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ABSTRACT

This is a companion paper to the previous work [P. H. Yoon, Phys. Plasmas **31**, 032309 (2024)] in which the nonlinear susceptibilities of weakly turbulent magnetized plasma are derived under a simplifying assumption of electrostatic interaction. The present paper extends the analysis to a general situation of electromagnetic interaction. The main novelty of the previous and present papers is that by employing the Bessel function addition theorem, the mathematical definitions for the susceptibilities are substantially simplified, a procedure that has not been discussed in the existing literature. In the present paper, a full set of Maxwell's equations are considered in conjunction with the nonlinear Vlasov equation, which is solved by a perturbative method. The result is a fully general nonlinear susceptibility, given in tensorial form, which is applicable for weakly turbulent magnetized plasmas.

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I. INTRODUCTION

As already alluded to in the Introduction of the previous paper,[1] the plasma weak turbulence theory is a powerful theoretical research tool, which has been proven to be valid especially when compared against the particle-in-cell simulations.[2] For a historical backdrop and for relevant literature, see the references cited in Ref. 1. As noted in Ref. 1, the existing weak turbulence theory is quite fully developed for unmagnetized plasmas and the applications thereof are also largely confined to the unmagnetized plasma systems.[3–14] Recently, the application of weak turbulence theory to the whistler and lower-hybrid frequency regimes has been discussed as an important dynamical process in the near-Earth environment.[15–22] Moreover, Ref. 23 reports circular polarizations for the type III radio bursts close to the Sun as measured using the contemporary Parker Solar Probe (PSP). This implies that the standard plasma emission theory, which is based on the weak turbulence theory for unmagnetized plasmas, should be extended to include the effects of the ambient magnetic field.[24–29] That is, this calls for the weak turbulence theory for magnetized plasmas.

In the literature, the weak turbulence theory for magnetized plasmas is occasionally discussed,[12,16–18,22–25,28–48] but it is by no means complete. A fundamental issue to be addressed is the computation of nonlinear susceptibilities. One may find some discussions thereof in the literature; among them are Refs. 12, 25, 32, 34, and 48. However, as it will be discussed in detail as we move along, the formal definitions that are found in these studies are quite cumbersome in that these definitions involve multiple Bessel function/cyclotron-harmonic series. Among the major issues is the mathematical complexity that arises as it relates to the Bessel function/cyclotron harmonic series expansion. The second- and third-order nonlinear susceptibilities as found in the literature, at first glance, appear to involve quadruple and sextuple infinite Bessel function/gyro-harmonic series, respectively. However, in the previous paper,[1] it was found that some of these apparent multiple infinite series can be succinctly expressed in closed forms by making use of the Bessel addition theorem, or the sum rule. This procedure leads to non-insignificant simplifications, albeit the final forms of nonlinear susceptibilities are still quite complex. Nevertheless, the final forms of these susceptibilities provide a useful platform for further approximate treatments and applications to practical problems.

The present paper constitutes an extension of the earlier paper,[1] where a simplifying assumption of electrostatic interaction was imposed for the sake of simplicity. In the present paper, we remove the assumption of electrostatic interaction and consider the full set of Maxwell's equations combined with the Vlasov equation. In

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the remainder of the present paper, the derivation of the results is systematically expounded with sufficient details.

Before moving on, we should note that an alternative approach of deriving the linear and nonlinear susceptibility tensors for unmagnetized or magnetized plasmas has been devised by Melrose.[49–51] This approach is termed the "forward-scattering method" by the author, and its early form can be found in Ref. 49. Fully developed versions can be found in the author's later monographs.[50,51] In this alternative approach, the linear and nonlinear response tensors are computed based upon the single charged particle current, with the particle orbit computed by iterative means—hence the conceptual similarity with the weak turbulence expansion of Vlasov equation. Then, the result is integrated over the spatial volume and, subsequently, multiplied with the velocity distribution function. The final form is integrated over the velocity space, as well as summed over all charged particle species. When compared to the traditional approach based upon the iterative solution of the Vlasov equation, which is the subject of the present paper, in the forward-scattering method, the second- and third-order "nonlinear response tensors are derived relatively simply" (quoting Ref. 50, p. 162). However, the equivalence of the forward-scattering method and the so-called Vlasov method seems to have been proven only for the simple case of linear response tensor (see Ref. 49, p. 20; Ref. 50, p. 163; and Ref. 51, p. 59). Despite this, the forward-scattering method is potentially an extremely useful method. The present paper, on the other hand, pays attention to the Vlasov approach, and further discussion of the forward-scattering method and its equivalence to the Vlasov approach as it relates to the nonlinear response belongs to a separate treatise, which is beyond the scope of the present paper.

II. NONLINEAR ELECTRODYNAMIC EQUATIONS IN PLASMAS

We start from the Vlasov–Maxwell system of equations,

$$\left[\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}}+\frac{e_a}{m_a}\left(\mathbf{E}(\mathbf{r},t)+\frac{\mathbf{v}}{c}\times\mathbf{B}(\mathbf{r},t)\right)\cdot\frac{\partial}{\partial \mathbf{v}}\right]f_a(\mathbf{r},\mathbf{v},t)=0, \tag{1a}$$

$$\frac{\partial}{\partial \mathbf{r}}\times\mathbf{E}(\mathbf{r},t)+\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)=0, \tag{1b}$$

$$\frac{\partial}{\partial \mathbf{r}}\cdot\mathbf{E}(\mathbf{r},t)=4\pi\sum_a e_a\hat{n}_a\int d\mathbf{v}\, f_a(\mathbf{r},\mathbf{v},t), \tag{1c}$$

$$\frac{\partial}{\partial \mathbf{r}}\cdot\mathbf{B}(\mathbf{r},t)=0, \tag{1d}$$

$$\frac{\partial}{\partial \mathbf{r}}\times\mathbf{B}(\mathbf{r},t)-\frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t)=\frac{4\pi}{c}\sum_a e_a\hat{n}_a\int d\mathbf{v}\,\mathbf{v}\, f_a(\mathbf{r},\mathbf{v},t), \tag{1e}$$

where $f_a(\mathbf{r},\mathbf{v},t)$ is the one-particle distribution function [normalized to unity, $\int d\mathbf{v} f_a(\mathbf{r},\mathbf{v},t)=1$] for particle species labeled a ($a=e$ for electrons and $a=i$ for ions); $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ stand for the electric and magnetic field vectors, respectively; $\hat{n}_a=\hat{n}_e=\hat{n}_i\equiv\hat{n}$ represents the ambient density that satisfies the charge neutrality; and e_a denotes the unit electric charge ($=e$ for protons and $=-e$ for electrons). For a uniform plasma immersed in an ambient magnetic field, and for a system with no net average electric field, charge, or current, we separate the physical quantities into average and fluctuating parts: $F_a(\mathbf{r},\mathbf{v},t)=f_a(\mathbf{v},t)+\delta f_a(\mathbf{r},\mathbf{v},t)$, $\mathbf{E}(\mathbf{r},t)=\delta\mathbf{E}(\mathbf{r},t)$, and $\mathbf{B}(\mathbf{r},t)=\mathbf{B}_0+\delta\mathbf{B}(\mathbf{r},t)$, where $\mathbf{B}_0=B_0\hat{\mathbf{e}}_z$ is the ambient magnetic field vector, which is assumed to lie along the z axis. Upon implementing this scheme of separating averaged and fluctuating quantities to the Vlasov–Maxwell equation (1), and taking the ensemble average of the particle equation over the random phases of the fluctuations, we obtain the formal particle kinetic equation. Upon subtracting the averaged particle equation from the original particle equation, one obtains the equation for the perturbed distribution function. These procedures are standard, which results in the set of equations that form the basis of subsequent analysis,

$$\frac{\partial F_a(\mathbf{v},t)}{\partial t}=-\frac{\partial}{\partial \mathbf{v}}\cdot\langle\mathbf{a}(\mathbf{r},\mathbf{v},t)\,\delta f_a(\mathbf{r},\mathbf{v},t)\rangle, \tag{2a}$$

$$\frac{D}{Dt}\delta f_a(\mathbf{r},\mathbf{v},t)=-\mathbf{a}(\mathbf{r},\mathbf{v},t)\cdot\frac{\partial F_a(\mathbf{v},t)}{\partial \mathbf{v}}-\left(\mathbf{a}(\mathbf{r},\mathbf{v},t)\cdot\frac{\partial}{\partial \mathbf{v}}\delta f_a(\mathbf{r},\mathbf{v},t)-\left\langle\mathbf{a}(\mathbf{r},\mathbf{v},t)\cdot\frac{\partial}{\partial \mathbf{v}}\delta f_a(\mathbf{r},\mathbf{v},t)\right\rangle\right), \tag{2b}$$

