb{V}^{\pi_0(\mathbf{L})}$). By assumption, $\mathcal{Y}H_{\mathrm{KR}}(\mathbf{L})$ is free over $\mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}]$, so the injectivity of the action of T follows from the injectivity of the action of T_0. □

Note that if $\mathbf{L}$ is the closure of a *pure* colored braid $\beta_{\boldsymbol{b}}$, then each of its components are unknots. Thus, any link splitting map gives a map

$$\mathcal{Y}H_{\mathrm{KR}}(\mathbf{L}) \to E_{\boldsymbol{b}}\,.$$

Further, if $\mathbf{L}$ is parity, Theorem 10.10 implies that this map is injective, and thus identifies the former with an ideal in the latter. As such, Theorem 10.10 generalizes Corollary 8.7.

We conclude this section with an application to the undeformed colored Khovanov–Rozansky homology of parity links.

Theorem 10.11. *Let $\mathbf{L}$ be a colored link that is parity. Upon collapsing the trigrading $(\deg_{\mathbf{a}}, \deg_{\mathbf{q}}, \deg_{\mathbf{t}})$ to the bigrading $(\deg_{\mathbf{a}}, \deg_{\mathbf{q}} + \deg_{\mathbf{t}})$, there is an isomorphism*

$$H_{\mathrm{KR}}(\mathbf{L}) \cong H_{\mathrm{KR}}(\mathrm{split}(\mathbf{L}))$$

of bigraded vector spaces.

Proof. Let $\mathbf{L} = \mathbf{L}_1 \cup \cdots \cup \mathbf{L}_\ell$ be an ℓ-component colored link, presented as the closure of a colored braid $\beta_{\boldsymbol{b}}$, and consider $\mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}})$. View $M = \mathbb{Q}$ as a module over $\mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}]$ on which $v_{c,k}$ act by zero if $k > 1$, and $v_{1,1}, \ldots, v_{\ell,1}$ act by distinct scalars. This implies that $v_{c,1} - v_{c',1}$ is invertible for $c \neq c'$ in $\pi_0(\mathbf{L})$, thus Example 10.3 implies that the transparifers associated with distinct link components are invertible. Since $\mathrm{wt}(v_{c,1}) = \mathbf{q}^{-2}\mathbf{t}^2$, we have that $\deg_{\mathbf{q}}(v_{c,1}) + \deg_{\mathbf{t}}(v_{c,1}) = 0$. Thus, we may regard M as a bigraded $\mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}]$-module, with gradings $\deg_{\mathbf{a}}$ and $\deg_{\mathbf{q}} + \deg_{\mathbf{t}}$.

Tensoring Lemma 5.50 with $M = \mathbb{Q}$ gives

$$\begin{aligned}
\mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}}, \mathbb{Q}) &= \mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}}) \otimes_{\mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}]} \mathbb{Q} \\
&\simeq \mathrm{tw}_{\Delta''}(H_{\mathrm{KR}}(\mathbf{L}) \otimes \mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}]) \otimes_{\mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}]} \mathbb{Q} \\
&\cong \mathrm{tw}_{\Delta''}(H_{\mathrm{KR}}(\mathbf{L}) \otimes \mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}] \otimes_{\mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}]} \mathbb{Q}) \cong \mathrm{tw}_{\Delta''}(H_{\mathrm{KR}}(\mathbf{L}))
\end{aligned}$$

for some twist Δ''. Since $\mathbf{L}$ is parity, the twist Δ'' is zero on $H_{\mathrm{KR}}(\mathbf{L})$ for degree reasons, thus $\mathrm{tw}_{\Delta''}(H_{\mathrm{KR}}(\beta_{\boldsymbol{b}})) = H_{\mathrm{KR}}(\mathbf{L})$. On the other hand, Corollary 10.8 and Proposition 5.33 imply that

$$\begin{aligned}
\mathcal{Y}C_{\mathrm{KR}}(\beta_{\boldsymbol{b}}, \mathbb{Q}) &\simeq \mathcal{Y}C_{\mathrm{KR}}(\mathrm{split}(\beta_{\boldsymbol{b}}), \mathbb{Q}) \\
&= \mathcal{Y}C_{\mathrm{KR}}(\mathrm{split}(\beta_{\boldsymbol{b}})) \otimes_{\mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}]} \mathbb{Q} \\
&\simeq C_{\mathrm{KR}}(\mathrm{split}(\beta_{\boldsymbol{b}})) \otimes \mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}] \otimes_{\mathbb{Q}[\mathbb{V}^{\pi_0(\mathbf{L})}]} \mathbb{Q} = C_{\mathrm{KR}}(\mathrm{split}(\beta_{\boldsymbol{b}}))\,.
\end{aligned}$$

