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Research article

L^2 diffusive expansion for neutron transport equation

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Abstract: One of the most classical and fundamental mathematical problems in kinetic theory is to study the diffusive limit of the neutron transport equation. As $\varepsilon \to 0$, the phase space density $u^\varepsilon(x,w)$

$$w\cdot\nabla_x u^\varepsilon + \varepsilon^{-1}\left(u^\varepsilon - \overline{u^\varepsilon}\right) = 0, \quad u^\varepsilon\Big|_{x\in\partial\Omega, w\cdot n<0} = g, \quad \overline{u^\varepsilon}(x) := \frac{1}{4\pi}\int_{\mathbb{S}^2} u^\varepsilon(x,w)\mathrm{d}w, \tag{0.1}$$

converges to the interior solution $U_0(x)$:

$$-\Delta_x U_0 = 0, \quad U_0\Big|_{\partial\Omega} = U^B_{0,\infty}, \tag{0.2}$$

in which $U^B_{0,\infty}$ is obtained by solving the Milne problem for the celebrated boundary layer correction U^B_0. The function g represents the inflow data, and n is the unit outward normal to the smooth bounded domain Ω. Surprisingly, we found [1, 2] that the expected L^∞ expansion

$$\left\|u^\varepsilon - U_0 - U^B_0\right\|_{L^\infty} \lesssim \varepsilon \tag{0.3}$$

is invalid due to the grazing singularity of U^B_0. As a result, the corresponding well-known mathematical theory breaks down, and the diffusive limit has remained an outstanding question. A satisfactory theory was developed for convex domains [1–6] by constructing new boundary layers with favorable ε-geometric corrections. However, this approach is inapplicable in non-convex domains. In this paper, we settle this open question affirmatively in the L^2 sense. The convergence

$$\|u^\varepsilon - U_0\|_{L^2} \lesssim \varepsilon^{\frac{1}{2}} \tag{0.4}$$

holds for general smooth domains, including non-convex ones. We achieve this by discovering a novel and optimal L^2 expansion theory that reveals a surprising $\varepsilon^{\frac{1}{2}}$ gain for the average of the remainder, and by choosing a test function with a new cancellation via conservation of mass. We also introduce a cutoff boundary layer U^B_0 and investigate its delicate regularity estimates to control the source terms of the remainder equation with the help of Hardy's inequality. Notably, our new cutoff boundary layer U^B_0 determines U_0, despite its absence in the estimate.

Keywords: non-convex domains; transport equation; cutoff boundary layer; diffusive limit
Mathematics Subject Classification: 35Q49, 82D75, 35Q62, 35Q20

1. Introduction

1.1. Problem formulation

We consider the steady neutron transport equation in a three-dimensional C^3 bounded domain (convex or non-convex) with in-flow boundary condition. In the spatial domain $\Omega \ni x = (x_1, x_2, x_3)$ and the velocity domain $\mathbb{S}^2 \ni w = (w_1, w_2, w_3)$, the neutron density $u^\varepsilon(x, w)$ satisfies (0.1) with the Knudsen number $0 < \varepsilon \ll 1$. We intend to study the asymptotic behavior of u^ε as $\varepsilon \to 0$.

Based on the flow direction, we can divide the boundary $\gamma := \{(x_0, w) : \ x_0 \in \partial\Omega, w \in \mathbb{S}^2\}$ into the incoming boundary γ_-, the outgoing boundary γ_+, and the grazing set γ_0 according to the sign of $w \cdot n(x_0)$. In particular, the boundary condition of (0.1) is only given on γ_-.

1.2. Normal chart near boundary

We follow the approach in [4, 6] to define the geometric quantities, and the details can be found in Section 2.2. For smooth manifold $\partial\Omega$, there exists an orthogonal curvilinear coordinates system (ι_1, ι_2) such that the coordinate lines coincide with the principal directions at any $x_0 \in \partial\Omega$. Assume that $\partial\Omega$ is parameterized by $\mathbf{r} = \mathbf{r}(\iota_1, \iota_2)$. Let the vector length be $L_i := \left|\partial_{\iota_i}\mathbf{r}\right|$ and let the unit vector be $\varsigma_i := L_i^{-1}\partial_{\iota_i}\mathbf{r}$ for $i = 1, 2$.

Consider the corresponding new coordinate system (μ, ι_1, ι_2), where μ denotes the normal distance to the boundary surface $\partial\Omega$, i.e.

$$x = \mathbf{r} - \mu n. \tag{1.1}$$

Define the orthogonal velocity substitution for $\mathfrak{w} := (\varphi, \psi)$ as

$$-w \cdot n = \sin\varphi, \quad w \cdot \varsigma_1 = \cos\varphi \sin\psi, \quad w \cdot \varsigma_2 = \cos\varphi\cos\psi. \tag{1.2}$$

Finally, we define the scaled normal variable $\eta = \dfrac{\mu}{\varepsilon}$, which implies $\dfrac{\partial}{\partial\mu} = \dfrac{1}{\varepsilon}\dfrac{\partial}{\partial\eta}$. We then write $\mathfrak{x} := (\eta, \iota_1, \iota_2)$.

1.3. Asymptotic expansion and remainder equation

We seek a solution to (0.1) in the form

$$u^\varepsilon = U + U^B + R = \left(U_0 + \varepsilon U_1 + \varepsilon^2 U_2\right) + U_0^B + R, \tag{1.3}$$

where the interior solution is

$$U(x, w) := U_0(x) + \varepsilon U_1(x, w) + \varepsilon^2 U_2(x, w), \tag{1.4}$$

and the boundary layer is

$$U^B(\mathfrak{x},\mathfrak{w}) := U_0^B(\mathfrak{x},\mathfrak{w}). \tag{1.5}$$

Here U_0, U_1, U_2, and U_0^B are constructed in Section 2.1 and Section 2.2, and $R(x,w)$ is the remainder satisfying

$$w\cdot\nabla_x R + \varepsilon^{-1}(R-\overline{R}) = S, \quad R\Big|_{x\in\partial\Omega, w\cdot n<0} = h, \quad \overline{R}(x) = \frac{1}{4\pi}\int_{\mathbb{S}^2} R(x,w)\mathrm{d}w, \tag{1.6}$$

where h and S are as defined in (3.4) and (3.6)–(3.9).

1.4. Literature

The study of the neutron transport equation in bounded domains has received a lot of attention since the dawn of the atomic age. This equation is not only significant in nuclear sciences and medical imaging but is also regarded as a linear prototype of the more important and complicated nonlinear Boltzmann equation. As a result, it serves as an ideal starting point to develop new theories and techniques.

For the formal expansion with respect to ε and its explicit solution, [7–15] provide relevant literature. The discussion on the bounded domain and half-space cases can be found in [16–23]. In the more general context, we refer to [24–34] for the hydrodynamic limits of Boltzmann equations in bounded domains, and the recent progress on the diffusive limit of the transfer equation (which is a coupled system of the transport equation and the heat equation) [35, 36].

The classical boundary layer analysis of the neutron transport equation leads to the Milne problem, which dictates that $U_0^B(\eta,\iota_1,\iota_2,\mathfrak{w})$ satisfies the equation given by

$$\sin\varphi\frac{\partial U_0^B}{\partial\eta} + U_0^B - \overline{U_0^B} = 0. \tag{1.7}$$

Based on the formal expansion in ε (see (2.6)), it is natural to expect the following remainder estimate [16]:

$$\|R\|_{L^\infty} = \left\|u^\varepsilon - U_0 - U_0^B\right\|_{L^\infty} \lesssim \varepsilon. \tag{1.8}$$

While this estimate holds for domains with flat boundaries, a surprising counter-example was constructed [1] that shows (1.8) to be invalid for a two-dimensional (2D) disk due to the grazing set singularity.

To provide more specific details, demonstrating the validity of the remainder estimates (1.8), necessitates the use of the higher-order boundary layer expansion $U_1^B \in L^\infty$. In this case, the bound $\partial_{\iota_i}U_0^B \in L^\infty$ is required, and even though $U_0^B \in L^\infty$, it has been proven that the normal derivative $\partial_\eta U_0^B$ is singular at the grazing set $\varphi = 0$. This singularity is then transferred to $\partial_{\iota_i}U_0^B \notin L^\infty$. A meticulous construction of the boundary data [1] reveals that both the method and the result of the boundary layer (1.7) are problematic, which justifies this invalidity.

A new approach to constructing the boundary layer has been proposed in recent works [1, 3–6]. It is based on the ε-Milne problem with geometric corrections for $\widetilde{U_0^B}(\mathfrak{x},\mathfrak{w})$, given by

$$\sin\varphi\frac{\partial\widetilde{U_0^B}}{\partial\eta} - \frac{\varepsilon}{\mathcal{R}_\kappa - \varepsilon\eta}\cos\varphi\frac{\partial\widetilde{U_0^B}}{\partial\varphi} + \widetilde{U_0^B} - \overline{\widetilde{U_0^B}} = 0, \tag{1.9}$$

where $\mathcal{R}_\kappa(\iota_1, \iota_2) > 0$ denotes the radius of curvature on $\partial\Omega$. This new construction has been shown to provide a satisfactory characterization of the L^∞ diffusive expansion in two-dimension (2D) or three-dimensional (3D) convex domains. The proof relies on a detailed analysis of $W^{1,\infty}$ regularity and boundary layer decomposition techniques for (1.9).