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}}+\frac{e_a}{m_a c}\mathbf{v}\times\mathbf{B}_0\cdot\frac{\partial}{\partial \mathbf{v}}, \tag{2c}$$

$$\mathbf{a}(\mathbf{r},\mathbf{v},t)=\frac{e_a}{m_a}\left(\delta\mathbf{E}(\mathbf{r},t)+\frac{\mathbf{v}}{c}\times\delta\mathbf{B}(\mathbf{r},t)\right). \tag{2d}$$

In the above, we have assumed that the averaged particle distribution function is independent of the velocity space gyro phase angle, $\partial f_a(\mathbf{v},t)/\partial\varphi=0$. To solve the nonlinear equation for the perturbed distribution function, $\delta f_a(\mathbf{r},\mathbf{v},t)$, we expand the solution in series with each term proportional to the wave field amplitude, $\delta f_a(\mathbf{r},\mathbf{v},t)=\varepsilon f_a^{(1)}(\mathbf{r},\mathbf{v},t)+\varepsilon^2 f_a^{(2)}(\mathbf{r},\mathbf{v},t)+\varepsilon^3 f_a^{(3)}(\mathbf{r},\mathbf{v},t)+\cdots$, where $\varepsilon=\mathcal{O}(\delta E)\sim\mathcal{O}(\delta B)$. Then, we have order-by-order equations,

$$\frac{D}{Dt}f_a^{(1)}=-\delta\mathbf{a}\cdot\frac{\partial F_a}{\partial \mathbf{v}}, \tag{3a}$$

$$\frac{D}{Dt}f_a^{(2)}=-\left(\delta\mathbf{a}\cdot\frac{\partial}{\partial \mathbf{v}}f_a^{(1)}-\left\langle\delta\mathbf{a}\cdot\frac{\partial}{\partial \mathbf{v}}f_a^{(1)}\right\rangle\right), \tag{3b}$$

$$\frac{D}{Dt}f_a^{(3)}=-\left(\delta\mathbf{a}\cdot\frac{\partial}{\partial \mathbf{v}}f_a^{(2)}-\left\langle\delta\mathbf{a}\cdot\frac{\partial}{\partial \mathbf{v}}f_a^{(2)}\right\rangle\right),\ldots. \tag{3c}$$

Making use of $[e_a/(m_a c)]\mathbf{v}\times\mathbf{B}_0\cdot(\partial/\partial\mathbf{v})=-\Omega_a(\partial/\partial\varphi)$, where $\Omega_e=e_aB_0/(m_a c)$ denotes the cyclotron frequency, we write the basic set of equations in spectral form,

$$\left(\frac{\partial}{\partial\varphi}+\frac{i(\omega-\mathbf{k}\cdot\mathbf{v})}{\Omega_a}\right)f_q^{a(1)}=\frac{e_a}{m_a\Omega_a}\left(\frac{\omega-\mathbf{k}\cdot\mathbf{v}}{\omega}\mathbf{E}_q\cdot\frac{\partial F_a}{\partial \mathbf{v}}+\frac{\mathbf{v}\cdot\mathbf{E}_q}{\omega}\mathbf{k}\cdot\frac{\partial F_a}{\partial \mathbf{v}}\right), \tag{4a}$$

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$$\left(\frac{\partial}{\partial\varphi}+\frac{i(\omega-\mathbf{k}\cdot\mathbf{v})}{\Omega_a}\right)f_q^{a(n)}$$
$$=\frac{e_a}{m_a\Omega_a}\sum_{q'}(1-A)$$
$$\times\left(\frac{\omega'-\mathbf{k}'\cdot\mathbf{v}}{\omega'}\,\mathbf{E}_{q'}\cdot\frac{\partial}{\partial\mathbf{v}}+\frac{\mathbf{v}\cdot\mathbf{E}_{q'}}{\omega'}\,\mathbf{k}'\cdot\frac{\partial}{\partial\mathbf{v}}\right)f_{q-q'}^{a(n-1)},\qquad(4b)$$

$n=2,3,\ldots$, where we have made use of the short-hand notation, $q=(\mathbf{k},\omega)$, $q'=(\mathbf{k}',\omega')$, $\sum_{q'}=\int d\mathbf{k}'\int d\omega'$, and have introduced the ensemble average operator $A[\cdots]\equiv\langle[\cdots]\rangle$.

Following Ref. 1, we assume that both $\mathbf{k}$ and $\mathbf{k}'$ lie in the xz plane. In this simplifying scheme, all the dynamical processes are computed in the xz plane, and the result is effectively revolved around the z axis parallel to the ambient magnetic field. As such, this arrangement of wave vector orientations effectively assumes the azimuthal symmetry for the turbulent dynamics. If we are concerned with a two-dimensional turbulence, then, of course, this scheme is exact. For a general situation, this picture may be considered as representing an idealization, however, since the basic assumption is that there is no interference between the dynamics that involves the particle gyro-phase angle φ and the wave azimuthal angle, which may be denoted as γ. Any physical process that may result from the interference, or the beating, between the particle velocity space gyro-phase angle φ and the wave azimuthal angle γ, that is, terms that relate to their difference, $\varphi-\gamma$, are assumed to be unimportant and, therefore, ignorable. Under this procedure, after some lengthy but otherwise straightforward intermediate manipulations, it is possible to write down the order-by-order equations for the iterative solutions as follows:

$$\frac{\partial Pf_q^{a(1)}}{\partial\varphi}=\frac{e_a}{m_a\omega\Omega_a}P\Bigg[(\cos\varphi E_q^x+\sin\varphi E_q^y)\left((\omega-k_\parallel v_\parallel)\frac{\partial}{\partial v_\perp}+k_\parallel v_\perp\frac{\partial}{\partial v_\parallel}\right)$$
$$+E_q^\parallel\left(k_\perp v_\parallel\cos\varphi\frac{\partial}{\partial v_\perp}+(\omega-k_\perp v_\perp\cos\varphi)\frac{\partial}{\partial v_\parallel}\right)\Bigg]F_a,\qquad(5a)$$

$$\frac{\partial Pf_q^{a(n)}}{\partial\varphi}=\sum_{q'}\frac{e_a}{m_a\omega'\Omega_a}(1-A)\exp\left(\frac{i(\omega-k_\parallel v_\parallel)}{\Omega_a}\varphi\right)\Bigg[(\cos\varphi E_{q'}^x+\sin\varphi E_{q'}^y)\left(\frac{\omega'-k'_\parallel v_\parallel}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp+k'_\parallel v_\perp\frac{\partial}{\partial v_\parallel}\right)$$
$$+E_{q'}^\parallel\left(k'_\perp v_\parallel\cos\varphi\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp+(\omega'-k'_\perp v_\perp\cos\varphi)\frac{\partial}{\partial v_\parallel}\right)$$
$$-\frac{\partial}{\partial\varphi}\left(\frac{\omega'-k'_\parallel v_\parallel}{v_\perp}(\sin\varphi E_{q'}^x-\cos\varphi E_{q'}^y)+k'_\perp E_{q'}^y+k'_\perp\sin\varphi\frac{v_\parallel}{v_\perp}E_{q'}^\parallel\right)$$
$$+ib\frac{\omega'-k'_\parallel v_\parallel-k'_\perp v_\perp\cos\varphi}{v_\perp}E_{q'}^y\Bigg]e^{-ib\sin\varphi}f_{q-q'}^{a(n-1)},$$
$$P=\exp\left(\frac{i(\omega-k_\parallel v_\parallel)}{\Omega_a}\varphi-ib\sin\varphi\right),\qquad b=\frac{k_\perp v_\perp}{\Omega_a}\qquad(n=2,3,\cdots).\qquad(5b)$$

A. First-order solution

The first-order solution is obtained upon making use of the well-known Bessel function property $e^{\pm ib\sin\varphi}=\sum_n J_n(b)e^{\pm in\varphi}$ together with the associated properties $\sum_n J_n(b)e^{\pm in\varphi}\cos\varphi=\sum_n[nJ_n(b)/b]e^{\pm in\varphi}$ and $\sum_n J_n(b)e^{\pm in\varphi}\sin\varphi=\mp i\sum_n J_n'(b)e^{\pm in\varphi}$. This results in the familiar textbook expression,

$$f_q^{a(1)}(\varphi)=\frac{-ie_a}{m_a\omega}\sum_n\frac{e^{ib\sin\varphi-in\varphi}}{\omega-k_\parallel v_\parallel-n\Omega_a}\Bigg[\left(\frac{nJ_n(b)}{b}E_q^x+iJ_n'(b)E_q^y\right)$$
$$\times\left((\omega-k_\parallel v_\parallel)\frac{\partial F_a}{\partial v_\perp}+k_\parallel v_\perp\frac{\partial F_a}{\partial v_\parallel}\right)$$
$$+J_n(b)E_q^\parallel\left(n\Omega_a\frac{v_\parallel}{v_\perp}\frac{\partial F_a}{\partial v_\perp}+(\omega-n\Omega_a)\frac{\partial F_a}{\partial v_\parallel}\right)\Bigg].\qquad(6)$$

