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Diffusive Limit of the Boltzmann Equation in Bounded Domains

Zhimeng Ouyang[1], **Lei Wu**[2]

[1] Department of Mathematics, University of Chicago, Chicago, IL 60637, USA.
E-mail: ouyangzm9386@uchicago.edu
[2] Department of Mathematics, Lehigh University, Bethlehem, PA 18015, USA. E-mail: lew218@lehigh.edu

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Abstract: The investigation of rigorous justification of the hydrodynamic limits in bounded domains has seen significant progress in recent years. While some headway has been made for the diffuse-reflection boundary case (Esposito et al. in Ann PDE 4:1–119, 2018; Ghost effect from Boltzmann theory. arXiv:2301.09427, 2023; Jang and Kim in Ann PDE 7:103, 2021), the more intricate in-flow boundary case, where the leading-order boundary layer effect cannot be neglected, still poses an unresolved challenge. In this study, we tackle the stationary and evolutionary Boltzmann equations, considering the in-flow boundary conditions within both convex and non-convex bounded domains, and demonstrate their diffusive limits in L^2. Our approach hinges on a groundbreaking insight: a remarkable gain of $\varepsilon^{\frac{1}{2}}$ in the kernel estimate, which arises from a meticulous selection of test functions and the careful application of conservation laws. Additionally, we introduce a boundary layer with a grazing-set cutoff and investigate its BV regularity estimates to effectively control the source terms in the remainder equation with the help of the Hardy's inequality.

Contents

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1. Introduction

Hydrodynamic limits concern the derivation of fluid equations, such as the Euler equations or the Navier–Stokes equations, from the kinetic equations such as the Boltzmann equation or the Landau equation. Hydrodynamic limits in non-convex domains (including the so-called exterior domains) play a significant role in the science and engineering problems: e.g. gas dynamics around airplane wings or high-rise buildings, water dynamics near ships or bridge pier, plasma dynamics inside the Tokamak, etc. However, due to the intrinsic singularity of kinetic equations in non-convex domains [28,60,82], the rigorous justification of hydrodynamic limits remains largely open so far.

In this work, we will study the diffusive limits of both the stationary and evolutionary Boltzmann equations in general (convex or non-convex) smooth bounded domains in the presence of boundary layer corrections. We will show that the solutions to the Boltzmann equations converge to a global Maxwellian plus a small perturbation given by the incompressible Navier–Stokes–Fourier system. Due to the complexity of the problems, we will put most of the notation in the Appendix for the convenience of the reader.

1.1. Stationary problem. We consider the stationary Boltzmann equation in a three-dimensional smooth bounded domain $\Omega \ni x = (x_1, x_2, x_3)$ and velocity domain $\mathbb{R}^3 \ni v = (v_1, v_2, v_3)$. The stationary density function $\mathfrak{F}(x, v)$ satisfies

$$\begin{cases} v\cdot\nabla_x\mathfrak{F} = \varepsilon^{-1}Q\left[\mathfrak{F},\mathfrak{F}\right] \ \text{ in } \ \Omega\times\mathbb{R}^3, \\ \mathfrak{F}(x_0,v) = \mathfrak{F}_b(x_0,v) \ \text{ for } \ x_0\in\partial\Omega \ \text{ and } \ v\cdot n<0, \end{cases} \tag{1.1}$$

where $n(x_0)$ is the unit outward normal vector at $x_0 \in \partial\Omega$, and the Knudsen number $0 < \varepsilon \ll 1$ characterizes the average distance a particle might travel between two collisions.

Here Q is the hard-sphere collision operator

$$Q[F,G] := \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} q(\omega, |\mathfrak{u}-v|)\Big[F(\mathfrak{u}_*)G(v_*) - F(\mathfrak{u})G(v)\Big]\mathrm{d}\omega\mathrm{d}\mathfrak{u}, \tag{1.2}$$

with $\mathfrak{u}_* := \mathfrak{u} + \omega\big((v-\mathfrak{u})\cdot\omega\big)$, $v_* := v - \omega\big((v-\mathfrak{u})\cdot\omega\big)$, and the hard-sphere collision kernel $q(\omega, |\mathfrak{u}-v|) := q_0\,|\omega\cdot(v-\mathfrak{u})|$ for a positive constant q_0.

We intend to study the asymptotic limit of $\mathfrak{F}(x,v)$ as $\varepsilon\to 0$.

1.1.1. Setup and assumptions The function spaces and norms used in this paper are introduced in "Appendix B". Assume the in-flow boundary data

$$\mathfrak{F}_b(x_0,v) := \mu(v) + \varepsilon\mu^{\frac{1}{2}}(v)\mathfrak{f}_b(x_0,v) \geq 0, \tag{1.3}$$

where μ denotes the global Maxwellian

$$\mu(v) := (2\pi)^{-\frac{3}{2}}\mathrm{e}^{-\frac{|v|^2}{2}}, \tag{1.4}$$

and $\mathfrak{f}_b(x_0,v)$ is a small perturbation satisfying

$$|\mathfrak{f}_b|_{W^{3,\infty}W^{1,\infty}_{\gamma_-,\varrho,\vartheta}} = o(1). \tag{1.5}$$

Here the subscript γ_- which denotes the in-flow boundary is also defined in "Appendix B".

1.1.2. Normal chart near boundary We follow the approach in [79] to define the geometric quantities.

For smooth manifold $\partial\Omega$, there exists an orthogonal curvilinear coordinates system (ι_1,ι_2) such that the coordinate lines coincide with the principal directions at any $x_0\in\partial\Omega$ (at least locally). Assume $\partial\Omega$ is parameterized by $\mathbf{r}=\mathbf{r}(\iota_1,\iota_2)$. Let the vector length be $L_i = \big|\partial_{\iota_i}\mathbf{r}\big|$ and unit vector $\varsigma_i = L_i^{-1}\partial_{\iota_i}\mathbf{r}$.

Consider the corresponding new coordinate system $(\iota_1,\iota_2,\mathfrak{n})$, where $\mathfrak{n}$ denotes the normal distance to boundary surface $\partial\Omega$, i.e. $x=\mathbf{r}-\mathfrak{n}n$. Define the scaled variable $\eta=\varepsilon^{-1}\mathfrak{n}$, which implies $\frac{\partial}{\partial\mathfrak{n}} = \frac{1}{\varepsilon}\frac{\partial}{\partial\eta}$. Denote $\mathfrak{x} := (\eta,\iota_1,\iota_2)$. Finally, we define the orthogonal velocity substitution for $\mathfrak{v} := (v_\eta, v_{\iota_1}, v_{\iota_2})$ as

$$-v\cdot n := v_\eta, \qquad -v\cdot\varsigma_1 := v_{\iota_1}, \qquad -v\cdot\varsigma_2 := v_{\iota_2}. \tag{1.6}$$

1.1.3. Asymptotic expansion We seek a solution to (1.1) in the form

$$\mathfrak{F}(x,v) = \mu + f + f^B + \varepsilon\mu^{\frac{1}{2}}R = \mu + \mu^{\frac{1}{2}}\Big(\varepsilon f_1 + \varepsilon^2 f_2\Big) + \mu^{\frac{1}{2}}\Big(\varepsilon f_1^B\Big) + \varepsilon\mu^{\frac{1}{2}}R, \tag{1.7}$$

where the interior solution is

$$f(x,v) := \mu^{\frac{1}{2}}(v)\Big(\varepsilon f_1(x,v) + \varepsilon^2 f_2(x,v)\Big), \tag{1.8}$$

and the boundary layer is

$$f^B(\mathfrak{x},\mathfrak{v}) := \mu^{\frac{1}{2}}(\mathfrak{v})\Big(\varepsilon f_1^B(\mathfrak{x},\mathfrak{v})\Big). \tag{1.9}$$

Here f and f^B are defined in Sect. 3.1 and $R(x,v)$ is the remainder satisfying

$$\begin{cases} v\cdot\nabla_x R + \varepsilon^{-1}\mathcal{L}[R] = S \ \text{ in } \ \Omega\times\mathbb{R}^3, \\ R(x_0,v) = h(x_0,v) \ \text{ for } \ v\cdot n<0 \ \text{ and } \ x_0\in\partial\Omega, \end{cases} \tag{1.10}$$

where the linearized Boltzmann operator $\mathcal{L}$ is defined in (A.2), h and S are defined in (3.71)–(3.78).

Let $\mathbf{P}[R]$ be the projection of R onto the null space $\mathcal{N}$ of $\mathcal{L}$ as introduced in Section A. Then we split for some $p(x)$, $\mathbf{b}(x)$, $c(x)$:

$$R = \mathbf{P}[R] + (\mathbf{I}-\mathbf{P})[R] := \mu^{\frac{1}{2}}(v)\left(p(x) + v\cdot\mathbf{b}(x) + \frac{|v|^2-5}{2}c(x)\right) + (\mathbf{I}-\mathbf{P})[R]. \tag{1.11}$$

Define the working space equipped with the norm

$$\begin{aligned}\|R\|_X &:= \varepsilon^{-\frac{1}{2}}\|\mathbf{P}[R]\|_{L^2} + \varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})[R]\|_{L^2_\nu} + \|R\|_{L^6} + \varepsilon^{-\frac{1}{2}}|R|_{L^2_{\gamma+}} + \left|\mu^{\frac{1}{4}}R\right|_{L^4_{\gamma+}}\\ &\quad + \varepsilon^{\frac{1}{2}}\|R\|_{L^\infty_{\varrho,\vartheta}} + \varepsilon^{\frac{1}{2}}|R|_{L^\infty_{\gamma,\varrho,\vartheta}}.\end{aligned} \tag{1.12}$$

1.1.4. Main result Let o_T be a sufficiently small constant depending on $\mathfrak{f}_b$ satisfying

$$o_T \to 0 \quad \text{as} \quad |\mathfrak{f}_b|_{W^{3,\infty}W^{1,\infty}_{\gamma_-,\varrho,\vartheta}} \to 0. \tag{1.13}$$

Theorem 1.1 (Stationary Problem). *Assume that Ω is a bounded C^3 domain and (1.5) holds. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, there exists a nonnegative solution $\mathfrak{F}(x, v)$ to the equation (1.1) represented by (1.7) satisfying*

$$\|R\|_X \lesssim o_T, \tag{1.14}$$

where the X norm is defined in (1.12). Such a solution is unique among all solutions satisfying (1.40). This further yields

$$\left\|\mu^{-\frac{1}{2}}\mathfrak{F} - \mu^{\frac{1}{2}} - \varepsilon\mu^{\frac{1}{2}}\left(\rho_1 + v\cdot\mathbf{u}_1 + \frac{|v|^2-3}{2}T_1\right)\right\|_{L^2} \lesssim o_T\varepsilon^{\frac{3}{2}}, \tag{1.15}$$

where $\big(\rho_1(x), \mathbf{u}_1(x), T_1(x), \mathfrak{p}_1(x)\big)$ satisfies the steady Navier–Stokes–Fourier system

$$\begin{cases}\nabla_x(\rho_1 + T_1) = 0,\\ \mathbf{u}_1\cdot\nabla_x\mathbf{u}_1 - \gamma_1\Delta_x\mathbf{u}_1 + \nabla_x\mathfrak{p}_1 = 0,\\ \nabla_x\cdot\mathbf{u}_1 = 0,\\ \mathbf{u}_1\cdot\nabla_x T_1 - \gamma_2\Delta_x T_1 = 0,\end{cases} \tag{1.16}$$

for constants $\gamma_1 > 0$ and $\gamma_2 > 0$. The boundary condition

$$\Big(\rho_1(x_0), \mathbf{u}_1(x_0), T_1(x_0)\Big) = \Big(\rho^B(x_0), \mathbf{u}^B(x_0), T^B(x_0)\Big) \tag{1.17}$$

is given by

$$\Phi_\infty(x_0, v) = \Phi_\infty(\iota_1, \iota_2, \mathfrak{v}) := \mu^{\frac{1}{2}}\left(\rho^B(\iota_1,\iota_2) + v\cdot\mathbf{u}^B(\iota_1,\iota_2) + \frac{|v|^2-3}{2}T^B(\iota_1,\iota_2)\right) \in \mathcal{N}, \tag{1.18}$$

which is solved from the Milne problem for $\Phi(\mathfrak{x}, \mathfrak{v})$*:*

$$\begin{cases} v_\eta \dfrac{\partial \Phi}{\partial \eta} + \mathcal{L}[\Phi] = 0, \\ \Phi(0, \iota_1, \iota_2, \mathfrak{v}) = \mathfrak{f}_b(\iota_1, \iota_2, \mathfrak{v}) \ \text{for} \ v_\eta > 0, \\ \displaystyle\int_{\mathbb{R}^3} v_\eta \mu^{\frac{1}{2}}(\mathfrak{v}) \Phi(0, \iota_1, \iota_2, \mathfrak{v}) \mathrm{d}\mathfrak{v} = M_f, \\ \displaystyle\lim_{\eta \to \infty} \Phi(\eta, \iota_1, \iota_2, \mathfrak{v}) = \Phi_\infty(\iota_1, \iota_2, \mathfrak{v}). \end{cases} \tag{1.19}$$

Here $M_f(\iota_1, \iota_2)$ *is chosen such that* $\rho^B + T^B = \text{constant}$ *and* $\int_{\partial\Omega} \left(\mathbf{u}^B \cdot n\right) \mathrm{d}S_x = 0.$

Remark 1.2. In the fluid system (1.16), ρ_1 denotes the density, $\mathbf{u}_1$ the velocity, and T_1 the temperature. From (1.15), they are related to the perturbation around the global Maxwellian μ. $\mathfrak{p}_1$ represents the pressure in the Navier–Stokes equations and can be determined as a byproduct of solving (1.16). Further expansion to $O(\varepsilon^2)$ as in Sect. 3.1 reveals that $\mathfrak{p}_1 = \rho_2 + T_2 + \rho_1 T_1$, where ρ_2, T_2 represent the next-order density and temperature.

Remark 1.3. From (1.40) and Theorem 3.9 (which provide the bounds of f_1, f_2 and f_1^B), we may deduce that for $o_T \varepsilon$ small enough

$$\left\| \mu^{-\frac{1}{2}} \mathfrak{F} - \mu^{\frac{1}{2}} \right\|_{L^2} \lesssim o_T \varepsilon, \quad \left\| \mu^{-\frac{1}{2}} \mathfrak{F} - \mu^{\frac{1}{2}} \right\|_{L^\infty_{\varrho,\vartheta}} \lesssim o_T \varepsilon^{\frac{1}{2}}. \tag{1.20}$$

For fixed $\varepsilon > 0$, we know that the solution $\mathfrak{F}$ belongs to the desired function space $L^2 \cap L^\infty_{\varrho,\vartheta}$ so that the well-posedness of the Boltzmann equation is guaranteed (see [41] and [26]). In particular, [41] merely requires a polynomial weight since a contradiction argument is employed in the proof of the L^∞ estimate. Unfortunately, such an approach is inapplicable to the asymptotic problems and thus we have to resort to the argument in [26] which requires a Gaussian-type weight.

Remark 1.4. The assumption that the domain Ω is bounded and C^3 is mainly used in the construction of the asymptotic expansion and remainder estimates (see Sect. 3.1 and 3.2). In detail, in order to define the radius of curvature as in (3.9) and the boundary layer f_1^B, we need at least C^2. Further, in the remainder estimates, the bound of S_2 in Lemma 3.16 relies on $\partial_{\iota_1} f_1^B$, $\partial_{\iota_2} f_1^B$, $\partial_{v_\eta} f_1^B$, $\partial_{v_{\iota_1}} f_1^B$, $\partial_{v_{\iota_2}} f_1^B$, which requires one more order of regularity and thus calls for C^3 domains. In addition, C^3 domain is a natural requirement to make sense of (1.5), which is crucial for Theorem 3.9 to justify the well-posedness of the asymptotic expansion. The assumption that Ω is bounded is also crucial in the proof of Proposition 3.34. The justification of (3.267) requires the finiteness of domain volume.

Remark 1.5. The similar result as Theorem 1.1 also holds in two dimensions. Actually, most of the proofs are analogous or even easier (e.g. in (3.177), the assumption $2 \leq r \leq 6$ is necessary for 3D, but it may be relaxed to $2 \leq r < \infty$ for 2D). We also refer the reader to the discussion on this dimension issue in [76,80,81].

1.2. Evolutionary problem. We consider the evolutionary Boltzmann equation in a three-dimensional smooth bounded domain $\Omega \ni x = (x_1, x_2, x_3)$ and velocity domain

$\mathbb{R}^3 \ni v = (v_1, v_2, v_3)$ with time $t \in \mathbb{R}_+$. The evolutionary density function $\mathfrak{F}(t, x, v)$ satisfies

$$\begin{cases} \varepsilon\partial_t\mathfrak{F} + v \cdot \nabla_x\mathfrak{F} = \varepsilon^{-1}Q[\mathfrak{F}, \mathfrak{F}] \text{ in } \mathbb{R}_+ \times \Omega \times \mathbb{R}^3, \\ \mathfrak{F}(0, x, v) = \mathfrak{F}_i(x, v) \text{ in } \Omega \times \mathbb{R}^3, \\ \mathfrak{F}(t, x_0, v) = \mathfrak{F}_b(t, x_0, v) \text{ for } t \in \mathbb{R}_+, \ x_0 \in \partial\Omega \text{ and } v \cdot n(x_0) < 0. \end{cases} \tag{1.21}$$

We intend to study the asymptotic limit of $\mathfrak{F}(t, x, v)$ as $\varepsilon \to 0$.

1.2.1. Setup and assumptions Assume the in-flow boundary data

$$\mathfrak{F}_b(t, x_0, v) := \mu(v) + \varepsilon\mu^{\frac{1}{2}}(v)\mathfrak{f}_b(t, x_0, v) \geq 0, \tag{1.22}$$

where $\mathfrak{f}_b(t, x_0, v)$ is a small perturbation satisfying

$$\|\mathfrak{f}_b\|_{W^{1,\infty}W^{3,\infty}W^{1,\infty}_{\gamma_-,\varrho,\vartheta}} = o(1). \tag{1.23}$$

Assume the initial data

$$\mathfrak{F}_i(x, v) := \mu(v) + \varepsilon\mu^{\frac{1}{2}}(v)\mathfrak{f}_i(x, v) = \mu(v) + \sum_{k=1}^{4}\varepsilon^k\mu^{\frac{1}{2}}(v)\mathfrak{f}_i^{[k]}(x, v) \geq 0 \tag{1.24}$$

where for some $\big(\rho^I(x), \mathbf{u}^I(x), T^I(x)\big)$ satisfying $\rho^I + T^I = \text{constant}$, $\nabla_x \cdot \mathbf{u}^I = 0$ and $\nabla_x \times \big(\mathbf{u}^I \cdot \nabla_x\mathbf{u}^I - \gamma_1\Delta_x\mathbf{u}^I\big) = \mathbf{0}$:

$$\mathfrak{f}_i^{[1]}(x, v) := \mu^{\frac{1}{2}}(v)\left(\rho^I(x) + v \cdot \mathbf{u}^I(x) + \frac{|v|^2 - 3}{2}T^I(x)\right), \tag{1.25}$$

$$\mathfrak{f}_i^{[2]}(x, v) := \mathcal{L}^{-1}\Big[- v \cdot \nabla_x\mathfrak{f}_i^{[1]} + \Gamma\big[\mathfrak{f}_i^{[1]}, \mathfrak{f}_i^{[1]}\big]\Big], \tag{1.26}$$

$$\mathfrak{f}_i^{[3]}(x, v) := \mathcal{L}^{-1}\Big[- v \cdot \nabla_x\mathfrak{f}_i^{[2]} + 2\Gamma\big[\mathfrak{f}_i^{[1]}, \mathfrak{f}_i^{[2]}\big]\Big], \tag{1.27}$$

and $\mathfrak{f}_i^{[4]}(x, v) \in L^2_v \cap C^1$ can be an arbitrary function. We assume that $\mathfrak{f}_i(x, v)$ is a small perturbation term satisfying

$$\|\mathfrak{f}_i\|_{W^{1,\infty}L^\infty_{\varrho,\vartheta}} \leq \sum_{k=1}^{4}\left\|\mathfrak{f}_i^{[k]}\right\|_{W^{1,\infty}L^\infty_{\varrho,\vartheta}} = o(1). \tag{1.28}$$

Remark 1.6. Solving $\partial_t\mathfrak{F}$ from (1.21), our definition of $\mathfrak{f}_i$ and (1.28) guarantee that

$$\left\|\varepsilon^{-1}\mu^{-\frac{1}{2}}\partial_t\mathfrak{F}\big|_{t=0}\right\|_{L^\infty_{\varrho,\vartheta}} = o(1)\varepsilon, \qquad \left\|\partial_t f_1\big|_{t=0}\right\|_{L^\infty_{\varrho,\vartheta}} = 0, \tag{1.29}$$

which will play a significant role in the remainder estimates. As a matter of fact, our proof still holds with even weaker assumptions: for f_1, f_2 introduced in (1.7)

$$\left\|\varepsilon^{-1}\mu^{-\frac{1}{2}}\partial_t\mathfrak{F}\big|_{t=0} - \partial_t f_1\big|_{t=0} - \varepsilon\partial_t f_2\big|_{t=0}\right\|_{L^\infty_{\varrho,\vartheta}} = o(1)\varepsilon^{\frac{1}{2}}. \tag{1.30}$$

Based on the analysis in [27], this weaker requirement is very sharp to guarantee that the initial time derivative of the remainder $\left\|\partial_t R\big|_{t=0}\right\|_{L^\infty_{\varrho,\vartheta}} \lesssim o_T\varepsilon^{\frac{1}{2}}$.

In addition, assume that $\mathfrak{f}_b$ and $\mathfrak{f}_i$ satisfy the compatibility condition at $t = 0, x_0 \in \partial\Omega$ and $v \cdot n < 0$:

$$\mathfrak{f}_b(0, x_0, v) = \mathfrak{f}_i(x_0, v) = 0, \qquad \partial_t \mathfrak{f}_b(0, x_0, v) = 0. \tag{1.31}$$

Remark 1.7. The compatibility condition (1.31) guarantees that there will be no boundary layer at $t = 0$.

1.2.2. Asymptotic expansions We seek a solution to (1.21) in the form

$$\mathfrak{F}(t, x, v) = \mu + f + f^B + \varepsilon\mu^{\frac{1}{2}} R = \mu + \mu^{\frac{1}{2}}\Big(\varepsilon f_1 + \varepsilon^2 f_2\Big) + \mu^{\frac{1}{2}}\Big(\varepsilon f_1^B\Big) + \varepsilon\mu^{\frac{1}{2}} R, \tag{1.32}$$

where the interior solution is

$$f(t, x, v) := \mu^{\frac{1}{2}}(v)\Big(\varepsilon f_1(t, x, v) + \varepsilon^2 f_2(t, x, v)\Big), \tag{1.33}$$

and the boundary layer is

$$f^B(t, \mathfrak{x}, \mathfrak{v}) := \mu^{\frac{1}{2}}(\mathfrak{v})\Big(\varepsilon f_1^B(t, \mathfrak{x}, \mathfrak{v})\Big). \tag{1.34}$$

Here f and f^B are defined in Sect. 4.1 and $R(t, x, v)$ is the remainder satisfying

$$\begin{cases} \varepsilon\partial_t R + v \cdot \nabla_x R + \varepsilon^{-1}\mathcal{L}[R] = S \ \text{ in } \ \mathbb{R}_+ \times \Omega \times \mathbb{R}^3, \\ R(0, x, v) = z(x, v) \ \text{ in } \ \Omega \times \mathbb{R}^3, \\ R(t, x_0, v) = h(t, x_0, v) \ \text{ for } \ v \cdot n < 0 \ \text{ and } \ x_0 \in \partial\Omega, \end{cases} \tag{1.35}$$

where z, h and S are defined in (4.25)–(4.33).

As for (1.11), we split

$$R = \mathbf{P}[R] + (\mathbf{I} - \mathbf{P})[R] = \mu^{\frac{1}{2}}(v)\left(p(t, x) + v \cdot \mathbf{b}(t, x) + \frac{|v|^2 - 5}{2} c(t, x)\right) + (\mathbf{I} - \mathbf{P})[R]. \tag{1.36}$$

Define the working space equipped with the norm

$$\begin{aligned} |||R|||_X := {} & |||R|||_{L^\infty_t L^2_{x,v}} + \varepsilon^{-\frac{1}{2}} \|R\|_{L^2_{\overline{\gamma}_+}} + \varepsilon^{-\frac{1}{2}} |||\mathbf{P}[R]|||_{L^2} + \varepsilon^{-1} |||(\mathbf{I} - \mathbf{P})[R]|||_{L^2_\nu} \\ & |||\partial_t R|||_{L^\infty_t L^2_{x,v}} + \varepsilon^{-\frac{1}{2}} \|\partial_t R\|_{L^2_{\overline{\gamma}_+}} + \varepsilon^{-\frac{1}{2}} |||\partial_t \mathbf{P}[R]|||_{L^2} + \varepsilon^{-1} |||\partial_t (\mathbf{I} - \mathbf{P})[R]|||_{L^2_\nu} \\ & + \varepsilon^{-\frac{1}{2}} \|R\|_{L^\infty_t L^2_{\gamma_+}} + \left\|\mu^{\frac{1}{4}} R\right\|_{L^\infty_t L^4_{\gamma_+}} + \varepsilon^{-1} |||(\mathbf{I} - \mathbf{P})[R]|||_{L^\infty_t L^2_\nu} + |||R|||_{L^\infty_t L^6_{x,v}} \\ & + \varepsilon^{\frac{1}{2}} |||R|||_{L^\infty_{\varrho,\vartheta}} + \varepsilon^{\frac{1}{2}} \|R\|_{L^\infty_{\overline{\gamma}_+,\varrho,\vartheta}} . \end{aligned} \tag{1.37}$$

1.2.3. Main result Let o_T be a sufficiently small constant depending on $\mathfrak{f}_i$ and $\mathfrak{f}_b$ satisfying

$$o_T \to 0 \ \text{ as } \ \|\mathfrak{f}_i\|_{W^{1,\infty}L^\infty_{\varrho,\vartheta}} + \|\mathfrak{f}_b\|_{W^{1,\infty}W^{3,\infty}W^{1,\infty}_{\gamma_-,\varrho,\vartheta}} \to 0. \tag{1.38}$$

Theorem 1.8 (Evolutionary Problem). *Assume that Ω is a bounded C^3 domain and (1.23), (1.28), (1.31) hold. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and any prescribed constant $\mathfrak{T} > 0$, there exists a nonnegative solution $\mathfrak{F}(t, x, v)$ defined on $(t, x, v) \in [0, \mathfrak{T}] \times \Omega \times \mathbb{R}^3$ to the equation (1.21) represented by (1.32) satisfying*

$$|||R|||_X \lesssim o_T, \tag{1.39}$$

where the X norm is defined in (1.37). Such a solution is unique among all solutions satisfying (1.39). This further yields

$$\left|\left|\left| \mu^{-\frac{1}{2}}\mathfrak{F} - \mu^{\frac{1}{2}} - \varepsilon\mu^{\frac{1}{2}}\left(\rho_1 + v\cdot\mathbf{u}_1 + \frac{|v|^2-3}{2}T_1\right)\right|\right|\right|_{L^2} \lesssim o_T\varepsilon^{\frac{3}{2}}, \tag{1.40}$$

where $\big(\rho_1(t,x), \mathbf{u}_1(t,x), T_1(t,x), \mathfrak{p}_1(t,x)\big)$ satisfies the unsteady Navier–Stokes–Fourier system

$$\begin{cases} \nabla_x(\rho_1 + T_1) = 0,\\ \partial_t\mathbf{u}_1 + \mathbf{u}_1\cdot\nabla_x\mathbf{u}_1 - \gamma_1\Delta_x\mathbf{u}_1 + \nabla_x\mathfrak{p}_1 = 0,\\ \nabla_x\cdot\mathbf{u}_1 = 0,\\ \partial_t T_1 + \mathbf{u}_1\cdot\nabla_x T_1 - \gamma_2\Delta_x T_1 = 0. \end{cases} \tag{1.41}$$

The initial condition

$$\Big(\rho_1(0,x), \mathbf{u}_1(0,x), T_1(0,x)\Big) = \Big(\rho^I(x), \mathbf{u}^I(x), T^I(x)\Big) \tag{1.42}$$

is given by (1.25), and the boundary condition

$$\Big(\rho_1(t,x_0), \mathbf{u}_1(t,x_0), T_1(t,x_0)\Big) = \Big(\rho^B(t,x_0), \mathbf{u}^B(t,x_0), T^B(t,x_0)\Big) \tag{1.43}$$

is given by (1.18) solved from the Milne problem (1.19) for $\Phi(t, \mathfrak{x}, \mathfrak{v})$.

Remark 1.9. The implicit constants in the estimates (1.39) and (1.40) may depend on $\mathfrak{T}$.

Remark 1.10. If we strengthen the boundary data assumption (1.23) such that for some constant $K_0 > 0$

$$\left\|\mathrm{e}^{K_0 t}\mathfrak{f}_b\right\|_{W^{1,\infty}W^{3,\infty}W^{1,\infty}_{\gamma_-,\varrho,\vartheta}} = o(1), \tag{1.44}$$

then by a similar argument, Theorem 1.8 still holds with $\mathfrak{T} = \infty$ and the improved (1.40) states that for some constant $K \in (0, K_0)$

$$\left|\left|\left|\mathrm{e}^{Kt}R\right|\right|\right|_X \lesssim o_T, \tag{1.45}$$

and

$$\left|\left|\left|\mathrm{e}^{Kt}\left\{\mu^{-\frac{1}{2}}\mathfrak{F} - \mu^{\frac{1}{2}} - \varepsilon\mu^{\frac{1}{2}}\left(\rho_1 + v\cdot\mathbf{u}_1 + \frac{|v|^2-3}{2}T_1\right)\right\}\right|\right|\right|_{L^2} \lesssim o_T\varepsilon^{\frac{3}{2}}. \tag{1.46}$$

Particularly, the constants in the estimates (1.45) and (1.46) are independent of $\mathfrak{T}$.

2. Background and Methodology

2.1. Literature review. The hydrodynamic limit of the Boltzmann equation is a key ingredient to tackle Hilbert's famous sixth problem [48] on the axiomization of physics, which has attracted a lot of attention since the pioneering work [49,50]. Most of the important fluid models can be derived by asymptotic expansion with respect to the Knudsen number ε (the so-called Hilbert expansion) at least formally. The rigorous justification of this asymptotic convergence has been well studied in many different settings (domains, scalings, notions of solutions) and it is almost impossible for us to provide an extensive literature review.

The classical dimensionless Boltzmann equation is given by:

$$\delta \partial_t \mathfrak{F} + v \cdot \nabla_x \mathfrak{F} = \varepsilon^{-1} Q\left[\mathfrak{F}, \mathfrak{F}\right], \tag{2.1}$$

where the Strouhal number δ quantifies the rate at which the particle system undergoes variation (relaxation time), while the Knudsen number ε measures the relative distance a particle can travel between two collisions (scattering strength). These dimensionless quantities play a crucial role in characterizing the scales of the problem. The fundamental problem in hydrodynamic limits is to study the asymptotic behavior of $\mathfrak{F}(t, x, v)$ in (2.1) as $\varepsilon \to 0$ and/or $\delta \to 0$.

Notably, the well-known von Kármán relation [71] provides a useful guideline for understanding the expected hydrodynamic limits in the context of kinetic theory. The von Kármán relation states that the Knudsen number (Kn) is proportional to the ratio of the Mach number (Ma) to the Reynolds number (Re). Depending on the different combinations of Kn, Ma and Re, the solution to (2.1) may result in different fluid equations.

When Ma $= O(1)$ and thus Re $= O\left(\varepsilon^{-1}\right)$, the solution of the Boltzmann equation will converge to a local Maxwellian which depends on solutions of the compressible Euler equations. Such result was obtained by Caflisch [16] and Lachowicz [63], while Nishida [68], Asano-Ukai [5] proved the similar results with a different approach. For the convergence in the presence of singularities for the Euler equations, we refer to Yu [83] and Huang-Wang-Yang [52,53]. The bounded domain case with boundary effects was considered in Huang-Wang-Yang [54], Huang-Wang-Wang-Yang [51] and Guo-Huang-Wang [42]. The relativistic Euler limit has been studied in Speck-Strain [73]. Notice that the local Maxwellian $\mu_\ell(t, x, v) = \frac{\rho(t,x)}{(2\pi T(t,x))^{\frac{3}{2}}} \exp\left(-\frac{|v-u(t,x)|^2}{2T(t,x)}\right)$ leads to an additional term $\left[(\partial_t + v \cdot \partial_x)\mu_\ell^{\frac{1}{2}}\right]\mu_\ell^{-\frac{1}{2}} R$ in the remainder equation (compared with (1.10) or (1.35)), which greatly distorts the energy-dissipation structure. Due to this intrinsic difficulty from the local Maxwellian and the possible singularity from solving the Euler equations, most of the results above are local in time. Also, due to the inapplicability of almost all mature techniques in the whole space $\mathbb{R}^n$ or the periodic domain $\mathbb{T}^n$, the bounded domain case remains largely open.

When Ma $= O(\varepsilon)$ and thus Re $= O(1)$, the diffusion effects become significant, and the solution of the Boltzmann equation will converge to a global Maxwellian plus an $O(\varepsilon)$ perturbation solving the incompressible Navier–Stokes equations. We refer to Bardos-Ukai [11], DeMasi-Esposito-Lebowitz [23], Guo [40], Guo-Jang [43] for smooth solutions, and Bardos-Golse-Levermore [7–10], Lions-Masmoudi [66], Jiang-Masmoudi [57], Masmoudi-Saint-Raymond [67], Golse-Saint-Raymond [38] for renormalized solutions. For more references and related topics and developments, we refer to Villani [74], Desvillettes-Villani [24,25], Carlen-Carvalho [18], Arkeryd-Nouri [4], Arkeryd-Esposito-Marra-Nouri [2,3], Esposito-Lebowitz-Marra [31,32]. We also refer to the review and survey by Saint-Raymond [70], Golse [36], Esposito-Marra [33], and the references therein.

When Ma $= O(\varepsilon^{\alpha})$ for $0 < \alpha < 1$ and thus Re $= O(\varepsilon^{\alpha-1})$, the solution of the Boltzmann equation will converge to a global Maxwellian plus an $O(\varepsilon)$ perturbation solving the incompressible Euler equations. Besides the overlapped references as above, we also refer to the recent development in Jang-Kim [56], Cao-Jang-Kim [17], Kim-La [61].

There is a surprising new phenomenon arising from the hydrodynamic limits. When Ma $= O(\varepsilon)$ and thus Re $= O(1)$, if the density/temperature is $O(1)$ instead of $O(\varepsilon)$, as Sone [71,72] predicted, a new type of mixed fluid system (the so-called ghost-effect equations) emerges as the hydrodynamic limit of the Boltzmann equation. We refer to the recent development [29,30].

Notice that the von Kármán relation only dictates the behavior of Knudsen numbers. The limiting fluid systems also depend on the scale of the Strouhal number δ (which is a word borrowed from the fluid mechanics). Unfortunately, there is no a priori knowledge of what the "correct" scaling $\delta = O\left(\varepsilon^{\varkappa}\right)$ should be, but usually a properly chosen $\varkappa$ would balance the particle collisions and time variation such that neither of them is negligible. For example, in the diffusion regime Ma $= O(\varepsilon)$ and thus Re $= O(1)$, if $\varkappa = 1$, then the balance is achieved and the corresponding fluid system is the evolutionary incompressible Navier–Stokes equations; if $\varkappa > 1$, then the time varies too slowly and the corresponding fluid system is the stationary incompressible Navier–Stokes equations.

In this paper, we will focus on the diffusive limit of the Boltzmann equation in bounded domains, under both stationary and evolutionary settings with balanced Strouhal/Knudsen numbers (i.e. Ma $= O(\varepsilon)$, Re $= O(1)$ and $\delta = O(\varepsilon)$). Our work is closely related to the recent development of $L^2 - L^6 - L^{\infty}$ framework and the kinetic boundary layer with geometric effects. We refer to Esposito-Guo-Kim-Marra [26,27], and Wu [76], Wu-Ouyang [79–81] for the diffuse-reflection boundary. In particular, [26,27] justify the L^2 convergence (for both convex and non-convex domains) relying on an improved $L^2 - L^6 - L^{\infty}$ framework without boundary layer expansion, and [76,79–81] show the L^{∞} convergence for convex domains with the boundary layer expansion. As [47,60,82] reveal, the delicately designed boundary layer with geometric correction cannot attain $W^{1,\infty}$ regularity in non-convex domains, and thus the L^{∞} convergence for non-convex domains is far from reach. The case of specular-reflection and bounce-back boundary remain largely open (except on half-space or channel domains), mainly due to the lack of explicit kernel estimate to track the ε dependence [14,41]. We also refer to the recent development [22].

Boundary effect and grazing singularity play a more significant role for the in-flow boundary, since this is the only case that the leading-order boundary layer does not vanish. The L^{∞} convergence requires $W^{2,\infty}$ regularity of the boundary layer, which is way too far from reach at this stage. As a matter of fact, even for L^2 convergence, we cannot see any viable option to completely avoid the introduction of the boundary layer expansion. As far as we are aware of, the best result for the in-flow case is the L^{∞} convergence for unit disk or unit ball domains [75], but there is no clear path to extend the techniques to cover general smooth convex domains, let alone the non-convex ones.

In addition, we include some recent papers on the diffusive limit of the Boltzmann equation and related models [14,15,55,58]. We also list some recent developments along $L^2 - L^p - L^{\infty}$ framework [41,43–46,59,60,73,77,78].

In this paper, we utilize a different approach and design a cutoff boundary layer combined with a novel remainder estimates to obtain the L^2 convergence in general smooth bounded (including convex and non-convex) domains.

2.2. Major difficulty. In the following, we will utilize the stationary remainder equation (1.10) to illustrate the key ideas. Rooted from the basic energy estimate (via multiplying $\varepsilon^{-1}R$ on both sides of (1.10) and integrating over $\Omega\times\mathbb{R}^3$ as in [27,75,79]) and the coercivity of $\big\langle\mathcal{L}[R], R\big\rangle$ in (A.5), we may bound $\|(\mathbf{I}-\mathbf{P})[R]\|_{L^2_\nu}$:

$$\varepsilon^{-\frac{1}{2}}\left|R\right|_{L^2_{\gamma_+}}+\varepsilon^{-1}\left\|(\mathbf{I}-\mathbf{P})[R]\right\|_{L^2_\nu}\lesssim o(1)\left\|\mathbf{P}[R]\right\|_{L^2}+o_T\varepsilon^{-\frac{1}{2}}. \tag{2.2}$$

Then by testing (1.10) against functions $\nabla_x\varphi\cdot\overline{\mathscr{A}}$ with $\varphi\sim\Delta_x^{-1}p,\ \Delta_x^{-1}c$ and $\nabla_x\psi:\overline{\mathscr{B}}$ with $\psi\sim\Delta_x^{-1}\mathbf{b}$ for $\overline{\mathscr{A}},\overline{\mathscr{B}}$ defined in (C.1) and (C.2), we may in turn bound $\mathbf{P}[R]$ in terms of $(\mathbf{I}-\mathbf{P})[R]$:

$$\left\|\mathbf{P}[R]\right\|_{L^2}\lesssim\varepsilon^{-\frac{1}{2}}\left|R\right|_{L^2_{\gamma_+}}+\varepsilon^{-1}\left\|(\mathbf{I}-\mathbf{P})[R]\right\|_{L^2_\nu}+o_T\varepsilon^{-\frac{1}{2}}. \tag{2.3}$$

Clearly, (2.2) and (2.3) lead to

$$\varepsilon^{-\frac{1}{2}}\left|R\right|_{L^2_{\gamma_+}}+\varepsilon^{-1}\left\|(\mathbf{I}-\mathbf{P})[R]\right\|_{L^2_\nu}+\left\|\mathbf{P}[R]\right\|_{L^2}\lesssim o_T\varepsilon^{-\frac{1}{2}}. \tag{2.4}$$

However, due to the negative power of ε on the RHS, (2.4) is insufficient to justify the desired L^2 convergence

$$\lim_{\varepsilon\to 0}\left\|R\right\|_{L^2}=0. \tag{2.5}$$

For the diffuse-reflection boundary in convex domains [27,81], the general strategy to overcome the above difficulty is to expand the interior solution and boundary layer to sufficiently high order, i.e. compared with (1.7), we redefine

$$\mathfrak{F}(x,v)=\mu+f+f^B+\varepsilon\mu^{\frac{1}{2}}R=\mu+\mu^{\frac{1}{2}}\left(\varepsilon f_1+\varepsilon^2 f_2+\varepsilon^3 f_3\right)+\mu^{\frac{1}{2}}\left(\varepsilon f_1^B+\varepsilon^2 f_2^B\right)+\varepsilon\mu^{\frac{1}{2}}R. \tag{2.6}$$

The diffuse-reflection boundary condition implies that $f_1^B=0$ and thus there is no difficulties caused by the regularity of the boundary layer [75]. The extra terms $\varepsilon^3 f_3\mu^{\frac{1}{2}}$ and $\varepsilon^2 f_2^B\mu^{\frac{1}{2}}$ help improve the bounds of S and h in (1.10). Eventually, this leads to the improved version of (2.4):

$$\varepsilon^{-\frac{1}{2}}\left|R\right|_{L^2_{\gamma_+}}+\varepsilon^{-1}\left\|(\mathbf{I}-\mathbf{P})[R]\right\|_{L^2_\nu}+\left\|\mathbf{P}[R]\right\|_{L^2}\lesssim o_T\varepsilon^{\alpha}, \tag{2.7}$$

for some $\alpha>0$ and thus (2.5) follows.

Unfortunately, this strategy does not work for the in-flow boundary, in which f_1^B is not necessarily zero and $\frac{\partial f_1^B}{\partial v_\eta}\notin L^\infty$ [75]. Hence, we cannot expect to expand to f_2^B which satisfies the Milne problem

$$v_\eta\frac{\partial f_2^B}{\partial\eta}+\mathcal{L}\left[f_2^B\right]\approx\frac{\partial f_1^B}{\partial v_\eta}. \tag{2.8}$$

The loss of f_1^B regularity makes it impossible [75] to justify the well-posedness of (2.8), and thus (2.5) is not attainable.

On the other hand, in non-convex domains, as illustrated in a similar scenario for the neutron transport equation [82], the boundary layer f_1^B is already problematic. Traditionally, there are two approaches to design the boundary layer. If f_1^B is defined satisfying the flat Milne problem [12] (which is designed for flat domains)

$$v_\eta \frac{\partial f_1^B}{\partial \eta} + \mathcal{L}\left[f_1^B\right] \approx 0, \tag{2.9}$$

then the remainder R estimates requires the control of $\frac{\partial f_1^B}{\partial v_\eta}$ which is not in L^∞ as the previous paragraph stated. If f_1^B is defined using the geometrically corrected Milne problem [75,76,80,81] (which is designed for curved convex domains)

$$v_\eta \frac{\partial f_1^B}{\partial \eta} + \frac{\varepsilon}{\mathcal{R}_1 - \varepsilon\eta}\left(v_{\iota_1}^2 \frac{\partial f_1^B}{\partial v_\eta}\right) + \frac{\varepsilon}{\mathcal{R}_2 - \varepsilon\eta}\left(v_{\iota_2}^2 \frac{\partial f_1^B}{\partial v_\eta}\right) + \mathcal{L}\left[f_1^B\right] \approx 0, \tag{2.10}$$

where $\mathcal{R}_i$ denotes the radii of principal curvatures, then the well-posedness of f_1^B is not attainable in non-convex domains. This adds additional difficulty in the construction of asymptotic expansion. In this work, we will utilize the so-called cutoff Milne problem to define f_1^B as (3.51) which contains a crucial cutoff near the grazing set $v_\eta = 0$. The usage of this cutoff is explained in the next two subsections and Lemma 3.16, and the rigorous well-posedness and regularity theory of f_1^B is given in Sect. 3.1.2.

2.3. Methodology: stationary problem. As the above analysis reveals, with the expansion (1.7) for the in-flow boundary, the bottleneck of (2.4) lies in the kernel bound $\|\mathbf{P}[R]\|_{L^2}$ and the source terms estimates for both (2.2) and (2.3). In this paper, we will design several delicate test functions to obtain

$$\varepsilon^{-\frac{1}{2}} |R|_{L^2_{\gamma+}} + \varepsilon^{-1} \|(\mathbf{I}-\mathbf{P})[R]\|_{L^2_\nu} \lesssim o(1)\varepsilon^{-\frac{1}{2}} \|\mathbf{P}[R]\|_{L^2} + o_T, \tag{2.11}$$

and

$$\varepsilon^{-\frac{1}{2}} \|\mathbf{P}[R]\|_{L^2} \lesssim \varepsilon^{-\frac{1}{2}} |R|_{L^2_{\gamma+}} + \varepsilon^{-1} \|(\mathbf{I}-\mathbf{P})[R]\|_{L^2_\nu} + o_T, \tag{2.12}$$

which introduce a crucial gain of half-order ε compared with (2.3):

$$\varepsilon^{-\frac{1}{2}} |R|_{L^2_{\gamma+}} + \varepsilon^{-1} \|(\mathbf{I}-\mathbf{P})[R]\|_{L^2_\nu} + \varepsilon^{-\frac{1}{2}} \|\mathbf{P}[R]\|_{L^2} \lesssim o_T, \tag{2.13}$$

and lead to (2.5).

Our key idea is a set of tricky combinations of weak formulations and conservation laws to eliminate the worst term $\varepsilon^{-1} |||(\mathbf{I}-\mathbf{P})[R]|||_{L^2}$ on the RHS of (2.3) and greatly improve the source term estimates in (2.2) and (2.3). We will illustrate more precise statement of the argument in the following.

Energy Estimate Testing (1.10) against $\varepsilon^{-1}R$ and utilizing the coercivity and orthogonality yield

$$\varepsilon^{-1} |R|^2_{L^2_{\gamma+}} + \varepsilon^{-2} \|(\mathbf{I}-\mathbf{P})[R]\|^2_{L^2_\nu} \lesssim \left|\varepsilon^{-1}\langle S, R\rangle\right|. \tag{2.14}$$

Here, the most difficult term in S is the normal velocity derivative of f_1^B:

$$\left|\varepsilon^{-1}\left\langle\frac{\partial f_1^B}{\partial v_\eta},\mathbf{P}[R]\right\rangle\right|\lesssim\varepsilon^{-1}\left|\left\langle f_1^B,\mathbf{P}[R]\right\rangle\right|\lesssim\varepsilon^{-1}\left\|f_1^B\right\|_{L^2_xL^1_v}\|\mathbf{P}[R]\|_{L^2_xL^\infty_v}\lesssim o_T\varepsilon^{-\frac{1}{2}}\|\mathbf{P}[R]\|_{L^2}, \tag{2.15}$$

which relies on a key integration by parts with respect to v_η and taking the full advantage of the rescaling $\eta=\varepsilon^{-1}\mathfrak{n}$. Here the inner product $\langle\cdot,\ \cdot\rangle$ is defined in "Appendix B".

Therefore, we arrive at

$$\varepsilon^{-\frac{1}{2}}|R|_{L^2_{\gamma_+}}+\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})[R]\|_{L^2_v}\lesssim o_T\varepsilon^{-\frac{1}{2}}\|\mathbf{P}[R]\|_{L^2}+o_T+\text{nonlinear terms}. \tag{2.16}$$

Estimate of p Test (1.10) against the smooth function $\psi=\mu^{\frac{1}{2}}\big(v\cdot\nabla_x\varphi\big)$ with $\varphi\sim\Delta_x^{-1}p$ yields

$$\int_\gamma R\psi(v\cdot n)-\big\langle R,v\cdot\nabla_x\psi\big\rangle=\big\langle S,\psi\big\rangle. \tag{2.17}$$

By oddness and orthogonality, we eliminate the worst contribution $\varepsilon^{-1}\big\langle\mathcal{L}[R],\psi\big\rangle=0$. Then a straightforward estimate for the source term $\big\langle S,\psi\big\rangle$ reveals the $\varepsilon^{\frac{1}{2}}$ gain:

$$\varepsilon^{-\frac{1}{2}}\|p\|_{L^2}\lesssim\varepsilon^{-\frac{1}{2}}|R|_{L^2_{\gamma_+}}+o_T. \tag{2.18}$$

Estimate of c For smooth function $\varphi\sim\Delta_x^{-1}c$ with $\varphi\big|_{\partial\Omega}=0$, we test (1.10) against $\varphi\big(|v|^2-5\big)\mu^{\frac{1}{2}}$ to obtain

$$-\big\langle\nabla_x\varphi,\varsigma\big\rangle_x+\int_{\partial\Omega}\varphi\varsigma\cdot n=\left\langle\varphi\left(|v|^2-5\right)\mu^{\frac{1}{2}},S\right\rangle, \tag{2.19}$$

and against $\nabla_x\varphi\cdot\mathscr{A}$ to obtain

$$\begin{aligned}&-\kappa\big\langle\Delta_x\varphi,c\big\rangle_x+\varepsilon^{-1}\big\langle\nabla_x\varphi,\varsigma\big\rangle\\&\quad=\big\langle\nabla_x\varphi\cdot\mathscr{A},h\big\rangle_{\gamma_-}-\big\langle\nabla_x\varphi\cdot\mathscr{A},R\big\rangle_{\gamma_+}+\Big\langle v\cdot\nabla_x\Big(\nabla_x\varphi\cdot\mathscr{A}\Big),(\mathbf{I}-\mathbf{P})[R]\Big\rangle+\big\langle\nabla_x\varphi\cdot\mathscr{A},S\big\rangle,\end{aligned} \tag{2.20}$$

where $\varsigma:=\displaystyle\int_{\mathbb{R}^3}\mu^{\frac{1}{2}}v\,|v|^2\,(\mathbf{I}-\mathbf{P})[R]$.