In non-convex domains, where $\mathcal{R}_\kappa(\iota_1, \iota_2) < 0$, the boundary layer with geometric correction is described in [2] as follows:

$$\sin\varphi \frac{\partial \widetilde{U_0^B}}{\partial \eta} + \frac{\varepsilon}{|\mathcal{R}_\kappa| + \varepsilon\eta} \cos\varphi \frac{\partial \widetilde{U_0^B}}{\partial \varphi} + \widetilde{U_0^B} - \overline{\widetilde{U_0^B}} = 0. \tag{1.10}$$

This sign flipping of the geometric correction term in contrast to (1.9) dramatically changes the characteristics of the boundary layer.

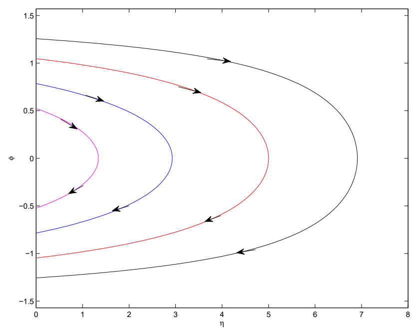

Figure 1. Characteristics in convex domains.

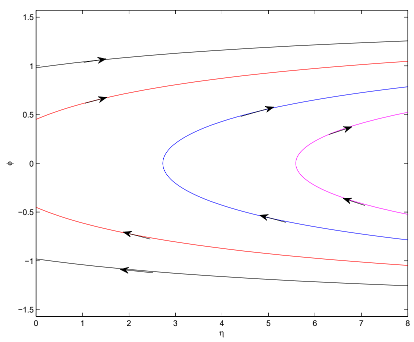

Figure 2. Characteristics in non-convex domains.

In Figure 1 and Figure 2 [2], the horizontal axis represents the scaled normal variable η, while the vertical axis represents the velocity φ. The inflow boundary is located on the left boundary where $\eta = 0$ and $\varphi > 0$. It is apparent from Figure 2 that there exists a "hollow" region where the characteristics may never track back to the inflow boundary. This discrepancy in the information source results in a strong discontinuity across the boundary of the "hollow" region, making it impossible to obtain $W^{1,\infty}$ estimates, which, in turn, prevents higher-order boundary layer expansion.

In this paper, we employ a fresh approach to design a cutoff boundary layer without the geometric correction and justify the L^2 diffusive expansion in smooth non-convex domains.

1.5. Notation and convention

Let $\langle \cdot , \cdot \rangle_w$ denote the inner product for $w \in \mathbb{S}^2$, $\langle \cdot , \cdot \rangle_x$ for $x \in \Omega$, and $\langle \cdot , \cdot \rangle$ for $(x, w) \in \Omega \times \mathbb{S}^2$. Moreover, let $\langle \cdot , \cdot \rangle_{\gamma_\pm}$ denote the inner product on $\gamma_\pm$ with the measure $\mathrm{d}\gamma := |w \cdot n| \,\mathrm{d}w\mathrm{d}S_x = |\sin \varphi| \cos \varphi \mathrm{d}\mathfrak{w}\mathrm{d}S_x$. Denote the bulk and boundary norms as follows:

$$\|f\|_{L^2} := \left(\iint_{\Omega\times\mathbb{S}^2} |f(x, w)|^2 \,\mathrm{d}w\mathrm{d}x \right)^{\frac{1}{2}}, \quad |f|_{L^2_{\gamma_\pm}} := \left(\int_{\gamma_\pm} |f(x, w)|^2 \,\mathrm{d}\gamma \right)^{\frac{1}{2}}. \tag{1.11}$$

Define the L^∞ norms as follows:

$$\|f\|_{L^\infty} := \operatorname*{ess\,sup}_{(x,w)\in\Omega\times\mathbb{S}^2} \big|f(x, w)\big|, \quad |f|_{L^\infty_{\gamma_\pm}} := \operatorname*{ess\,sup}_{(x,w)\in\gamma_\pm} \big|f(x, w)\big|. \tag{1.12}$$

Let $\|\cdot\|_{W^{k,p}_x}$ denote the usual Sobolev norm for $x \in \Omega$ and $|\cdot|_{W^{k,p}_x}$ for $x \in \partial\Omega$, and $\|\cdot\|_{W^{k,p}_x L^q_w}$ denote the $W^{k,p}$ norm for $x \in \Omega$ and the L^q norm for $w \in \mathbb{S}^2$. Similar notation also applies when we replace L^q by L^q_γ. When there is no possibility of confusion, we will ignore the (x, w) variables in the norms.

Throughout this paper, $C > 0$ denotes a constant that only depends on the domain Ω, but does not depend on the data or ε. It is referred to as universal and can change from one inequality to another. We write $a \lesssim b$ to denote $a \le Cb$ and $a \gtrsim b$ to denote $a \ge Cb$. We also write $a \simeq b$ if $a \lesssim b$ and $a \gtrsim b$. We use $o(1)$ to denote a sufficiently small constant that is independent of the data.

1.6. Main results

Theorem 1.1. *Under the assumption*

$$|g|_{W^{3,\infty}W^{1,\infty}_{\gamma_-}} \lesssim 1, \tag{1.13}$$

there exists a unique solution $u^\varepsilon(x, w) \in L^\infty(\Omega \times \mathbb{S}^2)$ *to* (0.1). *Moreover, the solution obeys the estimate*

$$\|u^\varepsilon - U_0\|_{L^2} \lesssim \varepsilon^{\frac{1}{2}}. \tag{1.14}$$

Here, $U_0(x)$ *satisfies the Laplace equation with the Dirichlet boundary condition*

$$\begin{cases} \Delta_x U_0(x) = 0 \ \text{ in } \ \Omega, \\ U_0(x_0) = \Phi_\infty(x_0) \ \text{ on } \ \partial\Omega, \end{cases} \tag{1.15}$$

in which $\Phi_\infty(\iota_1, \iota_2) = \Phi_\infty(x_0)$ *for* $x_0 \in \partial\Omega$ *is given by solving the Milne problem for* $\Phi(\mathfrak{x}, \mathfrak{w})$

$$\begin{cases} \sin\varphi \dfrac{\partial\Phi}{\partial\eta} + \Phi - \overline{\Phi} = 0, \\ \Phi(0, \iota_1, \iota_2, \mathfrak{w}) = g(\iota_1, \iota_2, \mathfrak{w}) \ \text{ for } \ \sin\varphi > 0, \\ \lim\limits_{\eta\to\infty} \Phi(\eta, \iota_1, \iota_2, \mathfrak{w}) = \Phi_\infty(\iota_1, \iota_2). \end{cases} \tag{1.16}$$

Remark 1.2. *In [1, 5, 6] for 2D/3D convex domains, as well as [2] for a 2D annulus domain, it is justified that for any* $0 < \delta \ll 1$

$$\left\| u^\varepsilon - \widetilde{U_0} - \widetilde{U_0^B} \right\|_{L^2} \lesssim \varepsilon^{\frac{5}{6}-\delta}, \tag{1.17}$$

where $\widetilde{U_0^B}(\mathfrak{x}, \mathfrak{w})$ *is the boundary layer with geometric correction defined in* (1.9)*, and* $\widetilde{U_0}$ *is the corresponding interior solution. Previous work [23, Theorem 2.1] has revealed that the difference between two types of interior solutions satisfies*

$$\left\|\widetilde{U_0} - U_0\right\|_{L^2} \lesssim \varepsilon^{\frac{2}{3}}. \tag{1.18}$$

Due to the rescaling $\eta = \varepsilon^{-1}\mu$*, for the general in-flow boundary data* g*, the boundary layer* $\widetilde{U_0^B} \neq 0$ *satisfies*

$$\left\|\widetilde{U_0^B}\right\|_{L^2} \simeq \varepsilon^{\frac{1}{2}}. \tag{1.19}$$

Hence, we conclude that

$$\|u^\varepsilon - U_0\|_{L^2} \simeq \varepsilon^{\frac{1}{2}}. \tag{1.20}$$

Therefore, this indicates that (1.14) *in Theorem 1.1 achieves the optimal* L^2 *bound of the diffusive approximation.*

1.7. Methodology

It is well-known that the key of the remainder estimate is to control $\overline{R}$. In a series of works [1–6], it has been shown that the kernel estimate

$$\left\|\overline{R}\right\|_{L^2} \lesssim \varepsilon^{-1}\left\|R - \overline{R}\right\|_{L^2} + \varepsilon^{-\frac{1}{2}}\,|R|_{L^2_{\gamma_-}} + 1 \tag{1.21}$$

and the basic energy (entropy production) bound

$$\varepsilon^{-1}\left\|R - \overline{R}\right\|_{L^2} + \varepsilon^{-\frac{1}{2}}\,|R|_{L^2_{\gamma_-}} \lesssim o(1)\left\|\overline{R}\right\|_{L^2} + 1. \tag{1.22}$$

provide a full control of the remainder R:

$$\|R\|_{L^2} \lesssim \varepsilon^{-1}\left\|R - \overline{R}\right\|_{L^2} + \left\|\overline{R}\right\|_{L^2} \lesssim 1. \tag{1.23}$$

Although (1.23) alone is not enough to justify

$$\lim_{\varepsilon\to 0}\left\|u^\varepsilon - U_0 - U_0^B\right\|_{L^2} = 0, \tag{1.24}$$

expanding the boundary layer approximation beyond the leading order U_0^B further improves the righ-hand side (RHS) of (1.23) to ε^α with $\alpha > 0$, and leads to (1.24). While this technique works well for convex domains, as our previous analysis revealed, it is impossible to expand to U_1^B for non-convex domains due to the lack of the $W^{1,\infty}$ estimate in (1.10).