B. Second-order solution

The second-order solution is obtained from the first-order solution. By substituting $f_{q-q'}^{a(1)}$, which follows from Eq. (6), to the second-order equation (5b) (with $n=2$), we obtain

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$$\frac{\partial P f_q^{a(2)}}{\partial \varphi} = \sum_{q'} \frac{-ie_a^2}{m_a^2\omega'(\omega-\omega')\Omega_a}(1-A)\sum_n\sum_m \exp\left(\frac{i(\omega-k_\parallel v_\parallel)}{\Omega_a}\varphi\right)\Bigg[(\cos\varphi E^x_{q'} + \sin\varphi E^y_{q'})\left(\frac{\omega'-k'_\parallel v_\parallel}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp + k'_\parallel v_\perp \frac{\partial}{\partial v_\parallel}\right)$$
$$+ E^\parallel_{q'}\left(k'_\perp v_\parallel \cos\varphi \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp + (\omega'-k'_\perp v_\perp\cos\varphi)\frac{\partial}{\partial v_\parallel}\right)$$
$$- \frac{\partial}{\partial\varphi}\left(\frac{\omega'-k'_\parallel v_\parallel}{v_\perp}(\sin\varphi E^x_{q'} - \cos\varphi E^y_{q'}) + k'_\perp E^y_{q'} + k'_\perp \sin\varphi \frac{v_\parallel}{v_\perp}E^\parallel_{q'}\right) + ib\frac{\omega'-k'_\parallel v_\parallel - k'_\perp v_\perp\cos\varphi}{v_\perp}E^y_{q'}\Bigg]\sum_l J_l(b)e^{-il\varphi}$$
$$\times \frac{J_m(b-b')e^{-i(n-m)\varphi}}{\omega-\omega'-(k_\parallel-k'_\parallel)v_\parallel - n\Omega_a}\Bigg[\left(\frac{nJ_n(b-b')}{b-b'}E^x_{q-q'} + iJ'_n(b-b')E^y_{q-q'}\right)$$
$$\times\left([\omega-\omega'-(k_\parallel-k'_\parallel)v_\parallel]\frac{\partial}{\partial v_\perp} + (k_\parallel-k'_\parallel)v_\perp\frac{\partial}{\partial v_\parallel}\right)F_a$$
$$+ J_n(b-b')E^\parallel_{q-q'}\left(n\Omega_a\frac{v_\parallel}{v_\perp}\frac{\partial}{\partial v_\perp} + (\omega-\omega'-n\Omega_a)\frac{\partial}{\partial v_\parallel}\right)F_a\Bigg]. \tag{7}$$

We next make use of $\sum_l J_l(b)e^{-il\varphi}\cos\varphi = \sum_l [lJ_l(b)/b]e^{-il\varphi}$ and $\sum_l J_l(b)e^{-il\varphi}\sin\varphi = i\sum_l J'_l(b)e^{-il\varphi}$ to obtain

$$f_q^{a(2)} = \sum_{q'}\frac{-e_a^2}{m_a^2\omega'(\omega-\omega')}(1-A)\sum_n\sum_m\sum_p\frac{e^{ib\sin\varphi - ip\varphi}}{\omega-k_\parallel v_\parallel - p\Omega_a}$$
$$\times\Bigg\{\left(\frac{\omega'-k'_\parallel v_\parallel}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp + k'_\parallel v_\perp\frac{\partial}{\partial v_\parallel}\right)\left(\frac{(p-n+m)J_{p-n+m}(b)}{b}E^x_{q'} + iJ'_{p-n+m}(b)E^y_{q'}\right)$$
$$+\left[(p-n+m)\Omega_a\left(\frac{v_\parallel}{v_\perp}\frac{\partial}{\partial v_\perp} - \frac{\partial}{\partial v_\parallel}\right)\frac{b'}{b} + \omega'\frac{\partial}{\partial v_\parallel}\right]J_{p-n+m}(b)E^\parallel_{q'}$$
$$+ ip\left[\frac{\omega'-k'_\parallel v_\parallel}{v_\perp}\left(iJ'_{p-n+m}(b)E^x_{q'} - \frac{(p-n+m)J_{p-n+m}(b)}{b}E^y_{q'}\right) + k'_\perp J_{p-n+m}(b)E^y_{q'} + ik'_\perp J'_{p-n+m}(b)\frac{v_\parallel}{v_\perp}E^\parallel_{q'}\right]$$
$$+\frac{ib}{v_\perp}\left(\omega'-k'_\parallel v_\parallel - (p-n+m)\Omega_a\frac{b'}{b}\right)J_{p-n+m}(b)E^y_{q'}\Bigg\}$$
$$\times\frac{J_m(b-b')}{\omega-\omega'-(k_\parallel-k'_\parallel)v_\parallel - n\Omega_a}\Bigg[\left(\frac{nJ_n(b-b')}{b-b'}E^x_{q-q'} + iJ'_n(b-b')E^y_{q-q'}\right)$$
$$\times\left([\omega-\omega'-(k_\parallel-k'_\parallel)v_\parallel]\frac{\partial}{\partial v_\perp} + (k_\parallel-k'_\parallel)v_\perp\frac{\partial}{\partial v_\parallel}\right)$$
$$+ J_n(b-b')E^\parallel_{q-q'}\left(n\Omega_a\frac{v_\parallel}{v_\perp}\frac{\partial}{\partial v_\perp} + (\omega-\omega'-n\Omega_a)\frac{\partial}{\partial v_\parallel}\right)\Bigg]F_a. \tag{8}$$

As noted in the previous paper,[1] the triple Bessel function summations can be reduced to a double sum upon making use of the addition theorem $\sum_m J_{r+m}(b)J_m(b-b') = J_r(b')$. In particular, we find that the following summations over index m can be carried out in closed forms:

$$\sum_m J_{p-n+m}(b)J_m(b-b') = J_{p-n}(b'),$$
$$\sum_m mJ_{p-n+m}(b)J_m(b-b') = \frac{b-b'}{b'}(p-n)J_{p-n}(b'),$$
$$\sum_m J'_{p-n+m}(b)J_m(b-b') = J'_{p-n}(b').$$

This is possible because the index m does not appear in the gyro-harmonic resonance denominators. Making use of this, we obtain a simplified expression,

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$$
\begin{aligned}
f_q^{a(2)} = \sum_{q'} \frac{-e_a^2}{m_a^2\omega'(\omega-\omega')}(1-A)\sum_n\sum_m \frac{e^{ib\,\sin\varphi - in\varphi}}{\omega - k_\parallel v_\parallel - n\Omega_a}
&\times\left\{\left(\frac{\omega'-k'_\parallel v_\parallel}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp + k'_\parallel v_\perp\frac{\partial}{\partial v_\parallel}\right)\left(\frac{(n-m)J_{n-m}(b')}{b'}E^x_{q'} + iJ'_{n-m}(b')E^y_{q'}\right)\right.\\
&+\left((n-m)\Omega_a\frac{v_\parallel}{v_\perp}\frac{\partial}{\partial v_\perp} + [\omega'-(n-m)\Omega_a]\frac{\partial}{\partial v_\parallel}\right)J_{n-m}(b')E^\parallel_{q'}\\
&+in\left[\frac{\omega'-k'_\parallel v_\parallel}{v_\perp}\left(iJ'_{n-m}(b')E^x_{q'} - \frac{(n-m)J_{n-m}(b')}{b'}E^y_{q'}\right) + k'_\perp J_{n-m}(b')E^y_{q'} + ik'_\perp J'_{n-m}(b')\frac{v_\parallel}{v_\perp}E^\parallel_{q'}\right]\\
&\left.+\frac{ib}{v_\perp}[\omega'-k'_\parallel v_\parallel-(n-m)\Omega_a]J_{n-m}(b')E^y_{q'}\right\}\frac{1}{\omega-\omega'-(k_\parallel-k'_\parallel)v_\parallel-m\Omega_a}\\
&\times\left[\left(\frac{mJ_m(b-b')}{b-b'}E^x_{q-q'} + iJ'_m(b-b')E^y_{q-q'}\right)\left([\omega-\omega'-(k_\parallel-k'_\parallel)v_\parallel]\frac{\partial}{\partial v_\perp} + (k_\parallel-k'_\parallel)v_\perp\frac{\partial}{\partial v_\parallel}\right)\right.\\
&\left.+J_m(b-b')E^\parallel_{q-q'}\left(m\Omega_a\frac{v_\parallel}{v_\perp}\frac{\partial}{\partial v_\perp} + (\omega-\omega'-m\Omega_a)\frac{\partial}{\partial v_\parallel}\right)\right]F_a. \qquad (10)
\end{aligned}
$$