Therefore, adding $\varepsilon^{-1}\times$(2.19) and (2.20) exactly eliminates the troublesome terms $\varepsilon^{-1}\big\langle\nabla_x\varphi,\varsigma\big\rangle_x$ and $\varepsilon^{-1}\displaystyle\int_{\partial\Omega}\varphi\varsigma\cdot n$ whose presence leads to the worst $\varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})[R]\|^2_{L^2_v}$ contribution. These new conservation laws will be the backbone of c estimates. Hence, we obtain the $\varepsilon^{\frac{1}{2}}$ gain:

$$\varepsilon^{-\frac{1}{2}}\|c\|_{L^2}\lesssim\varepsilon^{-\frac{1}{2}}|R|_{L^2_{\gamma_+}}+o_T\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})[R]\|_{L^2_v}+o_T+\text{nonlinear terms}, \tag{2.21}$$

which relies on the careful estimates for the source terms $\varepsilon^{-1}\left\langle\varphi\left(|v|^2-5\right)\mu^{\frac{1}{2}}, S\right\rangle$ and $\left\langle\nabla_x\varphi\cdot\mathscr{A}, S\right\rangle$. Here we will take full advantage of the cutoff boundary layer (see the key bounds in Lemma 3.16) and Hardy's inequality

$$\begin{aligned}
&\left|\varepsilon^{-1}\left\langle\varphi\left(|v|^2-5\right)\mu^{\frac{1}{2}}, \frac{\partial f_1^B}{\partial v_\eta}\right\rangle\right| \lesssim \left|\varepsilon^{-1}\left\langle\varphi\left(|v|^2-5\right)\mu^{\frac{1}{2}}, f_1^B\right\rangle\right| \\
&= \left|\left\langle\frac{1}{\mathfrak{n}}\int_0^{\mathfrak{n}}\frac{\partial\varphi}{\partial\mathfrak{n}}\left(|v|^2-5\right)\mu^{\frac{1}{2}}, \eta f_1^B\right\rangle\right| \\
&\lesssim \left\|\frac{1}{\mathfrak{n}}\int_0^{\mathfrak{n}}\frac{\partial\varphi}{\partial\mathfrak{n}}\left(|v|^2-5\right)\mu^{\frac{1}{2}}\right\|_{L^2}\left\|\eta f_1^B\right\|_{L^2} \lesssim o_T\varepsilon^{\frac{1}{2}}\|\varphi\|_{H^1} \lesssim o_T\varepsilon^{\frac{1}{2}}\|c\|_{L^2}. \qquad (2.22)
\end{aligned}$$

Estimate of $\mathbf{b}$

For smooth function $\psi\sim\Delta_x^{-1}\mathbf{b}$ with $\nabla_x\cdot\psi=0$ and $\psi\big|_{\partial\Omega}=0$ (solved from the Stokes problem), we test (1.10) against $\psi\cdot v\mu^{\frac{1}{2}}$ to obtain

$$-\left\langle\nabla_x\cdot\psi, p\right\rangle_x - \left\langle\nabla_x\psi, \varpi\right\rangle_x + \int_{\partial\Omega}\left(p\psi+\psi\cdot\varpi\right)\cdot n = \left\langle\psi\cdot v\mu^{\frac{1}{2}}, S\right\rangle, \qquad (2.23)$$

and against $\nabla_x\psi:\mathscr{B}$ to obtain

$$\begin{aligned}
&-\lambda\left\langle\Delta_x\psi, \mathbf{b}\right\rangle_x + \varepsilon^{-1}\left\langle\nabla_x\psi, \varpi\right\rangle \\
&= \left\langle\nabla_x\psi\cdot\mathscr{B}, h\right\rangle_{\gamma_-} - \left\langle\nabla_x\psi\cdot\mathscr{B}, R\right\rangle_{\gamma_+} + \left\langle v\cdot\nabla_x\left(\nabla_x\psi:\mathscr{B}\right), (\mathbf{I}-\mathbf{P})[R]\right\rangle + \left\langle\nabla_x\psi:\mathscr{B}, S\right\rangle, \qquad (2.24)
\end{aligned}$$

where $\varpi := \int_{\mathbb{R}^3}\mu^{\frac{1}{2}}\left(v\otimes v\right)(\mathbf{I}-\mathbf{P})[R]$.

Therefore, adding $\varepsilon^{-1}\times$(2.23) and (2.24) exactly eliminates the worrisome terms $\varepsilon^{-1}\left\langle\nabla_x\psi, \varpi\right\rangle_x$, $\varepsilon^{-1}\left\langle\nabla_x\cdot\psi, p\right\rangle_x$ and $\varepsilon^{-1}\int_{\partial\Omega}\left(p\psi+\psi\cdot\varpi\right)\cdot n$ which yields the worst $\varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})[R]\|_{L^2_v}^2$ contribution. Then similar to c estimates, after the careful estimates for the source terms $\varepsilon^{-1}\left\langle\psi\cdot v\mu^{\frac{1}{2}}, S\right\rangle$ and $\left\langle\nabla_x\psi:\mathscr{B}, S\right\rangle$, we obtain $O(\varepsilon^{\frac{1}{2}})$ gain of RHS:

$$\varepsilon^{-\frac{1}{2}}\|\mathbf{b}\|_{L^2} \lesssim \varepsilon^{-\frac{1}{2}}|R|_{L^2_{\gamma_+}} + o_T\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})[R]\|_{L^2_v} + o_T + \text{nonlinear terms}. \qquad (2.25)$$

These new conservation laws will sit at the center stage of $\mathbf{b}$ estimates.

L^∞ Estimate Due to the presence of the nonlinear term $\Gamma[R,R]$ in (1.10), we need to bound $L^2-L^6-L^\infty$ norms for both $\mathbf{P}[R]$ and $(\mathbf{I}-\mathbf{P})[R]$. After extending the above techniques from L^2 to L^6, we will employ the L^6-L^∞ framework to obtain

$$\varepsilon^{\frac{1}{2}}\|R\|_{L^\infty_{\varrho,\vartheta}} + \varepsilon^{\frac{1}{2}}|R|_{L^\infty_{\gamma_+,\varrho,\vartheta}} \lesssim \|\mathbf{P}[R]\|_{L^6} + \varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})[R]\|_{L^2_v} + o_T. \qquad (2.26)$$

One crucial step in proving the result above is the estimate of contributions related to the normal velocity derivative $\frac{\partial f_1^B}{\partial\eta}$. In view of Lemma 3.16 and the grazing-set cutoff, we may deduce $\frac{\partial f_1^B}{\partial\eta}\approx o_T\varepsilon^{-1}$, which leads to the bound (3.250).

2.4. Methodology: evolutionary problem. Besides the difficulties and methodology mentioned in the stationary problem, we have an additional obstacle in the evolutionary settings for the remainder equation (1.35). The L^∞ estimate

$$\begin{aligned}\varepsilon^{\frac{1}{2}} \, |||R|||_{L^\infty_{\varrho,\vartheta}} + \varepsilon^{\frac{1}{2}} \, \|R\|_{L^\infty_{\overline{\gamma}_+,\varrho,\vartheta}} &\lesssim |||\mathbf{P}[R]|||_{L^\infty_t L^6_{x,v}} \\ &+ \varepsilon^{-1} \, |||(\mathbf{I}-\mathbf{P})[R]|||_{L^\infty_t L^2_{x,v}} + o_T \, \|R\|_X + \|R\|_X^2 + o_T \end{aligned} \tag{2.27}$$

calls for the control of the instantaneous bound $|||\mathbf{P}[R]|||_{L^\infty_t L^6_{x,v}}$, instead of the accumulative bound $|||\mathbf{P}[R]|||_{L^6}$ from the energy and kernel estimates. Hence, we have to carefully study the interplay of the accumulative and instantaneous norms and estimate both $L^2 - L^6$ versions of them.

Accumulative Estimates Based on a delicate choice of test functions and the analogous cancellation with weak formulations and conservation laws, we obtain the energy estimate

$$\|R(t)\|_{L^2} + \varepsilon^{-\frac{1}{2}} \, \|R\|_{L^2_{\overline{\gamma}_+}} + \varepsilon^{-1} \, |||(\mathbf{I}-\mathbf{P})[R]|||_{L^2} \lesssim o_T \, |||R|||_X + |||R|||_X^2 + o_T, \tag{2.28}$$

$$\|\partial_t R(t)\|_{L^2} + \varepsilon^{-\frac{1}{2}} \, \|\partial_t R\|_{L^2_{\overline{\gamma}_+}} + \varepsilon^{-1} \, |||(\mathbf{I}-\mathbf{P})[\partial_t R]|||_{L^2} \lesssim o_T \, |||R|||_X + |||R|||_X^2 + o_T, \tag{2.29}$$

and the kernel estimate

$$\varepsilon^{-\frac{1}{2}} \, |||\mathbf{P}[R]|||_{L^2} \lesssim o_T \, |||R|||_X + |||R|||_X^2 + o_T, \tag{2.30}$$

$$\varepsilon^{-\frac{1}{2}} \, |||\partial_t \mathbf{P}[R]|||_{L^2} \lesssim o_T \, |||R|||_X + |||R|||_X^2 + o_T. \tag{2.31}$$

Notice that we need to bound both R and its time derivative $\partial_t R$ for the convenience of instantaneous estimates. Notably, the $\partial_t R$ estimates calls for $\|\partial_t R(0)\|_{L^2} \lesssim \varepsilon^{\frac{1}{2}}$ which is the key reason that our argument only applies to the well-prepared initial data (1.24), and cannot include the discussion of the initial layer (as opposed to the case of transport equation [69]).

In the analysis of evolutionary conservation laws, we also need a careful bound of the time-derivative terms which provide a favorable sign and separate estimates of $\partial_t \nabla_x \varphi$ and $\partial_t \nabla_x \psi$ to close the proof.

Instantaneous Estimates We rewrite (1.35) by moving $\partial_t R$ to the RHS

$$v \cdot \nabla_x R + \varepsilon^{-1} \mathcal{L}[R] = S - \varepsilon \partial_t R, \tag{2.32}$$

where we regard $\partial_t R$ as a source term in the stationary remainder equation. Hence, by a similar estimate as the stationary case, we obtain the energy estimate

$$\varepsilon^{-\frac{1}{2}} \, |R(t)|_{L^2_{\gamma_+}} + \varepsilon^{-1} \, \|(\mathbf{I}-\mathbf{P})[R](t)\|_{L^2} \lesssim o_T \, |||R|||_X + |||R|||_X^2 + o_T, \tag{2.33}$$

$$\|(\mathbf{I}-\mathbf{P})[R](t)\|_{L^6} + \left|\mu^{\frac{1}{4}} R(t)\right|_{L^4_{\gamma_+}} \lesssim o_T \, |||R|||_X + |||R|||_X^2 + o_T, \tag{2.34}$$

and the kernel estimate

$$\|\mathbf{P}[R](t)\|_{L^6} \lesssim o_T \, |||R|||_X + |||R|||_X^2 + o_T. \tag{2.35}$$

Notice that these estimates heavily rely on the accumulative L^2 bound of $\partial_t R$ in (2.29). Then we may proceed to the L^∞ estimate (2.27) and close the proof.

3. Stationary Problem

3.1. Asymptotic analysis.

3.1.1. Interior solution The derivation of the interior solution is classical. We refer to [36,71,72] and the references therein. By inserting (1.8) into (1.1) and equating the order of ε, we require that

$$0 = 2\mu^{-\frac{1}{2}} Q^*\big[\mu, \mu^{\frac{1}{2}} f_1\big], \tag{3.1}$$

$$v \cdot \nabla_x f_1 = 2\mu^{-\frac{1}{2}} Q^*\big[\mu, \mu^{\frac{1}{2}} f_2\big] + \mu^{-\frac{1}{2}} Q^*\big[\mu^{\frac{1}{2}} f_1, \mu^{\frac{1}{2}} f_1\big], \tag{3.2}$$

which are equivalent to

$$\mathcal{L}[f_1] = 0, \tag{3.3}$$

$$v \cdot \nabla_x f_1 + \mathcal{L}[f_2] = \Gamma[f_1, f_1]. \tag{3.4}$$

Considering the further expansion, we additionally require

$$v \cdot \nabla_x f_2 \perp \mathcal{N}. \tag{3.5}$$

Hence, we conclude that

$$f_1(x, v) = \mu^{\frac{1}{2}}(v)\left(\rho_1(x) + v \cdot \mathbf{u}_1(x) + \frac{|v|^2 - 3}{2} T_1(x)\right), \tag{3.6}$$

where $(\rho_1, \mathbf{u}_1, T_1)$ satisfies the incompressible Navier–Stokes–Fourier system (1.16).

Also, we have

$$\begin{aligned} f_2(x, v) = {}& \mu^{\frac{1}{2}}(v)\left(\rho_2(x) + v \cdot \mathbf{u}_2(x) + \frac{|v|^2 - 3}{2} T_2(x)\right) \\ & + \mu^{\frac{1}{2}}(v)\left(\rho_1(v \cdot \mathbf{u}_1) + \left(\rho_1 T_1 + \frac{|v|^2 - 3}{2}\,|\mathbf{u}_1|^2\right)\right) + \mathcal{L}^{-1}\Big[-v \cdot \nabla_x f_1 + \Gamma[f_1, f_1]\Big] \end{aligned} \tag{3.7}$$

where $(\rho_2, \mathbf{u}_2, T_2)$ satisfies the fluid system

$$\begin{cases} \rho_2 + T_2 + \rho_1 T_1 = \mathfrak{p}_1, \\ \mathbf{u}_1 \cdot \nabla_x \mathbf{u}_2 + (\rho_1 \mathbf{u}_1 + \mathbf{u}_2) \cdot \nabla_x \mathbf{u}_1 - \gamma_1 \Delta_x \mathbf{u}_2 + \nabla_x \mathfrak{p}_2 = -\gamma_2 \nabla_x \cdot \Delta_x T_1 - \gamma_4 \nabla_x \cdot \\ \Big(T_1\big(\nabla_x \mathbf{u}_1 + (\nabla_x \mathbf{u}_1)^T\big)\Big), \\ \nabla_x \cdot \mathbf{u}_2 = -\mathbf{u}_1 \cdot \nabla_x \rho_1, \\ \mathbf{u}_1 \cdot \nabla_x T_2 + (\rho_1 \mathbf{u}_1 + \mathbf{u}_2) \cdot \nabla_x T_1 - \mathbf{u}_1 \cdot \nabla_x \mathfrak{p}_1 = \gamma_1 \Big(\nabla_x \mathbf{u}_1 + (\nabla_x \mathbf{u}_1)^T\Big)^2 + \Delta_x\big(\gamma_2 T_2 + \gamma_5 T_1^2\big), \end{cases} \tag{3.8}$$

for constants $\gamma_3, \gamma_4, \gamma_5$. Here $\mathfrak{p}_1, \mathfrak{p}_2$ represent the pressures in the fluid equations and they are related to the density and temperature in various levels of expansion. In the equations above, we see that $\mathfrak{p}_1 = \rho_2 + T_2 + \rho_1 T_1$. If we further expand the interior solution to $O(\varepsilon^3)$, then $\mathfrak{p}_2$ will also be related to ρ_3 and T_3.

3.1.2. Milne problem The normal chart defined in Sect. 1.1.2 was introduced in [29,30], and was designed to split the normal and tangential variables for the convenience of defining boundary layers. Under the substitution $(x, v) \to (\mathfrak{x}, \mathfrak{v})$, we have (letting $\mathcal{R}_i$ be the radii of principal curvatures)

$$\begin{aligned} v\cdot\nabla_x =& \frac{1}{\varepsilon}v_\eta\frac{\partial}{\partial\eta} - \frac{1}{\mathcal{R}_1-\varepsilon\eta}\left(v_{\iota_1}^2\frac{\partial}{\partial v_\eta} - v_\eta v_{\iota_1}\frac{\partial}{\partial v_{\iota_1}}\right) - \frac{1}{\mathcal{R}_2-\varepsilon\eta}\left(v_{\iota_2}^2\frac{\partial}{\partial v_\eta} - v_\eta v_{\iota_2}\frac{\partial}{\partial v_{\iota_2}}\right)\\ &+\frac{1}{L_1L_2}\left(\frac{\mathcal{R}_1\partial_{\iota_1\iota_1}\mathbf{r}\cdot\partial_{\iota_2}\mathbf{r}}{L_1(\mathcal{R}_1-\varepsilon\eta)}v_{\iota_1}v_{\iota_2} + \frac{\mathcal{R}_2\partial_{\iota_1\iota_2}\mathbf{r}\cdot\partial_{\iota_2}\mathbf{r}}{L_2(\mathcal{R}_2-\varepsilon\eta)}v_{\iota_2}^2\right)\frac{\partial}{\partial v_{\iota_1}}\\ &+\frac{1}{L_1L_2}\left(\frac{\mathcal{R}_2\partial_{\iota_2\iota_2}\mathbf{r}\cdot\partial_{\iota_1}\mathbf{r}}{L_2(\mathcal{R}_2-\varepsilon\eta)}v_{\iota_1}v_{\iota_2} + \frac{\mathcal{R}_1\partial_{\iota_1\iota_2}\mathbf{r}\cdot\partial_{\iota_1}\mathbf{r}}{L_1(\mathcal{R}_1-\varepsilon\eta)}v_{\iota_1}^2\right)\frac{\partial}{\partial v_{\iota_2}}\\ &+\left(\frac{\mathcal{R}_1v_{\iota_1}}{L_1(\mathcal{R}_1-\varepsilon\eta)}\frac{\partial}{\partial\iota_1} + \frac{\mathcal{R}_2v_{\iota_2}}{L_2(\mathcal{R}_2-\varepsilon\eta)}\frac{\partial}{\partial\iota_2}\right). \end{aligned}\tag{3.9}$$

Well-Posedness and Regularity Now we discuss the well-posedness and regularity of the Milne problem for $\mathcal{G}(\mathfrak{x}, \mathfrak{v})$ (for generality, we use the notation $\mathcal{G}$ here instead of Φ in (1.19)):

$$\begin{cases} v_\eta\dfrac{\partial\mathcal{G}}{\partial\eta} + \nu\mathcal{G} - K[\mathcal{G}] = 0,\\ \mathcal{G}(0,\iota_1,\iota_2,\mathfrak{v}) = \overline{h}(\iota_1,\iota_2,\mathfrak{v}) \ \text{ for } \ v_\eta>0,\\ \displaystyle\int_{\mathbb{R}^3} v_\eta\mu^{\frac{1}{2}}(\mathfrak{v})\mathcal{G}(0,\iota_1,\iota_2,\mathfrak{v})\mathrm{d}\mathfrak{v} = M_f. \end{cases}\tag{3.10}$$

Here the function $\nu(v)$ and the operator K are defined via (A.2) and (A.3).

Remark 3.1. In the asymptotic problems, given $\overline{h}$ and M_f, for fixed (ι_1, ι_2), we are mostly concerned with the solution $\mathcal{G}$ that satisfies

$$\mathcal{G}(\eta,\mathfrak{v}) \to \mathcal{G}_\infty(\mathfrak{v}) \in \mathcal{N} \ \text{ as } \ \eta\to\infty,\tag{3.11}$$

where $\mathcal{G}_\infty$ can be determined from $\overline{h}$ and M_f. Since $\eta \in [0, \infty)$, for nonzero $\mathcal{G}_\infty$, it is quite hard to define proper function spaces in $(\eta, \mathfrak{v})$ to describe $\mathcal{G}$ (e.g. the integral in η may become infinity). Thus we will first confirm the existence and estimates of $\mathcal{G}_\infty$, and then investigate the estimates of the difference $\mathcal{G} - \mathcal{G}_\infty$. This is a well-known strategy as illustrated in [6,21]. In addition, $\mathcal{G}_\infty$ and $\mathcal{G} - \mathcal{G}_\infty$ play different roles in the matching procedure.

The following result is a generalization of [29, Theorem 3.1] and [6,21,75,79] when we consider nonzero mass flux M_f:

Proposition 3.2. *Assume the boundary data* $\overline{h} \in W^{k,\infty}_{\iota_1,\iota_2}$ *for some* $k \in \mathbb{N}$ *and the mass flux* $M_f \in W^{k,\infty}_{\iota_1,\iota_2}$ *are given. Then there exists a unique solution* $\mathcal{G}(\mathfrak{x}, \mathfrak{v}) \in L^\infty_{\varrho,\vartheta}$ *to* (3.10) *such that*

$$\mathcal{G}(\mathfrak{x},\mathfrak{v}) = \mathcal{G}_\infty(\iota_1,\iota_2,\mathfrak{v}) + \Big(\mathcal{G}(\mathfrak{x},\mathfrak{v}) - \mathcal{G}_\infty(\iota_1,\iota_2,\mathfrak{v})\Big),\tag{3.12}$$

where

$$\mathcal{G}_\infty(\iota_1,\iota_2,\mathfrak{v}) = \mu^{\frac{1}{2}}\left(\rho^\infty(\iota_1,\iota_2) + v\cdot\mathbf{u}^\infty(\iota_1,\iota_2) + \frac{|v|^2-3}{2}T^\infty(\iota_1,\iota_2)\right) \in \mathcal{N},\tag{3.13}$$

satisfies

$$\left|\mathcal{G}_\infty - v_\eta \mu^{\frac{1}{2}} M_f\right|_{L^\infty_{\iota_1,\iota_2}} \lesssim \left|\overline{h}\right|_{L^\infty_{\gamma_-,\varrho,\vartheta}} + \left|M_f\right|_{L^\infty_{\iota_1,\iota_2}}. \tag{3.14}$$

Also, $\mathfrak{G} := \mathcal{G} - \mathcal{G}_\infty$ *solves*

$$\begin{cases} v_\eta \dfrac{\partial \mathfrak{G}}{\partial \eta} + \nu \mathfrak{G} - K[\mathfrak{G}] = 0, \\ \mathfrak{G}(0, \iota_1, \iota_2, \mathfrak{v}) = \overline{h}(\iota_1, \iota_2, \mathfrak{v}) - \mathcal{G}_\infty(\iota_1, \iota_2, \mathfrak{v}) := \mathfrak{h}(\iota_1, \iota_2, \mathfrak{v}) \ \textit{for}\ \ v_\eta > 0, \\ \displaystyle\int_{\mathbb{R}^3} v_\eta \mu^{\frac{1}{2}}(\mathfrak{v}) \mathfrak{G}(0, \iota_1, \iota_2, \mathfrak{v}) \mathrm{d}\mathfrak{v} = 0, \end{cases} \tag{3.15}$$

and satisfies for some $K_0 > 0$ *and any* $0 < s \leq k$

$$\left\|\mathrm{e}^{K_0\eta} \mathfrak{G}\right\|_{L^\infty_{\varrho,\vartheta}} \lesssim \left|\overline{h}\right|_{L^\infty_{\gamma_-,\varrho,\vartheta}} + \left|M_f\right|_{L^\infty_{\iota_1,\iota_2}}, \tag{3.16}$$

and

$$\left\|\mathrm{e}^{K_0\eta} v_\eta \partial_\eta \mathfrak{G}\right\|_{L^\infty_{\varrho,\vartheta}} + \left\|\mathrm{e}^{K_0\eta} v_\eta \partial_{v_\eta} \mathfrak{G}\right\|_{L^\infty_{\varrho,\vartheta}} \lesssim \left|\overline{h}\right|_{L^\infty_{\gamma_-,\varrho,\vartheta}} + \left|\nabla_{\mathfrak{v}} \overline{h}\right|_{L^\infty_{\gamma_-,\varrho,\vartheta}} + \left|M_f\right|_{L^\infty_{\iota_1,\iota_2}}, \tag{3.17}$$

$$\left\|\mathrm{e}^{K_0\eta} \partial_{v_{\iota_1}} \mathfrak{G}\right\|_{L^\infty_{\varrho,\vartheta}} + \left\|\mathrm{e}^{K_0\eta} \partial_{v_{\iota_2}} \mathfrak{G}\right\|_{L^\infty_{\varrho,\vartheta}} \lesssim \left|\overline{h}\right|_{L^\infty_{\gamma_-,\varrho,\vartheta}} + \left|\nabla_{\mathfrak{v}} \overline{h}\right|_{L^\infty_{\gamma_-,\varrho,\vartheta}} + \left|M_f\right|_{L^\infty_{\iota_1,\iota_2}}, \tag{3.18}$$

$$\begin{aligned} \left\|\mathrm{e}^{K_0\eta} \partial^s_{\iota_1} \mathfrak{G}\right\|_{L^\infty_{\varrho,\vartheta}} + \left\|\mathrm{e}^{K_0\eta} \partial^s_{\iota_2} \mathfrak{G}\right\|_{L^\infty_{\varrho,\vartheta}} &\lesssim \left|\overline{h}\right|_{L^\infty_{\gamma_-,\varrho,\vartheta}} + \sum_{j=1}^{s} \left|\partial^j_{\iota_1} \overline{h}\right|_{L^\infty_{\gamma_-,\varrho,\vartheta}} + \sum_{j=1}^{s} \left|\partial^j_{\iota_2} \overline{h}\right|_{L^\infty_{\gamma_-,\varrho,\vartheta}} \\ &+ \left|M_f\right|_{L^\infty_{\iota_1,\iota_2}} + \sum_{j=1}^{s} \left|\partial^j_{\iota_1} M_f\right|_{L^\infty_{\iota_1,\iota_2}} + \sum_{j=1}^{s} \left|\partial^j_{\iota_2} M_f\right|_{L^\infty_{\iota_1,\iota_2}}. \end{aligned} \tag{3.19}$$

Proof. The estimates (3.16)–(3.19) with $M_f = 0$ have been obtained in [29, Theorem 2.1] and [6,75,79], so it suffices to consider the case $M_f \neq 0$. Directly integrating over $\mathbb{R}^3$ in (3.15), we obtain for any $\eta \in [0, \infty]$

$$M_f = \int_{\mathbb{R}^3} v_\eta \mu^{\frac{1}{2}}(\mathfrak{v}) \mathcal{G}(0, \mathfrak{v}) = \int_{\mathbb{R}^3} v_\eta \mu^{\frac{1}{2}}(\mathfrak{v}) \mathcal{G}(\eta, \mathfrak{v}). \tag{3.20}$$

In other words, the mass flux M_f is a conserved quantity for all η. Hence, $\mathcal{G} - v_\eta \mu^{\frac{1}{2}} M_f$ satisfies (3.10)-type equation with zero mass flux. Then we may directly apply the zero mass-flux results from [29, Theorem 2.1] and [6,75,79] to obtain (3.16)–(3.19). □

Remark 3.3. Suppose that $\overline{h}$ and M_f are given as in Proposition 3.2. Following a similar argument as the derivation of $\mathcal{G}_\infty$ and the proofs of (3.14) and (3.19), we may easily obtain that $\rho^\infty, T^\infty \in W^{k,\infty}_{\iota_1,\iota_2}$. Denote $\mathbf{u}^\infty = \left(\mathbf{u}^\infty_n, \mathbf{u}^\infty_{\iota_1}, \mathbf{u}^\infty_{\iota_2}\right)$, for the normal component and the two tangential components. From the proof of Proposition 3.2, we know

$$M_f = \int_{\mathbb{R}^3} v_\eta \mu^{\frac{1}{2}}(\mathfrak{v}) \mathcal{G}_\infty(\mathfrak{v}) = \mathbf{u}^\infty_n, \tag{3.21}$$

and thus from an analogous argument to derive (3.14), we deduce that $\mathbf{u}^\infty_{\iota_1}, \mathbf{u}^\infty_{\iota_2} \in W^{k,\infty}_{\iota_1,\iota_2}$.

BV Estimates For function $f(\eta, \mathfrak{v})$, denote the semi-norm

$$\|f\|_{\widetilde{\mathrm{BV}}} := \sup\left\{\iint_{\eta,\mathfrak{v}} f(\nabla_{\eta,\mathfrak{v}}\cdot\psi)\mathrm{d}\eta\mathrm{d}\mathfrak{v} : \ \psi \in C_c^1 \text{ and } \|\psi\|_{L^\infty} \leq 1\right\}, \tag{3.22}$$

and thus the BV norm can be defined as

$$\|f\|_{\mathrm{BV}} := \|f\|_{L^1} + \|f\|_{\widetilde{\mathrm{BV}}}\,. \tag{3.23}$$

It is classical that $W^{1,1} \hookrightarrow \mathrm{BV}$. The following result comes from [29, Theorem 2.16]:

Proposition 3.4. *Fixing (ι_1, ι_2), we have*

$$\|\nu\mathfrak{G}\|_{BV} \lesssim |\mathfrak{h}|_{L^\infty_{\gamma_-,\varrho,\vartheta}} + \int_{v_\eta>0}\left|\partial_{v_\eta}\mathfrak{h}\right| v_\eta \mathrm{d}\mathfrak{v} + \int_{v_\eta>0}|\mathfrak{h}|\,\mathrm{d}\mathfrak{v} + \left|M_f\right|. \tag{3.24}$$

Particularly, the constant in the above estimate is uniform in (ι_1, ι_2).

Mass Flux In the hydrodynamic limit problems, usually the boundary data $\overline{h}$ is determined a priori (see Sect. 3.1.4), but we still have the freedom of the mass flux M_f to manipulate. Next we plan to prove that the mass flux M_f can be well chosen such that the solution to (3.10) satisfies certain "desired properties" used in the matching procedure. In particular, M_f has an interesting interaction with $\rho^\infty + T^\infty$ defined in the expression of $\mathcal{G}_\infty$ as (3.13):

Proposition 3.5. *Suppose that $\overline{h}$ is given as in Proposition 3.2. For any given constant $P \in \mathbb{R}$, there exists a mass flux $M_f \in W^{k,\infty}_{\iota_1,\iota_2}$ such that $\mathcal{G}_\infty$ in Proposition 3.2 satisfies*

$$\rho^\infty(\iota_1, \iota_2) + T^\infty(\iota_1, \iota_2) = P. \tag{3.25}$$

In addition, M_f satisfies

$$\left\|M_f\right\|_{W^{k,\infty}_{\iota_1,\iota_2}} \lesssim \left|\overline{h}\right|_{W^{k,\infty}_{\iota_1,\iota_2}} + |P|\,. \tag{3.26}$$

Proof. We first solve a zero-mass flux problem for $\overline{\mathcal{G}}$:

$$\begin{cases} v_\eta\dfrac{\partial\overline{\mathcal{G}}}{\partial\eta} + \nu\overline{\mathcal{G}} - K\left[\overline{\mathcal{G}}\right] = 0, \\ \overline{\mathcal{G}}(0, \iota_1, \iota_2, \mathfrak{v}) = \overline{h}(\iota_1, \iota_2, \mathfrak{v}) \ \text{ for } \ v_\eta > 0, \\ \displaystyle\int_{\mathbb{R}^3} v_\eta\mu^{\frac{1}{2}}(\mathfrak{v})\overline{\mathcal{G}}(0, \iota_1, \iota_2, \mathfrak{v})\mathrm{d}\mathfrak{v} = 0. \end{cases} \tag{3.27}$$

Based on Proposition 3.2 and Remark 3.3, there exists a limit function

$$\overline{\mathcal{G}}_\infty(\iota_1, \iota_2, \mathfrak{v}) = \mu^{\frac{1}{2}}\left(\overline{\rho}^\infty + v\cdot\overline{\mathbf{u}}^\infty + \frac{|v|^2-3}{2}\overline{T}^\infty\right) \in \mathcal{N}. \tag{3.28}$$

which satisfies

$$\left\|\overline{\rho}^\infty\right\|_{W^{k,\infty}_{\iota_1,\iota_2}} + \left\|\overline{T}^\infty\right\|_{W^{k,\infty}_{\iota_1,\iota_2}} \lesssim \left|\overline{h}\right|_{W^{k,\infty}_{\iota_1,\iota_2}}, \tag{3.29}$$

If we already have $P = \overline{\rho}^{\infty} + \overline{T}^{\infty}$, then simply take $M_f = 0$. Otherwise, we consider the following auxiliary problem for $\widetilde{\mathcal{G}}$

$$\begin{cases} v_\eta \dfrac{\partial \widetilde{\mathcal{G}}}{\partial \eta} + \nu \widetilde{\mathcal{G}} - K[\widetilde{\mathcal{G}}] = 0, \\ \widetilde{\mathcal{G}}(0, \iota_1, \iota_2, \mathfrak{v}) = 0 \ \text{ for } \ v_\eta > 0, \\ \displaystyle\int_{\mathbb{R}^3} v_\eta \mu^{\frac{1}{2}}(\mathfrak{v}) \widetilde{\mathcal{G}}(0, \iota_1, \iota_2, \mathfrak{v}) \mathrm{d}\mathfrak{v} = \widetilde{M}_f. \end{cases} \tag{3.30}$$

Based on Proposition 3.2, there exists a limit function

$$\widetilde{\mathcal{G}}_\infty(\mathfrak{v}) = \mu^{\frac{1}{2}} \left(\widetilde{\rho}^{\infty} + v \cdot \widetilde{\mathbf{u}}^{\infty} + \frac{|v|^2 - 3}{2} \widetilde{T}^{\infty} \right) \in \mathcal{N}. \tag{3.31}$$

Without loss of generality, we may consider (3.30) for fixed (ι_1, ι_2). Multiplying $\widetilde{\mathcal{G}}$ on both sides of (3.30) and integrating over $(\eta, \mathfrak{v}) \in \mathbb{R}_+ \times \mathbb{R}^3$, we obtain

$$\int_{\mathbb{R}^3} v_\eta \left| \widetilde{\mathcal{G}}_\infty \right|^2 \mathrm{d}\mathfrak{v} - \int_{\mathbb{R}^3} v_\eta \left| \widetilde{\mathcal{G}}(0) \right|^2 \mathrm{d}\mathfrak{v} + \int_0^\infty \int_{\mathbb{R}^3} \widetilde{\mathcal{G}} \Big(\nu \widetilde{\mathcal{G}} - K[\widetilde{\mathcal{G}}] \Big) \mathrm{d}\mathfrak{v} \mathrm{d}\eta = 0. \tag{3.32}$$

When $M_f \neq 0$, direct computation reveals that $\widetilde{\mathcal{G}}$ cannot be in the kernel $\mathcal{N}$. Then based on the proof of Proposition 3.2, we know

$$\int_0^\infty \int_{\mathbb{R}^3} \widetilde{\mathcal{G}} \Big(\nu \widetilde{\mathcal{G}} - K[\widetilde{\mathcal{G}}] \Big) \mathrm{d}\mathfrak{v} \mathrm{d}\eta > 0 \tag{3.33}$$

Clearly, from the boundary condition in (3.30), we know

$$- \int_{\mathbb{R}^3} v_\eta \left| \widetilde{\mathcal{G}}(0) \right|^2 \mathrm{d}\mathfrak{v} = - \int_{v_\eta < 0} v_\eta \left| \widetilde{\mathcal{G}}(0) \right|^2 \mathrm{d}\mathfrak{v} \geq 0. \tag{3.34}$$

Hence, we must have

$$\int_{\mathbb{R}^3} v_\eta \left| \widetilde{\mathcal{G}}_\infty \right|^2 \mathrm{d}\mathfrak{v} = 2 \widetilde{\mathbf{u}}_n^{\infty} \left(\widetilde{\rho}^{\infty} + \widetilde{T}^{\infty} \right) = 2 \widetilde{M}_f \left(\widetilde{\rho}^{\infty} + \widetilde{T}^{\infty} \right) < 0. \tag{3.35}$$

Hence, when $\widetilde{M}_f \neq 0$, we must also have $\widetilde{\rho}^{\infty} + \widetilde{T}^{\infty} \neq 0$. Since (3.30) is a linear equation, we know that $\widetilde{M}_f$ is proportional to $\widetilde{\rho}^{\infty} + \widetilde{T}^{\infty}$, i.e. there exists a nonzero constant D such that

$$\widetilde{M}_f = D \left(\widetilde{\rho}^{\infty} + \widetilde{T}^{\infty} \right). \tag{3.36}$$

Then consider the sum of (3.27) and $\dfrac{P - \overline{\rho}^{\infty} - \overline{T}^{\infty}}{\widetilde{\rho}^{\infty} + \widetilde{T}^{\infty}} \times$(3.30), we know that the limit function is

$$\left(\overline{\rho}^{\infty} + \overline{T}^{\infty} \right) + \frac{P - \overline{\rho}^{\infty} - \overline{T}^{\infty}}{\widetilde{\rho}^{\infty} + \widetilde{T}^{\infty}} \cdot \left(\widetilde{\rho}^{\infty} + \widetilde{T}^{\infty} \right) = P \tag{3.37}$$

which satisfies the requirement. In this case, the mass flux is

$$M_f = \frac{P - \overline{\rho}^{\infty} - \overline{T}^{\infty}}{\widetilde{\rho}^{\infty} + \widetilde{T}^{\infty}} \cdot \widetilde{M}_f = D \Big(P - \overline{\rho}^{\infty} - \overline{T}^{\infty} \Big). \tag{3.38}$$

Hence, from (3.29), we know $M_f \in W^{k,\infty}_{\iota_1,\iota_2}$ satisfying

$$\left\| M_f \right\|_{W^{k,\infty}_{\iota_1,\iota_2}} \lesssim \left| \overline{h} \right|_{W^{k,\infty}_{\iota_1,\iota_2}} + |P| \,. \tag{3.39}$$

□

Remark 3.6. Suppose that $\overline{h}$ is given as in Proposition 3.2. From Remark 3.3 and the proof of Proposition 3.5, we know that $\mathbf{u}_n^\infty \in W^{k,\infty}_{\iota_1,\iota_2}$. Further, based on Remark 3.3, we have $\mathbf{u}^\infty \in W^{k,\infty}_{\iota_1,\iota_2}$.

The following corollary tells us what the "desired properties" are and confirms the existence of M_f.

Corollary 3.7. *Suppose that $\overline{h}$ is given as in Proposition 3.2. There exists a constant $P \in \mathbb{R}$ and a mass flux $M_f \in W^{k,\infty}_{\iota_1,\iota_2}$ such that $\mathcal{G}_\infty$ in Proposition 3.2 satisfies*

$$\rho^\infty(\iota_1, \iota_2) + T^\infty(\iota_1, \iota_2) = P, \tag{3.40}$$

and

$$\int_{\partial\Omega} M_f \, \mathrm{d}S_x = \int_{\partial\Omega} \mathbf{u}_n^\infty \, \mathrm{d}S_x = 0. \tag{3.41}$$

In addition, M_f and P satisfy

$$\left\| M_f \right\|_{W^{k,\infty}_{\iota_1,\iota_2}} \lesssim \left| \overline{h} \right|_{W^{k,\infty}_{\iota_1,\iota_2}} , \tag{3.42}$$

and

$$|P| = \left\| \rho^\infty + T^\infty \right\|_{L^\infty_{\iota_1,\iota_2}} \lesssim \left| \overline{h} \right|_{W^{k,\infty}_{\iota_1,\iota_2}} \tag{3.43}$$

Proof. Based on the proof of Proposition 3.5, in order to guarantee that

$$\int_{\partial\Omega} M_f \, \mathrm{d}S_x = \int_{\partial\Omega} D\Big(P - \overline{\rho}^\infty - \overline{T}^\infty\Big) \, \mathrm{d}S_x = 0, \tag{3.44}$$

we must take

$$P = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} \Big(\overline{\rho}^\infty + \overline{T}^\infty\Big) \, \mathrm{d}S_x. \tag{3.45}$$

Then the desired result follows from Proposition 3.5. □

Remark 3.8. Suppose that $\overline{h}$ is given as in Proposition 3.2. From Remark 3.3 and Remark 3.6, we know that $\mathcal{G}_\infty$ constructed in Corollary 3.7 satisfies

$$\left\| \rho^\infty \right\|_{W^{k,\infty}_{\iota_1,\iota_2}} + \left\| \mathbf{u}^\infty \right\|_{W^{k,\infty}_{\iota_1,\iota_2}} + \left\| T^\infty \right\|_{W^{k,\infty}_{\iota_1,\iota_2}} \lesssim \left| \overline{h} \right|_{W^{k,\infty}_{\iota_1,\iota_2}} . \tag{3.46}$$

3.1.3. Boundary layer Let Φ be solution to the Milne problem (1.19) with M_f determined from Corollary 3.7 with $\overline{h} = \mathfrak{f}_b$. Based on Proposition 3.2, Remark 3.8 and (1.5), we know that there exists

$$\Phi_\infty(\iota_1, \iota_2, \mathfrak{v}) := \mu^{\frac{1}{2}} \left(\rho^B(\iota_1, \iota_2) + v \cdot \mathbf{u}^B(\iota_1, \iota_2) + \frac{|v|^2 - 3}{2} T^B(\iota_1, \iota_2) \right) \in \mathcal{N}, \quad (3.47)$$

satisfying

$$\left\| \mathbf{u}^B \right\|_{L^\infty_{\iota_1,\iota_2}} \lesssim o_T, \quad |P| = \left\| \rho^B + T^B \right\|_{L^\infty_{\iota_1,\iota_2}} \lesssim o_T, \quad (3.48)$$

and for some $K_0 > 0$, $\overline{\Phi}(\mathfrak{x}, \mathfrak{v}) := \Phi(\mathfrak{x}, \mathfrak{v}) - \Phi_\infty(\iota_1, \iota_2, \mathfrak{v})$ satisfies

$$|\Phi_\infty|_{L^\infty_{\iota_1,\iota_2}} + \left\| \mathrm{e}^{K_0 \eta} \overline{\Phi} \right\|_{L^\infty_{\varrho,\vartheta}} \lesssim |\mathfrak{f}_b|_{L^\infty_{\gamma_-,\varrho,\vartheta}} \lesssim o_T. \quad (3.49)$$

Let $\chi(y) \in C^\infty(\mathbb{R})$ be smooth cut-off functions satisfying

$$\chi(y) = \begin{cases} 1 & \text{if} \quad |y| \leq 1, \\ 0 & \text{if} \quad |y| \geq 2, \end{cases} \quad (3.50)$$

and $\overline{\chi}(y) = 1 - \chi(y)$. We define a cutoff boundary layer f_1^B. Denote

$$f_1^B(\mathfrak{x}, \mathfrak{v}) := \overline{\chi}\left(\varepsilon^{-1} v_\eta\right) \chi(\varepsilon\eta) \overline{\Phi}(\mathfrak{x}, \mathfrak{v}). \quad (3.51)$$

We may verify that f_1^B satisfies

$$v_\eta \frac{\partial f_1^B}{\partial \eta} + \mathcal{L}\left[f_1^B\right] = v_\eta \overline{\chi}(\varepsilon^{-1} v_\eta) \frac{\partial \chi(\varepsilon\eta)}{\partial \eta} \overline{\Phi} + \chi(\varepsilon\eta) \Big(\overline{\chi}(\varepsilon^{-1} v_\eta) K g[\overline{\Phi}] - K\left[\overline{\chi}(\varepsilon^{-1} v_\eta) \overline{\Phi}\right] \Big), \quad (3.52)$$

with

$$f_1^B(0, \iota_1, \iota_2, \mathfrak{v}) = \overline{\chi}\left(\varepsilon^{-1} v_\eta\right) \Big(\mathfrak{f}_b(\iota_1, \iota_2, \mathfrak{v}) - \Phi_\infty(\iota_1, \iota_2, \mathfrak{v}) \Big) \quad \text{for} \quad v_\eta > 0. \quad (3.53)$$

Also, based on (3.51) and Proposition 3.2 as well as (1.5), we know that for some $K_0 > 0$ and any $0 < s \leq 3$

$$\left\| \mathrm{e}^{K_0 \eta} f_1^B \right\|_{L^\infty_{\varrho,\vartheta}} + \left\| \mathrm{e}^{K_0 \eta} \frac{\partial^s f_1^B}{\partial \iota_1^s} \right\|_{L^\infty_{\varrho,\vartheta}} + \left\| \mathrm{e}^{K_0 \eta} \frac{\partial^s f_1^B}{\partial \iota_2^s} \right\|_{L^\infty_{\varrho,\vartheta}} \lesssim o_T. \quad (3.54)$$

3.1.4. Matching procedure Considering the boundary condition in (1.1) and the expansion (1.7), we require the matching condition for $x_0 \in \partial\Omega$ and $v \cdot n < 0$:

$$\mu^{\frac{1}{2}} \left(f_1 + f_1^B \right) \Big|_{v \cdot n < 0} = \mathfrak{f}_b. \tag{3.55}$$

Hence, it suffices to define

$$\rho_1 \Big|_{\partial\Omega} = \rho^B, \quad \mathbf{u}_1 \Big|_{\partial\Omega} = \mathbf{u}^B, \quad T_1 \Big|_{\partial\Omega} = T^B. \tag{3.56}$$

Therefore, from (3.54) and (3.56), we know the boundary estimates

$$|\rho_1|_{W_x^{3,\infty}} + |\mathbf{u}_1|_{W_x^{3,\infty}} + |T_1|_{W_x^{3,\infty}} \lesssim o_T. \tag{3.57}$$

In particular, we know

$$\left(\rho_1 + T_1\right)\Big|_{\partial\Omega} = \rho^B + T^B = P, \quad \int_\Omega \left(\nabla_x \cdot \mathbf{u}_1\right) = \int_{\partial\Omega} \left(\mathbf{u}_1 \cdot n\right) = \int_{\partial\Omega} M_f = 0. \tag{3.58}$$

By standard fluid theory [13,19] for the steady Navier–Stokes equations (1.16), we have for any $s \in [2, \infty)$

$$\|\rho_1\|_{W_x^{3,s}} + \|\mathbf{u}_1\|_{W_x^{3,s}} + \|T_1\|_{W_x^{3,s}} \lesssim o_T. \tag{3.59}$$

Then for f_2, there is no corresponding boundary layer, and thus we may simply take

$$\rho_2 \Big|_{\partial\Omega} = 0, \quad \mathbf{u}_2 \Big|_{\partial\Omega} = -\frac{1}{|\partial\Omega|} \int_\Omega \left(\mathbf{u}_1 \cdot \nabla_x \rho_1\right), \quad T_2 \Big|_{\partial\Omega} = 0. \tag{3.60}$$

By standard fluid theory [13,19] for the linear steady Navier–Stokes equations (3.8), we have for any $s \in [2, \infty)$

$$\|\rho_2\|_{W_x^{2,s}} + \|\mathbf{u}_2\|_{W_x^{2,s}} + \|T_2\|_{W_x^{2,s}} \lesssim o_T. \tag{3.61}$$

Theorem 3.9. *Under the assumption* (1.5)*, there exists a unique solution* $(\rho_1, \mathbf{u}_1, T_1)$ *to the steady Navier–Stokes equations* (1.16) *and* $(\rho_2, \mathbf{u}_2, T_2)$ *to* (3.8) *satisfying for any* $s \in [2, \infty)$

$$\|\rho_1\|_{W_x^{3,s}} + \|\mathbf{u}_1\|_{W_x^{2,s}} + \|T_1\|_{W_x^{3,s}} \lesssim o_T, \tag{3.62}$$

$$\|\rho_2\|_{W_x^{2,s}} + \|\mathbf{u}_2\|_{W_x^{2,s}} + \|T_2\|_{W_x^{2,s}} \lesssim o_T. \tag{3.63}$$

Thus, we can construct f_1, f_2 *and* f_1^B *such that*

$$\|f_1\|_{W_x^{3,s} L^\infty_{v,\varrho,\vartheta}} + |f_1|_{W_x^{3-\frac{1}{s},s} L^\infty_{\gamma,\varrho,\vartheta}} \lesssim o_T, \tag{3.64}$$

$$\|f_2\|_{W_x^{2,s} L^\infty_{v,\varrho,\vartheta}} + |f_2|_{W_x^{2-\frac{1}{s},s} L^\infty_{\gamma,\varrho,\vartheta}} \lesssim o_T, \tag{3.65}$$

and for some $K_0 > 0$ *and any* $0 < s \leq 3$

$$\left\| \mathrm{e}^{K_0 \eta} f_1^B \right\|_{L^\infty_{\varrho,\vartheta}} + \left\| \mathrm{e}^{K_0 \eta} \frac{\partial^s f_1^B}{\partial \iota_1^s} \right\|_{L^\infty_{\varrho,\vartheta}} + \left\| \mathrm{e}^{K_0 \eta} \frac{\partial^s f_1^B}{\partial \iota_2^s} \right\|_{L^\infty_{\varrho,\vartheta}} \lesssim o_T. \tag{3.66}$$

3.2. Remainder equation. Inserting (1.7) into (1.1), we have

$$v\cdot\nabla_x\left(\mu+f+f^B+\varepsilon\mu^{\frac12}R\right)=\varepsilon^{-1}Q^*\left[\mu+f+f^B+\varepsilon\mu^{\frac12}R,\mu+f+f^B+\varepsilon\mu^{\frac12}R\right] \tag{3.67}$$

or equivalently

$$\begin{aligned}v\cdot\nabla_x R-2\varepsilon^{-1}\mu^{-\frac12}Q^*\left[\mu,\mu^{\frac12}R\right]=&-\varepsilon^{-1}\mu^{-\frac12}\Big(v\cdot\nabla_x\left(f+f^B\right)\Big)+\mu^{-\frac12}Q^*\left[\mu^{\frac12}R,\mu^{\frac12}R\right]\\&+2\varepsilon^{-1}\mu^{-\frac12}Q^*\left[f+f^B,\mu^{\frac12}R\right]+\varepsilon^{-2}\mu^{-\frac12}Q^*\left[\mu+f+f^B,\mu+f+f^B\right].\end{aligned} \tag{3.68}$$

Also, we have the boundary condition

$$\left(\mu+f+f^B+\varepsilon\mu^{\frac12}R\right)\Big|_{\gamma_-}=\mu+\varepsilon\mu^{\frac12}\mathfrak{f}_b, \tag{3.69}$$

which is equivalent to

$$R\big|_{\gamma_-}=\mathfrak{f}_b-\varepsilon^{-1}\mu^{-\frac12}\big(f+f^B\big). \tag{3.70}$$

Therefore, we need to consider the remainder equation (1.10). Here the boundary data is given by

$$h=\Big(-\varepsilon f_2+\chi\big(\varepsilon^{-1}v_\eta\big)\overline{\Phi}\Big)\Big|_{\gamma_-}, \tag{3.71}$$

and

$$S:=S_1+S_2+S_3+S_4+S_5+S_6, \tag{3.72}$$

where

$$S_1:=-\varepsilon v\cdot\nabla_x f_2, \tag{3.73}$$

$$\begin{aligned}S_2:=&\frac{1}{\mathcal{R}_1-\varepsilon\eta}\left(v_{\iota_1}^2\frac{\partial f_1^B}{\partial v_\eta}-v_\eta v_{\iota_1}\frac{\partial f_1^B}{\partial v_{\iota_1}}\right)+\frac{1}{\mathcal{R}_2-\varepsilon\eta}\left(v_{\iota_2}^2\frac{\partial f_1^B}{\partial v_\eta}-v_\eta v_{\iota_2}\frac{\partial f_1^B}{\partial v_{\iota_2}}\right)\\&-\frac{1}{L_1L_2}\left(\frac{\mathcal{R}_1\partial_{\iota_1\iota_1}\mathbf{r}\cdot\partial_{\iota_2}\mathbf{r}}{L_1(\mathcal{R}_1-\varepsilon\eta)}v_{\iota_1}v_{\iota_2}+\frac{\mathcal{R}_2\partial_{\iota_1\iota_2}\mathbf{r}\cdot\partial_{\iota_2}\mathbf{r}}{L_2(\mathcal{R}_2-\varepsilon\eta)}v_{\iota_2}^2\right)\frac{\partial f_1^B}{\partial v_{\iota_1}}\\&-\frac{1}{L_1L_2}\left(\frac{\mathcal{R}_2\partial_{\iota_2\iota_2}\mathbf{r}\cdot\partial_{\iota_1}\mathbf{r}}{L_2(\mathcal{R}_2-\varepsilon\eta)}v_{\iota_1}v_{\iota_2}+\frac{\mathcal{R}_1\partial_{\iota_1\iota_2}\mathbf{r}\cdot\partial_{\iota_1}\mathbf{r}}{L_1(\mathcal{R}_1-\varepsilon\eta)}v_{\iota_1}^2\right)\frac{\partial f_1^B}{\partial v_{\iota_2}}\\&-\left(\frac{\mathcal{R}_1v_{\iota_1}}{L_1(\mathcal{R}_1-\varepsilon\eta)}\frac{\partial f_1^B}{\partial\iota_1}+\frac{\mathcal{R}_2v_{\iota_2}}{L_2(\mathcal{R}_2-\varepsilon\eta)}\frac{\partial f_1^B}{\partial\iota_2}\right)\\&+\varepsilon^{-1}v_\eta\overline{\chi}(\varepsilon^{-1}v_\eta)\frac{\partial\chi(\varepsilon\eta)}{\partial\eta}\overline{\Phi}-\varepsilon^{-1}\Big(K\left[\overline{\Phi}\right]\chi(\varepsilon^{-1}v_\eta)\chi(\varepsilon\eta)-K\Big[\overline{\Phi}\chi(\varepsilon^{-1}v_\eta)\chi(\varepsilon\eta)\Big]\Big),\end{aligned} \tag{3.74}$$

$$S_3:=2\mu^{-\frac12}Q^*\left[\mu^{\frac12}f_1+\varepsilon\mu^{\frac12}f_2,\mu^{\frac12}R\right]=2\Gamma[f_1+\varepsilon f_2,R], \tag{3.75}$$

$$S_4:=2\mu^{-\frac12}Q^*\left[\mu^{\frac12}f_1^B,\mu^{\frac12}R\right]=2\Gamma\left[f_1^B,R\right], \tag{3.76}$$

$$\begin{aligned}S_5:=&\varepsilon\mu^{-\frac12}Q^*\left[\mu^{\frac12}f_2,\mu^{\frac12}\big(2f_1+\varepsilon f_2\big)\right]+2\mu^{-\frac12}Q^*\left[\mu^{\frac12}\big(2f_1+2\varepsilon f_2+f_1^B\big),\mu^{\frac12}f_1^B\right]\\=&\varepsilon\Gamma\left[f_2,2f_1+\varepsilon f_2\right]+2\Gamma\left[2f_1+2\varepsilon f_2+f_1^B,f_1^B\right],\end{aligned} \tag{3.77}$$

$$S_6 := \mu^{-\frac{1}{2}} Q^*\left[\mu^{\frac{1}{2}}R, \mu^{\frac{1}{2}}R\right] = \Gamma[R, R]. \tag{3.78}$$

In particular, we may further split S_2:

$$S_{2a} := \frac{1}{\mathcal{R}_1 - \varepsilon\eta}\left(v_{\iota_1}^2 \frac{\partial f_1^B}{\partial v_\eta}\right) + \frac{1}{\mathcal{R}_2 - \varepsilon\eta}\left(v_{\iota_2}^2 \frac{\partial f_1^B}{\partial v_\eta}\right), \tag{3.79}$$

$$\begin{aligned} S_{2b} :=& -\frac{1}{\mathcal{R}_1 - \varepsilon\eta}\left(v_\eta v_{\iota_1} \frac{\partial f_1^B}{\partial v_{\iota_1}}\right) - \frac{1}{\mathcal{R}_2 - \varepsilon\eta}\left(v_\eta v_{\iota_2} \frac{\partial f_1^B}{\partial v_{\iota_2}}\right) \\ & - \frac{1}{L_1L_2}\left(\frac{\mathcal{R}_1\partial_{\iota_1\iota_1}\mathbf{r}\cdot\partial_{\iota_2}\mathbf{r}}{L_1(\mathcal{R}_1-\varepsilon\eta)} v_{\iota_1}v_{\iota_2} + \frac{\mathcal{R}_2\partial_{\iota_1\iota_2}\mathbf{r}\cdot\partial_{\iota_2}\mathbf{r}}{L_2(\mathcal{R}_2-\varepsilon\eta)} v_{\iota_2}^2\right)\frac{\partial f_1^B}{\partial v_{\iota_1}} \\ & - \frac{1}{L_1L_2}\left(\frac{\mathcal{R}_2\partial_{\iota_2\iota_2}\mathbf{r}\cdot\partial_{\iota_1}\mathbf{r}}{L_2(\mathcal{R}_2-\varepsilon\eta)} v_{\iota_1}v_{\iota_2} + \frac{\mathcal{R}_1\partial_{\iota_1\iota_2}\mathbf{r}\cdot\partial_{\iota_1}\mathbf{r}}{L_1(\mathcal{R}_1-\varepsilon\eta)} v_{\iota_1}^2\right)\frac{\partial f_1^B}{\partial v_{\iota_2}} \\ & - \left(\frac{\mathcal{R}_1 v_{\iota_1}}{L_1(\mathcal{R}_1-\varepsilon\eta)}\frac{\partial f_1^B}{\partial \iota_1} + \frac{\mathcal{R}_2 v_{\iota_2}}{L_2(\mathcal{R}_2-\varepsilon\eta)}\frac{\partial f_1^B}{\partial \iota_2}\right) + \varepsilon^{-1} v_\eta \overline{\chi}(\varepsilon^{-1}v_\eta)\frac{\partial\chi(\varepsilon\eta)}{\partial\eta}\overline{\Phi}, \end{aligned} \tag{3.80}$$

$$S_{2c} := -\varepsilon^{-1}\Big(K\left[\overline{\Phi}\right]\chi(\varepsilon^{-1}v_\eta)\chi(\varepsilon\eta) - K\Big[\overline{\Phi}\chi(\varepsilon^{-1}v_\eta)\chi(\varepsilon\eta)\Big]\Big). \tag{3.81}$$

Here, S_{2a} will be the most tricky term in the later analysis since it involves the normal derivatives of the boundary layer f_1^B. Additionally, the non-local term S_{2c} needs some special handling. Lemma 3.16 will be devoted to these issues.