The bottleneck of the L^2 bound (1.23) lies in the kernel estimate (1.21), which stems from the weak formulation of (1.6) with the test function $w \cdot \nabla_x\xi$

$$\int_\gamma R(w\cdot\nabla_x\xi)(w\cdot n) - \left\langle R, w\cdot\nabla_x(w\cdot\nabla_x\xi)\right\rangle + \varepsilon^{-1}\left\langle R - \overline{R}, w\cdot\nabla_x\xi\right\rangle = \left\langle S, w\cdot\nabla_x\xi\right\rangle. \tag{1.25}$$

Here, the auxiliary function $\xi(x)$ satisfies

$$-\Delta_x \xi = \overline{R}, \quad \xi\big|_{\partial\Omega} = 0. \tag{1.26}$$

In (1.25), $\left\langle R, w\cdot\nabla_x(w\cdot\nabla_x\xi)\right\rangle \simeq \left\|\overline{R}\right\|_{L^2}^2$, and $\varepsilon^{-1}\left\langle R-\overline{R}, w\cdot\nabla_x\xi\right\rangle$ leads to the worst and critical contribution $\varepsilon^{-1}\left\|R-\overline{R}\right\|_{L^2}$.

The key improvement in our work is an upgraded (compared with (1.21)) kernel estimate

$$\left\|\overline{R}\right\|_{L^2} \lesssim \varepsilon^{-\frac{1}{2}}\left\|R-\overline{R}\right\|_{L^2} + |R|_{L^2_{\gamma_-}} + \varepsilon^{\frac{1}{2}} \tag{1.27}$$

which, combined with (1.22), leads to

$$\left\|\overline{R}\right\|_{L^2} \lesssim \varepsilon^{-\frac{1}{2}}\left\|R-\overline{R}\right\|_{L^2} + \left\|\overline{R}\right\|_{L^2} \lesssim \varepsilon^{\frac{1}{2}}. \tag{1.28}$$

The extra $\varepsilon^{\frac{1}{2}}$ gain in (1.27) follows from a crucial observation. We consider the conservation law of (1.6) with the test function ξ defined in (1.26):

$$-\left\langle R, w\cdot\nabla_x\xi\right\rangle = -\left\langle R-\overline{R}, w\cdot\nabla_x\xi\right\rangle = \left\langle S,\xi\right\rangle. \tag{1.29}$$

We discover that the combination $\varepsilon^{-1}\times$(1.29) and (1.25) yields

$$\int_\gamma R(w\cdot\nabla_x\xi)(w\cdot n) - \left\langle R, w\cdot\nabla_x(w\cdot\nabla_x\xi)\right\rangle = \varepsilon^{-1}\left\langle S,\xi\right\rangle + \left\langle S, w\cdot\nabla_x\xi\right\rangle, \tag{1.30}$$

which exactly cancels out the worst contribution $\varepsilon^{-1}\left\langle R-\overline{R}, w\cdot\nabla_x\xi\right\rangle$. This key cancellation leads to an additional crucial gain of $\varepsilon^{\frac{1}{2}}$ in (1.27). Consequently, we can deduce the remainder estimate (1.28) without any further expansion of the (singular) boundary layer approximation.

Technically, to estimate the source terms $\varepsilon^{-1}\left\langle S,\xi\right\rangle$ and $\left\langle S, w\cdot\nabla_x\xi\right\rangle$ in (1.30), and $\varepsilon^{-1}\left\langle S,R\right\rangle$ in deriving (1.22), particularly the derivatives of the boundary layers, we construct a new cut-off boundary layer

$$U_0^B(\mathfrak{x},\mathfrak{w}) := \widetilde{\chi}(\varepsilon^{-1}\varphi)\chi(\varepsilon\eta)\Psi(\mathfrak{x},\mathfrak{w}), \tag{1.31}$$

where Ψ solves the Milne problem. With the grazing set cutoff $\widetilde{\chi}(\varepsilon^{-1}\varphi)$, we are able to perform delicate and precise estimates to control the resulting complex forcing term S (see (3.6)–(3.9)). In addition, with the help of integration by parts in φ and Hardy's inequality [37, 38] in the μ direction, we have

$$\begin{aligned}\left|\varepsilon^{-1}\left\langle\cos\varphi\frac{\partial U_0^B}{\partial\varphi},\xi\right\rangle\right| &\lesssim \left|\varepsilon^{-1}\left\langle U_0^B,\xi\right\rangle\right| \lesssim \left|\left\langle\eta U_0^B, \frac{1}{\mu}\int_0^\mu\frac{\partial\xi}{\partial\mu}\right\rangle\right| \lesssim \left\|\eta U_0^B\right\|_{L^2_xL^1_w}\left\|\frac{1}{\mu}\int_0^\mu\frac{\partial\xi}{\partial\mu}\right\|_{L^2} \\ &\lesssim \left\|\eta U_0^B\right\|_{L^2_xL^1_w}\left\|\frac{\partial\xi}{\partial\mu}\right\|_{L^2} \lesssim \varepsilon^{\frac{1}{2}}\|\xi\|_{H^1} \lesssim \varepsilon^{\frac{1}{2}}\left\|\overline{R}\right\|_{L^2},\end{aligned} \tag{1.32}$$

we can bound all source contributions in terms of the desired order of ε for closure.

2. Asymptotic analysis

2.1. Interior solution

Inserting (1.4) into (0.1) and comparing the order of ε, and following the analysis in [4,6], we deduce that

$$U_0 = \overline{U}_0, \quad \Delta_x \overline{U}_0 = 0, \tag{2.1}$$

$$U_1 = \overline{U}_1 - w \cdot \nabla_x U_0, \quad \Delta_x \overline{U}_1 = 0, \tag{2.2}$$

$$U_2 = \overline{U}_2 - w \cdot \nabla_x U_1, \quad \Delta_x \overline{U}_2 = 0. \tag{2.3}$$

We need the boundary layer to determine the boundary conditions for $\overline{U}_0$, $\overline{U}_1$, and $\overline{U}_2$.

2.2. Boundary layer

2.2.1. Geometric substitutions

The construction of the boundary layer requires a local description in a neighborhood of the physical boundary $\partial\Omega$. We follow the procedure in [4,6].

Substitution 1: Spatial Substitution Following the notation in Section 1.2, under the coordinate system (μ, ι_1, ι_2), we have

$$w \cdot \nabla_x = -(w \cdot n)\frac{\partial}{\partial \mu} - \frac{w \cdot \varsigma_1}{L_1(\kappa_1 \mu - 1)}\frac{\partial}{\partial \iota_1} - \frac{w \cdot \varsigma_2}{L_2(\kappa_2 \mu - 1)}\frac{\partial}{\partial \iota_2}, \tag{2.4}$$

where $\kappa_i(\iota_1, \iota_2)$ for $i = 1, 2$ is the principal curvature.

Substitution 2: Velocity Substitution Under the orthogonal velocity substitution (1.2) for $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\psi \in [-\pi, \pi]$, we have

$$\begin{aligned} w \cdot \nabla_x = {} & \sin\varphi \frac{\partial}{\partial \mu} - \left(\frac{\sin^2\psi}{R_1 - \mu} + \frac{\cos^2\psi}{R_2 - \mu}\right)\cos\varphi \frac{\partial}{\partial \varphi} + \frac{\cos\varphi \sin\psi}{L_1(1 - \kappa_1 \mu)}\frac{\partial}{\partial \iota_1} + \frac{\cos\varphi\cos\psi}{L_2(1 - \kappa_2\mu)}\frac{\partial}{\partial \iota_2} \\ & + \frac{\sin\psi}{R_1 - \mu}\left\{\frac{R_1 \cos\varphi}{L_1 L_2}\Big(\varsigma_1 \cdot \big(\varsigma_2 \times (\partial_{\iota_1\iota_2}\mathbf{r} \times \varsigma_2)\big)\Big) - \sin\varphi\cos\psi\right\}\frac{\partial}{\partial \psi} \\ & - \frac{\cos\psi}{R_2 - \mu}\left\{\frac{R_2 \cos\varphi}{L_1 L_2}\Big(\varsigma_2 \cdot \big(\varsigma_1 \times (\partial_{\iota_1\iota_2}\mathbf{r} \times \varsigma_1)\big)\Big) - \sin\varphi\sin\psi\right\}\frac{\partial}{\partial \psi}, \end{aligned} \tag{2.5}$$

where $R_i = \kappa_i^{-1}$ represents the radius of curvature. Note that the Jacobian $\mathrm{d}w = \cos\varphi \mathrm{d}\varphi \mathrm{d}\psi$ will be present when we perform integration.