C. Third-order solution

The third-order solution can be constructed from the second-order solution. Substituting the second-order solution (10) to the equation for the third-order perturbed distribution function, i.e., Eq. (5b) with $n = 3$, and following the same manipulative procedures that we have employed in the derivation for the second-order solution (10), we obtain

$$
\begin{aligned}
f_q^{a(3)} = \sum_{q'}\sum_{q''}\frac{ie_a^3}{m_a^3\omega'\omega''(\omega-\omega'-\omega'')}(1-A)\sum_n\sum_m\sum_l\sum_r\frac{e^{ib\,\sin\varphi - ir\varphi}}{\omega-k_\parallel v_\parallel - r\Omega_a}
&\times\left\{\left(\frac{\omega'-k'_\parallel v_\parallel}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp + k'_\parallel v_\perp\frac{\partial}{\partial v_\parallel}\right)\left(\frac{(r-n+l)J_{r-n+l}(b)}{b}E^x_{q'} + iJ'_{r-n+l}(b)E^y_{q'}\right)\right.\\
&+\left[(r-n+l)\Omega_a\frac{v_\parallel}{v_\perp}\frac{\partial}{\partial v_\perp}\frac{b'}{b} + \left(\omega'-(r-n+l)\Omega_a\frac{b'}{b}\right)\frac{\partial}{\partial v_\parallel}\right]J_{r-n+l}(b)E^\parallel_{q'}\\
&+ir\left[\frac{\omega'-k'_\parallel v_\parallel}{v_\perp}\left(iJ'_{r-n+l}(b)E^x_{q'} - \frac{(r-n+l)J_{r-n+l}(b)}{b}E^y_{q'}\right) + k'_\perp J_{r-n+l}(b)E^y_{q'} + ik'_\perp J'_{r-n+l}(b)\frac{v_\parallel}{v_\perp}E^\parallel_{q'}\right]\\
&\left.+\frac{ib}{v_\perp}\left(\omega'-k'_\parallel v_\parallel-(r-n+l)\Omega_a\frac{b'}{b}\right)J_{r-n+l}(b)E^y_{q'}\right\}\frac{J_l(b-b')}{\omega-\omega'-(k_\parallel-k'_\parallel)v_\parallel-n\Omega_a}\\
&\times(1-A)\left\{\left(\frac{\omega''-k''_\parallel v_\parallel}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp + k''_\parallel v_\perp\frac{\partial}{\partial v_\parallel}\right)\left(\frac{(n-m)J_{n-m}(b'')}{b''}E^x_{q''} + iJ'_{n-m}(b'')E^y_{q''}\right)\right.\\
&+\left((n-m)\Omega_a\frac{v_\parallel}{v_\perp}\frac{\partial}{\partial v_\perp} + [\omega''-(n-m)\Omega_a]\frac{\partial}{\partial v_\parallel}\right)J_{n-m}(b'')E^\parallel_{q''}\\
&+in\left[\frac{\omega''-k''_\parallel v_\parallel}{v_\perp}\left(iJ'_{n-m}(b'')E^x_{q''} - \frac{(n-m)J_{n-m}(b'')}{b''}E^y_{q''}\right) + k''_\perp J_{n-m}(b'')E^y_{q''} + ik''_\perp J'_{n-m}(b'')\frac{v_\parallel}{v_\perp}E^\parallel_{q''}\right]\\
&\left.+\frac{i(b-b')}{v_\perp}[\omega''-k''_\parallel v_\parallel-(n-m)\Omega_a]J_{n-m}(b'')E^y_{q''}\right\}\frac{1}{\omega-\omega'-\omega''-(k_\parallel-k'_\parallel-k''_\parallel)v_\parallel-m\Omega_a}\\
&\times\left[\left(\frac{mJ_m(b-b'-b'')}{b-b'-b''}E^x_{q-q'-q''} + iJ'_m(b-b'-b'')E^y_{q-q'-q''}\right)\right.\\
&\times\left([\omega-\omega'-\omega''-(k_\parallel-k'_\parallel-k''_\parallel)v_\parallel]\frac{\partial}{\partial v_\perp} + (k_\parallel-k'_\parallel-k''_\parallel)v_\perp\frac{\partial}{\partial v_\parallel}\right)\\
&\left.+J_m(b-b'-b'')E^\parallel_{q-q'-q''}\left(m\Omega_a\frac{v_\parallel}{v_\perp}\frac{\partial}{\partial v_\perp} + (\omega-\omega'-\omega''-m\Omega_a)\frac{\partial}{\partial v_\parallel}\right)\right]F_a. \qquad (11)
\end{aligned}
$$

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We again make use of the Bessel function addition theorem, particularly, for summations involving index l,

$$\sum_l J_{r-n+l}(b)J_l(b-b') = J_{r-n}(b'),$$
$$\sum_l \frac{(r-n+l)J_{r-n+l}(b)}{b} J_l(b-b') = \frac{(r-n)J_{r-n}(b')}{b'}, \tag{12}$$
$$\sum_l J'_{r-n+l}(b)J_l(b-b') = J'_{r-n}(b'),$$

which again becomes possible because l does not appear in the resonant denominators. Then, we obtain

$$\begin{aligned}
f_q^{a(3)} = \sum_{q'}\sum_{q''} \frac{ie_a^3}{m_a^3\omega'\omega''(\omega-\omega'-\omega'')}(1-A)\sum_n\sum_m\sum_l \frac{e^{ib\,\sin\,\varphi-in\varphi}}{\omega-k_\parallel v_\parallel - n\Omega_a}\\
\times\left\{\left(\frac{\omega'-k'_\parallel v_\parallel}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp + k'_\parallel v_\perp\frac{\partial}{\partial v_\parallel}\right)\left(\frac{(n-m)J_{n-m}(b)}{b'}E^x_{q'} + iJ'_{n-m}(b)E^y_{q'}\right)\right.\\
+\left((n-m)\Omega_a\frac{v_\parallel}{v_\perp}\frac{\partial}{\partial v_\perp} + [\omega'-(n-m)\Omega_a]\frac{\partial}{\partial v_\parallel}\right)J_{n-m}(b)E^\parallel_{q'}\\
+ in\left[\frac{\omega'-k'_\parallel v_\parallel}{v_\perp}\left(iJ'_{n-m}(b)E^x_{q'} - \frac{(n-m)J_{n-m}(b)}{b'}E^y_{q'}\right) + k'_\perp J_{n-m}(b)E^y_{q'} + ik'_\perp J'_{n-m}(b)\frac{v_\parallel}{v_\perp}E^\parallel_{q'}\right]\\
\left.+\frac{ib}{v_\perp}[\omega'-k'_\parallel v_\parallel-(n-m)\Omega_a]J_{n-m}(b)E^y_{q'}\right\}\frac{1}{\omega-\omega'-(k_\parallel-k'_\parallel)v_\parallel-m\Omega_a}\\
\times(1-A)\left\{\left(\frac{\omega''-k''_\parallel v_\parallel}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp + k''_\parallel v_\perp\frac{\partial}{\partial v_\parallel}\right)\left(\frac{(m-l)J_{m-l}(b'')}{b''}E^x_{q''} + iJ'_{m-l}(b'')E^y_{q''}\right)\right.\\
+\left((m-l)\Omega_a\frac{v_\parallel}{v_\perp}\frac{\partial}{\partial v_\perp} + [\omega''-(m-l)\Omega_a]\frac{\partial}{\partial v_\parallel}\right)J_{m-l}(b'')E^\parallel_{q''}\\
+ im\left[\frac{\omega''-k''_\parallel v_\parallel}{v_\perp}\left(iJ'_{m-l}(b'')E^x_{q''} - \frac{(m-l)J_{m-l}(b'')}{b''}E^y_{q''}\right) + k''_\perp J_{m-l}(b'')E^y_{q''} + ik''_\perp J'_{m-l}(b'')\frac{v_\parallel}{v_\perp}E^\parallel_{q''}\right]\\
\left.+\frac{i(b-b')}{v_\perp}[\omega''-k''_\parallel v_\parallel-(m-l)\Omega_a]J_{m-l}(b'')E^y_{q''}\right\}\frac{1}{\omega-\omega'-\omega''-(k_\parallel-k'_\parallel-k''_\parallel)v_\parallel-l\Omega_a}\\
\times\left[\left(\frac{lJ_l(b-b'-b'')}{b-b'-b''}E^x_{q-q'-q''} + iJ'_l(b-b'-b'')E^y_{q-q'-q''}\right)\right.\\
\times\left([\omega-\omega'-\omega''-(k_\parallel-k'_\parallel-k''_\parallel)v_\parallel]\frac{\partial}{\partial v_\perp} + (k_\parallel-k'_\parallel-k''_\parallel)v_\perp\frac{\partial}{\partial v_\parallel}\right)\\
\left.+J_l(b-b'-b'')E^\parallel_{q-q'-q''}\left(l\Omega_a\frac{v_\parallel}{v_\perp}\frac{\partial}{\partial v_\perp} + (\omega-\omega'-\omega''-l\Omega_a)\frac{\partial}{\partial v_\parallel}\right)\right]F_a.
\end{aligned} \tag{13}$$

