Received: 1 June 2023 | Revised: 10 April 2024 | Accepted: 16 May 2024
DOI: 10.1112/topo.12350

RESEARCH ARTICLE
Journal of Topology

Stated SL(n)-skein modules and algebras

Thang T. Q. Lê[1] | Adam S. Sikora[2]

[1]School of Mathematics, Georgia Tech, Atlanta, Georgia, USA
[2]Department of Mathematics, University at Buffalo, Buffalo, New York, USA

Correspondence
Adam S. Sikora, Department of Mathematics, University at Buffalo, Buffalo, New York, USA.
Email: asikora@buffalo.edu

Funding information
National Science Foundation, Grant/Award Number: DMS-1811114; Centre International de Mathématiques et d'Informatique's Excellence; National Science Foundation, Grant/Award Number: DMS-1811114; Centre International de Mathématiques et d'Informatique's Excellence

Abstract
We develop a theory of stated SL(n)-skein modules, $\mathcal{S}_n(M, \mathcal{N})$, of 3-manifolds M marked with intervals $\mathcal{N}$ in their boundaries. These skein modules, generalizing stated SL(2)-modules of the first author, stated SL(3)-modules of Higgins', and SU(n)-skein modules of the second author, consist of linear combinations of framed, oriented graphs, called n-webs, with ends in $\mathcal{N}$, considered up to skein relations of the $U_q(sl_n)$-Reshetikhin–Turaev functor on tangles, involving coupons representing the anti-symmetrizer and its dual. We prove the Splitting Theorem asserting that cutting of a marked 3-manifold M along a disk resulting in a 3-manifold M' yields a homomorphism $\mathcal{S}_n(M) \to \mathcal{S}_n(M')$ for all n. That result allows to analyze the skein modules of 3-manifolds through the skein modules of their pieces. The theory of stated skein modules is particularly rich for thickened surfaces $M = \Sigma \times (-1, 1)$, in whose case, $\mathcal{S}_n(M)$ is an algebra, denoted by $\mathcal{S}_n(\Sigma)$. One of the main results of this paper asserts that the skein algebra of the ideal bigon is isomorphic with $\mathcal{O}_q(SL(n))$ and it provides simple geometric interpretations of the product, coproduct, counit, the antipode, and the cobraided structure on $\mathcal{O}_q(SL(n))$. (In particular, the coproduct is given by a splitting homomorphism.) We show that for surfaces with boundary Σ every splitting homomorphism is injective and that $\mathcal{S}_n(\Sigma)$ is a free module with a basis induced

from the Kashiwara–Lusztig canonical bases. Additionally, we show that a splitting of a thickened bigon near a marking defines a right $\mathcal{O}_q(SL(n))$-comodule structure on $\mathcal{S}_n(M)$, or dually, a left $U_q(sl_n)$-module structure. Furthermore, we show that the skein algebra of surfaces Σ_1, Σ_2 glued along two sides of a triangle is isomorphic with the braided tensor product $\mathcal{S}_n(\Sigma_1)\underline{\otimes}\mathcal{S}_n(\Sigma_2)$ of Majid. These results allow for geometric interpretation of further concepts in the theory of quantum groups, for example, of the braided products and of Majid's transmutation operation. Building upon the above results, we prove that the factorization homology with coefficients in the category of representations of $U_q(sl_n)$ is equivalent to the category of left modules over $\mathcal{S}_n(\Sigma)$ for surfaces Σ with $\partial\Sigma = S^1$. We also establish isomorphisms of our skein algebras with the quantum moduli spaces of Alekseev–Schomerus and with the internal algebras of the skein categories for these surfaces and $\mathfrak{g} = sl(n)$.

MSC 2020
57K31 (primary), 57K16, 20G42, 17B37 (secondary)

Contents

1 | INTRODUCTION

1.1 | Motivation

Moduli spaces of flat connections on surfaces and their quantizations play a pivotal role in quantum field theory. For example, they appear as the classical phase spaces of the Yang–Mills and Chern–Simons theories, [5, 67], and are quantized by these theories. More rigorous quantizations are achieved in mathematics through the Topological Quantum Field Theories, [65], Kauffman bracket skein algebras, [55, 56, 64], the lattice gauge theory, [3, 8], and more recently, through (quantum) cluster algebras, [23, 29], and factorization homology, [7]. These quantizations and relations between them are a subject of current active research and are of central importance to Quantum Topology.

Based on ideas of [12], the first author extended the notion of the Kauffman bracket skein algebras (quantizing $SL(2,\mathbb{C})$-character varieties) to their stated version, built of links and arcs with stated ends, [41]. His approach made it possible to analyze skein algebras of surfaces through surface triangulations and provided a conceptual framework for the Bonahon–Wang theory, [12, 42].

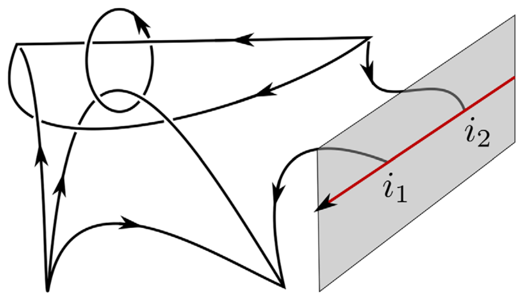


FIGURE 1 An example of a 3-web with two endpoints in a marking (in red) stated by i_1, i_2.

On the other hand, the second author introduced the notion of SL(n)-skein modules of 3-manifolds and proved that they quantize their SL(n)-character varieties, [58]. Based on these two developments, we develop a theory of stated SL(n)-skein modules of 3-manifolds. Our theory generalizes the recent work of Higgins for $n = 3$, [28], however, it is not a straightforward generalization of the $n = 2, 3$ cases, because the SL(n)-skein theories for $n = 2$ and 3 rely on explicit bases of skein algebras that are unknown for higher n. In fact, one of the main achievements of this work is a construction of bases for the SL(n)-skein algebras of surfaces with nonempty boundary for all n.

We discuss the relation between our stated SL(n)-skein algebras and other quantizations of the $SL(n)$-character varieties of surfaces in Subsection 1.14 and Section 10.

1.2 | Skein modules of marked 3-manifolds

In this paper, we will work with a commutative ring of coefficients R with a distinguished invertible $v = q^{\frac{1}{2n}}$. A marked 3-manifold is a pair $(M, \mathcal{N})$, where M is a smooth oriented 3-manifold with (possibly empty) boundary ∂M and $\mathcal{N} \subset \partial M$ consists of open intervals, called *markings*.

An n-web α in $(M, \mathcal{N})$ is a disjoint union of an oriented link and a directed ribbon graph whose every vertex is either 1-valent end in $\mathcal{N}$ or an internal n-valent sink or source, see Figure 1. Each web is equipped with a transversal vector field called its framing, which at each end e points in the direction of the marking containing e, see Subsection 4.1.

A *state* of a web α is an assignment of a label from $\{1, \dots, n\}$ to each of its ends.

The stated $SL(n)$-skein module, $\mathcal{S}_n(M, \mathcal{N})$, of $(M, \mathcal{N})$ is the space of all R-linear combination of stated n webs in $(M, \mathcal{N})$, up to internal skein relations (43)–(46) and boundary skein relations (47)-(50). These relations mimic those satisfied by the Reshetikhin–Turaev functor on tangles, with n-vertices representing the anti-symmetrizer tensor and its dual. More specifically, the internal relations are based on the skein relations of [58]. (It may be useful to recall here the premise of [58]: that although n-webs seem unnecessary from the point of view of study of quantum invariants of links in manifolds, they allow for a very efficient formulation of the necessary skein relations.)

However, our specific relations involve a novel sign modification that leads to a major technical benefit: the half-edges around each n-valent vertex have a cyclic ordering only, rather than a linear ordering required in [58], see Subsections 3.3 and 3.9. An additional benefit of this modification is that it makes skein relations invariant under the orientation reversal of the webs. (That is reversal of all loop orientations and edge directions.)

The boundary skein relations of $\mathcal{S}_n(M, \mathcal{N})$ are new and generalize those of [41] and [28].

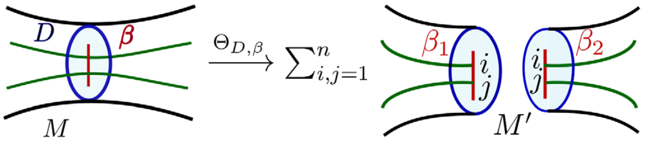


FIGURE 2 An example of a splitting of an n-web (in green) intersecting the splitting disk D twice.

1.3 | Splitting homomorphisms

An important property of stated skein modules is that they behave in a simple manner under the splitting of 3-manifolds along disks. Specifically, for a marked 3-manifold $(M, \mathcal{N})$ with a properly embedded closed disk D in $M - \mathcal{N}$, let M' be M with an open collar neighborhood of D removed. Then M' is a 3-manifolds with copies D_1, D_2 of D in its boundary, whose gluing together leads to an epimorphism $pr : M' \to M$. Given an oriented open interval $\beta \subset D$, the splitting of $(M, \mathcal{N})$ along (D, β), denoted by $\mathrm{cut}_{(D,\beta)}(M, \mathcal{N})$, is the marked 3-manifold $(M', \mathcal{N}')$, where $\mathcal{N}' = \mathcal{N} \cup \beta_1 \cup \beta_2$, where $\beta_i \subset D_i$ are the connected components of $pr^{-1}(\beta)$, see Figure 2. Note that for any stated n-web α in $(M, \mathcal{N})$ transversal to D with $\alpha \cap D \subset \beta$, the inverse image $pr^{-1}(\alpha)$ is an n-web in $(M', \mathcal{N}')$ stated at all its ends except those at $\beta_1 \cup \beta_2$. Given any map $s : \alpha \cap \beta \to \{1, \dots, n\}$ let $\alpha(s)$ be $\mathrm{pr}^{-1}(\alpha)$ with each of its ends $x \in \mathrm{pr}^{-1}(\alpha) \cap (\beta_1 \cup \beta_2)$ stated by $s(\mathrm{pr}(x))$.

The following result generalizes that of [11, 41] for the Kauffman bracket skein modules ($n = 2$) and of [28] for $n = 3$:

Theorem (Splitting Theorem 4.5). *There is a unique R-module homomorphism*

$$\Theta_{(D,\beta)} : \mathcal{S}_n(M, \mathcal{N}) \to \mathcal{S}_n(\mathrm{cut}_{(D,\beta)}(M, \mathcal{N}))$$

sending every stated (D, β)-transverse n-web α in $(M, \mathcal{N})$ to the sum of all of its lifts,

$$\Theta_{(D,\beta)}(\alpha) = \sum_{s : \alpha \cap \beta \to \{\pm\}} \alpha(s).$$

1.4 | Basic properties of stated skein modules

We discuss symmetries and other properties of stated skein modules of marked 3-manifolds in Subsections 4.9–4.10. In particular, we observe that for every marked 3-manifold $(M, \mathcal{N})$, the orientation reversal of webs $\alpha \to \overleftarrow{\alpha}$ defines an R-module automorphism:

$$\overleftarrow{\cdot} : \mathcal{S}_n(M, \mathcal{N}) \to \mathcal{S}_n(M, \mathcal{N}),$$

where an orientation of a web consists of orientations of all its loop components and directions of all its edges.

Let $\overline{(M, \mathcal{N})}$ denote M and $\mathcal{N}$ with reversed orientations. Let $\bar{R}$ be the ring R with the distinguished element v^{-1} instead of v. For an n-web α of $(M, \mathcal{N})$, let $\overline{\alpha}$ be the n-web in $\overline{(M, \mathcal{N})}$ obtained from α by negating its framing, $f \to -f$.

Theorem (Theorem 4.9).

(1) *Any ring isomorphism* $\varkappa : R \to \bar{R}$ *sending* v *to* v^{-1} *extends to an isomorphism of* R*-modules* $\varkappa_{(M,\mathcal{N})} : \mathcal{S}_n(M,\mathcal{N},R) \xrightarrow{\cong} \mathcal{S}_n(\overline{M,\mathcal{N}},\bar{R})$ *sending every stated* n*-web* α *to* $\overline{\alpha}$*, where* $\mathcal{S}_n(\overline{M,\mathcal{N}},\bar{R})$ *is an* R*-module via* $\varkappa : R \to \bar{R}$.

(2) *The composition* $\varkappa_{\overline{(M,\mathcal{N})}} \circ \varkappa_{(M,\mathcal{N})}$ *is the identity on* $\mathcal{S}_n(M,\mathcal{N},R)$.

For every marking β of $(M,\mathcal{N})$, there is an R-module automorphism g_β of $\mathcal{S}_n(M,\mathcal{N})$, called a *marking automorphism*, sending stated n-webs α to

$$g_\beta(\alpha) = \prod_{x\in\alpha\cap\beta} (-1)^{n-1} q^{2s(x)-n-1} \cdot \alpha,$$

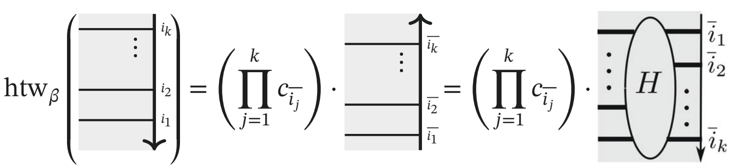
where $s(x)$ is the state of the endpoint x of α.

There is an additional automorphism of $\mathcal{S}_n(M,\mathcal{N})$ associated with each marking of $(M,\mathcal{N})$:

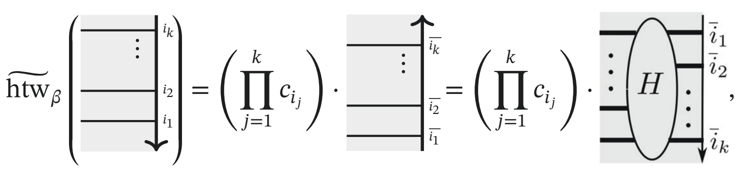
Proposition (Proposition 4.11). *For any marking* β *in* $\mathcal{N}$ *there exist unique* R*-linear isomorphisms, called the* half-twist automorphisms*,*

$$\mathrm{htw}_\beta, \widetilde{\mathrm{htw}}_\beta : \mathcal{S}_n(M,\mathcal{N}) \to \mathcal{S}_n(M,\mathcal{N})$$

sending any stated n*-web* α *in* $(M,\mathcal{N})$ *with* k *endpoints on* β *to*



and to



where $c_i \in R$*'s are defined in (2) in Subsection 2.1, and* $\bar{i} = n+1-i$*.* H *is the positive half-twist – see further details in Subsection 4.11. (The directions of the horizontal edges are arbitrary.)*

1.5 | Stated SL(n)-skein algebras of surfaces

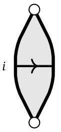
The theory of stated SL(n)-skein modules is particularly rich for thickened surfaces $M = \Sigma \times (-1,1)$. It is most convenient to consider it in the context of *punctured bordered surfaces* (*pb surfaces* for short) which are of the form $\Sigma = \bar{\Sigma} - \mathcal{P}$, where $\bar{\Sigma}$ is a compact oriented surface and $\mathcal{P} \subset \bar{\Sigma}$ is a finite set, called the *ideal points* of Σ, which meets each connected component of $\partial\bar{\Sigma}$. Then $\partial\Sigma$ is a union of open intervals. These intervals are called *boundary edges*.

In each boundary edge e, choose a point b_e. Let $\mathcal{S}_n(\Sigma) = \mathcal{S}_n(\Sigma \times (-1,1), \mathcal{N})$, where $\mathcal{N}$ is the union of all $b_e \times (-1,1)$, each having the natural orientation of the interval $(-1,1)$.

For the monogon $\mathfrak{M}$, which is the closed disk with a boundary point removed, we have

Theorem (Theorem 6.1). *The map $\mu : R \to S_n(\mathfrak{M})$ given by $\mu(r) = r \cdot \emptyset$ is an R-algebra isomorphism.*

Despite its simple statement, the proof of the above result is nontrivial; see the comment at the end of this subsection.

The bigon, $\mathfrak{B}$, is a closed disk with two boundary points removed. In Lemma 6.5, we show that the R-algebra $\mathcal{S}_n(\mathfrak{B})$ is generated by the arcs $a_{ij} =$ i  j for $1 \leqslant i, j \leqslant n$. Splitting $\mathfrak{B}$ along an interior ideal arc connecting its two ideal vertices defines an algebra R-homomorphism

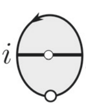
$$\Delta : \mathcal{S}_n(\mathfrak{B}) \to \mathcal{S}_n(\mathfrak{B}) \otimes \mathcal{S}_n(\mathfrak{B}).$$

Let $\epsilon : \mathcal{S}_n(\mathfrak{B}) \to R$ be the composition

$$\epsilon : \mathcal{S}_n(\mathfrak{B}) \xrightarrow{\widetilde{\mathrm{htw}}_{e_r}} \mathcal{S}_n(\mathfrak{B}) \xrightarrow{\iota_*} \mathcal{S}_n(\mathfrak{M}) \simeq R,$$

where htw_{e_r} is the half-twist automorphism defined above and ι_* is the algebra homomorphism induced by an embedding $\mathfrak{B} \hookrightarrow \mathfrak{M}$ filling in one of the two ideal points of $\mathfrak{B}$, (depicted always on top of $\mathfrak{B}$ in this paper).

On generators,

$$\epsilon(a^i_j) = \epsilon(\overleftarrow{a}^i_j) = c_{\bar{j}} \; i$$  $$\bar{j} = \delta_{i,j}.$$

Let $\mathcal{O}_q(SL(n))$ be the quantized coordinate ring algebra of sl_n. This Hopf algebra is the restricted dual of the quantized enveloping algebra, $U_q(sl_n)$, see [35, 9.2.2]. For technical convenience, we consider $\mathcal{O}_q(SL(n))$ as defined over $\mathbb{Q}[v^{\pm 1}]$.

Theorem (Theorem 6.3).

(a) *The algebra $\mathcal{S}_n(\mathfrak{B})$ has the structure of a Hopf algebra over R with the coproduct Δ, the counit ϵ, and the antipode S such that*

$$S(a^i_j) = (-q)^{i-j} \, \overleftarrow{a}^{\bar{j}}_{\bar{i}} \; for \; i, j = 1, \ldots, n.$$

(b) *The map $\Psi(u^i_j) = a^i_j$ extends to a unique Hopf algebra isomorphism*

$$\mathcal{O}_q(sl_n; R) := \mathcal{O}_q(SL(n)) \otimes R \xrightarrow{\Psi} \mathcal{S}_n(\mathfrak{B}).$$

The above theorems generalize statements for $n = 2$ in [15] and $n = 3$ in [28]. However, it is not a straightforward generalization. The proofs of [15, 28] relied on specific bases of $S_n(\Sigma)$ for $n = 2, 3$ that can be obtained through the confluence method, see [60]. That method does not work for higher n and a construction of bases for $n > 3$ is an important and still open problem. (We discuss our progress on that problem below.) We were able to establish the above theorem without constructing a basis of $\mathcal{S}_n(\mathfrak{B})$.

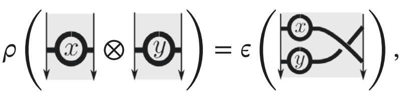
1.6 | Geometric interpretation of the cobraided structure on $\mathcal{O}_q(SL(n))$

The Hopf algebra $\mathcal{O}_q(sl_n; R)$ is *dual quasitriangular* (see [52, section 2.2], [35, section 10], [20, section 10.3]), also known as *cobraided* (see, e.g., [33, section VIII.5]). This means it has an R-form (also known as co-R-matrix), which is a bilinear form

$$\rho : \mathcal{O}_q(sl_n; R) \otimes \mathcal{O}_q(sl_n; R) \to R$$

satisfying certain properties, with the help of which one can make the category of $\mathcal{O}_q(sl_n; R)$-modules a braided category. The following generalizes [15, Theorem 3.5] from $n = 2$ to all n:

Theorem (Theorem 6.4). *Under the above identification $\mathcal{S}_n(\mathcal{B}) \simeq \mathcal{O}_q(sl_n; R)$ the R-form ρ has the following geometric description*



for any $x, y \in \mathcal{O}_q(sl_n; R)$.

Above we identified $\mathfrak{B}$ with $[-1, 1] \times (-1, 1)$ by stretching its top and bottom ideal points into horizontal intervals.

1.7 | Relation to Reshetikhin–Turaev theory

Sections 2–3 mostly summarize the background in quantum groups and in Reshetikhin–Turaev theory necessary for this paper. Section 2, however, also introduces a novel modification of the Reshetikhin–Turaev functor, utilized throughout the paper. Let us briefly describe its connection to stated skein modules.

When the bigon $\mathfrak{B}$ is identified with $[-1, 1] \times (-1, 1)$, the webs on $\mathfrak{B}$ can be thought as oriented framed tangles with coupons given by n-valent sinks and sources.

The *sign* $\mathrm{sgn}(e)$ of an endpoint $e \in \partial\alpha$ is positive if the direction of α goes from left to right at e, and negative otherwise. Let $\mathrm{sgn}_l(\alpha)$ (respectively, $\mathrm{sgn}_r(\alpha)$) be the sequence of signs of left (respectively, right) endpoints of α appearing from the bottom to the top.

Let $V = \mathbb{Q}(v)^n$ be the defining representation of $U_q(sl_n)$ with its standard basis $\{e_1, \dots, e_n\}$. Let $\{e_1^*, \dots, e_n^*\}$ be the dual basis of V^* and let $\{f_1, \dots, f_n\}$ be a basis of V^* given by $f_i = (-1)^{i-1} q^{i - \frac{n+1}{2n}} e^{\bar{i}}$

for $i = 1, \dots, n$. For any sign sequence $\boldsymbol{\eta} = (\eta_1, \dots, \eta_k)$, let

$$V^{\boldsymbol{\eta}} := V^{\eta_1} \otimes \cdots \otimes V^{\eta_k},$$

where $V^+ = V$ and $V^- = V^*$. The above bases $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_n\}$ induce the *tensor basis* of $V^{\boldsymbol{\eta}}$ indexed by the elements of $\{1, \dots, n\}^k$, see Subsection 3.4.

Note that $V^{\boldsymbol{\eta}}$ is a $U_q(sl_n)$-module for every sign sequence $\boldsymbol{\eta}$. The key to our work is a modified version of the Reshetikhin–Turaev functor, introduced in Section 3, which associates to each web α in $\mathfrak{B}$ a $U_q(sl_n)$-module homomorphism $\mathsf{RT}(\alpha) : V^{\mathrm{sgn}_l(\alpha)} \to V^{\mathrm{sgn}_r(\alpha)}$. (It is a sign modification of the standard $U_q(sl_n)$-Reshetikhin–Turaev functor, [57], which requires that the half-edges incident to each n-valent vertex in α are linearly ordered.)

We show that the benefit of utilizing the basis $\{f_1, \dots, f_n\}$ of V^* rather than the dual basis to $\{e_1, \dots, e_n\}$ is that it makes the modified R-T functor values $\mathsf{RT}(\alpha)$ independent of the orientation of tangles.

The following result relates our skein algebra of $\mathfrak{B}$ to the Reshetikhin–Turaev theory:

Proposition (Proposition 6.6). *Let $\boldsymbol{\alpha}$ be an n-web α on $\mathfrak{B}$ stated by $\mathbf{i} = (i_1, \dots, i_k)$ on the left and $\boldsymbol{j} = (j_1, \dots, j_l)$ on the right. Then $\epsilon(\boldsymbol{\alpha})$ is equal to the $(\mathbf{i}, \boldsymbol{j})$-entry of the matrix of the modified Reshetikhin–Turaev operator $\mathsf{RT}(\alpha)$ of α in the above tensor bases.*

1.8 | Module and Co-module structures

Given a marking β of a marked 3-manifold $(M, \mathcal{N})$, consider its closed disk neighborhood D in ∂M, disjoint from the other markings of $(M, \mathcal{N})$. By pushing the interior of D inside M we get a new disk D' which is properly embedded in M. Splitting $(M, \mathcal{N})$ along D', we get a new marked 3-manifold $(M', \mathcal{N}')$ isomorphic to $(M, \mathcal{N})$, and another marked 3-manifold bounded by D and D'. The latter, after removing the common boundary of D and D', is isomorphic to the thickening of the bigon. Hence, this construction yields an R-linear splitting map

$$\Delta_\beta : \mathcal{S}_n(M, \mathcal{N}) \to \mathcal{S}_n(M, \mathcal{N}) \otimes \mathcal{O}_q(SL(n)),$$

which defines a right coaction of $\mathcal{O}_q(SL(n))$ on $\mathcal{S}_n(M, \mathcal{N})$. Such coactions at different markings commute.