Substitution 3: Scaling Substitution Considering the scaled normal variable $\eta = \varepsilon^{-1}\mu$, we have

$$
\begin{aligned}
w\cdot\nabla_x =&\varepsilon^{-1}\sin\varphi\frac{\partial}{\partial\eta} - \left(\frac{\sin^2\psi}{R_1-\varepsilon\eta} + \frac{\cos^2\psi}{R_2-\varepsilon\eta}\right)\cos\varphi\frac{\partial}{\partial\varphi} + \frac{R_1\cos\varphi\sin\psi}{L_1(R_1-\varepsilon\eta)}\frac{\partial}{\partial\iota_1} + \frac{R_2\cos\varphi\cos\psi}{L_2(R_2-\varepsilon\eta)}\frac{\partial}{\partial\iota_2} \\
&+ \frac{\sin\psi}{R_1-\varepsilon\eta}\left\{\frac{R_1\cos\varphi}{L_1L_2}\Big(\varsigma_1\cdot\big(\varsigma_2\times(\partial_{\iota_1\iota_2}\mathbf{r}\times\varsigma_2)\big)\Big) - \sin\varphi\cos\psi\right\}\frac{\partial}{\partial\psi} \\
&- \frac{\cos\psi}{R_2-\varepsilon\eta}\left\{\frac{R_2\cos\varphi}{L_1L_2}\Big(\varsigma_2\cdot\big(\varsigma_1\times(\partial_{\iota_1\iota_2}\mathbf{r}\times\varsigma_1)\big)\Big) - \sin\varphi\sin\psi\right\}\frac{\partial}{\partial\psi}.
\end{aligned}
\tag{2.6}
$$

2.2.2. Milne problem

Let $\Phi(\mathfrak{x},\mathfrak{w})$ be the solution to the Milne problem

$$
\sin\varphi\frac{\partial\Phi}{\partial\eta} + \Phi - \overline{\Phi} =0, \quad \overline{\Phi}(\mathfrak{x}) = \frac{1}{4\pi}\int_{-\pi}^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\Phi(\mathfrak{x},\mathfrak{w})\cos\varphi \mathrm{d}\varphi\mathrm{d}\psi, \tag{2.7}
$$

with the boundary condition

$$
\Phi(0,\iota_1,\iota_2,\mathfrak{w}) = g(\iota_1,\iota_2,\mathfrak{w}) \ \text{ for } \ \sin\varphi > 0. \tag{2.8}
$$

We are interested in the solution that satisfies

$$
\lim_{\eta\to\infty}\Phi(\eta,\iota_1,\iota_2,\mathfrak{w}) = \Phi_\infty(\iota_1,\iota_2) \tag{2.9}
$$

for some $\Phi_\infty(\iota_1,\iota_2)$. Based on [4, Section 4], we have the well-posedness and regularity of (2.7).

Proposition 2.1. *Under the assumption* (1.13)*, there exist* $\Phi_\infty(\iota_1,\iota_2)$ *and a unique solution* Φ *to* (2.7)*such that* $\Psi := \Phi - \Phi_\infty$ *satisfies*

$$
\begin{cases}
\sin\varphi\dfrac{\partial\Psi}{\partial\eta} + \Psi - \overline{\Psi} = 0,\\
\Psi(0,\iota_1,\iota_2,\mathfrak{w}) = g(\iota_1,\iota_2,\mathfrak{w}) - \Phi_\infty(\iota_1,\iota_2),\\
\lim\limits_{\eta\to\infty}\Psi(\eta,\iota_1,\iota_2,\mathfrak{w}) = 0,
\end{cases}
\tag{2.10}
$$

and for some constant $K > 0$ *and any* $0 < r \le 3$*, we have*

$$
|\Phi_\infty|_{W^{3,\infty}_{\iota_1,\iota_2}} + \left\|\mathrm{e}^{K\eta}\Psi\right\|_{L^\infty} \lesssim 1, \tag{2.11}
$$

$$
\left\|\mathrm{e}^{K\eta}\sin\varphi\frac{\partial\Psi}{\partial\eta}\right\|_{L^\infty} + \left\|\mathrm{e}^{K\eta}\sin\varphi\frac{\partial\Psi}{\partial\varphi}\right\|_{L^\infty} + \left\|\mathrm{e}^{K\eta}\frac{\partial\Psi}{\partial\psi}\right\|_{L^\infty} \lesssim 1, \tag{2.12}
$$

$$
\left\|\mathrm{e}^{K\eta}\frac{\partial^r\Psi}{\partial\iota_1^r}\right\|_{L^\infty} + \left\|\mathrm{e}^{K\eta}\frac{\partial^r\Psi}{\partial\iota_2^r}\right\|_{L^\infty} \lesssim 1. \tag{2.13}
$$

Let $\chi(y) \in C^\infty(\mathbb{R})$ and $\widetilde{\chi}(y) = 1 - \chi(y)$ be smooth cut-off functions satisfying $\chi(y) = 1$ if $|y| \le 1$ and $\chi(y) = 0$ if $|y| \ge 2$. We define the boundary layer as follows:

$$
U_0^B(\mathfrak{x},\mathfrak{w}) := \widetilde{\chi}(\varepsilon^{-1}\varphi)\chi(\varepsilon\eta)\Psi(\mathfrak{x},\mathfrak{w}). \tag{2.14}
$$

Remark 2.2. *Due to the cutoff in* (2.14)*, we have*

$$U_0^B(0,\iota_1,\iota_2,\mathfrak{w}) = \widetilde{\chi}(\varepsilon^{-1}\varphi)\Big(g(\iota_1,\iota_2,\mathfrak{w}) - \Phi_\infty(\iota_1,\iota_2)\Big) = \widetilde{\chi}(\varepsilon^{-1}\varphi)\Psi(0,\iota_1,\iota_2,\mathfrak{w}), \tag{2.15}$$

and

$$\sin\varphi\frac{\partial U_0^B}{\partial\eta} + U_0^B - \overline{U_0^B} = -\overline{\widetilde{\chi}(\varepsilon^{-1}\varphi)\chi(\varepsilon\eta)\Psi} + \overline{\Psi}\widetilde{\chi}(\varepsilon^{-1}\varphi)\chi(\varepsilon\eta) + \sin\phi\widetilde{\chi}(\varepsilon^{-1}\varphi)\frac{\partial\chi(\varepsilon\eta)}{\partial\eta}\Psi. \tag{2.16}$$

2.3. Matching procedure

We plan to enforce the matching condition for $x_0 \in \partial\Omega$ and $w \cdot n < 0$

$$U_0(x_0) + U_0^B(x_0, w) = g(x_0, w) + O(\varepsilon). \tag{2.17}$$

Considering (2.15), it suffices to require

$$U_0(x_0) = \Phi_\infty(x_0) := \Phi_\infty(\iota_1,\iota_2), \tag{2.18}$$

which yields

$$U_0(x_0) + \Psi(x_0, w) = g(x_0, w). \tag{2.19}$$

Hence, we obtain

$$U_0(x_0, w) + U_0^B(x_0, w) = g(x_0, w) - \chi(\varepsilon^{-1}\varphi)\Psi(0,\iota_1,\iota_2,\mathfrak{w}). \tag{2.20}$$

Construction of U_0 Based on (2.1) and (2.18), we define $U_0(x)$ satisfying

$$U_0 = \overline{U}_0, \quad \Delta_x\overline{U}_0 = 0, \quad U_0(x_0) = \Phi_\infty(x_0). \tag{2.21}$$

From standard elliptic estimates [39, Chapter 9: Section 2], trace theorem, and Proposition 2.1, we have for any $s \in [2,\infty)$

$$\|U_0\|_{W^{3+\frac{1}{s},s}} + |U_0|_{W^{3,s}} \lesssim 1. \tag{2.22}$$

Construction of U_1 Based on (2.2), we define $U_1(x, w)$ satisfying

$$U_1 = \overline{U}_1 - w\cdot\nabla_x U_0, \quad \Delta_x\overline{U}_1 = 0, \quad \overline{U}_1(x_0) = 0. \tag{2.23}$$

From (2.22), we have that for any $s \in [2,\infty)$

$$\|U_1\|_{W^{2+\frac{1}{s},s}L^\infty} + |U_1|_{W^{2,s}L^\infty} \lesssim 1. \tag{2.24}$$

<u>**Construction of U_2**</u> Based on (2.2), define $U_2(x,w)$ satisfying

$$U_2 = \overline{U}_2 - w\cdot\nabla_x U_1, \quad \Delta_x\overline{U}_2 = 0, \quad \overline{U}_2(x_0) = 0. \tag{2.25}$$

From (2.24), we have for any $s \in [2,\infty)$

$$\|U_2\|_{W^{1+\frac{1}{s},s}L^\infty} + |U_2|_{W^{1,s}L^\infty} \lesssim 1. \tag{2.26}$$

Summarizing the analysis above, we have the well-posedness and regularity estimates of the interior solution and boundary layer.