Since complexes of vector spaces are homotopy equivalent to their homologies, we have

$$H_{\mathrm{KR}}(\mathrm{split}(\mathbf{L})) \simeq C_{\mathrm{KR}}(\mathrm{split}(\beta_{\boldsymbol{b}})) \simeq H_{\mathrm{KR}}(\mathbf{L})\,.$$

□

10.2. **Effective thickness.** In §10.1, we focused on specializations of the deformation parameters that inverted the transparifer; however, there are various other specializations of interest. This is already apparent in the description of $\mathcal{V}\mathsf{FT}_{a,b}$ afforded by Proposition 6.11 (see Example 6.14 for the $a = 2 = b$ case). Indeed, upon inverting the elements

$$\sum_{l=k+1}^{b} h_{l-k-1}(\mathbb{M}^{(k+i)})\bar{v}_l$$

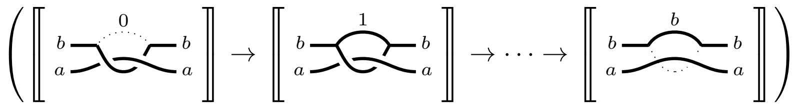
acting in the $(k+i)^{th}$ column of $\mathcal{V}\mathsf{FT}_{a,b}$ for all $i \geq 1$ (e.g. by inverting $\bar{v}_{k+1}$ and setting $\bar{v}_l = 0$ for $l > k+1$), each of these columns becomes null-homotopic. In turn, this forces $\mathcal{V}\mathsf{FT}_{a,b}$ to closely resemble the complex $\mathcal{V}\mathsf{FT}_{a,k}$.

A more-precise formulation of this phenomenon is as follows. Let $C(\beta_{\boldsymbol{b}})$ be the (uncurved) complex of singular Soergel bimodules associated to a colored braid $\beta_{\boldsymbol{b}}$. Let s be a connected component of $\beta_{\boldsymbol{b}}$ (i.e. a strand), and let $\mathbb{X}$ and $\mathbb{X}'$ be the associated alphabets on the left and right, respectively.

Intuitively speaking, "turning on" strand-wise curvature of the form $h_{a'+1}(\mathbb{X}-\mathbb{X}')v_{a'+1}$ with $v_{a'+1}$ invertible makes s behave as if its color is $\leq a'$. We might say that such a strand now has *effective thickness* $\leq a'$.

Let us illustrate this with a concrete example. Extending our notation from §7.3, let $\mathrm{TD}_b(a)$ denote the complex of (uncurved) threaded digons appearing on the left-hand side in the *undeformed* colored skein relation (95). Omitting all explicit grading shifts, this is

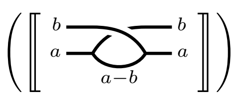
(150) $\mathrm{TD}_b(a) :=$



By [HRW21, Proposition 2.31 and Theorem 3.4], $\mathrm{TD}_b(a) \simeq 0$ when $a < b$, so (150) detects the "thickness" of the a-labeled strand. If instead $a \geq b$, this complex is not contractible, but becomes so after deforming to give the a-labeled strand effective thickness smaller than b. Thus, this deformation forces the a-labeled strand to "act" as if its label was smaller than b.

To see this, it is convenient to replace $\mathrm{TD}_b(a)$ by the (homotopy equivalent) complex on the right-hand side of the colored skein relation (95)

$\mathrm{KMCS}_{a,b} = K$



$= \mathrm{tw}_\delta\left(\mathrm{MCS}_{a,b} \otimes \wedge[\xi_1, \ldots, \xi_b]\right).$