D. Summary of iterative solutions

We recapitulate the series solution for the perturbed distribution function,

$$f_q^a = f_q^{a(1)} + f_q^{a(2)} + f_q^{a(3)} + \cdots, \tag{14a}$$

$$f_q^{a(1)} = \frac{-ie_a}{m_a\omega}\sum_n e^{ib\,\sin\,\varphi-in\varphi}\,G_q^n\,\alpha_n^j(q)F_aE_q^j, \tag{14b}$$

$$\begin{aligned}
f_q^{a(2)} = \sum_{q'}\frac{-e_a^2}{m_a^2\omega'(\omega-\omega')}\sum_n\sum_m e^{ib\,\sin\,\varphi-in\varphi}\,G_q^n\,\beta_{n,m}^j(q|q')\\
\times G^m_{q-q'}\,\alpha_m^k(q-q')F_a\left(E^j_{q'}E^k_{q-q'} - \langle E^j_{q'}E^k_{q-q'}\rangle\right),
\end{aligned} \tag{14c}$$

$$\begin{aligned}
f_q^{a(3)} = \sum_{q'}\sum_{q''}\frac{ie_a^3}{m_a^3\omega'\omega''(\omega-\omega'-\omega'')}\sum_n\sum_m\sum_l e^{ib\,\sin\,\varphi-in\varphi}\\
\times G_q^n\,\beta^j_{n,m}(q|q')\,G^m_{q-q'}\,\beta^k_{m,l}(q-q'|q'')\\
\times G^l_{q-q'-q''}\,\alpha_l^p(q-q'-q'')F_a\\
\times\left(E^j_{q'}E^k_{q''}E^p_{q-q'-q''} - E^j_{q'}\langle E^k_{q''}E^p_{q-q'-q''}\rangle\right.\\
\left.-\langle E^j_{q'}E^k_{q''}E^p_{q-q'-q''}\rangle\right),
\end{aligned} \tag{14d}$$

where

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$$V_q^i(n) = \left(v_\perp \frac{nJ_n(b)}{b},\ iv_\perp J_n'(b),\ v_\parallel J_n(b)\right), \tag{15a}$$

$$\alpha_n^i(q) = \left[V_q^x(n)\,\mathscr{L}_q,\ V_q^y(n)\,\mathscr{L}_q,\ V_q^z(n)\,\mathscr{D}_q^n\right], \tag{15b}$$

$$\begin{aligned}
\beta_{n,m}^x(q|q') &= \mathscr{L}_{q'} v_\perp \frac{(n-m)J_{n-m}(b')}{b'} - \frac{\omega' - k_\parallel' v_\parallel}{v_\perp} nJ_{n-m}'(b'),\\
\beta_{n,m}^y(q|q') &= i\left[\mathscr{L}_{q'} v_\perp J_{n-m}'(b') - \frac{\omega' - k_\parallel' v_\parallel}{v_\perp}\left(\frac{n(n-m)}{b'} - b\right)\right.\\
&\quad \left.\times J_{n-m}(b') + \frac{\Omega_a}{v_\perp}\left[nb' - (n-m)b\right]J_{n-m}(b')\right],\\
\beta_{n,m}^z(q|q') &= \frac{v_\parallel}{v_\perp}\left(\mathscr{D}_{q'}^{n-m} v_\perp J_{n-m}(b') - \frac{n\Omega_a}{v_\perp} b' J_{n-m}'(b')\right),
\end{aligned} \tag{15c}$$

$$\begin{aligned}
G_q^n &= \frac{1}{\omega - k_\parallel v_\parallel - n\Omega_a}, \qquad \mathscr{L}_q = \frac{\omega - k_\parallel v_\parallel}{v_\perp}\frac{\partial}{\partial v_\perp} + k_\parallel \frac{\partial}{\partial v_\parallel},\\
\mathscr{D}_q^n &= \frac{n\Omega_a}{v_\perp}\frac{\partial}{\partial v_\perp} + \frac{\omega - n\Omega_a}{v_\parallel}\frac{\partial}{\partial v_\parallel}.
\end{aligned} \tag{15d}$$

III. NONLINEAR WAVE EQUATION

The wave equation is given by

$$\left(1 - \frac{c^2k^2}{\omega^2}\right)\delta\mathbf{E}_{\mathbf{k}\omega} + \frac{c^2}{\omega^2}\mathbf{k}\left(\mathbf{k}\cdot\delta\mathbf{E}_{\mathbf{k}\omega}\right) = -\mathbf{J}_q = -\mathbf{J}_q^{(1)} - \mathbf{J}_q^{(2)} - \mathbf{J}_q^{(3)} - \cdots, \tag{16}$$

where $\mathbf{J}_q^{(1)}$, $\mathbf{J}_q^{(2)}$, and $\mathbf{J}_q^{(3)}$ are the first, second, and third-order normalized perturbed currents, respectively, and are given by

$$\begin{aligned}
\mathbf{J}_q^{(1)} &= -\frac{4\pi i}{\omega}\sum_a e_a \hat{n}\int d\mathbf{v}\,\mathbf{v}\,f_q^{a(1)},\\
\mathbf{J}_q^{(2)} &= -\frac{4\pi i}{\omega}\sum_a e_a \hat{n}\int d\mathbf{v}\,\mathbf{v}\,f_q^{a(2)},\\
\mathbf{J}_q^{(3)} &= -\frac{4\pi i}{\omega}\sum_a e_a \hat{n}\int d\mathbf{v}\,\mathbf{v}\,f_q^{a(3)}.
\end{aligned} \tag{17}$$

A. First-order perturbed current

The normalized first-order perturbed current is computed in a straightforward manner,

$$\begin{aligned}
\mathbf{J}_q^{(1)} &= -\sum_a \frac{\omega_{pa}^2}{\omega^2}\int d\mathbf{v}\left(\hat{\mathbf{x}}v_\perp\cos\varphi + \hat{\mathbf{y}}v_\perp\sin\varphi + \hat{\mathbf{z}}v_\parallel\right)\\
&\quad\times\sum_n e^{ib\sin\varphi - in\varphi} G_q^n \alpha_n^j(q) F_a E_q^j\\
&= -\sum_a \frac{\omega_{pa}^2}{\omega^2}\int d\mathbf{v}\sum_n\left(\hat{\mathbf{x}}v_\perp\frac{nJ_n(b)}{b} - i\hat{\mathbf{y}}v_\perp J_n'(b) + \hat{\mathbf{z}}v_\parallel J_n(b)\right)\\
&\quad\times G_q^n\,\alpha_n^j(q)F_a E_q^j.
\end{aligned} \tag{18}$$

In terms of $\mathbf{V}_q(n)$ defined in Eq. (15), we may write the first-order perturbed current compactly as follows:

$$J_q^{(1)i} = -\sum_a \frac{\omega_{pa}^2}{\omega^2}\sum_n \int d\mathbf{v}\,V_q^{i*}(n)\,G_q^n\,\alpha_n^j(q)F_a E_q^j = -\chi_{ij}^{(1)}(q)E_q^j, \tag{19}$$

where we have defined the linear dielectric susceptibility tensor,

$$\chi_{ij}^{(1)}(q) = \sum_a \frac{\omega_{pa}^2}{\omega^2}\sum_n \int d\mathbf{v}\,V_q^{i*}(n)\,G_q^n\,\alpha_n^j(q)F_a. \tag{20}$$