Proposition 2.3. *Under the assumption* (1.13)*, we can construct* U_0, U_1, U_2*, and* U_0^B *as in* (2.21)*,* (2.23)*,* (2.25)*, and* (2.14) *satisfying for any* $s \in [2,\infty)$

$$\|U_0\|_{W^{3+\frac{1}{s},s}} + |U_0|_{W^{3,s}} \lesssim 1, \tag{2.27}$$

$$\|U_1\|_{W^{2+\frac{1}{s},s}L^\infty} + |U_1|_{W^{2,s}L^\infty} \lesssim 1, \tag{2.28}$$

$$\|U_2\|_{W^{1+\frac{1}{s},s}L^\infty} + |U_2|_{W^{1,s}L^\infty} \lesssim 1, \tag{2.29}$$

and, for some constant $K > 0$ *and any* $0 < r \leq 3$*, we have*

$$\left\|\mathrm{e}^{K\eta}U_0^B\right\|_{L^\infty} + \left\|\mathrm{e}^{K\eta}\frac{\partial^r U_0^B}{\partial\iota_1^r}\right\|_{L^\infty} + \left\|\mathrm{e}^{K\eta}\frac{\partial^r U_0^B}{\partial\iota_2^r}\right\|_{L^\infty} \lesssim 1. \tag{2.30}$$

3. Remainder equation

Denote the approximate solution

$$u_a := \left(U_0 + \varepsilon U_1 + \varepsilon^2 U_2\right) + U_0^B. \tag{3.1}$$

Inserting (1.3) into (0.1), we have

$$w\cdot\nabla_x(u_a+R) + \varepsilon^{-1}\left(u_a+R\right) - \varepsilon^{-1}\left(\overline{u_a}+\overline{R}\right) = 0, \quad (u_a+R)\Big|_{\gamma_-} = g, \tag{3.2}$$

which yields

$$w\cdot\nabla_x R + \varepsilon^{-1}\left(R-\overline{R}\right) = -w\cdot\nabla_x u_a - \varepsilon^{-1}\left(u_a - \overline{u_a}\right), \quad R\Big|_{\gamma_-} = (g-u_a)\Big|_{\gamma_-}. \tag{3.3}$$

3.1. Formulation of remainder equation

Now we consider the remainder equation (1.6), where the boundary data h is given by

$$h := -\varepsilon w\cdot\nabla_x U_0 - \varepsilon^2 w\cdot\nabla_x U_1 - \chi(\varepsilon^{-1}\varphi)\Psi(0), \tag{3.4}$$

and the source term S is given by

$$S := S_0 + S_1 + S_2 + S_3, \tag{3.5}$$

where

$$S_0 := -\varepsilon^2 w\cdot\nabla_x U_2, \tag{3.6}$$

$$S_1 := \Big(\frac{\sin^2\psi}{R_1-\varepsilon\eta}+\frac{\cos^2\psi}{R_2-\varepsilon\eta}\Big)\cos\varphi\frac{\partial U_0^B}{\partial\varphi}, \tag{3.7}$$

$$\begin{aligned} S_2 :=& \varepsilon^{-1}\sin\phi\widetilde{\chi}(\varepsilon^{-1}\varphi)\frac{\partial\chi(\varepsilon\eta)}{\partial\eta}\Psi+\frac{R_1\cos\varphi\sin\psi}{L_1(R_1-\varepsilon\eta)}\frac{\partial U_0^B}{\partial\iota_1}+\frac{R_2\cos\varphi\cos\psi}{L_2(R_2-\varepsilon\eta)}\frac{\partial U_0^B}{\partial\iota_2} \\ &+\frac{\sin\psi}{R_1-\varepsilon\eta}\Big\{\frac{R_1\cos\varphi}{L_1L_2}\Big(\varsigma_1\cdot\Big(\varsigma_2\times(\partial_{\iota_1\iota_2}\mathbf{r}\times\varsigma_2)\Big)\Big)-\sin\varphi\cos\psi\Big\}\frac{\partial U_0^B}{\partial\psi} \\ &-\frac{\cos\psi}{R_2-\varepsilon\eta}\Big\{\frac{R_2\cos\varphi}{L_1L_2}\Big(\varsigma_2\cdot\Big(\varsigma_1\times(\partial_{\iota_1\iota_2}\mathbf{r}\times\varsigma_1)\Big)\Big)-\sin\varphi\sin\psi\Big\}\frac{\partial U_0^B}{\partial\psi}, \end{aligned} \tag{3.8}$$

$$S_3 := \varepsilon^{-1}\Big(\overline{\widetilde{\chi}(\varepsilon^{-1}\varphi)\chi(\varepsilon\eta)\Psi}-\overline{\Psi}\widetilde{\chi}(\varepsilon^{-1}\varphi)\chi(\varepsilon\eta)\Big). \tag{3.9}$$

3.2. Weak formulation

Lemma 3.1 (Green's identity, Lemma 2.2 of [40])**.** *Assume* $f(x,w),\ g(x,w)\in L^2(\Omega\times\mathbb{S}^2)$*, and* $w\cdot\nabla_x f,\ w\cdot\nabla_x g\in L^2(\Omega\times\mathbb{S}^2)$ *with* $f,\ g\in L^2_\gamma$*. Then*

$$\iint_{\Omega\times\mathbb{S}^2}\Big((w\cdot\nabla_x f)g+(w\cdot\nabla_x g)f\Big)\mathrm{d}x\mathrm{d}w=\int_\gamma fg(w\cdot n)=\int_{\gamma_+}fg\mathrm{d}\gamma-\int_{\gamma_-}fg\mathrm{d}\gamma. \tag{3.10}$$

Using Lemma 3.1, we can derive the weak formulation of (1.6). For any test function $\mathfrak{g}(x,w)\in L^2(\Omega\times\mathbb{S}^2)$ with $w\cdot\nabla_x\mathfrak{g}\in L^2(\Omega\times\mathbb{S}^2)$ with $\mathfrak{g}\in L^2_\gamma$, we have

$$\int_\gamma R\mathfrak{g}(w\cdot n)-\iint_{\Omega\times\mathbb{S}^2}R(w\cdot\nabla_x\mathfrak{g})+\varepsilon^{-1}\iint_{\Omega\times\mathbb{S}^2}\Big(R-\overline{R}\Big)\mathfrak{g}=\iint_{\Omega\times\mathbb{S}^2}S\,\mathfrak{g}. \tag{3.11}$$

3.3. Estimates of boundary and source terms

Lemma 3.2. *Under the assumption* (1.13)*, for h defined in* (3.4)*, we have*

$$|h|_{L^2_{\gamma_-}}\lesssim\varepsilon. \tag{3.12}$$

Proof. Based on Proposition 2.3, we have

$$|\varepsilon w\cdot\nabla_x U_0|_{L^2_{\gamma_-}}+\Big|\varepsilon^2 w\cdot\nabla_x U_1\Big|_{L^2_{\gamma_-}}\lesssim\varepsilon. \tag{3.13}$$

Noting that the cutoff $\chi(\varepsilon^{-1}\varphi)$ restricts the support to $|\varphi|\lesssim\varepsilon$ and that the $\mathrm{d}\gamma$ measure contributes an extra $\sin\varphi$, we have

$$\Big|\chi(\varepsilon^{-1}\varphi)\Psi(0)\Big|_{L^2_{\gamma_-}}\lesssim\varepsilon. \tag{3.14}$$

Hence, our result follows. □

Lemma 3.3. *Under the assumption* (1.13)*, for* S_0 *defined in* (3.6)*, we have*

$$\|S_0\|_{L^2}\lesssim\varepsilon^2. \tag{3.15}$$

Proof. This follows from Proposition 2.3. □

Lemma 3.4. *Under the assumption* (1.13)*, for* S_1 *defined in* (3.7)*, we have*

$$\left\|(1+\eta)S_1\right\|_{L^2} \lesssim 1. \tag{3.16}$$

Also, for the boundary layer U_0^B *defined in* (2.14)*, we have*

$$\left\|(1+\eta)U_0^B\right\|_{L^2} \lesssim \varepsilon^{\frac{1}{2}}, \quad \left\|(1+\eta)U_0^B\right\|_{L^2_x L^1_w} \lesssim \varepsilon^{\frac{1}{2}}, \tag{3.17}$$

and

$$\left|\left\langle(1+\eta)S_1, g\right\rangle\right| \lesssim \left\|(1+\eta)\left\langle w\right\rangle^2 U_0^B\right\|_{L^2} \|\nabla_w g\|_{L^2} \lesssim \varepsilon^{\frac{1}{2}} \|\nabla_w g\|_{L^2}. \tag{3.18}$$

Proof. We split

$$\begin{aligned} S_1 = S_{11} + S_{12} :=& \left(\frac{\sin^2\psi}{R_1-\varepsilon\eta} + \frac{\cos^2\psi}{R_2-\varepsilon\eta}\right)\cos\varphi\frac{\partial\Psi}{\partial\varphi}\widetilde{\chi}(\varepsilon^{-1}\varphi)\chi(\varepsilon\eta) \\ &+ \left(\frac{\sin^2\psi}{R_1-\varepsilon\eta} + \frac{\cos^2\psi}{R_2-\varepsilon\eta}\right)\cos\varphi\frac{\partial\widetilde{\chi}(\varepsilon^{-1}\varphi)}{\partial\varphi}\chi(\varepsilon\eta)\Psi. \end{aligned} \tag{3.19}$$

Note that S_{11} is nonzero only when $|\varphi| \geq \varepsilon$ and thus based on Proposition 2.1, we know $\left|\frac{\partial\Psi}{\partial\varphi}\right| \leq |\sin\varphi|^{-1}|\Psi| \lesssim \varepsilon^{-1}$. Hence, using $\mathrm{d}\mu = \varepsilon\mathrm{d}\eta$, we have

$$\begin{aligned} \|S_{11}\|_{L^2} \lesssim& \left(\iint_{|\varphi|\geq\varepsilon}\left|\frac{\partial\Psi}{\partial\varphi}\right|^2 \mathrm{d}\varphi\mathrm{d}\mu\right)^{\frac{1}{2}} \lesssim \left(\iint_{|\varphi|\geq\varepsilon}|\sin\varphi|^{-2}|\Psi|^2\mathrm{d}\varphi\mathrm{d}\mu\right)^{\frac{1}{2}} \\ \lesssim& \left(\iint_{|\varphi|\geq\varepsilon}|\sin\varphi|^{-2}\mathrm{e}^{-2K\eta}\mathrm{d}\varphi\mathrm{d}\mu\right)^{\frac{1}{2}} \lesssim \left(\varepsilon\iint_{|\varphi|\geq\varepsilon}|\sin\varphi|^{-2}\mathrm{e}^{-2K\eta}\mathrm{d}\varphi\mathrm{d}\eta\right)^{\frac{1}{2}} \lesssim \left(\varepsilon\varepsilon^{-1}\right)^{\frac{1}{2}} = 1. \end{aligned} \tag{3.20}$$