The Hopf algebra $\mathcal{O}_q(SL(n); \mathbb{Z}[v^{\pm 1}])$ has a Hopf dual given by a completion $\widetilde{U_q(sl_n)}$ of the Lusztig's integral version, $\widetilde{U^L}$, of U^L, which is a Hopf algebra over $\mathbb{Z}[v^{\pm 1}]$, see Subsection 3.9 and [48, section 1.3]. The duality between these two Hopf algebras turns any right $\mathcal{O}_q(SL(n))$-comodule W to a left $\widetilde{U^L}$-module as follows: For $u \in \widetilde{U^L}$ and $x \in W$,

$$u * x = \sum x_{(1)} \langle f_{(2)}, u \rangle, \quad \text{where } \Delta(x) = \sum x_{(1)} \otimes f_{(2)}$$

(in the Sweedler notation) is the $\mathcal{O}_q(SL(n))$-coaction map. We make this left $\widetilde{U^L}$-action on $\mathcal{S}_n(M, \mathcal{N})$ explicit in Subsection 7.1.

The Hopf algebra $\widetilde{U^L}$ contains distinguished charmed element g and the half-ribbon element $X \in \widetilde{U^L}$. We prove that the action of these elements on $\mathcal{S}_n(M, \mathcal{N})$ at a marking β is exactly the marking automorphism g_β and the half-twist homomorphism htw_β of Subsection 1.4.

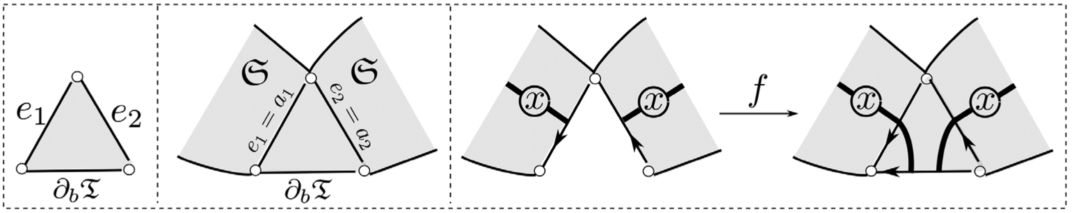


FIGURE 3 Left: The standard ideal triangle $\mathfrak{T}$. Middle: Glueing Σ and $\mathfrak{T}$ by $a_1 = e_1$ and $a_2 = e_2$ to get $\Sigma_{a_1\triangle a_2}$. Right: Tangle diagram $x \in \mathcal{S}_n(\Sigma)$ and its image $\mathsf{glue}_{a_1,a_2}(x) \in \mathcal{S}_n(\Sigma_{a_1\triangle a_2})$.

1.9 | Glueing over an ideal triangle

The standard ideal triangle $\mathfrak{T} \subset \mathbb{R}^2$ is the closed triangle with vertices $(-1,0), (1,0)$ and $(0,1)$ with these vertices removed. We will denote its sides by e_1, e_2, and $\partial_b\mathfrak{T}$ as in Figure 3. Suppose a_1, a_2 are two distinct boundary edges of a (possibly disconnected) pb surface. Define

$$\Sigma_{a_1\triangle a_2} = (\Sigma \sqcup \mathfrak{T})/(e_1 = a_1, e_2 = a_2),$$

as in Figure 3. Define the R-linear homomorphism $\mathsf{glue}_{a_1,a_2} : \mathcal{S}_n(\Sigma) \to \mathcal{S}_n(\Sigma_{a_1\triangle a_2})$ by continuing the strands of any web α with endpoints on a_1 and a_2 until they reach $\partial_b\mathfrak{T}$, as in Figure 3 (right).

Proposition (Proposition 8.1, see [15] for $n = 2$ and [28] for $n = 3$). *The map*

$$\mathsf{glue}_{a_1,a_2} : \mathcal{S}_n(\Sigma) \to \mathcal{S}_n(\Sigma_{a_1\triangle a_2})$$

is an R-linear isomorphism.

We construct an explicit inverse map to glue_{a_1,a_2} in Subsection 8.1.

Although the bijective map glue_{a_1,a_2} is not an algebra isomorphism with respect to the standard skein algebra product on $\mathcal{S}_n(\Sigma_{a_1\triangle a_2})$, we show in Subsection 8.1 that it is one with respect to the *self-braided tensor product* which we will define right now.

There are two right $\mathcal{O}_q(SL(n))$-comodule algebra structures on $\mathcal{S}_n(\Sigma)$ given by

$$\Delta_i := \Delta_{a_i} : \mathcal{S}_n(\Sigma) \to \mathcal{S}_n(\Sigma) \otimes \mathcal{O}_q(SL(n)), \quad i = 1, 2,$$

which commute. Define the R-linear map $\underline{\Delta} : \mathcal{S}_n(\Sigma) \to \mathcal{S}_n(\Sigma) \otimes \mathcal{O}_q(SL(n))$ by

$$\underline{\Delta}(x) = \sum x' \otimes u_1 u_2,$$

in Sweedler's notation, where

$$(\Delta_1 \otimes \mathrm{Id}_{\mathcal{O}_q(SL(n))}) \circ \Delta_2(x) = \sum x_{(1)} \otimes u_{(2)} \otimes u_{(3)}.$$

For $x, y \in \mathcal{S}_n(\Sigma)$ define a new product by

$$y \underline{*} x = \sum y_{(1)} x_{(1)} \rho(u_{(2)} \otimes w_{(2)}),$$

where

$$\Delta_2(y) = \sum y_{(1)} \otimes u_{(2)}, \quad \Delta_1(x) = \sum x_{(1)} \otimes w_{(2)},$$

and ρ is the R-form.

It is proved in [15] that $\underline{\Delta}$ and $\underline{*}$ together give $\mathcal{S}_n(\Sigma)$ a right $\mathcal{O}_q(SL(n))$-comodule algebra structure for $n = 2$. That proof extends to all n.

Denote by $\underline{\otimes}\mathcal{S}_n(\Sigma)$ the R-module $\mathcal{S}_n(\Sigma)$ with this $\mathcal{O}_q(SL(n))$-comodule algebra structure. On the other hand, $\mathcal{S}_n(\Sigma_{a_1 \triangle a_2})$ has a right $\mathcal{O}_q(SL(n))$-comodule algebra structure coming from the boundary edge $\partial_b \mathfrak{T}$. Here is a stronger version of the proposition above.

Theorem (Theorem 8.2). *The map* $\mathsf{glue}_{a_1,a_2} : \underline{\otimes}\mathcal{S}_n(\Sigma) \to \mathcal{S}_n(\Sigma_{a_1 \triangle a_2})$ *is an isomorphism of right* $\mathcal{O}_q(SL(n))$*-comodule algebras.*

When $\Sigma = \Sigma_1 \sqcup \Sigma_2$ and $a_i \subset \Sigma_i$ for $i = 1, 2$ then each $\mathcal{S}_n(\Sigma_i)$ is a right $\mathcal{O}_q(SL(n))$-comodule algebra via the coaction coming from the edge a_i and $\underline{\otimes}(\mathcal{S}_n(\Sigma))$ is the well-known *braided tensor product* $\mathcal{S}_n(\Sigma_1)$ and $\mathcal{S}_n(\Sigma_2)$ of the two $\mathcal{O}_q(SL(n))$-module algebras $\mathcal{S}_n(\Sigma_1)$ and $\mathcal{S}_n(\Sigma_2)$ of Majid, see [52, Lemma 9.2.12]. An analogous braided tensor product in the context of lattice gauge theory appears in [3] (quantizing [24]) and, in the context of factorization homology, in [7, Corollary 6.11], see Subsection 1.14.

1.10 | On injectivity of splitting homomorphism

A pb surface Σ is *essentially bordered* if every connected component of it has nonempty boundary.

Proposition (Proposition 8.6). *Suppose* Σ *is an essentially bordered pb surface. Then for any interior ideal arc* c *of* Σ*, the splitting homomorphism* $\Theta_c : \mathcal{S}_n(\Sigma) \to \mathcal{S}_n(\mathsf{cut}_c\Sigma)$ *is injective.*

Conjecture (Conjecture 8.7). *For any punctured bordered surface* Σ *and any interior ideal arc* c *the splitting homomorphism* Θ_c *is injective as well.*

The conjecture is true when $n = 2$ by [15] and for $n = 3$ by Higgins [28]. In both cases, the proofs rely on explicit bases of $\mathcal{S}_n(\Sigma)$. Proposition 8.6 shows the conjecture is true if Σ has nontrivial boundary. We will establish an alternative, weaker version of this conjecture for all pb surfaces in Subsection 1.12.

1.11 | Skein algebras of surfaces with boundary

Let $\Sigma_{g,p}$ denote the surface of genus g with $p-1$ punctures and $\partial\Sigma_{g,p} = S^1$. Let $\Sigma^*_{g,p}$ be $\Sigma_{g,p}$ with a boundary point removed. Hence, $\Sigma^*_{0,2}$ is a punctured monogon.

Utilizing the results of Subsection 1.9, we show in Proposition 8.5 that $\mathcal{S}_n(\Sigma^*_{0,2}) \simeq \mathcal{O}_q(SL(n))$ as an R-module with the $\mathcal{O}_q(SL(n))$-comodule structure, $\Delta_{\partial \mathcal{S}_n(\Sigma^*_{0,2})}$, coinciding with the adjoint $\mathcal{O}_q(SL(n))$-coaction on $\mathcal{O}_q(SL(n))$, [52, Example 1.6.14]. Furthermore, we prove that the product on $\mathcal{S}_n(\Sigma^*_{0,2})$ coincides with the *braided* (or, *covantarised*) *product* of Majid, [52, Example 9.4.10]. (That result was shown for $n = 2$ in [15].) Consequently, our theory provides simple geometric proofs of the associativity of the braided product on $\mathcal{O}_q(SL(n))$ and of $\mathcal{O}_q(SL(n))$ being an

FIGURE 4 From Σ to Σ_p. Here p is an interior ideal point. The picture when p is a boundary ideal point is similar.

$\mathcal{O}_q(SL(n))$-comodule algebra. (The proofs of these facts are quite technical and involved in [52].) Furthermore, our theory generalizes these statements to the boundary $\mathcal{O}_q(SL(n))$-coaction on the skein algebra of any essentially bordered punctured surface. We discuss a finite presentation of $\mathcal{S}_n(\Sigma^*_{0,2})$ in Subsection 8.2.

Let Σ be now any essentially bordered pb surface. A collection $A = \{a_1, \dots, a_r\}$ of disjoint closed oriented arcs properly embedded in Σ is *saturated* if

(i) each connected component of $\Sigma \setminus \bigcup_{i=1}^r a_i$ contains exactly one ideal point (interior or boundary) of Σ, and
(ii) A is maximal with respect to the above condition.

Let $U(a_1), \dots, U(a_n)$ be a collection of disjoint open tubular neighborhood of a saturated collection of arcs $a_1, \dots, a_n$, respectively. Each $U(a_i)$ is homeomorphic with $a_i \times (-1,1)$ (by an orientation preserving homeomorphism) and we require that $(\partial a_i) \times (-1,1) \subset \partial\Sigma$. Let $U(A) = \bigcup_{i=1}^k U(a_i)$.

Theorem (Theorem 8.8).

(1) *We have $r = r(\Sigma) := \#\partial\Sigma - \chi(\Sigma)$, where $\#\partial\Sigma$ is the number of boundary components of Σ and χ denotes the Euler characteristics.*
(2) *The embedding $U(A) \hookrightarrow \Sigma$ induces an R-module isomorphism $f_A : \mathcal{S}_n(U(A)) \to \mathcal{S}_n(\Sigma)$.*

Note that each $U(a_i) = a_i \times (-1,1)$ is naturally a directed bigon, with its sides $(\partial a_i) \times (-1,1)$ oriented in the direction of $(-1,1)$ and that we have an R-linear isomorphism

$$\mathcal{O}_q(SL(n))^{\otimes r} \xrightarrow{\Psi^{\otimes r}} \mathcal{S}_n(U(A)) \xrightarrow{f_A} \mathcal{S}_n(\Sigma).$$

As $\mathcal{O}_q(SL(n))$ has a Kashiwara–Lusztig's canonical basis over $\mathbb{Z}[v^{\pm 1}]$, see [32, 49], we have

Corollary. *$\mathcal{S}_n(\Sigma)$ is a free R-module with a basis given by the image of tensor product of Kashiwara–Lusztig's canonical bases on $\mathcal{O}_q(SL(n))^{\otimes r}$ under $f_A \circ \Psi^{\otimes r}$.*

We apply the above method to show that $\mathcal{S}_n(\Sigma^*_{1,1}) \simeq \mathcal{O}_q(SL(n))^{\otimes 2}$ (as an R-module) and to describe the product on it in Subsection 8.6. Furthermore, we explain a construction of finite presentations for $\mathcal{S}_n(\Sigma)$, for every essentially bordered Σ.

1.12 | Kernel of the splitting homomorphism

Suppose Σ is a connected pb surface with an ideal point p. Then a trivial ideal arc c_p at p cuts Σ into a monogon and a new pb surface Σ_p which has c_p as its boundary edge, see Figure 4.

Theorem 9.1 shows that the kernel of

$$\Theta_p : \mathcal{S}_n(\Sigma) \xrightarrow{\Theta_{c_p}} \mathcal{S}_n(\Sigma_p) \otimes_R \mathcal{S}_n(\mathfrak{M}) \xrightarrow{\cong} \mathcal{S}_n(\Sigma_p)$$

does not depend on the choice of p. Let us denote it by $\mathcal{K}(\Sigma)$. Then the quotient $\bar{\mathcal{S}}_n(\Sigma) := \mathcal{S}_n(\Sigma)/\mathcal{K}(\Sigma)$ is called the *projected stated skein algebra of* Σ. By Proposition 8.6, $\mathcal{K}$ is trivial and $\bar{\mathcal{S}}_n(\Sigma) = \mathcal{S}_n(\Sigma)$ if $\partial\Sigma \neq \emptyset$,

Corollary (Corollary 9.2). *The splitting homomorphism descends to an injective algebra homomorphism*

$$\bar{\Theta}_c : \bar{\mathcal{S}}_n(\Sigma) \to \bar{\mathcal{S}}_n(\mathrm{cut}_c\,\Sigma) = \mathcal{S}_n(\mathrm{cut}_c\,\Sigma).$$

The following is an alternative characterization of projected skein algebras:

Theorem. *For any Σ, p and c_p as above, $\bar{\mathcal{S}}_n(\Sigma)$ coincides with the subalgebra of $\mathcal{S}_n(\Sigma_p)$ coinvariant under the coaction $\Delta_{c_p} : \mathcal{S}_n(\Sigma_p) \to \mathcal{S}_n(\Sigma_p) \otimes \mathcal{S}_n(\mathfrak{B})$ at c_p:*

$$\bar{\mathcal{S}}_n(\Sigma) = \{x \in \mathcal{S}_n(\Sigma_p) : \Delta_{c_p}(x) = x \otimes 1\}.$$

Let $\Sigma = \overline{\Sigma} - \mathcal{P}$, where $\mathcal{P}$ is a finite subset of compact surface $\overline{\Sigma}$, as in Subsection 1.6. Generalizing the setup of Subsection 1.11, consider a collection A of disjoint, oriented, arcs in Σ, each with endpoints in $\partial\Sigma \cup \mathcal{P}$, satisfying the conditions (i) and (ii) above. We show in Subsection 9.1 that such A defines an identification of $\bar{\mathcal{S}}_n(\Sigma)$ with $\mathcal{O}_q(SL(n))^{\otimes r}$ and, hence, it determines a basis of $\bar{\mathcal{S}}_n(\Sigma)$.

1.13 | The image of the splitting homomorphism

Let c be an interior oriented ideal arc of a pb surface Σ. Denote the two copies of c in $\mathrm{cut}_c\,\Sigma$ by a_1 and a_2. We have the splitting R-algebra homomorphism

$$\Theta_c : \mathcal{S}_n(\Sigma) \to \mathcal{S}_n(\mathrm{cut}_c\,\Sigma)$$

and $\mathcal{S}_n(\mathrm{cut}_c\,\Sigma)$ is a $\mathcal{O}_q(SL(n))$-bi-comodule with the right and left coactions

$$\Delta_{a_1} : \mathcal{S}_n(\mathrm{cut}_c\,\Sigma) \to \mathcal{S}_n(\mathrm{cut}_c\,\Sigma) \otimes \mathcal{O}_q(SL(n))$$

$${}_{a_2}\Delta : \mathcal{S}_n(\mathrm{cut}_c\,\Sigma) \to \mathcal{O}_q(SL(n)) \otimes \mathcal{S}_n(\mathrm{cut}_c\,\Sigma),$$

respectively, where $\mathcal{O}_q(SL(n))$ is identified with the skein algebra of the bigon directed by the orientation of c. Recall that the Hochschild cohomology module is defined by

$$HH^0(\mathcal{S}_n(\mathrm{cut}_c\,\Sigma)) = \{x \in \mathcal{S}_n(\mathrm{cut}_c\,\Sigma) \mid \Delta_{a_1}(x) = \mathrm{fl} \circ {}_{a_2}\Delta(x)\},$$

where fl is the transposition

$$\mathrm{fl} : \mathcal{O}_q(SL(n)) \otimes \mathcal{S}_n(\Sigma) \to \mathcal{S}_n(\Sigma) \otimes \mathcal{O}_q(SL(n)), \quad \mathrm{fl}(x \otimes y) = y \otimes x.$$

Theorem 1.1 ([15, 38] for $n = 2$ and [28] for $n = 3$). *The image of* Θ_c *is equal to* $HH^0(\mathcal{S}_n(\mathrm{cut}_c \Sigma))$.

We prove it by considering the projected version $\Theta_c : \bar{\mathcal{S}}_n(\Sigma) \to HH^0(\mathcal{S}_n(\mathrm{cut}_c \Sigma))$, which has the same image as nonprojected one. Furthermore, we construct an explicit inverse map $\nabla : HH^0(\mathcal{S}_n(\mathrm{cut}_c \Sigma)) \to \bar{\mathcal{S}}_n(\Sigma)$.

1.14 | Relation to factorization homology, skein categories, and lattice field theory

Factorization homology is an invariant of oriented n-dimensional manifolds introduced by Beilinson and Drinfeld [6] in the setting of conformal field theory and then in [1, 2, 47] in the topological context. For $n = 2$, it associates with every surface Σ and every balanced braided category $\mathcal{A}$, a certain category denoted by $\int_\Sigma \mathcal{A}$. When the base ring is a *field*, using reconstruction theory Ben-Zvi, Brochier, and Jordan [7] showed that $\int_\Sigma \mathcal{A}$ is equivalent to the category of left modules over a certain algebra $\mathcal{A}_\Sigma$, which is isomorphic to the quantum moduli spaces of Alekseev–Grosse–Schomerus and Buffenoir–Roche [3, 4, 8–10], and also to the internal algebras of the skein categories of Walker and Johnson-Freyd, [14, 30, 66].

Building upon the above theory of stated skein algebras, we prove that for surfaces Σ with $\partial\Sigma = S^1$, the algebra A_Σ, for the representation category $\mathcal{A}$ of $U_q(sl(n))$, is isomorphic with $\mathcal{S}_n(\Sigma^*)$, where Σ^* is Σ with a boundary point removed. The $n = 2$ case follows also from the results of [21, 37, 43]; see further discussion in Subsection 10.2.

Although our construction of the stated skein modules was motivated by its rich theory developed in this paper, the above result provides a further justification for that construction.

On the one hand, factorization homology of [7] is more general in that it can defined for all semi-simple Lie algebras $\mathfrak{g}$ and it can be viewed as quantizing the entire moduli stacks of representations, rather than just the character varieties.

On the other hand, our approach has its own advantages. First our theory is defined over any ground ring, a commutative domain, while the factorization homology approach defines the algebra $\mathcal{A}_\Sigma$ over a field. For example, our theory works over the cyclotomic ring (and of course cyclotomic field), an important quantization case. Over the ring $\mathbb{Z}[q, q^{-1}]$ our theory highlights some integral results in quantum group theory, like relations to canonical basis. Furthermore, we define the stated skein module not only for surfaces but also for all 3-dimensional manifolds, and we worked out the theory for surfaces with multiple boundary components and multiple markings.

The stated skein algebra of a surface is defined explicitly, via generators (which are geometric objects and relations, making the theory elementary) while in factorization homology the algebra A_Σ is defined up to an isomorphism only.

The cutting homomorphism in our theory, though related to the excision in factorization homology, is different from the latter. Our cutting homomorphism and gluing over triangle operations make the study of the stated skein algebra easy by cutting surfaces into triangles.

An important application of our approach is it allows the first author and Yu [45] to prove the existence of the quantum trace map that quantizes the classical Fock and Goncharov trace map [22] and generalizes the quantum SL_2-trace map of Bonahon and Wong [12] to all SL_n. The concrete geometric nature of the generators of the stated skein algebra allows in many cases to present a set of elements that generates a quantum space inside the stated skein algebra. This eventually leads to various versions of quantum traces.

The above works relate our algebras to the theory of quantum cluster algebras, which provide alternative quantizations of character varieties. Further connections to quantum cluster algebras are through [13, 29, 59].

1.15 | Compatibility with stated Kauffman bracket skein modules

The stated Kauffman bracket skein algebras (of surfaces) of the first author [41] were generalized to stated skein modules of marked 3-manifolds in [11] (cf. also [43]). We are going to prove that these modules are isomorphic with our $SL(2)$-skein modules, $\mathcal{S}_2(M, \mathcal{N})$.

To relate these modules to ours, let us replace the variable q of [41] with $q^{1/2}$ and denote the resulted stated Kauffman bracket skein module by $\mathscr{S}(M, \mathcal{N})_{q^{1/2}}$. Recall that it is built of nonoriented 2-webs without sinks nor sources, stated by sings $\pm$.

Theorem (Theorem 11.1). *Suppose* $(M, \mathcal{N})$ *is a marked 3-manifold.*

(1) *There is a unique R-linear isomorphism* $\Lambda : \mathscr{S}(M, \mathcal{N})_{q^{1/2}} \to \mathcal{S}_2(M, \mathcal{N})$ *that maps framed links* α *to a stated 2-webs by assigning arbitrary orientation to them, and changing the minus state to 1 and the plus state to 2.*
(2) *The splitting homomorphism of [11, 41] coincides with ours through* Λ.

1.16 | Compatibility with the SU(n)-skein modules

As mentioned already, a partial motivation for our definition of $\mathcal{S}_n(M, \mathcal{N})$ were the $SU(n)$-skein modules of 3-manifolds introduced by the second author in [58]. These skein modules are built of *based n-webs* in M that are defined as our n-webs in $(M, \emptyset)$, except that the half-edges incident to any of their n-valent vertices are linearly ordered. In particular, the based n-webs have no end-points and $SU(n)$-skein modules have no boundary skein relations. In Subsection 11.2 we show that for any 3-manifold M and any n our $\mathcal{S}_n(M, \emptyset)$ is isomorphic with the SU(n)-skein module of M. That isomorphism is straightforward for n odd, but it requires a choice of a spin structure on M for n even.