It is instructive to express the linear dielectric susceptibility tensor in an explicit form,

$$\chi_{ij}^{(1)}(q) = \sum_a \frac{\omega_{pa}^2}{\omega^2}\sum_n \int d\mathbf{v}\,\frac{1}{\omega - k_\parallel v_\parallel - n\Omega_a}\begin{pmatrix}
v_\perp^2\frac{n^2J_n^2(b)}{b^2}\mathscr{L}_q & iv_\perp^2\frac{nJ_n(b)J_n'(b)}{b}\mathscr{L}_q & v_\perp v_\parallel\frac{nJ_n^2(b)}{b}\mathscr{D}_q^n\\
-iv_\perp^2\frac{nJ_n(b)J_n'(b)}{b}\mathscr{L}_q & v_\perp^2[J_n'(b)]^2\mathscr{L}_q & -iv_\perp v_\parallel J_n(b)J_n'(b)\mathscr{D}_q^n\\
v_\perp v_\parallel\frac{nJ_n^2(b)}{b}\mathscr{L}_q & iv_\perp v_\parallel J_n(b)J_n'(b)\mathscr{L}_q & v_\parallel^2 J_n^2(b)\mathscr{D}_q^n
\end{pmatrix}F_a, \tag{21}$$

where $\mathscr{L}_q$ and $\mathscr{D}_q^n$ are defined in Eq. (15). This is, of course, the well-known textbook result. One may further write

$$\mathscr{D}_q^n = -\frac{\omega - k_\parallel v_\parallel - n\Omega_a}{v_\perp}\left(\frac{\partial}{\partial v_\perp} - \frac{v_\perp}{v_\parallel}\frac{\partial}{\partial v_\parallel}\right) + \mathscr{L}_q \tag{22}$$

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and obtain an alternative version, which is also well known in the literature,

$$\chi_{ij}^{(1)}(q) = \sum_a \frac{\omega_{pa}^2}{\omega^2} \sum_n \int d\mathbf{v}\, v_\parallel \left(\frac{\partial F_a}{\partial v_\parallel} - \frac{v_\parallel}{v_\perp} \frac{\partial F_a}{\partial v_\perp} \right) b_i b_j + \sum_a \frac{\omega_{pa}^2}{\omega^2} \sum_n \int d\mathbf{v}\, \frac{V_q^{i*}(n) V_q^i(n)}{\omega - k_\parallel v_\parallel - n\Omega_a} \times \left((\omega - k_\parallel v_\parallel) \frac{\partial F_a}{v_\perp \partial v_\perp} + k_\parallel \frac{\partial F_a}{\partial v_\parallel} \right), \tag{23}$$

where $\mathbf{b} = \mathbf{B}_0/|\mathbf{B}_0| = \hat{\mathbf{z}}$. In deriving this result, we have made use of $\sum_{n=-\infty}^{\infty} nJ_n^2(b)/b = 0$, $\sum_{n=-\infty}^{\infty} J_n(b)J_n'(b) = 0$, and $\sum_{n=-\infty}^{\infty} J_n^2(b) = 1$.

B. Second-order perturbed current

The second-order current is computed next,

$$\mathbf{J}_q^{(2)} = -\sum_{q'} \sum_a \frac{-ie_a}{m_a} \sum_n \sum_m \frac{\omega_{pa}^2}{\omega\omega'(\omega - \omega')} \int d\mathbf{v}\, \mathbf{V}_q^*(n) \times G_q^n \beta_{n,m}^j(q|q') G_{q-q'}^m \alpha_m^k(q - q') \times F_a \Big(E_{q'}^j E_{q-q'}^k - \langle E_{q'}^j E_{q-q'}^k \rangle \Big). \tag{24}$$

We may define the second-order nonlinear susceptibility tensor of rank three,

$$\chi_{ijk}^{(2)}(q'|q - q') = \sum_a \frac{-ie_a}{m_a} \sum_n \sum_m \frac{\omega_{pa}^2}{\omega\omega'(\omega - \omega')} \times \int d\mathbf{v}\, V_q^{i*}(n)\, G_q^n \beta_{n,m}^j(q|q') G_{q-q'}^m \alpha_m^k(q - q') F_a. \tag{25}$$

Then, the second-order perturbed current is expressed as

$$J_q^{(2)i} = -\sum_{q'} \chi_{ijk}^{(2)}(q'|q - q') \Big(E_{q'}^j E_{q-q'}^k - \langle E_{q'}^j E_{q-q'}^k \rangle \Big). \tag{26}$$

We may rewrite the above in a symmetric way by interchanging dummy indices and integral variables,

$$J_q^{(2)i} = -\frac{1}{2} \sum_{q'} \left[\chi_{ijk}^{(2)}(q'|q - q') + \chi_{ikj}^{(2)}(q - q'|q') \right] \times \Big(E_{q'}^j E_{q-q'}^k - \langle E_{q'}^j E_{q-q'}^k \rangle \Big). \tag{27}$$

This leads to the symmetrical second-order susceptibility tensor,

$$\begin{aligned} \chi_{ijk}^{(2)\mathrm{symm}}(q'|q - q') &= \frac{1}{2} \left[\chi_{ijk}^{(2)}(q'|q - q') + \chi_{ikj}^{(2)}(q - q'|q') \right] \\ &= \frac{1}{2} \sum_a \frac{-ie_a}{m_a} \sum_n \sum_m \frac{\omega_{pa}^2}{\omega\omega'(\omega - \omega')} \int d\mathbf{v}\, V_q^{i*}(n)\, G_q^n \\ &\quad \times \left[\beta_{n,m}^j(q|q') G_{q-q'}^m \alpha_m^k(q - q') + \beta_{n,m}^k(q|q - q') G_{q'}^m \alpha_m^j(q') \right] F_a. \end{aligned} \tag{28}$$

Henceforth, we drop the superscript "symm" and simply redefine the second-order susceptibility by the symmetric form,

$$\chi_{ijk}^{(2)}(q'|q - q') = \sum_a \frac{-ie_a}{2m_a} \sum_n \sum_m \frac{\omega_{pa}^2}{\omega\omega'(\omega - \omega')} \int d\mathbf{v}\, V_q^{i*}(n)\, G_q^n \times \left[\beta_{n,m}^j(q|q') G_{q-q'}^m \alpha_m^k(q - q') + \beta_{n,m}^k(q|q - q') G_{q'}^m \alpha_m^j(q') \right] F_a. \tag{29}$$

Then, the second-order current given in the form of (26) is applicable.

C. Third-order perturbed current

The third-order current is computed likewise,

$$\begin{aligned} \mathbf{J}_q^{(3)} &= -\sum_{q'} \sum_{q''} \sum_a \frac{(-ie_a)^2}{m_a^2} \frac{\omega_{pa}^2}{\omega\omega'\omega''(\omega - \omega' - \omega'')} \\ &\quad \times \sum_n \sum_m \sum_p \int d\mathbf{v}\, \mathbf{V}_q^*(n) \\ &\quad \times G_q^n \beta_{n,m}^j(q|q') G_{q-q'}^m \beta_{m,p}^k(q - q'|q'') \\ &\quad \times G_{q-q'-q''}^p \alpha_p^l(q - q' - q'') F_a \\ &\quad \times \Big(E_{q'}^j E_{q''}^k E_{q-q'-q''}^l - E_{q'}^j \langle E_{q''}^k E_{q-q'-q''}^l \rangle \\ &\quad - \langle E_{q'}^j E_{q''}^k E_{q-q'-q''}^l \rangle \Big). \end{aligned} \tag{30}$$

We may define the un-symmetrized third-order nonlinear susceptibility tensor of rank four,

$$\begin{aligned} \tilde{\chi}_{ijkl}^{(3)}(q'|q''|q - q' - q'') &= \sum_a \frac{(-ie_a)^2}{m_a^2} \frac{\omega_{pa}^2}{\omega\omega'\omega''(\omega - \omega' - \omega'')} \\ &\quad \times \sum_n \sum_m \sum_p \int d\mathbf{v}\, V_q^{i*}(n) \\ &\quad \times G_q^n \beta_{n,m}^j(q|q') G_{q-q'}^m \beta_{m,p}^k(q - q'|q'') \\ &\quad \times G_{q-q'-q''}^p \alpha_p^j(q - q' - q'') F_a. \end{aligned} \tag{31}$$