Noticing that $\frac{\partial\widetilde{\chi}(\varepsilon^{-1}\varphi)}{\partial\varphi} = \varepsilon^{-1}\widetilde{\chi}'(\varepsilon^{-1}\varphi)$ and $\widetilde{\chi}'(\varepsilon^{-1}\varphi)$ is nonzero only when $\varepsilon < |\varphi| < 2\varepsilon$, based on Proposition 2.1, we have

$$\begin{aligned} \|S_{12}\|_{L^2} \lesssim& \varepsilon^{-1}\left(\iint_{\varepsilon<|\varphi|<2\varepsilon}|\Psi|^2\mathrm{d}\varphi\mathrm{d}\mu\right)^{\frac{1}{2}} \lesssim \varepsilon^{-1}\left(\iint_{\varepsilon<|\varphi|<2\varepsilon}\mathrm{e}^{-2K\eta}\mathrm{d}\varphi\mathrm{d}\mu\right)^{\frac{1}{2}} \\ \lesssim& \varepsilon^{-1}\left(\varepsilon\iint_{\varepsilon<|\varphi|<2\varepsilon}\mathrm{e}^{-2K\eta}\mathrm{d}\varphi\mathrm{d}\eta\right)^{\frac{1}{2}} \lesssim \varepsilon^{-1}(\varepsilon\varepsilon)^{\frac{1}{2}} = 1. \end{aligned} \tag{3.21}$$

Combining (3.20) and (3.21), we have (3.16). Note that $\mathrm{e}^{-K\eta}$ will suppress the growth from the pre-factor $1+\eta$.

Naturally, (3.17) comes from Proposition 2.1. We now turn to (3.18). The most difficult term in $\left|\langle S_1, g\rangle\right|$ is essentially $\left|\left\langle\frac{\partial U_0^B}{\partial\varphi}, g\right\rangle\right|$. Integration by parts with respect to φ implies that

$$\left|\left\langle\frac{\partial U_0^B}{\partial\varphi}, g\right\rangle\right| \lesssim \left|\left\langle U_0^B, \frac{\partial g}{\partial\varphi}\right\rangle\right| \lesssim \left\|U_0^B\right\|_{L^2}\left\|\frac{\partial g}{\partial\varphi}\right\|_{L^2}. \tag{3.22}$$

From (1.2) and $\frac{\partial x}{\partial \varphi} = 0$, we know that the substitution $(\mu, \iota_1, \iota_2, w) \to (\mu, \iota_1, \iota_2, \mathfrak{w})$ implies

$$-\frac{\partial w}{\partial \varphi} \cdot n = \cos\varphi, \quad \frac{\partial w}{\partial \varphi} \cdot \varsigma_1 = -\sin\varphi \sin\psi, \quad \frac{\partial w}{\partial \varphi} \cdot \varsigma_2 = -\sin\varphi\cos\psi. \tag{3.23}$$

Hence, we know $\left|\frac{\partial w}{\partial \varphi}\right| \lesssim 1$, and thus

$$\left|\frac{\partial g}{\partial \varphi}\right| \lesssim |\nabla_w g| \left|\frac{\partial w}{\partial \varphi}\right| \lesssim |\nabla_w g|. \tag{3.24}$$

Hence, we know that

$$\left|\left\langle \frac{\partial U_0^B}{\partial \varphi}, g\right\rangle\right| \lesssim \left\|U_0^B\right\|_{L^2} \|\nabla_w g\|_{L^2} \lesssim \varepsilon^{\frac{1}{2}} \|\nabla_w g\|_{L^2}. \tag{3.25}$$

□

Lemma 3.5. *Under the assumption* (1.13)*, for* S_2 *defined in* (3.8)*, we have*

$$\left\|(1+\eta)S_2\right\|_{L^2} \lesssim \varepsilon^{\frac{1}{2}}, \quad \left\|(1+\eta)S_2\right\|_{L^2_x L^1_w} \lesssim \varepsilon^{\frac{1}{2}}. \tag{3.26}$$

Proof. Notice that $\left|\varepsilon^{-1}\sin\phi\widetilde{\chi}(\varepsilon^{-1}\varphi)\frac{\partial\chi(\varepsilon\eta)}{\partial\eta}\right| \lesssim 1$. Based on Proposition 2.1 and Proposition 2.3, we directly bound

$$\begin{aligned} \|S_2\|_{L^2} &\lesssim \left(\iint \left(|\Phi|^2 + \left|\frac{\partial\Phi}{\partial\iota_1}\right|^2 + \left|\frac{\partial\Phi}{\partial\iota_2}\right|^2 + \left|\frac{\partial\Phi}{\partial\psi}\right|^2\right) \mathrm{d}\varphi \mathrm{d}\mu\right)^{\frac{1}{2}} \\ &\lesssim \left(\iint \mathrm{e}^{-2K\eta}\mathrm{d}\varphi\mathrm{d}\mu\right)^{\frac{1}{2}} \lesssim \left(\varepsilon \iint \mathrm{e}^{-2K\eta}\mathrm{d}\varphi\mathrm{d}\eta\right)^{\frac{1}{2}} \lesssim \varepsilon^{\frac{1}{2}}. \end{aligned} \tag{3.27}$$

Then the $L^2_x L^1_w$ estimate follows from a similar argument, noting that there is no rescaling in w variables.

□

Lemma 3.6. *Under the assumption* (1.13)*, for* S_3 *defined in* (3.9)*, we have*

$$\left\|(1+\eta)S_3\right\|_{L^2} \lesssim 1, \quad \left\|(1+\eta)S_3\right\|_{L^2_x L^1_w} \lesssim \varepsilon^{\frac{1}{2}}. \tag{3.28}$$

Proof. Using $\chi = 1 - \widetilde{\chi}$, we split

$$S_3 = S_{31} + S_{32} := \varepsilon^{-1}\overline{\Psi}\chi(\varepsilon^{-1}\varphi)\chi(\varepsilon\eta) - \varepsilon^{-1}\overline{\chi(\varepsilon^{-1}\varphi)\chi(\varepsilon\eta)\Psi}. \tag{3.29}$$

Noting that S_{31} is nonzero only when $|\varphi| \leq \varepsilon$, based on Proposition 2.1, we have

$$\|S_{31}\|_{L^2} \lesssim \left(\iint_{|\varphi|\leq\varepsilon} \left|\varepsilon^{-1}\overline{\Psi}\right|^2 \mathrm{d}\varphi\mathrm{d}\mu\right)^{\frac{1}{2}} \lesssim \left(\varepsilon^{-2}\iint_{|\varphi|\leq\varepsilon} \mathrm{e}^{-2K\eta}\mathrm{d}\varphi\mathrm{d}\mu\right)^{\frac{1}{2}} \tag{3.30}$$

$$\lesssim\left(\varepsilon^{-1}\iint_{|\varphi|\leq\varepsilon}\mathrm{e}^{-2K\eta}\mathrm{d}\varphi\mathrm{d}\eta\right)^{\frac{1}{2}}\lesssim\left(\varepsilon^{-1}\varepsilon\right)^{\frac{1}{2}}\lesssim 1.$$

Analogously, noting that S_{32} contains a w integral, we have

$$\begin{aligned}\|S_{32}\|_{L^2}&\lesssim\left(\iint\left|\varepsilon^{-1}\overline{\Psi\chi(\varepsilon^{-1}\varphi)}\right|^2\mathrm{d}\varphi\mathrm{d}\mu\right)^{\frac{1}{2}}\lesssim\left(\varepsilon^{-2}\iint\left|\int_{|\varphi|\leq\varepsilon}\Psi\mathrm{d}\varphi\right|^2\mathrm{d}\varphi\mathrm{d}\mu\right)^{\frac{1}{2}}\\&\lesssim\left(\varepsilon^{-2}\iint\left|\int_{|\varphi|\leq\varepsilon}\mathrm{e}^{-K\eta}\mathrm{d}\varphi\right|^2\mathrm{d}\varphi\mathrm{d}\mu\right)^{\frac{1}{2}}\lesssim\left(\varepsilon^{-2}\iint\varepsilon^2\mathrm{e}^{-2K\eta}\mathrm{d}\varphi\mathrm{d}\mu\right)^{\frac{1}{2}}\\&\lesssim\left(\iint\mathrm{e}^{-2K\eta}\mathrm{d}\varphi\mathrm{d}\mu\right)^{\frac{1}{2}}\lesssim\left(\varepsilon\iint\mathrm{e}^{-2K\eta}\mathrm{d}\varphi\mathrm{d}\eta\right)^{\frac{1}{2}}\lesssim\varepsilon^{\frac{1}{2}}.\end{aligned}\tag{3.31}$$