1.17 | Compatibility with Higgins' SL3 skein algebras

In his recent work [28], Higgins introduced his version of stated SL_3-skein algebras, denoted by $\mathcal{S}_q^{SL_3}(\Sigma)$, of punctured bordered surfaces Σ. His skein algebra is the R-module freely generated by 3-webs stated by $-1, 0, 1$, subject to his system of skein relations. We show that in that our theory recovers Higgins' for $n = 3$ in Subsection 11.3.

1.18 | Relation to the Frohman–Sikora SU(3)-skein algebras

C. Frohman and the second author considered in [25] the "reduced $SU(3)$-skein algebra" of marked surfaces built of unstated 3-webs, subject to the $SU(3)$-skein relations of [40], extended by certain boundary skein relations, which depend on an invertible parameter $a \in R$. We denote that algebra by $\mathcal{S}_{FS}(\Sigma)$ for the value 1 of that parameter.

For an unstated 3-web α in Σ, let $\eta_+(\alpha)$ (respectively, $\eta_-(\alpha)$) denote α stated with threes (respectively, ones) at all its ends. In Subsection 11.4 we show that these operations extend to injective homomorphisms $\eta_+, \eta_- : \mathcal{S}_{FS}(\Sigma) \to \mathcal{S}_3(\Sigma)$. Furthermore, their images are direct summands of $\mathcal{S}_3(\Sigma)$.

2 | QUANTUM GROUPS ASSOCIATED TO sl_n

2.1 | Notations and conventions

We use the notations $\mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}$ for, respectively, the sets of complex numbers, reals, rationals numbers, integers, and nonnegative integers. We emphasize that our $\mathbb{N}$ contains 0. The number n (in sl_n) is a fixed integer $\geqslant 2$.

The ground ring R is a commutative ring with unit containing a distinguished invertible element v. The element $q = v^{2n}$ is the usual quantum parameter. The basic example is $R = \mathbb{Z}[v^{\pm 1}]$, the ring of Laurent polynomials in v with integer coefficients. All fractional powers of q in our papers are defined via the obvious integral powers of $v = q^{1/2n}$.

For a nonnegative integer m, we define the quantum integer $[m]$ and its factorials by

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}}, \quad [m]! = \prod_{i=1}^{m} [i], \quad [0]! = 1.$$

We will often use the following scalars:

$$t_0^{1/2} = q^{\frac{n^2-1}{2n}} := v^{n^2-1}, \quad t = (-1)^{n-1} t_0 = (-1)^{n-1} q^{\frac{n^2-1}{n}}, \; t^{n/2} = (-1)^{\frac{(n-1)n}{2}} q^{\frac{n^2-1}{2}} \tag{1}$$

$$a = (-v)^{n(n-1)/2} t^{-n/2} = q^{(1-n)(2n+1)/4} = q^{\frac{1-n^2}{4} - \frac{n^2-n}{4}}, \quad d_i = i - \frac{n+1}{2} \tag{2}$$

$$c_i = (-1)^{n-i} q^{\frac{n+1}{2} - i} q^{\frac{n^2-1}{2n}} = (-1)^{n-i} q^{-d_i} t_0^{1/2}. \tag{3}$$

Note that

$$\prod_{i=1}^{n} c_i = t^{n/2} = (-1)^{\binom{n}{2}} q^{\frac{n^2-1}{2}} \text{ and } c_i \cdot c_{\bar{i}} = t, \text{ for } i = 1, \dots, n, \tag{4}$$

where $\bar{i}$ is the *conjugate of* i, defined as $n + 1 - i$.

In fact, our entire theory works (up to a normalization) for any invertible $a, t, c_1, \dots, c_n$ satisfying Equation (4), see Subsection 4.12. However, our particular choice of these constants makes our theory invariant under the orientation reversal of 3-manifolds and the orientation reversal of webs in it, see Corollary 4.8 and Theorem 4.9.

Let S_n be the group of permutations of $\{1, \dots, n\}$. The *length* of $\sigma \in S_n$ is the minimal number of factors in the decomposition of σ into elementary transpositions $(i, i+1)$, $i = 1, \dots, n-1$. Alternatively, it is

$$\ell(\sigma) = |\{(i,j) | 1 \leqslant i < j \leqslant n,\ \sigma(i) > \sigma(j)\}|. \tag{5}$$

The longest element $w_0 \in S_n$ is the permutation $w_0(i) = \bar{i}$.

We use the convention that

$$(-q)^{\ell(\sigma)} = 0 \text{ if } \sigma : \{1, \dots, n\} \to \{1, \dots, n\} \text{ is not a permutation.} \tag{6}$$

We also use Kronecker's delta notation and its sibling:

$$\delta_{i,j} = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}, \qquad \delta_{j>i} = \begin{cases} 1 & \text{if } j > i \\ 0 & \text{if } j \leqslant i. \end{cases}$$

2.2 | Quantized enveloping algebra $U_q(sl_n)$

The quantized enveloping algebra $U_q(sl_n)$ is a Hopf algebra over the field $\mathbb{Q}(v)$ with explicit presentation given in [35, section 6.1.2].

Let $V = \mathbb{Q}(v)^n$ be the defining representation of $U_q(sl_n)$. Its dual V^* is the simple $U_q(sl_n)$-module with highest weight the $(n-1)$-st fundamental weight. Let $e_1, \dots, e_n$ be the standard basis of V with e_n being the highest weight vector, see [35, section 8.4.1]. Let $e^1, \dots, e^n$ be the dual basis of V^*, defined by $e^i(e_j) = \delta_{i,j}$. As $U_q(sl_n)$ is a Hopf algebra, the category of finite-dimensional $U_q(sl_n)$-modules is monoidal. Let C_n be the full subcategory of $U_q(sl_n)$-modules consisting of objects isomorphic to tensor products of copies of V and V^*.

It is known that $\mathrm{Hom}_{U_q(sl_n)}(V^{\otimes n}, \mathbb{Q}(v))$ has dimension 1 and is generated by the q-antisymmetrizer $\mathcal{A}_- : V^{\otimes n} \to \mathbb{Q}(v)$ defined by

$$\mathcal{A}_-(e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(n)}) = t^{n/2} a (-q)^{\ell(\sigma)}, \text{ for any } \sigma : \{1, \dots, n\} \to \{1, \dots, n\}, \tag{7}$$

where $(-q)^{\ell(\sigma)} = 0$ if σ is not a permutation (according to Equation 6), and a, t are given in Subsection 2.1, see [26]. Similarly, $\mathrm{Hom}_{U_q(sl_n)}(\mathbb{Q}(v), V^{\otimes n})$ has dimension 1 and is generated by $\mathcal{A}_+ : \mathbb{Q}(v) \to V^{\otimes n}$, given by

$$\mathcal{A}_+(1) = a \sum_{\sigma \in S_n} (-q)^{l(\sigma)} e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes \dots \otimes e_{\sigma(n)}, \tag{8}$$

2.3 | Braiding and the Iwahori–Hecke algebra

The algebra $U_q(sl_n)$ has a topological completion which is a topological ribbon Hopf algebra, making the category of finite-dimensional $U_q(sl_n)$-modules a ribbon category, see [16, 64]. The ribbon structure defines (through the universal R-matrix) for any two $U_q(sl_n)$-modules V_1, V_2 a braiding $\widehat{\mathcal{R}}_{V_1,V_2} : V_1 \otimes V_2 \to V_2 \otimes V_1$, which is an invertible $U_q(sl_n)$-morphism satisfying certain conditions discussed below. Let us record here the formula for $\widehat{\mathcal{R}}_{V,V}$, which will be simply denoted by $\widehat{\mathcal{R}}$. An operator $A : V \otimes V \to V \otimes V$ is given by its matrix entries $A^{ij}_{mk} \in \mathbb{Q}(v)$, with $i, j, m, k \in \{1, \dots, n\}$, which are defined such that

$$A(e_m \otimes e_k) = \sum_{i,j} A^{ij}_{mk} e_i \otimes e_j.$$

The braiding $\widehat{\mathcal{R}} : V \otimes V \to V \otimes V$ and the matrix $\mathcal{R}$ are given by

$$\widehat{\mathcal{R}}^{ji}_{lk} = \mathcal{R}^{ij}_{lk} = q^{-\frac{1}{n}}\left(q^{\delta_{i,j}}\delta_{j,k}\delta_{i,l} + (q - q^{-1})\delta_{j<i}\delta_{j,l}\delta_{i,k}\right), \tag{9}$$

see [35, section 8.4.2(60) and section 9.2], where $\delta_{j<i} = 1$ if $j < i$ and $\delta_{j<i} = 0$ otherwise, as in Subsection 2.1.

For an integer $k \geqslant 2$ and $i = 1, \dots, k-1$, we define $\widehat{R}_i : V^{\otimes k} \to V^{\otimes k}$ by

$$\widehat{R}_i = \mathrm{id}^{\otimes i-1} \otimes \widehat{\mathcal{R}} \otimes \mathrm{id}^{\otimes^{k-i-1}}.$$

Then the operators $\widehat{R}_i$ satisfy the following relations

$$q^{\frac{1}{n}}\widehat{R}_i - q^{-\frac{1}{n}}\widehat{R}_i = (q - q^{-1})\mathrm{id}, \quad \text{for } i = 1, \dots, k-1$$
$$\widehat{R}_i\widehat{R}_j = \widehat{R}_j\widehat{R}_i \quad \text{for } 1 \leqslant i < j-1 \leqslant k-1$$
$$\widehat{R}_i\widehat{R}_{i+1}\widehat{R}_i = \widehat{R}_{i+1}\widehat{R}_i\widehat{R}_{i+1} \quad \text{for } i = 1, \dots, k-2.$$

The last equation, known as the braid relation, is a consequence of $\widehat{\mathcal{R}}$ being induced by the R-matrix of $U_q(sl_n)$.

By [26, 58],

$$\widehat{R}_i \circ \mathcal{A}_+ = -q^{-\frac{1}{n}-1}\mathcal{A}_+, \quad \mathcal{A}_- \circ \widehat{R}_i = -q^{-\frac{1}{n}-1}\mathcal{A}_-. \tag{10}$$

2.4 | The dual module

For $x \in V^*$ and $y \in V$ let $\langle x, y\rangle$ denote $x(y) \in \mathbb{Q}(v)$. There is an invertible element $g_0 \in U_q(sl_n)$ called the charmed element, whose action on V is given by

$$g_0(e_i) = q^{2i-n-1}e_i = q^{2d_i}e_i.$$

The ribbon structure implies that the following $\mathbb{Q}(v)$-linear maps are $U_q(sl_n)$-morphisms:

$$\mathrm{ev} : V^* \otimes V \to \mathbb{Q}(v), \qquad \mathrm{ev}(e^i \otimes e_j) = \langle e^i, e_j\rangle = \delta_{i,j}$$

$$\mathrm{coev} : \mathbb{Q}(v) \to V \otimes V^*, \qquad \mathrm{coev}(1) = \sum_{i=1}^n e_i \otimes e^i$$

$$\widetilde{\mathrm{ev}}_0 : V \otimes V^* \to \mathbb{Q}(v), \qquad \widetilde{\mathrm{ev}}_0(e_i \otimes e^j) = q^{2i-n-1}\delta_{i,j} = \langle e^j, g_0(e_i)\rangle$$

$$\widetilde{\mathrm{coev}}_0 : \mathbb{Q}(v) \to V^* \otimes V, \qquad \widetilde{\mathrm{coev}}_0(1) = \sum_{i=1}^n q^{n+1-2i}e^i \otimes e_i = \sum_{i=1}^n e^i \otimes (g_0)^{-1}e_i.$$

2.5 The quantized coordinate algebra $\mathcal{O}_q(SL(n))$

The *algebra of quantum matrices* $\mathcal{O}_q(M(n))$ is the associative $\mathbb{Z}[v^{\pm 1}]$-algebra generated by elements u^i_j, for $i, j = 1, 2, \dots, n$, subject to relations

$$(\mathbf{u} \otimes \mathbf{u})\mathcal{R} = \mathcal{R}(\mathbf{u} \otimes \mathbf{u}), \tag{11}$$

where $\mathcal{R}$ is the R-matrix given by Equation (9), and $\mathbf{u} \otimes \mathbf{u}$ is the $n^2 \times n^2$ matrix with entries $(\mathbf{u} \otimes \mathbf{u})^{ik}_{jl} = u^i_j u^k_l$ for $i, j, k, l \in \{1, \dots, n\}$. We call $\mathbf{u} := (u^i_j)$ as well as its images under algebra homomorphisms *quantum matrices*. Any square submatrix of $\mathbf{u}$ is a quantum matrix (of smaller size). The element

$$\det_q(\mathbf{u}) := \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} u^{\sigma(1)}_1 \cdots u^{\sigma(n)}_n = \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} u^1_{\sigma(1)} \cdots u^n_{\sigma(n)}$$

is central and called the *quantum determinant* of the quantum matrix $\mathbf{u}$, see [35, 9.2.2].

The *quantized coordinate algebra of* $SL(n)$ is the quotient

$$\mathcal{O}_q(SL(n)) = \mathcal{O}_q(M(n))/(\det_q \mathbf{u} - 1).$$

It is a Hopf algebra with comultiplication, counit, and antipode given by

$$\Delta(u^i_j) = \sum_k u^i_k \otimes u^k_j, \quad \epsilon(u^i_j) = \delta_{i,j}, \quad S(u^i_j) = (-q)^{i-j} M^j_i(\mathbf{u}), \tag{12}$$

where $M^j_i(\mathbf{u})$ is the quantum determinant of the minor of $\mathbf{u}$ obtained by removing the jth row and ith column of the matrix $\mathbf{u}$, see, for example, [35, 9.2.2] or [63].

For technical convenience, we have defined $\mathcal{O}_q(SL(n))$ over the ring $\mathbb{Z}[v^{\pm 1}]$. The dual $U_q(sl_n)^*$, consisting of all $\mathbb{Q}(v)$-linear maps $U_q(sl_n) \to \mathbb{Q}(v)$, has a $\mathbb{Q}(v)$-algebra structure, dual to the coalgebra structure of $U_q(sl_n)$, and is considered as a $\mathbb{Z}[v^{\pm 1}]$-algebra via $\mathbb{Z}[v^{\pm 1}] \hookrightarrow \mathbb{Q}(v)$.

Proposition 2.1 (Hopf duality between $\mathcal{O}_q(SL(n))$ and $U_q(sl_n)$, [62, section 4]). *There is a unique pairing* $\langle \cdot, \cdot \rangle : \mathcal{O}_q(SL(n)) \times U_q(sl_n) \to \mathbb{Q}(v)$, *such that* $\langle u^i_j, x \rangle = e^i(x(e_j))$ *for* $x \in U_q(sl_n)$ *and* $i, j = 1, \dots, n$. *It is a Hopf pairing that induces an embedding of* $\mathbb{Z}[v^{\pm 1}]$*-algebras* $\mathcal{O}_q(SL(n)) \hookrightarrow U_q(sl_n)^*$.

For the convenience of the reader, we recall that $\langle \cdot, \cdot \rangle$ being a Hopf pairing means that $\mathcal{O}_q(SL(n))$ and $U_q(sl_n)$ are in Hopf duality: for all $u, u' \in \mathcal{O}_q(SL(n))$ and $x, x' \in U_q(sl_n)$

$$\langle uu', x \rangle = \sum \langle u, x_{(1)} \rangle \langle u', x_{(2)} \rangle, \quad \text{where } \Delta(x) = \sum x_{(1)} \otimes x_{(2)}$$
$$\langle u, xx' \rangle = \sum \langle u_{(1)}, x \rangle \langle u_{(2)}, x' \rangle, \quad \text{where } \Delta(u) = \sum u_{(1)} \otimes u_{(2)}$$
$$\langle 1, x \rangle = \varepsilon(x), \quad \langle u, 1 \rangle = \varepsilon(u), \quad \langle S(u), x \rangle = \langle u, S(x) \rangle.$$

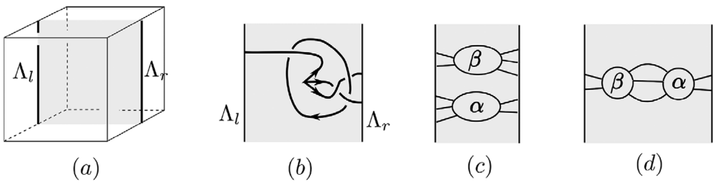


FIGURE 5 (a) The cube Q, the square S (shadowed) and its sides Λ_r, Λ_l. (b) An example of a diagram of a based 3-tangle. The order at the source is counterclockwise, beginning with lowest branch. (c) The tensor product $\alpha \otimes \beta$. (d) The composition $\beta \circ \alpha$. (Orientations of α, β are not shown.).

3 | REVIEW OF THE RESHETIKHIN–TURAEV THEORY

3.1 | Based *n*-tangles and their operator invariants

Reshetikhin–Turaev theory associates with every ribbon category an operator invariant of ribbon graphs, see [65]. Let us make this construction explicit for the category of left $U_q(sl_n)$-modules and a special class of ribbon graphs, called based n-tangles, defined below.

The cube $Q := [-1,1] \times (-1,1)^2$ in the 3-space, see Figure 5a, has boundary consisting of the right face $\{1\} \times (-1,1)^2$ and the left face $\{-1\} \times (-1,1)^2$. The intersection of Q and the XY-plane is $S = [-1,1] \times (-1,1) \times \{0\}$. It is depicted as the shaded square in Figure 5a, with its sides $\Lambda_l = \{-1\} \times \{0\} \times (-1,1)$ and $\Lambda_r = \{1\} \times \{0\} \times (-1,1)$. We will say that vectors of the form $(0,0,z)$ with $z > 0$ are in the *Z-direction*.

For the sake of the definition below and for later use, we say that directed graph α is *properly embedded* into a 3-manifold M, if its set of 1-valent vertices, $\partial\alpha$, coincides with $\alpha \cap \partial M$ and α is transversal to ∂M.

Definition 3.1. A *based n-tangle* α is a disjoint union of finitely many oriented circles and directed graphs properly embedded into the cube Q, such that

(1) The graphs of α have finitely many vertices only. Every vertex of α is either a sink or a source and either 1-valent or n-valent. We denote the set of 1-valent vertices, called *endpoints* of α, by $\partial\alpha$.
(2) Each edge of the graph is a smooth embedding of the closed interval [0,1] into Q.
(3) α is equipped with a *framing* that is a continuous nonvanishing vector field transversal to α. In particular, the framing at a vertex is transversal to all incident edges.
(4) The set of half-edges at every n-valent vertex is linearly ordered.
(5) The endpoints of α lie in $\Lambda_l \cup \Lambda_r$, and the framing at these endpoints has a Z-direction.

We consider based n-tangles up to *isotopies* that are continuous deformations of n-webs in their class.

Remark 3.2. Our notion of an n-tangle is broader than that of a traditional tangle, because it allows for n-valent vertices. It is a version of the notion of an n-web of [58]. However, we use a different name for it here to distinguish it from n-webs that we will introduce in Subsection 4.1 and that are unbased and have a different framing setup near their boundaries.

Suppose α is a based n-tangle. The *sign* $\mathrm{sgn}(e)$ of an endpoint $e \in \partial\alpha$ is positive if the direction of α goes from left to right at e, and negative otherwise. Let $\mathrm{sgn}_l(\alpha)$ (respectively, $\mathrm{sgn}_l(\alpha)$) be the sequence of signs of endpoints of α on Λ_l (respectively, Λ_r) appearing from the bottom to the top.

Let ${\mathfrak{C}_n}^b$ be the $\mathbb{Z}[v^{\pm 1}]$-linear monoidal category whose objects $\boldsymbol{\eta}$ are finite sequences of signs $\pm$, and the set of morphisms $\mathrm{Hom}_{{\mathfrak{C}_n}^b}(\boldsymbol{\eta}, \boldsymbol{\mu})$ is the $\mathbb{Z}[v^{\pm 1}]$-module freely spanned by isotopy classes of based n-tangles α such that $\mathrm{sgn}_r(\alpha) = \boldsymbol{\eta}$ and $\mathrm{sgn}_l(\alpha) = \boldsymbol{\mu}$. Here the tensor product of two sequences $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$ is the concatenation of $\boldsymbol{\eta}$ followed by $\boldsymbol{\mu}$. The empty sequence is the unit. The tensor product $\alpha \otimes \beta$ of two based n-tangles is the result of stacking β above α, as in Figure 5c. If $\mathrm{sgn}_l(\alpha) = \mathrm{sgn}_r(\beta)$ then the composition $\beta \circ \alpha$ is obtained by placing β to the left of α (after an isotopy to match endpoints), as in Figure 5d.

Let C_n be the category of left $U_q(sl_n)$-modules isomorphic to tensor products of finite numbers of modules V and V^*. The morphisms of C_n are $U_q(sl_n)$-module homomorphisms. It is a ribbon category. As based tangles can be viewed as tangles with coupons, in the sense of Reshetikhin–Turaev, their theory [57] defines a monoidal functor $\mathsf{RT}_0 : {\mathfrak{C}_n}^b \to C_n$ constructed as follows. For a sequence $\boldsymbol{\eta} = (\eta_1, \dots, \eta_k)$ of signs $\pm$ let

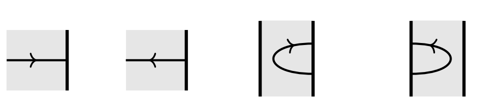
$$\mathsf{RT}_0(\boldsymbol{\eta}) = V^{\boldsymbol{\eta}} := V^{\eta_1} \otimes \cdots \otimes V^{\eta_k},$$

where $V^+ = V$ and $V^- = V^*$.

We will define values of RT_0 for based n-tangles through their diagrams. For that purpose, we will identify the XY-plane in $\mathbb{R}^3$ with the pages of this paper and we will point the z-axis toward the reader, as in Figure 5a. Any based n-tangle α can be represented by its diagram obtained by isotoping α first so that its framing is in the Z-direction everywhere and then by putting it in a general position with respect to the projection $p : \mathbb{R}^3 \to \mathbb{R}^2$ onto the XY-plane, see Figure 5b. We further assume that the projection of α near an n-valent vertex consists of n lines directed from left to right, and that their linear order is counterclockwise beginning from the lowest line to the highest for the source, and from the highest to the lowest line, for the sink.

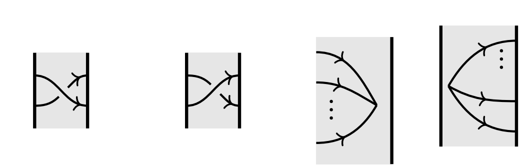
Such a projection of α onto the square $S = [-1,1] \times (-1,1) \times \{0\}$ (shaded in Figure 5a), together with the over/under information at every crossing is called a *diagram of* α.

In Equations (13)–(15), we list *elementary* based n-tangles α and the corresponding operators $\mathsf{RT}_0(\alpha)$. The associated operators $\mathrm{ev}, \widetilde{\mathrm{ev}}_0, \mathrm{coev}, \widetilde{\mathrm{coev}}_0$ were defined in Subsection 2.4, while $\mathcal{A}_-, \mathcal{A}_+$ and $\widehat{\mathcal{R}}$ were given by Equations (7)–(9), respectively. As every based n-tangle can be built of them through tensor products and compositions, these operators totally determine RT_0.



(13)

(14)



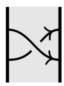
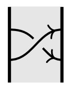
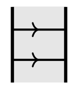
(15)

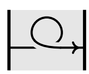
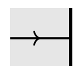
Based n-tangles and the Reshetikhin–Turaev functor on them were considered in [58], albeit with a different normalization of $\mathcal{A}_{\pm}$.

3.2 | Kernel of functor RT_0

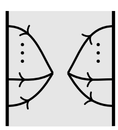
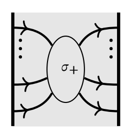
Recall that a monoidal ideal in a monoidal category C is a subset $I \subset \mathrm{Hom}(C)$ such that for $x \in I$ and $y \in \mathrm{Hom}(C)$ we have $x \otimes y, y \otimes x \in I$ and $x \circ y, y \circ x \in I$, whenever such compositions can be defined.