Lemma 3.10 *(Green's Identity, Lemma 2.2 of [26]). Assume* $f(x,v),\ g(x,v) \in L^2(\Omega\times\mathbb{R}^3)$ *and* $v\cdot\nabla_x f,\ v\cdot\nabla_x g \in L^2(\Omega\times\mathbb{R}^3)$ *with* $f,\ g \in L^2_\gamma$. *Then*

$$\big\langle v\cdot\nabla_x f, g\big\rangle + \big\langle v\cdot\nabla_x g, f\big\rangle = \int_\gamma fg\big(v\cdot n\big). \tag{3.82}$$

Using Lemma 3.10, we can derive the weak formulation of (1.10). For any test function $g(x,v) \in L^2_\nu(\Omega\times\mathbb{R}^3)$ with $v\cdot\nabla_x g \in L^2(\Omega\times\mathbb{R}^3)$ with $g \in L^2_\gamma$, we have

$$\int_\gamma Rg\big(v\cdot n\big) - \big\langle v\cdot\nabla_x g, R\big\rangle + \varepsilon^{-1}\big\langle\mathcal{L}[R], g\big\rangle = \big\langle S, g\big\rangle. \tag{3.83}$$

3.2.1. Estimates of boundary and source terms The estimates below in Lemma 3.13 to Lemma 3.21 follow from analogous arguments as in [29, Section 4]. To keep this article self-contained and avoid the notational confusions, we will include brief proofs highlighting the key steps. Particularly, the bounds heavily rely on the estimates presented in Sect. 3.1. In the following, assume that g is a given function and $2 \le r \le 6$. In what follows, $\int_{\mathbb{R}^3}$ denotes integration with respect to the measure $\mathrm{d}v$.

Preliminary Estimates Here we present some lemmas regarding Γ.

Lemma 3.11 (Lemma 2.3 of [39]). *Let* $\Gamma[f,g]$ *be given by* (A.6). *We have*

$$\begin{aligned} \left|\big\langle\Gamma[g_1,g_2],g_3\big\rangle_v\right| \lesssim & \Bigg\{\left(\int_{\mathbb{R}^3}\nu|g_1|^2\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}|g_2|^2\right)^{\frac{1}{2}} \\ & + \left(\int_{\mathbb{R}^3}\nu|g_2|^2\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}|g_1|^2\right)^{\frac{1}{2}}\Bigg\}\left(\int_{\mathbb{R}^3}\nu|g_3|^2\right)^{\frac{1}{2}}, \end{aligned} \tag{3.84}$$

$$\left\|\left\langle \Gamma[g_1,g_2],g_3\right\rangle_v\right\|_{L^2_x} \lesssim \Big(\sup_{x,v}\left|\nu^3 g_3\right|\Big)\min\left\{\sup_x\left(\int_{\mathbb{R}^3}|g_1|^2\right)^{\frac12}\|g_2\|_{L^2},\ \sup_x\left(\int_{\mathbb{R}^3}|g_2|^2\right)^{\frac12}\|g_1\|_{L^2}\right\}, \tag{3.85}$$

$$\|\Gamma[g_1,g_2]g_3\|_{L^2} \lesssim \Big(\sup_{x,v}|\nu g_3|\Big)\min\left\{\sup_x\left(\int_{\mathbb{R}^3}|g_1|^2\right)^{\frac12}\|g_2\|_{L^2},\ \sup_x\left(\int_{\mathbb{R}^3}|g_2|^2\right)^{\frac12}\|g_1\|_{L^2}\right\}. \tag{3.86}$$

Lemma 3.12 (Lemma 5 of [41]). *Let $\Gamma[f,g]$ be given by (A.6). We have*

$$\left\|\nu^{-1}\Gamma[g_1,g_2]\right\|_{L^\infty_{\varrho,\vartheta}} \lesssim \|g_1\|_{L^\infty_{\varrho,\vartheta}}\|g_2\|_{L^\infty_{\varrho,\vartheta}}. \tag{3.87}$$

Estimates of/, h

Lemma 3.13. *Under the assumption (1.5), for h defined in (3.71), we have*

$$|h|_{L^2_{\gamma_-}} \lesssim o_T\varepsilon,\quad |h|_{L^{\frac{2r}{3}}_{\gamma_-}} \lesssim o_T\varepsilon^{\frac{3}{r}},\quad |h|_{L^\infty_{\gamma_-,\varrho,\vartheta}} \lesssim o_T,\quad \sup_{\iota_1,\iota_2}\int_{v\cdot n<0}|h|\,|v\cdot n|\,\mathrm{d}v \lesssim o_T\varepsilon. \tag{3.88}$$

Proof. Note that from Theorem 3.9, it holds that $|f_2|_{L^\infty_{\gamma,\varrho,\vartheta}} \lesssim \|f_2\|_{W^{1,s}L^\infty_{\varrho,\vartheta}} \lesssim o_T$. Then we know

$$|\varepsilon f_2|_{L^\infty_{\gamma_-,\varrho,\vartheta}} \lesssim \varepsilon\,|f_2|_{L^\infty_{\gamma,\varrho,\vartheta}} \lesssim o_T\varepsilon. \tag{3.89}$$

Then we obtain the similar estimates for $L^2_{\gamma_-}$ and $L^{\frac{2r}{3}}_{\gamma_-}$ norms.

On the other hand, noticing that $\left|\overline{\Phi}\right|_{L^\infty_{\gamma,\varrho,\vartheta}} \lesssim o_T$ from (3.49) and Proposition 3.2, the cutoff $\chi\left(\varepsilon^{-1}v_\eta\right)$ implies a restriction to the domain $\left|v_\eta\right| \leq \varepsilon$, and the γ norm has an extra v_η, we have

$$\left|\chi\left(\varepsilon^{-1}v_\eta\right)\overline{\Phi}\right|_{L^{\frac{2r}{3}}_{\gamma_-}} \lesssim o_T\varepsilon^{\frac{3}{r}}, \tag{3.90}$$

and the $L^2_{\gamma_-}$ estimate follows when $r=3$, and

$$\int_{v\cdot n<0}\left|\chi\left(\varepsilon^{-1}v_\eta\right)\overline{\Phi}\right|\left|v_\eta\right|\mathrm{d}v \lesssim \varepsilon. \tag{3.91}$$

Then our estimates follow. □

Remark 3.14. We may directly compute that for $x_0\in\partial\Omega$

$$\mathbf{b}(x_0)\cdot n = \int_{\mathbb{R}^3}R(x_0)\mu^{\frac12}(v\cdot n)\mathrm{d}v = \int_{v\cdot n<0}h(x_0)\mu^{\frac12}(v\cdot n)\mathrm{d}v + \int_{v\cdot n>0}R(x_0)\mu^{\frac12}(v\cdot n)\mathrm{d}v. \tag{3.92}$$

Estimates of S_1

Lemma 3.15. *Under the assumption* (1.5)*, for* S_1 *defined in* (3.73)*, we have*

$$\left\| \langle v\rangle^2 S_1 \right\|_{L^2} \lesssim o_T\varepsilon, \qquad \|S_1\|_{L^r} \lesssim o_T\varepsilon, \qquad \|S_1\|_{L^\infty_{\varrho,\vartheta}} \lesssim o_T\varepsilon. \tag{3.93}$$

Also, we have the orthogonality property

$$\left\langle \mu^{\frac12}, S_1\right\rangle_v = 0, \qquad \left\langle \mu^{\frac12} v, S_1\right\rangle_v = \mathbf{0}, \qquad \left\langle \mu^{\frac12}|v|^2, S_1\right\rangle_v = 0. \tag{3.94}$$

Proof. The desired estimates follow from Lemma 3.11, Lemma 3.12, and

$$\|\nabla_x f_2\|_{L^\infty_{\varrho,\vartheta}} \lesssim \|f_2\|_{W^{2,s}_x L^\infty_{v,\varrho,\vartheta}} \lesssim o_T, \tag{3.95}$$

which is derived from Theorem 3.9. Also, the orthogonality property follows from (3.5). □

Estimates of S_2

Lemma 3.16. *Under the assumption* (1.5)*, for* S_2 *defined in* (3.74)*, we have*

$$\|S_2\|_{L^1} + \|\eta\,(S_{2b}+S_{2c})\|_{L^1} + \left\|\eta^2\,(S_{2b}+S_{2c})\right\|_{L^1} \lesssim o_T\varepsilon, \tag{3.96}$$

$$\left\|\langle v\rangle^2\, S_2\right\|_{L^2} + \|\eta\,(S_{2b}+S_{2c})\|_{L^2} + \left\|\eta^2\,(S_{2b}+S_{2c})\right\|_{L^2} \lesssim o_T, \tag{3.97}$$

$$\|S_2\|_{L^r} + \|\eta\,(S_{2b}+S_{2c})\|_{L^r} + \left\|\eta^2\,(S_{2b}+S_{2c})\right\|_{L^r} \lesssim o_T\varepsilon^{\frac{2}{r}-1}, \tag{3.98}$$

$$\|S_2\|_{L^r_{\iota_1\iota_2}L^1_{\mathfrak{n}}L^1_v} + \|\eta\,(S_{2b}+S_{2c})\|_{L^r_{\iota_1\iota_2}L^1_{\mathfrak{n}}L^1_v} \lesssim o_T\varepsilon, , \tag{3.99}$$

and

$$\|S_{2b}+S_{2c}\|_{L^r_xL^1_v} + \|\eta\,(S_{2b}+S_{2c})\|_{L^r_xL^1_v} \lesssim o_T\varepsilon^{\frac1r}, \tag{3.100}$$

$$|\langle S_{2a}, g\rangle| + |\langle \eta S_{2a}, g\rangle| + \left|\left\langle \eta^2 S_{2a}, g\right\rangle\right| \lesssim \left\|\langle v\rangle^2 f_1^B\right\|_{L^{\frac{r}{r-1}}} \|\nabla_v g\|_{L^r} \lesssim o_T\varepsilon^{1-\frac1r}\|\nabla_v g\|_{L^r}. \tag{3.101}$$

Also, we have

$$\|S_2\|_{L^\infty_{\varrho,\vartheta}} \lesssim o_T\varepsilon^{-1}. \tag{3.102}$$

Proof. We start from the L^r estimate (3.98), and then (3.96) and (3.97) will naturally follow. We first focus on S_{2a}. Notice that

$$\frac{\partial f_1^B}{\partial v_\eta}(\eta,\mathfrak{v}) = \varepsilon^{-1}\overline{\chi}'\left(\varepsilon^{-1}v_\eta\right)\chi(\varepsilon\eta)\overline{\Phi}(\eta,\mathfrak{v}) + \overline{\chi}\left(\varepsilon^{-1}v_\eta\right)\chi(\varepsilon\eta)\frac{\partial\overline{\Phi}(\eta,\mathfrak{v})}{\partial v_\eta}. \tag{3.103}$$

From Propositions 3.2 and 3.4, we have

$$\begin{aligned}\left\|\overline{\chi}\left(\varepsilon^{-1}v_\eta\right)\chi(\varepsilon\eta)\frac{\partial\overline{\Phi}(\eta,\mathfrak{v})}{\partial v_\eta}\right\|_{L^\infty_{\varrho,\vartheta}} &\lesssim \varepsilon^{-1}\left\|v_\eta\frac{\partial\overline{\Phi}(\eta,\mathfrak{v})}{\partial v_\eta}\right\|_{L^\infty_{\varrho,\vartheta}}\\ &\lesssim \varepsilon^{-1}, \quad \left\|\overline{\chi}\left(\varepsilon^{-1}v_\eta\right)\chi(\varepsilon\eta)\frac{\partial\overline{\Phi}(\eta,\mathfrak{v})}{\partial v_\eta}\right\|_{L^1} \lesssim 1.\end{aligned} \tag{3.104}$$

Then by change of variable $\eta = \varepsilon^{-1}\mathfrak{n}$, we have

$$\begin{aligned}\left\|\varepsilon^{-1}\overline{\chi}'\left(\varepsilon^{-1}v_\eta\right)\chi(\varepsilon\eta)\overline{\Phi}(\eta,\mathfrak{v})\right\|_{L^r} &= \left(\iint_{\Omega\times\mathbb{R}^3}\left|\varepsilon^{-1}\overline{\chi}'\left(\varepsilon^{-1}v_\eta\right)\chi(\varepsilon\eta)\overline{\Phi}(\eta,\mathfrak{v})\right|^r\right)^{\frac{1}{r}}\\ &\lesssim \varepsilon^{-1}\left(\iint_{\Omega\times\mathbb{R}^3}\left|\overline{\chi}'\left(\varepsilon^{-1}v_\eta\right)\chi(\varepsilon\eta)\overline{\Phi}(\eta,\mathfrak{v})\right|^r\right)^{\frac{1}{r}} \lesssim \varepsilon^{\frac{2}{r}-1},\end{aligned} \tag{3.105}$$

and

$$\begin{aligned}&\left\|\overline{\chi}\left(\varepsilon^{-1}v_\eta\right)\chi(\varepsilon\eta)\frac{\partial\overline{\Phi}(\eta,\mathfrak{v})}{\partial v_\eta}\right\|_{L^r} = \left(\iint_{\Omega\times\mathbb{R}^3}\left|\overline{\chi}\left(\varepsilon^{-1}v_\eta\right)\chi(\varepsilon\eta)\frac{\partial\overline{\Phi}(\eta,\mathfrak{v})}{\partial v_\eta}\right|^r\right)^{\frac{1}{r}}\\ &\lesssim \left(\left\|\overline{\chi}\left(\varepsilon^{-1}v_\eta\right)\frac{\partial\overline{\Phi}(\eta,\mathfrak{v})}{\partial v_\eta}\right\|^{r-1}_{L^\infty_{\varrho,\vartheta}}\iint_{\Omega\times\mathbb{R}^3}\left|\overline{\chi}\left(\varepsilon^{-1}v_\eta\right)\chi(\varepsilon\eta)\frac{\partial\overline{\Phi}(\eta,\mathfrak{v})}{\partial v_\eta}\right|\right)^{\frac{1}{r}}\\ &\lesssim \left(\varepsilon^{-(r-1)}\varepsilon\right)^{\frac{1}{r}} \lesssim \varepsilon^{\frac{2}{r}-1}.\end{aligned} \tag{3.106}$$

Hence, we know $\|S_{2a}\|_{L^r} \lesssim \varepsilon^{\frac{2}{r}-1}$. Similarly S_{2b} estimates follow from Proposition 3.2 and Proposition 3.4.

Noticing that

$$S_{2c} = \varepsilon^{-1}\mu^{-\frac{1}{2}}\mu_w^{\frac{1}{2}}\chi(\varepsilon\eta)\Big(\chi(\varepsilon^{-1}v_\eta)K[\overline{\Phi}] - K\Big[\chi(\varepsilon^{-1}v_\eta)\overline{\Phi}\Big]\Big), \tag{3.107}$$

which has one less ε-power but contains an ε-size cutoff $\chi(\varepsilon^{-1}v_\eta)$. Clearly, the term $\chi(\varepsilon^{-1}v_\eta)K[\overline{\Phi}]$ can be estimated by a similar argument as (3.106). Then noticing that by the change of variable $w_\eta = \varepsilon^{-1}\mathfrak{u}_\eta$

$$\begin{aligned}\int_{\mathbb{R}^3}\left|K\Big[\chi(\varepsilon^{-1}v_\eta)\overline{\Phi}\Big]\right|\mathrm{d}v &= \int_{\mathbb{R}^3}\left|\int_{\mathbb{R}^3}k(\mathfrak{u},v)\chi(\varepsilon^{-1}\mathfrak{u}_\eta)\overline{\Phi}(\mathfrak{u})\mathrm{d}\mathfrak{u}\right|\mathrm{d}v\\ &\lesssim \sup_{\mathfrak{u}}\int_{\mathbb{R}^3}|k(\mathfrak{u},v)|\,\mathrm{d}v\left|\int_{\mathbb{R}^3}\chi(\varepsilon^{-1}\mathfrak{u}_\eta)\overline{\Phi}(\mathfrak{u})\mathrm{d}\mathfrak{u}\right|\\ &\lesssim \left|\int_{\mathbb{R}^3}\chi(\varepsilon^{-1}\mathfrak{u}_\eta)\overline{\Phi}(\mathfrak{u})\mathrm{d}\mathfrak{u}\right| \lesssim \varepsilon,\end{aligned} \tag{3.108}$$

we can bound $K\Big[\chi(\varepsilon^{-1}v_\eta)\overline{\Phi}\Big]$ by a similar argument as (3.106). Hence, we complete the proof of (3.98).

Noting the rescaling $\eta = \varepsilon^{-1}\mathfrak{n}$ and the cutoff $\chi(\varepsilon^{-1}v_\eta)$, using Proposition 3.2 and Proposition 3.4, (3.99) and (3.100) follow from substitution in the integral.

Then we turn to (3.101). The most difficult term in $|\langle S_{2a}, g\rangle|$ is $\left|\left\langle\frac{\partial f_1^B}{\partial v_\eta}, g\right\rangle\right|$. Note that $\frac{\partial f_1^B}{\partial v_\eta} = 0$ for $\left|v_\eta\right| \leq \varepsilon$ due to the cutoff in f_1^B. Integration by parts with respect to v_η implies

$$\left|\left\langle\frac{\partial f_1^B}{\partial v_\eta}, g\right\rangle\right| \lesssim \left|\left\langle f_1^B, \frac{\partial g}{\partial v_\eta}\right\rangle\right| \lesssim \left\|f_1^B\right\|_{L^r}\left\|\frac{\partial g}{\partial v_\eta}\right\|_{L^{\frac{r}{r-1}}}. \tag{3.109}$$

From (1.6) and $\dfrac{\partial x}{\partial v_\eta} \equiv \mathbf{0}$, we know the substitution $(\mathfrak{n}, \iota_1, \iota_2, v) \to (\mathfrak{n}, \iota_1, \iota_2, \mathfrak{v})$ implies $-\frac{\partial v}{\partial v_\eta} \cdot n = 1$, $-\frac{\partial v}{\partial v_\eta} \cdot \varsigma_1 = 0$, $-\frac{\partial v}{\partial v_\eta} \cdot \varsigma_2 = 0$. Hence, we know $\left|\frac{\partial v}{\partial v_\eta}\right| \lesssim 1$, and thus $\left|\frac{\partial g}{\partial v_\eta}\right| \lesssim |\nabla_v g| \left|\frac{\partial v}{\partial v_\eta}\right| \lesssim |\nabla_v g|$. Hence, we know that

$$\left|\left\langle \frac{\partial f_1^B}{\partial v_\eta}, g \right\rangle\right| \lesssim \left\| f_1^B \right\|_{L^r} \|\nabla_v g\|_{L^{\frac{r}{r-1}}} \lesssim o_T \varepsilon^{\frac{1}{r}} \|\nabla_v g\|_{L^{\frac{r}{r-1}}}. \tag{3.110}$$

Finally, (3.102) holds due to the cutoff $\overline{\chi}$. □

Remark 3.17. Notice that the BV estimate in Proposition 3.4 does not contain exponential decay in η, and thus we cannot directly bound ηS_{2a} and $\eta^2 S_{2a}$. Instead, we should first integrate by parts with respect to v_η as in (3.101) to study f_1^B:

$$\left\| f_1^B \right\|_{L^r} + \left\| \eta f_1^B \right\|_{L^r} + \left\| \eta^2 f_1^B \right\|_{L^r} \lesssim o_T \varepsilon^{\frac{2}{r}-1}, \tag{3.111}$$

$$\left\| f_1^B \right\|_{L^r_{\iota_1\iota_2} L^1_{\mathfrak{n}} L^1_v} + \left\| \eta f_1^B \right\|_{L^r_{\iota_1\iota_2} L^1_{\mathfrak{n}} L^1_v} \lesssim o_T \varepsilon, , \tag{3.112}$$

$$\left\| f_1^B \right\|_{L^r_x L^1_v} + \left\| \eta f_1^B \right\|_{L^r_x L^1_v} \lesssim o_T \varepsilon^{\frac{1}{r}}. \tag{3.113}$$

Estimates of S_3

Lemma 3.18. *Under the assumption* (1.5)*, for S_3 defined in* (3.75)*, we have*

$$\left|\langle S_3, g\rangle_v\right| \lesssim o_T \left(\int_{\mathbb{R}^3} \nu |g|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} \nu |R|^2\right)^{\frac{1}{2}}, \tag{3.114}$$

and thus

$$|\langle S_3, g\rangle| \lesssim o_T \|g\|_{L^2_\nu} \|R\|_{L^2_\nu} \lesssim o_T \|g\|_{L^2_\nu} \left(\|\mathbf{P}[R]\|_{L^2} + \|(\mathbf{I}-\mathbf{P})[R]\|_{L^2_\nu}\right). \tag{3.115}$$

Also, we have

$$\|S_3\|_{L^2} \lesssim o_T \|R\|_{L^2_\nu}, \quad \left\|\nu^{-1} S_3\right\|_{L^\infty_{\varrho,\vartheta}} \lesssim o_T \|R\|_{L^\infty_{\varrho,\vartheta}}. \tag{3.116}$$

Proof. The desired estimates follow from Lemma 3.11, Lemma 3.12, and noting the fact that

$$\|f_1\|_{L^\infty_{\varrho,\vartheta}} + \|f_2\|_{L^\infty_{\varrho,\vartheta}} \lesssim \|f_1\|_{W^{1,s}_x L^\infty_{v,\varrho,\vartheta}} + \|f_2\|_{W^{1,s}_x L^\infty_{v,\varrho,\vartheta}} \lesssim o_T, \tag{3.117}$$

derived from Theorem 3.9. □

Estimates of S_4

Lemma 3.19. *Under the assumption* (1.5)*, for* S_4 *defined in* (3.76)*, we have*

$$\left|\langle S_4, g\rangle_v\right| \lesssim \left(\int_{\mathbb{R}^3} \nu\,|g|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} \nu\left|f_1^B\right|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} \nu\,|R|^2\right)^{\frac{1}{2}}, \tag{3.118}$$

and thus

$$|\langle S_4, g\rangle| \lesssim o_T\,\|g\|_{L^2_\nu}\,\|R\|_{L^2_\nu} \lesssim o_T\,\|g\|_{L^2_\nu}\left(\|\mathbf{P}[R]\|_{L^2} + \|(\mathbf{I}-\mathbf{P})[R]\|_{L^2_\nu}\right), \tag{3.119}$$

$$|\langle S_4, g\rangle| \lesssim o_T\left\|f_1^B\right\|_{L^2_\nu}\|g\|_{L^\infty_{\varrho,\vartheta}}\|R\|_{L^2_\nu} \lesssim o_T\,\varepsilon^{\frac{1}{2}}\,\|g\|_{L^\infty_{\varrho,\vartheta}}\left(\|\mathbf{P}[R]\|_{L^2} + \|(\mathbf{I}-\mathbf{P})[R]\|_{L^2_\nu}\right). \tag{3.120}$$

Also, we have

$$\|S_4\|_{L^2} \lesssim o_T\,\|R\|_{L^2_\nu}, \quad \left\|\nu^{-1}S_4\right\|_{L^\infty_{\varrho,\vartheta}} \lesssim o_T\,\|R\|_{L^\infty_{\varrho,\vartheta}}. \tag{3.121}$$

Proof. The desired estimates follow from Lemma 3.11, Lemma 3.12, and the fact that

$$\left\|f_1^B\right\|_{L^\infty_{\varrho,\vartheta}} \lesssim o_T, \quad \left\|f_1^B\right\|_{L^2_\nu} \lesssim o_T\,\varepsilon^{\frac{1}{2}}, \tag{3.122}$$

derived from Theorem 3.9. □

Estimates of S_5

Lemma 3.20. *Under the assumption* (1.5)*, for* S_5 *defined in* (3.77)*, we have*

$$\left|\langle S_5, g\rangle_v\right| \lesssim o_T\left(\int_{\mathbb{R}^3} \nu\,|g|^2\right)^{\frac{1}{2}}, \tag{3.123}$$

and thus

$$|\langle S_5, g\rangle| \lesssim o_T\,\varepsilon^{\frac{1}{2}}\,\|g\|_{L^2_\nu}, \quad |\langle S_5, g\rangle| \lesssim o_T\,\varepsilon\,\|g\|_{L^\infty_{\varrho,\vartheta}}. \tag{3.124}$$

Also, we have

$$\|S_5\|_{L^2} \lesssim o_T\,\varepsilon^{\frac{1}{2}}, \quad \left\|\nu^{-1}S_5\right\|_{L^\infty_{\varrho,\vartheta}} \lesssim o_T. \tag{3.125}$$

Proof. Similar to the proof of Lemma 3.18 and Lemma 3.19, this follows from Lemma 3.11 and Lemma 3.12 with the help of Theorem 3.9. □

Estimates of S_6

Lemma 3.21. *Under the assumption* (1.5)*, for* S_6 *defined in* (3.78)*, we have*

$$\left|\langle S_6, g\rangle_v\right| \lesssim \left(\int_{\mathbb{R}^3} \nu\,|g|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} \nu\,|R|^2\right), \tag{3.126}$$

and thus

$$\left|\langle S_6, g\rangle\right| \lesssim \|g\|_{L^2_\nu}\left(\|(\mathbf{I}-\mathbf{P})[R]\|_{L^2_\nu}\|R\|_{L^\infty_{\varrho,\vartheta}}+\|\mathbf{P}[R]\|_{L^3}\|\mathbf{P}[R]\|_{L^6}\right)\lesssim \|g\|_{L^2_\nu}\|R\|_X^2. \tag{3.127}$$

Also, we have

$$\|S_6\|_{L^2}\lesssim\left(\|(\mathbf{I}-\mathbf{P})[R]\|_{L^2_\nu}\|R\|_{L^\infty_{\varrho,\vartheta}}+\|\mathbf{P}[R]\|_{L^3}\|\mathbf{P}[R]\|_{L^6}\right)\lesssim\|R\|_X^2, \tag{3.128}$$

$$\left\|\nu^{-1}S_6\right\|_{L^\infty_{\varrho,\vartheta}}\lesssim\|R\|^2_{L^\infty_{\varrho,\vartheta}}. \tag{3.129}$$

Proof. The estimate (3.126) follows from Lemma 3.11. Then using Hölder's inequality

$$\left|\int_{\mathbb{R}^3}\nu R(\mathbf{I}-\mathbf{P})[R]\right|\lesssim\|(\mathbf{I}-\mathbf{P})R\|_{L^2_\nu}\|R\|_{L^\infty_{\varrho,\vartheta}}, \tag{3.130}$$

$$\int_{\mathbb{R}^3}\nu\left|\mathbf{P}[R]\right|^2\lesssim\|\mathbf{P}[R]\|_{L^3}\|\mathbf{P}[R]\|_{L^6}, \tag{3.131}$$

we obtain (3.127). Then (3.128) follows from duality. Finally, (3.129) holds due to Lemma 3.12. □

3.2.2. Conservation laws This section is dedicated to the proof of the crucial conservation laws via a delicate design of a family of test functions.

Classical Conservation Laws

Lemma 3.22. *Let R be the solution to* (1.10). *Under the assumption* (1.5)*, we have the conservation laws*

$$\nabla_x\cdot\mathbf{b}=\left\langle\mu^{\frac12},S\right\rangle_v=\left\langle\mu^{\frac12},S_2\right\rangle_v, \tag{3.132}$$

$$\nabla_x p+\nabla_x\cdot\varpi=\left\langle v\mu^{\frac12},S\right\rangle_v=\left\langle v\mu^{\frac12},S_2\right\rangle_v, \tag{3.133}$$

$$5\nabla_x\cdot\mathbf{b}+\nabla_x\cdot\varsigma=\left\langle|v|^2\mu^{\frac12},S\right\rangle_v=\left\langle|v|^2\mu^{\frac12},S_2\right\rangle_v, \tag{3.134}$$

where $\varpi:=\int_{\mathbb{R}^3}\mu^{\frac12}\big(v\otimes v\big)(\mathbf{I}-\mathbf{P})[R]$ *and* $\varsigma:=\int_{\mathbb{R}^3}\mu^{\frac12}v\,|v|^2\,(\mathbf{I}-\mathbf{P})[R]$.

Proof. We multiply test functions $\mu^{\frac12}$, $v\mu^{\frac12}$, $|v|^2\mu^{\frac12}$ on both sides of (1.10) and integrate over $v\in\mathbb{R}^3$. Using the orthogonality of $\mathcal{L}$ and Lemma 3.15 (which comes from (3.5)), the results follow. □

Conservation Law with Test Function $\nabla_x\varphi\cdot\mathscr{A}$

Lemma 3.23. *Let R be the solution to* (1.10). *Under the assumption* (1.5)*, for any smooth function $\varphi(x)$, we have*

$$\begin{aligned}&-\kappa\big\langle\Delta_x\varphi,c\big\rangle_x+\varepsilon^{-1}\big\langle\nabla_x\varphi,\varsigma\big\rangle\\&=\big\langle\nabla_x\varphi\cdot\mathscr{A},h\big\rangle_{\gamma_-}-\big\langle\nabla_x\varphi\cdot\mathscr{A},R\big\rangle_{\gamma_+}+\Big\langle v\cdot\nabla_x\Big(\nabla_x\varphi\cdot\mathscr{A}\Big),(\mathbf{I}-\mathbf{P})[R]\Big\rangle+\big\langle\nabla_x\varphi\cdot\mathscr{A},S\big\rangle.\end{aligned} \tag{3.135}$$

Proof. Taking test function $g = \nabla_x\varphi\cdot\mathscr{A}$ in (3.83), we obtain

$$\int_\gamma \Big(\nabla_x\varphi\cdot\mathscr{A}\Big)R\big(v\cdot n\big) - \big\langle v\cdot\nabla_x\Big(\nabla_x\varphi\cdot\mathscr{A}\Big), R\big\rangle + \varepsilon^{-1}\big\langle\mathcal{L}[R], \nabla_x\varphi\cdot\mathscr{A}\big\rangle = \big\langle\nabla_x\varphi\cdot\mathscr{A}, S\big\rangle. \tag{3.136}$$

Using the splitting (1.11), oddness and orthogonality of $\mathscr{A}$, we deduce

$$\begin{aligned}&-\kappa\big\langle\Delta_x\varphi, c\big\rangle_x + \varepsilon^{-1}\big\langle\nabla_x\varphi, \varsigma\big\rangle\\ &\quad= -\int_\gamma \Big(\nabla_x\varphi\cdot\mathscr{A}\Big)R(v\cdot n) + \Big\langle v\cdot\nabla_x\Big(\nabla_x\varphi\cdot\mathscr{A}\Big), (\mathbf{I}-\mathbf{P})[R]\Big\rangle + \big\langle\nabla_x\varphi\cdot\mathscr{A}, S\big\rangle.\end{aligned} \tag{3.137}$$

Notice that

$$R\mathbf{1}_\gamma = R\mathbf{1}_{\gamma_+} + h\mathbf{1}_{\gamma_-}, \tag{3.138}$$

we have (3.135). □

<u>Conservation Law with Test Function $\nabla_x\psi : \mathscr{B}$</u>

Lemma 3.24. *Let R be the solution to (1.10). Under the assumption (1.5), for any smooth function $\psi(x)$ satisfying $\nabla_x\cdot\psi = 0$, we have*

$$\begin{aligned}&-\lambda\big\langle\Delta_x\psi, \mathbf{b}\big\rangle_x + \varepsilon^{-1}\big\langle\nabla_x\psi, \varpi\big\rangle\\ &\quad= \big\langle\nabla_x\psi\cdot\mathscr{B}, h\big\rangle_{\gamma_-} - \big\langle\nabla_x\psi\cdot\mathscr{B}, R\big\rangle_{\gamma_+} + \Big\langle v\cdot\nabla_x\Big(\nabla_x\psi : \mathscr{B}\Big), (\mathbf{I}-\mathbf{P})[R]\Big\rangle\\ &\qquad+ \big\langle\nabla_x\psi : \mathscr{B}, S\big\rangle.\end{aligned} \tag{3.139}$$

Proof. Taking test function $g = \nabla_x\psi : \mathscr{B}$ in (3.83), we obtain

$$\begin{aligned}&\int_\gamma \Big(\nabla_x\psi : \mathscr{B}\Big)R\big(v\cdot n\big) - \big\langle v\cdot\nabla_x\Big(\nabla_x\psi : \mathscr{B}\Big), R\big\rangle + \varepsilon^{-1}\big\langle\mathcal{L}[R], \nabla_x\psi : \mathscr{B}\big\rangle\\ &\quad= \big\langle\nabla_x\psi : \mathscr{B}, S\big\rangle.\end{aligned} \tag{3.140}$$

Using the splitting (1.11), oddness and orthogonality of $\mathscr{B}$, we deduce

$$\begin{aligned}&-\Big\langle v\cdot\nabla_x\Big(\nabla_x\psi : \mathscr{B}\Big), v\mu^{\frac{1}{2}}\cdot\mathbf{b}\Big\rangle + \varepsilon^{-1}\big\langle\nabla_x\psi, \varpi\big\rangle\\ &= -\int_\gamma \Big(\nabla_x\psi : \mathscr{B}\Big)R(v\cdot n) + \Big\langle v\cdot\nabla_x\Big(\nabla_x\psi : \mathscr{B}\Big), (\mathbf{I}-\mathbf{P})[R]\Big\rangle + \big\langle\nabla_x\psi : \mathscr{B}, S\big\rangle.\end{aligned} \tag{3.141}$$

Here, we may further compute

$$\Big\langle v\cdot\nabla_x\Big(\nabla_x\psi : \mathscr{B}\Big), v\mu^{\frac{1}{2}}\cdot\mathbf{b}\Big\rangle = \big\langle\overline{\mathscr{B}}\cdot\nabla_x\big(\nabla_x\psi : \mathscr{B}\big), \mathbf{b}\big\rangle, \tag{3.142}$$

and use $\nabla_x\cdot\psi = 0$ to obtain

$$\int_{\mathbb{R}^3} \overline{\mathscr{B}}\cdot\nabla_x\big(\nabla_x\psi : \mathscr{B}\big) = \begin{pmatrix}\alpha\partial_{11}\psi_1 + (\overline{\alpha}+\lambda)\partial_{12}\psi_2 + (\overline{\alpha}+\lambda)\partial_{13}\psi_3 + \lambda\partial_{22}\psi_1 + \lambda\partial_{33}\psi_1\\ \alpha\partial_{22}\psi_2 + (\overline{\alpha}+\lambda)\partial_{12}\psi_1 + (\overline{\alpha}+\lambda)\partial_{23}\psi_3 + \lambda\partial_{11}\psi_2 + \lambda\partial_{33}\psi_2\\ \alpha\partial_{33}\psi_3 + (\overline{\alpha}+\lambda)\partial_{13}\psi_1 + (\overline{\alpha}+\lambda)\partial_{23}\psi_2 + \lambda\partial_{11}\psi_3 + \lambda\partial_{22}\psi_3\end{pmatrix}$$

$$= \begin{pmatrix} (\alpha - \overline{\alpha} - \lambda)\partial_{11}\psi_1 + \lambda\partial_{22}\psi_1 + \lambda\partial_{33}\psi_1 \\ (\alpha - \overline{\alpha} - \lambda)\partial_{22}\psi_2 + \lambda\partial_{11}\psi_2 + \lambda\partial_{33}\psi_2 \\ (\alpha - \overline{\alpha} - \lambda)\partial_{33}\psi_3 + \lambda\partial_{11}\psi_3 + \lambda\partial_{22}\psi_3 \end{pmatrix} = \lambda \begin{pmatrix} \Delta_x\psi_1 \\ \Delta_x\psi_2 \\ \Delta_x\psi_3 \end{pmatrix}. \tag{3.143}$$

Here, we use the fact that $\mathscr{B} = \Upsilon\big(|v|\big)\overline{\mathscr{B}}$ for some function Υ that only depend on $|v|$ (see [37, Lemma 14]). The constants α, $\overline{\alpha}$ and λ are defined in Section C. Then direct computation shows that $\alpha - \overline{\alpha} = 2\lambda$: for $i \neq j$

$$\begin{aligned} \alpha - \overline{\alpha} - 2\lambda &= \int_{\mathbb{R}^3} \Upsilon\big(|v|\big)\bigg(\Big(v_i^2 - \frac{1}{3}|v|^2\Big)^2 - \Big(v_i^2 - \frac{1}{3}|v|^2\Big)\Big(v_j^2 - \frac{1}{3}|v|^2\Big) \\ &\quad - 2v_i^2v_j^2\bigg)\mu(v)\mathrm{d}v \\ &= \int_{\mathbb{R}^3} \Upsilon\big(|v|\big)\Big(v_i^4 - 3v_i^2v_j^2\Big)\mu(v)\mathrm{d}v. \end{aligned} \tag{3.144}$$

Then we use the spherical coordinates

$$v_i = |v|\sin\theta\sin\varphi, \qquad v_j = |v|\sin\theta\cos\varphi, \tag{3.145}$$

to estimate

$$\begin{aligned} \alpha - \overline{\alpha} - 2\lambda = \int_0^\infty |v|^2\,\Upsilon\big(|v|\big)\mu\big(|v|\big)\mathrm{d}\,|v| \int_0^\pi \sin^5\theta\mathrm{d}\theta \\ \int_0^{2\pi}\Big(\sin^4\varphi - 3\sin^2\varphi\cos^2\varphi\Big)\mathrm{d}\varphi = 0. \end{aligned} \tag{3.146}$$

Using (3.138), we obtain (3.139). □

<u>Conservation Law with Test Function $\nabla_x\varphi \cdot \mathscr{A} + \varepsilon^{-1}\varphi\big(|v|^2 - 5\big)\mu^{\frac{1}{2}}$</u>

Lemma 3.25. *Let R be the solution to* (1.10)*. Under the assumption* (1.5)*, for any smooth function $\varphi(x)$ satisfying $\varphi\big|_{\partial\Omega} = 0$, we have*

$$\begin{aligned} -\kappa\big\langle\Delta_x\varphi, c\big\rangle_x &= \big\langle\nabla_x\varphi\cdot\mathscr{A}, h\big\rangle_{\gamma_-} - \big\langle\nabla_x\varphi\cdot\mathscr{A}, R\big\rangle_{\gamma_+} + \Big\langle v\cdot\nabla_x\Big(\nabla_x\varphi\cdot\mathscr{A}\Big), (\mathbf{I}-\mathbf{P})[R]\Big\rangle \\ &\quad + \varepsilon^{-1}\Big\langle\varphi\Big(|v|^2 - 5\Big)\mu^{\frac{1}{2}}, S\Big\rangle + \big\langle\nabla_x\varphi\cdot\mathscr{A}, S\big\rangle. \end{aligned} \tag{3.147}$$

Proof. From (3.132) and (3.134), we have

$$\nabla_x\cdot\varsigma = \Big\langle\Big(|v|^2 - 5\Big)\mu^{\frac{1}{2}}, S\Big\rangle_v. \tag{3.148}$$

Multiplying $\varphi(x) \in \mathbb{R}$ on both sides of (3.148) and integrating over $x \in \Omega$, we obtain

$$-\big\langle\nabla_x\varphi, \varsigma\big\rangle_x + \int_{\partial\Omega}\varphi\varsigma\cdot n = \Big\langle\varphi\Big(|v|^2 - 5\Big)\mu^{\frac{1}{2}}, S\Big\rangle. \tag{3.149}$$

Hence, adding $\varepsilon^{-1}\times$(3.149) and (3.135) to eliminate $\varepsilon^{-1}\big\langle\nabla_x\varphi, \varsigma\big\rangle_x$ yields

$$-\kappa\big\langle\Delta_x\varphi, c\big\rangle_x + \varepsilon^{-1}\int_{\partial\Omega}\varphi\varsigma\cdot n$$

$$= \big\langle \nabla_x \varphi \cdot \mathscr{A}, h \big\rangle_{\gamma_-} - \big\langle \nabla_x \varphi \cdot \mathscr{A}, R \big\rangle_{\gamma_+} + \Big\langle v \cdot \nabla_x \Big(\nabla_x \varphi \cdot \mathscr{A} \Big), (\mathbf{I} - \mathbf{P})[R] \Big\rangle \\ + \varepsilon^{-1} \Big\langle \varphi \Big(|v|^2 - 5 \Big) \mu^{\frac{1}{2}}, S \Big\rangle + \big\langle \nabla_x \varphi \cdot \mathscr{A}, S \big\rangle. \tag{3.150}$$

The assumption $\varphi\big|_{\partial\Omega} = 0$ completely eliminates the boundary term $\varepsilon^{-1} \int_{\partial\Omega} \varphi \varsigma \cdot n$ in (3.150). Hence, we have (3.147). □

Conservation Law with Test Function $\nabla_x \psi : \mathscr{B} + \varepsilon^{-1} \psi \cdot v \mu^{\frac{1}{2}}$

Lemma 3.26. *Let R be the solution to* (1.10)*. Under the assumption* (1.5)*, for any smooth function $\psi(x)$ satisfying $\nabla_x \cdot \psi = 0$, $\psi\big|_{\partial\Omega} = 0$, we have*

$$-\lambda \big\langle \Delta_x \psi, \mathbf{b} \big\rangle_x = \big\langle \nabla_x \psi : \mathscr{B}, h \big\rangle_{\gamma_-} - \big\langle \nabla_x \psi : \mathscr{B}, R \big\rangle_{\gamma_+} + \Big\langle v \cdot \nabla_x \Big(\nabla_x \psi : \mathscr{B} \Big), (\mathbf{I} - \mathbf{P})[R] \Big\rangle \\ + \varepsilon^{-1} \Big\langle \psi \cdot v \mu^{\frac{1}{2}}, S \Big\rangle + \big\langle \nabla_x \psi : \mathscr{B}, S \big\rangle. \tag{3.151}$$

Proof. Multiplying $\psi(x) \in \mathbb{R}^3$ on both sides of (3.133) and integrating over $x \in \Omega$, we obtain

$$-\big\langle \nabla_x \cdot \psi, p \big\rangle_x - \big\langle \nabla_x \psi, \varpi \big\rangle_x + \int_{\partial\Omega} \Big(p\psi + \psi \cdot \varpi \Big) \cdot n = \Big\langle \psi \cdot v \mu^{\frac{1}{2}}, S \Big\rangle. \tag{3.152}$$

Hence, adding $\varepsilon^{-1} \times$(3.152) and (3.139) to eliminate $\varepsilon^{-1} \big\langle \nabla_x \psi, \varpi \big\rangle_x$ yields

$$- \lambda \big\langle \Delta_x \psi, \mathbf{b} \big\rangle - \varepsilon^{-1} \big\langle \nabla_x \cdot \psi, p \big\rangle_x + \varepsilon^{-1} \int_{\partial\Omega} \Big(p\psi + \psi \cdot \varpi \Big) \cdot n \\ = \big\langle \nabla_x \psi : \mathscr{B}, h \big\rangle_{\gamma_-} - \big\langle \nabla_x \psi : \mathscr{B}, R \big\rangle_{\gamma_+} + \Big\langle v \cdot \nabla_x \Big(\nabla_x \psi : \mathscr{B} \Big), (\mathbf{I} - \mathbf{P})[R] \Big\rangle \\ + \varepsilon^{-1} \Big\langle \psi \cdot v \mu^{\frac{1}{2}}, S \Big\rangle + \big\langle \nabla_x \psi : \mathscr{B}, S \big\rangle. \tag{3.153}$$