Combining (3.30) and (3.31), we have the L^2 estimate. Similarly, we derive the $L^2_xL^1_w$ bound:

$$\|S_{31}\|_{L^2_xL^1_w}\lesssim\left(\int\left(\int_{|\varphi|\leq\varepsilon}\left|\varepsilon^{-1}\overline{\Psi}\right|\mathrm{d}\varphi\right)^2\mathrm{d}\mu\right)^{\frac{1}{2}}\lesssim\left(\int\mathrm{e}^{-2K\eta}\mathrm{d}\mu\right)^{\frac{1}{2}}\lesssim\left(\varepsilon\int\mathrm{e}^{-2K\eta}\mathrm{d}\eta\right)^{\frac{1}{2}}\lesssim\varepsilon^{\frac{1}{2}},\tag{3.32}$$

$$\begin{aligned}\|S_{32}\|_{L^2_xL^1_w}&\lesssim\left(\int\left(\int\left|\varepsilon^{-1}\overline{\Psi\chi(\varepsilon^{-1}\varphi)}\right|\mathrm{d}\varphi\right)^2\mathrm{d}\mu\right)^{\frac{1}{2}}\lesssim\left(\varepsilon^{-2}\int\left(\int\left|\int_{|\varphi|\leq\varepsilon}\Psi\mathrm{d}\varphi\right|\mathrm{d}\varphi\right)^2\mathrm{d}\mu\right)^{\frac{1}{2}}\\&\lesssim\left(\varepsilon^{-2}\int\left(\int\varepsilon\mathrm{e}^{-K\eta}\mathrm{d}\varphi\right)^2\mathrm{d}\mu\right)^{\frac{1}{2}}\lesssim\left(\int\mathrm{e}^{-2K\eta}\mathrm{d}\mu\right)^{\frac{1}{2}}\lesssim\left(\varepsilon\int\mathrm{e}^{-2K\eta}\mathrm{d}\eta\right)^{\frac{1}{2}}\lesssim\varepsilon^{\frac{1}{2}}.\end{aligned}\tag{3.33}$$

□

4. Remainder estimate

4.1. Basic energy estimate

Lemma 4.1. *Under the assumption* (1.13)*, we have*

$$\varepsilon^{-1}|R|^2_{L^2_{\gamma+}}+\varepsilon^{-2}\left\|R-\overline{R}\right\|^2_{L^2}\lesssim o(1)\varepsilon^{-1}\left\|\overline{R}\right\|^2_{L^2}+1.\tag{4.1}$$

Proof. Taking $\mathfrak{g}=\varepsilon^{-1}R$ in (3.11), we obtain

$$\frac{\varepsilon^{-1}}{2}\int_\gamma|R|^2(w\cdot n)+\varepsilon^{-2}\left\langle R,R-\overline{R}\right\rangle=\varepsilon^{-1}\left\langle R,S\right\rangle.\tag{4.2}$$

By using the orthogonality of $\overline{R}$ and $R-\overline{R}$, we have

$$\frac{\varepsilon^{-1}}{2}|R|^2_{L^2_{\gamma+}}+\varepsilon^{-2}\left\|R-\overline{R}\right\|^2_{L^2}=\varepsilon^{-1}\left\langle R,S\right\rangle+\frac{\varepsilon^{-1}}{2}|h|^2_{L^2_{\gamma-}}.\tag{4.3}$$

Using Lemma 3.2, we know that

$$\varepsilon^{-1}|R|^2_{L^2_{\gamma+}}+\varepsilon^{-2}\left\|R-\overline{R}\right\|^2_{L^2}\lesssim\varepsilon+\varepsilon^{-1}\left\langle R,S_0+S_1+S_2+S_3\right\rangle.\tag{4.4}$$

Using Lemma 3.3, we have

$$\left|\varepsilon^{-1}\left\langle R, S_0\right\rangle\right| \lesssim \varepsilon^{-1} \|R\|_{L^2} \|S_0\|_{L^2} \lesssim \varepsilon \|R\|_{L^2} \lesssim o(1) \|R\|_{L^2}^2 + \varepsilon^2. \tag{4.5}$$

Using Lemma 3.4, Lemma 3.5, and Lemma 3.6, we have

$$\begin{aligned}\left|\varepsilon^{-1}\left\langle R-\overline{R}, S_1+S_2+S_3\right\rangle\right| \lesssim& \varepsilon^{-1} \left\|R-\overline{R}\right\|_{L^2} \|S_1+S_2+S_3\|_{L^2} \\ \lesssim& \varepsilon^{-1} \left\|R-\overline{R}\right\|_{L^2} \lesssim o(1)\varepsilon^{-2} \left\|R-\overline{R}\right\|_{L^2}^2 + 1.\end{aligned} \tag{4.6}$$

Finally, we turn to $\varepsilon^{-1}\left\langle\overline{R}, S_1+S_2+S_3\right\rangle$. For S_1, we integrate by parts with respect to φ and use Lemma 3.4 to obtain

$$\begin{aligned}\left|\varepsilon^{-1}\left\langle\overline{R}, S_1\right\rangle\right| =& \varepsilon^{-1} \left|\left\langle\overline{R}, \Big(\frac{\sin^2\psi}{R_1-\varepsilon\eta} + \frac{\cos^2\psi}{R_2-\varepsilon\eta}\Big)\cos\varphi\frac{\partial U_0^B}{\partial\varphi}\right\rangle\right| \\ =& \varepsilon^{-1} \left|\left\langle\overline{R}, \Big(\frac{\sin^2\psi}{R_1-\varepsilon\eta} + \frac{\cos^2\psi}{R_2-\varepsilon\eta}\Big)U_0^B\sin\varphi\right\rangle\right| \\ \lesssim& \varepsilon^{-1} \left\|\overline{R}\right\|_{L^2} \left\|U_0^B\right\|_{L_x^2L_w^1} \lesssim \varepsilon^{-\frac{1}{2}} \left\|\overline{R}\right\|_{L^2} \lesssim o(1)\varepsilon^{-1} \left\|\overline{R}\right\|_{L^2}^2 + 1.\end{aligned} \tag{4.7}$$

In addition, Lemma 3.5 and Lemma 3.6 yield

$$\left|\varepsilon^{-1}\left\langle\overline{R}, S_2+S_3\right\rangle\right| \lesssim \varepsilon^{-1} \left\|\overline{R}\right\|_{L^2} \left(\|S_2\|_{L_x^2L_w^1} + \|S_3\|_{L_x^2L_w^1} \right) \lesssim \varepsilon^{-\frac{1}{2}} \left\|\overline{R}\right\|_{L^2} \lesssim o(1)\varepsilon^{-1} \left\|\overline{R}\right\|_{L^2}^2 + 1. \tag{4.8}$$

Combining (4.5)–(4.8), we obtain

$$\left|\varepsilon^{-1}\left\langle R, S_0+S_1+S_2+S_3\right\rangle\right| \lesssim o(1)\varepsilon^{-2} \left\|R-\overline{R}\right\|_{L^2}^2 + o(1)\varepsilon^{-1} \|R\|_{L^2}^2 + 1. \tag{4.9}$$

Combining (4.9) and (4.4), we have (4.1). □

4.2. Kernel estimate

Lemma 4.2. *Under the assumption* (1.13)*, we have*

$$\left\|\overline{R}\right\|_{L^2}^2 \lesssim \left\|R-\overline{R}\right\|_{L^2}^2 + |R|_{L^2_{\gamma_+}}^2 + \varepsilon. \tag{4.10}$$

Proof. Denote $\xi(x)$ satisfying

$$\begin{cases} -\Delta_x\xi = \overline{R} \ \text{ in } \ \Omega, \\ \xi(x_0) = 0 \ \text{ on } \ \partial\Omega. \end{cases} \tag{4.11}$$

Based on standard elliptic estimates and trace estimates [41, Chapter 6: Section 6.3], we have

$$\|\xi\|_{H^2} + |\nabla_x\xi|_{H^{\frac{1}{2}}} \lesssim \left\|\overline{R}\right\|_{L^2}. \tag{4.12}$$

Taking $\mathfrak{g} = \xi$ in (3.11), we have

$$\int_\gamma R\xi(w\cdot n) - \left\langle R, w\cdot\nabla_x\xi\right\rangle + \varepsilon^{-1}\left\langle R-\overline{R}, \xi\right\rangle = \left\langle S, \xi\right\rangle. \tag{4.13}$$

Using oddness, orthogonality, and $\xi\big|_{\partial\Omega} = 0$, we obtain (1.29)

$$-\Big\langle R, w\cdot\nabla_x\xi\Big\rangle = -\Big\langle R-\overline{R}, w\cdot\nabla_x\xi\Big\rangle = \Big\langle S,\xi\Big\rangle, \tag{4.14}$$

Then taking $\mathrm{g} = w\cdot\nabla_x\xi$ in (3.11), we obtain (1.25)

$$\int_\gamma R(w\cdot\nabla_x\xi)(w\cdot n) - \Big\langle R, w\cdot\nabla_x(w\cdot\nabla_x\xi)\Big\rangle + \varepsilon^{-1}\Big\langle R-\overline{R}, w\cdot\nabla_x\xi\Big\rangle = \Big\langle S, w\cdot\nabla_x\xi\Big\rangle. \tag{4.15}$$