The following are well-known elements of $\ker \mathsf{RT}_0$ (see [58]):

$q^{\frac{1}{n}}$  $- q^{-\frac{1}{n}}$  $\overset{\mathsf{RT}_0}{=} (q - q^{-1})$  (16)

 $\overset{\mathsf{RT}_0}{=} t_0$  (17)

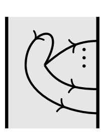
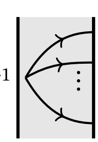
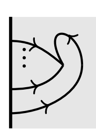
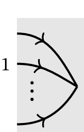
 $\overset{\mathsf{RT}_0}{=} [n]$  (18)

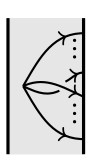
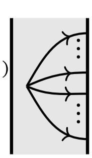
 $\overset{\mathsf{RT}_0}{=} (-q)^{\binom{n}{2}} \sum_{\sigma \in S_n} \left(-q^{\frac{1}{n}-1}\right)^{\ell(\sigma)}$  , (19)

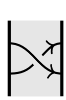
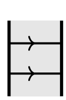
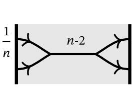
where t_0 is given by Equation (1), σ_+ is the minimum crossing positive braid representing a permutation σ and $\ell(\sigma)$ is the length of σ defined in Subsection 2.1. Here $x \overset{\mathsf{RT}_0}{=} y$ means $x - y \in \ker(\mathsf{RT}_0)$.

Conjecture 3.3. *The kernel* $\ker \mathsf{RT}_0$ *is the monoidal ideal* I_0 *generated by elements given in Equation (16)–(19).*

For example, the arguments in [58] indeed imply that the following identities are consequences of (16)–(19):

 $\overset{\mathsf{RT}_0}{=} (-1)^{n-1}$  ,  $\overset{\mathsf{RT}_0}{=} (-1)^{n-1}$  , (20)

 $\overset{\mathsf{RT}_0}{=} -q^{-(1+\frac{1}{n})}$  (21)

$[n-2]!$  $\overset{\mathsf{RT}_0}{=} [n-2]!\, q^{\frac{n-1}{n}}$  $- (-1)^{\binom{n}{2}} q^{-\frac{1}{n}}$  , (22)

where the tangle on the right has $n-2$ parallel edges in the middle.

Conjecture 3.3 is analogous to Morrison's conjecture [53, section 5.5] that was proved in [17]. Note that our formalism of n-webs leads to a simpler set of kernel generators than that in [53]. After this manuscript was posted on arxiv, Poudel announced a proof of the above conjecture [54].

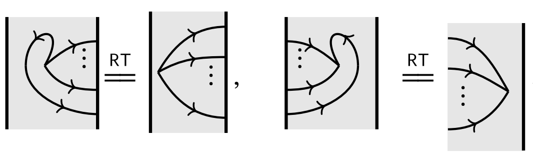
3.3 | (Unbased) n-tangles

Now we will consider a modified ribbon structure on the category of $U_q(sl_n)$-modules, giving rise to a new Reshetikhin–Turaev functor, denoted by RT, with simpler skein properties. We will explain in Subsection 3.9 how this new ribbon structure comes from the theory of quantized enveloping algebras. For now we simply declare RT to coincide on all elementary based n-tangles with RT_0, except that we multiply the values of $\widetilde{\mathrm{ev}}_0$ and $\widetilde{\mathrm{coev}}_0$ of (14) by $(-1)^{n-1}$. Thus, if D is a diagram of a based n-tangle α and $\# \downarrow (D)$ is the number of its downward critical points, that is, points where the tangent is parallel to the vertical y-axis and pointing downward, then

$$\mathsf{RT}(D) = (-1)^{(n-1)\#\downarrow(D)}\mathsf{RT}_0(\alpha). \tag{23}$$

It is an easy exercise to show that $\mathsf{RT}(D)$ is an isotopy invariant of based n-tangles.

From Equation (20) it follows that RT is invariant under cyclic changes of an order at a vertex:



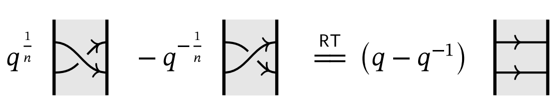
(24)

where $x \overset{\mathsf{RT}}{=} y$ means $\mathsf{RT}(x) = \mathsf{RT}(y)$.

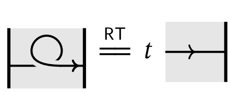
An (unbased) *n-tangle* is defined exactly as a based n-tangle except that half-edges incident to its every n-valent vertex are required to be cyclically ordered only. (Such cyclic orderings of edges around each vertex are called a ribbon structure.) Equation (24) shows that RT is an invariant of n-tangles. In a diagram of an n-tangle, the cyclic order at every n-valent vertex is the counterclockwise order.

Let $\mathfrak{C}_n$ be a monoidal category obtained from $\mathfrak{C}_n^b$ by replacing based n-tangles with n-tangles. Then RT is a $\mathbb{Z}[v^{\pm 1}]$-linear monoidal function from $\mathfrak{C}_n$ to C_n.

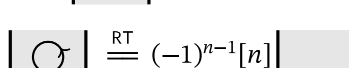
By Equations (16)–(19), we have



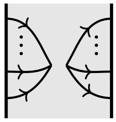
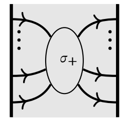
(25)



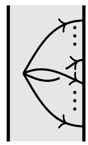
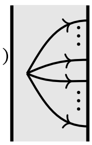
(26)

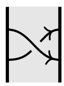
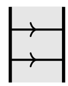
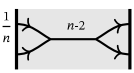


(27)

 $\overset{\mathsf{RT}}{=} (-q)^{\binom{n}{2}} \sum_{\sigma \in S_n} \left(-q^{\frac{1}{n}-1}\right)^{\ell(\sigma)}$ , (28)

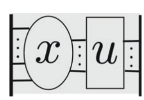
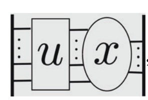
The following two are consequences of Equations (25)–(28):

 $\overset{\mathsf{RT}}{=} -q^{-(1+\frac{1}{n})}$ , (29)

$[n-2]!$  $\overset{\mathsf{RT}}{=} [n-2]!\,q^{\frac{n-1}{n}}$  $- (-1)^{\binom{n}{2}} q^{-\frac{1}{n}}$ , (30)

3.4 | Linear functionals on $U_q(sl_n)$ from n-tangles

For $x \in \mathrm{Hom}_{\mathfrak{C}_n}(\boldsymbol{\eta}, \boldsymbol{\mu})$ and an element $u \in U_q(sl_n)$ define $\mathsf{RT}(x)\circ u : V^{\boldsymbol{\eta}} \to V^{\boldsymbol{\mu}}$ as the composition of the action of u on $V^{\boldsymbol{\eta}}$ with $\mathsf{RT}(x)$. Similarly, one defines $u\circ\mathsf{RT}(\alpha) : V^{\boldsymbol{\eta}} \to V^{\boldsymbol{\mu}}$ where u acts on $V^{\boldsymbol{\mu}}$ now. We will use ovals for depicting morphisms of $\mathfrak{C}_n$ and rectangles for elements of $U_q(sl_n)$ throughout the paper. As $\mathsf{RT}(x)$ is a $U_q(sl_n)$-morphism, we have

$\mathsf{RT}(x)\circ u = u\circ\mathsf{RT}(x)$, or, pictorially,  $\overset{\mathsf{RT}}{=}$ , (31)

A vector space equipped with a basis is called *based*. The tensor product of based vector spaces V_i with bases B_i for $i = 1, \dots, k$, has the natural tensor product basis $\prod_{i=1}^k B_i$. A linear operator $A : V_1 \to V_2$ between based vector spaces with bases $\{e_i^{(1)}\}$ and $\{e_j^{(2)}\}$ defines a matrix with elements A_i^j, also denoted by $\langle j|A|i\rangle$, such that

$$A\left(e_i^{(1)}\right) = \sum_j A_i^j e_j^{(2)}, \quad \langle j|A|i\rangle := A_i^j.$$

We consider the $U_q(sl_n)$-module V as a based $\mathbb{Q}(v)$-vector space with basis $\{e_1, \dots, e_n\}$, defined in Subsection 2.2. The dual V^* will be considered as a based vector space with the basis $\{f^1, \dots, f^n\}$ where

$$f^i = c_{\bar{i}} e^{\bar{i}}, \text{ for } i = 1, \dots, n, \text{ where } c_i \text{ are given by (3).} \tag{32}$$

This basis allows for a simplification of our theory and, in particular, makes the RT functor independent of web orientations, see Subsection 3.5. Now for any sign sequence $\boldsymbol{\eta}$, the vector space $V^{\boldsymbol{\eta}}$ is based with the *tensor basis*, indexed by $\{1, \dots, n\}^{|\boldsymbol{\eta}|}$, induced by the above bases of V and V^*.

A *right state* (respectively, a *left state*) of a morphism $x \in \mathrm{Hom}_{\mathfrak{C}_n}(\boldsymbol{\eta}, \boldsymbol{\mu})$, for some $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$, is an assignment of $i_1, \dots, i_{|\boldsymbol{\eta}|} \in \{1, \dots, n\}$ (respectively, $j_1, \dots, j_{|\boldsymbol{\mu}|} \in \{1, \dots, n\}$) to the right (respectively, left) endpoints of x, listing them from the bottom to the top. A right (respectively, a left) stated morphism x as above will be denoted by $\mathbf{x} = (x, \mathbf{i})$ (respectively, $\mathbf{x} = (\boldsymbol{j}, x)$), where $\mathbf{i} = \{i_1, \dots, i_{|\eta|}\} \in \{1, \dots, n\}^{|\eta|}$ and $\boldsymbol{j} = \{j_1, \dots, j_{|\mu|}\} \in \{1, \dots, n\}^{|\mu|}$. Similarly, a *stated morphism* is a triple $\mathbf{x} = (\boldsymbol{j}, x, \mathbf{i})$, as above. For such $\mathbf{x}$ define a $\mathbb{Q}(v)$-linear function $\Gamma(\mathbf{x}) : U_q(sl_n) \to \mathbb{Q}(v)$ whose value at $u \in U_q(sl_n)$ is

$$\langle \Gamma(\mathbf{x}), u \rangle := \langle \boldsymbol{j} | \mathsf{RT}(x) \circ u | \mathbf{i} \rangle = \langle \boldsymbol{j} | u \circ \mathsf{RT}(x) | \mathbf{i} \rangle.$$

For example, for $x = \emptyset$

$$\Gamma(\emptyset) = \epsilon : U_q(sl_n) \to \mathbb{Q}(v), \quad \text{is the counit.} \tag{33}$$

Let $\mathcal{T}$ be the set of isotopy classes of stated n-tangles. The module $\mathbb{Z}[v^{\pm 1}]\mathcal{T}$ over $\mathbb{Z}[v^{\pm 1}]$ freely spanned by $\mathcal{T}$ is an algebra with the product $\boldsymbol{\alpha}\boldsymbol{\beta} = \boldsymbol{\alpha} \otimes \boldsymbol{\beta}$ obtained by placing $\boldsymbol{\alpha}$ below $\boldsymbol{\beta}$ and concatenating the states. We extend Γ linearly onto a $\mathbb{Z}[v^{\pm 1}]$-linear map

$$\Gamma : \mathbb{Z}[v^{\pm 1}]\mathcal{T} \to U_q(sl_n)^*,$$

Proposition 3.4. *The map Γ is a $\mathbb{Z}[v^{\pm 1}]$-algebra homomorphism.*

Proof. As the $U_q(sl_n)$-action on tensor product of $U_q(sl_n)$-modules is given by the coproduct on $U_q(sl_n)$, for any n-tangles α and β and any $u \in U_q(sl_n)$ we have

$$(\alpha \otimes \beta) \circ u = \sum (\alpha \circ u') \otimes (\beta \circ u''), \quad \text{where } \Delta(u) = \sum u' \otimes u''.$$

By assigning states, we get

$$\langle \boldsymbol{\alpha} \otimes \boldsymbol{\beta}, u \rangle = \sum \langle \boldsymbol{\alpha}, u' \rangle \langle \boldsymbol{\beta}, u'' \rangle,$$

which means that Γ maps the product $\boldsymbol{\alpha} \otimes \boldsymbol{\beta}$ to the dual of the coproduct in $U_q(sl_n)$, which is the product on $U_q(sl_n)^*$. Hence, Γ is an algebra homomorphism. □

3.5 | Dual operator, orientation reversal invariance

For a stated morphism $\mathbf{x} = (\mathbf{i}, x, \boldsymbol{j})$ let $\langle \mathbf{x} \rangle$ be the $(\mathbf{i}, \boldsymbol{j})$-element of the matrix of $\mathsf{RT}(x)$,

$$\langle \mathbf{x} \rangle = \langle \mathbf{i} \mid \mathsf{RT}(x) \mid \boldsymbol{j} \rangle \in \mathbb{Q}(v).$$

For a stated n-tangle, $\boldsymbol{\alpha} = (\mathbf{i}, \alpha, \boldsymbol{j})$, let $\overleftarrow{\boldsymbol{\alpha}} = (\mathbf{i}, \overleftarrow{\alpha}, \boldsymbol{j})$, where $\overleftarrow{\alpha}$ is obtained from α by reversing the orientation of all its edges and and of all circle components.

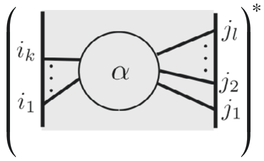
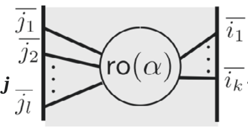
Furthermore, let

$$\boldsymbol{\alpha}^* = (c_{\mathbf{i}})^{-1} c_{\boldsymbol{j}} (\boldsymbol{j}^*, \mathsf{ro}(\alpha), \mathbf{i}^*),$$

where $\mathsf{ro}(\alpha)$ is obtained by $180°$ rotation of α,

$$c_{\mathbf{i}} = \prod_{m=1}^{k} c_{i_m} \ \text{ and } \ (i_1, \dots, i_k)^* = (\bar{i}_k, \dots, \bar{i}_1), \ \text{ where } \bar{i} = n + 1 - i.$$

Pictorially,

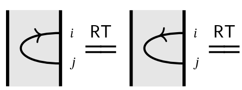
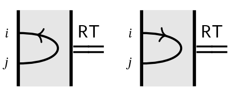
$\boldsymbol{\alpha}^* =$  $= (c_{\mathbf{i}})^{-1} c_{\boldsymbol{j}}$ 

The above operations extend linearly onto $\mathbb{Z}[v^{\pm 1}]\mathcal{T}$.

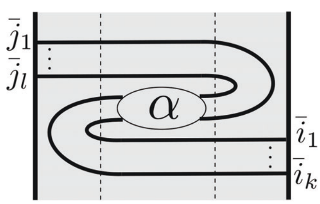
It should be noted that for an n-tangle α the operators $\mathsf{RT}(\alpha)$, $\mathsf{RT}(\overleftarrow{\alpha})$, and $\mathsf{RT}(\mathsf{ro}(\alpha))$ have different domains and different target spaces. The following statement shows an important benefit the basis $\{f^1, \ldots, f^n\}$ of V^*:

Proposition 3.5. *For any stated morphism* $\mathbf{x} = (\boldsymbol{i}, x, \boldsymbol{j})$ *one has* $\langle \mathbf{x} \rangle = \langle \overleftarrow{\mathbf{x}} \rangle = \langle \mathbf{x}^* \rangle$.

Proof. Checking $\langle \overleftarrow{\alpha} \rangle = \langle \alpha \rangle$ for cups and caps is quite easy:

 $\delta_{i,\bar{j}} c_i$,  $\delta_{i,\bar{j}} (c_{\bar{i}})^{-1}$. (34)

(1) The identity $\langle \alpha^* \rangle = \langle \alpha \rangle$ follows from

$\langle \boldsymbol{j}^* | \mathsf{RT}(\mathsf{ro}(\alpha)) | \mathbf{i}^* \rangle \overset{\mathsf{RT}}{=}$  $= \left(\prod_{m=1}^{k} c_{i_m} \right) \left(\prod_{m=1}^{l} c_{j_m} \right)^{-1} \langle \mathbf{i} | \mathsf{RT}(\alpha) | \boldsymbol{j} \rangle,$

where the first identity is by an isotopy. For the second identity, we decompose the tangle above along the dashed lines and use the values of cups and caps in Equation (34).

(2) To prove $\langle \overleftarrow{\alpha} \rangle = \langle \alpha \rangle$ one needs to check it for the elementary n-tangles. For cups and caps, it have been done in Equation (34). The statement for the positive crossing (Equation (15)) follows from part (1) and the fact that the R-matrix formula (9) is preserved under the involution $i \leftrightarrow \bar{l}$, $j \leftrightarrow \bar{k}$. The statement for the sink and the source follows by a straightforward computation. □

3.6 | Annihilators

In this and the next two subsections, we will analyze the kernel of $\Gamma : \mathbb{Z}[v^{\pm 1}]\mathcal{T} \to U_q(sl_n)^*$.

An *internal annihilator* is a $\mathfrak{C}_n$-morphism x such that $\mathsf{RT}(x) = 0$. From the definition, we have:

Proposition 3.6.

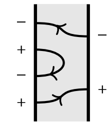
(a) *Internal annihilators form a monoidal ideal in* $\mathfrak{C}_n$.

(b) *For any stating* $(\boldsymbol{i}, x, \boldsymbol{j})$ *of an internal annihilator* x, *we have* $\Gamma((\boldsymbol{i}, x, \boldsymbol{j})) = 0$.

Equations (25)–(28) are internal annihilators, called the *basic internal annihilators*.

For a sequence $\boldsymbol{\eta}$ of signs $\{\pm\}$ let $\mathcal{H}_r(\boldsymbol{\eta})$ be the $\mathbb{Z}[v^{\pm 1}]$-module freely spanned by right stated n-tangles $(\alpha, \boldsymbol{i})$ such that the sequence of left ends of α has type $\boldsymbol{\eta}$. Note that $\mathcal{H}_r(\boldsymbol{\eta})$ contains $\mathrm{Hom}_{\mathfrak{C}_n}(\emptyset, \boldsymbol{\eta})$, and we extend the map RT to

$$\mathsf{RT} : \mathcal{H}_r(\boldsymbol{\eta}) \to \mathrm{Hom}_{\mathbb{Q}(v)}(\mathbb{Q}(v), V^{\eta}) \text{ by } \mathsf{RT}((\alpha, \boldsymbol{i}))(1) = \mathsf{RT}(\alpha)(v_{\boldsymbol{i}}),$$

where $v_{\boldsymbol{i}}$ is the basis vector of $V_r(\alpha)$ with index $\boldsymbol{i}$. For example, for $\eta = (+, -, +, -)$, $\alpha =$ 

and $\boldsymbol{i} = (i_1, i_2)$, we have

$$\mathsf{RT}((\alpha, \boldsymbol{i})) = e_{i_1} \otimes \widetilde{\mathrm{coev}}_0(1) \otimes f^{i_2} \in V \otimes V^* \otimes V \otimes V^*.$$

Note that for $x \in \mathcal{H}_r(\boldsymbol{\eta}) \setminus \mathrm{Hom}(\emptyset, \boldsymbol{\eta})$, the operator $\mathsf{RT}(x)$ might not be a $U_q(sl_n)$-morphism.

For $y \in \mathcal{H}_r(\boldsymbol{\eta}')$ define $x \otimes y \in \mathcal{H}_r(\boldsymbol{\eta} \otimes \boldsymbol{\eta}')$ by placing y atop x and by concatenating the right states. For $z \in \mathrm{Hom}_{\mathfrak{C}_n}(\boldsymbol{\eta}, \boldsymbol{\mu})$, we can define the composition $z \circ x \in \mathcal{H}_r(\boldsymbol{\mu})$. Clearly,

$$\mathsf{RT}(x \otimes y) = \mathsf{RT}(x) \otimes \mathsf{RT}(y) \quad \text{and} \quad \mathsf{RT}(x \circ y) = \mathsf{RT}(x) \circ \mathsf{RT}(y).$$

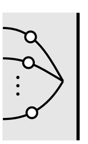
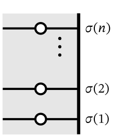
A *right annihilator* is an element $x \in \mathcal{H}_r(\boldsymbol{\eta})$, for certain $\boldsymbol{\eta}$, such that $\mathsf{RT}(x) = 0$. By the above equation and by Equation (31), we have

Proposition 3.7. *Let $x \in \mathcal{H}_r(\boldsymbol{\eta})$, $y \in \mathcal{H}_r(\boldsymbol{\eta}')$ be right annihilators and let $z \in \mathrm{Hom}(\boldsymbol{\eta}, \boldsymbol{\mu})$.*

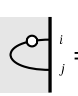
(a) *For any left state $\boldsymbol{i}$ of x, we have $\Gamma((\boldsymbol{i}, x)) = 0$.*
(b) *All $x \otimes y$, $y \otimes x$, and $z \circ x$ are right annihilators.*

We do not draw the left boundary vertical edge in pictures of right annihilators, to indicate that $\overset{\mathsf{RT}}{=}$ holds for their composition with any web on the left (for which such composition is possible).

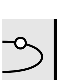
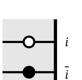
Proposition 3.8. *We have the following identities for the values of the function* RT*:*

 $\overset{\mathsf{RT}}{=} a \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)}$ 

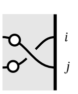
(35)

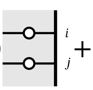


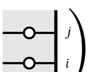
$\overset{\mathsf{RT}}{=} \delta_{\bar{j},i}\, c_i$  , (36)

 $\overset{\mathsf{RT}}{=} \sum_{i=1}^{n} (c_{\bar{i}})^{-1}$ 

(37)



$\overset{\mathsf{RT}}{=} q^{-\frac{1}{n}} \Bigg(\delta_{j<i}(q - q^{-1})$ 

$+ q^{\delta_{i,j}}$ 

$\Bigg)$, (38)

for any $i, j = 1, \dots, n$, where the white circle represents the orientation, left-to-right or right-to-left, and is the same for all white circles in one identity. The black circle stands for the opposite orientation of the white one. The values $\delta_{j<i}, \delta_{i,j}, a, c_1, \dots, c_n$ were defined in Subsection 2.1.

By subtracting the left side from the right side in the equations above we obtain right annihilators which we call *basic*.

Proof of Proposition 3.8. By applying the total orientation reversion and Proposition 3.5 if necessary, we can assume all the white circles indicate the left-to-right orientation. Identity (35) is the defining equation (8) of the operator $\mathcal{A}_+$, while Identity (38) is the defining equation (9) of the braiding. Identities (36) and (37) are consequences of Equation (34). □

We now define left annihilators. For a sign sequence $\boldsymbol{\eta}$, let $\mathcal{H}_l(\boldsymbol{\eta})$ be the $\mathbb{Z}[v^{\pm 1}]$-module freely spanned by left-stated n-tangles $(\boldsymbol{i}, \alpha)$ such that $V_r(\alpha) = V^{\boldsymbol{\eta}}$. Then $\mathcal{H}_l(\boldsymbol{\eta})$ contains $\mathrm{Hom}_{\mathfrak{C}_n}(\boldsymbol{\eta}, \emptyset)$, and we extend RT to

$$\mathsf{RT} : \mathcal{H}_l(\boldsymbol{\eta}) \to \mathrm{Hom}_{\mathbb{Q}(v)}(V^{\boldsymbol{\eta}}, \mathbb{Q}(v)) \text{ by } \mathsf{RT}((\boldsymbol{i}, \alpha))(z) = \mathrm{coeff}_{\boldsymbol{i}}(\mathsf{RT}(\alpha)(z)),$$

for $z \in V^{\eta}$, where $\mathrm{coeff}_{\boldsymbol{i}} : V^{\mu} \to \mathbb{Q}(v)$ is coefficient of the basis vector of V^{μ} indexed by $\boldsymbol{i}$.