The assumptions $\nabla_x \cdot \psi = 0$ and $\psi\big|_{\partial\Omega} = 0$ eliminate $\varepsilon^{-1} \big\langle \nabla_x \cdot \psi, p \big\rangle_x$ and $\varepsilon^{-1} \int_{\partial\Omega} \Big(p\psi + \psi \cdot \varpi \Big) \cdot n$ in (3.153). Hence, we have (3.151). □

3.3. Energy estimate.

Proposition 3.27. *Let R be the solution to (1.10). Under the assumption (1.5), we have*

$$\varepsilon^{-\frac{1}{2}} \left| R \right|_{L^2_{\gamma_+}} + \varepsilon^{-1} \left\| (\mathbf{I} - \mathbf{P})[R] \right\|_{L^2_\nu} \lesssim o_T \left\| R \right\|_X + \left\| R \right\|_X^2 + o_T. \tag{3.154}$$

Proof. It suffices to justify

$$\varepsilon^{-\frac{1}{2}} \left| R \right|_{L^2_{\gamma_+}} + \varepsilon^{-1} \left\| (\mathbf{I} - \mathbf{P})[R] \right\|_{L^2_\nu} \lesssim o_T \varepsilon^{-\frac{1}{2}} \left\| \mathbf{P}[R] \right\|_{L^2} + o_T \left\| R \right\|_X + \left\| R \right\|_X^2 + o_T. \tag{3.155}$$

Weak Formulation Taking test function $g = \varepsilon^{-1} R$ in (3.83), we obtain

$$\frac{\varepsilon^{-1}}{2} \int_\gamma R^2 (v \cdot n) + \varepsilon^{-2} \big\langle \mathcal{L}[R], R \big\rangle = \varepsilon^{-1} \big\langle S, R \big\rangle. \tag{3.156}$$

Notice that

$$\int_\gamma R^2(v\cdot n) = |R|^2_{L^2_{\gamma+}} - |R|^2_{L^2_{\gamma-}} = |R|^2_{L^2_{\gamma+}} - |h|^2_{L^2_\gamma}, \tag{3.157}$$

and

$$\big\langle \mathcal{L}[R], R\big\rangle \gtrsim \|(\mathbf{I}-\mathbf{P})[R]\|^2_{L^2_\nu}. \tag{3.158}$$

Then we know

$$\varepsilon^{-1}|R|^2_{L^2_{\gamma+}} + \varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})[R]\|^2_{L^2_\nu} \lesssim \Big|\varepsilon^{-1}\big\langle S, R\big\rangle\Big| + \varepsilon^{-1}|h|^2_{L^2_\gamma}. \tag{3.159}$$

Using Lemma 3.13, we have

$$\varepsilon^{-1}|R|^2_{L^2_{\gamma+}} + \varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})[R]\|^2_{L^2_\nu} \lesssim \Big|\varepsilon^{-1}\big\langle S, R\big\rangle\Big| + o_T\varepsilon. \tag{3.160}$$

Source Term Estimates We split

$$\varepsilon^{-1}\big\langle S, R\big\rangle = \varepsilon^{-1}\big\langle S, \mathbf{P}[R]\big\rangle + \varepsilon^{-1}\big\langle S, (\mathbf{I}-\mathbf{P})[R]\big\rangle. \tag{3.161}$$

We may directly bound using Lemmas 3.15–3.21

$$\begin{aligned}\Big|\varepsilon^{-1}\big\langle S, (\mathbf{I}-\mathbf{P})[R]\big\rangle\Big| &\lesssim \varepsilon^{-1}\|S\|_{L^2}\|(\mathbf{I}-\mathbf{P})[R]\|_{L^2_\nu}\\ &\lesssim \big(o(1)+o_T\big)\varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})[R]\|^2_{L^2_\nu} + o_T\|R\|^2_X + \|R\|^4_X + o_T.\end{aligned} \tag{3.162}$$

Using the orthogonality property of Γ, we have

$$\varepsilon^{-1}\big\langle S, \mathbf{P}[R]\big\rangle = \varepsilon^{-1}\big\langle S_1+S_2, \mathbf{P}[R]\big\rangle. \tag{3.163}$$

From Lemma 3.15, we know

$$\Big|\varepsilon^{-1}\big\langle S_1, \mathbf{P}[R]\big\rangle\Big| \lesssim \varepsilon^{-1}\|S_1\|_{L^2}\|\mathbf{P}[R]\|_{L^2} \lesssim o_T\|\mathbf{P}[R]\|^2_{L^2} + o_T. \tag{3.164}$$

Also, from Lemma 3.16 and Remark 3.17, we have

$$\varepsilon^{-1}\big\langle S_2, \mathbf{P}[R]\big\rangle = \varepsilon^{-1}\big\langle S_{2a}, \mathbf{P}[R]\big\rangle + \varepsilon^{-1}\big\langle S_{2b}+S_{2c}, \mathbf{P}[R]\big\rangle. \tag{3.165}$$

After integrating by parts with respect to v_η in the S_{2a} term, we obtain

$$\begin{aligned}\Big|\varepsilon^{-1}\big\langle S_2, \mathbf{P}[R]\big\rangle\Big| &\lesssim \varepsilon^{-1}\Big(\Big\|f_1^B\Big\|_{L^2_xL^1_v} + \|S_{2b}+S_{2c}\|_{L^2_xL^1_v}\Big)\|\mathbf{P}[R]\|_{L^2_xL^\infty_v}\\ &\lesssim o_T\varepsilon^{-\frac12}\|\mathbf{P}[R]\|_{L^2} \lesssim o_T\varepsilon^{-1}\|\mathbf{P}[R]\|^2_{L^2} + o_T.\end{aligned} \tag{3.166}$$

In total, we have

$$\big|\varepsilon^{-1}\big\langle S, R\big\rangle\big| \lesssim o_T\varepsilon^{-1}\|\mathbf{P}[R]\|^2_{L^2} + \big(o(1)+o_T\big)\varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})[R]\|^2_{L^2_\nu} + o_T\|R\|^2_X + \|R\|^4_X + o_T. \tag{3.167}$$

Synthesis Inserting (3.167) into (3.160), we have

$$\varepsilon^{-1}|R|^2_{L^2_{\gamma+}} + \varepsilon^{-2}\Big\|(\mathbf{I}-\mathbf{P})[R]\Big\|^2_{L^2} \lesssim o_T\varepsilon^{-1}\|\mathbf{P}[R]\|^2_{L^2} + o_T\|R\|^2_X + \|R\|^4_X + o_T. \tag{3.168}$$

Then we have (3.155). □

Corollary 3.28. *Let R be the solution to (1.10). Under the assumption (1.5), we have*

$$\|(\mathbf{I}-\mathbf{P})[R]\|_{L^6}+\left|\mu^{\frac{1}{4}}R\right|_{L^4_{\gamma+}}\lesssim o_T\,\|R\|_X+\|R\|_X^2+o_T. \tag{3.169}$$

Proof. By interpolation and Proposition 3.27, we obtain

$$\begin{aligned}\left\|(\mathbf{I}-\mathbf{P})[R]\right\|_{L^6}&\lesssim\left\|(\mathbf{I}-\mathbf{P})[R]\right\|_{L^2}^{\frac{1}{3}}\left\|(\mathbf{I}-\mathbf{P})[R]\right\|_{L^\infty_{\varrho,\vartheta}}^{\frac{2}{3}}\\&\lesssim\left(o_T\,\|R\|_X^{\frac{1}{3}}+\|R\|_X^{\frac{1}{3}}+o_T\right)\left(\varepsilon^{\frac{1}{2}}\,\|R\|_{L^\infty_{\varrho,\vartheta}}\right)^{\frac{2}{3}}\lesssim o_T\,\|R\|_X+\|R\|_X^2+o_T,\end{aligned} \tag{3.170}$$

$$\begin{aligned}\left|\mu^{\frac{1}{4}}R\right|_{L^4_{\gamma+}}&\lesssim|R|_{L^2_{\gamma+}}^{\frac{1}{2}}\;|R|_{L^\infty_{\gamma+,\varrho,\vartheta}}^{\frac{1}{2}}\\&\lesssim\left(o_T\,\|R\|_X^{\frac{1}{2}}+\|R\|_X^{\frac{1}{2}}+o_T\right)\left(\varepsilon^{\frac{1}{2}}\,|R|_{L^\infty_{\gamma+,\varrho,\vartheta}}\right)^{\frac{1}{2}}\lesssim o_T\,\|R\|_X+\|R\|_X^2+o_T.\end{aligned} \tag{3.171}$$

Then the desired result follows from (3.170), (3.171). □

3.4. Kernel estimate.

3.4.1. Estimate of p

Proposition 3.29. *Let R be the solution to (1.10). Under the assumption (1.5), we have*

$$\varepsilon^{-\frac{1}{2}}\,\|p\|_{L^2}+\|p\|_{L^6}\lesssim o_T\,\|R\|_X+\|R\|_X^2+o_T. \tag{3.172}$$

Proof. It suffices to show for $2\leq r\leq 6$

$$\|p\|_{L^r}\lesssim\left|\mu^{\frac{1}{4}}R\right|_{L^{\frac{2r}{3}}_{\gamma+}}+o_T\varepsilon^{\frac{2}{r}}. \tag{3.173}$$

Weak Formulation Denote

$$\psi(x,v):=\mu^{\frac{1}{2}}(v)\Big(v\cdot\nabla_x\varphi(x)\Big), \tag{3.174}$$

where $\varphi(x)$ is defined via solving the elliptic problem

$$\begin{cases}-\Delta_x\varphi=p\,|p|^{r-2}&\text{in }\ \Omega,\\ \varphi=0\ \text{ on }\ \partial\Omega.\end{cases} \tag{3.175}$$

Based on standard elliptic estimates [62], there exists a solution φ satisfying

$$\|\psi\|_{W^{1,\frac{r}{r-1}}L^\infty_{\varrho,\vartheta}}\lesssim\|\varphi\|_{W^{2,\frac{r}{r-1}}}\lesssim\left\|p\,|p|^{r-2}\right\|_{L^{\frac{r}{r-1}}}\lesssim\|p\|_{L^r}^{r-1}. \tag{3.176}$$

Based on Sobolev embedding and trace estimate, we have for $2\leq r\leq 6$

$$\|\psi\|_{L^2}+|\psi|_{L^{\frac{2r}{2r-3}}_{\gamma}}\lesssim\|p\|_{L^r}^{r-1}. \tag{3.177}$$

Taking test function $g=\psi$ in (3.83), we obtain

$$\int_{\gamma} R\psi \mathrm{d}\gamma - \big\langle R, v\cdot\nabla_x\psi\big\rangle = \big\langle S, \psi\big\rangle. \tag{3.178}$$

From Lemma 3.13, we know

$$\begin{aligned}\left|\int_{\gamma} R\psi \mathrm{d}\gamma\right| &\leq \left|\int_{\gamma_+} R\psi \mathrm{d}\gamma\right| + \left|\int_{\gamma_-} h\psi \mathrm{d}\gamma\right| \lesssim \left|\mu^{\frac{1}{4}}R\right|_{L^{\frac{2r}{3}}_{\gamma_+}} |\psi|_{L^{\frac{2r}{2r-3}}_{\gamma_+}} + |h|_{L^{\frac{2r}{3}}_{\gamma_-}} |\psi|_{L^{\frac{2r}{2r-3}}_{\gamma_-}} \\ &\lesssim o(1)\,|\psi|^{\frac{r}{r-1}}_{L^{\frac{2r}{2r-3}}_{\gamma}} + \left|\mu^{\frac{1}{4}}R\right|^r_{L^{\frac{2r}{3}}_{\gamma_+}} + |h|^r_{L^{\frac{2r}{3}}_{\gamma_-}} \lesssim o(1)\,\|p\|^r_{L^r} + \left|\mu^{\frac{1}{4}}R\right|^r_{L^{\frac{2r}{3}}_{\gamma_+}} + o_T\varepsilon^3.\end{aligned} \tag{3.179}$$

Due to oddness and orthogonality, we have

$$\Big\langle\mu^{\frac{1}{2}}\big(v\cdot\mathbf{b}\big), v\cdot\nabla_x\psi\Big\rangle = \big\langle(\mathbf{I}-\mathbf{P})[R], v\cdot\nabla_x\psi\big\rangle = 0. \tag{3.180}$$

Due to orthogonality of $\overline{\mathscr{A}}$, we know

$$\left\langle\mu^{\frac{1}{2}}\frac{|v|^2-5}{2}c, v\cdot\nabla_x\psi\right\rangle = \big\langle c, \mu^{\frac{1}{2}}\overline{\mathscr{A}}\cdot\nabla_x\psi\big\rangle = 0. \tag{3.181}$$

Also, we have

$$-\Big\langle\mu^{\frac{1}{2}}p, v\cdot\nabla_x\psi\Big\rangle = -\Big\langle p\mu, v\cdot\nabla_x\Big(v\cdot\nabla_x\varphi\Big)\Big\rangle = -\frac{1}{3}\int_{\Omega} p\big(\Delta_x\varphi\big)\int_{\mathbb{R}^3}\mu\,|v|^2 = \|p\|^r_{L^r}. \tag{3.182}$$

In summary, we have shown that

$$\|p\|^r_{L^r} \lesssim \left|\mu^{\frac{1}{4}}R\right|^r_{L^{\frac{2r}{3}}_{\gamma_+}} + o_T\varepsilon^3 + \left|\big\langle S, \psi\big\rangle\right|. \tag{3.183}$$

<u>Source Term Estimates</u> Due to the orthogonality property of Γ and Lemma 3.15, we know

$$\big\langle S, \psi\big\rangle = \big\langle S_2, \psi\big\rangle. \tag{3.184}$$

Using Hardy's inequality and integrating by parts with respect to v_η in S_{2a}, based on Lemma 3.16 and Remark 3.17, we have

$$\begin{aligned}&\left|\big\langle S_2, \psi\big\rangle\right| \leq \left|\Big\langle S_2, \psi\Big|_{\mathfrak{n}=0}\Big\rangle\right| + \left|\left\langle S_2, \int_0^{\mathfrak{n}}\partial_{\mathfrak{n}}\psi\right\rangle\right| = \left|\Big\langle S_2, \psi\Big|_{\mathfrak{n}=0}\Big\rangle\right| + \left|\varepsilon\left\langle \eta S_2, \frac{1}{\mathfrak{n}}\int_0^{\mathfrak{n}}\partial_{\mathfrak{n}}\psi\right\rangle\right| \\ &\lesssim \left\|f_1^B + S_{2b} + S_{2c}\right\|_{L^2_{\iota_1\iota_2}L^1_{\mathfrak{n}}L^1_v} |\psi|_{L^2_\gamma} + \varepsilon\left\|\eta\big(f_1^B + S_{2b} + S_{2c}\big)\right\|_{L^r}\left\|\frac{1}{\mathfrak{n}}\int_0^{\mathfrak{n}}\partial_{\mathfrak{n}}\psi\right\|_{L^{\frac{r}{r-1}}} \\ &\lesssim \left\|f_1^B + S_{2b} + S_{2c}\right\|_{L^2_{\iota_1\iota_2}L^1_{\mathfrak{n}}L^1_v} |\psi|_{L^2_\gamma} + \varepsilon\left\|\eta\big(f_1^B + S_{2b} + S_{2c}\big)\right\|_{L^r}\|\partial_{\mathfrak{n}}\psi\|_{L^{\frac{r}{r-1}}} \\ &\lesssim o_T\varepsilon\,|\psi|_{L^2_\gamma} + o_T\varepsilon^{\frac{2}{r}}\,\|\partial_{\mathfrak{n}}\psi\|_{L^{\frac{r}{r-1}}} \lesssim o_T\varepsilon^{\frac{2}{r}}\,\|p\|^{r-1}_{L^r} \lesssim o_T\,\|p\|^r_{L^r} + o_T\varepsilon^2.\end{aligned} \tag{3.185}$$

Inserting (3.185) into (3.183), we have shown

$$\|p\|^r_{L^r} \lesssim \left|\mu^{\frac{1}{4}}R\right|^r_{L^{\frac{2r}{3}}_{\gamma_+}} + o_T\varepsilon^2. \tag{3.186}$$

Hence, we have (3.173). □

3.4.2. Estimate of c

Proposition 3.30. *Let R be the solution to (1.10). Under the assumption (1.5), we have*

$$\varepsilon^{-\frac{1}{2}}\,\|c\|_{L^2}+\|c\|_{L^6}\lesssim o_T\,\|R\|_X+\|R\|_X^2+o_T. \tag{3.187}$$

Proof. It suffices to justify for $2\leq r\leq 6$

$$\|c\|_{L^r}\lesssim\varepsilon^{\frac{1}{4r}}\,\|R\|_X^{\frac{1}{r}}\,\|c\|_{L^r}+\left|\mu^{\frac{1}{4}}R\right|_{L^{\frac{2r}{3}}_{\gamma_+}}+\|(\mathbf{I}-\mathbf{P})[R]\|_{L^r}$$
$$+\,o_T\varepsilon^{\frac{1}{2}}\,\|R\|_X+\varepsilon^{\frac{1}{2}}\,\|R\|_X^2+o_T\left(\varepsilon^{\frac{1}{2}}+\varepsilon^{\frac{2}{r}}\right). \tag{3.188}$$

Weak Formulation We consider the conservation law (3.147) where the smooth test function $\varphi(x)$ satisfies

$$\begin{cases}-\Delta_x\varphi=c\,|c|^{r-2} & \text{in } \Omega,\\ \varphi=0 & \text{on } \partial\Omega.\end{cases} \tag{3.189}$$

Based on standard elliptic estimates [62], there exists a solution φ satisfying

$$\|\varphi\|_{W^{2,\frac{r}{r-1}}}\lesssim\left\|c\,|c|^{r-2}\right\|_{L^{\frac{r}{r-1}}}\lesssim\|c\|_{L^r}^{r-1}. \tag{3.190}$$

Based on Sobolev embedding and trace estimate, we have for $2\leq r\leq 6$

$$\|\varphi\|_{H^1}+|\nabla_x\varphi|_{L^{\frac{2r}{2r-3}}_{\partial\Omega}}\lesssim\|c\|_{L^r}^{r-1}. \tag{3.191}$$

Hence, from (3.147), we have

$$\kappa\,\|c\|_{L^r}^r=\left\langle\nabla_x\varphi\cdot\mathscr{A},h\right\rangle_{\gamma_-}-\left\langle\nabla_x\varphi\cdot\mathscr{A},R\right\rangle_{\gamma_+}+\left\langle v\cdot\nabla_x\left(\nabla_x\varphi\cdot\mathscr{A}\right),(\mathbf{I}-\mathbf{P})[R]\right\rangle$$
$$+\,\varepsilon^{-1}\left\langle\varphi\left(|v|^2-5\right)\mu^{\frac{1}{2}},S\right\rangle+\left\langle\nabla_x\varphi\cdot\mathscr{A},S\right\rangle. \tag{3.192}$$

From Lemma 3.13, we have

$$\left|\left\langle\nabla_x\varphi\cdot\mathscr{A},h\right\rangle_{\gamma_-}\right|\lesssim|\nabla_x\varphi\cdot\mathscr{A}|_{L^{\frac{2r}{2r-3}}_{\gamma_-}}|h|_{L^{\frac{2r}{3}}_{\gamma_-}}\lesssim o_T\,\|c\|_{L^r}^r+o_T\varepsilon^3, \tag{3.193}$$

$$\left|\left\langle\nabla_x\varphi\cdot\mathscr{A},R\right\rangle_{\gamma_+}\right|\lesssim|\nabla_x\varphi\cdot\mathscr{A}|_{L^{\frac{2r}{2r-3}}_{\gamma_+}}|R|_{L^{\frac{2r}{3}}_{\gamma_+}}\lesssim o(1)\,\|c\|_{L^r}^r+\left|\mu^{\frac{1}{4}}R\right|_{L^{\frac{2r}{3}}_{\gamma_+}}^r, \tag{3.194}$$

and

$$\left|\left\langle v\cdot\nabla_x\left(\nabla_x\varphi\cdot\mathscr{A}\right),(\mathbf{I}-\mathbf{P})[R]\right\rangle\right|\lesssim\left\|v\cdot\nabla_x\left(\nabla_x\varphi\cdot\mathscr{A}\right)\right\|_{L^{\frac{r}{r-1}}}\|(\mathbf{I}-\mathbf{P})[R]\|_{L^r}$$
$$\lesssim o(1)\,\|c\|_{L^r}^r+\|(\mathbf{I}-\mathbf{P})[R]\|_{L^r}^r. \tag{3.195}$$

In summary, we have shown that

$$\|c\|_{L^r}^r\lesssim\left|\mu^{\frac{1}{4}}R\right|_{L^{\frac{2r}{3}}_{\gamma_+}}^r+\|(\mathbf{I}-\mathbf{P})[R]\|_{L^r}^r+o_T\varepsilon^3+\left|\varepsilon^{-1}\left\langle\varphi\big(|v|^2-5\big)\mu^{\frac{1}{2}},S\right\rangle\right|+\left|\left\langle\nabla_x\varphi\cdot\mathscr{A},S\right\rangle\right|. \tag{3.196}$$

Source Term Estimates Due to the orthogonality of Γ and Lemma 3.15, we have

$$\varepsilon^{-1}\left\langle\varphi\left(|v|^2-5\right)\mu^{\frac{1}{2}},S\right\rangle=\varepsilon^{-1}\left\langle\varphi\left(|v|^2-5\right)\mu^{\frac{1}{2}},S_2\right\rangle. \tag{3.197}$$

Similar to (3.185), based on Lemma 3.16, Remark 3.17 and Hardy's inequality, we have

$$\begin{aligned}
&\left|\varepsilon^{-1}\left\langle\varphi\left(|v|^2-5\right)\mu^{\frac{1}{2}},S_2\right\rangle\right|\lesssim\varepsilon^{-1}\left|\left\langle S_2,\int_0^{\mathfrak{n}}\partial_{\mathfrak{n}}\varphi\right\rangle\right|\\
&\lesssim\left|\left\langle\eta S_2,\frac{1}{\mathfrak{n}}\int_0^{\mathfrak{n}}\partial_{\mathfrak{n}}\varphi\right\rangle\right|\lesssim\left\|\eta\left(f_1^B+S_{2b}+S_{2c}\right)\right\|_{L^2_xL^1_v}\left\|\frac{1}{\mathfrak{n}}\int_0^{\mathfrak{n}}\partial_{\mathfrak{n}}\varphi\right\|_{L^2}\\
&\lesssim\left\|\eta\left(f_1^B+S_{2b}+S_{2c}\right)\right\|_{L^2_xL^1_v}\|\partial_{\mathfrak{n}}\varphi\|_{L^2}\lesssim o_T\|c\|_{L^r}^r+o_T\varepsilon^{\frac{r}{2}}.
\end{aligned}\tag{3.198}$$

From Lemma 3.15, we directly bound

$$\left|\left\langle\nabla_x\varphi\cdot\mathscr{A},S_1\right\rangle\right|\lesssim\|\nabla_x\varphi\|_{L^2}\|S_1\|_{L^2}\lesssim o_T\|c\|_{L^r}^r+o_T\varepsilon^r. \tag{3.199}$$

Similar to (3.185), based on Lemma 3.16, Remark 3.17 and Hardy's inequality, we have

$$\begin{aligned}
&\left|\left\langle\nabla_x\varphi\cdot\mathscr{A},S_2\right\rangle\right|\leq\left|\left\langle S_2,\nabla_x\varphi\Big|_{\mathfrak{n}=0}\right\rangle\right|+\left|\varepsilon\left\langle\eta S_2,\frac{1}{\mathfrak{n}}\int_0^{\mathfrak{n}}\partial_{\mathfrak{n}}\nabla_x\varphi\right\rangle\right|\\
&\lesssim\left\|f_1^B+S_{2b}+S_{2c}\right\|_{L^2_{\iota_1\iota_2}L^1_{\mathfrak{n}}L^1_v}|\nabla_x\varphi|_{L^2_\gamma}+\varepsilon\left\|\eta\left(f_1^B+S_{2b}+S_{2c}\right)\right\|_{L^r}\left\|\frac{1}{\mathfrak{n}}\int_0^{\mathfrak{n}}\partial_{\mathfrak{n}}\nabla_x\varphi\right\|_{L^{\frac{r}{r-1}}}\\
&\lesssim\left\|f_1^B+S_{2b}+S_{2c}\right\|_{L^2_{\iota_1\iota_2}L^1_{\mathfrak{n}}L^1_v}|\nabla_x\varphi|_{L^2_\gamma}+\varepsilon\left\|\eta\left(f_1^B+S_{2b}+S_{2c}\right)\right\|_{L^r}\|\partial_{\mathfrak{n}}\nabla_x\varphi\|_{L^{\frac{r}{r-1}}}\\
&\lesssim o_T\varepsilon^{\frac{2}{r}}\|c\|_{L^r}^{r-1}\lesssim o_T\|c\|_{L^r}^r+o_T\varepsilon^2.
\end{aligned}\tag{3.200}$$

Based on Lemma 3.18, Lemma 3.19, and Lemma 3.20, we have

$$\begin{aligned}
&\left|\left\langle\nabla_x\varphi\cdot\mathscr{A},S_3+S_4+S_5\right\rangle\right|\lesssim\|\nabla_x\varphi\|_{L^2}\|S_3+S_4+S_5\|_{L^2}\\
&\quad\lesssim o_T\|c\|_{L^r}^r+o_T\|R\|_{L^2}^r+o_T\varepsilon^{\frac{r}{2}}\lesssim o_T\|c\|_{L^r}^r+o_T\varepsilon^{\frac{r}{2}}\|R\|_X^r+o_T\varepsilon^{\frac{r}{2}}.
\end{aligned}\tag{3.201}$$

Finally, based on Lemma 3.21, we have

$$\left|\left\langle\nabla_x\varphi\cdot\mathscr{A},S_6\right\rangle\right|\lesssim\left|\left\langle\nabla_x\varphi\cdot\mathscr{A},\Gamma\big[\mathbf{P}[R],\mathbf{P}[R]\big]\right\rangle\right|+\left|\left\langle\nabla_x\varphi\cdot\mathscr{A},\Gamma\big[R,(\mathbf{I}-\mathbf{P})[R]\big]\right\rangle\right|. \tag{3.202}$$

The oddness and orthogonality of the elements in (A.4), combined with the interpolation $\|\mathbf{b}\|_{L^3}\lesssim\|\mathbf{b}\|_{L^2}^{\frac{1}{2}}\|\mathbf{b}\|_{L^6}^{\frac{1}{2}}\lesssim\varepsilon^{\frac{1}{4}}\|R\|_X$, imply that

$$\begin{aligned}
\left|\left\langle\nabla_x\varphi\cdot\mathscr{A},\Gamma\big[\mathbf{P}[R],\mathbf{P}[R]\big]\right\rangle\right|&\lesssim\left|\left\langle\nabla_x\varphi\cdot\mathscr{A},\Gamma\left[\mu^{\frac{1}{2}}(v\cdot\mathbf{b}),\mu^{\frac{1}{2}}\left(\frac{|v|^2-5}{2}c\right)\right]\right\rangle\right|\\
&\lesssim\|\nabla_x\varphi\|_{L^{\frac{3r}{2r-3}}}\|\mathbf{b}\|_{L^3}\|c\|_{L^r}\lesssim\|\varphi\|_{W^{2,\frac{r}{r-1}}}\|\mathbf{b}\|_{L^3}\|c\|_{L^r}\\
&\lesssim\varepsilon^{\frac{1}{4}}\|R\|_X\|c\|_{L^r}^r.
\end{aligned}\tag{3.203}$$

In addition, using the interpolation $\|(\mathbf{I}-\mathbf{P})[R]\|_{L^3} \lesssim \|(\mathbf{I}-\mathbf{P})[R]\|_{L^2_\nu}^{\frac{1}{2}} \|(\mathbf{I}-\mathbf{P})[R]\|_{L^6}^{\frac{1}{2}} \lesssim \varepsilon^{\frac{1}{2}} \|R\|_X$, we have

$$\left|\left\langle \nabla_x\varphi \cdot \mathscr{A}, \Gamma\left[R, (\mathbf{I}-\mathbf{P})[R]\right]\right\rangle\right| \lesssim \|\nabla_x\varphi\|_{L^2} \|R\|_{L^6} \|(\mathbf{I}-\mathbf{P})[R]\|_{L^3}$$
$$\lesssim \varepsilon^{\frac{1}{2}} \|c\|_{L^r}^{r-1} \|R\|_X^2 \lesssim o(1) \|c\|_{L^r}^r + \varepsilon^{\frac{r}{2}} \|R\|_X^{2r}. \tag{3.204}$$

Hence, we know

$$\left|\left\langle \nabla_x\varphi \cdot \mathscr{A}, S_6\right\rangle\right| \lesssim \varepsilon^{\frac{1}{4}} \|R\|_X \|c\|_{L^r}^r + o(1) \|c\|_{L^r}^r + \varepsilon^{\frac{r}{2}} \|R\|_X^{2r}. \tag{3.205}$$

Summarizing, we have found that

$$\left|\varepsilon^{-1}\left\langle \varphi\left(|v|^2-5\right)\mu^{\frac{1}{2}}, S\right\rangle\right| + \left|\left\langle \nabla_x\varphi \cdot \mathscr{A}, S\right\rangle\right|$$
$$\lesssim \varepsilon^{\frac{1}{4}} \|R\|_X \|c\|_{L^r}^r + \left(o(1)+o_T\right) \|c\|_{L^r}^r + o_T\varepsilon^{\frac{r}{2}} \|R\|_X^r + \varepsilon^{\frac{r}{2}} \|R\|_X^{2r} + o_T\left(\varepsilon^{\frac{r}{2}} + \varepsilon^2\right). \tag{3.206}$$

Inserting (3.206) into (3.196), we have

$$\|c\|_{L^r}^r \lesssim \varepsilon^{\frac{1}{4}} \|R\|_X \|c\|_{L^r}^r + \left|\mu^{\frac{1}{4}}R\right|_{L^{\frac{2r}{3}}_{\gamma+}}^r + \|(\mathbf{I}-\mathbf{P})[R]\|_{L^r}^r + o_T\varepsilon^{\frac{r}{2}} \|R\|_X^r$$
$$+ \varepsilon^{\frac{r}{2}} \|R\|_X^{2r} + o_T\left(\varepsilon^{\frac{r}{2}} + \varepsilon^2\right). \tag{3.207}$$

Hence, (3.188) follows. □

3.4.3. *Estimate of* **b**

Proposition 3.31. *Let R be the solution to (1.10). Under the assumption (1.5), we have*

$$\varepsilon^{-\frac{1}{2}} \|\mathbf{b}\|_{L^2} + \|\mathbf{b}\|_{L^6} \lesssim o_T \|R\|_X + \|R\|_X^2 + o_T. \tag{3.208}$$

Proof. It suffices to justify for $2 \le r \le 6$

$$\|\mathbf{b}\|_{L^r} \lesssim \varepsilon^{\frac{1}{4r}} \|R\|_X^{\frac{1}{r}} \|\mathbf{b}\|_{L^r} + \left|\mu^{\frac{1}{4}}R\right|_{L^{\frac{2r}{3}}_{\gamma+}} + \|(\mathbf{I}-\mathbf{P})[R]\|_{L^r} + o_T\varepsilon^{\frac{1}{2}} \|R\|_X + \varepsilon^{\frac{1}{2}} \|R\|_X^2$$
$$+ o_T\left(\varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{2}{r}}\right). \tag{3.209}$$

Weak Formulation Assume $(\psi, q) : \Omega \to \mathbb{R}^3 \times \mathbb{R}$ (where q has zero average) is the unique strong solution to the Stokes problem

$$\begin{cases} -\lambda\Delta_x\psi + \nabla_x q = \mathbf{b}\,|\mathbf{b}|^{r-2} & \text{in } \Omega, \\ \nabla_x \cdot \psi = 0 & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.210}$$

We have the standard estimate [19]

$$\|\psi\|_{W^{2,\frac{r}{r-1}}} + \|q\|_{W^{1,\frac{r}{r-1}}} \lesssim \left\|\mathbf{b}\,|\mathbf{b}|^{r-2}\right\|_{L^{\frac{r}{r-1}}} \lesssim \|\mathbf{b}\|_{L^r}^{r-1}. \tag{3.211}$$

Based on Sobolev embedding and trace estimate, we have for $2 \leq r \leq 6$

$$\|\psi\|_{H^1} + |\nabla_x \psi|_{L^{\frac{2r}{2r-3}}_{\partial\Omega}} + \|q\|_{L^2} + |q|_{L^{\frac{2r}{2r-3}}_{\partial\Omega}} \lesssim \|\mathbf{b}\|_{L^r}^{r-1}. \tag{3.212}$$

Multiplying $\mathbf{b}$ on both sides of (3.210) and integrating by parts for $\langle \nabla_x q, \mathbf{b} \rangle_x$, we have

$$-\langle \lambda \Delta_x \psi, \mathbf{b} \rangle_x - \langle q, \nabla_x \cdot \mathbf{b} \rangle_x + \int_{\partial\Omega} q(\mathbf{b} \cdot n) = \|\mathbf{b}\|_{L^r}^r, \tag{3.213}$$

which, by combining (3.132) and Remark 3.14, implies

$$-\langle \lambda \Delta_x \psi, \mathbf{b} \rangle_x - \left\langle q\mu^{\frac{1}{2}}, S \right\rangle + \langle q\mu^{\frac{1}{2}}, R \rangle_{\gamma_+} - \langle q\mu^{\frac{1}{2}}, h \rangle_{\gamma_-} = \|\mathbf{b}\|_{L^r}^r. \tag{3.214}$$

Inserting (3.214) into (3.151) to replace $-\langle \lambda \Delta_x \psi, \mathbf{b} \rangle_x$, we obtain

$$\begin{aligned} \|\mathbf{b}\|_{L^r}^r = &- \langle q\mu^{\frac{1}{2}}, h \rangle_{\gamma_-} + \langle q\mu^{\frac{1}{2}}, R \rangle_{\gamma_+} + \langle \nabla_x \psi : \mathscr{B}, h \rangle_{\gamma_-} - \langle \nabla_x \psi : \mathscr{B}, R \rangle_{\gamma_+} \\ &+ \left\langle v \cdot \nabla_x \left(\nabla_x \psi : \mathscr{B} \right), (\mathbf{I} - \mathbf{P})[R] \right\rangle - \left\langle q\mu^{\frac{1}{2}}, S \right\rangle + \varepsilon^{-1} \left\langle \psi \cdot v\mu^{\frac{1}{2}}, S \right\rangle + \langle \nabla_x \psi : \mathscr{B}, S \rangle. \end{aligned} \tag{3.215}$$

From Lemma 3.13, we have

$$\begin{aligned} \left| \langle q\mu^{\frac{1}{2}}, h \rangle_{\gamma_-} \right| + \left| \langle \nabla_x \psi : \mathscr{B}, h \rangle_{\gamma_-} \right| &\lesssim \left(|q|_{L^{\frac{2r}{2r-3}}_{\partial\Omega}} + |\nabla_x \psi|_{L^{\frac{2r}{2r-3}}_{\partial\Omega}} \right) |h|_{L^{\frac{2r}{3}}_{\gamma_-}} \\ &\lesssim o_T \|\mathbf{b}\|_{L^r}^r + o_T \varepsilon^3, \end{aligned} \tag{3.216}$$

$$\begin{aligned} \left| \langle q\mu^{\frac{1}{2}}, R \rangle_{\gamma_+} \right| + \left| \langle \nabla_x \psi : \mathscr{B}, R \rangle_{\gamma_+} \right| &\lesssim \left(|q|_{L^{\frac{2r}{2r-3}}_{\partial\Omega}} + |\nabla_x \psi|_{L^{\frac{2r}{2r-3}}_{\partial\Omega}} \right) |R|_{L^{\frac{2r}{3}}_{\gamma_+}} \\ &\lesssim o(1) \|\mathbf{b}\|_{L^r}^r + \left| \mu^{\frac{1}{4}} R \right|_{L^{\frac{2r}{3}}_{\gamma_+}}^r, \end{aligned} \tag{3.217}$$

and

$$\begin{aligned} \left| \left\langle v \cdot \nabla_x \left(\nabla_x \psi : \mathscr{B} \right), (\mathbf{I} - \mathbf{P})[R] \right\rangle \right| &\lesssim \left\| v \cdot \nabla_x \left(\nabla_x \psi : \mathscr{B} \right) \right\|_{L^{\frac{r}{r-1}}} \|(\mathbf{I} - \mathbf{P})[R]\|_{L^r} \\ &\lesssim o(1) \|\mathbf{b}\|_{L^r}^r + \|(\mathbf{I} - \mathbf{P})[R]\|_{L^r}^r. \end{aligned} \tag{3.218}$$

In summary, we have shown that

$$\begin{aligned} \|\mathbf{b}\|_{L^r}^r \lesssim &\left| \mu^{\frac{1}{4}} R \right|_{L^{\frac{2r}{3}}_{\gamma_+}}^r + \|(\mathbf{I} - \mathbf{P})[R]\|_{L^r}^r + o_T \varepsilon^3 + \left| \left\langle q\mu^{\frac{1}{2}}, S \right\rangle \right| \\ &+ \left| \varepsilon^{-1} \left\langle \psi \cdot v\mu^{\frac{1}{2}}, S \right\rangle \right| + \left| \langle \nabla_x \psi : \mathscr{B}, S \rangle \right|. \end{aligned} \tag{3.219}$$

<u>Source Term Estimates</u> Due to orthogonality of Γ and Lemma 3.15, we have

$$\left| \left\langle q\mu^{\frac{1}{2}}, S \right\rangle \right| + \left| \varepsilon^{-1} \left\langle \psi \cdot v\mu^{\frac{1}{2}}, S \right\rangle \right| = \left| \left\langle q, \mu^{\frac{1}{2}} S_2 \right\rangle \right| + \left| \varepsilon^{-1} \left\langle \psi \cdot v\mu^{\frac{1}{2}}, S_2 \right\rangle \right|. \tag{3.220}$$

Using Lemma 3.16 and Remark 3.17, integrating by parts in v_η for S_{2a}, we obtain

$$\left| \left\langle q\mu^{\frac{1}{2}}, S_2 \right\rangle \right| \lesssim \|q\|_{L^2} \left\| f_1^B + S_{2b} + S_{2c} \right\|_{L^2_x L^1_v} \lesssim o_T \varepsilon^{\frac{1}{2}} \|q\|_{L^2} \lesssim o_T \|\mathbf{b}\|_{L^r}^r + o_T \varepsilon^{\frac{r}{2}}. \tag{3.221}$$

Similar to (3.185), we have

$$\begin{aligned}&\left|\varepsilon^{-1}\left\langle\psi\cdot v\mu^{\frac{1}{2}},\,S_2\right\rangle\right|\lesssim\varepsilon^{-1}\left|\left\langle S_2,\,\int_0^{\mathfrak{n}}\partial_{\mathfrak{n}}\psi\right\rangle\right|\lesssim\left|\left\langle\eta S_2,\,\frac{1}{\mathfrak{n}}\int_0^{\mathfrak{n}}\partial_{\mathfrak{n}}\psi\right\rangle\right|\\&\lesssim\left\|\eta\big(f_1^B+S_{2b}+S_{2c}\big)\right\|_{L^2_xL^1_v}\left\|\frac{1}{\mathfrak{n}}\int_0^{\mathfrak{n}}\partial_{\mathfrak{n}}\psi\right\|_{L^2}\\&\lesssim\left\|\eta\big(f_1^B+S_{2b}+S_{2c}\big)\right\|_{L^2_xL^1_v}\left\|\partial_{\mathfrak{n}}\psi\right\|_{L^2}\lesssim o_T\left\|\mathbf{b}\right\|_{L^r}^r+o_T\varepsilon^{\frac{r}{2}}.\end{aligned}\tag{3.222}$$

From Lemma 3.15, we directly bound

$$\left|\left\langle\nabla_x\psi:\mathscr{B},\,S_1\right\rangle\right|\lesssim\left\|\nabla_x\psi\right\|_{L^2}\left\|S_1\right\|_{L^2}\lesssim o_T\left\|\mathbf{b}\right\|_{L^r}^r+o_T\varepsilon^r.\tag{3.223}$$

Similar to (3.185), based on Lemma 3.16, Remark 3.17 and Hardy's inequality, we have

$$\begin{aligned}&\left|\left\langle\nabla_x\psi:\mathscr{B},\,S_2\right\rangle\right|\le\left|\left\langle S_2,\,\nabla_x\psi\Big|_{\mathfrak{n}=0}\right\rangle\right|+\left|\varepsilon\left\langle\eta S_2,\,\frac{1}{\mathfrak{n}}\int_0^{\mathfrak{n}}\partial_{\mathfrak{n}}\nabla_x\psi\right\rangle\right|\\&\lesssim\left\|f_1^B+S_{2b}+S_{2c}\right\|_{L^2_{\iota_1\iota_2}L^1_{\mathfrak{n}}L^1_v}\left|\nabla_x\psi\right|_{L^2_\gamma}+\varepsilon\left\|\eta\big(f_1^B+S_{2b}+S_{2c}\big)\right\|_{L^r}\left\|\frac{1}{\mathfrak{n}}\int_0^{\mathfrak{n}}\partial_{\mathfrak{n}}\nabla_x\psi\right\|_{L^{\frac{r}{r-1}}}\\&\lesssim\left\|f_1^B+S_{2b}+S_{2c}\right\|_{L^2_{\iota_1\iota_2}L^1_{\mathfrak{n}}L^1_v}\left|\nabla_x\psi\right|_{L^2_\gamma}+\varepsilon\left\|\eta\big(f_1^B+S_{2b}+S_{2c}\big)\right\|_{L^r}\left\|\partial_{\mathfrak{n}}\nabla_x\psi\right\|_{L^{\frac{r}{r-1}}}\\&\lesssim o_T\varepsilon^{\frac{2}{r}}\left\|\mathbf{b}\right\|_{L^r}^{r-1}\lesssim o_T\left\|\mathbf{b}\right\|_{L^r}^r+o_T\varepsilon^2.\end{aligned}\tag{3.224}$$

Based on Lemma 3.18, Lemma 3.19, and Lemma 3.20, we have

$$\begin{aligned}\left|\left\langle\nabla_x\psi:\mathscr{B},\,S_3+S_4+S_5\right\rangle\right|&\lesssim\left\|\nabla_x\psi\right\|_{L^2}\left\|S_3+S_4+S_5\right\|_{L^2}\\&\lesssim o_T\left\|\mathbf{b}\right\|_{L^r}^r+o_T\left\|R\right\|_{L^2}^r+o_T\varepsilon^{\frac{r}{2}}\lesssim o_T\left\|\mathbf{b}\right\|_{L^r}^r+o_T\varepsilon^{\frac{r}{2}}\left\|R\right\|_X^r+o_T\varepsilon^{\frac{r}{2}}.\end{aligned}\tag{3.225}$$

Finally, based on Lemma 3.21, we have

$$\left|\left\langle\nabla_x\psi:\mathscr{B},\,S_6\right\rangle\right|\lesssim\left|\left\langle\nabla_x\psi:\mathscr{B},\,\Gamma\Big[\mathbf{P}[R],\mathbf{P}[R]\Big]\right\rangle\right|+\left|\left\langle\nabla_x\psi:\mathscr{B},\,\Gamma\Big[R,(\mathbf{I}-\mathbf{P})[R]\Big]\right\rangle\right|.\tag{3.226}$$

The oddness and orthogonality imply that

$$\begin{aligned}&\left|\left\langle\nabla_x\psi:\mathscr{B},\,\Gamma\Big[\mathbf{P}[R],\mathbf{P}[R]\Big]\right\rangle\right|\\&\lesssim\left|\left\langle\nabla_x\psi:\mathscr{B},\,\Gamma\left[\mu^{\frac{1}{2}}\left(v\cdot\mathbf{b}\right),\mu^{\frac{1}{2}}\left(v\cdot\mathbf{b}\right)\right]\right\rangle\right|\\&\quad+\left|\left\langle\nabla_x\psi:\mathscr{B},\,\Gamma\left[\mu^{\frac{1}{2}}\left(\frac{|v|^2-5}{2}c\right),\mu^{\frac{1}{2}}\left(\frac{|v|^2-5}{2}c\right)\right]\right\rangle\right|.\end{aligned}\tag{3.227}$$

We may directly bound

$$\begin{aligned}\left|\left\langle\nabla_x\psi:\mathscr{B},\,\Gamma\left[\mu^{\frac{1}{2}}\left(v\cdot\mathbf{b}\right),\mu^{\frac{1}{2}}\left(v\cdot\mathbf{b}\right)\right]\right\rangle\right|&\lesssim\left\|\nabla_x\psi\right\|_{L^{\frac{3r}{2r-3}}}\left\|\mathbf{b}\right\|_{L^3}\left\|\mathbf{b}\right\|_{L^r}\\&\lesssim\left\|\psi\right\|_{W^{2,\frac{r}{r-1}}}\left\|\mathbf{b}\right\|_{L^3}\left\|\mathbf{b}\right\|_{L^r}\lesssim\varepsilon^{\frac{1}{4}}\left\|R\right\|_X\left\|\mathbf{b}\right\|_{L^r}^r.\end{aligned}\tag{3.228}$$

Due to oddness and $\mathscr{B}_{ii} = \mathcal{L}^{-1}\left[\left(|v_i|^2 - \frac{1}{3}|v|^2\right)\mu^{\frac{1}{2}}\right]$, noting that $\Gamma\left[\mu^{\frac{1}{2}}\left(\frac{|v|^2-5}{2}c\right),\right.$ $\left.\mu^{\frac{1}{2}}\left(\frac{|v|^2-5}{2}c\right)\right]$ only depends on $|v|^2$, we have

$$\begin{aligned}
&\left|\left\langle \nabla_x\psi : \mathscr{B}, \Gamma\left[\mu^{\frac{1}{2}}\left(\frac{|v|^2-5}{2}c\right), \mu^{\frac{1}{2}}\left(\frac{|v|^2-5}{2}c\right)\right]\right\rangle\right| \\
&= \left|\left\langle \partial_1\psi_1\mathscr{B}_{11} + \partial_2\psi_2\mathscr{B}_{22} + \partial_3\psi_3\mathscr{B}_{33}, \Gamma\left[\mu^{\frac{1}{2}}\left(\frac{|v|^2-5}{2}c\right), \mu^{\frac{1}{2}}\left(\frac{|v|^2-5}{2}c\right)\right]\right\rangle\right| \\
&= \left|\left\langle \big(\nabla_x\cdot\psi\big)\mathscr{B}_{ii}, \Gamma\left[\mu^{\frac{1}{2}}\left(\frac{|v|^2-5}{2}c\right), \mu^{\frac{1}{2}}\left(\frac{|v|^2-5}{2}c\right)\right]\right\rangle\right| = 0. \qquad (3.229)
\end{aligned}$$

In addition, we have

$$\begin{aligned}
\left|\left\langle \nabla_x\psi : \mathscr{B}, \Gamma\Big[R, (\mathbf{I}-\mathbf{P})[R]\Big]\right\rangle\right| &\lesssim \|\nabla_x\psi\|_{L^2}\|R\|_{L^6}\|(\mathbf{I}-\mathbf{P})[R]\|_{L^3} \\
&\lesssim \varepsilon^{\frac{1}{2}}\|\mathbf{b}\|_{L^r}^{r-1}\|R\|_X^2 \lesssim o(1)\|\mathbf{b}\|_{L^r}^r + \varepsilon^{\frac{r}{2}}\|R\|_X^{2r}. \qquad (3.230)
\end{aligned}$$

Hence, we know

$$\left|\left\langle \nabla_x\psi : \mathscr{B}, S_6\right\rangle\right| \lesssim \varepsilon^{\frac{1}{4}}\|R\|_X\|\mathbf{b}\|_{L^r}^r + o(1)\|\mathbf{b}\|_{L^r}^r + \varepsilon^{\frac{r}{2}}\|R\|_X^{2r}. \qquad (3.231)$$

Summarizing the above, we have found that

$$\begin{aligned}
&\left|\left\langle q\mu^{\frac{1}{2}}, S\right\rangle\right| + \left|\varepsilon^{-1}\left\langle \psi\cdot v\mu^{\frac{1}{2}}, S\right\rangle\right| + \left|\left\langle \nabla_x\psi : \mathscr{B}, S\right\rangle\right| \\
&\lesssim \varepsilon^{\frac{1}{4}}\|R\|_X\|\mathbf{b}\|_{L^r}^r + \big(o(1) + o_T\big)\|\mathbf{b}\|_{L^r}^r + o_T\varepsilon^{\frac{r}{2}}\|R\|_X^r + \varepsilon^{\frac{r}{2}}\|R\|_X^{2r} + o_T\left(\varepsilon^{\frac{r}{2}} + \varepsilon^2\right). \qquad (3.232)
\end{aligned}$$

Inserting (3.232) into (3.219), we have

$$\begin{aligned}
\|\mathbf{b}\|_{L^r}^r &\lesssim \varepsilon^{\frac{1}{4}}\|R\|_X\|\mathbf{b}\|_{L^r}^r + \left|\mu^{\frac{1}{4}}R\right|_{L_{\gamma+}^{\frac{2r}{3}}}^r + \|(\mathbf{I}-\mathbf{P})[R]\|_{L^r}^r + o_T\varepsilon^{\frac{r}{2}}\|R\|_X^r + \varepsilon^{\frac{r}{2}}\|R\|_X^{2r} \\
&\quad + o_T\left(\varepsilon^{\frac{r}{2}} + \varepsilon^2\right). \qquad (3.233)
\end{aligned}$$

Hence, (3.209) follows. □

3.4.4. Synthesis of kernel estimates

Proposition 3.32. *Let R be the solution to (1.10). Under the assumption (1.5), we have*

$$\varepsilon^{-\frac{1}{2}}\|\mathbf{P}[R]\|_{L^2} + \|\mathbf{P}[R]\|_{L^6} \lesssim o_T\|R\|_X + \|R\|_X^2 + o_T. \qquad (3.234)$$

Proof. Collecting Proposition 3.29, Proposition 3.30 and Proposition 3.31, we obtain the desired result. □

3.5. L^∞ estimate. We define a weight function scaled with parameters $0 \le \varrho < \frac{1}{2}$ and $\vartheta \ge 0$,

$$w(v) := \langle v\rangle^{\vartheta}\,\mathrm{e}^{\varrho\frac{|v|^2}{2}}. \tag{3.235}$$

Lemma 3.33. *We have*

$$|k(\mathfrak{u},v)| \lesssim \Big(|\mathfrak{u}-v| + |\mathfrak{u}-v|^{-1}\Big)\,\mathrm{e}^{-\frac{1}{4}|\mathfrak{u}-v|^2-\frac{1}{4}\frac{\left||\mathfrak{u}|^2-|v|^2\right|^2}{|\mathfrak{u}-v|^2}}. \tag{3.236}$$

Let $0 \le \varrho < \frac{1}{2}$ and $\vartheta \ge 0$. Then for $\delta > 0$ sufficiently small and any $v \in \mathbb{R}^3$,

$$\int_{\mathbb{R}^3} \mathrm{e}^{\delta|\mathfrak{u}-v|^2}\,|k(\mathfrak{u},v)|\,\frac{\langle v\rangle^{\vartheta}\,\mathrm{e}^{\varrho\frac{|v|^2}{2}}}{\langle \mathfrak{u}\rangle^{\vartheta}\,\mathrm{e}^{\varrho\frac{|\mathfrak{u}|^2}{2}}}\mathrm{d}\mathfrak{u} \lesssim \nu^{-1}. \tag{3.237}$$

Proof. This is a rescaled version of [41, Lemma 3] and [44, Lemma 2.3]. □

Proposition 3.34. *Let R be the solution to (1.10). Under the assumption (1.5), we have*

$$\varepsilon^{\frac{1}{2}}\,\|R\|_{L^\infty_{\varrho,\vartheta}} + \varepsilon^{\frac{1}{2}}\,|R|_{L^\infty_{\gamma+,\varrho,\vartheta}} \lesssim o_T\,\|R\|_X + \|R\|_X^2 + o_T. \tag{3.238}$$

Proof. We will use the well-known $L^2 - L^6 - L^\infty$ framework.