Then, we have

$$\begin{aligned} J_q^{(3)i} &= -\sum_{q'} \sum_{q''} \tilde{\chi}_{ijkl}^{(3)}(q'|q''|q - q' - q'') \Big(E_{q'}^j E_{q''}^k E_{q-q'-q''}^l \\ &\quad - E_{q'}^j \langle E_{q''}^k E_{q-q'-q''}^l \rangle - \langle E_{q'}^j E_{q''}^k E_{q-q'-q''}^l \rangle \Big) \\ &= -\frac{1}{2} \sum_{q'} \sum_{q''} \Big[\tilde{\chi}_{ijkl}^{(3)}(q'|q''|q - q' - q'') \\ &\quad + \tilde{\chi}_{ijlk}^{(3)}(q'|q - q' - q''|q'') \Big] \\ &\quad \times \Big(E_{q'}^j E_{q-q'-q''}^k E_{q''}^l - E_{q'}^j \langle E_{q''}^k E_{q-q'-q''}^l \rangle \\ &\quad - \langle E_{q'}^j E_{q''}^k E_{q-q'-q''}^l \rangle \Big), \end{aligned} \tag{32}$$

where, in the second equality, we have switched $(q - q' - q'' \leftrightarrow q'')$ and $(k \leftrightarrow l)$. This leads to the symmetrized third-order tensor,

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$$\chi^{(3)}_{ijkl}(q'|q''|q-q'-q'') = \sum_a \frac{(-ie_a)^2}{2m_a^2}\frac{\omega_{pa}^2}{\omega\omega'\omega''(\omega-\omega'-\omega'')} \times \sum_n\sum_m\sum_p \int d\mathbf{v}\, V_q^{i*}(n)\, G_q^n\, \beta^j_{n,m}(q|q')\, G^m_{q-q'} \times \Big[\beta^k_{m,p}(q-q'|q'')\, G^p_{q-q'-q''}\, \alpha^j_p(q-q'-q'') + \beta^l_{m,p}(q-q'|q-q'-q'')\, G^p_{q''}\, \alpha^j_p(q'')\Big]F_a, \tag{33}$$

and the third-order current is expressed as

$$J^{(3)i}_q = -\sum_{q'}\sum_{q''}\chi^{(3)}_{ijkl}(q'|q''|q-q'-q'') \Big(E^j_{q'}E^k_{q-q'-q''}E^l_{q''} - E^j_{q'}\langle E^k_{q''}E^l_{q-q'-q''}\rangle - \langle E^j_{q'}E^k_{q''}E^l_{q-q'-q''}\rangle\Big). \tag{34}$$

D. Summary of nonlinear susceptibilities

To summarize the results that we have obtained thus far, let us write down the linear and second- and third-order nonlinear susceptibilities,

$$\chi^{(1)}_{ij}(\mathbf{k},\omega) = \sum_a \frac{\omega_{pa}^2}{\omega^2}\sum_{n=-\infty}^{\infty}\int d\mathbf{v}\, V_q^{i*}(n)\, G_q^n\, \alpha^j_n(q) F_a, \tag{35a}$$

$$\chi^{(2)}_{ijk}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2) = \sum_a \frac{-ie_a}{2m_a}\sum_n\sum_m \frac{\omega_{pa}^2}{\omega\omega_1\omega_2}\int d\mathbf{v}\, V_q^{i*}(n)\, G_q^n \times \Big[\beta^j_{n,m}(q|q_1)\, G^m_{q_2}\, \alpha^k_m(q_2) + \beta^k_{n,m}(q|q_2)\, G^m_{q_1}\, \alpha^j_m(q_1)\Big]F_a, \times (\omega=\omega_1+\omega_2,\ \mathbf{k}=\mathbf{k}_1+\mathbf{k}_2), \tag{35b}$$

$$\chi^{(3)}_{ijkl}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2|\mathbf{k}_3,\omega_3) = \sum_a \frac{(-ie_a)^2}{2m_a^2}\frac{\omega_{pa}^2}{\omega\omega_1\omega_2\omega_3} \times \sum_n\sum_m\sum_p \int d\mathbf{v}\, V_q^{i*}(n)\, G_q^n\, \beta^j_{n,m}(q|q_1)\, G^m_{q_2+q_3} \times \Big[\beta^k_{m,p}(q_2+q_3|q_2)\, G^p_{q_3}\, \alpha^j_p(q_3) + \beta^l_{m,p}(q_2+q_3|q_3)\, G^p_{q_2}\, \alpha^j_p(q_2)\Big]F_a, \times (\omega=\omega_1+\omega_2+\omega_3,\ \mathbf{k}=\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3), \tag{35c}$$

where the various quantities are defined in Eq. (15). This result, which is a direct generalization of the companion paper,[1] is the highlight of the present paper.

It is interesting to compare the present result (35) against the nonlinear susceptibilities for unmagnetized plasmas,[14]

$$\chi^{(1)}_{ij}(\mathbf{k},\omega) = \sum_a \frac{\omega_{pa}^2}{\omega^2}\int d\mathbf{v}\, \nu_i G_q L^j_q F_a, \tag{36a}$$

$$\chi^{(2)}_{ijk}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2) = \sum_a \frac{-ie_a}{2m_a}\frac{\omega_{pa}^2}{\omega\omega_1\omega_2} \times \int d\mathbf{v}\, \nu_i G_q\Big(L^j_{q_1}G_{q_2}L^k_{q_2} + L^k_{q_2}G_{q_1}L^j_{q_1}\Big)F_a, \quad (\omega=\omega_1+\omega_2,\ \mathbf{k}=\mathbf{k}_1+\mathbf{k}_2), \tag{36b}$$

$$\chi^{(3)}_{ijkl}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2|\mathbf{k}_3,\omega_3) = \sum_a \frac{(-ie_a)^2}{2m_a^2}\frac{\omega_{pa}^2}{\omega\omega_1\omega_2\omega_3} \times \int d\mathbf{v}\, \nu_i G_q L^j_{q_1} G_{q_2+q_3}\Big(L^k_{q_2}G_{q_3}L^l_{q_3} + L^l_{q_3}G_{q_2}L^k_{q_2}\Big)F_a, \quad (\omega=\omega_1+\omega_2+\omega_3,\ \mathbf{k}=\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3), \tag{36c}$$

$$G_{\mathbf{k},\omega} = \frac{1}{\omega-\mathbf{k}\cdot\mathbf{v}}, \qquad L^i_{\mathbf{k},\omega} = (\omega-\mathbf{k}\cdot\mathbf{v})\frac{\partial}{\partial\nu_i} + \nu_i\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}. \tag{36d}$$

One can clearly see the analogy and correspondence as well as the differences between the magnetized vs unmagnetized nonlinear susceptibilities.

E. Nonlinear wave equation up to third-order nonlinearity

Before we close, we now put everything together to arrive at

$$0 = \Lambda_{ij}(q)E^j_q + \sum_{q'}\chi^{(2)}_{ijk}(q'|q-q')\Big(E^j_{q'}E^k_{q-q'} - \langle E^j_{q'}E^k_{q-q'}\rangle\Big) + \sum_{q'}\sum_{q''}\chi^{(3)}_{ijkl}(q'|q''|q-q'-q'')\Big(E^j_{q'}E^k_{q-q'-q''}E^l_{q''} - E^j_{q'}\langle E^k_{q''}E^l_{q-q'-q''}\rangle - \langle E^j_{q'}E^k_{q''}E^l_{q-q'-q''}\rangle\Big),$$
$$\Lambda_{ij}(q) = \delta_{ij} + \chi^{(1)}_{ij}(q) - \frac{c^2k^2}{\omega^2}\Big(\delta_{ij} - \frac{k_ik_j}{k^2}\Big). \tag{37}$$

This is formally identical to the equation for unmagnetized plasma electromagnetic weak turbulence theory that is formulated in Ref. 14, except that, now, the definitions of linear and nonlinear susceptibilities pertain to fully magnetized plasmas. Consequently, we may immediately transport the final form of wave kinetic equation from this reference,

$$0 = \frac{i}{2}\frac{\partial\,\mathrm{Re}\,\Lambda_{ij}(q)}{\partial\omega}\frac{\partial}{\partial t}\langle E_iE_j\rangle_q + \mathrm{Re}\,\Lambda_{ij}(q)\langle E_iE_j\rangle_q + i\,\mathrm{Im}\,\Lambda_{ij}(q)\langle E_iE_j\rangle_q - 2\sum_{q'}\Big\{\chi^{(2)}_{ijk}(q'|q-q')\Big[\Lambda^{-1}_{jl}(q')\chi^{(2)}_{lmn}(-q+q'|q)\langle E_kE_m\rangle_{q-q'} + \Lambda^{-1}_{kl}(q-q')\chi^{(2)}_{lmn}(-q'|q)\langle E_jE_m\rangle_{q'}\Big]\langle E_iE_n\rangle_q + \chi^{(2)}_{ijk}(q'|q-q')\Lambda^{-1}_{il}(-q)\chi^{(2)}_{lmn}(-q'|-q+q')\langle E_jE_m\rangle_{q'}\langle E_kE_n\rangle_{q-q'}\Big\} + 2\sum_{q'}\chi^{(3)}_{ijkl}(q'|-q'|q)\langle E_iE_l\rangle_q\langle E_jE_k\rangle_{q'}. \tag{38}$$