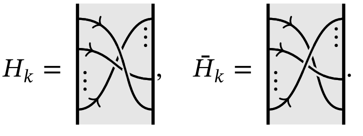
A *left annihilator* is an element $x \in \mathcal{H}_l(\boldsymbol{\eta})$, for a certain $\boldsymbol{\eta}$, such that $\mathsf{RT}(x) = 0$. In analogy to Proposition 3.7, we have:

Proposition 3.9. *Let $x \in \mathcal{H}_l(\boldsymbol{\eta}), y \in \mathcal{H}_l(\boldsymbol{\eta}')$ be left annihilators and let $z \in \mathrm{Hom}(\boldsymbol{\mu}, \boldsymbol{\eta})$.*

(a) *For any right state $\boldsymbol{i}$ of x, we have $\Gamma((x, \boldsymbol{i})) = 0$.*
(b) *All $x \otimes y, y \otimes x$, and $x \circ z$ are left annihilators.*

3.7 | Turning right annihilators to left ones

For an integer $k \geqslant 2$ let H_k be the positive half-twist of k strands and let $\bar{H}_k$ be its inverse:



(Note that H_k does not twist the framing, which always points toward the reader. This applies to all half-twists considered in this paper.) By abuse of notation, for any n-tangle α with k left endpoints, denote by $\bar{H}_k \circ \alpha$ the composition of α with a version of $\bar{H}_k$ in which the orientation of some of its components was reversed so that it is composable with α.

Let $\mathsf{hd} : \mathcal{H}_r(\boldsymbol{\eta}) \to \mathcal{H}_l(\boldsymbol{\eta})$ be a $\mathbb{Z}[v^{\pm 1}]$-linear map given by

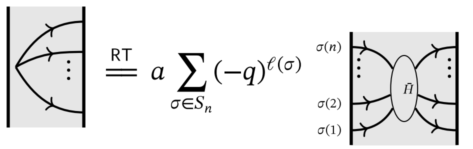
$$\mathsf{hd}((\alpha, \boldsymbol{i})) = (\boldsymbol{i}, \bar{H} \circ \mathsf{ro}(\alpha)),$$

where $\mathsf{ro}(\alpha)$ denotes the 180° rotation of α about the center, as before.

In Subsection 3.5, we showed that our basis $\{f^i\}$ of V^* makes the matrices $\mathsf{RT}(x)$ invariant under the total reversal of orientation of x. It also makes the following statement hold.

Proposition 3.10. *If x is a basic right annihilator then* $\mathsf{hd}(x)$ *is a left annihilator.*

Proof. The statement for each basic right annihilator can be checked by a direct computation. The calculation for Relations (36)–(37) follows from the framing change, Equation (26). The calculation for Relation (38) utilizes Equation (25). Finally, let us show that the image of (35) under hd, pictured below, is a left annihilator. (We assume its orientation to the right, as the statement for the opposite orientation follows from Proposition 3.5.)



Note that as H_k is invertible in $End(V^{\otimes n})$ we can consider the above equality composed with H on the right instead. Then, by Equation (10), it reduces to

$$\left(-q^{-\frac{1}{n}-1}\right)^{\frac{n(n-1)}{2}} \mathcal{A}_- = a \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} e^{\sigma(1)} \otimes \cdots \otimes e^{\sigma(n)},$$

which is indeed equivalent to the definition of $\mathcal{A}_-$, Equation (7), as $(-q^{-\frac{1}{n}-1})^{\frac{n(n-1)}{2}} = t^{-n/2}$. □

A stronger statement, valid for all right annihilators, will be shown in Subsection 3.9 using a more conceptual approach.

We call the left annihilators of Proposition 3.10 *basic*.

3.8 | Kernel of Γ

Theorem 3.11 (Proof in Subsection 6.10). *The kernel of* $\Gamma : \mathbb{Z}[v^{\pm 1}]\mathcal{T} \to U_q(sl_n)^*$ *is generated by internal basic annihilators, right basic annihilators, and left basic annihilators.*

This means that if we begin with the internal, left, and right basic annihilators, and use procedures described in Propositions 3.6, 3.7, and 3.9, we obtain the entire kernel $\ker \Gamma$. This theorem is analogous to Conjecture 3.3, except that it describes the kernel of a stated version of RT.

3.9 | Half-ribbon Hopf algebra

In this subsection, we explain conceptually some technical aspects of this paper. In particular, we interpret our sign modification of the Reshetikhin–Turaev functor, Equation (23), through a modification of the ribbon element in a completion of $U_q(sl_n)$. Coincidently, that modification leads to a "half-ribbon" element which makes it possible to prove a stronger version of Proposition 3.10. Although the content of this subsection is more technical, it is not necessary for the paper.

Proposition 3.12. *If x is a right annihilator, then* $\mathsf{hd}(x)$ *is a left annihilator.*

We precede the proof with a few preliminaries: the quantized enveloping algebra $U_q(sl_n)$ is a topological ribbon Hopf algebra, meaning it has an R-matrix $\mathcal{R}$ in a completion of $U_q(sl_n) \otimes U_q(sl_n)$ and a ribbon element ϑ_0 in a completion of $U_q(sl_n)$, satisfying certain conditions. It is proved in [61] that there is a completion $\widetilde{U_q(sl_n)}$ of $U_q(sl_n)$ having the same R-matrix but a new ribbon element ϑ, which acts on V and V^* the same way as $(-1)^{n-1}\vartheta_0$. (That completion was studied in different context also in [51].) Consequently, the new charmed element g is $(-1)^{n-1}g_0$ on V and V^*, explaining the sign correction in (23), which we used as the definition of RT.

Additionally, there is an invertible element $X \in \widetilde{U_q(sl_n)}$, called the half-twist, such that $X^2 = \vartheta$ and the universal R-matrix $\mathcal{R}$ satisfies

$$\mathcal{R} = (X^{-1} \otimes X^{-1})\Delta(X) = ((\mathrm{fl}\circ\Delta)(X))(X^{-1} \otimes X^{-1}), \text{ where } \mathrm{fl}(x \otimes y) := y \otimes x. \tag{39}$$

For a sign sequence $\boldsymbol{\eta} = (\eta_1, \dots, \eta_k)$ let $\overleftarrow{\boldsymbol{\eta}} = (\eta_k, \dots, \eta_1)$ and let $\mathrm{rev}_k : V^{\eta} \to V^{\overleftarrow{\boldsymbol{\eta}}}$ be the R-linear operator given by $\mathrm{rev}_k(x_1 \otimes \cdots \otimes x_k) = (x_k \otimes \cdots \otimes x_1)$. From Equation (39) by induction on k, we get the following, see [61, Proposition 4.18]: If $\bar{H}$ stands for $\bar{H}_k$ with an orientation on the strands on the right given by $\boldsymbol{\eta}$ then we have an equality of transformations $V^{\eta} \to V^{\overleftarrow{\boldsymbol{\eta}}}$:

$$\mathsf{RT}(\bar{H}) = \mathrm{rev}_k \circ X^{\otimes k} \circ \Delta^{[k]}(X^{-1}). \tag{40}$$

Here $\Delta^{[k]}$ is defined inductively by $\Delta^{[2]} = \Delta$ and $\Delta^{[k+1]} = (\Delta \otimes \mathrm{id}^{\otimes k})\circ\Delta^{[k]}$.

An additional special feature of the basis $\{f^i\}$ of V^* (besides those discussed already) is:

Proposition 3.13. *The actions of X on V and V^* are given by the same matrix*

$$X^i_j = \delta_{i,\bar{j}}\, c_i. \tag{41}$$

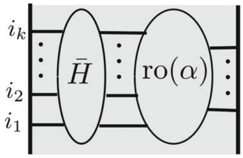
As this is not proved in [61], we give a proof of this result in the Appendix. Similarly, the actions of the charmed element g on both V and V^* are given by the same diagonal matrix with entries

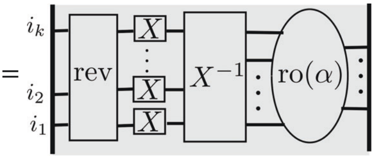
$$g^i_j = \delta_{i,j}\, g_i, \qquad \text{where } g_i = (-1)^{n-1}q^{2i-n-1} = (-1)^{n-1}q^{2d_i}. \tag{42}$$

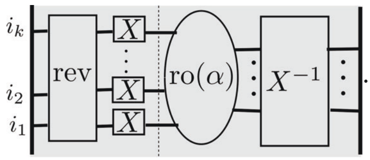
Proof of Proposition 3.12. Let $x \in \mathcal{H}_r(\boldsymbol{\eta})$ for some $\boldsymbol{\eta}$ be a right annihilator. We need to show that $\mathsf{RT}(\mathsf{hd}(x)) = 0$. As X is invertible, this is equivalent to $\mathsf{RT}(\mathsf{hd}(x))\circ X = 0$, which, in turn, is equivalent to:

$$\langle \mathsf{RT}(\mathsf{hd}(x))\circ X, \boldsymbol{j}\rangle = 0 \qquad \text{for all } \boldsymbol{j} = (j_1, \dots, j_l) \in \{1, \dots, n\}^l, l := |\boldsymbol{\eta}|.$$

Suppose $(\alpha, \boldsymbol{i})$ is a right stated n-tangle. By Equation (40) and then by Equation (31), we have

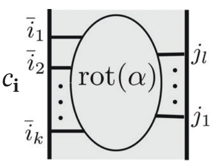
$\mathsf{hd}((\alpha, \mathbf{i}))$ = 

= 

= 

.

By composing with X on the right, then decomposing along the dashed line and using the values of X^i_j from Equation (41), we get

$\langle \mathsf{hd}((\alpha, \mathbf{i}))\circ X, \boldsymbol{j}\rangle \overset{\mathsf{RT}}{=\!=} c_{\mathbf{i}}$ 

$= c_{\boldsymbol{j}}\, \langle \boldsymbol{j}^* | (\alpha, \mathbf{i})\rangle$

where $c_{\boldsymbol{i}} = \prod_k c_{i_k}$. The second identity follows from Proposition 3.5. By linearity,

$$\langle \mathsf{hd}(x)\circ X, \boldsymbol{j}\rangle \overset{\mathsf{RT}}{=\!=} c_{\boldsymbol{j}} \langle \boldsymbol{j}^* | x\rangle = 0,$$

for every $\boldsymbol{j} \in \{1, \dots, n\}^{|\eta|}$. This proves $\mathsf{hd}(x) \overset{\mathsf{RT}}{=\!=} 0$. □

Remark 3.14. The use of half-ribbon element came up in a discussion of the first author with Costantino and Korinman. A full-fledged theory of stated skein algebra based on half-ribbon category will be developed in an upcoming work by Costantino, Korinman, and Lê.

4 | STATED SL(n)-SKEIN MODULES

4.1 | Marked 3-manifolds and n-webs

A *marked 3-manifold* is a pair $(M, \mathcal{N})$, where M is a smooth oriented 3-manifold with (possibly empty) boundary ∂M and $\mathcal{N} \subset \partial M$ consists of open intervals, called *markings*. The topological closure of each marking is required to be the closed interval [0,1], disjoint from the closure of other markings.

Roughly speaking, an n-web in $(M, \mathcal{N})$ is like n-tangle, except that the framing at boundary points is different. Here is the precise definition, where the first four requirements are the same as those in the definition of an n-tangle.

Definition 4.1. An *n-web* α in $(M, \mathcal{N})$ is a disjoint union of finite number of oriented circles and a finite directed graph properly embedded into M such that

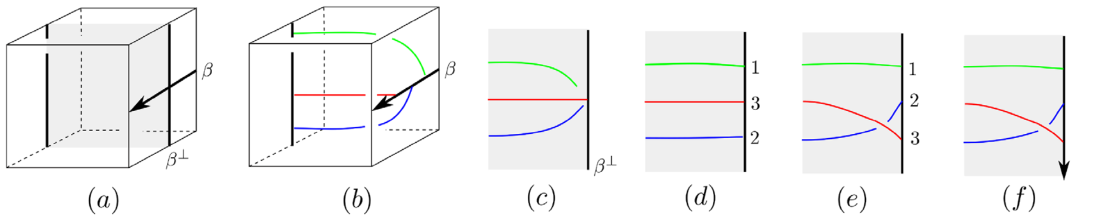


FIGURE 6 (a) Cube Q with a marking β and a perpendicular line $\beta^\perp$ on the right face. The shaded square S is in the XY-plane. (b) An example of a web α with three strands (depicted by different colors) in $\alpha \cap Q$. (c) The projection of $\alpha \cap Q$ onto S. (d) A diagram of $\alpha \cap Q$, with the height order indicated by numeric labels: $i > j$ means i is higher than j. (e) Another diagram of $\alpha \cap Q$ obtained by using a different height preserving deformation. (f) This is the same diagram of (e), with the height order indicated by the direction of the boundary line.

(1) Every vertex of α is either a sink or a source and either 1-valent or n-valent. We denote the set of 1-valent vertices, called *endpoints* of α, by $\partial\alpha$.
(2) Each edge of the graph is a smooth embedding of the closed interval [0,1] into M.
(3) α is equipped with a *framing* that is a continuous nonvanishing vector field transversal to α. In particular, the framing at a vertex is transversal to all incident edges.
(4') The set of half-edges at every n-valent vertex is cyclically ordered.
(5') The endpoints of α lie in $\mathcal{N}$ and the framing at these endpoints is a tangent vector of $\mathcal{N}$, pointing in the direction of the orientation of $\mathcal{N}$. We call such tangent vector *positive*.

Webs are considered up to continuous isotopy within their space.

Note that the only difference between the unbased n-tangles of Subsection 3.3 and n-webs in the cube Q marked with Λ_l, Λ_r is the framing at their endpoints. The difference explains why half-twists appear in our theory.

The *height order* on $\partial\alpha$ is the partial order in which two points $x, y \in \partial\alpha$ are comparable if and only if they belong to the same marking, and $x > y$, or x *is higher than* y, if going along the positive direction of the marking we encounter y first. We say x and y are *consecutive* if there is no $z \in \partial\alpha$ such that $x > z > y$ or $y > z > x$.

To depict a local part of an n-web α we consider the intersection of α with the cube $Q = [-1,1] \times (-1,1)^2$ embedded into M, presented in Figure 6a. The cube Q can be either in the interior of M or its right side, $\{1\} \times (-1,1)^2$, lies in ∂M.

- If Q is in the interior of M then we assume that $\alpha \cap Q$ is an n-tangle, and depict $\alpha \cap Q$ by its n-tangle diagrams on the shaded square, as in Subsection 3.3. In particular, for all drawn diagrams the framing is perpendicular to the page and pointing to the reader, and the cyclic order of half-edges at each n-valent vertex is counterclockwise.
- In the second case, we assume that $Q \cap \partial M$ is equal to the right face of Q, and $Q \cap \mathcal{N}$ is a subinterval of a marking β depicted pointing in the direction of the z-axis, as in Figure 6a. In the Q coordinates, $\beta \cap \partial M = \{1\} \times \{0\} \times (-1,1)$. Let $\beta^\perp = \Lambda_r$, the right side of the shaded square. As in the previous case, we assume that the framing points to the reader. (Note the difference between webs in M and tangles in Q: the boundary points of α in Q are in β, while the right endpoints tangles in Q are in $\Lambda_r = \beta^\perp$.) By an isotopy, we can bring $\alpha \cap Q$ to a general position with respect to the projection $p : \mathbb{R}^3 \to \mathbb{R}^2$, except that all the points in $\partial\alpha \cap \beta$ project to the same point, see Figure 6c. To resolve this issue, first we define a *height-preserving deformation of* Q as a continuous family of diffeomorphisms $\phi_t : Q \to Q, t \in [0,1]$, supported in a small

neighborhood of the right face of Q and preserving the z-coordinate, that is, the height above the page. We use such a height-preserving deformation to bring α to α' whose endpoints on the right face of Q have distinct projections (through p). The image $p(\alpha' \cap Q)$ together with the usual over- and undercrossing data and with the linear order of its boundary points on $\beta^{\perp}$ (induced from the height order) is a *diagram of* $\alpha \cap Q$. For example, Figure 6d shows a diagram of Figure 6b. Note however that a different height-preserving deformation can give rise to different diagram, see for example Figure 6e. Note that α' is not an n-web because its endpoints are not in $\mathcal{N}$ in general.

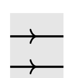
Although the height order of web ends in $\alpha \cap Q$ can be always indicated by integers as in Figure 6d–e, in this paper we will always use the following convention: when presenting $\alpha \cap Q \subset M$ diagrammatically, we will choose a direction of $\beta^{\perp}$ (indicating it by an arrow down or up) and arrange for the height order of web ends in $\alpha \cap Q$ to increase monotonically (without gaps) in the indicated direction. For example, Figure 6f indicates the web in part (e). Note that the height order of endpoints of the α outside the drawn part can be arbitrary.

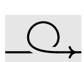
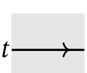
4.2 | Skein relations for n-webs

A *state* of an n-web α is a map $s : \partial\alpha \to \{1, 2, \dots, n\}$. The value $s(x)$, for $x \in \partial\alpha$, is called the *state of* x. A web with a state s is called *stated*.

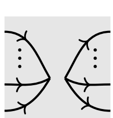
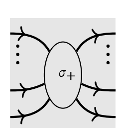
We will consider stated n-webs up to isotopy (in the space of all stated n-webs) and denote the set of their isotopy classes by $\mathcal{W}_n(M, \mathcal{N})$.

Recall that the ground ring R is commutative and it comes with a distinguished invertible $v = q^{1/2n} \in R$. The *stated SL*(n)-skein module of $(M, \mathcal{N})$, denoted by $\mathcal{S}_n(M, \mathcal{N})$, is the quotient of the free R-module $R\mathcal{W}_n(M, \mathcal{N})$ by the submodule $SkRel_n(M, \mathcal{N})$ generated by the following *internal relations* (43)–(50), which are the basic internal annihilators, and *boundary relations* (47)–(50), which comes from the basic right annihilators:

$q^{\frac{1}{n}}$  $- q^{-\frac{1}{n}}$  $= (q - q^{-1})$  (43)

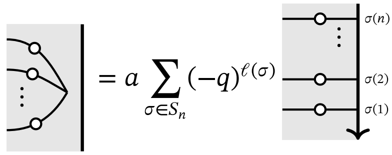
 $= t$  , where $t = (-1)^{n-1} q^{n-\frac{1}{n}}$ (44)

 $= (-1)^{n-1}[n]_q$, where $[n]_q = \dfrac{q^n - q^{-n}}{q - q^{-1}}$, (45)

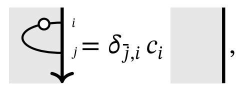
 $= (-q)^{\binom{n}{2}} \cdot \sum_{\sigma \in S_n} \left(-q^{\frac{1-n}{n}}\right)^{\ell(\sigma)}$  (46)

where the ellipse enclosing σ_+ is the minimum crossing positive braid representing a permutation $\sigma \in S_n$ and $\ell(\sigma)$ is the length of $\sigma \in S_n$, as before.

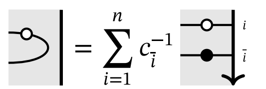
The remaining relations in $SkRel_n(M, \mathcal{N})$ take place near markings where we use the convention in Subsection 4.1 about height order. Thus, the bold boundary line of a shaded rectangle is a part of $\beta^{\perp}$, orthogonal to a marking β, and if it has a direction, then the endpoints on that part are consecutive in the height order, given by the direction. The height order outside the drawn part of $\beta^{\perp}$ can be arbitrary. Here are the boundary relations:



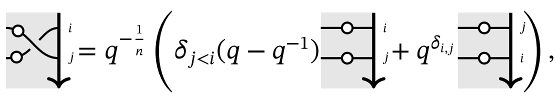
(47)



(48)



(49)



(50)

where the values $\delta_{j<i}, \delta_{i,j}, a, c_1, \dots, c_n$ were defined in Subsection 2.1 and the small white circles represent an arbitrary direction of the edges (left-to-right or right-to-left), consistent for the entire equation, as before. The black circle represents the opposite direction.

We show in Proposition 7.5 that if $[n-2]!$ is invertible in R then Relation (50) is a consequence of Relations (43)–(49).

4.3 | Eliminating sinks and sources

Proposition 4.2. *For any marked 3-manifold, $\mathcal{S}_n(M, \mathcal{N})$ is spanned by stated n-webs with no sinks nor sources.*

Proof. If $\mathcal{N} = \emptyset$ then the numbers of sinks and sources in any n-web coincide and they can be eliminated by Relation (46). If $\mathcal{N} \neq \emptyset$, then sinks and sources can be eliminated by Relation (47). □

Nonetheless, the use of sinks and sources in our theory makes it much more manageable.

4.4 | Change of ground ring

We will use the notation $\mathcal{S}_n(M, \mathcal{N}, R)$ when we need to make the coefficient ring R explicit. By our assumptions, R is an algebra over $\mathbb{Z}[v^{\pm 1}]$. The right exactness of tensor product gives a natural isomorphism

$$\mathcal{S}_n(M, \mathcal{N}, \mathbb{Z}[v^{\pm 1}]) \otimes_{\mathbb{Z}[v^{\pm 1}]} R \xrightarrow{\cong} \mathcal{S}_n(M, \mathcal{N}, R).$$

Therefore, many properties of $\mathcal{S}_n(M, \mathcal{N}, R)$ follow from those of $\mathcal{S}_n(M, \mathcal{N}, \mathbb{Z}[v^{\pm 1}])$.

4.5 | Functoriality

An *embedding* of a marked 3-manifold $(M, \mathcal{N})$ into a marked 3-manifold $(M', \mathcal{N}')$ is an orientation preserving proper embedding $f : M \hookrightarrow M'$ that maps $\mathcal{N}$ into $\mathcal{N}'$ preserving their orientations. Clearly, f induces an R-module homomorphism $\mathcal{S}_n(f) : \mathcal{S}_n(M, \mathcal{N}) \to \mathcal{S}_n(M', \mathcal{N}')$ mapping each n-web α to $f(\alpha)$ with its framing transformed by the differential $f_* : TM \to TM'$. That homomorphism depends only on the isotopy class of f (in the embeddings). A *morphism* from $(M, \mathcal{N})$ to $(M', \mathcal{N}')$ is an isotopy class of embeddings from $(M, \mathcal{N})$ to $(M', \mathcal{N}')$. Hence, $\mathcal{S}_n(\cdot)$ defines a functor from the category of marked 3-manifolds to the category of R-modules.

Example 4.3. Let $(M, \mathcal{N})$ be a marked 3-manifold. For any closed subset X of $\partial M - \mathcal{N}$, its complement $(M - X, \mathcal{N})$ is a marked 3-manifold as well and the natural embedding $\iota : (M - X, \mathcal{N}) \hookrightarrow (M, \mathcal{N})$ is a morphism called a *pseudo-isomorphism*. It induces an R-module isomorphism $\iota_* : \mathcal{S}_n(M', \mathcal{N}) \xrightarrow{\simeq} \mathcal{S}_n(M, \mathcal{N})$.

In this paper, we will consider certain geometric operations on 3-manifolds, like cutting and gluing them along disks, which produce new manifolds defined up a diffeomorphisms only. We will address this issue with the aid of the following notion:

A *strict isomorphism class of marked 3-manifolds* is a family of marked 3-manifolds $(M_i, \mathcal{N}_i)$, $i \in I$ equipped with isomorphisms $f_{ij} : (M_i, \mathcal{N}_i) \to (M_j, \mathcal{N}_j)$ for any two indices i, j such that $f_{ii} = \mathrm{id}$ and $f_{jk} \circ f_{ij} = f_{ik}$. For a strict isomorphism class of marked 3-manifolds we can identify all R-modules $\mathcal{S}_n(M_i, \mathcal{N}_i)$ via the isomorphisms $\mathcal{S}_n(f_{ij})$.

For example, to glue a pair of boundary edges e_1 and e_2 we first fix an orientation reversing diffeomorphism $\phi : e_1 \to e_2$ and then identify $x \equiv \phi(x)$ for all $x \in e_1$. Various ϕ's give various surfaces, but they belong to the same strict isomorphism class.