Step 1: Mild Formulation Denote the weighted solution

$$R_w(x,v) := w(v)R(x,v), \tag{3.239}$$

and the weighted non-local operator

$$K_{w(v)}[R_w](v) := w(v)K\left[\frac{R_w}{w}\right](v) = \int_{\mathbb{R}^3} k_{w(v)}(v,\mathfrak{u})R_w(\mathfrak{u})\mathrm{d}\mathfrak{u}, \tag{3.240}$$

where

$$k_{w(v)}(v,\mathfrak{u}) := k(v,\mathfrak{u})\frac{w(v)}{w(\mathfrak{u})}. \tag{3.241}$$

Multiplying εw on both sides of (1.10), we have

$$\begin{cases} \varepsilon v\cdot\nabla_x R_w + \nu R_w = K_w[R_w](x,v) + \varepsilon w(v)S(x,v) \ \text{ in } \ \Omega\times\mathbb{R}^3,\\ R_w(x_0,v) = wh(x_0,v) \ \text{ for } \ x_0\in\partial\Omega \ \text{ and } \ v\cdot n<0, \end{cases} \tag{3.242}$$

We can rewrite the solution of the equation (3.242) along the characteristics by Duhamel's principle as

$$\begin{aligned} R_w(x,v) =& w(v)h(x_b,v)\mathrm{e}^{-\nu(v)t_b} + \int_0^{t_b} w(v)\varepsilon S\Big(x-\varepsilon(t_b-s)v,\,v\Big)\mathrm{e}^{-\nu(v)(t_b-s)}\mathrm{d}s\\ &+ \int_0^{t_b}\int_{\mathbb{R}^3} k_{w(v)}(v,\mathfrak{u})R_w\Big(x-\varepsilon(t_b-s)\mathfrak{u},\,\mathfrak{u}\Big)\mathrm{e}^{-\nu(v)(t_b-s)}\mathrm{d}\mathfrak{u}\mathrm{d}s, \end{aligned} \tag{3.243}$$

where

$$t_b(x,v) := \inf\Big\{t>0 : x-\varepsilon tv\notin\Omega\Big\}, \tag{3.244}$$

and

$$x_b(x,v) := x - \varepsilon t_b(x,v)v \notin \Omega. \tag{3.245}$$

We further rewrite the non-local term along the characteristics as

$$\begin{aligned}
R_w(x,v) =& w(v)h(x_b,v)\mathrm{e}^{-\nu(v)t_b} + \int_0^{t_b} w(v)\varepsilon S\Big(x-\varepsilon(t_b-s)v, v\Big)\mathrm{e}^{-\nu(v)(t_b-s)}\mathrm{d}s\\
&+ \int_0^{t_b}\int_{\mathbb{R}^3} k_{w(v)}(v,\mathfrak{u})w(\mathfrak{u})h(x_b',v)\mathrm{e}^{-\nu(\mathfrak{u})t_b'}\mathrm{e}^{-\nu(v)(t_b-s)}\mathrm{d}\mathfrak{u}\mathrm{d}s\\
&+ \int_0^{t_b}\int_{\mathbb{R}^3} k_{w(v)}(v,\mathfrak{u})\int_0^{t_b'} \varepsilon S\Big(x-\varepsilon(t_b-s)\mathfrak{u}-\varepsilon(t_b'-r)\mathfrak{u},\mathfrak{u}\Big)\\
&\mathrm{e}^{-\nu(\mathfrak{u})(t_b'-r)}\mathrm{e}^{-\nu(v)(t_b-s)}\mathrm{d}r\mathrm{d}\mathfrak{u}\mathrm{d}s\\
&+ \int_0^{t_b}\int_{\mathbb{R}^3} k_{w(v)}(v,\mathfrak{u})\int_0^{t_b'}\int_{\mathbb{R}^3} k_{w(\mathfrak{u})}(\mathfrak{u},\mathfrak{u}')R_w\Big(x-\varepsilon(t_b-s)\mathfrak{u}-\varepsilon(t_b'-r)\mathfrak{u}',\mathfrak{u}'\Big)\\
&\mathrm{e}^{-\nu(\mathfrak{u})(t_b'-r)}\mathrm{e}^{-\nu(v)(t_b-s)}\mathrm{d}\mathfrak{u}'\mathrm{d}r\mathrm{d}\mathfrak{u}\mathrm{d}s,
\end{aligned} \tag{3.246}$$

where

$$t_b'(x,v;s,\mathfrak{u}) := \inf\Big\{t>0 : x-\varepsilon(t_b-s)-\varepsilon t\mathfrak{u}\notin\Omega\Big\}, \tag{3.247}$$

and

$$x_b'(x,v;s,\mathfrak{u}) := x-\varepsilon(t_b-s)-\varepsilon t_b'(x,v;s,\mathfrak{u})\mathfrak{u}\notin\Omega. \tag{3.248}$$

Step 2: Estimates of Source Terms and Boundary Terms Based on Lemma 3.13 – Lemma 3.21, we have

$$\begin{aligned}
&\left|w(v)h(x_b,v)\mathrm{e}^{-\nu(v)t_b}\right| + \left|\int_0^{t_b}\int_{\mathbb{R}^3} k_{w(v)}(v,\mathfrak{u})w(\mathfrak{u})h(x_b',v)\mathrm{e}^{-\nu(\mathfrak{u})t_b'}\mathrm{e}^{-\nu(v)(t_b-s)}\mathrm{d}\mathfrak{u}\mathrm{d}s\right|\\
&\lesssim |h|_{L^\infty_{\gamma_-,\varrho,\vartheta}} \lesssim o_T,
\end{aligned} \tag{3.249}$$

and

$$\begin{aligned}
&\left|\int_0^{t_b} w(v)\varepsilon S\Big(x-\varepsilon(t_b-s)v,v\Big)\mathrm{e}^{-\nu(v)(t_b-s)}\mathrm{d}s\right|\\
&+\left|\int_0^{t_b}\int_{\mathbb{R}^3} k_{w(v)}(v,\mathfrak{u})\int_0^{t_b'}\varepsilon S\Big(x-\varepsilon(t_b-s)\mathfrak{u}-\varepsilon(t_b'-r)\mathfrak{u},\mathfrak{u}\Big)\right.\\
&\left.\mathrm{e}^{-\nu(\mathfrak{u})(t_b'-r)}\mathrm{e}^{-\nu(v)(t_b-s)}\mathrm{d}r\mathrm{d}\mathfrak{u}\mathrm{d}s\right|\\
\lesssim&\varepsilon\left\|\nu^{-1}S\right\|_{L^\infty_{\varrho,\vartheta}} \lesssim o_T + o_T\varepsilon\left\|R\right\|_{L^\infty_{\varrho,\vartheta}} + \varepsilon\left\|R\right\|^2_{L^\infty_{\varrho,\vartheta}} \lesssim o_T\varepsilon^{\frac12}\left\|R\right\|_X + \left\|R\right\|_X^2 + o_T.
\end{aligned} \tag{3.250}$$

Step 3: Estimates of Non-Local Terms The only remaining term in (3.246) is the non-local term

$$I := \int_0^{t_b}\int_{\mathbb{R}^3} k_{w(v)}(v,\mathfrak{u})\int_0^{t_b'}\int_{\mathbb{R}^3} k_{w(\mathfrak{u})}(\mathfrak{u},\mathfrak{u}')R_w\Big(x-\varepsilon(t_b-s)\mathfrak{u}-\varepsilon(t_b'-r)\mathfrak{u}',\mathfrak{u}'\Big)$$

$$\mathrm{e}^{-\nu(\mathfrak{u})(t_b'-r)}\mathrm{e}^{-\nu(v)(t_b-s)}\mathrm{d}\mathfrak{u}'\mathrm{d}r\mathrm{d}\mathfrak{u}\mathrm{d}s, \tag{3.251}$$

which will be estimated in five cases:

$$I := I_1 + I_2 + I_3 + I_4 + I_5. \tag{3.252}$$

In the following, we assume that $\delta \ll 1$ and $N \gg 1$ are constants that will be determined later.

Case I: $I_1 : |v| \geq N$

Based on Lemma 3.33, we have

$$\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} k_{w(v)}(v,\mathfrak{u})k_{w(\mathfrak{u})}(\mathfrak{u},\mathfrak{u}')\mathrm{d}\mathfrak{u}\mathrm{d}\mathfrak{u}'\right| \lesssim \frac{1}{1+|v|} \lesssim \frac{1}{N}. \tag{3.253}$$

Hence, we get

$$|I_1| \lesssim \frac{1}{N}\left\|R_w\right\|_{L^\infty}. \tag{3.254}$$

Case II: $I_2 : |v| \leq N, |\mathfrak{u}| \geq 2N$, or $|\mathfrak{u}| \leq 2N, \left|\mathfrak{u}'\right| \geq 3N$ Notice this implies either $|\mathfrak{u}-v| \geq N$ or $\left|\mathfrak{u}-\mathfrak{u}'\right| \geq N$. Hence, at least one of the following is valid:

$$\left|k_{w(v)}(v,\mathfrak{u})\right| \leq C\mathrm{e}^{-\delta N^2}\left|k_{w(v)}(v,\mathfrak{u})\right|\mathrm{e}^{\delta|v-\mathfrak{u}|^2}, \tag{3.255}$$

$$\left|k_{w(\mathfrak{u})}(\mathfrak{u},\mathfrak{u}')\right| \leq C\mathrm{e}^{-\delta N^2}\left|k_{w(\mathfrak{u})}(\mathfrak{u},\mathfrak{u}')\right|\mathrm{e}^{\delta\left|\mathfrak{u}-\mathfrak{u}'\right|^2}. \tag{3.256}$$

Correspondingly, based on Lemma 3.33, we know

$$\int_{\mathbb{R}^3}\left|k_{w(v)}(v,\mathfrak{u})\right|\mathrm{e}^{\delta|v-\mathfrak{u}|^2}\mathrm{d}\mathfrak{u} < \infty \quad \text{or} \quad \int_{\mathbb{R}^3}\left|k_{w(\mathfrak{u})}(\mathfrak{u},\mathfrak{u}')\right|\mathrm{e}^{\delta\left|\mathfrak{u}-\mathfrak{u}'\right|^2}\mathrm{d}\mathfrak{u}' < \infty. \tag{3.257}$$

Hence, we have

$$|I_2| \lesssim \mathrm{e}^{-\delta N^2}\left\|R_w\right\|_{L^\infty}. \tag{3.258}$$

Case III: $I_3 : t_b'-r \leq \delta$ and $|v| \leq N, |\mathfrak{u}| \leq 2N, \left|\mathfrak{u}'\right| \leq 3N$ In this case, since the integral with respect to r is restricted in a very short interval, there is a small contribution as

$$|I_3| \lesssim \left|\int_{t_b'-\delta}^{t_b'}\mathrm{e}^{-(t_b'-r)}\mathrm{d}r\right|\left\|R_w\right\|_{L^\infty} \lesssim \delta\left\|R_w\right\|_{L^\infty}. \tag{3.259}$$

Case IV: $I_4 : t_b'-r \geq |\ln(\delta)|$ and $|v| \leq N, |\mathfrak{u}| \leq 2N, \left|\mathfrak{u}'\right| \leq 3N$ In this case, $t_b'-r$ is significantly large, so $\mathrm{e}^{-(t_b'-r)} \leq \delta$ is very small. Hence, the contribution is small

$$|I_4| \lesssim \left|\int_{|\ln(\delta)|}^{\infty}\mathrm{e}^{-(t_b'-r)}\mathrm{d}r\right|\left\|R_w\right\|_{L^\infty} \lesssim \delta\left\|R_w\right\|_{L^\infty}. \tag{3.260}$$

Case V: $I_5 : \delta \leq t_b'-r \leq |\ln(\delta)|$ and $|v| \leq N, |\mathfrak{u}| \leq 2N, \left|\mathfrak{u}'\right| \leq 3N$

This is the most complicated case. Since $k_{w(v)}(v,\mathfrak{u})$ has integrable singularity of type $|v-\mathfrak{u}|^{-1}$, we can introduce the truncated kernel $k_N(v,\mathfrak{u})$ which is smooth and has compactly supported range such that

$$\sup_{|v|\leq 3N}\int_{|\mathfrak{u}|\leq 3N}\left|k_N(v,\mathfrak{u})-k_{w(v)}(v,\mathfrak{u})\right|\mathrm{d}\mathfrak{u}\leq\frac{1}{N}. \tag{3.261}$$

Then we can split

$$\begin{aligned}k_{w(v)}(v,\mathfrak{u})k_{w(\mathfrak{u})}(\mathfrak{u},\mathfrak{u}') =&k_N(v,\mathfrak{u})k_N(\mathfrak{u},\mathfrak{u})+\Big(k_{w(v)}(v,\mathfrak{u})-k_N(v,\mathfrak{u})\Big)k_{w(\mathfrak{u})}(\mathfrak{u},\mathfrak{u}')\\&+\Big(k_{w(\mathfrak{u})}(\mathfrak{u},\mathfrak{u}')-k_N(\mathfrak{u},\mathfrak{u}')\Big)k_N(v,\mathfrak{u}).\end{aligned} \tag{3.262}$$

This means that we further split I_5 into

$$I_5:=I_{5,1}+I_{5,2}+I_{5,3}. \tag{3.263}$$

Based on (3.261), we have

$$\left|I_{5,2}\right|\lesssim\frac{1}{N}\left\|R_w\right\|_{L^\infty},\quad \left|I_{5,3}\right|\lesssim\frac{1}{N}\left\|R_w\right\|_{L^\infty}. \tag{3.264}$$

Therefore, the only remaining term is $I_{5,1}$. Note that we always have $x-\varepsilon(t_b-s)v-\varepsilon(t_b'-r)\mathfrak{u}\in\Omega$. Hence, we define the change of variable $\mathfrak{u}\to y$ as $y=(y_1,y_2,y_3)=x-\varepsilon(t_b-s)v-\varepsilon(t_b'-r)\mathfrak{u}$. Then the Jacobian

$$\left|\frac{\mathrm{d}y}{\mathrm{d}\mathfrak{u}}\right|=\left\|\begin{matrix}\varepsilon(t_b'-r)&0&0\\0&\varepsilon(t_b'-r)&0\\0&0&\varepsilon(t_b'-r)\end{matrix}\right\|=\varepsilon^3(t_b'-r)^3\geq\varepsilon^3\delta^3. \tag{3.265}$$

Considering $|v|,|\mathfrak{u}|,\left|\mathfrak{u}'\right|\leq 3N$, we know $|R_w|\simeq|R|$. Also, since k_N is bounded, we estimate

$$\begin{aligned}\left|I_{5,1}\right|\lesssim&\int_{|\mathfrak{u}|\leq 2N}\int_{|\mathfrak{u}'|\leq 3N}\int_0^{t_b'}\mathbf{1}_{\{x-\varepsilon(t_b-s)v-\varepsilon(t_b'-r)\mathfrak{u}\in\Omega\}}\left|R\Big(x-\varepsilon(t_b-s)v-\varepsilon(t_b'-r)\mathfrak{u},\mathfrak{u}'\Big)\right|\\&\mathrm{e}^{-\nu(\mathfrak{u})(t_b'-r)}\mathrm{d}r\mathrm{d}\mathfrak{u}\mathrm{d}\mathfrak{u}'.\end{aligned} \tag{3.266}$$

Using Hölder's inequality, we estimate

$$\begin{aligned}&\int_{|\mathfrak{u}|\leq 2N}\int_{|\mathfrak{u}'|\leq 3N}\int_0^{t_b'}\mathbf{1}_{\{x-\varepsilon(t_b-s)v-\varepsilon(t_b'-r)\mathfrak{u}\in\Omega\}}\left|R\Big(x-\varepsilon(t_b-s)v-\varepsilon(t_b'-r)\mathfrak{u},\mathfrak{u}'\Big)\right|\\&\quad\mathrm{e}^{-\nu(\mathfrak{u})(t_b'-r)}\mathrm{d}r\mathrm{d}\mathfrak{u}\mathrm{d}\mathfrak{u}'\\&\leq\bigg(\int_{|\mathfrak{u}|\leq 2N}\int_{|\mathfrak{u}'|\leq 3N}\int_0^{t_b'}\mathbf{1}_{\{x-\varepsilon(t_b-s)v-\varepsilon(t_b'-r)\mathfrak{u}\in\Omega\}}\mathrm{e}^{-\nu(\mathfrak{u})(t_b'-r)}\mathrm{d}r\mathrm{d}\mathfrak{u}\mathrm{d}\mathfrak{u}'\bigg)^{\frac{5}{6}}\\&\quad\times\bigg(\int_{|\mathfrak{u}|\leq 2N}\int_{|\mathfrak{u}'|\leq 3N}\int_0^{t_b'}\mathbf{1}_{\{x-\varepsilon(t_b-s)v-\varepsilon(t_b'-r)\mathfrak{u}\in\Omega\}}\left|R\Big(x-\varepsilon(t_b-s)v-\varepsilon(t_b'-r)\mathfrak{u},\mathfrak{u}'\Big)\right|^6\\&\quad\mathrm{e}^{-\nu(\mathfrak{u})(t_b'-r)}\mathrm{d}r\mathrm{d}\mathfrak{u}\mathrm{d}\mathfrak{u}'\bigg)^{\frac{1}{6}}\end{aligned}$$

$$\lesssim \left| \int_0^{t'_b} \frac{1}{\varepsilon^3\delta^3} \int_{|\mathfrak{u}'|\leq 3N} \int_\Omega \mathbf{1}_{\{y\in\Omega\}} \left|R(y,\mathfrak{u}')\right|^6 \mathrm{e}^{-(t'_b-r)} \mathrm{d}y\mathrm{d}\mathfrak{u}'\mathrm{d}r \right|^{\frac{1}{6}} \lesssim \frac{1}{\varepsilon^{\frac{1}{2}}\delta^{\frac{1}{2}}} \|R\|_{L^6} . \tag{3.267}$$

Inserting (3.267) into (3.266), we obtain

$$\left|I_{5,1}\right| \lesssim \frac{1}{\varepsilon^{\frac{1}{2}}\delta^{\frac{1}{2}}} \|R\|_{L^6} . \tag{3.268}$$

Combined with (3.264), we know

$$I_5 \lesssim \frac{1}{N} \|R_w\|_{L^\infty} + \frac{1}{\varepsilon^{\frac{1}{2}}\delta^{\frac{1}{2}}} \|R\|_{L^6} . \tag{3.269}$$

Summarizing all five cases in (3.254),(3.258),(3.259),(3.260),(3.269), we obtain

$$|I| \lesssim \left(\frac{1}{N} + \mathrm{e}^{-\delta N^2} + \delta\right) \|R_w\|_{L^\infty} + \frac{1}{\varepsilon^{\frac{1}{2}}\delta^{\frac{1}{2}}} \|R\|_{L^6} . \tag{3.270}$$

Choosing $\delta \ll 1$ sufficiently small, and then taking N sufficiently large satisfying $N^{-1} \leq \delta$ and $\mathrm{e}^{-\delta N^2} \leq \delta$, we have

$$|I| \lesssim \delta \|R_w\|_{L^\infty} + \frac{1}{\varepsilon^{\frac{1}{2}}\delta^{\frac{1}{2}}} \|R\|_{L^6} . \tag{3.271}$$

Step 4: Synthesis Summarizing all above, we obtain for any $(x, v) \in \overline{\Omega} \times \mathbb{R}^3$,

$$\left|R_w(x,v)\right| \lesssim \delta \|R_w\|_{L^\infty} + \frac{1}{\varepsilon^{\frac{1}{2}}\delta^{\frac{1}{2}}} \|R\|_{L^6} + o_T\varepsilon^{\frac{1}{2}} \|R\|_X + \|R\|_X^2 + o_T. \tag{3.272}$$

Since $\delta \ll 1$, we obtain

$$\left|R_w(x,v)\right| \lesssim \varepsilon^{-\frac{1}{2}} \|R\|_{L^6} + o_T\varepsilon^{\frac{1}{2}} \|R\|_X + \|R\|_X^2 + o_T, \tag{3.273}$$

and thus the desired result follows from Proposition 3.32. □

3.6. Remainder estimate.

Theorem 3.35. *Let R be the solution to (1.10). Under the assumption (1.5), we have*

$$\|R\|_X \lesssim o_T. \tag{3.274}$$

Proof. Based on Proposition 3.27 and Corollary 3.28, we have

$$\varepsilon^{-\frac{1}{2}} |R|_{L^2_{\gamma+}} + \left|\mu^{\frac{1}{4}}R\right|_{L^4_{\gamma+}} + \varepsilon^{-1} \|(\mathbf{I}-\mathbf{P})[R]\|_{L^2_\nu} + \|(\mathbf{I}-\mathbf{P})[R]\|_{L^6} \lesssim o_T \|R\|_X + \|R\|_X^2 + o_T. \tag{3.275}$$

Based on Proposition 3.32, we have

$$\varepsilon^{-\frac{1}{2}} \|\mathbf{P}[R]\|_{L^2} + \|\mathbf{P}[R]\|_{L^6} \lesssim o_T \|R\|_X + \|R\|_X^2 + o_T. \tag{3.276}$$

Combining both of them, we have

$$\varepsilon^{-\frac{1}{2}} |R|_{L^2_{\gamma+}} + \left|\mu^{\frac{1}{4}}R\right|_{L^4_{\gamma+}} + \varepsilon^{-\frac{1}{2}} \|\mathbf{P}[R]\|_{L^2} + \varepsilon^{-1} \|(\mathbf{I}-\mathbf{P})[R]\|_{L^2_\nu} + \|R\|_{L^6}$$

$$\lesssim o_T \|R\|_X + \|R\|_X^2 + o_T. \tag{3.277}$$

Based on Proposition 3.34, we have

$$\varepsilon^{\frac{1}{2}} \|R\|_{L^\infty_{\varrho,\vartheta}} + \varepsilon^{\frac{1}{2}} |R|_{L^\infty_{\gamma+,\varrho,\vartheta}} \lesssim o_T \|R\|_X + \|R\|_X^2 + o_T. \tag{3.278}$$

Collecting (3.277) and (3.278), we have

$$\|R\|_X \lesssim o_T \|R\|_X + \|R\|_X^2 + o_T. \tag{3.279}$$

Hence, we have

$$\|R\|_X \lesssim \|R\|_X^2 + o_T. \tag{3.280}$$

By a standard iteration/fixed-point argument, our desired result follows. □

Proof of Theorem 1.1. The estimate (1.40) follows from Theorem 3.35. The construction and positivity of $\mathfrak{F}$ based on the expansion (1.7) is standard and we refer to [27,29], so we will focus on the proof of (1.15). From Theorem 3.35, we have

$$\varepsilon^{-\frac{1}{2}} \|\mathbf{P}[R]\|_{L^2} + \varepsilon^{-1} \|(\mathbf{I}-\mathbf{P})[R]\|_{L^2_\nu} \lesssim o_T, \tag{3.281}$$

which yields

$$\|R\|_{L^2} \lesssim o_T \varepsilon^{\frac{1}{2}}. \tag{3.282}$$

From (1.7), we know

$$\left\|\mu^{-\frac{1}{2}}\mathfrak{F} - \mu^{\frac{1}{2}} - \varepsilon f_1 - \varepsilon^2 f_2 - \varepsilon f_1^B\right\|_{L^2} = \|\varepsilon R\|_{L^2} \lesssim o_T \varepsilon^{\frac{3}{2}}. \tag{3.283}$$

From Theorem 3.9 and the rescaling $\eta = \varepsilon^{-1}\mathfrak{n}$, we have

$$\left\|\varepsilon^2 f_2\right\|_{L^2} \lesssim o_T \varepsilon^2, \quad \left\|\varepsilon f_1^B\right\|_{L^2} \lesssim o_T \varepsilon^{\frac{3}{2}}. \tag{3.284}$$

Hence, we have

$$\left\|\mu^{-\frac{1}{2}}\mathfrak{F} - \mu^{\frac{1}{2}} - \varepsilon f_1\right\|_{L^2} \lesssim o_T \varepsilon^{\frac{3}{2}}. \tag{3.285}$$

Therefore, (1.15) follows. □

4. Evolutionary Problem

4.1. Asymptotic analysis.

4.1.1. Interior solution The derivation of the interior solution is classical. We refer to [36,71,72] and the references therein. By inserting (1.33) into (1.21) and comparing the order of ε, we require that

$$0 = 2\mu^{-\frac{1}{2}} Q^*\big[\mu, \mu^{\frac{1}{2}} f_1\big], \tag{4.1}$$

$$v \cdot \nabla_x f_1 = 2\mu^{-\frac{1}{2}} Q^*\big[\mu, \mu^{\frac{1}{2}} f_2\big] + \mu^{-\frac{1}{2}} Q^*\big[\mu^{\frac{1}{2}} f_1, \mu^{\frac{1}{2}} f_1\big], \tag{4.2}$$

which are equivalent to

$$\mathcal{L}[f_1] = 0, \tag{4.3}$$

$$v \cdot \nabla_x f_1 + \mathcal{L}[f_2] = \Gamma[f_1, f_1]. \tag{4.4}$$

Considering the further expansion, we additionally require

$$\partial_t f_1 + v \cdot \nabla_x f_2 \perp \mathcal{N}. \tag{4.5}$$

Hence, we conclude

$$f_1(t,x,v) = \mu^{\frac{1}{2}}(v)\left(\rho_1(t,x) + v\cdot \mathbf{u}_1(t,x) + \frac{|v|^2-3}{2} T_1(t,x)\right), \tag{4.6}$$

where $(\rho_1, \mathbf{u}_1, T_1)$ satisfies the incompressible Navier–Stokes–Fourier system (3.8).
Also, we have

$$\begin{aligned} f_2(t,x,v) =\; & \mu^{\frac{1}{2}}(v)\left(\rho_2(t,x) + v\cdot \mathbf{u}_2(t,x) + \frac{|v|^2-3}{2} T_2(t,x)\right) \\ & + \mu^{\frac{1}{2}}(v)\left(\rho_1(v\cdot\mathbf{u}_1) + \left(\rho_1 T_1 + \frac{|v|^2-3}{2}|\mathbf{u}_1|^2\right)\right) \\ & + \mathcal{L}^{-1}\Big[-v\cdot\nabla_x f_1 + \Gamma[f_1,f_1]\Big] \end{aligned} \tag{4.7}$$

where $(\rho_2, \mathbf{u}_2, T_2)$ satisfies the fluid system

$$\begin{cases} \rho_2 + T_2 + \rho_1 T_1 = \mathfrak{p}_1, \\ \partial_t \mathbf{u}_2 + \mathbf{u}_1\cdot\nabla_x\mathbf{u}_2 + (\rho_1\mathbf{u}_1 + \mathbf{u}_2)\cdot\nabla_x\mathbf{u}_1 - \gamma_1\Delta_x\mathbf{u}_2 + \nabla_x\mathfrak{p}_2 \\ \quad = -\gamma_2\nabla_x\cdot\Delta_x T_1 - \gamma_4\nabla_x\cdot\Big(T_1\big(\nabla_x\mathbf{u}_1 + (\nabla_x\mathbf{u}_1)^T\big)\Big), \\ \nabla_x\cdot\mathbf{u}_2 = -\mathbf{u}_1\cdot\nabla_x\rho_1, \\ \partial_t T_2 + \mathbf{u}_1\cdot\nabla_x T_2 + (\rho_1\mathbf{u}_1 + \mathbf{u}_2)\cdot\nabla_x T_1 - \mathbf{u}_1\cdot\nabla_x\mathfrak{p}_1 \\ \quad = \gamma_1\Big(\nabla_x\mathbf{u}_1 + (\nabla_x\mathbf{u}_1)^T\Big)^2 + \Delta_x\big(\gamma_2 T_2 + \gamma_5 T_1^2\big). \end{cases} \tag{4.8}$$

4.1.2. Boundary layer We define a cutoff boundary layer f_1^B as (3.51). Denote

$$f_1^B(t,\mathfrak{x},\mathfrak{v}) = \overline{\chi}\left(\varepsilon^{-1}v_\eta\right)\chi(\varepsilon\eta)\overline{\Phi}(t,\mathfrak{x},\mathfrak{v}). \tag{4.9}$$

We may verify that f_1^B satisfies

$$v_\eta\frac{\partial f_1^B}{\partial\eta}+\mathcal{L}\left[f_1^B\right]=v_\eta\overline{\chi}(\varepsilon^{-1}v_\eta)\frac{\partial\chi(\varepsilon\eta)}{\partial\eta}\overline{\Phi}+\chi(\varepsilon\eta)\left(\overline{\chi}(\varepsilon^{-1}v_\eta)K\left[\overline{\Phi}\right]-K\left[\overline{\chi}(\varepsilon^{-1}v_\eta)\overline{\Phi}\right]\right), \tag{4.10}$$

with

$$f_1^B(t,0,\iota_1,\iota_2,\mathfrak{v}) = \overline{\chi}\left(\varepsilon^{-1}v_\eta\right)\left(\mathfrak{f}_b(t,\iota_1,\iota_2,\mathfrak{v})-\Phi_\infty(t,\iota_1,\iota_2,\mathfrak{v})\right)\ \text{ for }\ v_\eta>0. \tag{4.11}$$

4.1.3. Matching procedure The construction of boundary layer and the boundary condition of the interior solution is exactly the same as in Sect. 3.1.4, so we only discuss the initial condition of the interior solution.

Using (1.25), we require the matching condition for $t=0$:

$$\rho_1\Big|_{t=0}=\rho^I,\quad \mathbf{u}_1\Big|_{t=0}=\mathbf{u}^I,\quad T_1\Big|_{t=0}=T^I. \tag{4.12}$$

By standard fluid theory [13,19] for the unsteady Navier–Stokes equations (3.8), we have for any $s\in[2,\infty)$

$$\|\rho_1\|_{W_t^{1,\infty}W_x^{3,s}}+\|\mathbf{u}_1\|_{W_t^{1,\infty}W_x^{3,s}}+\|T_1\|_{W_t^{1,\infty}W_x^{3,s}}\lesssim o_T. \tag{4.13}$$

Also, for f_2, since there is no initial layer, we may simply take

$$\rho_2\Big|_{t=0}=0,\quad \mathbf{u}_2\Big|_{t=0}=\mathbf{0},\quad T_2\Big|_{t=0}=0. \tag{4.14}$$

By standard fluid theory [13,19] for the linear unsteady Navier–Stokes equations (4.8), we have for any $s\in[2,\infty)$

$$\|\rho_2\|_{W_t^{1,\infty}W_x^{2,s}}+\|\mathbf{u}_2\|_{W_t^{1,\infty}W_x^{2,s}}+\|T_2\|_{W_t^{1,\infty}W_x^{2,s}}\lesssim o_T. \tag{4.15}$$

Theorem 4.1. *Under the assumptions (1.23),(1.28),(1.31), there exists a unique solution* $(\rho_1,\mathbf{u}_1,T_1)$ *to the unsteady Navier–Stokes equations (3.8) and* $(\rho_2,\mathbf{u}_2,T_2)$ *to (4.8) satisfying for any* $s\in[2,\infty)$

$$\|\rho_1\|_{W_t^{1,\infty}W_x^{3,s}}+\|\mathbf{u}_1\|_{W_t^{1,\infty}W_x^{2,s}}+\|T_1\|_{W_t^{1,\infty}W_x^{3,s}}\lesssim o_T, \tag{4.16}$$

$$\|\rho_2\|_{W_t^{1,\infty}W_x^{2,s}}+\|\mathbf{u}_2\|_{W_t^{1,\infty}W_x^{2,s}}+\|T_2\|_{W_t^{1,\infty}W_x^{2,s}}\lesssim o_T. \tag{4.17}$$

Thus, we can construct f_1, f_2 *and* f_1^B *such that*

$$|||f_1|||_{W_t^{1,\infty}W_x^{3,s}L^\infty_{v,\varrho,\vartheta}}+\|f_1\|_{W^{1,\infty}W^{3-\frac{1}{s},s}L^\infty_{\overline{\gamma},\varrho,\vartheta}}\lesssim o_T, \tag{4.18}$$

$$|||f_2|||_{W_t^{1,\infty}W_x^{2,s}L^\infty_{v,\varrho,\vartheta}}+\|f_2\|_{W_t^{1,\infty}W_x^{2-\frac{1}{s},s}L^\infty_{\overline{\gamma},\varrho,\vartheta}}\lesssim o_T, \tag{4.19}$$

and for some $K_0>0$ *and any* $0<s\leq 3$

$$\left|\left|\left|\mathrm{e}^{K_0\eta}f_1^B\right|\right|\right|_{W_t^{1,\infty}L^\infty_{x,v,\varrho,\vartheta}}+\left|\left|\left|\mathrm{e}^{K_0\eta}\frac{\partial^s f_1^B}{\partial\iota_1^s}\right|\right|\right|_{W_t^{1,\infty}L^\infty_{x,v,\varrho,\vartheta}}+\left|\left|\left|\mathrm{e}^{K_0\eta}\frac{\partial^s f_1^B}{\partial\iota_2^s}\right|\right|\right|_{W_t^{1,\infty}L^\infty_{x,v,\varrho,\vartheta}}\lesssim o_T. \tag{4.20}$$

4.2. Remainder equation. Inserting (2.6) into (1.21), we have

$$\begin{aligned}&\varepsilon\partial_t\Big(\mu+f+f^B+\varepsilon\mu^{\frac12}R\Big)+v\cdot\nabla_x\Big(\mu+f+f^B+\varepsilon\mu^{\frac12}R\Big)\\&=\varepsilon^{-1}Q^*\Big[\mu+f+f^B+\varepsilon\mu^{\frac12}R,\mu+f+f^B+\varepsilon\mu^{\frac12}R\Big],\end{aligned}\tag{4.21}$$

or equivalently

$$\begin{aligned}&\varepsilon\partial_t R+v\cdot\nabla_x R-2\varepsilon^{-1}\mu^{-\frac12}Q^*\big[\mu,\mu^{\frac12}R\big]\\&=-\mu^{-\frac12}\partial_t\Big(f+f^B\Big)-\varepsilon^{-1}\mu^{-\frac12}\Big(v\cdot\nabla_x\Big(f+f^B\Big)\Big)+\mu^{-\frac12}Q^*\Big[\mu^{\frac12}R,\mu^{\frac12}R\Big]\\&+2\varepsilon^{-1}\mu^{-\frac12}Q^*\Big[f+f^B,\mu^{\frac12}R\Big]+\varepsilon^{-2}\mu^{-\frac12}Q^*\Big[\mu+f+f^B,\mu+f+f^B\Big].\end{aligned}\tag{4.22}$$

Also, we have the initial and boundary conditions

$$\Big(\mu+f+\varepsilon\mu^{\frac12}R\Big)\Big|_{t=0}=\mu+\varepsilon\mu^{\frac12}\mathfrak{f}_i,\qquad\Big(\mu+f+f^B+\varepsilon\mu^{\frac12}R\Big)\Big|_{\overline{\gamma}_-}=\mu+\varepsilon\mu^{\frac12}\mathfrak{f}_b,\tag{4.23}$$

which are equivalent to

$$R\big|_{t=0}=\mathfrak{f}_i-\varepsilon^{-1}\mu^{-\frac12}f,\qquad R\big|_{\overline{\gamma}_-}=\mathfrak{f}_b-\varepsilon^{-1}\mu^{-\frac12}\big(f+f^B\big).\tag{4.24}$$

Note that due to the compatibility condition (1.31), the boundary layer has no influence on the initial data.

Therefore, we need to consider the remainder equation (1.35). Here the initial data is given by

$$z:=-\varepsilon f_2,\tag{4.25}$$

the boundary data is given by

$$h=-\varepsilon f_2+\chi\big(\varepsilon^{-1}v_\eta\big)\overline{\Phi},\tag{4.26}$$

and

$$S:=S_1+S_2+S_3+S_4+S_5+S_6,\tag{4.27}$$

where

$$S_1:=-\varepsilon\partial_t f_1-\varepsilon^2\partial_t f_2-\varepsilon v\cdot\nabla_x f_2,\tag{4.28}$$

$$\begin{aligned}S_2:=&-\varepsilon\partial_t f_1^B+\frac{1}{\mathcal{R}_1-\varepsilon\eta}\left(v_{\iota_1}^2\frac{\partial f_1^B}{\partial v_\eta}-v_\eta v_{\iota_1}\frac{\partial f_1^B}{\partial v_{\iota_1}}\right)+\frac{1}{\mathcal{R}_2-\varepsilon\eta}\left(v_{\iota_2}^2\frac{\partial f_1^B}{\partial v_\eta}-v_\eta v_{\iota_2}\frac{\partial f_1^B}{\partial v_{\iota_2}}\right)\\&-\frac{1}{L_1L_2}\left(\frac{\mathcal{R}_1\partial_{\iota_1\iota_1}\mathbf{r}\cdot\partial_{\iota_2}\mathbf{r}}{L_1(\mathcal{R}_1-\varepsilon\eta)}v_{\iota_1}v_{\iota_2}+\frac{\mathcal{R}_2\partial_{\iota_1\iota_2}\mathbf{r}\cdot\partial_{\iota_2}\mathbf{r}}{L_2(\mathcal{R}_2-\varepsilon\eta)}v_{\iota_2}^2\right)\frac{\partial f_1^B}{\partial v_{\iota_1}}\\&-\frac{1}{L_1L_2}\left(\frac{\mathcal{R}_2\partial_{\iota_2\iota_2}\mathbf{r}\cdot\partial_{\iota_1}\mathbf{r}}{L_2(\mathcal{R}_2-\varepsilon\eta)}v_{\iota_1}v_{\iota_2}+\frac{\mathcal{R}_1\partial_{\iota_1\iota_2}\mathbf{r}\cdot\partial_{\iota_1}\mathbf{r}}{L_1(\mathcal{R}_1-\varepsilon\eta)}v_{\iota_1}^2\right)\frac{\partial f_1^B}{\partial v_{\iota_2}}\\&-\left(\frac{\mathcal{R}_1v_{\iota_1}}{L_1(\mathcal{R}_1-\varepsilon\eta)}\frac{\partial f_1^B}{\partial\iota_1}+\frac{\mathcal{R}_2v_{\iota_2}}{L_2(\mathcal{R}_2-\varepsilon\eta)}\frac{\partial f_1^B}{\partial\iota_2}\right)\end{aligned}\tag{4.29}$$

$$+\varepsilon^{-1}v_\eta\overline{\chi}(\varepsilon^{-1}v_\eta)\frac{\partial\chi(\varepsilon\eta)}{\partial\eta}\overline{\Phi}-\varepsilon^{-1}\Big(K\left[\overline{\Phi}\right]\chi(\varepsilon^{-1}v_\eta)\chi(\varepsilon\eta)-K\Big[\overline{\Phi}\chi(\varepsilon^{-1}v_\eta)\chi(\varepsilon\eta)\Big]\Big),$$

$$S_3:=2\mu^{-\frac{1}{2}}Q^*\left[\mu^{\frac{1}{2}}f_1+\varepsilon\mu^{\frac{1}{2}}f_2,\mu^{\frac{1}{2}}R\right]=2\Gamma[f_1+\varepsilon f_2,R], \tag{4.30}$$

$$S_4:=2\mu^{-\frac{1}{2}}Q^*\left[\mu^{\frac{1}{2}}f_1^B,\mu^{\frac{1}{2}}R\right]=2\Gamma\left[f_1^B,R\right], \tag{4.31}$$

$$\begin{aligned}S_5:=&\,\varepsilon\mu^{-\frac{1}{2}}Q^*\left[\mu^{\frac{1}{2}}f_2,\mu^{\frac{1}{2}}\big(2f_1+\varepsilon f_2\big)\right]+2\mu^{-\frac{1}{2}}Q^*\left[\mu^{\frac{1}{2}}\big(2f_1+2\varepsilon f_2+f_1^B\big),\mu^{\frac{1}{2}}f_1^B\right]\\ &=\varepsilon\Gamma\left[f_2,2f_1+\varepsilon f_2\right]+2\Gamma\left[2f_1+2\varepsilon f_2+f_1^B,f_1^B\right],\end{aligned} \tag{4.32}$$

$$S_6:=\mu^{-\frac{1}{2}}Q^*\left[\mu^{\frac{1}{2}}R,\mu^{\frac{1}{2}}R\right]=\Gamma[R,R]. \tag{4.33}$$

In particular, we may further split S_2:

$$S_{2a}:=\frac{1}{\mathcal{R}_1-\varepsilon\eta}\left(v_{\iota_1}^2\frac{\partial f_1^B}{\partial v_\eta}\right)+\frac{1}{\mathcal{R}_2-\varepsilon\eta}\left(v_{\iota_2}^2\frac{\partial f_1^B}{\partial v_\eta}\right), \tag{4.34}$$

$$\begin{aligned}S_{2b}:=&-\varepsilon\partial_t f_1^B-\frac{1}{\mathcal{R}_1-\varepsilon\eta}\left(v_\eta v_{\iota_1}\frac{\partial f_1^B}{\partial v_{\iota_1}}\right)-\frac{1}{\mathcal{R}_2-\varepsilon\eta}\left(v_\eta v_{\iota_2}\frac{\partial f_1^B}{\partial v_{\iota_2}}\right)\\ &-\frac{1}{L_1L_2}\left(\frac{\mathcal{R}_1\partial_{\iota_1\iota_1}\mathbf{r}\cdot\partial_{\iota_2}\mathbf{r}}{L_1(\mathcal{R}_1-\varepsilon\eta)}v_{\iota_1}v_{\iota_2}+\frac{\mathcal{R}_2\partial_{\iota_1\iota_2}\mathbf{r}\cdot\partial_{\iota_2}\mathbf{r}}{L_2(\mathcal{R}_2-\varepsilon\eta)}v_{\iota_2}^2\right)\frac{\partial f_1^B}{\partial v_{\iota_1}}\\ &-\frac{1}{L_1L_2}\left(\frac{\mathcal{R}_2\partial_{\iota_2\iota_2}\mathbf{r}\cdot\partial_{\iota_1}\mathbf{r}}{L_2(\mathcal{R}_2-\varepsilon\eta)}v_{\iota_1}v_{\iota_2}+\frac{\mathcal{R}_1\partial_{\iota_1\iota_2}\mathbf{r}\cdot\partial_{\iota_1}\mathbf{r}}{L_1(\mathcal{R}_1-\varepsilon\eta)}v_{\iota_1}^2\right)\frac{\partial f_1^B}{\partial v_{\iota_2}}\\ &-\left(\frac{\mathcal{R}_1v_{\iota_1}}{L_1(\mathcal{R}_1-\varepsilon\eta)}\frac{\partial f_1^B}{\partial\iota_1}+\frac{\mathcal{R}_2v_{\iota_2}}{L_2(\mathcal{R}_2-\varepsilon\eta)}\frac{\partial f_1^B}{\partial\iota_2}\right)+\varepsilon^{-1}v_\eta\overline{\chi}(\varepsilon^{-1}v_\eta)\frac{\partial\chi(\varepsilon\eta)}{\partial\eta}\overline{\Phi},\end{aligned} \tag{4.35}$$

$$S_{2c}:=-\varepsilon^{-1}\Big(K\left[\overline{\Phi}\right]\chi(\varepsilon^{-1}v_\eta)\chi(\varepsilon\eta)-K\Big[\overline{\Phi}\chi(\varepsilon^{-1}v_\eta)\chi(\varepsilon\eta)\Big]\Big). \tag{4.36}$$

We also consider the time derivative of the remainder equation

$$\begin{cases}\varepsilon\partial_t\big(\partial_tR\big)+v\cdot\nabla_x\big(\partial_tR\big)+\varepsilon^{-1}\mathcal{L}[\partial_tR]=\partial_tS\ \text{ in }\ \mathbb{R}_+\times\Omega\times\mathbb{R}^3,\\ \partial_tR(0,x,v)=\partial_tz(x,v)\ \text{ in }\ \Omega\times\mathbb{R}^3,\\ \partial_tR(t,x_0,v)=\partial_th(t,x_0,v)\ \text{ for }\ v\cdot n<0\ \text{ and }\ x_0\in\partial\Omega.\end{cases} \tag{4.37}$$

Here the initial data $\partial_t z$ is solved from (2.6) and Remark 1.6:

$$\partial_tz:=\partial_tR\big|_{t=0}=\Big(\varepsilon^{-1}\mu^{-\frac{1}{2}}\partial_t\mathfrak{F}-\partial_tf_1-\varepsilon\partial_tf_2\Big)\Big|_{t=0}. \tag{4.38}$$

For any fixed $t\in\mathbb{R}_+$, we may also rewrite (1.35) as a stationary remainder equation

$$\begin{cases}v\cdot\nabla_xR(t)+\varepsilon^{-1}\mathcal{L}[R(t)]=S(t)-\varepsilon\partial_tR(t)\ \text{ in }\ \Omega\times\mathbb{R}^3,\\ R(t,x_0,v)=h(t,x_0,v)\ \text{ for }\ v\cdot n<0\ \text{ and }\ x_0\in\partial\Omega.\end{cases} \tag{4.39}$$

Lemma 4.2 (Green's Identity, Lemma 2.2 of [26]). *Assume $f(t,x,v),\ g(t,x,v)\in L^\infty([0,T];L^2(\Omega\times\mathbb{R}^3))$ and $\partial_tf+v\cdot\nabla_xf,\ \partial_tg+v\cdot\nabla_xg\in L^2([0,T]\times\Omega\times\mathbb{R}^3)$ with $f,\ g\in L^2_\gamma$. Then for almost all $t,s\in[0,T]$*

$$\begin{aligned}&\int_s^t\iint_{\Omega\times\mathbb{R}^3}\big(\partial_tf+v\cdot\nabla_xf\big)g+\int_s^t\iint_{\Omega\times\mathbb{R}^3}\big(\partial_tg+v\cdot\nabla_xg\big)f\\ &=\iint_{\Omega\times\mathbb{R}^3}f(t)g(t)-\iint_{\Omega\times\mathbb{R}^3}f(s)g(s)+\int_s^t\int_\gamma fg(v\cdot n).\end{aligned} \tag{4.40}$$

Using Lemma 4.2, we can derive the weak formulation of (1.35). For any test function $g(t,x,v) \in L^\infty([0,T]; L^2_\nu(\Omega\times\mathbb{R}^3))$ with $\partial_t g + v\cdot\nabla_x g \in L^2([0,T]\times\Omega\times\mathbb{R}^3)$ with $g \in L^2_{\overline{\gamma}}$, we have

$$\varepsilon\big\langle R(t), g(t)\big\rangle - \varepsilon\big\langle z, g(0)\big\rangle - \varepsilon\langle\!\langle R, \partial_t g\rangle\!\rangle + \iint_{\overline{\gamma}} Rg(v\cdot n) - \langle\!\langle v\cdot\nabla_x g, R\rangle\!\rangle$$
$$+\,\varepsilon^{-1}\langle\!\langle \mathcal{L}[R], g\rangle\!\rangle = \langle\!\langle S, g\rangle\!\rangle. \tag{4.41}$$

4.2.1. Estimates of initial, boundary and source terms The estimates below follow from analogous argument as in [29, Section 4] and Sect. 3.2.1, so we omit the details and only highlight the key differences. In particular, for S_1–S_6 estimates, we need both the accumulative L^p_t and instantaneous L^∞_t versions. To present them uniformly, we only write down the estimates for each fixed t (which is similar to those in Sect. 3.2.1) and simply ignore the t variable. Then the instantaneous estimates (by taking $\sup_{t\in[0,\mathfrak{T}]}$) and the accumulative estimates (by integrating over $t\in[0,\mathfrak{T}]$) will naturally follow.