(Again, we are suppressing grading shifts.) Recall from Definition 6.1 that $\delta = \sum_{1\leq i\leq b} h_i(\mathbb{X}_2 - \mathbb{X}_2') \otimes \xi_i^*$. Fix an integer $a' \geq 0$ and "turn on" a curved Maurer–Cartan element Δ with curvature $h_{a'+1}(\mathbb{X}_1 - \mathbb{X}_1')v_{a'+1}$ with $v_{a'+1}$ invertible. If $a' < b$, we claim that this deformed complex is contractible. Indeed, since $\mathbb{X}_1 + \mathbb{X}_2 = \mathbb{X}_1' + \mathbb{X}_2'$ on $\mathrm{KMCS}_{a,b}$, equation (20) gives that

$$h_r(\mathbb{X}_1 - \mathbb{X}_1') = h_r(\mathbb{X}_2' - \mathbb{X}_2) = -\sum_{j=1}^{r} h_{r-j}(\mathbb{X}_2' - \mathbb{X}_2) h_j(\mathbb{X}_2 - \mathbb{X}_2')$$

so we may take

$$\Delta = -v_{a'+1} \sum_{j=1}^{a'+1} h_{a'+1-j}(\mathbb{X}_2' - \mathbb{X}_2)\xi_j = -v_{a'+1}\xi_{a'+1} - \sum_{j=1}^{a'} h_{a'+1-j}(\mathbb{X}_2' - \mathbb{X}_2)\xi_j\,.$$

It follows that

$$\mathrm{tw}_\Delta(\mathrm{KMCS}_{a,b}) \cong \mathrm{tw}_{-\sum_{j=1}^{a'} h_{a'+1-j}(\mathbb{X}_2'-\mathbb{X}_2)\xi_j}\left(\mathrm{tw}_{-v_{a'+1}\xi_{a'+1}}(\mathrm{KMCS}_{a,b})\right) \simeq 0\,.$$

Here, we have used that $\mathrm{tw}_{-v_{a'+1}\xi_{a'+1}}(\mathrm{KMCS}_{a,b})$ is contractible when $v_{a'+1}$ is invertible, together with Proposition 4.1, which show that further twisting by $-\sum_{j=1}^{a'} h_{a'+1-j}(\mathbb{X}_2' - \mathbb{X}_2)\xi_j$ does not break contractibility.

Since the present paper is already quite long, we save further investigations along these lines for future work. However, to come full circle, we do comment that our discussion here informs Conjecture 1.25 from the introduction. Recall that part of this conjecture asserts that a generalization of (150) should be interpreted as the b^{th} "elementary symmetric function" of certain braids associated with the strands threading the digons. Some motivation is thus: as with (150), when the (effective) size of the inputs to the elementary symmetric function $e_b(-)$ is smaller than b, it vanishes.

Appendix A. Some Hom-space computations

We first recall the following.

Lemma A.1 ([RT21, Lemma 4.10])**.** *Let $b, c \geq 0$, then there is an isomorphism*

$$\mathrm{Tr}^c({}_{b,c}S_{b+c} \star {}_{b+c}M_{b,c}) \cong \prod_{i=1}^{c} (\mathbf{q}^b + \mathbf{a}\mathbf{q}^{-b-2i}) \cdot \mathbf{1}_b \otimes \mathrm{End}_{\mathrm{SSBim}}(\mathbf{1}_c)$$

of $\mathrm{End}_{\mathrm{SSBim}}(\mathbf{1}_c)$-modules. (The $\mathrm{End}_{\mathrm{SSBim}}(\mathbf{1}_c)$ action on the left-hand side is induced from the action on the left c-labeled boundary of ${}_{b,c}S_{b+c} \star {}_{b+c}M_{b,c}$.)

We now use the colored partial trace to give quick proofs of the following results. To save space, we abbreviate $\mathcal{C} = \mathcal{C}(\mathrm{SSBim})$, $\mathcal{C}(\mathcal{D}) = \mathcal{C}(\mathcal{D}(\mathrm{Bim}))$, and ${}_{b,c}S_{b+c}M_{b,c} = {}_{b,c}S_{b+c} \star {}_{b+c}M_{b,c}$.

Corollary A.2. *Let $a, b, c \geq 0$ and let $X, Y \in \mathcal{C}({}_{a,b}\mathrm{SSBim}_{a,,b})$ be complexes of singular Soergel bimodules. There is an isomorphism of $\mathrm{End}_{\mathrm{SSBim}}(\mathbf{1}_c)$-modules*

$$\mathrm{Hom}_{\mathcal{C}}\left(X \boxtimes \mathbf{1}_c, (\mathbf{1}_a \boxtimes {}_{b,c}S_{b+c}M_{b,c}) \star (Y \boxtimes \mathbf{1}_c)\right) \cong \mathbf{q}^{bc} \mathrm{Hom}_{\mathcal{C}}(X, Y) \otimes \mathrm{End}_{\mathcal{C}}(\mathbf{1}_c)$$

natural in both X and Y.