For a disjoint union of M_1 and M_2, the map

$$\mathcal{S}_n(M_1, \mathcal{N}_1) \otimes \mathcal{S}_n(M_2, \mathcal{N}_2) \to \mathcal{S}_n(M_1 \sqcup M_2, \mathcal{N}_1 \sqcup \mathcal{N}_2)$$

sending $\alpha_1 \otimes \alpha_2$ to $\alpha_1 \sqcup \alpha_2$ is an isomorphism. We will identify $\mathcal{S}_n(M_1 \sqcup M_2, \mathcal{N}_1 \sqcup \mathcal{N}_2)$ with $\mathcal{S}_n(M_1, \mathcal{N}_1) \otimes \mathcal{S}_n(M_2, \mathcal{N}_2)$ though this map.

4.6 | Grading

For a stated n-web α in $(M, \mathcal{N})$ and a marking $\beta \subset \mathcal{N}$ we define the β-*degree*

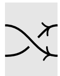
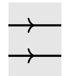
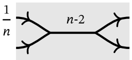
$$\deg_\beta(\alpha) = \sum_{x \in \alpha \cap \beta} d_{s(x)} = \sum_{x \in \alpha \cap \beta} \left(s(x) - \frac{n+1}{2} \right) \in \frac{1}{2}\mathbb{Z},$$

where $s(x)$ denotes the state of α at x. Note that the β-degree is preserved by the skein relations (43)–(50) and, therefore, it descends to $\frac{1}{2}\mathbb{Z}$-valued grading on $\mathcal{S}_n(M, \mathcal{N})$.

4.7 | Useful identities

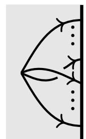
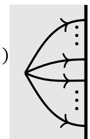
Recall that a, t, c_i were defined in Subsection 2.1.

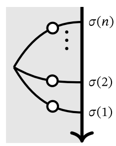
Proposition 4.4. *The following identities hold in any stated skein module $\mathcal{S}_n(M,\mathcal{N})$:*

$[n-2]!$  $= [n-2]!\,q^{\frac{n-1}{n}}$  $-(-1)^{\binom{n}{2}}q^{-\frac{1}{n}}$ 

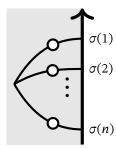
, (51)

where the label in the diagram on the right indicates $n-2$ parallel horizontal edges.

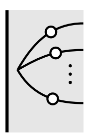
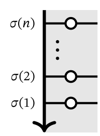
 $= -q^{-(1+\frac{1}{n})}$  , (52)



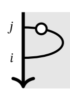
$= at^{n/2}(-q)^{\ell(\sigma)}$ (53)

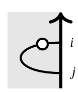


$= a(-q)^{\ell(\sigma)}$ (54)

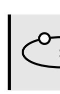
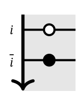
 $= at^{n/2}\sum_{\sigma\in S_n}(-q)^{\ell(\sigma)}$ 

. (55)

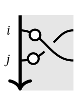


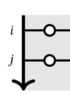
$=$ 

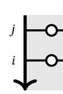
$= \delta_{\bar{i},j}c_i^{-1}$ (56)

 $= \sum_{i=1}^{n} c_i$ 

(57)

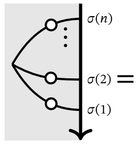


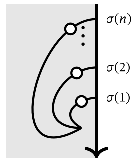
$= q^{-\frac{1}{n}}\Big(\delta_{j<i}(q-q^{-1})$ 

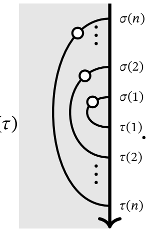
$+ q^{\delta_{i,j}}$ 

$\Big)$ (58)

Proof. Identities (51) and (52) are, respectively, (30) and (29). As remarked in Subsection 3.3, these identities are consequences of the basic internal annihilators, which are skein relations for $\mathcal{S}_n(M,\mathcal{N})$.

Proof of (53): 

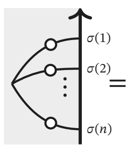
$=$ 

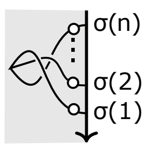
$= a \sum_{\tau\in S_n} (-q)^{\ell(\tau)}$ 

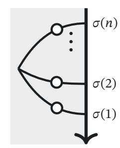
. The web on the right is nonzero only for $\tau(i) = \overline{\sigma(\bar{i})}$ for $i = 1, \dots n$. As $\ell(\tau) = \ell(\sigma)$ then, the above equals

$$a(-q)^{\ell(\sigma)} \cdot c_1 \cdot \dots \cdot c_n = at^{n/2}(-q)^{\ell(\sigma)},$$

by (4).

Proof of (54): 

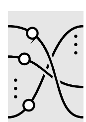
$=$ 

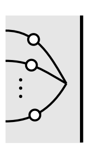
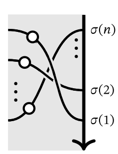
$= \left(-q^{-\left(1+\frac{1}{n}\right)}\right)^{\binom{n}{2}}$ 

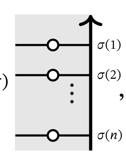
by (52). As

$$\left(-q^{-\left(1+\frac{1}{n}\right)}\right)^{\binom{n}{2}} = (-1)^{\binom{n}{2}} q^{-\frac{n^2-1}{2}}, \tag{59}$$

the statement follows by (53).

Proof of (55): By composing (47) with  on the left, we obtain

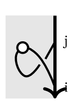
$$\left(-q^{-\left(1+\frac{1}{n}\right)}\right)^{\binom{n}{2}}$$  $= \sum_{\sigma\in S_n} a\,(-q)^{\ell(\sigma)}$ 

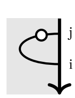
$= \sum_{\sigma\in S_n} a\,(-q)^{\ell(\sigma)}$ 

,

by (52). Now the statement follows by (59) and by rotating these skeins 180°.

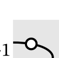
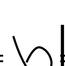
Proof of (56): The left side equals



$= t^{-1} \cdot$ 

$= t^{-1}\delta_{\bar{i},j} c_j = t^{-1}\delta_{\bar{i},j} c_{\bar{i}} = \delta_{\bar{i},j} c_i^{-1},$

by (44).

Proof of (57): By (44) and (49), t^{-1}  $=$  $= \sum_{i=1}^{n} (c_{\bar{i}})^{-1}$ 

.

Now the statement follows by 180° rotation and the fact that $tc_{\bar{i}}^{-1} = c_i$, see (4).

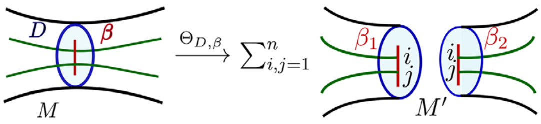
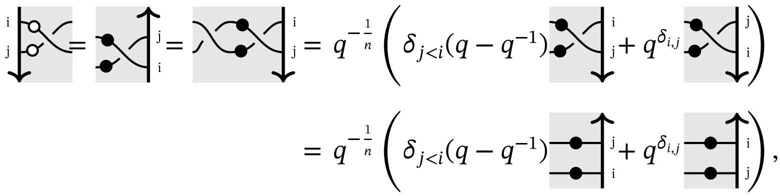


FIGURE 7 An example of a splitting of an n-web (in green) intersecting the splitting disk D twice.

Proof of (58):



by (50). ☐

4.8 | Splitting homomorphism

As mentioned in the introduction, an important property of stated skein modules is that they behave in a simple manner under the splitting of 3-manifolds along disks. This property, known as the Splitting Theorem, was first proved by [11, 41] for the Kauffman bracket skein modules ($n = 2$) and then by [28] for $n = 3$. We formulate it now and prove for all n below.

Suppose $(M, \mathcal{N})$ is a marked 3-manifold and D is a properly embedded closed disk in M (and, hence, $\partial D \subset \partial M$), disjoint from the closure of $\mathcal{N}$. By removing a collar neighborhood of D we get a topological 3-manifold M' whose boundary contains two copies D_1 and D_2 of D such that gluing D_1 to D_2 yields M together with a surjective homomorphism $\mathrm{pr} : M' \to M$. The manifold M' has a smooth structure with corners. However, these corners can be smoothed out uniquely up to isotopy.

Let $\beta \subset D$ be an oriented open interval, and $\beta_1 \subset D_1$ and $\beta_2 \subset D_2$ be preimages of β. The *splitting of* $(M, \mathcal{N})$ along (D, β), denoted by $\mathrm{cut}_{(D,\beta)}(M, \mathcal{N})$, is the marked 3-manifold $(M', \mathcal{N}')$, where $\mathcal{N}' = \mathcal{N} \cup \beta_1 \cup \beta_2$. It is easy to see that $\mathrm{cut}_{(D,\beta)}(M, \mathcal{N})$ is defined uniquely as a strict isomorphism class, see Subsection 4.5.

Let α be a stated n-web in $(M, \mathcal{N})$. This web in this subsection is given by a specific embedding and, hence, not considered up to isotopy. It is said to be (D, β)-*transverse* if the vertices of α are not in D, α is transverse to D, $\alpha \cap D \subset \beta$, and the framing at every point of $\alpha \cap \beta$ is a positive tangent vector of β. Note that every web α in $(M, \mathcal{N})$ can be isotoped so that it is (D, β)-transverse. Suppose in addition that α is stated. Then the n-web $\mathrm{pr}^{-1}(\alpha)$ of $(M', \mathcal{N}')$ is stated everywhere except for its endpoints in $\beta_1 \cup \beta_2$, see Figure 7. Given any map $s : \alpha \cap \beta \to \{\pm\}$, let $\alpha(s)$ denote the (partially stated) n-web $\mathrm{pr}^{-1}(\alpha)$ in $(M', \mathcal{N}')$ with additional states $s(\mathrm{pr}(x))$ for $x \in \mathrm{pr}^{-1}(\alpha) \cap (\beta_1 \cup \beta_2)$. Hence, $\alpha(s)$ is fully stated. We call $\alpha(s)$ a *lift* of α. If $|\alpha \cap \beta| = k$ then α has n^k distinct lifts.

Theorem 4.5. *Let D be a closed disk properly embedded in a marked 3-manifold $(M, \mathcal{N})$ and let β be an oriented open arc in D. Let* $\mathrm{cut}_{(D,\beta)}(M, \mathcal{N})$ *be the splitting of $(M, \mathcal{N})$ along (D, β), as described*

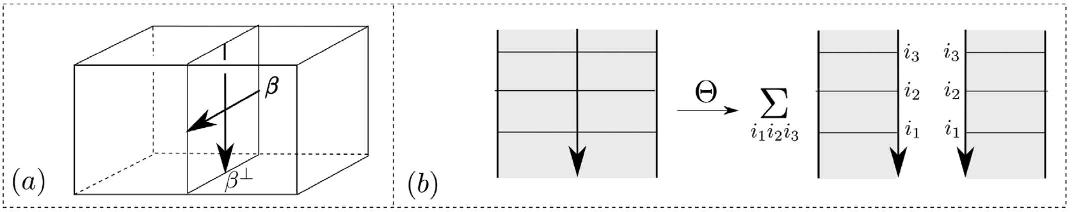


FIGURE 8 (a) The cube $\bar{Q}$. The disk D is the middle square containing the lines β and $\beta^\perp$. (b) The splitting homomorphism Θ.

above. Then there is a unique R-module homomorphism

$$\Theta_{(D,\beta)} : \mathcal{S}_n(M,\mathcal{N}) \to \mathcal{S}_n(\mathsf{cut}_{(D,\beta)}(M,\mathcal{N}))$$

sending every stated (D,β)-transverse n-web α in $(M,\mathcal{N})$ to the sum of all of its lifts,

$$\Theta_{(D,\beta)}(\alpha) = \sum_{s\colon \alpha\cap\beta\to\{\pm\}} \alpha(s). \tag{60}$$

Note that for any arcs β,β' in D there is an isomorphism between marked 3-manifolds $\mathsf{cut}_{(D,\beta)}(M,\mathcal{N}) \simeq \mathsf{cut}_{(D,\beta')}(M,\mathcal{N})$, inducing an isomorphism of stated skein modules that commutes with the splitting homomorphisms. Consequently, we will often denote $\Theta_{(D,\beta)}$ and $\mathsf{cut}_{(D,\beta)}(M,\mathcal{N})$ by Θ_D and $\mathsf{cut}_D(M,\mathcal{N})$ when it does not lead to confusion.

Remark 4.6. It is easy to see that splitting homomorphisms along any two disjoint splitting disks D_1 and D_2 commute,

$$\Theta_{D_1}\circ\Theta_{D_2} = \Theta_{D_2}\circ\Theta_{D_1}.$$

Remark 4.7. By removing a closed subset of ∂M disjoint from $\mathcal{N}$ and using the pseudo-isomorphism of Example 4.3, we can apply the theorem to many cases when D is a closed disk with some closed intervals on its boundary removed. This fact will be useful in Section 5 where we will apply the Splitting Theorem to thickened surfaces $\Sigma\times(-1,1)$ cut along open disks, $(-1,1)\times(-1,1)$.

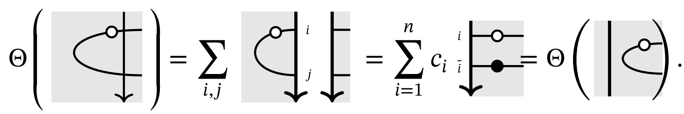
In general, the splitting homomorphism is not injective. For an example in the $n=2$ case, see a forthcoming paper by Costantino and Lê. We will discuss the injectivity and the image of the splitting homomorphism for thickened surfaces in Section 5.

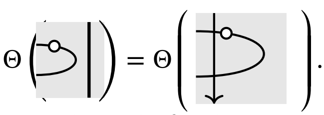
Proof of the Splitting Theorem. We identify a closed collar neighborhood of D with the closed cube $\overline{Q} = [-1,1]^3$ so that $D = \{0\}\times[-1,1]^2$ and β is an open interval subset of $\{0\}\times\{0\}\times[-1,1]$, as in Figure 8a. For a stated (D,β)-transverse n-web α let $\Theta(\alpha)\in\mathcal{S}_n(\mathsf{cut}_{(D,\beta)}(M,\mathcal{N}))$ be the right side of Equation (60). To prove the theorem, we need to show $\Theta(\alpha)$ is invariant under isotopies of α.

An ambient isotopy of α in M can be decomposed into a sequence of isotopies, each of which is supported in a small neighborhood of D or supported outside of D. The latter clearly preserves Θ, so we only need to check invariance of Θ under isotopies with support in the interior of $\bar{Q}$. By an isotopy outside D we can assume that $\alpha\cap Q$ is an n-tangle. To get a diagram of $\alpha\cap Q$ we first use height preserving deformation near D to move α to a general position with respect to the projection onto $[-1,1]^2 \times \{0\}$ (as always considered in page). The points in $\alpha \cap \beta$, after that deformation, project to points on $\beta^\perp$. We will always choose a height preserving deformation such that the height order on $\beta^\perp$ is given by the direction from the top to the bottom, as in Figure 8b.

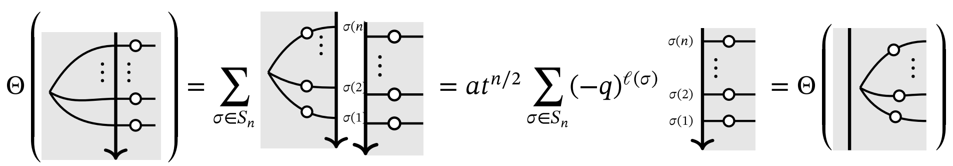
Now we can decompose a diagram of α into elementary tangle diagrams listed in (13)–(15). If α' is isotopic to α by an isotopy in Q, then its diagram can be obtained by a sequence of operations moving elementary tangles through β, and the height exchange move, discussed in point (d) below. Therefore, it is enough to verify that $\Theta(\alpha)$ is preserved by the following four moves.

(a) Passing a cap through β. The invariance of $\Theta(\alpha)$ under this move is a consequence of skein relations (48) and (57):

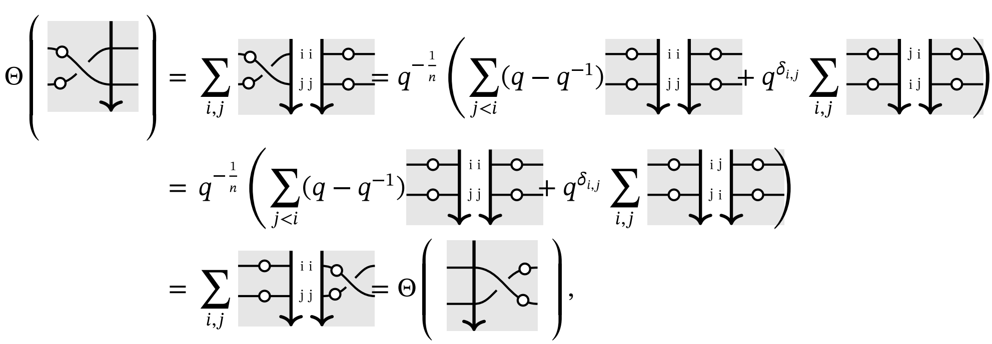


By the same argument, 

(b) Passing a sink or a source through β. The invariance of Θ under this move is a direct consequence of skein relations (53) and (55):



(c) Passing a positive crossing through β:



by (50) and (58).

(d) Passing a negative crossing through β follows from (c) by composing the fragments of diagrams on the left and the right side of the above identity with a negative crossing on their both sides.

(e) Height exchange of two consecutive points of $\alpha \cap \beta$ as in Figure 9. The invariance of Θ under this moves follows from the move in (c) or (d) if the arcs involved have coinciding orientations. If Figure 9 involves arcs in opposite directions, then the left side of the diagram on the right

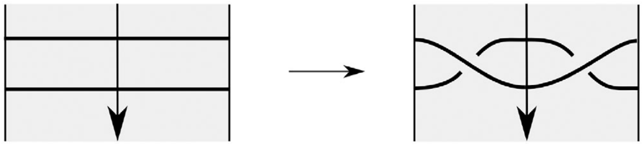


FIGURE 9 Height exchange move.

can be decomposed into elementary diagrams and all of them can be moved to the right side by (a)–(d).

The above argument shows that $\Theta(\alpha)$ is preserved by isotopies of α. To finish the proof off, observe that Θ maps the defining relations (43)–(50) to 0 in $\mathcal{S}_n(M', N')$, because they are all local and can be moved away from D. □

4.9 | Reversing orientations of 3-manifolds and of webs

An *orientation* of a web consists of orientations of all its loop components and directions of all its edges. Let $\overleftarrow{\alpha}$ denote an n-web α with its orientation reversed (and unchanged framing). As the defining relations (43)–(50) of $\mathcal{S}_n(M, \mathcal{N})$ are invariant under the total orientation inversion, we have

Corollary 4.8.

$$\overleftarrow{\cdot} : \mathcal{S}_n(M, \mathcal{N}) \to \mathcal{S}_n(M, \mathcal{N})$$

is a well-defined R-module automorphism.

Let $\overline{(M, \mathcal{N})}$ denote M and $\mathcal{N}$ with reversed orientations. Let $\bar{R}$ be the ring R with the distinguished element v^{-1} instead of v. For an n-web α of $(M, \mathcal{N})$ let $\overline{\alpha}$ be the n-web in $\overline{(M, \mathcal{N})}$ obtained from α by negating its framing, $f \to -f$, but retaining the orientation.

Theorem 4.9.

(1) *Any ring isomorphism $\varkappa : R \to \bar{R}$ sending v to v^{-1} extends to an isomorphism of R-modules $\varkappa_{(M,\mathcal{N})} : \mathcal{S}_n(M, \mathcal{N}, R) \xrightarrow{\cong} \mathcal{S}_n(\overline{M, \mathcal{N}}, \bar{R})$ sending every stated n-web α to $\overline{\alpha}$, where $\mathcal{S}_n(\overline{M, \mathcal{N}}, \bar{R})$ is an R-module via $\varkappa : R \to \bar{R}$.*

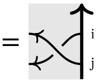
(2) *The composition $\varkappa_{\overline{(M,\mathcal{N})}} \circ \varkappa_{(M,\mathcal{N})}$ is the identity on $\mathcal{S}_n(M, \mathcal{N}, R)$.*

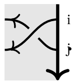
The above isomorphism is called the *orientation reversion isomorphism*. (Note that for some rings R, an isomorphism $\varkappa : R \to \bar{R}$ as above may not exist or be nonunique.)

Proof of Theorem 4.9: By abuse of notation, we define an R-linear map $\varkappa_{(M,\mathcal{N})}$ first as

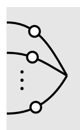
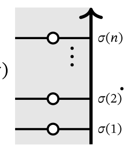
$$\varkappa_{(M,\mathcal{N})} : R\mathcal{W}_n(M, \mathcal{N}) \to \mathcal{S}_n(\overline{M, \mathcal{N}}), \quad \varkappa_{(M,\mathcal{N})}(\alpha) = \overline{\alpha}.$$

One checks immediately that map factors through Relations (43)–(45).

By our graphical convention, a diagram of $\overline{\alpha}$ near a marking is given by switching all crossings in a diagram α and by reversing of the direction of the vertical line $\beta^{\perp}$. For example, if $\alpha =$ 

then $\overline{\alpha} =$  It is clear that $\varkappa_{(M,\mathcal{N})}$ maps (46) to the equality of Lemma 4.10 and, therefore, it preserves that relation.

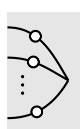
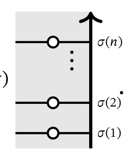
To see that $\varkappa_{(M,\mathcal{N})}$ factors through (47) substitute σ' for σ in (55), where $\sigma'(i) = \sigma(\bar{i})$, for $i = 1, \dots, n$, and rotate that equation 180°. As $\ell(\sigma') = \binom{n}{2} - \ell(\sigma)$, we get

 $= t^{n/2} a \sum_{\sigma \in S_n} (-q)^{\binom{n}{2} - \ell(\sigma)}$ 

As

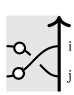
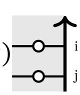
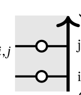
$$(-q)^{\binom{n}{2}} t^{n/2} = q^{\binom{n}{2} + \frac{n^2-1}{2}} = a^{-2},$$

by Equation (2), we get the desired relation

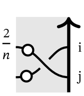
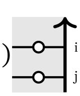
 $= a^{-1} \sum_{\sigma \in S_n} (-q)^{-\ell(\sigma)}$ 

Furthermore, $\varkappa_{(M,\mathcal{N})}$ maps (48) and (49) to (56) and (57).

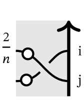
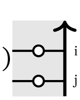
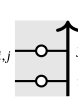
Let us show now that $\varkappa_{(M,\mathcal{N})}$ factors through (50). We need to verify that

 $= q^{\frac{1}{n}} \Big(\delta_{j<i} (q^{-1} - q)$  $+ q^{-\delta_{i,j}}$  $\Big).$

By (43), the left side is

$q^{\frac{2}{n}}$  $- q^{\frac{1}{n}} (q - q^{-1})$ 

and as $1 - \delta_{j<i} = \delta_{\bar{j}<\bar{i}} + \delta_{i,j}$, the above equation reduces to

$q^{\frac{2}{n}}$  $= q^{\frac{1}{n}} \Big(\delta_{\bar{j}<\bar{i}} (q - q^{-1})$  $+ q^{\delta_{i,j}}$  $\Big),$

which is (58) rotated 180° (and with i and j interchanged).

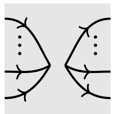
Hence, we have shown that the above map factors to

$$\varkappa_{(M,\mathcal{N})} : \mathcal{S}_n(M, \mathcal{N}) \to \mathcal{S}_n(\overline{M, \mathcal{N}}).$$

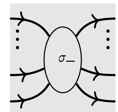
It is an R-module homomorphism by definition.

Part (2) is obvious.