Estimates of z

Lemma 4.3. *Under the assumptions (1.23),(1.28),(1.31), for z defined in (4.25), we have*

$$\|z\|_{L^2} \lesssim o_T\varepsilon, \qquad \|z\|_{L^\infty_{\varrho,\vartheta}} \lesssim o_T\varepsilon. \tag{4.42}$$

In addition, for $\partial_t z$ defined in (4.38), we have

$$\|\partial_t z\|_{L^2} \lesssim o_T\varepsilon, \qquad \|\partial_t z\|_{L^\infty_{\varrho,\vartheta}} \lesssim o_T\varepsilon. \tag{4.43}$$

Proof. The estimates follow from Remark 1.6 and Theorem 4.1. □

Estimates of h

Lemma 4.4. *Under the assumptions (1.23),(1.28),(1.31), for h defined in (4.26), we have*

$$|h|_{L^2_{\gamma_-}} \lesssim o_T\varepsilon, \quad |h|_{L^{\frac{2r}{3}}_{\gamma_-}} \lesssim o_T\varepsilon^{\frac{3}{r}}, \quad |h|_{L^\infty_{\gamma_-,\varrho,\vartheta}} \lesssim o_T, \quad \sup_{\iota_1,\iota_2}\int_{v\cdot n<0}|h|\,|v\cdot n|\,\mathrm{d}v \lesssim o_T\varepsilon. \tag{4.44}$$

In addition, the estimates in (4.44) still hold with h replaced by $\partial_t h$.

Estimatesof S_1

Lemma 4.5. *Under the assumptions (1.23),(1.28),(1.31), for S_1 defined in (4.28), we have*

$$\big\|\langle v\rangle^2 S_1\big\|_{L^2} \lesssim o_T\varepsilon, \qquad \|S_1\|_{L^r} \lesssim o_T\varepsilon, \qquad \|S_1\|_{L^\infty_{\varrho,\vartheta}} \lesssim o_T\varepsilon. \tag{4.45}$$

Also, we have the property

$$\left\langle \mu^{\frac12}, S_1\right\rangle_v = \varepsilon^2\left\langle \mu^{\frac12}, \partial_t f_2\right\rangle_v, \quad \left\langle \mu^{\frac12}v, S_1\right\rangle_v = \varepsilon^2\left\langle \mu^{\frac12}v, \partial_t f_2\right\rangle_v,$$
$$\left\langle \mu^{\frac12}|v|^2, S_1\right\rangle_v = \varepsilon^2\left\langle \mu^{\frac12}|v|^2, \partial_t f_2\right\rangle_v \tag{4.46}$$

In addition, the estimates in (4.45) still hold with S_1 replaced by $\partial_t S_1$.

Estimates of S_2

Lemma 4.6. *Under the assumptions (1.23),(1.28),(1.31), for S_2 defined in (4.29), we have*

$$\|S_2\|_{L^1} + \|\eta\,(S_{2b}+S_{2c})\|_{L^1} + \left\|\eta^2\,(S_{2b}+S_{2c})\right\|_{L^1} \lesssim o_T\varepsilon, \tag{4.47}$$

$$\left\|\langle v\rangle^2\,S_2\right\|_{L^2} + \|\eta\,(S_{2b}+S_{2c})\|_{L^2} + \left\|\eta^2\,(S_{2b}+S_{2c})\right\|_{L^2} \lesssim o_T, \tag{4.48}$$

$$\|S_2\|_{L^r} + \|\eta\,(S_{2b}+S_{2c})\|_{L^r} + \left\|\eta^2\,(S_{2b}+S_{2c})\right\|_{L^r} \lesssim o_T\varepsilon^{\frac{2}{r}-1}, \tag{4.49}$$

$$\|S_2\|_{L^r_{\iota_1\iota_2}L^1_{\mathfrak{n}}L^1_v} + \|\eta\,(S_{2b}+S_{2c})\|_{L^r_{\iota_1\iota_2}L^1_{\mathfrak{n}}L^1_v} \lesssim o_T\varepsilon,\,, \tag{4.50}$$

and

$$\|S_{2b}+S_{2c}\|_{L^r_xL^1_v} + \|\eta\,(S_{2b}+S_{2c})\|_{L^r_xL^1_v} \lesssim o_T\varepsilon^{\frac{1}{r}}, \tag{4.51}$$

$$|\langle S_{2a},g\rangle| + |\langle \eta S_{2a},g\rangle| + \left|\langle \eta^2 S_{2a},g\rangle\right| \lesssim \left\|\langle v\rangle^2\,f_1^B\right\|_{L^{\frac{r}{r-1}}} \|\nabla_v g\|_{L^r} \lesssim o_T\varepsilon^{1-\frac{1}{r}}\,\|\nabla_v g\|_{L^r}\,. \tag{4.52}$$

Also, we have

$$\|S_2\|_{L^\infty_{\varrho,\vartheta}} \lesssim o_T\varepsilon^{-1}. \tag{4.53}$$

In addition, the estimates in (4.47)–(4.53) still hold with S_2 replaced by $\partial_t S_2$.

Remark 4.7. Notice that the BV estimate in Proposition 3.4 does not contain exponential decay in η, and thus we cannot directly bound ηS_{2a} and $\eta^2 S_{2a}$. Instead, we should first integrate by parts with respect to v_η as in (4.52) to study f_1^B:

$$\left\|f_1^B\right\|_{L^r} + \left\|\eta f_1^B\right\|_{L^r} + \left\|\eta^2 f_1^B\right\|_{L^r} \lesssim o_T\varepsilon^{\frac{2}{r}-1}, \tag{4.54}$$

$$\left\|f_1^B\right\|_{L^r_{\iota_1\iota_2}L^1_{\mathfrak{n}}L^1_v} + \left\|\eta f_1^B\right\|_{L^r_{\iota_1\iota_2}L^1_{\mathfrak{n}}L^1_v} \lesssim o_T\varepsilon,\,, \tag{4.55}$$

$$\left\|f_1^B\right\|_{L^r_xL^1_v} + \left\|\eta f_1^B\right\|_{L^r_xL^1_v} \lesssim o_T\varepsilon^{\frac{1}{r}}. \tag{4.56}$$

Estimates of S_3

Lemma 4.8. *Under the assumptions (1.23),(1.28),(1.31), for S_3 defined in (4.30), we have*

$$\left|\langle S_3,g\rangle_v\right| \lesssim o_T\varepsilon\left(\int_{\mathbb{R}^3}\nu\,|g|^2\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}\nu\,|R|^2\right)^{\frac{1}{2}}, \tag{4.57}$$

and thus

$$|\langle S_3,g\rangle| \lesssim o_T\varepsilon\,\|g\|_{L^2_\nu}\,\|R\|_{L^2_\nu} \lesssim o_T\varepsilon\,\|g\|_{L^2_\nu}\left(\|\mathbf{P}[R]\|_{L^2} + \|(\mathbf{I}-\mathbf{P})[R]\|_{L^2_\nu}\right). \tag{4.58}$$

Also, we have

$$\|S_3\|_{L^2} \lesssim o_T\varepsilon\,\|R\|_{L^2_\nu}\,,\qquad \left\|\nu^{-1}S_3\right\|_{L^\infty_{\varrho,\vartheta}} \lesssim o_T\varepsilon\,\|R\|_{L^\infty_{\varrho,\vartheta}}\,. \tag{4.59}$$

In addition, the estimates in (4.57)–(4.59) still hold with S_3 replaced by $\partial_t S_3$ and R replaced by $\partial_t R$.

Estimates of S_4

Lemma 4.9. *Under the assumptions (1.23),(1.28),(1.31), for S_4 defined in (4.31), we have*

$$\left|\langle S_4, g\rangle_v\right| \lesssim \left(\int_{\mathbb{R}^3} \nu\,|g|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} \nu\left|f_1^B\right|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} \nu\,|R|^2\right)^{\frac{1}{2}}, \tag{4.60}$$

and thus

$$|\langle S_4, g\rangle| \lesssim o_T\,\|g\|_{L^2_\nu}\,\|R\|_{L^2_\nu} \lesssim o_T\,\|g\|_{L^2_\nu}\left(\|\mathbf{P}[R]\|_{L^2} + \|(\mathbf{I}-\mathbf{P})[R]\|_{L^2_\nu}\right), \tag{4.61}$$

$$|\langle S_4, g\rangle| \lesssim o_T\left\|f_1^B\right\|_{L^2_\nu}\|g\|_{L^\infty_{\varrho,\vartheta}}\|R\|_{L^2_\nu} \lesssim o_T\varepsilon^{\frac{1}{2}}\,\|g\|_{L^\infty_{\varrho,\vartheta}}\left(\|\mathbf{P}[R]\|_{L^2} + \|(\mathbf{I}-\mathbf{P})[R]\|_{L^2_\nu}\right). \tag{4.62}$$

Also, we have

$$\|S_4\|_{L^2} \lesssim o_T\,\|R\|_{L^2_\nu}, \qquad \left\|\nu^{-1}S_4\right\|_{L^\infty_{\varrho,\vartheta}} \lesssim o_T\,\|R\|_{L^\infty_{\varrho,\vartheta}}. \tag{4.63}$$

In addition, the estimates in (4.60)–(4.63) still hold with S_4 replaced by $\partial_t S_4$ and R replaced by $\partial_t R$.

Estimates of S_5

Lemma 4.10. *Under the assumptions (1.23),(1.28),(1.31), for S_5 defined in (4.32), we have*

$$\left|\langle S_5, g\rangle_v\right| \lesssim o_T\left(\int_{\mathbb{R}^3} \nu\,|g|^2\right)^{\frac{1}{2}}, \tag{4.64}$$

and thus

$$|\langle S_5, g\rangle| \lesssim o_T\varepsilon^{\frac{1}{2}}\,\|g\|_{L^2_\nu}, \qquad |\langle S_5, g\rangle| \lesssim o_T\varepsilon\,\|g\|_{L^\infty_{\varrho,\vartheta}}. \tag{4.65}$$

Also, we have

$$\|S_5\|_{L^2} \lesssim o_T\varepsilon^{\frac{1}{2}}, \qquad \left\|\nu^{-1}S_5\right\|_{L^\infty_{\varrho,\vartheta}} \lesssim o_T. \tag{4.66}$$

In addition, the estimates in (4.64)–(4.66) still hold with S_5 replaced by $\partial_t S_5$ and R replaced by $\partial_t R$.

Estimates of S_6

Note that $\partial_t\Gamma[R,R] = 2\Gamma[R,\partial_t R]$. Then the proof follows from that of Lemma 3.21.

Lemma 4.11. *Under the assumptions (1.23),(1.28),(1.31), for S_6 defined in (4.33), we have*

$$\left|\langle S_6, g\rangle_v\right| \lesssim \left(\int_{\mathbb{R}^3} \nu\,|g|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} \nu\,|R|^2\right), \tag{4.67}$$

and thus

$$\left|\langle S_6, g\rangle\right| \lesssim \|g\|_{L^2_\nu} \|R\|_{L^2_\nu} \|R\|_{L^\infty_{\varrho,\vartheta}} . \tag{4.68}$$

Also, we have

$$\|S_6\|_{L^2} \lesssim \|R\|_{L^2_\nu} \|R\|_{L^\infty_{\varrho,\vartheta}} , \tag{4.69}$$

$$\left\|\nu^{-1} S_6\right\|_{L^\infty_{\varrho,\vartheta}} \lesssim \|R\|^2_{L^\infty_{\varrho,\vartheta}} . \tag{4.70}$$

In addition, we have

$$\left|\langle \partial_t S_6, g\rangle_v\right| \lesssim \left(\int_{\mathbb{R}^3} \nu\, |g|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} \nu\, |R|\, |\partial_t R|\right), \tag{4.71}$$

and thus

$$\left|\langle \partial_t S_6, g\rangle\right| \lesssim \|g\|_{L^2_\nu} \|\partial_t R\|_{L^2_\nu} \|R\|_{L^\infty_{\varrho,\vartheta}} . \tag{4.72}$$

Also, we have

$$\|\partial_t S_6\|_{L^2} \lesssim \|\partial_t R\|_{L^2_\nu} \|R\|_{L^\infty_{\varrho,\vartheta}} , \tag{4.73}$$

$$\left\|\nu^{-1} \partial_t S_6\right\|_{L^\infty_{\varrho,\vartheta}} \lesssim \|\partial_t R\|_{L^\infty_{\varrho,\vartheta}} \|R\|_{L^\infty_{\varrho,\vartheta}} . \tag{4.74}$$

4.2.2. Conservation laws Classical Conservation Laws

Lemma 4.12. *Let R be the solution to (1.35). Under the assumptions (1.23),(1.28),(1.31), we have the conservation laws*

$$\varepsilon \partial_t \big(p - c\big) + \nabla_x \cdot \mathbf{b} = \left\langle \mu^{\frac{1}{2}}, S_1 + S_2 \right\rangle_v , \tag{4.75}$$

$$\varepsilon \partial_t \mathbf{b} + \nabla_x p + \nabla_x \cdot \varpi = \left\langle v\mu^{\frac{1}{2}}, S_1 + S_2 \right\rangle_v , \tag{4.76}$$

$$\varepsilon \partial_t \big(3p\big) + 5\nabla_x \cdot \mathbf{b} + \nabla_x \cdot \varsigma = \left\langle |v|^2 \mu^{\frac{1}{2}}, S_1 + S_2 \right\rangle_v , \tag{4.77}$$

where ϖ and ς are defined in Lemma 3.22.

Proof. We multiply test functions $\mu^{\frac{1}{2}}, v\mu^{\frac{1}{2}}, |v|^2 \mu^{\frac{1}{2}}$ on both sides of (1.35) and integrate over $v \in \mathbb{R}^3$. Using the orthogonality of $\mathcal{L}$ and noticing

$$\int_{\mathbb{R}^3} \mu^{\frac{1}{2}} R = p - c, \qquad \int_{\mathbb{R}^3} v\mu^{\frac{1}{2}} R = \mathbf{b}, \qquad \int_{\mathbb{R}^3} |v|^2 \mu^{\frac{1}{2}} R = 3p. \tag{4.78}$$

the results follow. □

Conservation Law with Test Function $\nabla_x \varphi \cdot \mathscr{A}$

Lemma 4.13. *Let R be the solution to (1.35). Under the assumptions (1.23),(1.28),(1.31), for smooth test function $\varphi(t,x)$, we have*

$$\begin{aligned}&\varepsilon\big\langle R(t),\nabla_x\varphi(t)\cdot\mathscr{A}\big\rangle-\varepsilon\big\langle z,\nabla_x\varphi(0)\cdot\mathscr{A}\big\rangle-\varepsilon\langle\!\langle R,\partial_t\nabla_x\varphi\cdot\mathscr{A}\rangle\!\rangle-\kappa\langle\!\langle\Delta_x\varphi,c\rangle\!\rangle_{tx}\\&+\varepsilon^{-1}\langle\!\langle\nabla_x\varphi,\varsigma\rangle\!\rangle=\langle\!\langle\nabla_x\varphi\cdot\mathscr{A},h\rangle\!\rangle_{\overline{\gamma}_-}-\langle\!\langle\nabla_x\varphi\cdot\mathscr{A},R\rangle\!\rangle_{\overline{\gamma}_+}\\&+\Big\langle\!\!\Big\langle v\cdot\nabla_x\Big(\nabla_x\varphi\cdot\mathscr{A}\Big),(\mathbf{I}-\mathbf{P})[R]\Big\rangle\!\!\Big\rangle+\langle\!\langle\nabla_x\varphi\cdot\mathscr{A},S\rangle\!\rangle.\end{aligned}\tag{4.79}$$

Proof. Taking test function $g=\nabla_x\varphi\cdot\mathscr{A}$ in (4.41), we obtain

$$\begin{aligned}&\varepsilon\big\langle R(t),\nabla_x\varphi(t)\cdot\mathscr{A}\big\rangle-\varepsilon\big\langle z,\nabla_x\varphi(0)\cdot\mathscr{A}\big\rangle-\varepsilon\langle\!\langle R,\partial_t\nabla_x\varphi\cdot\mathscr{A}\rangle\!\rangle\\&+\iint_{\overline{\gamma}}\Big(\nabla_x\varphi\cdot\mathscr{A}\Big)R\big(v\cdot n\big)-\langle\!\langle v\cdot\nabla_x\Big(\nabla_x\varphi\cdot\mathscr{A}\Big),R\rangle\!\rangle+\varepsilon^{-1}\langle\!\langle\mathcal{L}[R],\nabla_x\varphi\cdot\mathscr{A}\rangle\!\rangle=\langle\!\langle\nabla_x\varphi\cdot\mathscr{A},S\rangle\!\rangle.\end{aligned}\tag{4.80}$$

Then following a similar argument as the proof of Lemma 3.23, we have (4.79). □

Conservation Law with Test Function $\nabla_x\psi:\mathscr{B}$

Lemma 4.14. *Let R be the solution to (1.35). Under the assumptions (1.23),(1.28),(1.31), for smooth test function $\psi(t,x)$ satisfying $\nabla_x\cdot\psi=0$, we have*

$$\begin{aligned}&\varepsilon\big\langle R(t),\nabla_x\psi(t):\mathscr{B}\big\rangle-\varepsilon\big\langle z,\nabla_x\psi(0):\mathscr{B}\big\rangle-\varepsilon\langle\!\langle R,\partial_t\nabla_x\psi:\mathscr{B}\rangle\!\rangle-\lambda\langle\!\langle\Delta_x\psi,\mathbf{b}\rangle\!\rangle_{tx}\\&+\varepsilon^{-1}\langle\!\langle\nabla_x\psi,\varpi\rangle\!\rangle=\langle\!\langle\nabla_x\psi\cdot\mathscr{B},h\rangle\!\rangle_{\overline{\gamma}_-}-\langle\!\langle\nabla_x\psi\cdot\mathscr{B},R\rangle\!\rangle_{\overline{\gamma}_+}\\&+\Big\langle\!\!\Big\langle v\cdot\nabla_x\Big(\nabla_x\psi:\mathscr{B}\Big),(\mathbf{I}-\mathbf{P})[R]\Big\rangle\!\!\Big\rangle+\langle\!\langle\nabla_x\psi:\mathscr{B},S\rangle\!\rangle.\end{aligned}\tag{4.81}$$

Proof. Taking test function $g=\nabla_x\psi:\mathscr{B}$ in (4.41), we obtain

$$\begin{aligned}&\varepsilon\big\langle R(t),\nabla_x\psi(t):\mathscr{B}\big\rangle-\varepsilon\big\langle z,\nabla_x\psi(0):\mathscr{B}\big\rangle-\varepsilon\langle\!\langle R,\partial_t\nabla_x\psi:\mathscr{B}\rangle\!\rangle\\&+\iint_{\overline{\gamma}}\Big(\nabla_x\psi:\mathscr{B}\Big)R\big(v\cdot n\big)-\langle\!\langle v\cdot\nabla_x\Big(\nabla_x\psi:\mathscr{B}\Big),R\rangle\!\rangle+\varepsilon^{-1}\langle\!\langle\mathcal{L}[R],\nabla_x\psi:\mathscr{B}\rangle\!\rangle\\&=\langle\!\langle\nabla_x\psi:\mathscr{B},S\rangle\!\rangle.\end{aligned}\tag{4.82}$$

Then following a similar argument as the proof of Lemma 3.24, we have (4.81). □

Conservation Law with Test Function, $\nabla_x\varphi\cdot\mathscr{A}+\varepsilon^{-1}\varphi\big(|v|^2-5\big)\mu^{\frac{1}{2}}$

Lemma 4.15. *Let R be the solution to (1.35). Under the assumptions (1.23),(1.28),(1.31), for smooth test function $\varphi(t,x)$ satisfying $\varphi\big|_{\partial\Omega}=0$, we have*

$$\begin{aligned}&\big\langle 5c(t)-2p(t),\varphi(t)\big\rangle_x-\big\langle 5c(0)-2p(0),\varphi(0)\big\rangle_x-\langle\!\langle 5c-2p,\partial_t\varphi\rangle\!\rangle_{tx}\\&+\varepsilon\big\langle R(t),\nabla_x\varphi(t)\cdot\mathscr{A}\big\rangle-\varepsilon\big\langle z,\nabla_x\varphi(0)\cdot\mathscr{A}\big\rangle-\varepsilon\langle\!\langle R,\partial_t\nabla_x\varphi\cdot\mathscr{A}\rangle\!\rangle-\kappa\langle\!\langle\Delta_x\varphi,c\rangle\!\rangle_{tx}\\=&\langle\!\langle\nabla_x\varphi\cdot\mathscr{A},h\rangle\!\rangle_{\overline{\gamma}_-}-\langle\!\langle\nabla_x\varphi\cdot\mathscr{A},R\rangle\!\rangle_{\overline{\gamma}_+}+\Big\langle\!\!\Big\langle v\cdot\nabla_x\Big(\nabla_x\varphi\cdot\mathscr{A}\Big),(\mathbf{I}-\mathbf{P})[R]\Big\rangle\!\!\Big\rangle\\&+\varepsilon^{-1}\Big\langle\!\!\Big\langle\varphi\Big(|v|^2-5\Big)\mu^{\frac{1}{2}},S\Big\rangle\!\!\Big\rangle+\langle\!\langle\nabla_x\varphi\cdot\mathscr{A},S\rangle\!\rangle.\end{aligned}\tag{4.83}$$

Proof. From (4.75) and (4.77), we have

$$\varepsilon\partial_t\big(5c-2p\big)+\nabla_x\cdot\varsigma=\Big\langle\Big(|v|^2-5\Big)\mu^{\frac{1}{2}},S\Big\rangle_v. \tag{4.84}$$

Multiplying $\varphi(t,x)\in\mathbb{R}$ on both sides of (4.84) and integrating over $[0,t]\times\Omega$, we obtain

$$\begin{aligned}&\varepsilon\big\langle 5c(t)-2p(t),\varphi(t)\big\rangle_x-\varepsilon\big\langle 5c(0)-2p(0),\varphi(0)\big\rangle_x-\varepsilon\langle\!\langle 5c-2p,\partial_t\varphi\rangle\!\rangle_{tx}\\&\quad-\langle\!\langle\nabla_x\varphi,\varsigma\rangle\!\rangle_{tx}+\int_0^t\int_{\partial\Omega}\varphi\varsigma\cdot\\&\quad n=\Big\langle\!\Big\langle\varphi\Big(|v|^2-5\Big)\mu^{\frac{1}{2}},S\Big\rangle\!\Big\rangle.\end{aligned}\tag{4.85}$$

Hence, adding $\varepsilon^{-1}\times$(4.85) and (4.79) to eliminate $\varepsilon^{-1}\langle\!\langle\nabla_x\varphi,\varsigma\rangle\!\rangle_{tx}$ yields

$$\begin{aligned}&\big\langle 5c(t)-2p(t),\varphi(t)\big\rangle_x-\big\langle 5c(0)-2p(0),\varphi(0)\big\rangle_x-\langle\!\langle 5c-2p,\partial_t\varphi\rangle\!\rangle_{tx}\\&+\varepsilon\big\langle R(t),\nabla_x\varphi(t)\cdot\mathscr{A}\big\rangle-\varepsilon\big\langle z,\nabla_x\varphi(0)\cdot\mathscr{A}\big\rangle-\varepsilon\langle\!\langle R,\partial_t\nabla_x\varphi\cdot\mathscr{A}\rangle\!\rangle-\kappa\langle\!\langle\Delta_x\varphi,c\rangle\!\rangle_{tx}\\&+\varepsilon^{-1}\int_0^t\int_{\partial\Omega}\varphi\varsigma\cdot n\\&=\langle\!\langle\nabla_x\varphi\cdot\mathscr{A},h\rangle\!\rangle_{\overline{\gamma}_-}-\langle\!\langle\nabla_x\varphi\cdot\mathscr{A},R\rangle\!\rangle_{\overline{\gamma}_+}+\Big\langle\!\Big\langle v\cdot\nabla_x\Big(\nabla_x\varphi\cdot\mathscr{A}\Big),(\mathbf{I}-\mathbf{P})[R]\Big\rangle\!\Big\rangle\\&+\varepsilon^{-1}\Big\langle\!\Big\langle\varphi\Big(|v|^2-5\Big)\mu^{\frac{1}{2}},S\Big\rangle\!\Big\rangle+\langle\!\langle\nabla_x\varphi\cdot\mathscr{A},S\rangle\!\rangle.\end{aligned}\tag{4.86}$$

The assumption $\varphi\big|_{\partial\Omega}=0$ completely eliminates the boundary term $\varepsilon^{-1}\int_0^t\int_{\partial\Omega}\varphi\varsigma\cdot n$ in (4.86). Hence, we have (4.83).

□

Conservation Law with Test Function $\nabla_x\psi:\mathscr{B}+\varepsilon^{-1}\psi\cdot v\mu^{\frac{1}{2}}$

Lemma 4.16. *Let R be the solution to (1.35). Under the assumptions (1.23),(1.28),(1.31), for smooth test function $\psi(t,x)$ satisfying $\nabla_x\cdot\psi=0$, $\psi\big|_{\partial\Omega}=0$, we have*

$$\begin{aligned}&\big\langle\mathbf{b}(t),\psi(t)\big\rangle_x-\big\langle\mathbf{b}(0),\psi(0)\big\rangle_x-\langle\!\langle\mathbf{b},\partial_t\psi\rangle\!\rangle_{tx}\\&\quad+\varepsilon\big\langle R(t),\nabla_x\psi(t):\mathscr{B}\big\rangle-\varepsilon\big\langle z,\nabla_x\psi(0):\mathscr{B}\big\rangle-\varepsilon\langle\!\langle R,\partial_t\nabla_x\psi:\mathscr{B}\rangle\!\rangle-\lambda\langle\!\langle\Delta_x\psi,\mathbf{b}\rangle\!\rangle_{tx}\\&=\langle\!\langle\nabla_x\psi:\mathscr{B},h\rangle\!\rangle_{\overline{\gamma}_-}-\langle\!\langle\nabla_x\psi:\mathscr{B},R\rangle\!\rangle_{\overline{\gamma}_+}+\Big\langle\!\Big\langle v\cdot\nabla_x\Big(\nabla_x\psi:\mathscr{B}\Big),(\mathbf{I}-\mathbf{P})[R]\Big\rangle\!\Big\rangle\\&\quad+\varepsilon^{-1}\Big\langle\!\Big\langle\psi\cdot v\mu^{\frac{1}{2}},S\Big\rangle\!\Big\rangle+\langle\!\langle\nabla_x\psi:\mathscr{B},S\rangle\!\rangle.\end{aligned}\tag{4.87}$$

Proof. Multiplying $\psi(t,x)\in\mathbb{R}^3$ on both sides of (4.76) and integrating over $[0,t]\times\Omega$, we obtain

$$\begin{aligned}&\varepsilon\big\langle\mathbf{b}(t),\psi(t)\big\rangle_x-\varepsilon\big\langle\mathbf{b}(0),\psi(0)\big\rangle_x-\varepsilon\langle\!\langle\mathbf{b},\partial_t\psi\rangle\!\rangle_{tx}\\&-\langle\!\langle\nabla_x\cdot\psi,p\rangle\!\rangle_{tx}-\langle\!\langle\nabla_x\psi,\varpi\rangle\!\rangle_{tx}+\int_0^t\int_{\partial\Omega}\Big(p\psi+\psi\cdot\varpi\Big)\cdot n=\Big\langle\!\Big\langle\psi\cdot v\mu^{\frac{1}{2}},S\Big\rangle\!\Big\rangle.\end{aligned}\tag{4.88}$$

Hence, adding $\varepsilon^{-1}\times$(4.88) and (4.81) to eliminate $\varepsilon^{-1}\langle\!\langle \nabla_x\psi, \varpi\rangle\!\rangle_{tx}$ yields

$$\begin{aligned}
&\big\langle \mathbf{b}(t), \psi(t)\big\rangle_x - \big\langle \mathbf{b}(0), \psi(0)\big\rangle_x - \langle\!\langle \mathbf{b}, \partial_t\psi\rangle\!\rangle_{tx} \\
&+ \varepsilon\big\langle R(t), \nabla_x\psi(t) : \mathscr{B}\big\rangle - \varepsilon\big\langle z, \nabla_x\psi(0) : \mathscr{B}\big\rangle - \varepsilon\langle\!\langle R, \partial_t\nabla_x\psi : \mathscr{B}\rangle\!\rangle \\
&- \lambda\langle\!\langle \Delta_x\psi, \mathbf{b}\rangle\!\rangle_{tx} - \varepsilon^{-1}\langle\!\langle \nabla_x\cdot\psi, p\rangle\!\rangle_{tx} + \varepsilon^{-1}\int_0^t\int_{\partial\Omega}\Big(p\psi + \psi\cdot\varpi\Big)\cdot n \\
=&\langle\!\langle \nabla_x\psi : \mathscr{B}, h\rangle\!\rangle_{\overline{\gamma}_-} - \langle\!\langle \nabla_x\psi : \mathscr{B}, R\rangle\!\rangle_{\overline{\gamma}_+} + \Big\langle\!\Big\langle v\cdot\nabla_x\Big(\nabla_x\psi : \mathscr{B}\Big), (\mathbf{I}-\mathbf{P})[R]\Big\rangle\!\Big\rangle \\
&+ \varepsilon^{-1}\Big\langle\!\Big\langle \psi\cdot v\mu^{\frac{1}{2}}, S\Big\rangle\!\Big\rangle + \langle\!\langle \nabla_x\psi : \mathscr{B}, S\rangle\!\rangle.
\end{aligned} \tag{4.89}$$

The assumptions $\nabla_x\cdot\psi = 0$ and $\psi\big|_{\partial\Omega} = 0$ eliminates $\varepsilon^{-1}\langle\!\langle \nabla_x\cdot\psi, p\rangle\!\rangle_{tx}$ and $\varepsilon^{-1}\int_0^t\int_{\partial\Omega}\Big(p\psi + \psi\cdot\varpi\Big)\cdot n$ in (4.89). Hence, we have (4.87). □

4.3. Energy estimate: accumulative.

Proposition 4.17. *Let R be the solution to (1.35). Under the assumptions (1.23),(1.28), (1.31), we have*

$$\|R(t)\|_{L^2} + \varepsilon^{-\frac{1}{2}}\|R\|_{L^2_{\overline{\gamma}_+}} + \varepsilon^{-1}|||(\mathbf{I}-\mathbf{P})[R]|||_{L^2} \lesssim o_T\,|||R|||_X + |||R|||_X^2 + o_T. \tag{4.90}$$

Proof. It suffices to justify

$$\begin{aligned}
\|R(t)\|_{L^2} + \varepsilon^{-\frac{1}{2}}\|R\|_{L^2_{\overline{\gamma}_+}} + \varepsilon^{-1}|||(\mathbf{I}-\mathbf{P})[R]|||_{L^2} \lesssim{}& o_T\varepsilon^{-\frac{1}{2}}|||\mathbf{P}[R]|||_{L^2} + o_T\,|||R|||_X \\
&+ |||R|||_X^2 + o_T.
\end{aligned} \tag{4.91}$$

<u>Weak Formulation</u> Taking test function $g = \varepsilon^{-1}R$ in (4.41), we obtain

$$\frac{1}{2}\|R(t)\|_{L^2}^2 - \frac{1}{2}\|z\|_{L^2}^2 + \frac{\varepsilon^{-1}}{2}\iint_{\overline{\gamma}} R^2(v\cdot n) + \varepsilon^{-2}\langle\!\langle \mathcal{L}[R], R\rangle\!\rangle = \varepsilon^{-1}\langle\!\langle S, R\rangle\!\rangle. \tag{4.92}$$

Notice that

$$\iint_{\overline{\gamma}} R^2(v\cdot n) = \|R\|_{L^2_{\overline{\gamma}_+}}^2 - \|R\|_{L^2_{\overline{\gamma}_-}}^2 = \|R\|_{L^2_{\overline{\gamma}_+}}^2 - \|h\|_{L^2_{\overline{\gamma}}}^2, \tag{4.93}$$

and

$$\langle\!\langle \mathcal{L}[R], R\rangle\!\rangle \gtrsim |||(\mathbf{I}-\mathbf{P})[R]|||_{L^2_\nu}^2. \tag{4.94}$$

Then we know

$$\|R(t)\|_{L^2}^2 + \varepsilon^{-1}\|R\|_{L^2_{\overline{\gamma}_+}}^2 + \varepsilon^{-2}|||(\mathbf{I}-\mathbf{P})[R]|||_{L^2_\nu}^2 \lesssim \Big|\varepsilon^{-1}\langle\!\langle S, R\rangle\!\rangle\Big| + \varepsilon^{-1}\|h\|_{L^2_{\overline{\gamma}}}^2 + \|z\|_{L^2}^2. \tag{4.95}$$

Using Lemma 4.3 and Lemma 4.4, we have

$$\|R(t)\|_{L^2}^2 + \varepsilon^{-1}\|R\|_{L^2_{\overline{\gamma}_+}}^2 + \varepsilon^{-2}|||(\mathbf{I}-\mathbf{P})[R]|||_{L^2_\nu}^2 \lesssim \Big|\varepsilon^{-1}\langle\!\langle S, R\rangle\!\rangle\Big| + o_T\varepsilon. \tag{4.96}$$

Source Term Estimates We split

$$\varepsilon^{-1}\langle\!\langle S, R\rangle\!\rangle = \varepsilon^{-1}\langle\!\langle S, \mathbf{P}[R]\rangle\!\rangle + \varepsilon^{-1}\langle\!\langle S, (\mathbf{I}-\mathbf{P})[R]\rangle\!\rangle. \tag{4.97}$$

We may directly bound using Lemma 4.5 – Lemma 4.11

$$\begin{aligned}\left|\varepsilon^{-1}\langle\!\langle S, (\mathbf{I}-\mathbf{P})[R]\rangle\!\rangle\right| &\lesssim \varepsilon^{-1} |||S|||_{L^2} |||(\mathbf{I}-\mathbf{P})[R]|||_{L^2} \\ &\lesssim \big(o(1)+o_T\big)\varepsilon^{-2} |||(\mathbf{I}-\mathbf{P})[R]|||_{L^2}^2 + o_T |||R|||_X^2 + |||R|||_X^4 + o_T.\end{aligned} \tag{4.98}$$

Using orthogonality of Γ, we have

$$\varepsilon^{-1}\langle\!\langle S, \mathbf{P}[R]\rangle\!\rangle = \varepsilon^{-1}\langle\!\langle S_1 + S_2, \mathbf{P}[R]\rangle\!\rangle. \tag{4.99}$$

From Lemma 4.5, we know

$$\left|\varepsilon^{-1}\langle\!\langle S_1, \mathbf{P}[R]\rangle\!\rangle\right| = \varepsilon\left|\langle\!\langle \partial_t f_2, \mathbf{P}[R]\rangle\!\rangle\right| \lesssim o_T\varepsilon |||\mathbf{P}[R]|||_{L^2} \lesssim o_T |||\mathbf{P}[R]|||_{L^2}^2 + o_T\varepsilon^2. \tag{4.100}$$

Also, from Lemma 4.6 and Remark 4.7, after integrating by parts with respect to v_η in S_{2a} term, we obtain

$$\begin{aligned}\left|\varepsilon^{-1}\langle\!\langle S_2, \mathbf{P}[R]\rangle\!\rangle\right| &\lesssim \varepsilon^{-1} \left|\left|\left| f_1^B + S_{2b} + S_{2c}\right|\right|\right|_{L^2_{tx}L^1_v} |||\mathbf{P}[R]|||_{L^2_{tx}L^\infty_v} \\ &\lesssim o_T\varepsilon^{-\frac12} |||\mathbf{P}[R]|||_{L^2} \lesssim o_T\varepsilon^{-1} |||\mathbf{P}[R]|||_{L^2}^2 + o_T.\end{aligned} \tag{4.101}$$

In total, we have

$$\begin{aligned}\left|\varepsilon^{-1}\langle\!\langle S, R\rangle\!\rangle\right| \lesssim{}& o_T\varepsilon^{-1} |||\mathbf{P}[R]|||_{L^2}^2 + \big(o(1)+o_T\big)\varepsilon^{-2} |||(\mathbf{I}-\mathbf{P})[R]|||_{L^2}^2 \\ &+ o_T |||R|||_X^2 + |||R|||_X^4 + o_T.\end{aligned} \tag{4.102}$$

Synthesis Inserting (4.102) into (4.96), we have

$$\begin{aligned}\|R(t)\|_{L^2}^2 + \varepsilon^{-1}\|R\|_{L^2_{\gamma_+}}^2 + \varepsilon^{-2} |||(\mathbf{I}-\mathbf{P})[R]|||_{L^2_v}^2 \lesssim{}& o_T\varepsilon^{-1} |||\mathbf{P}[R]|||_{L^2}^2 + o_T |||R|||_X^2 \\ &+ |||R|||_X^4 + o_T.\end{aligned} \tag{4.103}$$

Then we have (4.91). □

Proposition 4.18. *Let R be the solution to (1.35). Under the assumptions (1.23),(1.28), (1.31), we have*

$$\|\partial_t R(t)\|_{L^2} + \varepsilon^{-\frac12}\|\partial_t R\|_{L^2_{\gamma_+}} + \varepsilon^{-1} |||(\mathbf{I}-\mathbf{P})[\partial_t R]|||_{L^2} \lesssim o_T |||R|||_X + |||R|||_X^2 + o_T. \tag{4.104}$$

Proof. Applying a similar argument as in the proof of Proposition 4.17 to the equation (4.37), using Lemma 4.3 – Lemma 4.11, we obtain the desired result. In particular, we should use $\partial_t z$ estimates in Lemma 4.3. □

4.4. Kernel estimate: accumulative.

4.4.1. Estimate of p

Proposition 4.19. *Let R be the solution to (1.35). Under the assumptions (1.23),(1.28), (1.31), we have*

$$\varepsilon^{-\frac{1}{2}} |||p|||_{L^2} \lesssim \varepsilon^{-\frac{1}{2}} |||\mathbf{b}|||_{L^2} + o_T |||R|||_X + |||R|||_X^2 + o_T. \tag{4.105}$$

Proof. It suffices to show

$$|||p|||_{L^2} \lesssim \varepsilon^{\frac{1}{2}} \|R(t)\|_{L^2} + \|R\|_{L^2_{\gamma_+}} + |||\mathbf{b}|||_{L^2} + |||(\mathbf{I}-\mathbf{P})[R]|||_{L^2} + o_T\varepsilon. \tag{4.106}$$

Weak Formulation Denote

$$\psi(t,x,v) := \mu^{\frac{1}{2}}(v)\Big(v\cdot\nabla_x\varphi(t,x)\Big), \tag{4.107}$$

where $\varphi(t,x)$ is defined via solving the elliptic problem

$$\begin{cases} -\Delta_x\varphi(t) = p(t) \ \text{ in } \ \Omega, \\ \varphi(t) = 0 \ \text{ on } \ \partial\Omega. \end{cases} \tag{4.108}$$

Based on standard elliptic estimates [62] and trace theorem, there exists a solution φ satisfying

$$|\psi(t)|_{L^2_\gamma} + \|\psi(t)\|_{H^1_x L^\infty_{v,\varrho,\vartheta}} \lesssim \|\varphi(t)\|_{H^2} \lesssim \|p(t)\|_{L^2}. \tag{4.109}$$

Taking test function $g = \psi$ in (4.41), we obtain

$$\varepsilon\big\langle R(t),\psi(t)\big\rangle - \varepsilon\big\langle z,\psi(0)\big\rangle - \varepsilon\langle\langle R,\partial_t\psi\rangle\rangle + \iint_{\gamma} R\psi(v\cdot n) - \langle\langle R, v\cdot\nabla_x\psi\rangle\rangle = \langle\langle S,\psi\rangle\rangle. \tag{4.110}$$

We may directly bound

$$\Big|\varepsilon\big\langle R(t),\psi(t)\big\rangle\Big| \lesssim \varepsilon\,\|R(t)\|_{L^2}\,\|\psi(t)\|_{L^2} \lesssim \varepsilon\,\|R(t)\|_{L^2}\,\|p(t)\|_{L^2} \lesssim \varepsilon\,\|R(t)\|_{L^2}^2. \tag{4.111}$$

From Lemma 4.3, we know

$$\Big|\varepsilon\big\langle R(0),\psi(0)\big\rangle\Big| \lesssim \varepsilon\,\|R(0)\|_{L^2}\,\|\psi(0)\|_{L^2} \lesssim \varepsilon\,\|z\|_{L^2}^2 \lesssim o_T\varepsilon^3. \tag{4.112}$$

And oddness and orthogonality lead to

$$\big|\varepsilon\langle\langle R,\partial_t\psi\rangle\rangle\big| = \Big|\varepsilon\big\langle\mu^{\frac{1}{2}}\big(v\cdot\mathbf{b}\big),\partial_t\psi\big\rangle\Big| \lesssim |||\mathbf{b}|||_{L^2}^2 + \varepsilon^2\,|||\partial_t\nabla_x\varphi|||_{L^2}^2. \tag{4.113}$$

Based on Lemma 4.4, we know

$$\begin{aligned} \left|\iint_{\gamma} R\psi(v\cdot n)\right| &\lesssim \|R\|_{L^2_{\gamma_+}}\,\|\psi\|_{L^2_{\gamma_+}} + \|h\|_{L^2_{\gamma_-}}\,\|\psi\|_{L^2_{\gamma_-}} \\ &\lesssim o(1)\,\|\psi\|_{L^2_\gamma}^2 + \|R\|_{L^2_{\gamma_+}}^2 + \|h\|_{L^2_{\gamma_-}}^2 \lesssim o(1)\,|||p|||_{L^2}^2 + \|R\|_{L^2_{\gamma_+}}^2 + o_T\varepsilon^2. \end{aligned} \tag{4.114}$$

Due to oddness and orthogonality, we have

$$\left\langle\!\left\langle \mu^{\frac{1}{2}}\big(v\cdot\mathbf{b}\big), v\cdot\nabla_x\psi\right\rangle\!\right\rangle = \langle\!\langle(\mathbf{I}-\mathbf{P})[R], v\cdot\nabla_x\psi\rangle\!\rangle = 0. \tag{4.115}$$

Due to orthogonality of $\overline{\mathscr{A}}$, we know

$$\left\langle\!\left\langle \mu^{\frac{1}{2}}\frac{|v|^2-5}{2}c, v\cdot\nabla_x\psi\right\rangle\!\right\rangle = \langle\!\langle c, \mu^{\frac{1}{2}}\overline{\mathscr{A}}\cdot\nabla_x\psi\rangle\!\rangle = 0. \tag{4.116}$$

Also, we have

$$\begin{aligned}-\left\langle\!\left\langle \mu^{\frac{1}{2}}p, v\cdot\nabla_x\psi\right\rangle\!\right\rangle &= -\left\langle\!\left\langle p\mu, v\cdot\nabla_x\Big(v\cdot\nabla_x\varphi\Big)\right\rangle\!\right\rangle\\ &= -\frac{1}{3}\int_0^t\int_\Omega p\big(\Delta_x\varphi\big)\int_{\mathbb{R}^3}\mu\,|v|^2 = |||p|||_{L^2}^2.\end{aligned} \tag{4.117}$$

In summary, we have shown that

$$|||p|||_{L^2}^2 \lesssim \varepsilon\,\|R(t)\|_{L^2}^2 + \|R\|_{L^2_{\gamma_+}}^2 + |||\mathbf{b}|||_{L^2}^2 + \varepsilon^2\,|||\partial_t\nabla_x\varphi|||_{L^2}^2 + o_T\varepsilon^2 + \big|\langle\!\langle S,\psi\rangle\!\rangle\big|. \tag{4.118}$$

<u>Source Term Estimates</u> Due to the orthogonality of Γ and Lemma 3.15, we know

$$\langle\!\langle S,\psi\rangle\!\rangle = \langle\!\langle S_1+S_2,\psi\rangle\!\rangle. \tag{4.119}$$

Using Lemma 4.5, we have

$$\big|\langle\!\langle S_1,\psi\rangle\!\rangle\big| = \varepsilon^2\,\big|\langle\!\langle\partial_t f_2,\psi\rangle\!\rangle\big| \lesssim o_T\,|||p|||_{L^2}^2 + o_T\varepsilon^4. \tag{4.120}$$

Using Hardy's inequality and integrating by parts with respect to v_η in S_{2a}, based on Lemma 4.6 and Remark 4.7, we have

$$\begin{aligned}&\big|\langle\!\langle S_2,\psi\rangle\!\rangle\big| \le \left|\left\langle\!\left\langle S_2, \psi\Big|_{\mathfrak{n}=0}\right\rangle\!\right\rangle\right| + \left|\left\langle\!\left\langle S_2, \int_0^{\mathfrak{n}}\partial_{\mathfrak{n}}\psi\right\rangle\!\right\rangle\right| = \left|\left\langle\!\left\langle S_2, \psi\Big|_{\mathfrak{n}=0}\right\rangle\!\right\rangle\right| + \left|\varepsilon\left\langle\!\left\langle \eta S_2, \frac{1}{\mathfrak{n}}\int_0^{\mathfrak{n}}\partial_{\mathfrak{n}}\psi\right\rangle\!\right\rangle\right|\\
&\lesssim \left|\left|\left| f_1^B + S_{2b} + S_{2c}\right|\right|\right|_{L^2_tL^2_{\iota_1\iota_2}L^1_{\mathfrak{n}}L^1_v}\|\psi\|_{L^2_{\gamma}} + \varepsilon\left|\left|\left|\eta\big(f_1^B+S_{2b}+S_{2c}\big)\right|\right|\right|_{L^2}\left|\left|\left|\frac{1}{\mathfrak{n}}\int_0^{\mathfrak{n}}\partial_{\mathfrak{n}}\psi\right|\right|\right|_{L^2}\\
&\lesssim \left|\left|\left| f_1^B + S_{2b} + S_{2c}\right|\right|\right|_{L^2_tL^2_{\iota_1\iota_2}L^1_{\mathfrak{n}}L^1_v}\|\psi\|_{L^2_{\gamma}} + \varepsilon\left|\left|\left|\eta\big(f_1^B+S_{2b}+S_{2c}\big)\right|\right|\right|_{L^2}|||\partial_{\mathfrak{n}}\psi|||_{L^2}\\
&\lesssim o_T\varepsilon\,\|\psi\|_{L^2_{\gamma}} + o_T\varepsilon\,|||\partial_{\mathfrak{n}}\psi|||_{L^2} \lesssim o_T\varepsilon\,|||p|||_{L^2} \lesssim o_T\,|||p|||_{L^2}^2 + o_T\varepsilon^2.\end{aligned} \tag{4.121}$$

In summary, we have shown that

$$\big|\langle\!\langle S,\psi\rangle\!\rangle\big| \lesssim o_T\,|||p|||_{L^2}^2 + o_T\varepsilon^2. \tag{4.122}$$

Inserting (4.122) into (4.118), we have

$$|||p|||_{L^2}^2 \lesssim \varepsilon\,\|R(t)\|_{L^2}^2 + \|R\|_{L^2_{\gamma_+}}^2 + |||\mathbf{b}|||_{L^2}^2 + \varepsilon^2\,|||\partial_t\nabla_x\varphi|||_{L^2}^2 + o_T\varepsilon^2. \tag{4.123}$$

<u>$Estimate of\ |||\partial_t\nabla_x\varphi|||_{L^2}$</u> Denote $\Phi = \partial_t\varphi$. Taking $g = \varepsilon\Phi\,|v|^2\,\mu^{\frac{1}{2}}$ in (4.41), due to orthogonality and $\Phi\big|_{\partial\Omega} = 0$, we obtain

$$\varepsilon^2\langle\!\langle\partial_t R, \Phi\,|v|^2\,\mu^{\frac{1}{2}}\rangle\!\rangle - \varepsilon\langle\!\langle R, v\cdot\nabla_x\Big(\Phi\,|v|^2\,\mu^{\frac{1}{2}}\Big)\rangle\!\rangle = \varepsilon\langle\!\langle S, \Phi\,|v|^2\,\mu^{\frac{1}{2}}\rangle\!\rangle. \tag{4.124}$$

Notice that

$$\varepsilon^2 \langle\langle \partial_t R, \Phi \rangle\rangle = 3\varepsilon^2 \langle\langle \partial_t p, \Phi \rangle\rangle = -3\varepsilon^2 \langle\langle \Delta_x \Phi, \Phi \rangle\rangle = 3\varepsilon^2 \langle\langle \nabla_x \Phi, \nabla_x \Phi \rangle\rangle = 3\varepsilon^2 \, |||\partial_t \nabla_x \varphi|||_{L^2}^2 . \tag{4.125}$$

Also, we know

$$\begin{aligned} \left| \varepsilon \langle\langle R, v \cdot \nabla_x \left(\Phi \, |v|^2 \mu^{\frac{1}{2}} \right) \rangle\rangle \right| &\leq \left| \varepsilon \langle\langle \mu^{\frac{1}{2}} (v \cdot \mathbf{b}), v \, |v|^2 \mu^{\frac{1}{2}} \cdot \nabla_x \Phi \rangle\rangle \right| \\ &\quad + \left| \varepsilon \langle\langle (\mathbf{I} - \mathbf{P})[R], v \, |v|^2 \mu^{\frac{1}{2}} \cdot \nabla_x \Phi \rangle\rangle \right| \\ &\lesssim |||\mathbf{b}|||_{L^2}^2 + |||(\mathbf{I} - \mathbf{P})[R]|||_{L^2}^2 + o(1)\varepsilon^2 \, |||\partial_t \nabla_x \varphi|||_{L^2}^2 . \end{aligned} \tag{4.126}$$

Then, by a similar argument as the above estimates for $\langle\langle S, \psi \rangle\rangle$, we have

$$\left| \varepsilon \langle\langle S, \Phi \, |v|^2 \mu^{\frac{1}{2}} \rangle\rangle \right| = \left| \varepsilon \langle S_1 + S_2, \Phi \, |v|^2 \mu^{\frac{1}{2}} \rangle \right| \lesssim o_T \varepsilon^2 \, |||\partial_t \nabla_x \varphi|||_{L^2}^2 + o_T \varepsilon^2 . \tag{4.127}$$

In summary, we have shown that

$$\varepsilon^2 \, |||\partial_t \nabla_x \varphi|||_{L^2}^2 \lesssim |||\mathbf{b}|||_{L^2}^2 + |||(\mathbf{I} - \mathbf{P})[R]|||_{L^2}^2 + o_T \varepsilon^2 . \tag{4.128}$$