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As is customary, if we ignore the nonlinear response for the real part of the above equation, then the dispersion relation is given by

$$\mathrm{Re}\,\Lambda_{ij}(q)\langle E_iE_j\rangle_q = 0. \tag{39}$$

The wave kinetic equation is then determined by the imaginary part of the above equation, or

$$\begin{aligned}\frac{\partial\,\mathrm{Re}\,\Lambda_{ij}(q)}{\partial\omega}\frac{\partial}{\partial t}\langle E_iE_j\rangle_q = &-2\,\mathrm{Im}\,\Lambda_{ij}(q)\langle E_iE_j\rangle_q + 4\,\mathrm{Im}\sum_{q'}\Big\{\chi^{(2)}_{ijk}(q'|q-q')\Big[\Lambda^{-1}_{jl}(q')\chi^{(2)}_{lmn}(-q+q'|q)\langle E_kE_m\rangle_{q-q'}\\ &+\Lambda^{-1}_{kl}(q-q')\chi^{(2)}_{lmn}(-q'|q)\langle E_jE_m\rangle_{q'}\Big]\langle E_iE_n\rangle_q\\ &+\chi^{(2)}_{ijk}(q'|q-q')\Lambda^{-1}_{il}(-q)\chi^{(2)}_{lmn}(-q'|-q+q')\langle E_jE_m\rangle_{q'}\langle E_kE_n\rangle_{q-q'}\Big\}\\ &-4\,\mathrm{Im}\sum_{q'}\chi^{(3)}_{ijkl}(q'|-q'|q)\langle E_iE_l\rangle_q\langle E_jE_k\rangle_{q'}.\end{aligned} \tag{40}$$

IV. SUMMARY AND DISCUSSION

In summary, we have extended the derivation of nonlinear plasma susceptibilities for magnetized plasmas to the fully electromagnetic regime. In the previous work,[1] a simplifying assumption of electrostatic approximation was imposed, but the present paper removed this assumption. The major implications of the present paper are that the mathematical definitions for the nonlinear susceptibility tensors are now given by sufficiently concrete forms such that they can be utilized for actual applications. This was made possible by the use of the Bessel function addition theorem, which helped simplify multiple Bessel function/gyro-harmonic series expansions. Apparently, this property has not been discussed in the standard literature. As discussed in the main body of the present paper, if one were to simply follow the standard definitions for the nonlinear susceptibilities found in the literature, then one would end up with cumbersome multiple Bessel function/cyclotron-harmonic series. However, the present paper demonstrated that some of these infinite series can be eliminated by the use of Bessel addition theorem, resulting in a substantially streamlined definition for these quantities.

A case in point may be to compare the present formalism to formal definitions of nonlinear susceptibilities found in the standard literature. A prime example is taken from Ref. 12, which is slightly modified for the sake of making the formalism consistent with the present arrangement where all the $\mathbf{k}$ vectors are taken to lie in the xz plane. According to Ref. 12, the second-order nonlinear susceptibility tensor for magnetized plasmas is defined by

$$\begin{aligned}&\chi^{(2)}_{ijk}(\mathbf{k}_1,\omega_1|\mathbf{k}_2,\omega_2)\\ &\quad=\sum_a\frac{-ie_a}{2m_a}\sum_n\sum_m\frac{\omega^2_{pa}}{\omega\omega_1\omega_2}\int d\mathbf{v}\,e^{in\varphi-ib\,\sin\varphi}G^n_qV^i_q(n)\\ &\quad\times\Big[Y^j_{q_1}e^{-im\varphi+ib_2\,\sin\varphi}G^m_{q_2}Z^k_{q_2}(m)\\ &\quad+Y^k_{q_2}e^{-im\varphi+ib_1\,\sin\varphi}G^m_{q_1}Z^j_{q_1}(m)\Big]F_a,\end{aligned} \tag{41a}$$

$$Y^i_q=[(\omega-\mathbf{k}\cdot\mathbf{v})\delta_{ij}+k_jv_i]\frac{\partial}{\partial v_j}, \tag{41b}$$

$$\begin{aligned}Z^j_q(m)=&\,V^{i*}_q(m)\left(\frac{\omega-k_\parallel v_\parallel}{v_\perp}\frac{\partial}{\partial v_\perp}+k_\parallel\frac{\partial}{\partial v_\parallel}\right)-\delta_{iz}J_m(b)\\ &\times(\omega-k_\parallel v_\parallel-m\Omega_a)\left(\frac{v_\parallel}{v_\perp}\frac{\partial}{\partial v_\perp}-\frac{\partial}{\partial v_\parallel}\right),\end{aligned} \tag{41c}$$

which is displayed in Ref. 12 without derivation. A similar expression for the third-order susceptibility tensor is also displayed in Ref. 12, again without derivation, but for the sake of present discussion, the second-order susceptibility (41) is sufficient. First, we note that the operator Y^i_q should be written out explicitly by making use of the definition for velocity in the cylindrical coordinate system, $\mathbf{v}=(v_\perp\cos\varphi, v_\perp\sin\varphi, v_\parallel)$. Similarly, the derivatives with respect to $\mathbf{v}$ should also be expressed in terms of the cylindrical coordinate representation. Second, the exponential factors $e^{in\varphi-ib\sin\varphi}$, $e^{im\varphi-ib_1\,\sin\varphi}$, and $e^{im\varphi-ib_2\,\sin\varphi}$ should be expressed as the customary Bessel function series, $e^{in\varphi-ib\sin\varphi}=\sum_lJ_l(b)e^{-i(n-l)\varphi}$, $e^{im\varphi-ib_1\,\sin\varphi}=\sum_pJ_p(b_1)e^{-i(n-p)\varphi}$, and $e^{im\varphi-ib_2\,\sin\varphi}=\sum_pJ_p(b_2)e^{-i(n-p)\varphi}$. These further manipulations are necessary if the formal expressions, such as those found in the standard literature, are to be turned into practical mathematical objects that readily lend themselves to a pragmatic use. However, doing so, one finds that in the end, the formalism leads to an overall factor,

$$\sum_n\sum_m\sum_l\sum_p e^{i(n-l+m-p)\varphi}[\cdots],$$

where $[\cdots]$ denotes the remaining terms. Even if one of the infinite series can be eliminated by integrating over the gyrophase angle, $\int_0^{2\pi}d\varphi e^{i(n-l+m-p)\varphi}=2\pi\delta_{n-l+m-p}$, one is still left with a triple Bessel function sum. Of course, one of them could, and should, be eliminated via the use of addition theorem, but obviously, the standard literature does not discuss such a procedure. A similar situation arises for the third-order susceptibility tensor. Following the standard definitions, e.g., from Ref. 12, one would obtain a sextuple infinite series. Even after eliminating one of them by virtue of the gyrophase angle integration, one is still left with a quintuplet sum. However, two of these can be eliminated by virtue of the sum rule, leaving in the end, a triple sum—which is what we have demonstrated in the present paper. Again, however, such a procedure is not discussed in the literature.

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Before we finally wrap up the discussion, we reiterate that an alternative approach of deriving the linear and nonlinear susceptibility tensors has been invented by Melrose.[49–51] Although the equivalence of this alternative method to the Vlasov method seems to have been demonstrated only for the linear response, this alternative method is potentially very useful in that the forms of response tensors are given in terms of velocity integral weighted by the distribution function, and no derivatives are involved. In many actual applications of the weak turbulence theories, the susceptibility tensors derived from the Vlasov theory require partial integrations. The forward-scattering formalism naturally yields the form of response tensors where such partial integrations have already been carried out. It is in this context where the alternative method can be very useful, but the full equivalence of the forward-scattering and Vlasov approaches as it relates to the nonlinear response tensors has yet to be fully demonstrated.

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AUTHOR DECLARATIONS

Conflict of Interest

The author has no conflicts to disclose.

Author Contributions

Peter H. Yoon: Conceptualization (equal); Formal analysis (equal); Funding acquisition (equal); Investigation (equal); Methodology (equal); Project administration (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

This paper deals with purely theoretical aspects. As such, no actual data are generated herewith.

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