Lemma 4.10. *One has*



$$= (-q)^{-\binom{n}{2}} \sum_{\sigma\in S_n} \left(-q^{-\frac{1-n}{n}}\right)^{\ell(\sigma)}$$

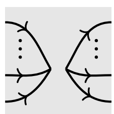


,

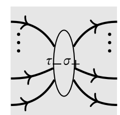
where σ_- is the minimal crossing negative braid representing $\sigma \in S_n$.

Proof. Let $\tau(i) = n + 1 - i$, for $i = 1, \dots, n$. Then τ_- is the negative half-twist n-braid. By applying it to the left side of (46), we obtain

$$\left(-q^{-\frac{n+1}{n}}\right)^{-\binom{n}{2}}$$



$$= (-q)^{\binom{n}{2}} \cdot \sum_{\sigma\in S_n} \left(-q^{\frac{1-n}{n}}\right)^{\ell(\sigma)}$$



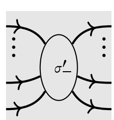
, (61)

by (52). Note that $\tau_-\sigma_+ = (\tau\sigma)_-$ for every $\sigma \in S_n$ and that by (5),

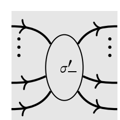
$$\ell(\sigma) + \ell(\tau\sigma) = \ell(\tau) = \binom{n}{2}.$$

Therefore, by denoting $\tau\sigma$ by σ', the right side of (61) reduces to

$$(-q)^{\binom{n}{2}} \cdot \sum_{\sigma'\in S_n} \left(-q^{\frac{1-n}{n}}\right)^{\binom{n}{2}-\ell(\sigma')}$$

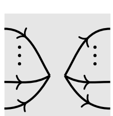


$$= q^{\binom{n}{2}} q^{\frac{1-n}{n}\cdot\binom{n}{2}} \sum_{\sigma'\in S_n} \left(-q^{\frac{1-n}{n}}\right)^{-\ell(\sigma')}$$

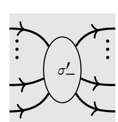


.

and, hence, (61) becomes



$$= (-1)^{\binom{n}{2}} q^D \cdot \sum_{\sigma'\in S_n} \left(-q^{\frac{1-n}{n}}\right)^{\ell(\sigma)}$$



,

where

$$D = -\frac{n+1}{n} \cdot \binom{n}{2} + \binom{n}{2} + \frac{1-n}{n}\binom{n}{2} = -\binom{n}{2}.$$

□

4.10 | Marking automorphisms

Consider a function $\eta : \{1, \dots, n\} \to R^*$ such that

$$\prod_{i=1}^{n} \eta(i) = 1 \text{ and } \eta(i)\eta(\bar{i}) = 1 \text{ for every } i,$$

where the bar denotes the conjugation, $\bar{i} = n + 1 - i$, as before. It is easy to see that for every such η and every marking β in $\mathcal{N}$ there is an R-module automorphism $\phi_{\eta,\beta}$ of $\mathcal{S}_n(M, \mathcal{N})$ sending stated n-webs α to

$$\phi_{\eta,\beta}(\alpha) = \prod_{x \in \alpha \cap \beta} \eta(s(x)) \cdot \alpha$$

where $s(x)$ is the state of the endpoint x of α. We call $\phi_{\eta,\beta}$ a *marking automorphism* of $\mathcal{S}_n(M, \mathcal{N})$.

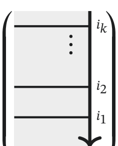
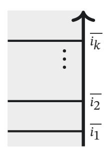
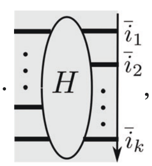
When $\eta_i = g_i = (-1)^{n-1} q^{2i-n-1}$, as in Equation (42), we denote $\phi_{\eta,\beta}$ by g_β.

4.11 | Half-twist automorphisms

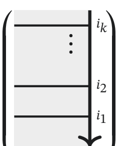
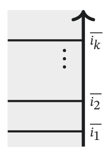
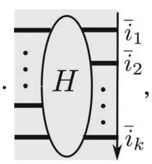
Proposition 4.11. *For any marking β in $\mathcal{N}$ there exist unique R-linear isomorphisms*

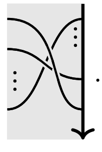
$$\mathrm{htw}_\beta, \widetilde{\mathrm{htw}}_\beta : \mathcal{S}_n(M, \mathcal{N}) \to \mathcal{S}_n(M, \mathcal{N})$$

sending any stated n-web α in $(M, \mathcal{N})$ with k endpoints on β to

htw_β () $= \left(\prod_{j=1}^{k} c_{\bar{i}_j}\right) \cdot$  $= \left(\prod_{j=1}^{k} c_{\bar{i}_j}\right) \cdot$  , (62)

and to

$\widetilde{\mathrm{htw}}_\beta$ () $= \left(\prod_{j=1}^{k} c_{i_j}\right) \cdot$  $= \left(\prod_{j=1}^{k} c_{i_j}\right) \cdot$  , (63)

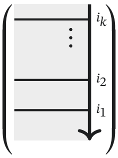
where H is the positive half-twist,  *. (The orientations of the horizontal edges are arbitrary.)*

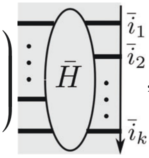
We call htw_β and $\widetilde{\mathrm{htw}}_\beta$ the *half-twist automorphisms*. Note that $\widetilde{\mathrm{htw}}_\beta$ coincides with htw_β composed with g_β of Subsection 4.10, because $c_i = c_{\bar{i}}(-1)^{n-1} q^{2\bar{i}-n-1}$.

Remark 4.12. We will show in Subsection 7.1 that every marking in $\mathcal{N}$ defines a left action of a completion $\widetilde{U^L}$ of the Lusztig integral version U^L of $U_q(sl_n)$. That completion contains the half-twist element X and the charmed element g of Subsection 3.9. The automorphisms htw_β and g_β coincide with the actions of X and g, respectively.

The map htw_β generalizes the inversion along an edge, for the stated Kauffman bracket skein algebras of thickened surfaces in [15]. However, it is the inverse of that map.

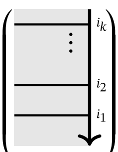
The inverse of htw_β is given by

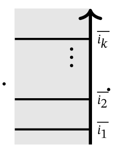
$$\mathrm{htw}_\beta^{-1}\left(\right.$$


$$\left.\right) = \left(\frac{1}{\prod_{j=1}^{k} c_{i_j}}\right)$$


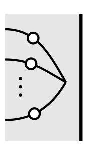
where $\bar{H}$ denotes the negative half-twist, as before.

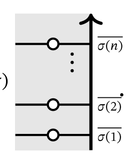
Proof of Proposition 4.11. By abuse of notation, let us first consider a map $\mathrm{htw}_\beta : \mathcal{W}_n(M,\mathcal{N}) \to \mathcal{S}_n(M,\mathcal{N})$ sending any stated n-web α in $(M,\mathcal{N})$ with k endpoints on β to

$$\mathrm{htw}_\beta\left(\right.$$


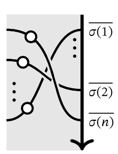
$$\left.\right) = \prod_{j=1}^{k} c_{\overline{i_j}} \cdot$$


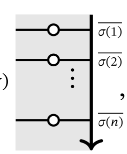
Obviously, htw_β preserves the internal skein relations, (43)–(46). It maps (47) to



$$= a \cdot \left(\prod_{i=1}^{n} c_i\right) \cdot \sum_{\sigma\in S_n} (-q)^{\ell(\sigma)}$$


The right side equals

$$a\left(\prod_{i=1}^{n} c_i\right)\sum_{\sigma\in S_n}(-q)^{\ell(\sigma)}$$


$$= a\left(\prod_{i=1}^{n} c_i\right)\left(-q^{-\frac{n+1}{n}}\right)^{\frac{n(n-1)}{2}}\sum_{\sigma\in S_n}(-q)^{\ell(\sigma)}$$


,

by (52). By (4), $\left(\prod_{i=1}^{n} c_i\right)\cdot\left(-q^{-\frac{n+1}{n}}\right)^{\frac{n(n-1)}{2}} = 1$ and, hence, the expression above coincides with the right side of (47) by the substitution $\sigma \to \sigma'$, where $\sigma'(i) \to \overline{\sigma(\bar{i})}$, for $i = 1, \dots, n$, which does not affect the permutation length. Consequently, htw_β preserves (47).

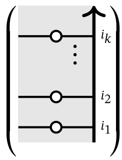
The preservation of the remaining relations, (48)–(50) by htw_β is an immediate consequence of the left boundary relations (56)–(58).

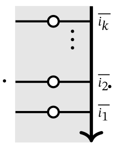
This shows that our map descends indeed to an R-module homomorphism

$$\mathrm{htw}_\beta : \mathcal{S}_n(M,\mathcal{N}) \to \mathcal{S}_n(M,\mathcal{N}).$$

Similarly, it is straightforward to show that there is a well-defined map $\mathrm{htw}'_\beta : \mathcal{S}_n(M,\mathcal{N}) \to \mathcal{S}_n(M,\mathcal{N})$ sending any n-web α in $(M,\mathcal{N})$ with k endpoints on β to

$$\mathrm{htw}'_\beta\left(\quad\right) = \left(\prod_{j=1}^{k} c_{i_j}\right)^{-1} \cdot$$





As htw'_β is an inverse of htw_β, both are isomorphisms. □

4.12 | Essential uniqueness of the skein relations of $\mathcal{S}_n(M,\mathcal{N})$

In the context of our theory, it is natural to ask how arbitrary are the constants a, c_i in Subsection 2.1. For a tuple $\boldsymbol{u} = (u, u_1, \ldots, u_n)$ of $n+1$ invertible elements of R let $\mathcal{S}_n(M,\mathcal{N};\boldsymbol{u})$ be defined the same as $\mathcal{S}_n(M,\mathcal{N})$, with c_i and a replaced, respectively, by $c'_i = c_i(u_i u_{\bar{i}})^{-1}$ and $a' = a(\prod_{i=1}^n u_i)/u$, and with the right side of (46) multiplied by u^2. We denote the set of n-valent vertices of α by $V_n(\alpha)$. Then it is easy to see that the map

$$\alpha \to \alpha\, u^{|V_n(\alpha)|} \prod_{x \in \partial\alpha} u_{s(x)},$$

defined on stated n-webs, extends to an R-linear isomorphism from $\mathcal{S}_n(M,\mathcal{N})$ to $\mathcal{S}_n(M,\mathcal{N};\boldsymbol{u})$.

One can show that the new stated skein module $\mathcal{S}_n(M,\mathcal{N};\boldsymbol{u})$ satisfies the splitting homomorphism if and only if the following holds:

$$u_i = \pm 1, \prod_{i=1}^{n} u_i = 1\ u_i u_{\bar{i}} = 1, \text{ for every } i.$$

Furthermore, all properties of $\mathcal{S}_n(M,\mathcal{N})$ formulated so far have their version for $\mathcal{S}_n(M,\mathcal{N};\boldsymbol{u})$.

5 | STATED SL(n)-SKEIN ALGEBRAS OF SURFACES

The theory of stated SL(n)-skein modules is particularly rich for thickened surfaces $M = \Sigma \times (-1,1)$. Note that any finite set $B \subset \partial\Sigma$ defines markings $\mathcal{N} = B \times (-1,1)$ for which $\mathcal{S}_n(M,\mathcal{N})$ is an R-algebra with the product of webs $\alpha_1 \cdot \alpha_2$ given by stacking α_1 on top of α_2. It is convenient, however, to represent unmarked boundary components of Σ by punctures and to separate points of B by ideal boundary points. That leads to the notion of a punctured bordered surface, considered, for example, in [15, 41] already. In particular, a punctured bordered surface encapsulates information about the points B in it.

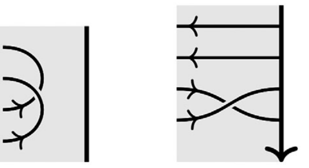


FIGURE 10 Eliminating crossings.

5.1 | Punctured bordered surface

A *punctured bordered surface* (a *pb surface* for short) Σ is an oriented surface with possibly empty boundary $\partial\Sigma$ such that each connected component of $\partial\Sigma$ is an open interval. These components are called *boundary edges*.

For simplicity, we will assume that Σ is of finite type in the sense that $\Sigma = \bar{\Sigma} \setminus \mathcal{P}$, where $\bar{\Sigma}$ is a compact oriented surface and $\mathcal{P} \subset \bar{\Sigma}$ is a finite set, called the *ideal points* of Σ. Note that each connected component of $\partial\bar{\Sigma}$ meets $\mathcal{P}$. (However, some of the points of $\mathcal{P}$ may be in the interior of Σ.)

An *ideal arc* in Σ is the image of a proper embedding $c : (0,1) \hookrightarrow \Sigma$. This means c can be extended to an immersion $\bar{c} : [0,1] \to \Sigma$ such that $\bar{c}(0), \bar{c}(1) \in \mathcal{P}$. An ideal arc is *trivial* if it bounds a disk in Σ.

In each boundary edge e choose a point b_e. Let $\mathcal{S}_n(\Sigma) = \mathcal{S}_n(M, \mathcal{N})$, where $M = \Sigma \times (-1,1)$ and $\mathcal{N}$ is the union of all $b_e \times (-1,1)$, each having the natural orientation of the interval $(-1,1)$. As up to a canonical isomorphism, $\mathcal{S}_n(\Sigma)$ does not depend on the specific choice of the points b_e, we do not specify them in our notation. An n-web in $(M, \mathcal{N})$ is simply called an n-web over Σ.

For stated n-webs α and β over Σ let their product $\alpha\beta \in \mathcal{S}_n(\Sigma)$ be the result of stacking α above β. This product turns $\mathcal{S}_n(\Sigma)$ into an R-algebra.

According to the graphical convention of Subsection 4.1, an n-web over Σ is presented by its diagram on Σ, which is the projection of α onto Σ with the over/undercrossing information at every double point. Before projecting, we use height-preserving deformation near the markings $b_e \times (-1,1)$ to make the projections of endpoints of α distinct. As before, the height order at endpoints of the diagram on each boundary edge is part of the diagram.

The orientation of Σ induces an orientation on its boundary. When part of Σ is drawn on a page of paper, which is identified with the standard XY-plane, the orientation of $\partial\Sigma$ is the counterclockwise direction. A diagram where the height order on a boundary edge e is given by the orientation of e induced from that of Σ (respectively, the opposite orientation) is called *positively (respectively, negatively) ordered on e*.

Given two edges e_1, e_2 of a pb surface Σ, not necessarily connected, the gluing $\Sigma/(e_1 = e_2)$ is the result of identifying e_1 with e_2 via a diffeomorphism $e_1 \to e_2$ such that the resulting surface has an orientation induced from that of Σ. Such a surface is defined uniquely up to strict isomorphisms.

A pb surface Σ is *essentially bordered* if every connected component of it has nonempty boundary.

Lemma 5.1. *If a pb surface Σ is essentially bordered then $\mathcal{S}_n(\Sigma)$ is spanned by stated n-web diagrams without any of: sinks, sources, crossings, trivial loops, and trivial arcs.*

Proof. Crossings can be eliminated by bringing them to near a boundary edge as in Figure 10 (left), then expressing them as linear combinations of webs of the form Figure 10 (right) by Relation (49), and finally eliminating them by Relation (50).

Sinks and sources, trivial loops, and trivial arcs can be eliminated by Relations (47), (45), (48), respectively. □

5.2 | Splitting homomorphism for surfaces

Let c be an ideal arc in the interior of a pb surface Σ. The splitting $\mathrm{cut}_c(\Sigma)$ is a pb surface having two boundary edges c_1, c_2 such that $\Sigma = \mathrm{cut}_c(\Sigma)/(c_1 = c_2)$. Let $\mathrm{pr} : \mathrm{cut}_c(\Sigma) \to \Sigma$ be the natural projection map. An n-web diagram D is *c-transverse* if n-valent vertices of D are not in c and D is transverse to c. Assume that D is a stated c-transverse n-web diagram. Let h be a linear order on the set $D \cap c$. For a map $s : D \cap c \to \{1, \dots, n\}$ let $D(h, s)$ be the stated n-web diagram over $\mathrm{cut}_c(\Sigma)$ which is $\mathrm{pr}^{-1}(D)$ with the height order on $c_1 \cup c_2$ induced (via pr) from h, and the states on $c_1 \cup c_2$ induced (via pr) from s. The Splitting Theorem (Theorem 4.5) for Σ becomes

Theorem 5.2. *Let c be an interior ideal arc of a pb surface Σ. There is a unique R-linear map $\Theta_c : \mathcal{S}_n(\Sigma) \to \mathcal{S}_n(\mathrm{cut}_c(\Sigma))$ such that if D is a diagram of a stated n-web α over Σ which is c-transverse and h is any linear order on $D \cap c$, then*

$$\Theta_c(\alpha) = \sum_{s : D\cap c \to \{1,\dots,n\}} D(h, s).$$

The map Θ_c is an R-algebra homomorphism.

Proof. The set $c \times (-1, 1)$ is not a closed disk but we can still use Theorem 4.5, see Remark 4.7. More precisely, let us enlarge the ideal points of Σ to open disks in $\bar{\Sigma}$, and embed Σ into $\mathbb{R}^3$. Let $\bar{M}$ be the topological closure of $\Sigma \times (-1, 1)$ in $\mathbb{R}^3$.

Then $(M, \mathcal{N})$ is pseudo-isomorphic to $(\bar{M}, \mathcal{N})$. Applying Theorem 4.5 to split $(\bar{M}, \mathcal{N})$ along the topological closure of $c \times (-1, 1)$ in $\bar{M}$, we get the R-linear map $\Theta_c : \mathcal{S}_n(\Sigma) \to \mathcal{S}_n(\mathrm{cut}_c(\Sigma))$ defined in the statement.

From the definition, it is clear that Θ_c is an algebra homomorphism. □

5.3 | Reflection anti-involution

For any pb surface Σ we have an involution $\tau : \Sigma \times (-1, 1) \to \Sigma \times (-1, 1)$, $(x, u) \to (x, -u)$ that maps webs α to $\tau(\alpha)$ (with their framing transformed by the tangent map $\tau_* : T(\Sigma \times (-1, 1)) \to T(\Sigma \times (-1, 1))$. Given a stated web α in $\Sigma \times (-1, 1)$ let $\bar{\alpha}$ be $\tau(\alpha)$ with its framing reversed, $f \to -f$.

Proposition 5.3. *For any commutative ring P and $R = P[v^{\pm 1}]$ and for any pb surface Σ, there is a unique P-algebra anti-involution $\bar{\cdot} : \mathcal{S}_n(\Sigma) \to \mathcal{S}_n(\Sigma)$ such that $\bar{v} = v^{-1}$ and $\bar{\alpha}$ for stated webs α is defined as above.*

We call $\bar{\cdot}$ the *mirror reflection* map.

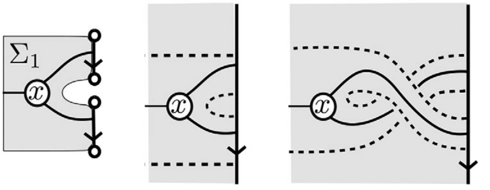


FIGURE 11 Left: A surface Σ_1 with a skein x. Middle: The image of x under the negative height order embedding of Σ_1 into Σ_2. Right: The image of x under the positive height order embedding of Σ_2.

Proof. Let $(M, \mathcal{N})$ be defined as in Subsection 5.1 and $\overline{(M, \mathcal{N})}$ be defined as in Subsection 4.9. Then the mirror reflection map is the composition of the orientation reversion $\varkappa_{(M,\mathcal{N})}$ with τ and hence, a P-linear isomorphism sending v to $\bar{v} = v^{-1}$ by Theorem 4.9. It is easy to see that

$$\bar{\bar{\alpha}} = \alpha \text{ and } \overline{\alpha \cdot \alpha'} = \bar{\alpha}' \cdot \bar{\alpha}$$

for stated webs α, α'. □

If α is a stated n-web diagram over Σ then $\bar{\alpha}$ is obtained from α by switching all the crossings and reversing the height order on each boundary edge.

5.4 | Embedding of punctured bordered surfaces

Consider a proper embedding of a pb surface Σ_1 into Σ_2. Note that it can map several boundary edges of Σ_1 into one boundary edge of Σ_2. For a boundary edge b of Σ_2 a linear order on the set of boundary edges of Σ_1 mapped into b is called a *b-order*. Fixing it for each b defines a *height ordered embedding* $f : \Sigma_1 \hookrightarrow \Sigma_2$, inducing an R-module homomorphism $f_* : \mathcal{S}_n(\Sigma_1) \to \mathcal{S}_n(\Sigma_2)$, where $f_*(\alpha)$ is α with its height order on each b determined by the b-order in addition to the height order of $\partial\alpha$. If the b-order is given by the positive (respectively, negative) orientation of b, we say f_* is positively (respectively, negatively) induced from f, see Figure 11.

Note that f_* is an R-algebra homomorphism if and only if each boundary edge of Σ_2 contains the image of at most one boundary edge of Σ_1.

6 | SKEIN ALGEBRAS OF BIGON AND QUANTUM GROUPS

In this section, we prove that the stated skein algebra $\mathcal{S}_n(\mathfrak{B})$ of the bigon, $\mathfrak{B}$, has a natural structure of a co-braided Hopf algebra which is naturally isomorphic to the quantized coordinate algebra $\mathcal{O}_q(SL(n))$. We also show that the stated skein algebra of the monogon, $\mathfrak{M}$, is the ground ring R. Finally, we prove Theorem 3.11 that identifies the kernel of the map Γ.

6.1 | Monogon

Let $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant 1\}$ be the standard disk with the counterclockwise orientation. The *monogon* $\mathfrak{M}$ is the pb surface obtained by removing the bottom point $(0, -1)$ from D.

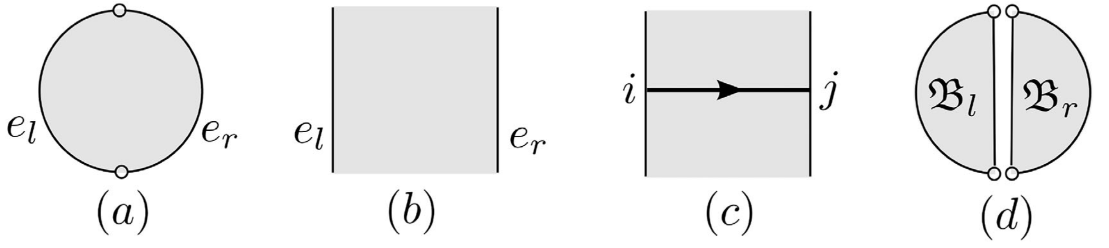


FIGURE 12 (a) and (b) Bigon $\mathfrak{B}$. (c) Stated arc a^i_j. (d) Splitting of $\mathfrak{B}$.

Theorem 6.1 (Proof in Subsection 6.6). *The stated skein algebra $\mathcal{S}_n(\mathfrak{M})$ of the monogon $\mathfrak{M}$ is isomorphic to the ground ring R via the map $\mu : R \to \mathcal{S}_n(\mathfrak{M})$ given by $\mu(r) = r \cdot \emptyset$.*

6.2 | Bigon

The *bigon* $\mathfrak{B}$ is the punctured bordered surface obtained from the standard disk D by removing the top and the bottom points, $(0,1), (0,-1)$. The two edges of $\mathfrak{B}$ are denoted by e_l and e_r as in Figure 12. Up to isotopy there are two orientation preserving auto-diffeomorphisms of $\mathfrak{B}$, the identity and the rotation rot by 180° about the center of $\mathfrak{B}$. The rotation rot induces an algebra involution $\mathrm{rot}_* : \mathcal{S}_n(\mathfrak{B}) \to \mathcal{S}_n(\mathfrak{B})$.