Proof. We compute using Propositions 5.8 and 5.10 and Lemma A.1:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}\left(X \boxtimes \mathbf{1}_c, (\mathbf{1}_a \boxtimes {}_{b,c}S_{b+c}M_{b,c}) \star (Y \boxtimes \mathbf{1}_c)\right) &= \mathrm{Hom}^{\mathbf{a}=0}_{\mathcal{C}(\mathcal{D})}\left(X \boxtimes \mathbf{1}_c, (\mathbf{1}_a \boxtimes {}_{b,c}S_{b+c}M_{b,c}) \star (Y \boxtimes \mathbf{1}_c)\right) \\
&\cong \mathrm{Hom}^{\mathbf{a}=0}_{\mathcal{C}(\mathcal{D})}\left(X, (\mathbf{1}_a \boxtimes \mathrm{Tr}^c({}_{b,c}S_{b+c}M_{b,c})) \star Y\right) \\
&\cong \mathrm{Hom}^{\mathbf{a}=0}_{\mathcal{C}(\mathcal{D})}\left(X, \prod_{i=1}^{c} (\mathbf{q}^b + \mathbf{a}\mathbf{q}^{-b-2i}) Y \otimes \mathrm{End}_{\mathcal{C}}(\mathbf{1}_c)\right) \\
&= \mathbf{q}^{bc} \mathrm{Hom}_{\mathcal{C}}(X, Y) \otimes \mathrm{End}_{\mathcal{C}}(\mathbf{1}_c)\,.
\end{aligned}$$

The result follows since all of the constituent isomorphisms are natural in X and Y. ☐

Corollary A.3. *Let $b, c \geq 0$, then the* Hom*-spaces*

$$\mathrm{Hom}_{\mathrm{SSBim}}(\mathbf{1}_{b,c}, {}_{b,c}S_{b+c} \star {}_{b+c}M_{b,c}) \quad \text{and} \quad \mathrm{Hom}_{\mathrm{SSBim}}({}_{b,c}S_{b+c} \star {}_{b+c}M_{b,c}, \mathbf{1}_{b,c})$$

are free $\mathrm{End}_{\mathrm{SSBim}}(\mathbf{1}_{b,c})$-modules generated by the zip and unzip morphisms, respectively.

Proof. Let $\mathbb{X}_1$ and $\mathbb{X}_2$ be alphabets with $|\mathbb{X}_1| = b$ and $|\mathbb{X}_2| = c$ and identify $\mathrm{Sym}(\mathbb{X}_1|\mathbb{X}_2) = \mathrm{End}_{\mathrm{SSBim}}(\mathbf{1}_{b,c})$. The map

$$\mathrm{End}_{\mathrm{SSBim}}(\mathbf{1}_{b,c}) \to \mathrm{Hom}_{\mathrm{SSBim}}(\mathbf{1}_{b,c}, {}_{b,c}S_{b+c}M_{b,c})\,, \quad f \mapsto \mathbf{zip} \circ f$$

is injective since $\mathbf{un} \circ \mathbf{zip} = \mathfrak{s}_{c^b}(\mathbb{X}_1 - \mathbb{X}_2)$ and $\mathrm{Sym}(\mathbb{X}_1|\mathbb{X}_2)$ is torsion-free. To see that it is surjective, we compare graded dimensions. For this, Corollary A.2 gives that

$$\mathrm{Hom}_{\mathrm{SSBim}}(\mathbf{1}_{b,c}, {}_{b,c}S_{b+c}M_{b,c}) = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}_{b,c}, {}_{b,c}S_{b+c}M_{b,c}) \cong \mathbf{q}^{bc} \mathrm{End}_{\mathcal{C}}(\mathbf{1}_b) \otimes \mathrm{End}_{\mathcal{C}}(\mathbf{1}_c) = \mathbf{q}^{bc} \mathrm{End}_{\mathrm{SSBim}}(\mathbf{1}_{b,c})$$

as desired.

The other assertion follows either using a similar argument (here, we use that $\mathbf{zip} \circ \mathbf{un} = \mathfrak{s}_{c^b}(\mathbb{X}_1 - \mathbb{X}_2')$ is injective on ${}_{b,c}S_{b+c} \star {}_{b+c}M_{b,c}$), or by applying duality in SSBim. ☐

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