Inserting (4.128) into (4.123), we have

$$|||p|||_{L^2}^2 \lesssim \varepsilon \, \|R(t)\|_{L^2}^2 + \|R\|_{L^2_{\gamma_+}}^2 + |||\mathbf{b}|||_{L^2}^2 + |||(\mathbf{I} - \mathbf{P})[R]|||_{L^2}^2 + o_T \varepsilon^2 . \tag{4.129}$$

Then (4.106) follows. □

Proposition 4.20. *Let R be the solution to (1.35). Under the assumptions (1.23), (1.28),(1.31), we have*

$$\varepsilon^{-\frac{1}{2}} \, |||\partial_t p|||_{L^2} \lesssim \varepsilon^{-\frac{1}{2}} \, |||\partial_t \mathbf{b}|||_{L^2} + o_T \, |||R|||_X + |||R|||_X^2 + o_T . \tag{4.130}$$

Proof. Applying a similar argument as in the proof of Proposition 4.19 to the equation (4.37), we obtain the desired result. □

4.4.2. Estimate of c

Proposition 4.21. *Let R be the solution to (1.35). Under the assumptions (1.23), (1.28),(1.31), we have*

$$\varepsilon^{-\frac{1}{2}} \, |||c|||_{L^2} \lesssim \varepsilon^{-\frac{1}{2}} \, |||p|||_{L^2} + o_T \, |||R|||_X + |||R|||_X^2 + o_T . \tag{4.131}$$

Proof. It suffices to justify

$$\begin{aligned} |||c|||_{L^2} &\lesssim \varepsilon^{\frac{1}{12}} \, |||R|||_X^{\frac{1}{2}} \, |||c|||_{L^2} + \varepsilon^{\frac{1}{2}} \, \|R(t)\|_{L^2} + \|R\|_{L^2_{\gamma_+}} + |||p|||_{L^2} + |||(\mathbf{I} - \mathbf{P})[R]|||_{L^2} \\ &\quad + o_T \varepsilon^{\frac{1}{2}} \, |||R|||_X + \varepsilon^{\frac{1}{2}} \, |||R|||_X^2 + o_T \varepsilon^{\frac{1}{2}} . \end{aligned} \tag{4.132}$$

Weak Formulation We consider the conservation law (4.83) where the smooth test function $\varphi(t, x)$ satisfies

$$\begin{cases} -\Delta_x \varphi(t) = 5c(t) - 2p(t) & \text{in } \Omega, \\ \varphi(t) = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.133}$$

Based on the standard elliptic estimates [62] and trace theorem, there exists a solution φ satisfying

$$|\nabla_x\varphi(t)|_{L^2_{\partial\Omega}} + \|\varphi(t)\|_{H^2} \lesssim \|c(t)\|_{L^2} + \|p(t)\|_{L^2}. \tag{4.134}$$

From (4.83), we have

$$\begin{aligned}
&\langle\!\langle \partial_t\big(5c-2p\big), \varphi\rangle\!\rangle_{tx} \\
&\quad + \varepsilon\big\langle R(t), \nabla_x\varphi(t)\cdot\mathscr{A}\big\rangle - \varepsilon\big\langle z, \nabla_x\varphi(0)\cdot\mathscr{A}\big\rangle - \varepsilon\langle\!\langle R, \partial_t\nabla_x\varphi\cdot\mathscr{A}\rangle\!\rangle - \kappa\langle\!\langle \Delta_x\varphi, c\rangle\!\rangle_{tx} \\
&= \langle\!\langle \nabla_x\varphi\cdot\mathscr{A}, h\rangle\!\rangle_{\overline{\gamma}_-} - \langle\!\langle \nabla_x\varphi\cdot\mathscr{A}, R\rangle\!\rangle_{\overline{\gamma}_+} + \Big\langle\!\!\Big\langle v\cdot\nabla_x\Big(\nabla_x\varphi\cdot\mathscr{A}\Big), (\mathbf{I}-\mathbf{P})[R]\Big\rangle\!\!\Big\rangle \\
&\quad + \varepsilon^{-1}\Big\langle\!\!\Big\langle \varphi\left(|v|^2-5\right)\mu^{\frac{1}{2}}, S\Big\rangle\!\!\Big\rangle + \langle\!\langle \nabla_x\varphi\cdot\mathscr{A}, S\rangle\!\rangle.
\end{aligned} \tag{4.135}$$

Direct computation reveals that

$$\begin{aligned}
\langle\!\langle \partial_t\big(5c-2p\big), \varphi\rangle\!\rangle_{tx} &= -\langle\!\langle \partial_t\Delta_x\varphi, \varphi\rangle\!\rangle_{tx} = \langle\!\langle \partial_t\nabla_x\varphi, \nabla_x\varphi\rangle\!\rangle_{tx} \\
&= \frac{1}{2}\|\nabla_x\varphi(t)\|_{L^2}^2 - \frac{1}{2}\|\nabla_x\varphi(0)\|_{L^2}^2,
\end{aligned} \tag{4.136}$$

and from Lemma 4.3

$$\|\nabla_x\varphi(0)\|_{L^2}^2 \lesssim \|p(0)\|_{L^2}^2 + \|c(0)\|_{L^2}^2 \lesssim \|z\|_{L^2}^2 \lesssim o_T\varepsilon^2. \tag{4.137}$$

Also, we have

$$-\kappa\langle\!\langle \Delta_x\varphi, c\rangle\!\rangle_{tx} = \kappa\langle\!\langle 5c-p, c\rangle\!\rangle_{tx} = 5\kappa\,|||c|||_{L^2}^2 - \kappa\langle\!\langle p, c\rangle\!\rangle_{tx} \tag{4.138}$$

with

$$\left|\kappa\langle\!\langle p, c\rangle\!\rangle_{tx}\right| \lesssim o(1)\,|||c|||_{L^2}^2 + |||p|||_{L^2}^2. \tag{4.139}$$

Using Lemma 4.3, we have

$$\begin{aligned}
\left|\varepsilon\big\langle R(t), \nabla_x\varphi(t)\cdot\mathscr{A}\big\rangle\right| &\lesssim \varepsilon\|R(t)\|_{L^2}\|\varphi(t)\|_{H^1} \lesssim \varepsilon\|R(t)\|_{L^2}\Big(\|c(t)\|_{L^2} + \|p(t)\|_{L^2}\Big) \\
&\lesssim \varepsilon\|R(t)\|_{L^2}^2,
\end{aligned} \tag{4.140}$$

$$\left|\varepsilon\big\langle R(0), \nabla_x\varphi(0)\cdot\mathscr{A}\big\rangle\right| \lesssim \varepsilon\|R(0)\|_{L^2}\|\varphi(0)\|_{H^1} \lesssim \varepsilon\|z\|_{L^2}^2 \lesssim o_T\varepsilon^3. \tag{4.141}$$

Due to the orthogonality of $\mathscr{A}$, we know

$$\begin{aligned}
\left|\varepsilon\langle\!\langle R, \partial_t\nabla_x\varphi\cdot\mathscr{A}\rangle\!\rangle\right| = \left|\varepsilon\langle\!\langle (\mathbf{I}-\mathbf{P})[R], \partial_t\nabla_x\varphi\cdot\mathscr{A}\rangle\!\rangle\right| &\lesssim |||(\mathbf{I}-\mathbf{P})[R]|||_{L^2}^2 \\
&\quad + o(1)\varepsilon^2\,|||\partial_t\nabla_x\varphi|||_{L^2}^2.
\end{aligned} \tag{4.142}$$

Using Lemma 4.4, we have

$$\left|\langle\!\langle \nabla_x\varphi\cdot\mathscr{A}, h\rangle\!\rangle_{\overline{\gamma}_-}\right| \lesssim \|\nabla_x\varphi\cdot\mathscr{A}\|_{L^2_{\overline{\gamma}_-}}\|h\|_{L^2_{\overline{\gamma}_-}} \lesssim o_T\,|||c|||_{L^2}^2 + o_T\,|||p|||_{L^2}^2 + o_T\varepsilon^2, \tag{4.143}$$

$$\left|\langle\!\langle \nabla_x\varphi\cdot\mathscr{A}, R\rangle\!\rangle_{\overline{\gamma}_+}\right| \lesssim \|\nabla_x\varphi\cdot\mathscr{A}\|_{L^2_{\overline{\gamma}_+}}\|R\|_{L^2_{\overline{\gamma}_+}} \lesssim o(1)\,|||c|||_{L^2}^2 + o(1)\,|||p|||_{L^2}^2 + \|R\|_{L^2_{\overline{\gamma}_+}}^2, \tag{4.144}$$

and

$$\left|\left\langle\!\left\langle v\cdot\nabla_x\Big(\nabla_x\varphi\cdot\mathscr{A}\Big),(\mathbf{I}-\mathbf{P})[R]\right\rangle\!\right\rangle\right|\lesssim\left\|\!\left|v\cdot\nabla_x\Big(\nabla_x\varphi\cdot\mathscr{A}\Big)\right|\!\right\|_{L^2}|||(\mathbf{I}-\mathbf{P})[R]|||_{L^2}$$
$$\lesssim o(1)\,|||c|||_{L^2}^2+o(1)\,|||p|||_{L^2}^2+|||(\mathbf{I}-\mathbf{P})[R]|||_{L^2}^2. \tag{4.145}$$

In summary, we have shown that

$$\begin{aligned}\|\nabla_x\varphi(t)\|_{L^2}^2+|||c|||_{L^2}^2\lesssim\varepsilon\,\|R(t)\|_{L^2}^2+\|R\|_{L^2_{\gamma_+}}^2+|||p|||_{L^2}^2&\\+|||(\mathbf{I}-\mathbf{P})[R]|||_{L^2}^2+o(1)\varepsilon^2\,|||\partial_t\nabla_x\varphi|||_{L^2}^2+o_T\varepsilon^2&\\+\left|\varepsilon^{-1}\left\langle\!\left\langle\varphi\big(|v|^2-5\big)\mu^{\frac12},S\right\rangle\!\right\rangle\right|+\left|\langle\!\langle\nabla_x\varphi\cdot\mathscr{A},S\rangle\!\rangle\right|.&\end{aligned} \tag{4.146}$$

<u>Source Term Estimates</u> Due to the orthogonality of Γ, we have

$$\varepsilon^{-1}\left\langle\!\left\langle\varphi\big(|v|^2-5\big)\mu^{\frac12},S\right\rangle\!\right\rangle=\varepsilon^{-1}\left\langle\!\left\langle\varphi\big(|v|^2-5\big)\mu^{\frac12},S_1+S_2\right\rangle\!\right\rangle. \tag{4.147}$$

Based on Lemma 4.5, we have

$$\begin{aligned}&\left|\varepsilon^{-1}\left\langle\!\left\langle\varphi\big(|v|^2-5\big)\mu^{\frac12},S_1\right\rangle\!\right\rangle\right|\\&=\left|\varepsilon\left\langle\!\left\langle\varphi\big(|v|^2-5\big)\mu^{\frac12},\partial_t f_2\right\rangle\!\right\rangle\right|\lesssim o_T\varepsilon\,|||c|||_{L^2}\lesssim o_T\,|||c|||_{L^2}^2+o_T\varepsilon^2.\end{aligned} \tag{4.148}$$

Similar to (4.121), based on Lemma 4.6, Remark 4.7 and Hardy's inequality, we have

$$\begin{aligned}&\left|\varepsilon^{-1}\left\langle\!\left\langle\varphi\big(|v|^2-5\big)\mu^{\frac12},S_2\right\rangle\!\right\rangle\right|\lesssim\varepsilon^{-1}\left|\left\langle\!\!\left\langle S_2,\int_0^{\mathfrak{n}}\partial_{\mathfrak{n}}\varphi\right\rangle\!\!\right\rangle\right|\\&\lesssim\left|\left\langle\!\!\left\langle\eta S_2,\frac{1}{\mathfrak{n}}\int_0^{\mathfrak{n}}\partial_{\mathfrak{n}}\varphi\right\rangle\!\!\right\rangle\right|\lesssim\left\|\eta\big(f_1^B+S_{2b}+S_{2c}\big)\right\|_{L^2_tL^2_xL^1_v}\left\|\!\left|\frac{1}{\mathfrak{n}}\int_0^{\mathfrak{n}}\partial_{\mathfrak{n}}\varphi\right|\!\right\|_{L^2}\\&\lesssim\left\|\eta\big(f_1^B+S_{2b}+S_{2c}\big)\right\|_{L^2_tL^2_xL^1_v}|||\partial_{\mathfrak{n}}\varphi|||_{L^2}\lesssim o_T\,|||c|||_{L^2}^2+o_T\varepsilon.\end{aligned} \tag{4.149}$$

From Lemma 4.5, we directly bound

$$\left|\langle\!\langle\nabla_x\varphi\cdot\mathscr{A},S_1\rangle\!\rangle\right|\lesssim|||\nabla_x\varphi|||_{L^2}\,|||S_1|||_{L^2}\lesssim o_T\,\|c\|_{L^2}^2+o_T\varepsilon^2. \tag{4.150}$$

By a similar argument as for deriving (4.149), we obtain

$$\begin{aligned}&\left|\langle\!\langle\nabla_x\varphi\cdot\mathscr{A},S_2\rangle\!\rangle\right|\leq\left|\left\langle\!\!\left\langle S_2,\nabla_x\varphi\Big|_{\mathfrak{n}=0}\right\rangle\!\!\right\rangle\right|+\left|\varepsilon\left\langle\!\!\left\langle\eta S_2,\frac{1}{\mathfrak{n}}\int_0^{\mathfrak{n}}\partial_{\mathfrak{n}}\nabla_x\varphi\right\rangle\!\!\right\rangle\right|\\&\lesssim\left\|f_1^B+S_{2b}+S_{2c}\right\|_{L^2_tL^2_{\iota_1\iota_2}L^1_{\mathfrak{n}}L^1_v}\|\nabla_x\varphi\|_{L^2_\gamma}+\varepsilon\left\|\!\left|\eta\big(f_1^B+S_{2b}+S_{2c}\big)\right|\!\right\|_{L^2}\left\|\!\left|\frac{1}{\mathfrak{n}}\int_0^{\mathfrak{n}}\partial_{\mathfrak{n}}\nabla_x\varphi\right|\!\right\|_{L^2}\\&\lesssim\left\|f_1^B+S_{2b}+S_{2c}\right\|_{L^2_tL^2_{\iota_1\iota_2}L^1_{\mathfrak{n}}L^1_v}\|\nabla_x\varphi\|_{L^2_\gamma}\\&\quad+\varepsilon\left\|\!\left|\eta\big(f_1^B+S_{2b}+S_{2c}\big)\right|\!\right\|_{L^2}|||\partial_{\mathfrak{n}}\nabla_x\varphi|||_{L^2}\lesssim o_T\varepsilon\,|||c|||_{L^2}\lesssim o_T\,|||c|||_{L^2}^2+o_T\varepsilon^2.\end{aligned} \tag{4.151}$$

Based on Lemma 4.8, Lemma 4.9, and Lemma 4.10, we have

$$\begin{aligned}\left|\langle\!\langle\nabla_x\varphi\cdot\mathscr{A},S_3+S_4+S_5\rangle\!\rangle\right|&\lesssim|||\nabla_x\varphi|||_{L^2}\,|||S_3+S_4+S_5|||_{L^2}\\&\lesssim o_T\,|||c|||_{L^2}^2+o_T\,|||R|||_{L^2}^2+o_T\end{aligned}$$

$$\varepsilon \lesssim o_T \, \|\|c\|\|_{L^2}^2 + o_T\varepsilon \, \|\|R\|\|_X^2 + o_T\varepsilon. \tag{4.152}$$

Finally, based on Lemma 4.11, we have

$$\left|\langle\langle \nabla_x\varphi \cdot \mathscr{A}, S_6\rangle\rangle\right| \lesssim \left|\langle\langle \nabla_x\varphi \cdot \mathscr{A}, \Gamma\Big[\mathbf{P}[R], \mathbf{P}[R]\Big]\rangle\rangle\right| + \left|\langle\langle \nabla_x\varphi \cdot \mathscr{A}, \Gamma\Big[R, (\mathbf{I}-\mathbf{P})[R]\Big]\rangle\rangle\right|. \tag{4.153}$$

The oddness and orthogonality, with the help of interpolation $\|\|\mathbf{b}\|\|_{L^3} \lesssim \|\|\mathbf{b}\|\|_{L^2}^{\frac{2}{3}} \|\|\mathbf{b}\|\|_{L^\infty_{\varrho,\vartheta}}^{\frac{1}{3}} \lesssim \varepsilon^{\frac{1}{6}} \|\|R\|\|_X$, imply that

$$\begin{aligned}\left|\left\langle\!\!\left\langle \nabla_x\varphi \cdot \mathscr{A}, \Gamma\Big[\mathbf{P}[R], \mathbf{P}[R]\Big]\right\rangle\!\!\right\rangle\right| &\lesssim \left|\left\langle\!\!\left\langle \nabla_x\varphi \cdot \mathscr{A}, \Gamma\left[\mu^{\frac{1}{2}}(v\cdot\mathbf{b}), \mu^{\frac{1}{2}}\left(\frac{|v|^2-5}{2}c\right)\right]\right\rangle\!\!\right\rangle\right| \\ &\lesssim \|\|\nabla_x\varphi\|\|_{L^6} \|\|\mathbf{b}\|\|_{L^3} \|\|c\|\|_{L^2} \lesssim \|\|\varphi\|\|_{L^2_t H^2_x} \|\|\mathbf{b}\|\|_{L^3} \|\|c\|\|_{L^2} \\ &\lesssim \varepsilon^{\frac{1}{6}} \|\|R\|\|_X \|\|c\|\|_{L^2}^2. \end{aligned}\tag{4.154}$$

In addition, with the help of $\|\|(\mathbf{I}-\mathbf{P})[R]\|\|_{L^3} \lesssim \|\|(\mathbf{I}-\mathbf{P})[R]\|\|_{L^2}^{\frac{2}{3}} \|\|(\mathbf{I}-\mathbf{P})[R]\|\|_{L^\infty_{\varrho,\vartheta}}^{\frac{1}{3}} \lesssim \varepsilon^{\frac{1}{2}} \|\|R\|\|_X$, we have

$$\begin{aligned}\left|\langle\langle \nabla_x\varphi \cdot \mathscr{A}, \Gamma\Big[R, (\mathbf{I}-\mathbf{P})[R]\Big]\rangle\rangle\right| &\lesssim \|\|\nabla_x\varphi\|\|_{L^6} \|\|R\|\|_{L^2} \|\|(\mathbf{I}-\mathbf{P})[R]\|\|_{L^3} \\ &\lesssim \varepsilon \|\|c\|\|_{L^2} \|\|R\|\|_X^2 \lesssim o(1) \|\|c\|\|_{L^2}^2 + \varepsilon^2 \|\|R\|\|_X^4. \end{aligned}\tag{4.155}$$

Hence, we know

$$\left|\langle\langle \nabla_x\varphi \cdot \mathscr{A}, S_6\rangle\rangle\right| \lesssim \varepsilon^{\frac{1}{6}} \|\|R\|\|_X \|\|c\|\|_{L^2}^2 + o(1) \|\|c\|\|_{L^2}^2 + \varepsilon^2 \|\|R\|\|_X^4. \tag{4.156}$$

In summary, we have shown that

$$\begin{aligned}&\left|\varepsilon^{-1}\left\langle\!\!\left\langle \varphi\big(|v|^2-5\big)\mu^{\frac{1}{2}}, S\right\rangle\!\!\right\rangle\right| + \left|\langle\langle \nabla_x\varphi \cdot \mathscr{A}, S\rangle\rangle\right| \\ &\lesssim \varepsilon^{\frac{1}{6}} \|\|R\|\|_X \|\|c\|\|_{L^2}^2 + \big(o(1)+o_T\big) \|\|c\|\|_{L^2}^2 + o_T\varepsilon \|\|R\|\|_X^2 + \varepsilon \|\|R\|\|_X^4 + o_T\varepsilon. \end{aligned}\tag{4.157}$$

Inserting (4.157) into (4.146), we have

$$\begin{aligned}\|\nabla_x\varphi(t)\|_{L^2}^2 + \|\|c\|\|_{L^2}^2 \lesssim \varepsilon^{\frac{1}{6}} \|\|R\|\|_X \|\|c\|\|_{L^2}^2 + \varepsilon \|R(t)\|_{L^2}^2 + \|R\|_{L^2_{\gamma_+}}^2 \\ + \|\|p\|\|_{L^2}^2 + \|\|(\mathbf{I}-\mathbf{P})[R]\|\|_{L^2}^2 + \varepsilon^2 \|\|\partial_t\nabla_x\varphi\|\|_{L^2}^2 \\ + o_T\varepsilon \|\|R\|\|_X^2 + \varepsilon \|\|R\|\|_X^4 + o_T\varepsilon. \end{aligned}\tag{4.158}$$

<u>Estimate of $\|\|\partial_t\nabla_x\varphi\|\|_{L^2}$</u> Denote $\Phi = \partial_t\varphi$. Taking $g = \varepsilon\Phi\big(|v|^2-5\big)\mu^{\frac{1}{2}}$ in (4.41), due to orthogonality and $\Phi\big|_{\partial\Omega} = 0$, we obtain

$$\varepsilon^2\langle\langle \partial_t R, \Phi\left(|v|^2-5\right)\mu^{\frac{1}{2}}\rangle\rangle - \varepsilon\langle\langle R, v\cdot\nabla_x\left(\Phi\left(|v|^2-5\right)\mu^{\frac{1}{2}}\right)\rangle\rangle = \varepsilon\langle\langle S, \Phi\left(|v|^2-5\right)\mu^{\frac{1}{2}}\rangle\rangle. \tag{4.159}$$

Notice that

$$\varepsilon^2 \langle\!\langle \partial_t R, \Phi \left(|v|^2 - 5\right) \mu^{\frac{1}{2}} \rangle\!\rangle = 2\varepsilon^2 \langle\!\langle \partial_t \big(5c - p\big), \Phi \rangle\!\rangle = -2\varepsilon^2 \langle\!\langle \Delta_x \Phi, \Phi \rangle\!\rangle = 2\varepsilon^2 \left\| \partial_t \nabla_x \varphi \right\|_{L^2}^2 . \tag{4.160}$$

Based on orthogonality, we have

$$\begin{aligned} \left| \varepsilon \langle\!\langle R, v \cdot \nabla_x \left(\Phi \left(|v|^2 - 5 \right) \mu^{\frac{1}{2}} \right) \rangle\!\rangle \right| &= \left| \varepsilon \langle\!\langle (\mathbf{I} - \mathbf{P})[R], v \cdot \nabla_x \left(\Phi \left(|v|^2 - 5 \right) \mu^{\frac{1}{2}} \right) \rangle\!\rangle \right| \\ &\lesssim \left\| (\mathbf{I} - \mathbf{P})[R] \right\|_{L^2}^2 + o(1)\varepsilon^2 \left\| \partial_t \nabla_x \varphi \right\|_{L^2}^2 . \end{aligned} \tag{4.161}$$

Then, by a similar argument as the above estimates for $\varepsilon^{-1} \left\langle\!\!\left\langle \varphi\big(|v|^2 - 5\big)\mu^{\frac{1}{2}}, S \right\rangle\!\!\right\rangle$, we have

$$\left| \varepsilon \langle\!\langle S, \Phi \left(|v|^2 - 5\right) \mu^{\frac{1}{2}} \rangle\!\rangle \right| = \varepsilon \left| \langle\!\langle S_1 + S_2, \Phi \left(|v|^2 - 5\right) \mu^{\frac{1}{2}} \rangle\!\rangle \right| \lesssim o(1)\varepsilon^2 \left\| \partial_t \nabla_x \varphi \right\|_{L^2}^2 + o_T \varepsilon. \tag{4.162}$$

In summary, we have shown that

$$\varepsilon^2 \left\| \partial_t \nabla_x \varphi \right\|_{L^2}^2 \lesssim \left\| (\mathbf{I} - \mathbf{P})[R] \right\|_{L^2_v}^2 + o_T \varepsilon. \tag{4.163}$$

Inserting (4.163) into (4.158), we have

$$\begin{aligned} \left\| \nabla_x \varphi(t) \right\|_{L^2}^2 + \left\| c \right\|_{L^2}^2 \lesssim{}& \varepsilon^{\frac{1}{6}} \left\| R \right\|_X \left\| c \right\|_{L^2}^2 + \varepsilon \left\| R(t) \right\|_{L^2}^2 \\ &+ \left\| R \right\|_{L^2_{\gamma_+}}^2 + \left\| p \right\|_{L^2}^2 + \left\| (\mathbf{I} - \mathbf{P})[R] \right\|_{L^2}^2 \\ &+ o_T \varepsilon \left\| R \right\|_X^2 + \varepsilon \left\| R \right\|_X^4 + o_T \varepsilon. \end{aligned} \tag{4.164}$$

Hence, (4.132) follows. □

Proposition 4.22. *Let R be the solution to (1.35). Under the assumptions (1.23),(1.28), (1.31), we have*

$$\varepsilon^{-\frac{1}{2}} \left\| \partial_t c \right\|_{L^2} \lesssim \varepsilon^{-\frac{1}{2}} \left\| \partial_t p \right\|_{L^2} + o_T \left\| R \right\|_X + \left\| R \right\|_X^2 + o_T. \tag{4.165}$$

Proof. Applying a similar argument as in the proof of Proposition 4.21 to the equation (4.37), we obtain the desired result. Notice that in the bounds (4.154) and (4.155), we should always assign L^2 norm to the time-derivative terms and L^3 to the no-derivative terms. □

4.4.3. Estimate of **b**

Proposition 4.23. *Let R be the solution to (1.35). Under the assumptions (1.23),(1.28), (1.31), we have*

$$\varepsilon^{-\frac{1}{2}} \left\| \mathbf{b} \right\|_{L^2} \lesssim o(1)\varepsilon^{-\frac{1}{2}} \left\| p \right\|_{L^2} + o_T \left\| R \right\|_X + \left\| R \right\|_X^2 + o_T. \tag{4.166}$$

Proof. It suffices to justify

$$
\|\|\|\mathbf{b}\|\|\|_{L^2} \lesssim \varepsilon^{\frac{1}{12}} \|\|\|R\|\|\|_X^{\frac{1}{2}} \|\|\|\mathbf{b}\|\|\|_{L^2} + \varepsilon^{\frac{1}{2}} \|R(t)\|_{L^2} + \|R\|_{L^2_{\overline{\gamma}_+}} + o(1) \|\|\|p\|\|\|_{L^2} + \|\|\|(\mathbf{I}-\mathbf{P})[R]\|\|\|_{L^2}
$$
$$
+ o_T \varepsilon^{\frac{1}{2}} \|\|\|R\|\|\|_X + \varepsilon^{\frac{1}{2}} \|\|\|R\|\|\|_X^2 + o_T \varepsilon^{\frac{1}{2}}. \tag{4.167}
$$

<u>Weak Formulation</u> Assume $\big(\psi(t), q(t)\big) : \Omega \to \mathbb{R}^3 \times \mathbb{R}$ (where $q(t)$ has zero average) is the unique strong solution to the Stokes problem

$$
\begin{cases} -\beta \Delta_x \psi(t) + \nabla_x q(t) = \mathbf{b}(t) & \text{in } \Omega, \\ \nabla_x \cdot \psi(t) = 0 & \text{in } \Omega, \\ \psi(t) = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.168}
$$

Based on the standard fluid estimates [19] and trace theorem, we have

$$
\|\psi(t)\|_{H^2} + |\nabla_x \psi(t)|_{L^2} + \|q(t)\|_{H^1} + |q(t)|_{L^2} \lesssim \|\mathbf{b}(t)\|_{L^2}. \tag{4.169}
$$

Multiplying $\mathbf{b}$ on both sides of (4.168) and integrating by parts for $\langle\langle \nabla_x q, \mathbf{b} \rangle\rangle_{tx}$, we have

$$
-\langle\langle \lambda \Delta_x \psi, \mathbf{b} \rangle\rangle_{tx} - \langle\langle q, \nabla_x \cdot \mathbf{b} \rangle\rangle_{tx} + \int_0^t \int_{\partial\Omega} q(\mathbf{b} \cdot n) = \|\|\|\mathbf{b}\|\|\|_{L^2}^2, \tag{4.170}
$$

which, by combining (4.75) and Remark 3.14, implies

$$
-\langle\langle \lambda \Delta_x \psi, \mathbf{b} \rangle\rangle_{tx} - \Big\langle\Big\langle q\mu^{\frac{1}{2}}, S \Big\rangle\Big\rangle + \langle\langle q\mu^{\frac{1}{2}}, R \rangle\rangle_{\overline{\gamma}_+} - \langle\langle q\mu^{\frac{1}{2}}, h \rangle\rangle_{\overline{\gamma}_-} = \|\|\|\mathbf{b}\|\|\|_{L^2}^2. \tag{4.171}
$$

Inserting (4.171) into (4.87) to replace $-\langle\langle \lambda \Delta_x \psi, \mathbf{b} \rangle\rangle_{tx}$, we obtain

$$
\begin{aligned}
&\langle\langle \partial_t \mathbf{b}, \psi \rangle\rangle_{tx} + \varepsilon\big\langle R(t), \nabla_x \psi(t) : \mathscr{B} \big\rangle - \varepsilon\big\langle z, \nabla_x \psi(0) : \mathscr{B} \big\rangle - \varepsilon\langle\langle R, \partial_t \nabla_x \psi : \mathscr{B} \rangle\rangle + \|\|\|\mathbf{b}\|\|\|_{L^2}^2 \\
= &- \langle\langle q\mu^{\frac{1}{2}}, h \rangle\rangle_{\overline{\gamma}_-} + \langle\langle q\mu^{\frac{1}{2}}, R \rangle\rangle_{\overline{\gamma}_+} + \langle\langle \nabla_x \psi : \mathscr{B}, h \rangle\rangle_{\overline{\gamma}_-} - \langle\langle \nabla_x \psi : \mathscr{B}, R \rangle\rangle_{\overline{\gamma}_+} \\
&+ \Big\langle\Big\langle v \cdot \nabla_x \Big(\nabla_x \psi : \mathscr{B}\Big), (\mathbf{I}-\mathbf{P})[R] \Big\rangle\Big\rangle - \Big\langle\Big\langle q\mu^{\frac{1}{2}}, S \Big\rangle\Big\rangle + \varepsilon^{-1} \Big\langle\Big\langle \psi \cdot v\mu^{\frac{1}{2}}, S \Big\rangle\Big\rangle + \langle\langle \nabla_x \psi : \mathscr{B}, S \rangle\rangle.
\end{aligned} \tag{4.172}
$$

Using the divergence-free of ψ and $\psi\big|_{\partial\Omega} = 0$, we have

$$
\begin{aligned}
\langle\langle \partial_t \mathbf{b}, \psi \rangle\rangle_{tx} &= \langle\langle -\lambda \partial_t \Delta_x \psi + \partial_t \nabla_x q, \psi \rangle\rangle_{tx} = \lambda \langle\langle \partial_t \nabla_x \psi, \nabla_x \psi \rangle\rangle_{tx} \\
&= \frac{\lambda}{2} \|\nabla_x \psi(t)\|_{L^2}^2 - \frac{\lambda}{2} \|\nabla_x \psi(0)\|_{L^2}^2,
\end{aligned} \tag{4.173}
$$

and from Lemma 4.3

$$
\frac{\lambda}{2} \|\nabla_x \psi(0)\|_{L^2}^2 \lesssim \|\mathbf{b}(0)\|_{L^2}^2 \lesssim \|z\|_{L^2}^2 \lesssim o_T \varepsilon^2. \tag{4.174}
$$

Similarly, based on Lemma 4.3, we know

$$
\Big|\varepsilon\big\langle z, \nabla_x \psi(0) : \mathscr{B} \big\rangle\Big| \lesssim \varepsilon \|z\|_{L^2} \|\nabla_x \psi(0)\|_{L^2} \lesssim \varepsilon \|z\|_{L^2}^2 \lesssim o_T \varepsilon^3, \tag{4.175}
$$

and direct bounds yield

$$
\Big|\varepsilon\big\langle R(t), \nabla_x \psi(t) : \mathscr{B} \big\rangle\Big| \lesssim \varepsilon \|R(t)\|_{L^2} \|\nabla_x \psi(t)\|_{L^2} \lesssim \varepsilon \|R(t)\|_{L^2} \|\mathbf{b}(t)\|_{L^2} \lesssim \varepsilon \|R(t)\|_{L^2}^2, \tag{4.176}
$$

and

$$\begin{aligned}&\left|\varepsilon\langle\!\langle R, \partial_t\nabla_x\psi : \mathscr{B}\rangle\!\rangle\right| = \left|\varepsilon\langle\!\langle(\mathbf{I}-\mathbf{P})[R], \partial_t\nabla_x\psi : \mathscr{B}\rangle\!\rangle\right|\\&\lesssim |||(\mathbf{I}-\mathbf{P})[R]|||_{L^2}^2 + o(1)\varepsilon^2\,|||\partial_t\nabla_x\psi|||_{L^2}^2 .\end{aligned}\tag{4.177}$$

In addition, based on Lemma 4.4, we have

$$\begin{aligned}\left|\langle\!\langle q\mu^{\frac12}, h\rangle\!\rangle_{\overline{\gamma}_-}\right| + \left|\langle\!\langle\nabla_x\psi : \mathscr{B}, h\rangle\!\rangle_{\overline{\gamma}_-}\right| &\lesssim o(1)\left\|q\mu^{\frac12}\right\|_{L^2_{\overline{\gamma}_-}}^2 + o(1)\left\|\nabla_x\psi : \mathscr{B}\right\|_{L^2_{\overline{\gamma}_-}}^2 + \|h\|_{L^2_{\overline{\gamma}_-}}^2\\&\lesssim o(1)\,|||\mathbf{b}|||_{L^2}^2 + o_T\varepsilon^2,\end{aligned}\tag{4.178}$$

$$\begin{aligned}\left|\langle\!\langle q\mu^{\frac12}, R\rangle\!\rangle_{\overline{\gamma}_+}\right| + \left|\langle\!\langle\nabla_x\psi : \mathscr{B}, R\rangle\!\rangle_{\overline{\gamma}_+}\right| &\lesssim o(1)\left\|q\mu^{\frac12}\right\|_{L^2_{\overline{\gamma}_+}}^2 + o(1)\left\|\nabla_x\psi : \mathscr{B}\right\|_{L^2_{\overline{\gamma}_+}}^2 + \|R\|_{L^2_{\overline{\gamma}_+}}^2\\&\lesssim o(1)\,|||\mathbf{b}|||_{L^2}^2 + \|R\|_{L^2_{\overline{\gamma}_+}}^2 ,\end{aligned}\tag{4.179}$$

and

$$\begin{aligned}\left|\left\langle\!\!\left\langle v\cdot\nabla_x\Big(\nabla_x\psi : \mathscr{B}\Big), (\mathbf{I}-\mathbf{P})[R]\right\rangle\!\!\right\rangle\right| &\lesssim o(1)\left|\left|\left|\nabla_x^2\psi\right|\right|\right|_{L^2}^2 + |||(\mathbf{I}-\mathbf{P})[R]|||_{L^2}^2\\&\lesssim o(1)\,|||\mathbf{b}|||_{L^2}^2 + |||(\mathbf{I}-\mathbf{P})[R]|||_{L^2}^2 .\end{aligned}\tag{4.180}$$

In summary, we have shown that

$$\begin{aligned}&\|\nabla_x\psi(t)\|_{L^2}^2 + |||\mathbf{b}|||_{L^2}^2 \lesssim \varepsilon\,\|R(t)\|_{L^2}^2 + \|R\|_{L^2_{\overline{\gamma}_+}}^2 + |||(\mathbf{I}-\mathbf{P})[R]|||_{L^2}^2 + o(1)\varepsilon^2\,|||\partial_t\nabla_x\psi|||_{L^2}^2 + o_T\varepsilon^2\\&\quad + \left|\left\langle\!\!\left\langle q\mu^{\frac12}, S\right\rangle\!\!\right\rangle\right| + \left|\varepsilon^{-1}\left\langle\!\!\left\langle\psi\cdot v\mu^{\frac12}, S\right\rangle\!\!\right\rangle\right| + \left|\langle\!\langle\nabla_x\psi : \mathscr{B}, S\rangle\!\rangle\right| .\end{aligned}\tag{4.181}$$

<u>Source Term Estimates</u> Due to orthogonality of Γ, we have

$$\left|\left\langle\!\!\left\langle q\mu^{\frac12}, S\right\rangle\!\!\right\rangle\right| + \left|\varepsilon^{-1}\left\langle\!\!\left\langle\psi\cdot v\mu^{\frac12}, S\right\rangle\!\!\right\rangle\right| = \left|\left\langle\!\!\left\langle q\mu^{\frac12}, S_1+S_2\right\rangle\!\!\right\rangle\right| + \left|\varepsilon^{-1}\left\langle\!\!\left\langle\psi\cdot v\mu^{\frac12}, S_1+S_2\right\rangle\!\!\right\rangle\right| .\tag{4.182}$$

Using Lemma 4.5, we have

$$\begin{aligned}\left|\left\langle\!\!\left\langle q\mu^{\frac12}, S_1\right\rangle\!\!\right\rangle\right| + \left|\varepsilon^{-1}\left\langle\!\!\left\langle\psi\cdot v\mu^{\frac12}, S_1\right\rangle\!\!\right\rangle\right| &= \varepsilon^2\left|\left\langle\!\!\left\langle q\mu^{\frac12}, \partial_t f_2\right\rangle\!\!\right\rangle\right| + \varepsilon\left|\left\langle\!\!\left\langle\psi\cdot v\mu^{\frac12}, \partial_t f_2\right\rangle\!\!\right\rangle\right|\\&\lesssim \varepsilon\Big(|||q|||_{L^2} + |||\nabla_x\psi|||_{L^2}\Big)|||\partial_t f_2|||_{L^2} \lesssim o_T\,|||\mathbf{b}|||_{L^2}^2 + o_T\varepsilon^2.\end{aligned}\tag{4.183}$$

Using Lemma 4.6 and Remark 4.7, integrating by parts in v_η for S_{2a}, we obtain

$$\left|\left\langle\!\!\left\langle q\mu^{\frac12}, S_2\right\rangle\!\!\right\rangle\right| \lesssim |||q|||_{L^2}\left|\left|\left|f_1^B + S_{2b} + S_{2c}\right|\right|\right|_{L^2_tL^2_xL^1_v} \lesssim o_T\varepsilon^{\frac12}\,|||q|||_{L^2} \lesssim o_T\,|||\mathbf{b}|||_{L^2}^2 + o_T\varepsilon.\tag{4.184}$$

Similar to (4.121), we have

$$\begin{aligned}&\left|\varepsilon^{-1}\left\langle\!\!\left\langle\psi\cdot v\mu^{\frac12}, S_2\right\rangle\!\!\right\rangle\right| \lesssim \varepsilon^{-1}\left|\left\langle\!\!\left\langle S_2, \int_0^{\mathfrak{n}}\partial_{\mathfrak{n}}\psi\right\rangle\!\!\right\rangle\right|\\&\lesssim \left|\left\langle\!\!\left\langle \eta S_2, \frac{1}{\mathfrak{n}}\int_0^{\mathfrak{n}}\partial_{\mathfrak{n}}\psi\right\rangle\!\!\right\rangle\right| \lesssim \left|\left|\left|\eta\big(f_1^B + S_{2b} + S_{2c}\big)\right|\right|\right|_{L^2_tL^2_xL^1_v}\left|\left|\left|\frac{1}{\mathfrak{n}}\int_0^{\mathfrak{n}}\partial_{\mathfrak{n}}\psi\right|\right|\right|_{L^2}\end{aligned}$$

$$\lesssim \left\|\left\| \eta\left(f_1^B + S_{2b} + S_{2c}\right)\right\|\right\|_{L_t^2 L_x^2 L_v^1} \left\|\left|\partial_{\mathfrak{n}}\psi\right|\right\|_{L^2} \lesssim o_T \left\|\left|\mathbf{b}\right|\right\|_{L^2}^2 + o_T \varepsilon. \tag{4.185}$$

From Lemma 4.5, we directly bound

$$\left|\langle\langle \nabla_x \psi : \mathscr{B}, S_1 \rangle\rangle\right| \lesssim \left\|\left|\nabla_x \psi\right|\right\|_{L^2} \left\|\left|S_1\right|\right\|_{L^2} \lesssim o_T \left\|\left|\mathbf{b}\right|\right\|_{L^2}^2 + o_T \varepsilon^2. \tag{4.186}$$

Similar to (4.121), based on Lemma 4.6, Remark 4.7 and Hardy's inequality, we have

$$\begin{aligned}
&\left|\langle\langle \nabla_x \psi : \mathscr{B}, S_2 \rangle\rangle\right| \leq \left|\left\langle\left\langle S_2, \nabla_x \psi\Big|_{\mathfrak{n}=0}\right\rangle\right\rangle\right| + \left|\varepsilon \left\langle\left\langle \eta S_2, \frac{1}{\mathfrak{n}}\int_0^{\mathfrak{n}} \partial_{\mathfrak{n}} \nabla_x \psi\right\rangle\right\rangle\right| \\
&\lesssim \left\|\left| f_1^B + S_{2b} + S_{2c}\right|\right\|_{L_t^2 L_{\iota_1\iota_2}^2 L_{\mathfrak{n}}^1 L_v^1} \left\|\nabla_x \psi\right\|_{L_\gamma^2} + \varepsilon \left\|\left| \eta\left(f_1^B + S_{2b} + S_{2c}\right)\right|\right\|_{L^2} \left\|\left| \frac{1}{\mathfrak{n}}\int_0^{\mathfrak{n}} \partial_{\mathfrak{n}} \nabla_x \psi\right|\right\|_{L^2} \\
&\lesssim \left\|\left| f_1^B + S_{2b} + S_{2c}\right|\right\|_{L_t^2 L_{\iota_1\iota_2}^2 L_{\mathfrak{n}}^1 L_v^1} \left\|\nabla_x \psi\right\|_{L_\gamma^2} \\
&\quad + \varepsilon \left\|\left| \eta\left(f_1^B + S_{2b} + S_{2c}\right)\right|\right\|_{L^2} \left\|\left|\partial_{\mathfrak{n}} \nabla_x \psi\right|\right\|_{L^2} \lesssim o_T \varepsilon \left\|\left|\mathbf{b}\right|\right\|_{L^2} \lesssim o_T \left\|\left|\mathbf{b}\right|\right\|_{L^2}^2 + o_T \varepsilon^2.
\end{aligned} \tag{4.187}$$

Based on Lemma 4.8, Lemma 4.9, and Lemma 4.10, we have

$$\begin{aligned}
\left|\langle\langle \nabla_x \psi : \mathscr{B}, S_3 + S_4 + S_5 \rangle\rangle\right| &\lesssim \left\|\left|\nabla_x \psi\right|\right\|_{L^2} \left\|\left|S_3 + S_4 + S_5\right|\right\|_{L^2} \\
&\lesssim o_T \left\|\left|\mathbf{b}\right|\right\|_{L^2}^2 + o_T \left\|\left|R\right|\right\|_{L^2}^2 + o_T \varepsilon \lesssim o_T \left\|\left|\mathbf{b}\right|\right\|_{L^2}^2 + o_T \varepsilon \left\|\left|R\right|\right\|_X^2 + o_T \varepsilon.
\end{aligned} \tag{4.188}$$

Finally, based on Lemma 4.11, we have

$$\left|\langle\langle \nabla_x \psi : \mathscr{B}, S_6 \rangle\rangle\right| \lesssim \left|\left\langle\left\langle \nabla_x \psi : \mathscr{B}, \Gamma\Big[\mathbf{P}[R], \mathbf{P}[R]\Big]\right\rangle\right\rangle\right| + \left|\left\langle\left\langle \nabla_x \psi : \mathscr{B}, \Gamma\Big[R, (\mathbf{I}-\mathbf{P})[R]\Big]\right\rangle\right\rangle\right|. \tag{4.189}$$

The oddness and orthogonality implies that

$$\begin{aligned}
\left|\left\langle\left\langle \nabla_x \psi : \mathscr{B}, \Gamma\Big[\mathbf{P}[R], \mathbf{P}[R]\Big]\right\rangle\right\rangle\right| &\lesssim \left|\left\langle\left\langle \nabla_x \psi : \mathscr{B}, \Gamma\left[\mu^{\frac{1}{2}}(v\cdot\mathbf{b}), \mu^{\frac{1}{2}}(v\cdot\mathbf{b})\right]\right\rangle\right\rangle\right| \\
&\quad + \left|\left\langle\left\langle \nabla_x \psi : \mathscr{B}, \Gamma\left[\mu^{\frac{1}{2}}\left(\frac{|v|^2-5}{2}c\right), \mu^{\frac{1}{2}}\left(\frac{|v|^2-5}{2}c\right)\right]\right\rangle\right\rangle\right|.
\end{aligned} \tag{4.190}$$

Similar to (4.154), we may directly bound

$$\begin{aligned}
\left|\left\langle\left\langle \nabla_x \psi : \mathscr{B}, \Gamma\left[\mu^{\frac{1}{2}}(v\cdot\mathbf{b}), \mu^{\frac{1}{2}}(v\cdot\mathbf{b})\right]\right\rangle\right\rangle\right| &\lesssim \left\|\left|\nabla_x \psi\right|\right\|_{L^6} \left\|\left|\mathbf{b}\right|\right\|_{L^3} \left\|\left|\mathbf{b}\right|\right\|_{L^2} \\
&\lesssim \left\|\left|\psi\right|\right\|_{L_t^2 H_x^2} \left\|\left|\mathbf{b}\right|\right\|_{L^3} \left\|\left|\mathbf{b}\right|\right\|_{L^2} \lesssim \varepsilon^{\frac{1}{6}} \left\|\left|R\right|\right\|_X \left\|\left|\mathbf{b}\right|\right\|_{L^2}^2.
\end{aligned} \tag{4.191}$$

Due to oddness and $\mathscr{B}_{ii} = \mathcal{L}^{-1}\left[\left(|v_i|^2 - \frac{1}{3}|v|^2\right)\mu^{\frac{1}{2}}\right]$, noting that $\Gamma\left[\mu^{\frac{1}{2}}\left(\frac{|v|^2-5}{2}c\right), \mu^{\frac{1}{2}}\left(\frac{|v|^2-5}{2}c\right)\right]$ only depends on $|v|^2$, we have

$$\left|\left\langle\left\langle \nabla_x \psi : \mathscr{B}, \Gamma\left[\mu^{\frac{1}{2}}\left(\frac{|v|^2-5}{2}c\right), \mu^{\frac{1}{2}}\left(\frac{|v|^2-5}{2}c\right)\right]\right\rangle\right\rangle\right|$$

$$
\begin{aligned}
&= \left| \left\langle\!\!\left\langle \partial_1\psi_1\mathscr{B}_{11} + \partial_2\psi_2\mathscr{B}_{22} + \partial_3\psi_3\mathscr{B}_{33}, \Gamma\left[\mu^{\frac12}\left(\frac{|v|^2-5}{2}c\right), \mu^{\frac12}\left(\frac{|v|^2-5}{2}c\right)\right] \right\rangle\!\!\right\rangle \right| \\
&= \left| \left\langle\!\!\left\langle (\nabla_x\cdot\psi)\mathscr{B}_{ii}, \Gamma\left[\mu^{\frac12}\left(\frac{|v|^2-5}{2}c\right), \mu^{\frac12}\left(\frac{|v|^2-5}{2}c\right)\right] \right\rangle\!\!\right\rangle \right| = 0. \qquad (4.192)
\end{aligned}
$$

In addition, similar to (4.155), we have

$$
\begin{aligned}
\left|\langle\!\langle \nabla_x\psi : \mathscr{B}, \Gamma\Big[R, (\mathbf{I}-\mathbf{P})[R]\Big]\rangle\!\rangle\right| &\lesssim |||\nabla_x\psi|||_{L^6}\, |||R|||_{L^2}\, |||(\mathbf{I}-\mathbf{P})[R]|||_{L^3} \\
&\lesssim \varepsilon\, |||\mathbf{b}|||_{L^2}\, |||R|||_X^2 \lesssim o(1)\, |||\mathbf{b}|||_{L^2}^2 + \varepsilon^2\, |||R|||_X^4 .
\end{aligned} \qquad (4.193)
$$

Hence, we know

$$
\left|\langle\!\langle \nabla_x\psi : \mathscr{B}, S_6\rangle\!\rangle\right| \lesssim \varepsilon^{\frac16}\, |||R|||_X\, |||\mathbf{b}|||_{L^2}^2 + o(1)\, |||\mathbf{b}|||_{L^2}^2 + \varepsilon^2\, |||R|||_X^4 . \qquad (4.194)
$$

In summary, we have shown that

$$
\begin{aligned}
&\left|\left\langle\!\!\left\langle q\mu^{\frac12}, S\right\rangle\!\!\right\rangle\right| + \left|\varepsilon^{-1}\left\langle\!\!\left\langle \psi\cdot v\mu^{\frac12}, S\right\rangle\!\!\right\rangle\right| \\
&+ \left|\langle\!\langle \nabla_x\psi : \mathscr{B}, S\rangle\!\rangle\right| \\
&\lesssim \varepsilon^{\frac16}\, |||R|||_X\, |||\mathbf{b}|||_{L^2}^2 + \big(o(1)+o_T\big)\, |||\mathbf{b}|||_{L^2}^2 + o_T\varepsilon\, |||R|||_X^2 + \varepsilon\, |||R|||_X^4 + o_T\varepsilon.
\end{aligned} \qquad (4.195)
$$

Inserting (4.195) into (4.181), we have

$$
\begin{aligned}
\|\nabla_x\psi(t)\|_{L^2}^2 + |||\mathbf{b}|||_{L^2}^2 \lesssim{}& \varepsilon^{\frac16}\, |||R|||_X\, |||\mathbf{b}|||_{L^2}^2 + \varepsilon\, \|R(t)\|_{L^2}^2 + \|R\|_{L^2_{\gamma_+}}^2 \\
&+ |||(\mathbf{I}-\mathbf{P})[R]|||_{L^2}^2 + o(1)\varepsilon^2\, |||\partial_t\nabla_x\psi|||_{L^2}^2 \\
&+ o_T\varepsilon\, |||R|||_X^2 + \varepsilon\, |||R|||_X^4 + o_T\varepsilon.
\end{aligned} \qquad (4.196)
$$