Adding $\varepsilon^{-1}\times$(4.14) and (4.15) to eliminate $\varepsilon^{-1}\Big\langle R-\overline{R}, w\cdot\nabla_x\xi\Big\rangle$, we obtain

$$\int_\gamma R(w\cdot\nabla_x\xi)(w\cdot n) - \Big\langle R, w\cdot\nabla_x(w\cdot\nabla_x\xi)\Big\rangle = \varepsilon^{-1}\Big\langle S,\xi\Big\rangle + \Big\langle S, w\cdot\nabla_x\xi\Big\rangle. \tag{4.16}$$

Notice that

$$-\Big\langle R, w\cdot\nabla_x(w\cdot\nabla_x\xi)\Big\rangle = -\Big\langle \overline{R}, w\cdot\nabla_x(w\cdot\nabla_x\xi)\Big\rangle - \Big\langle R-\overline{R}, w\cdot\nabla_x(w\cdot\nabla_x\xi)\Big\rangle, \tag{4.17}$$

where (4.12) and Cauchy's inequality yield

$$-\Big\langle \overline{R}, w\cdot\nabla_x(w\cdot\nabla_x\xi)\Big\rangle \simeq \left\|\overline{R}\right\|_{L^2}^2, \tag{4.18}$$

$$\left|\Big\langle R-\overline{R}, w\cdot\nabla_x(w\cdot\nabla_x\xi)\Big\rangle\right| \lesssim \left\|R-\overline{R}\right\|_{L^2}^2 + o(1)\left\|\overline{R}\right\|_{L^2}^2. \tag{4.19}$$

Also, using (4.12) and Lemma 3.2, we have

$$\left|\int_\gamma R(w\cdot\nabla_x\xi)(w\cdot n)\right| \lesssim \Big(|R|_{L^2_{\gamma_+}} + |h|_{L^2_{\gamma_-}}\Big)|\nabla_x\xi|_{L^2} \lesssim o(1)\left\|\overline{R}\right\|_{L^2}^2 + |R|_{L^2_{\gamma_+}}^2 + \varepsilon^2. \tag{4.20}$$

Inserting (4.17)–(4.20) into (4.16), we obtain

$$\left\|\overline{R}\right\|_{L^2}^2 \lesssim \varepsilon^2 + \left\|R-\overline{R}\right\|_{L^2}^2 + |R|_{L^2_{\gamma_+}}^2 + \left|\varepsilon^{-1}\Big\langle S,\xi\Big\rangle\right| + \left|\Big\langle S, w\cdot\nabla_x\xi\Big\rangle\right|. \tag{4.21}$$

Then we turn to the estimate of source terms in (4.21). Cauchy's inequality and Lemma 3.3 yield

$$\left|\varepsilon^{-1}\Big\langle S_0,\xi\Big\rangle\right| + \left|\Big\langle S_0, w\cdot\nabla_x\xi\Big\rangle\right| \lesssim \varepsilon^{-1}\|S_0\|_{L^2}\|\xi\|_{H^1} \lesssim \varepsilon\left\|\overline{R}\right\|_{L^2} \lesssim o(1)\left\|\overline{R}\right\|_{L^2}^2 + \varepsilon^2. \tag{4.22}$$

Similar to (4.7), we first integrate by parts with respect to φ in S_1. Using $\xi\big|_{\partial\Omega} = 0$, (4.12), Hardy's inequality, Lemma 3.4, Lemma 3.5, and Lemma 3.6, we have

$$\begin{aligned}\left|\varepsilon^{-1}\Big\langle S_1+S_2+S_3,\xi\Big\rangle\right| &\lesssim \left|\varepsilon^{-1}\Big\langle U_0^B+S_2+S_3, \int_0^\mu\frac{\partial\xi}{\partial\mu}\Big\rangle\right| = \left|\Big\langle \eta U_0^B+\eta S_2+\eta S_3, \frac{1}{\mu}\int_0^\mu\frac{\partial\xi}{\partial\mu}\Big\rangle\right| \\ &\lesssim \left\|\eta U_0^B+\eta S_2+\eta S_3\right\|_{L^2_xL^1_w}\left\|\frac{1}{\mu}\int_0^\mu\frac{\partial\xi}{\partial\mu}\right\|_{L^2} \lesssim \left\|\eta U_0^B+\eta S_2+\eta S_3\right\|_{L^2_xL^1_w}\left\|\frac{\partial\xi}{\partial\mu}\right\|_{L^2} \lesssim \varepsilon^{\frac12}\|\xi\|_{H^1} \\ &\lesssim \varepsilon^{\frac12}\left\|\overline{R}\right\|_{L^2} \lesssim o(1)\left\|\overline{R}\right\|_{L^2}^2 + \varepsilon.\end{aligned} \tag{4.23}$$

Analogously, we integrate by parts with respect to φ in S_1. Then using (4.12), fundamental theorem of calculus, Hardy's inequality, Lemma 3.4, Lemma 3.5, and Lemma 3.6, we bound

$$\begin{aligned}
&\left|\left\langle S_1 + S_2 + S_3, w \cdot \nabla_x \xi \right\rangle\right| \lesssim \left|\left\langle U_0^B + S_2 + S_3, \nabla_x \xi\Big|_{\mu=0} + \int_0^\mu \frac{\partial(\nabla_x \xi)}{\partial \mu}\right\rangle\right| \qquad (4.24)\\
&\lesssim \left|\left\langle U_0^B + S_2 + S_3, \nabla_x \xi\Big|_{\mu=0}\right\rangle\right| + \left|\varepsilon\left\langle \eta U_0^B + \eta S_2 + \eta S_3, \frac{1}{\mu}\int_0^\mu \frac{\partial(\nabla_x \xi)}{\partial \mu}\right\rangle\right|\\
&\lesssim \left\|U_0^B + S_2 + S_3\right\|_{L^2_x L^1_w} |\nabla_x \xi|_{L^2} + \varepsilon \left\|\eta U_0^B + \eta S_2 + \eta S_3\right\|_{L^2} \left\|\frac{\partial(\nabla_x \xi)}{\partial \mu}\right\|_{L^2}\\
&\lesssim \varepsilon^{\frac{1}{2}} |\nabla_x \xi|_{L^2_{\partial\Omega}} + \varepsilon \|\xi\|_{H^2} \lesssim \varepsilon^{\frac{1}{2}} \left\|\overline{R}\right\|_{L^2} \lesssim o(1) \left\|\overline{R}\right\|_{L^2}^2 + \varepsilon.
\end{aligned}$$

Hence, inserting (4.22), (4.23), and (4.24) into (4.21), we have shown (4.10). □

4.3. Synthesis

Proposition 4.3. *Under the assumption* (1.13)*, we have*

$$\varepsilon^{-\frac{1}{2}} |R|_{L^2_{\gamma_+}} + \varepsilon^{-\frac{1}{2}} \left\|\overline{R}\right\|_{L^2} + \varepsilon^{-1} \left\|R - \overline{R}\right\|_{L^2} \lesssim 1. \tag{4.25}$$

Proof. From (4.1), we have

$$\varepsilon^{-1} |R|^2_{L^2_{\gamma_+}} + \varepsilon^{-2} \left\|R - \overline{R}\right\|^2_{L^2} \lesssim o(1)\varepsilon^{-1} \left\|\overline{R}\right\|^2_{L^2} + 1. \tag{4.26}$$

From (4.10), we have

$$\left\|\overline{R}\right\|^2_{L^2} \lesssim \left\|R - \overline{R}\right\|^2_{L^2} + |R|^2_{L^2_{\gamma_+}} + \varepsilon. \tag{4.27}$$

Inserting (4.27) into (4.26), we have

$$\varepsilon^{-1} |R|^2_{L^2_{\gamma_+}} + \varepsilon^{-2} \left\|R - \overline{R}\right\|^2_{L^2} \lesssim 1. \tag{4.28}$$

Inserting (4.28) into (4.27), we have

$$\left\|\overline{R}\right\|^2_{L^2} \lesssim \varepsilon. \tag{4.29}$$

Hence, adding $\varepsilon^{-1}\times$(4.29) and (4.28), we have

$$\varepsilon^{-1} |R|^2_{L^2_{\gamma_+}} + \varepsilon^{-1} \left\|\overline{R}\right\|^2_{L^2} + \varepsilon^{-2} \left\|R - \overline{R}\right\|^2_{L^2} \lesssim 1. \tag{4.30}$$

Then our result follows. □

5. Proof of the main theorem

The well-posedness of (0.1) is well-known [1, 16, 17]. The construction of U_0, Φ, and Φ_∞ follows from Proposition 2.1 and Proposition 2.3, so we focus on the derivation of (1.14).

Based on Proposition 4.3 and (1.3), we have

$$\left\|u^{\varepsilon} - U_0 - \varepsilon U_1 - \varepsilon^2 U_2 - U_0^B\right\|_{L^2} \lesssim \varepsilon^{\frac{1}{2}}. \tag{5.1}$$

Using Proposition 2.3, we have

$$\left\|\varepsilon U_1 + \varepsilon^2 U_2\right\|_{L^2} \lesssim \varepsilon. \tag{5.2}$$

Using Proposition 2.3 and the rescaling $\eta = \varepsilon^{-1}\mu$, we have

$$\left\|U_0^B\right\|_{L^2} \lesssim \varepsilon^{\frac{1}{2}}. \tag{5.3}$$

Then (1.14) follows from inserting (5.2) and (5.3) into (5.1).

Author contributions

The authors have equal contribution to this paper.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors claim that there is no conflict of interest.

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