Estimate of $|||\partial_t\nabla_x\psi|||_{L^2}$ Denote $\Psi = \partial_t\psi$. Taking $g = \varepsilon\Psi\cdot v\mu^{\frac12}$ in (4.41), due to orthogonality and $\Psi\big|_{\partial\Omega} = 0$, we obtain

$$
\varepsilon^2\langle\!\langle \partial_t R, \Psi\cdot v\mu^{\frac12}\rangle\!\rangle - \varepsilon\langle\!\langle R, v\cdot\nabla_x\left(\Psi\cdot v\mu^{\frac12}\right)\rangle\!\rangle = \varepsilon\langle\!\langle S, \Psi\cdot v\mu^{\frac12}\rangle\!\rangle. \qquad (4.197)
$$

Noticing that Ψ is divergence-free and that $\Psi\big|_{\partial\Omega} = 0$, we find

$$
\begin{aligned}
\varepsilon^2\langle\!\langle \partial_t R, \Psi\cdot v\mu^{\frac12}\rangle\!\rangle &= \varepsilon^2\langle\!\langle \partial_t\mathbf{b}, \Psi\rangle\!\rangle = \varepsilon^2\langle\!\langle -\lambda\Delta_x\Psi + \partial_t\nabla_x q, \Psi\rangle\!\rangle \\
&= \varepsilon^2\langle\!\langle -\lambda\Delta_x\Psi, \Psi\rangle\!\rangle = \lambda\varepsilon^2\, \|\partial_t\nabla_x\psi\|_{L^2}^2 .
\end{aligned} \qquad (4.198)
$$

Also, using orthogonality, we have

$$
\begin{aligned}
\left|\varepsilon\langle\!\langle R, v\cdot\nabla_x\left(\Psi\cdot v\mu^{\frac12}\right)\rangle\!\rangle\right| &= \varepsilon\left|\left\langle\!\!\left\langle \mu^{\frac12}p + \mu^{\frac12}\left(\frac{|v|^2-5}{2}\right)c, v\cdot\nabla_x\left(\Psi\cdot v\mu^{\frac12}\right)\right\rangle\!\!\right\rangle\right| \\
&= \varepsilon\left|\left\langle\!\!\left\langle \mu^{\frac12}p, v\cdot\nabla_x\left(\Psi\cdot v\mu^{\frac12}\right)\right\rangle\!\!\right\rangle\right| \lesssim |||p|||_{L^2}^2 + o(1)\varepsilon^2\, |||\partial_t\nabla_x\psi|||_{L^2}^2 ,
\end{aligned} \qquad (4.199)
$$

Then, by a similar argument as the above estimates for $\varepsilon^{-1}\left\langle\!\left\langle \psi \cdot v\mu^{\frac{1}{2}}, S\right\rangle\!\right\rangle$, we have

$$\left|\varepsilon\langle\!\langle S, \Psi \cdot v\mu^{\frac{1}{2}}\rangle\!\rangle\right| = \left|\varepsilon\langle\!\langle S_1 + S_2, \Psi \cdot v\mu^{\frac{1}{2}}\rangle\!\rangle\right| \lesssim o(1)\varepsilon^2 \,|||\partial_t \nabla_x \varphi|||_{L^2}^2 + o_T\varepsilon. \tag{4.200}$$

In summary, we have shown that

$$\varepsilon^2 \,|||\partial_t \nabla_x \varphi|||_{L^2}^2 \lesssim |||p|||_{L^2}^2 + o_T\varepsilon. \tag{4.201}$$

Inserting (4.201) into (4.196), we have

$$\begin{aligned} \|\nabla_x \psi(t)\|_{L^2}^2 + |||\mathbf{b}|||_{L^2}^2 \lesssim \varepsilon^{\frac{1}{6}} \,|||R|||_X \,|||\mathbf{b}|||_{L^2}^2 + \varepsilon \,\|R(t)\|_{L^2}^2 \\ + \|R\|_{L^2_{\gamma_+}}^2 + o(1)\,|||p|||_{L^2}^2 + |||(\mathbf{I}-\mathbf{P})[R]|||_{L^2}^2 \\ + o_T\varepsilon \,|||R|||_X^2 + \varepsilon \,|||R|||_X^4 + o_T\varepsilon. \end{aligned} \tag{4.202}$$

Hence, (4.167) follows. □

Proposition 4.24. *Let R be the solution to (1.35). Under the assumptions (1.23),(1.28), (1.31), we have*

$$\varepsilon^{-\frac{1}{2}} \,|||\partial_t \mathbf{b}|||_{L^2} \lesssim o(1)\varepsilon^{-\frac{1}{2}} \,|||\partial_t p|||_{L^2} + o_T \,|||R|||_X + |||R|||_X^2 + o_T. \tag{4.203}$$

Proof. We may use a similar argument as proving Proposition 4.23 to the equation (4.37). Notice that in the bounds (4.191) and (4.193), we should always assign L^2 norm to the time-derivative terms and L^3 to the no-derivative terms. □

4.4.4. Summary of kernel estimates

Proposition 4.25. *Let R be the solution to (1.35). Under the assumptions (1.23),(1.28), (1.31), we have*

$$\varepsilon^{-\frac{1}{2}} \,|||\mathbf{P}[R]|||_{L^2} \lesssim o_T \,|||R|||_X + |||R|||_X^2 + o_T. \tag{4.204}$$

Proof. Summarizing Proposition 4.19, Proposition 4.21 and Proposition 4.23 leads to the desired result. □

Proposition 4.26. *Let R be the solution to (1.35). Under the assumptions (1.23),(1.28), (1.31), we have*

$$\varepsilon^{-\frac{1}{2}} \,|||\mathbf{P}[\partial_t R]|||_{L^2} \lesssim o_T \,|||R|||_X + |||R|||_X^2 + o_T. \tag{4.205}$$

Proof. Summarizing Proposition 4.20, Proposition 4.22 and Proposition 4.24 leads to the desired result. □

4.5. Energy estimate: instantaneous.

Proposition 4.27. *Let R be the solution to (1.35). Under the assumptions (1.23),(1.28), (1.31), we have*

$$\varepsilon^{-\frac{1}{2}} |R(t)|_{L^2_{\gamma+}} + \varepsilon^{-1} \|(\mathbf{I}-\mathbf{P})[R](t)\|_{L^2} \lesssim o_T \, |||R|||_X + |||R|||_X^2 + o_T. \tag{4.206}$$

Proof. For fixed $t \in \mathbb{R}_+$, we apply a similar argument as the proof of Proposition 3.27 to (4.39), and obtain

$$\varepsilon^{-1} |R(t)|^2_{L^2_{\gamma+}} + \varepsilon^{-2} \|(\mathbf{I}-\mathbf{P})[R](t)\|^2_{L^2} \lesssim o_T \, |||R|||_X^2 + |||R|||_X^4 + o_T + \left|\varepsilon^{-1}\langle \varepsilon\partial_t R(t), R(t)\rangle\right|. \tag{4.207}$$

Using Proposition 4.17 and Proposition 4.18, we have

$$\left|\varepsilon^{-1}\langle \varepsilon\partial_t R(t), R(t)\rangle\right| = \left|\langle \partial_t R(t), R(t)\rangle\right| \lesssim \|R(t)\|^2_{L^2} + \|\partial_t R(t)\|^2_{L^2} \lesssim o_T \, |||R|||_X^2 + |||R|||_X^4 + o_T. \tag{4.208}$$

□

Corollary 4.28. *Let R be the solution to (1.35). Under the assumptions (1.23),(1.28), (1.31), we have*

$$\|(\mathbf{I}-\mathbf{P})[R](t)\|_{L^6} + \left|\mu^{\frac{1}{4}} R(t)\right|_{L^4_{\gamma+}} \lesssim o_T \, |||R|||_X + |||R|||_X^2 + o_T. \tag{4.209}$$

Proof. This is similar to the proof of Corollary 3.28. □

4.6. Kernel estimate: instantaneous.

Proposition 4.29. *Let R be the solution to (1.35). Under the assumptions (1.23),(1.28), (1.31), we have*

$$\|p(t)\|_{L^6} \lesssim o_T \, |||R|||_X + |||R|||_X^2 + o_T. \tag{4.210}$$

Proof. For fixed $t \in \mathbb{R}_+$, we apply a similar argument as the proof of Proposition 3.29 with $r = 6$ to (4.39). We obtain for ψ defined in (3.174)

$$\|p(t)\|^6_{L^6} \lesssim o_T \, |||R|||_X^6 + |||R|||_X^{12} + o_T + \left|\langle \varepsilon\partial_t R(t), \psi\rangle\right|. \tag{4.211}$$

Using Proposition 4.18, we have

$$\begin{aligned} \left|\langle \varepsilon\partial_t R(t), \psi\rangle\right| &\lesssim \varepsilon \|\partial_t R(t)\|_{L^2} \|\psi\|_{L^2} \lesssim \varepsilon \|\partial_t R(t)\|_{L^2} \|\psi\|_{W^{1,\frac{6}{5}}} \\ &\lesssim \varepsilon\Big(o_T \, |||R|||_X + |||R|||_X^2 + o_T\Big) \|p(t)\|^5_{L^6} \lesssim \varepsilon^6 \|p(t)\|^6_{L^6} + o_T \, |||R|||_X^6 \\ &\quad + |||R|||_X^{12} + o_T. \end{aligned} \tag{4.212}$$

Hence, we have

$$\|p(t)\|^6_{L^6} \lesssim o_T \, |||R|||_X^6 + |||R|||_X^{12} + o_T, \tag{4.213}$$

and thus (4.210) follows. □

Proposition 4.30. *Let R be the solution to (1.35). Under the assumptions (1.23),(1.28), (1.31), we have*

$$\|c(t)\|_{L^6} \lesssim o_T \, |||R|||_X + |||R|||_X^2 + o_T. \tag{4.214}$$

Proof. For fixed $t \in \mathbb{R}_+$, we apply a similar argument as the proof of Proposition 3.30 with $r = 6$ to (4.39). We obtain for φ defined in (3.189)

$$\begin{aligned}\|c(t)\|_{L^6}^6 \lesssim o_T \, |||R|||_X^6 + |||R|||_X^{12} + o_T + \left|\varepsilon^{-1}\left\langle \varphi\big(|v|^2-5\big)\mu^{\frac12}, \varepsilon\partial_t R(t)\right\rangle\right| \\ + \left|\left\langle \nabla_x\varphi\cdot\mathscr{A}, \varepsilon\partial_t R(t)\right\rangle\right|.\end{aligned} \tag{4.215}$$

Using Proposition 4.18, we have

$$\begin{aligned}&\left|\varepsilon^{-1}\left\langle \varphi\big(|v|^2-5\big)\mu^{\frac12}, \varepsilon\partial_t R(t)\right\rangle\right| + \left|\left\langle \nabla_x\varphi\cdot\mathscr{A}, \varepsilon\partial_t R(t)\right\rangle\right| \\ &\lesssim \|\partial_t R(t)\|_{L^2}\,\|\varphi\|_{H^1} \lesssim \|\partial_t R(t)\|_{L^2}\,\|\varphi\|_{W^{2,\frac65}} \lesssim \Big(o_T \, |||R|||_X + |||R|||_X^2 + o_T\Big)\,\|c(t)\|_{L^6}^5 \\ &\lesssim o(1)\,\|c(t)\|_{L^6}^6 + o_T \, |||R|||_X^6 + |||R|||_X^{12} + o_T.\end{aligned} \tag{4.216}$$

Hence, we have

$$\|c(t)\|_{L^6}^6 \lesssim o_T \, |||R|||_X^6 + |||R|||_X^{12} + o_T, \tag{4.217}$$

and thus (4.214) follows. □

Proposition 4.31. *Let R be the solution to (1.35). Under the assumptions (1.23),(1.28), (1.31), we have*

$$\|\mathbf{b}(t)\|_{L^6} \lesssim o_T \, |||R|||_X + |||R|||_X^2 + o_T. \tag{4.218}$$

Proof. For fixed $t \in \mathbb{R}_+$, we apply a similar argument as the proof of Proposition 3.31 with $r = 6$ to (4.39). We obtain for ψ and q defined in (3.210)

$$\begin{aligned}&\|\mathbf{b}(t)\|_{L^6}^6 \lesssim o_T \, |||R|||_X^6 + |||R|||_X^{12} + o_T \\ &\quad + \left|\left\langle q\mu^{\frac12}, \varepsilon\partial_t R(t)\right\rangle\right| + \left|\varepsilon^{-1}\left\langle \psi\cdot v\mu^{\frac12}, \varepsilon\partial_t R(t)\right\rangle\right| + \left|\left\langle \nabla_x\psi : \mathscr{B}, \varepsilon\partial_t R(t)\right\rangle\right|.\end{aligned} \tag{4.219}$$

Using Proposition 4.18, we have

$$\begin{aligned}&\left|\left\langle q\mu^{\frac12}, \varepsilon\partial_t R(t)\right\rangle\right| + \left|\varepsilon^{-1}\left\langle \psi\cdot v\mu^{\frac12}, \varepsilon\partial_t R(t)\right\rangle\right| + \left|\left\langle \nabla_x\psi : \mathscr{B}, \varepsilon\partial_t R(t)\right\rangle\right| \\ &\lesssim \|\partial_t R(t)\|_{L^2}\Big(\|\psi\|_{H^1} + \|q\|_{L^2}\Big) \lesssim \varepsilon\,\|\partial_t R(t)\|_{L^2}\Big(\|\psi\|_{W^{2,\frac65}} + \|q\|_{W^{1,\frac65}}\Big) \\ &\lesssim \Big(o_T \, |||R|||_X + |||R|||_X^2 + o_T\Big)\,\|\mathbf{b}(t)\|_{L^6}^5 \lesssim o(1)\,\|\mathbf{b}(t)\|_{L^6}^6 + o_T \, |||R|||_X^6 + |||R|||_X^{12} + o_T.\end{aligned} \tag{4.220}$$

Hence, we have

$$\|\mathbf{b}(t)\|_{L^6}^6 \lesssim o_T \, |||R|||_X^6 + |||R|||_X^{12} + o_T, \tag{4.221}$$

and thus (4.218) follows. □

Proposition 4.32. *Let R be the solution to (1.35). Under the assumptions (1.23),(1.28), (1.31), we have*

$$\|\mathbf{P}[R](t)\|_{L^6} \lesssim o_T \, |||R|||_X + |||R|||_X^2 + o_T. \tag{4.222}$$

Proof. Summarizing Proposition 4.29, Proposition 4.30 and Proposition 4.31 leads to the desired result. □

4.7. L^∞ estimate. We define a weight function as (3.235).

Proposition 4.33. *Let R be the solution to (1.35). Under the assumptions (1.23),(1.28), (1.31), we have*

$$\varepsilon^{\frac{1}{2}} \left\| R \right\|_{L^\infty_{\varrho,\vartheta}} + \varepsilon^{\frac{1}{2}} \left\| R \right\|_{L^\infty_{\overline{\gamma}_+,\varrho,\vartheta}} \lesssim o_T \left\| R \right\|_X + \left\| R \right\|_X^2 + o_T. \tag{4.223}$$

Proof. We will use the well-known $L^2 - L^6 - L^\infty$ framework.

Step 1: Mild Formulation Denote the weighted solution

$$R_w(t,x,v) := w(v)R(t,x,v), \tag{4.224}$$

and the weighted non-local operator

$$K_{w(v)}[R_w](v) := w(v)K\left[\frac{R_w}{w}\right](v) = \int_{\mathbb{R}^3} k_{w(v)}(v,\mathfrak{u})R_w(\mathfrak{u})\mathrm{d}\mathfrak{u}, \tag{4.225}$$

where

$$k_{w(v)}(v,\mathfrak{u}) := k(v,\mathfrak{u})\frac{w(v)}{w(\mathfrak{u})}. \tag{4.226}$$

Multiplying εw on both sides of (1.10), we have

$$\begin{cases} \varepsilon^2\partial_t R_w + \varepsilon v\cdot\nabla_x R_w + \nu R_w = K_w[R_w](x,v) + \varepsilon w(v)S(t,x,v) \ \text{ in } \ \mathbb{R}_+\times\Omega\times\mathbb{R}^3,\\ R_w(0,x,v) = wz(x,v) \ \text{ in } \ \Omega\times\mathbb{R}^3,\\ R_w(t,x_0,v) = wh(t,x_0,v) \ \text{ for } \ x_0\in\partial\Omega \ \text{ and } \ v\cdot n<0, \end{cases} \tag{4.227}$$

We can rewrite the solution of the equation (4.227) along the characteristics by Duhamel's principle as

$$\begin{aligned} R_w(t,x,v) &= \mathbf{1}_{t_i<t_b} w(v)z(\overline{x},v)\mathrm{e}^{-\nu(v)t_i} + \mathbf{1}_{t_b<t_i} w(v)h(\overline{x},v)\mathrm{e}^{-\nu(v)t_b}\\ &\quad + \int_0^{\overline{t}} w(v)\varepsilon S\Big(t-\varepsilon^2 s, x-\varepsilon(\overline{t}-s)v, v\Big)\mathrm{e}^{-\nu(v)(\overline{t}-s)}\mathrm{d}s\\ &\quad + \int_0^{\overline{t}}\int_{\mathbb{R}^3} k_{w(v)}(v,\mathfrak{u})R_w\Big(t-\varepsilon^2 s, x-\varepsilon(\overline{t}-s)\mathfrak{u}, \mathfrak{u}\Big)\mathrm{e}^{-\nu(v)(\overline{t}-s)}\mathrm{d}\mathfrak{u}\mathrm{d}s, \end{aligned} \tag{4.228}$$

where

$$t_b(x,v) := \inf\Big\{t>0 : x-\varepsilon t v\notin\Omega\Big\}, \quad t_i(t) = \varepsilon^{-2}t, \quad \overline{t} = \min\Big\{t_i,t_b\Big\}, \tag{4.229}$$

and

$$\overline{x}(x,v) := x - \varepsilon\overline{t}(x,v)v. \tag{4.230}$$

We further rewrite the non-local term along the characteristics as

$$\begin{aligned} &R_w(t,x,v)\\ &= \mathbf{1}_{t_i<t_b} w(v)z(\overline{x},v)\mathrm{e}^{-\nu(v)t_i} + \mathbf{1}_{t_b<t_i} w(v)h(\overline{x},v)\mathrm{e}^{-\nu(v)t_b} \end{aligned}$$

$$
\begin{aligned}
&+ \int_0^{\bar{t}} w(v)\varepsilon S\Big(t-\varepsilon^2 s, x-\varepsilon(\bar{t}-s)v, v\Big)\mathrm{e}^{-\nu(v)(\bar{t}-s)}\mathrm{d}s \\
&+ \mathbf{1}_{t_i'<t_b'}\int_0^{\bar{t}}\int_{\mathbb{R}^3} k_{w(v)}(v,\mathfrak{u})w(\mathfrak{u})z(\overline{x}',v)\mathrm{e}^{-\nu(\mathfrak{u})t_i'}\mathrm{e}^{-\nu(v)(\bar{t}-s)}\mathrm{d}\mathfrak{u}\mathrm{d}s \\
&+ \mathbf{1}_{t_b'<t_i'}\int_0^{\bar{t}}\int_{\mathbb{R}^3} k_{w(v)}(v,\mathfrak{u})w(\mathfrak{u})h(\overline{x}',v)\mathrm{e}^{-\nu(\mathfrak{u})t_b'}\mathrm{e}^{-\nu(v)(\bar{t}-s)}\mathrm{d}\mathfrak{u}\mathrm{d}s \\
&+ \int_0^{\bar{t}}\int_{\mathbb{R}^3} k_{w(v)}(v,\mathfrak{u})\int_0^{\bar{t}'}\varepsilon S\Big(t-\varepsilon^2 s-\varepsilon^2 r, x-\varepsilon(\bar{t}-s)\mathfrak{u}-\varepsilon(\bar{t}'-r)\mathfrak{u},\mathfrak{u}\Big) \\
&\quad \mathrm{e}^{-\nu(\mathfrak{u})(\bar{t}'-r)}\mathrm{e}^{-\nu(v)(\bar{t}-s)}\mathrm{d}r\mathrm{d}\mathfrak{u}\mathrm{d}s \\
&+ \int_0^{\bar{t}}\int_{\mathbb{R}^3} k_{w(v)}(v,\mathfrak{u})\int_0^{\bar{t}'}\int_{\mathbb{R}^3} k_{w(\mathfrak{u})}(\mathfrak{u},\mathfrak{u}')R_w\Big(t-\varepsilon^2 s-\varepsilon^2 r, x-\varepsilon(\bar{t}-s)\mathfrak{u}-\varepsilon(\bar{t}'-r)\mathfrak{u}',\mathfrak{u}'\Big) \\
&\quad \mathrm{e}^{-\nu(\mathfrak{u})(\bar{t}'-r)}\mathrm{e}^{-\nu(v)(\bar{t}-s)}\mathrm{d}\mathfrak{u}'\mathrm{d}r\mathrm{d}\mathfrak{u}\mathrm{d}s, \qquad (4.231)
\end{aligned}
$$

where

$$
\begin{aligned}
t_b'(x,v;s,\mathfrak{u}) &:= \inf\Big\{t>0: x-\varepsilon(\bar{t}-s)-\varepsilon t\mathfrak{u}\notin\Omega\Big\}, \quad t_i'(t;s)=\varepsilon^{-2}t-s, \\
\bar{t}' &= \min\Big\{t_i', t_b'\Big\}, \qquad (4.232)
\end{aligned}
$$

and

$$
\overline{x}'(x,v;s,\mathfrak{u}) := x-\varepsilon(\bar{t}-s)-\varepsilon\bar{t}'(x,v;s,\mathfrak{u})\mathfrak{u}. \qquad (4.233)
$$

Step 2: Estimates of Source Terms and Boundary Terms Based on Lemma 4.3 – Lemma 4.11, we have

$$
\begin{aligned}
&\left|\mathbf{1}_{t_i<t_b} w(v)z(\overline{x},v)\mathrm{e}^{-\nu(v)t_i}\right| \\
&+\left|\mathbf{1}_{t_i'<t_b'}\int_0^{\bar{t}}\int_{\mathbb{R}^3} k_{w(v)}(v,\mathfrak{u})w(\mathfrak{u})z(\overline{x}',v)\mathrm{e}^{-\nu(\mathfrak{u})t_i'}\mathrm{e}^{-\nu(v)(\bar{t}-s)}\mathrm{d}\mathfrak{u}\mathrm{d}s\right| \\
&\lesssim \|z\|_{L^\infty_{\varrho,\vartheta}} \lesssim o_T\varepsilon, \qquad (4.234)
\end{aligned}
$$

$$
\begin{aligned}
&\left|\mathbf{1}_{t_b<t_i} w(v)h(\overline{x},v)\mathrm{e}^{-\nu(v)t_b}\right| \\
&+\left|\mathbf{1}_{t_b'<t_i'}\int_0^{\bar{t}}\int_{\mathbb{R}^3} k_{w(v)}(v,\mathfrak{u})w(\mathfrak{u})h(\overline{x}',v)\mathrm{e}^{-\nu(\mathfrak{u})t_b'}\mathrm{e}^{-\nu(v)(\bar{t}-s)}\mathrm{d}\mathfrak{u}\mathrm{d}s\right| \\
&\lesssim \|h\|_{L^\infty_{\overline{\gamma}_-,\varrho,\vartheta}} \lesssim o_T, \qquad (4.235)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\int_0^{\bar{t}} w(v)\varepsilon S\Big(t-\varepsilon^2 s, x-\varepsilon(\bar{t}-s)v, v\Big)\mathrm{e}^{-\nu(v)(\bar{t}-s)}\mathrm{d}s\right| \\
&+\left|\int_0^{\bar{t}}\int_{\mathbb{R}^3} k_{w(v)}(v,\mathfrak{u})\int_0^{\bar{t}'}\varepsilon S\Big(t-\varepsilon^2 s-\varepsilon^2 r, x-\varepsilon(\bar{t}-s)\mathfrak{u}-\varepsilon(\bar{t}'-r)\mathfrak{u},\mathfrak{u}\Big)\mathrm{e}^{-\nu(\mathfrak{u})(\bar{t}'-r)}\mathrm{e}^{-\nu(v)(\bar{t}-s)}\mathrm{d}r\mathrm{d}\mathfrak{u}\mathrm{d}s\right| \\
&\lesssim \varepsilon\left\|\nu^{-1}S\right\|_{L^\infty_{\varrho,\vartheta}} \lesssim o_T + o_T\varepsilon\|R\|_{L^\infty_{\varrho,\vartheta}} + \varepsilon\|R\|^2_{L^\infty_{\varrho,\vartheta}} \lesssim o_T\varepsilon^{\frac{1}{2}}\|R\|_X + \|R\|_X^2 + o_T. \qquad (4.236)
\end{aligned}
$$

Step 3: Estimates of Non-Local Terms The only remaining term in (4.231) is the non-local term

$$I := \int_0^{\bar{t}} \int_{\mathbb{R}^3} k_{w(v)}(v, \mathfrak{u}) \int_0^{\bar{t}'} \int_{\mathbb{R}^3} k_{w(\mathfrak{u})}(\mathfrak{u}, \mathfrak{u}') R_w \Big(t - \varepsilon^2 s - \varepsilon^2 r, x - \varepsilon(\bar{t} - s)\mathfrak{u} - \varepsilon(\bar{t}' - r)\mathfrak{u}', \mathfrak{u}'\Big) \mathrm{e}^{-\nu(\mathfrak{u})(\bar{t}'-r)} \mathrm{e}^{-\nu(v)(\bar{t}-s)} \mathrm{d}\mathfrak{u}' \mathrm{d}r \mathrm{d}\mathfrak{u} \mathrm{d}s. \tag{4.237}$$

The proof is very similar to Step 3 in the proof of Proposition 3.34 with t_b' replaced by $\bar{t}$, so we will skip the details. The only non-trivial step is in the estimate of $I_{5,1}$, we should have

$$\begin{aligned}
&\int_{|\mathfrak{u}|\leq 2N} \int_{|\mathfrak{u}'|\leq 3N} \int_0^{\bar{t}'} \mathbf{1}_{\{x-\varepsilon(\bar{t}-s)v-\varepsilon(\bar{t}'-r)\mathfrak{u}\in\Omega\}} \\
&\quad \Big|R\Big(t - \varepsilon^2 s - \varepsilon^2 r, x - \varepsilon(\bar{t} - s)v - \varepsilon(\bar{t}' - r)\mathfrak{u}, \mathfrak{u}'\Big)\Big| \mathrm{e}^{-\nu(\mathfrak{u})(\bar{t}'-r)} \mathrm{d}r \mathrm{d}\mathfrak{u} \mathrm{d}\mathfrak{u}' \\
\leq& \Big(\int_{|\mathfrak{u}|\leq 2N} \int_{|\mathfrak{u}'|\leq 3N} \int_0^{\bar{t}'} \mathbf{1}_{\{x-\varepsilon(\bar{t}-s)v-\varepsilon(\bar{t}'-r)\mathfrak{u}\in\Omega\}} \mathrm{e}^{-\nu(\mathfrak{u})(\bar{t}'-r)} \mathrm{d}r \mathrm{d}\mathfrak{u} \mathrm{d}\mathfrak{u}' \Big)^{\frac{5}{6}} \\
&\times \Big(\int_{|\mathfrak{u}|\leq 2N} \int_{|\mathfrak{u}'|\leq 3N} \int_0^{\bar{t}'} \mathbf{1}_{\{x-\varepsilon(\bar{t}-s)v-\varepsilon(\bar{t}'-r)\mathfrak{u}\in\Omega\}} \\
&\quad \Big|R\Big(t - \varepsilon^2 s - \varepsilon^2 r, x - \varepsilon(\bar{t} - s)v - \varepsilon(\bar{t}' - r)\mathfrak{u}, \mathfrak{u}'\Big)\Big|^6 \mathrm{e}^{-\nu(\mathfrak{u})(\bar{t}'-r)} \mathrm{d}r \mathrm{d}\mathfrak{u} \mathrm{d}\mathfrak{u}' \Big)^{\frac{1}{6}} \\
\lesssim& \left| \int_0^{\bar{t}'} \frac{1}{\varepsilon^3 \delta^3} \int_{|\mathfrak{u}'|\leq 3N} \int_\Omega \mathbf{1}_{\{y\in\Omega\}} \big|R(t - \varepsilon^2 s - \varepsilon^2 r, y, \mathfrak{u}')\big|^6 \mathrm{e}^{-(\bar{t}'-r)} \mathrm{d}y \mathrm{d}\mathfrak{u}' \mathrm{d}r \right|^{\frac{1}{6}} \lesssim \frac{1}{\varepsilon^{\frac{1}{2}} \delta^{\frac{1}{2}}} \left|\left|\left| R \right|\right|\right|_{L_t^\infty L_{x,v}^6}.
\end{aligned} \tag{4.238}$$

Therefore, we conclude that

$$|I| \lesssim \delta \left|\left|\left| R_w \right|\right|\right|_{L^\infty} + \frac{1}{\varepsilon^{\frac{1}{2}} \delta^{\frac{1}{2}}} \left|\left|\left| R \right|\right|\right|_{L_t^\infty L_{x,v}^6}. \tag{4.239}$$

Step 4: Synthesis Summarizing all above, we obtain for any $(t, x, v) \in \mathbb{R}_+ \times \overline{\Omega} \times \mathbb{R}^3$,

$$\big|R_w(t, x, v)\big| \lesssim \delta \left|\left|\left| R_w \right|\right|\right|_{L^\infty} + \frac{1}{\varepsilon^{\frac{1}{2}} \delta^{\frac{1}{2}}} \left|\left|\left| R \right|\right|\right|_{L_t^\infty L_{x,v}^6} + o_T \varepsilon^{\frac{1}{2}} \left\| R \right\|_X + \left\| R \right\|_X^2 + o_T. \tag{4.240}$$

Hence, when $\delta \ll 1$, we obtain

$$\big|R_w(t, x, v)\big| \lesssim \varepsilon^{-\frac{1}{2}} \left|\left|\left| R \right|\right|\right|_{L_t^\infty L_{x,v}^6} + o_T \varepsilon^{\frac{1}{2}} \left\| R \right\|_X + \left\| R \right\|_X^2 + o_T, \tag{4.241}$$

and thus the desired result follows. □

4.8. Remainder estimate.

Theorem 4.34. *Let R be the solution to (1.35). Under the assumptions (1.23),(1.28), (1.31), we have*

$$\left\|\left|R\right|\right\|_X \lesssim o_T. \tag{4.242}$$

Proof. Based on Proposition 4.17, we have

$$\|R\|_{L^\infty_t L^2_{xv}} + \varepsilon^{-\frac{1}{2}} \|R\|_{L^2_{\overline{\gamma}_+}} + \varepsilon^{-1} \left\|\left|(\mathbf{I}-\mathbf{P})[R]\right|\right\|_{L^2} \lesssim o_T \left\|\left|R\right|\right\|_X + \left\|\left|R\right|\right\|_X^2 + o_T. \tag{4.243}$$

Based on Proposition 4.25, we have

$$\varepsilon^{-\frac{1}{2}} \left\|\left|\mathbf{P}[R]\right|\right\|_{L^2} \lesssim o_T \left\|\left|R\right|\right\|_X + \left\|\left|R\right|\right\|_X^2 + o_T. \tag{4.244}$$

Combining both of them, we arrive at

$$\|R\|_{L^\infty_t L^2_{xv}} + \varepsilon^{-\frac{1}{2}} \|R\|_{L^2_{\overline{\gamma}_+}} + \varepsilon^{-\frac{1}{2}} \left\|\left|\mathbf{P}[R]\right|\right\|_{L^2} + \varepsilon^{-1} \left\|\left|(\mathbf{I}-\mathbf{P})[R]\right|\right\|_{L^2} \lesssim o_T \left\|\left|R\right|\right\|_X + \left\|\left|R\right|\right\|_X^2 + o_T. \tag{4.245}$$

Similarly, combining Proposition 4.18 and Proposition 4.26, we arrive at

$$\|\partial_t R\|_{L^\infty_t L^2_{xv}} + \varepsilon^{-\frac{1}{2}} \|\partial_t R\|_{L^2_{\overline{\gamma}_+}} + \varepsilon^{-\frac{1}{2}} \left\|\left|\partial_t \mathbf{P}[R]\right|\right\|_{L^2} + \varepsilon^{-1} \left\|\left|\partial_t(\mathbf{I}-\mathbf{P})[R]\right|\right\|_{L^2} \lesssim o_T \left\|\left|R\right|\right\|_X + \left\|\left|R\right|\right\|_X^2 + o_T. \tag{4.246}$$

Based on Proposition 4.27 and Corollary 4.28, we have

$$\begin{aligned}
&\varepsilon^{-\frac{1}{2}} \|R\|_{L^\infty_t L^2_{\gamma_+}} + \left\|\mu^{\frac{1}{4}} R\right\|_{L^\infty_t L^4_{\gamma_+}} + \varepsilon^{-1} \left\|\left|(\mathbf{I}-\mathbf{P})[R]\right|\right\|_{L^\infty_t L^2_v} + \left\|\left|(\mathbf{I}-\mathbf{P})[R]\right|\right\|_{L^\infty_t L^6_{xv}} \\
&\lesssim o_T \left\|\left|R\right|\right\|_X + \left\|\left|R\right|\right\|_X^2 + o_T.
\end{aligned} \tag{4.247}$$

Based on Proposition 4.32, we have

$$\|\mathbf{P}[R]\|_{L^\infty_t L^6_{xv}} \lesssim o_T \left\|\left|R\right|\right\|_X + \left\|\left|R\right|\right\|_X^2 + o_T. \tag{4.248}$$

Combining both of them, we arrive at

$$\begin{aligned}
&\varepsilon^{-\frac{1}{2}} \|R\|_{L^\infty_t L^2_{\gamma_+}} + \left\|\mu^{\frac{1}{4}} R\right\|_{L^\infty_t L^4_{\gamma_+}} + \varepsilon^{-1} \left\|\left|(\mathbf{I}-\mathbf{P})[R]\right|\right\|_{L^\infty_t L^2_v} + \left\|\left|R\right|\right\|_{L^\infty_t L^6_{xv}} \\
&\lesssim o_T \left\|\left|R\right|\right\|_X + \left\|\left|R\right|\right\|_X^2 + o_T.
\end{aligned} \tag{4.249}$$

Based on Proposition 4.33, we have

$$\varepsilon^{\frac{1}{2}} \left\|\left|R\right|\right\|_{L^\infty_t L^\infty_{\varrho,\vartheta}} + \varepsilon^{\frac{1}{2}} \|R\|_{L^\infty_t L^\infty_{\gamma_+}} + \varepsilon^{\frac{1}{2}} \left\|\left|R\right|\right\|_{L^\infty_{\varrho,\vartheta}} + \varepsilon^{\frac{1}{2}} \|R\|_{L^\infty_{\overline{\gamma}_+,\varrho,\vartheta}} \lesssim o_T \left\|\left|R\right|\right\|_X + \left\|\left|R\right|\right\|_X^2 + o_T. \tag{4.250}$$

Collecting (4.245),(4.246),(4.249),(4.250), we have

$$\left\|\left|R\right|\right\|_X \lesssim o_T \left\|\left|R\right|\right\|_X + \left\|\left|R\right|\right\|_X^2 + o_T. \tag{4.251}$$

Hence, we have

$$\left\|\left|R\right|\right\|_X \lesssim \left\|\left|R\right|\right\|_X^2 + o_T. \tag{4.252}$$

By a standard iteration/fixed-point argument, our desired result follows. □

Proof of Theorem 1.8. The estimate (1.39) follows from Theorem 4.34. The construction and positivity of $\mathfrak{F}$ based on the expansion (2.6) is standard and we refer to [27,29], so we will focus on the proof of (1.40). From Theorem 4.34, we have

$$\varepsilon^{-\frac{1}{2}} \left|\left|\left|\mathbf{P}[R]\right|\right|\right|_{L^2} + \varepsilon^{-1} \left|\left|\left|(\mathbf{I}-\mathbf{P})[R]\right|\right|\right|_{L^2} \lesssim o_T, \tag{4.253}$$

which yields

$$\left|\left|\left|R\right|\right|\right|_{L^2} \lesssim o_T \varepsilon^{\frac{1}{2}}. \tag{4.254}$$

From (2.6), we know

$$\left|\left|\left|\mu^{-\frac{1}{2}}\mathfrak{F} - \mu^{\frac{1}{2}} - \varepsilon f_1 - \varepsilon^2 f_2 - \varepsilon f_1^B\right|\right|\right|_{L^2} = \left|\left|\left|\varepsilon R\right|\right|\right|_{L^2} \lesssim o_T \varepsilon^{\frac{3}{2}}. \tag{4.255}$$

From Theorem 4.1 and the rescaling $\eta = \varepsilon^{-1}\mathfrak{n}$, we have

$$\left|\left|\left|\varepsilon^2 f_2\right|\right|\right|_{L^2} \lesssim o_T \varepsilon^2, \quad \left|\left|\left|\varepsilon f_1^B\right|\right|\right|_{L^2} \lesssim o_T \varepsilon^{\frac{3}{2}}. \tag{4.256}$$

Hence, we have

$$\left|\left|\left|\mu^{-\frac{1}{2}}\mathfrak{F} - \mu^{\frac{1}{2}} - \varepsilon f_1\right|\right|\right|_{L^2} \lesssim o_T \varepsilon^{\frac{3}{2}}. \tag{4.257}$$

Therefore, (1.40) follows. □

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Data Availability The authors declare that this research does not generate any data.

Appendix A. Linearized Boltzmann Operator

Based on [20, Chapter 7] and [35, Chapters 1&3], define the symmetrized version of Q in (1.2):

$$Q^*[F,G] := \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} q(\omega, |\mathfrak{u}-v|)\Big(F(\mathfrak{u}_*)G(v_*) + F(v_*)G(\mathfrak{u}_*) - F(\mathfrak{u})G(v) - F(v)G(\mathfrak{u})\Big)\mathrm{d}\omega\mathrm{d}\mathfrak{u}. \tag{A.1}$$

Clearly, $Q[F,F] = Q^*[F,F]$. Denote the linearized Boltzmann operator $\mathcal{L}$

$$\mathcal{L}[f] := -2\mu^{-\frac{1}{2}} Q^*\left[\mu, \mu^{\frac{1}{2}} f\right] := \nu f - K[f], \tag{A.2}$$

where for some kernels $k(\mathfrak{u}, v)$,

$$\nu(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} q(\omega, |\mathfrak{u}-v|)\mu(\mathfrak{u})\mathrm{d}\omega\mathrm{d}\mathfrak{u}, \qquad K[f](v) = \int_{\mathbb{R}^3} k(\mathfrak{u}, v) f(\mathfrak{u})\mathrm{d}\mathfrak{u}. \tag{A.3}$$

$\mathcal{L}$ is self-adjoint in $L^2_\nu(\mathbb{R}^3)$ with the null space

$$\mathcal{N} := \operatorname{span}\left\{\mu^{\frac{1}{2}}, v\mu^{\frac{1}{2}}, |v|^2 \mu^{\frac{1}{2}}\right\}. \tag{A.4}$$

Let $\mathcal{N}^\perp$ be the orthogonal complement of $\mathcal{N}$ in $L^2(\mathbb{R}^3)$. Denote $\mathbf{P}$ the orthogonal projection onto $\mathcal{N}$ and $\mathbf{I}-\mathbf{P}$ the complement. Then $\mathcal{L}$ satisfies the coercivity property

$$\int_{\mathbb{R}^3} f\mathcal{L}[f]\mathrm{d}v \gtrsim \int_{\mathbb{R}^3} \nu(v)\left|(\mathbf{I}-\mathbf{P})[f]\right|^2\mathrm{d}v. \tag{A.5}$$

Note that the validity of (A.5) and the fact $\nu(v) \gtrsim \langle v\rangle$ rely on the assumption of hard-sphere gas in (1.2). Denote $\mathcal{L}^{-1} : \mathcal{N}^\perp \to \mathcal{N}^\perp$ the quasi-inverse of $\mathcal{L}$. Also, denote the nonlinear Boltzmann operator Γ

$$\Gamma[f, g] := \mu^{-\frac{1}{2}} Q^*\left[\mu^{\frac{1}{2}} f, \mu^{\frac{1}{2}} g\right] \in \mathcal{N}^\perp. \tag{A.6}$$

Appendix B. Inner Products and Norms

Based on the flow direction, we can divide the boundary $\gamma := \left\{(x_0, v) : x_0 \in \partial\Omega, v \in \mathbb{R}^3\right\}$ into the incoming boundary γ_-, the outgoing boundary γ_+, and the grazing set γ_0 based on the sign of $v\cdot n(x_0)$. Similarly, we further divide the boundary $\overline{\gamma} := \left\{(t, x_0, v) : t \in \mathbb{R}_+, x_0 \in \partial\Omega, v \in \mathbb{R}^3\right\}$ into $\overline{\gamma}_-$, $\overline{\gamma}_+$, and $\overline{\gamma}_0$.

Let $\langle\,\cdot\,\cdot\,\rangle_v$ denote the inner product in $v \in \mathbb{R}^3$, $\langle\,\cdot\,\cdot\,\rangle_x$ the inner product in $x \in \Omega$, $\langle\,\cdot\,\cdot\,\rangle$ the inner product in $(x, v) \in \Omega \times \mathbb{R}^3$. Also, let $\langle\,\cdot\,\cdot\,\rangle_{\gamma_\pm}$ denote the inner product on $\gamma_\pm$ with measure $\mathrm{d}\gamma := |v\cdot n|\,\mathrm{d}v\mathrm{d}S_x$.

Denote the bulk and boundary norms

$$\|f\|_{L^r} := \left(\iint_{\Omega\times\mathbb{R}^3} |f(x, v)|^r \,\mathrm{d}v\mathrm{d}x\right)^{\frac{1}{r}}, \qquad |f|_{L^r_{\gamma_\pm}} := \left(\int_{\gamma_\pm} |f(x, v)|^r \,|v\cdot n|\,\mathrm{d}\gamma\right)^{\frac{1}{r}}. \tag{B.1}$$

Define the weighted L^∞ norms for $0 \le \varrho < \frac{1}{2}$ and $\vartheta \ge 0$

$$\|f\|_{L^\infty_{\varrho,\vartheta}} := \operatorname*{ess\,sup}_{(x,v)\in\Omega\times\mathbb{R}^3} \left(\langle v\rangle^\vartheta \,\mathrm{e}^{\varrho\frac{|v|^2}{2}}\,|f(x, v)|\right),$$
$$|f|_{L^\infty_{\gamma_\pm,\varrho,\vartheta}} := \operatorname*{ess\,sup}_{(x,v)\in\gamma_\pm} \left(\langle v\rangle^\vartheta \,\mathrm{e}^{\varrho\frac{|v|^2}{2}}\,|f(x, v)|\right). \tag{B.2}$$

Denote the ν-norm

$$\|f\|_{L^2_\nu} := \left(\iint_{\Omega\times\mathbb{R}^3} \nu(v)\,|f(x, v)|^2 \,\mathrm{d}v\mathrm{d}x\right)^{\frac{1}{2}}. \tag{B.3}$$

When the time integral is involved (usually on $[0, \mathfrak{T}]$ for some $\mathfrak{T} > 0$ from the context), we define the corresponding inner products $\langle\!\langle\,\cdot\,\cdot\,\rangle\!\rangle_{tv}$, $\langle\!\langle\,\cdot\,\cdot\,\rangle\!\rangle_{tx}$, $\langle\!\langle\,\cdot\,\cdot\,\rangle\!\rangle$ and $\langle\!\langle\,\cdot\,\cdot\,\rangle\!\rangle_{\overline{\gamma}_\pm}$. Also, we define the corresponding norms: $|\!|\!|f|\!|\!|_{L^r}$, $\|f\|_{L^r_{\overline{\gamma}_\pm}}$, $|\!|\!|f|\!|\!|_{L^\infty_{\varrho,\vartheta}}$, $\|f\|_{L^\infty_{\overline{\gamma}_\pm,\varrho,\vartheta}}$ and $|\!|\!|f|\!|\!|_{L^2_\nu}$.

We will also employ the standard Sobolev norms which are essentially the L^p-norms of the function together with its (weak) derivatives up to a given order: (for $1 < p < \infty$ and $s \in \mathbb{R}$)

$$\|f\|_{W^{s,p}(\mathbb{R}^d)} := \left\| \langle \nabla \rangle^s f \right\|_{L^p(\mathbb{R}^d)} \sim_{s\in\mathbb{N},p,d} \sum_{j=0}^{s} \left\| \nabla^j f \right\|_{L^p(\mathbb{R}^d)}. \tag{B.4}$$

For a more comprehensive discussion on the Sobolev spaces (including the definition of the Sobolev norms on bounded domains as well as for the endpoint cases), we refer the reader to [1,34,64,65]. In addition, we will specify the variable(s) in the subscript when necessary.

Sometimes, (x, v) or (t, x, v) may call for different norms on each variable. Let $\|\cdot\|_{W^{k,p}W^{\ell,q}}$ denote $W^{k,p}$ norm for $x \in \Omega$ and $W^{\ell,q}$ norm for $v \in \mathbb{R}^3$ and $|||\cdot|||_{W^{m,s}W^{k,p}W^{\ell,q}}$ denote $W^{m,s}$ norm for $t \in [0, \mathfrak{T}]$ with some $\mathfrak{T} > 0$, $W^{k,p}$ norm for $x \in \Omega$ and $W^{\ell,q}$ norm for $v \in \mathbb{R}^3$. The similar notation also applies when we replace $W^{\ell,q}$ by L^q, L^2_v, $L^\infty_{\varrho,\vartheta}$, or $W^{1,\infty}_{\varrho,\vartheta}$. When the boundary norms are considered, let $\|\cdot\|_{W^{k,p}W^{\ell,q}_{\gamma\pm}}$ denote $W^{k,p}$ norm for $x \in \partial\Omega$ and $W^{\ell,q}$ norm for $\{v \in \mathbb{R}^3 : v \cdot n \gtrless 0\}$. The similar notation also applies to $|||\cdot|||_{W^{m,s}W^{k,p}W^{\ell,q}_{\gamma\pm}}$.

We will only write the variables (t, x, v) explicitly when there is a possibility of confusion. For example, we may write $\|\cdot\|_{L^\infty_t L^2_{x,v}}$ to denote L^∞ norm for t and L^2 norm for (x, v) (instead of the longer notation $\|\cdot\|_{L^\infty L^2 L^2}$). Also, all variables will be explicitly written if we will further prescribe different norms on the normal $\mathfrak{n}$ or tangential (ι_1, ι_2) variables.

Appendix C. Symbols and Constants

Let $\mathbf{1}$ denote the 3×3 identity tensor. Define the quantities

$$\overline{\mathscr{A}}(v) := v \cdot \left(|v|^2 - 5T\right) \mu^{\frac{1}{2}} \in \mathbb{R}^3, \quad \mathscr{A}(v) := \mathcal{L}^{-1}\left[\overline{\mathscr{A}}\right] \in \mathbb{R}^3, \tag{C.1}$$

$$\overline{\mathscr{B}}(v) := \left(v \otimes v - \frac{|v|^2}{3}\mathbf{1}\right)\mu^{\frac{1}{2}} \in \mathbb{R}^{3\times3}, \quad \mathscr{B}(v) := \mathcal{L}^{-1}\left[\overline{\mathscr{B}}\right] \in \mathbb{R}^{3\times3}, \tag{C.2}$$

along with

$$\kappa := \int_{\mathbb{R}^3} \mathscr{A}_i \overline{\mathscr{A}}_i, \quad \sigma := \int_{\mathbb{R}^3} \left(|v|^2 - 5T\right)\left(\mathscr{A}_i \overline{\mathscr{A}}_i\right), \tag{C.3}$$

$$\lambda := \int_{\mathbb{R}^3} \mathscr{B}_{ij} \overline{\mathscr{B}}_{ij}, \quad \alpha := \int_{\mathbb{R}^3} \mathscr{B}_{ii} \overline{\mathscr{B}}_{ii}, \quad \overline{\alpha} := \int_{\mathbb{R}^3} \mathscr{B}_{ii} \overline{\mathscr{B}}_{jj}, \quad \text{for } i \neq j. \tag{C.4}$$

Notice that in (C.3), κ and σ remain constant for $i = 1, 2, 3$, and in (C.4), λ, α and γ remain constant for $i, j = 1, 2, 3$ with $i \neq j$.
For two tensors $\mathscr{M}, \mathscr{N} \in \mathbb{R}^{3\times3}$, we may define their double dot product (contraction) as

$$\mathscr{M} : \mathscr{N} = \sum_{i,j=1}^{3} \mathscr{M}_{ij} \mathscr{N}_{ij}. \tag{C.5}$$

Throughout this paper, $C > 0$ denotes a constant that only depends on the domain Ω, but does not depend on the data or ε. It is referred as universal and can change from one inequality to another. When we write $C(z)$, it means a certain positive constant depending on the quantity z. We write $a \lesssim b$ to denote $a \leq Cb$ and $a \gtrsim b$ to denote $a \geq Cb$. Also, we write $a \simeq b$ if $a \lesssim b$ and $a \gtrsim b$.

In this paper, we will use $o(1)$ to denote a sufficiently small constant independent of the data. Also, let o_T be a sufficiently small constant depending on the data. For the stationary problem, o_T depends on $\mathfrak{f}_b$ only satisfying

$$o_T \to 0 \;\text{ as }\; |\mathfrak{f}_b|_{W^{3,\infty}W^{1,\infty}_{\gamma_-,\varrho,\vartheta}} \to 0. \tag{C.6}$$

For the evolutionary problem, o_T depends on $\mathfrak{f}_i$ and $\mathfrak{f}_b$ satisfying

$$o_T \to 0 \;\text{ as }\; \|\mathfrak{f}_i\|_{W^{1,\infty}L^\infty_{\varrho,\vartheta}} + \|\mathfrak{f}_b\|_{W^{1,\infty}W^{3,\infty}W^{1,\infty}_{\gamma_-,\varrho,\vartheta}} \to 0. \tag{C.7}$$

In principle, while o_T is determined by data a priori, we are free to choose $o(1)$ in each estimate.

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