A *directed bigon* is an oriented surface diffeomorphic to $\mathfrak{B}$, with one ideal vertex designated as the *bottom vertex*. Equivalently the direction of a bigon can be specified by choosing the left (or right) edge. We often depict $\mathfrak{B}$ as the square $[-1,1] \times (-1,1)$, as in Figure 12b. Let a_{ij} be the stated n-web over $\mathfrak{B}$ given in Figure 12c, and let $\overleftarrow{a}^i_j$ be a^i_j with the reverse orientation.

We will now define a Hopf algebra structure on $\mathcal{S}_n(\mathfrak{B})$ geometrically. By splitting $\mathfrak{B}$ along an interior ideal arc connecting its two ideal vertices we get two directed bigons $\mathfrak{B}_l$ and $\mathfrak{B}_r$, for each the bottom vertex comes from the one of $\mathfrak{B}$. The splitting homomorphism becomes an algebra R-homomorphism

$$\Delta : \mathcal{S}_n(\mathfrak{B}) \to \mathcal{S}_n(\mathfrak{B}) \otimes \mathcal{S}_n(\mathfrak{B}).$$

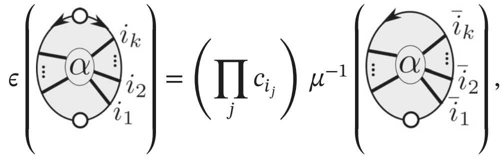
The commutativity of the splitting homomorphisms at disjoint ideal arcs shows that Δ is a coproduct. For example, from the definition one has

$$\Delta(a^i_j) = \sum_k a^i_k \otimes a^k_j, \quad \Delta(\overleftarrow{a}^i_j) = \sum_k \overleftarrow{a}^i_k \otimes \overleftarrow{a}^k_j. \tag{64}$$

The natural embedding $\iota : \mathfrak{B} \to \mathfrak{M}$ (filling in the top ideal point) induces an R-linear map $\iota_* : \mathcal{S}_n(\mathfrak{B}) \to \mathcal{S}_n(\mathfrak{M})$, where the left edge e_l is higher than the right edge e_r. Let $\epsilon : \mathcal{S}_n(\mathfrak{B}) \to R$ be the composition

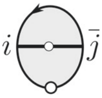
$$\epsilon : \mathcal{S}_n(\mathfrak{B}) \xrightarrow{\widetilde{\mathrm{htw}}_{e_r}} \mathcal{S}_n(\mathfrak{B}) \xrightarrow{\iota_*} \mathcal{S}_n(\mathfrak{M}) \simeq R, \tag{65}$$

where $\widetilde{\mathrm{htw}}_{e_r}$ is a half-twist automorphism of Subsection 4.11. Explicitly, for a stated diagram α,



where $\mu : R \to \mathcal{S}_n(\mathfrak{M})$ is the isomorphism of Theorem 6.1. For example,

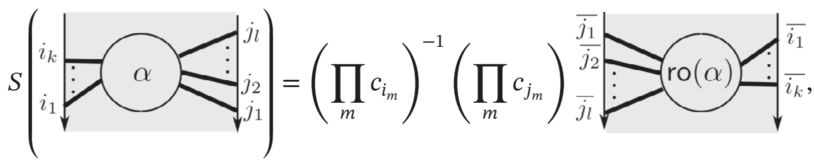
$$\epsilon(a^i_j) = \epsilon(\overleftarrow{a}^i_j) = c_j \quad (66)$$



Let the R-module automorphism $S : \mathcal{S}_n(\mathfrak{B}) \to \mathcal{S}_n(\mathfrak{B})$ be the composition

$$S = \mathrm{rot}_* \circ \mathrm{htw}_{e_l}^{-1} \circ \widetilde{\mathrm{htw}}_{e_r}.$$

Explicitly, for a stated diagram α,



where $\mathrm{ro}(\alpha)$ is the result of rotating the planar diagram α about the center of the square by 180°. (Here, we use the fact that $c_{\bar{j}} = (-1)^{n-1} q^{2j-n-1} c_j$.)

For example, we have

$$S(a^i_j) = (-q)^{i-j}\, \overleftarrow{a}^{\bar{j}}_{\bar{i}}. \quad (68)$$

Remark 6.2. Note that stated n-tangle diagrams can be identified with the diagrams of stated n-webs in $\mathfrak{B}$ with the downward ascending height order on $\partial_l \mathfrak{B}$ and $\partial_r \mathfrak{B}$, that is, the height order is positive on the left edge but negative on the right edge.

That leads to a natural identification of stated n-tangles with stated n-webs in the thickened bigon, for which the basic internal, right, and left annihilators of Subsection 3.6 correspond to defining skein relations (43)–(50) of $\mathcal{S}_n(\mathfrak{B})$.

The positive order on the left edge explains why there is a twist in the definition of the operation hd that turns right annihilators to basic annihilators of Subsection 3.7.

By this identification S corresponds to the dual operation $\alpha \to \alpha^*$ of Subsection 3.5.

Recall that $\mathfrak{B}$ is the standard bigon.

Theorem 6.3 (Proof in Subsection 6.8).

(a) *The algebra $\mathcal{S}_n(\mathfrak{B})$ has the structure of a Hopf algebra over R with the coproduct Δ, the counit ϵ, and the antipode S.*

(b) *The map* $\Psi(u^i_j) = a^i_j$ *extends to a unique Hopf algebra isomorphism*

$$\Psi : \mathcal{O}_q(sl_n; R) \xrightarrow{\cong} \mathcal{S}_n(\mathfrak{B}).$$

Here $\mathcal{O}_q(sl_n; R) := \mathcal{O}_q(SL(n)) \otimes_{\mathbb{Z}[v^{\pm 1}]} R$ is the algebra $\mathcal{O}_q(SL(n))$ of Subsection 2.5 with the ground ring R.

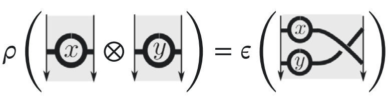
6.3 | Cobraided structure

The Hopf algebra $\mathcal{O}_q(sl_n; R)$ is *dual quasitriangular* (see [52, section 2.2], [35, section 10], [20, section 10.3]), also known as *cobraided* (see, e.g., [33, section VIII.5]). This means it has an R-form (i.e., a co-R-matrix), which is a bilinear form

$$\rho : \mathcal{O}_q(sl_n; R) \otimes \mathcal{O}_q(sl_n; R) \to R$$

satisfying certain properties, with the help of which one can make the category of $\mathcal{O}_q(sl_n; R)$-modules a braided category. The following generalizes [15, Theorem 3.5] from $n = 2$ to all n:

Theorem 6.4 (Proof in Subsection 6.9). *Under the identification of* $\mathcal{S}_n(\mathcal{B})$ *and* $\mathcal{O}_q(sl_n; R)$ *via the isomorphism* Ψ*, the* R*-form* ρ *has the following geometric description*



(69)

for any $x, y \in \mathcal{O}_q(sl_n; R)$.

6.4 | Ground ring

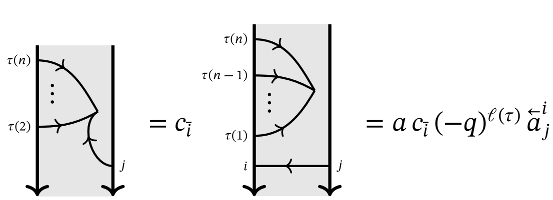
The remainder of this section is devoted to proving Theorems 6.1, 6.3, 6.4, and 3.11. As it is enough to do it for $R = \mathbb{Z}[v^{\pm 1}]$, we will assume this ground ring for the rest of this section.

6.5 | Algebra homomorphism $\mathcal{S}_n(\mathfrak{B}) \to \mathcal{O}_q(SL(n))$

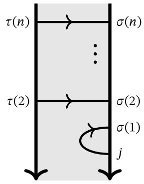
Lemma 6.5. *The webs* a^i_j *for* $i, j \in \{1, \dots, n\}$*, generate* $\mathcal{S}_n(\mathfrak{B})$ *as an* R*-algebra.*

Proof. By Lemma 5.1, $\mathcal{S}_n(\mathfrak{B})$ is generated by a^i_j and $\overleftarrow{a}^i_j$ for $i, j = 1, \dots, n$.

Fix i, j and choose a permutation $\tau \in S_n$ with $\tau(1) = \bar{i}$. By (57) and (54),



On the other hand, Equation (47) expresses the left side in terms of a_l^k's:

$$a \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)}$$  $$= a \cdot c_{\bar{j}} \cdot \sum_{\sigma \in S_n \,:\, \sigma(1)=\bar{j}} (-q)^{\ell(\sigma)} a_{\sigma(2)}^{\tau(2)} \cdot \dots \cdot a_{\sigma(n)}^{\tau(n)}.$$

As $c_{\bar{j}} c_{\bar{i}}^{-1} = (-q)^{j-i}$, by comparing the two equalities, we have

$$\overleftarrow{a}_j^i = (-q)^{j-i} \sum_{\sigma \in S_n : \sigma(1)=\bar{j}} (-q)^{\ell(\sigma)-\ell(\tau)} a_{\sigma(2)}^{\tau(2)} \cdot \dots \cdot a_{\sigma(n)}^{\tau(n)} \tag{70}$$

which shows $\overleftarrow{a}_j^i$ is in the subalgebra generated by a_j^i. □

As the first step toward proving the theorems of this section, we will construct an R-algebra homomorphism $\Phi : \mathscr{S}_n(\mathfrak{B}) \to \mathcal{O}_q(SL(n)) \subset U_q(sl_n)^*$.

Let $\alpha \to T(\alpha)$ be the bijection of Remark 6.2 between the set of isotopy classes of stated n-webs over $\mathfrak{B}$ and stated n-tangles. It extends to an R-algebra isomorphism $T : R\mathcal{W}_n(\mathfrak{B}) \to R\mathcal{T}$.

By Remark 6.2, the composition $\Gamma \circ T : R\mathcal{W}_n(\mathfrak{B}) \to U_q(sl_n)^*$ of T with Γ defined in Subsection 3.4 preserves all the defining relations of $\mathscr{S}_n(\mathfrak{B})$. Hence, $\Gamma \circ T$ descends to an R-linear homomorphism $\Phi : \mathscr{S}_n(\mathfrak{B}) \to U_q(sl_n)^*$, which by Proposition 3.4 is an algebra homomorphism. From Equation (33) and Proposition 2.1, we have

$$\Phi(\emptyset) = \epsilon, \quad \text{counit of } U_q(sl_n), \tag{71}$$

$$\Phi(a_j^i) = u_j^i, \quad \text{generators of } \mathcal{O}_q(SL(n)), \tag{72}$$

for $i, j = 1, \dots, n$. As u_j^i generate $\mathcal{O}_q(SL(n))$, Lemma 6.5 and Equation (72) show that

$$\Phi(\mathscr{S}_n(\mathfrak{B})) = \mathcal{O}_q(SL(n)).$$

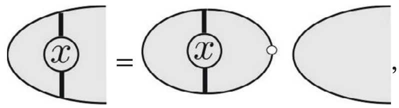
6.6 | Proof of Theorem 6.1

By Lemma 5.1, $\mathscr{S}_n(\mathfrak{M})$ is spanned by the empty n-web. Therefore, the map $\mu : R \to \mathscr{S}_n(\mathfrak{M})$ given by $\mu(r) = r \cdot \emptyset$ is surjective. By removing the left edge of $\mathfrak{B}$, we get a monogon. This gives an embedding $\iota : \mathfrak{M} \hookrightarrow \mathfrak{B}$, which induces an R-algebra homomorphism $\iota_* : \mathscr{S}_n(\mathfrak{M}) \to \mathscr{S}_n(\mathfrak{B})$. By Equation (71), the composition

$$R \xrightarrow{\mu} \mathscr{S}_n(\mathfrak{M}) \xrightarrow{\iota_*} \mathscr{S}_n(\mathfrak{B}) \xrightarrow{\Phi} U_q(sl_n)^*$$

is an R-linear map sending 1 to ϵ. As the free R-module generated by ϵ is a submodule of $U_q(sl_n)^*$, the composition is injective. Thus, μ is injective, and hence, bijective.

Note that we have

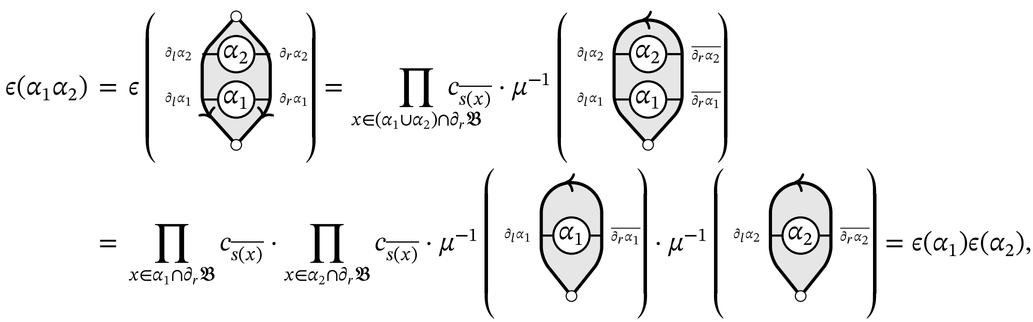


(73)

as the skein x can be brought to a scalar with the same skein relations on the left as on the right.

6.7 | Proof that $\mathcal{S}_n(\mathfrak{B})$ is a Hopf algebra

We already noted that Δ is a coproduct. As μ is an isomorphism, ϵ is well-defined by Equation (65). A version of the argument of [15] shows that ϵ is an R-algebra homomorphism as well: for any webs α_1, α_2,



where $\partial_l\alpha, \partial_r\alpha$ denote the sequences of left and right side states of α and the third identity follows from (73).

We also have

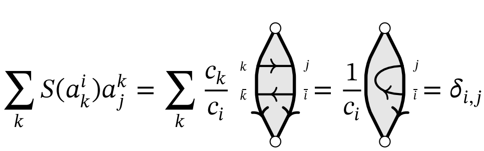
$$(\epsilon \otimes id)\circ\Delta(x) = x = (id \otimes \epsilon)\circ\Delta(x)$$

for all $x \in \mathcal{S}_n(\mathfrak{B})$. Indeed, as Δ and ϵ are algebra homomorphisms, it is enough to verify it for the generators $x = a^i_j$ and that follows from the explicit values of $\Delta(a^i_j)$ and of $\epsilon(a^i_j)$ given by Equations (64) and (66).

Consequently, $(\mathcal{S}_n(\mathfrak{B}), \Delta, \epsilon)$ is an R-bialgebra. By (67), S is R-algebra anti-isomorphism. Therefore, to prove that S is an antipode for $(\mathcal{S}_n(\mathfrak{B}), \Delta, \epsilon)$ it remains to to be shown that

$$\sum S(x_{(1)})x_{(2)} = \varepsilon(x) = \sum x_{(1)}S(x_{(2)}), \quad \text{where } \Delta x = \sum x_{(1)} \otimes x_{(2)}, \tag{74}$$

As before, it suffices to be verified for the generators a^i_j, $i, j = 1, \dots, n$, only. As $\Delta(a^i_j) = \sum_k a^i_k \otimes a^k_j$, the left side of (74) reduces to:



by Equations (68), (57), and (48). The proof of the right identity of (74) is analogous.

This completes the proof that $(\mathcal{S}_n(\mathfrak{B}), \Delta, \epsilon, S)$ is a Hopf algebra.

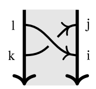
6.8 | Proof of Theorem 6.3

Proposition 6.6. *Suppose $\boldsymbol{\alpha} = (\boldsymbol{i}, \alpha, \boldsymbol{j})$ is a stated n-web on $\mathfrak{B}$. Then $\epsilon(\boldsymbol{\alpha})$ is equal to the matrix element of the corresponding modified Reshetikhin–Turaev operator:*

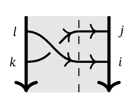
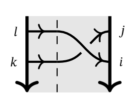
$$\epsilon(\boldsymbol{\alpha}) = \langle \boldsymbol{i} \mid \mathsf{RT}(T(\alpha)) \mid \boldsymbol{j} \rangle. \tag{75}$$

Proof. The map $\mathcal{S}_n(\mathfrak{B}) \to \mathbb{Q}(v)$ given by $\boldsymbol{\alpha} \to \langle \boldsymbol{i} \mid \mathsf{RT}(T(\alpha)) \mid \boldsymbol{j} \rangle$ is clearly an R-algebra homomorphism whose values on a^i_j coincide with those of ϵ, by Equation (71). □

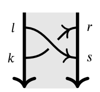
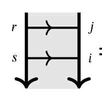
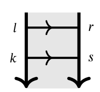
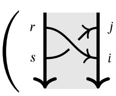
In particular, we have

$$\epsilon\Big($$  $$\Big) = \hat{\mathcal{R}}^{kl}_{ij} = \mathcal{R}^{lk}_{ij}. \tag{76}$$

Let us show that $\mathbf{a} = (a^i_j)$ is a quantum matrix. By isotopy,

 $=$ 

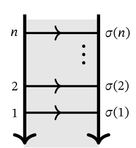
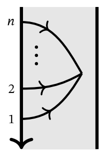
Split along the dashed lines (coproduct), then apply the counit

$\epsilon\Big($  $\Big) \cdot$  $=$  $\cdot\, \epsilon\Big($  $\Big),$

Using the value of $\mathcal{R}$ in Equation (76), the above identity becomes

$$\mathcal{R}(\mathbf{a} \otimes \mathbf{a}) = (\mathbf{a} \otimes \mathbf{a})\mathcal{R},$$

which is the defining relation Equation (11) of a quantum matrix. Besides

$\det_q(\mathbf{a}) = \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)}$  $= a^{-1}$  $= 1$

where the second equality is from Equation (47) and the third is from Equation (54). Hence, the algebra map $\Psi : \mathcal{O}_q(SL(n)) \to \mathcal{S}_n(\mathfrak{B})$ given by $\Psi(u^i_j) = a^i_j$ is well-defined.

As $\Phi \circ \Psi(a^i_j) = a^i_j$, we have $\Phi \circ \Psi = \mathrm{id}$. This shows Φ is injective, and hence $\Phi : \mathcal{S}_n(\mathfrak{B}) \to \mathcal{O}_q(SL(n))$ is an algebra isomorphism. By checking the values of Δ, ϵ, and S on the generators a^i_j we see that Φ is a Hopf algebra homomorphism. This completes the proof of Theorem 6.3.

6.9 | Proof of Theorem 6.4

Proof. The R-form satisfies the following equalities (stated with Sweedler's notation for the coproduct):

$$\rho(xy \otimes z) = \sum \rho(x \otimes z')\rho(y \otimes z'') \tag{77}$$

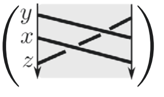
$$\rho(x \otimes yz) = \sum \rho(x' \otimes z)\rho(x'' \otimes y). \tag{78}$$

For $\mathcal{O}_q(SL(n))$, the values of ρ are given by (see [52, section 2.2] or [35, section 10.1.2]):

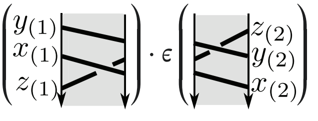
$$\rho(u^i_j \otimes 1) = \rho(1 \otimes u^i_j) = \delta_{ij},\ \rho(u^i_j \otimes u^k_l) = R^{ik}_{jl}, \tag{79}$$

which, together with Relations (77) and (78), totally determine ρ.

The first part of the proof follows that of [15]. Let ρ' be the map defined by the right side of (69); we will show that $\rho' = \rho$. It is enough to show that ρ' satisfies (77), (78), and the initial values (79), all with ρ replaced by ρ'. We have, where a line labeled by, say x, stands for the stated n-tangle diagram x,

$\rho'(xy \otimes z) = \epsilon$ 

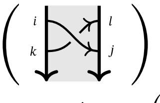
Splitting the bigon by the vertical middle ideal arc, then using $\epsilon(u) = \sum \epsilon(u_{(1)})\epsilon(u_{(2)})$),

$\rho'(xy \otimes z) = \sum \epsilon$ 

$$= \sum \rho'(x_{(1)} \otimes z_{(1)})\rho'(y_{(2)} \otimes z_{(2)})\epsilon(x_{(2)})\epsilon(y_{(1)}) = \sum \rho'(x \otimes z_{(1)})\rho'(y \otimes z_{(2)}).$$

This proves (77) for ρ'. The proof of (78) is similar.

Under the isomorphism, u^i_j becomes a^i_j. Using Equation (76), we have

$\rho'(a^i_j \otimes a^k_l) = \epsilon$  $= \hat{\mathcal{R}}^{ki}_{jl} = \mathcal{R}^{ik}_{jl},$

$$\rho'(a^i_j \otimes 1) = \rho'(1 \otimes a^i_j) = \epsilon\left(a^i_j\right) = \delta_{i,j}$$

which proves (79), completing the proof of the theorem. □

6.10 | Proof of Theorem 3.11

Proof. The proof of Theorem 6.3 shows that the kernel of the map $\Gamma \circ T : R\mathcal{W}_n(\mathfrak{B}) \to U_q(sl_n)^*$ is generated by internal relations, boundary relations on the right side, and boundary relations on the left side. Transferring back to $\Gamma : R\mathcal{T} \to U_q(sl_n)^*$ via the isomorphism $T : R\mathcal{W}_n(\mathfrak{B}) \to$

$R\mathcal{T}$ we conclude that the kernel of Γ is generated by the basic internal annihilators, basic right annihilators, and basic left annihilators. □

6.11 | Additional facts

(1) Formula (70), after a simple manipulation, has the form

$$\overleftarrow{a}^i_j = \det_q M^{\bar{i}}_{\bar{j}}(\mathbf{a}),$$

where $M^{\bar{i}}_{\bar{j}}(\mathbf{a})$ is the submatrix of $\mathbf{a}$ obtained my removing the $\bar{i}$-row and $\bar{j}$-column. Alternatively, this formula is a consequence of the (geometric) antipode formula, $S(a^i_j) = (-q)^{i-j}\overleftarrow{a}^{\bar{j}}_{\bar{i}}$, combined with the (algebraic) antipode formula, (12), in $\mathcal{O}_q(SL(n))$.

(2) The antipode, given by (67), is equivalent to the dual map of Subsection 3.5 via T:

$$T(S(x)) = (T(x))^*.$$

(3) The algebra involution $\mathrm{rot}_* : \mathcal{O}_q(SL(n)) \to \mathcal{O}_q(SL(n))$, induced from the rotation by 180°, is a coalgebra anti-homomorphism. It is easy to show that its dual restrict to an algebra anti-involution $\mathrm{rot}^* : U_q(sl_n) \to U_q(sl_n)$. One can check that rot^* is equal the anti-involution ρ introduced by Lusztig [50, chapter 19] in his study of canonical bases of quantized enveloping algebras.

7 | COACTION OF $\mathcal{O}_q(SL(n))$ ON STATED SKEIN MODULES

Similarly to the case $n = 2$ considered in [11, 15], we are going to show that every marking β of a marked 3-manifold $(M, \mathcal{N})$ defines a right coaction of $\mathcal{O}_q(SL(n))$ on $\mathcal{S}_n(M, \mathcal{N})$. Dually, it defines a left-$\widetilde{U^L}$ module structure on $\mathcal{S}_n(M, \mathcal{N})$, where $\widetilde{U^L}$ is a completion of the Lusztig integral version U^L of the quantum group $U_q(sl_n)$. We will observe that the actions of the charmed and the half-ribbon elements on $\mathcal{S}_n(M, \mathcal{N})$ coincide with the marking automorphism g_β and the half-twist automorphism htw_β of Subsections 4.10–4.11, respectively.

The above $\mathcal{O}_q(SL(n))$-coaction will be very important for the further development of the theory of stated skein algebras in the remainder of this paper. For simplicity, we assume $R = \mathbb{Z}[v^{\pm 1}]$ in this section.

7.1 | Module and Co-module structures

Suppose Σ is a punctured bordered surface and b is a boundary edge. Let c be an interior ideal arc isotopic to b. This means that b and c cobound a bigon. By splitting Σ along c, we get a surface Σ' and a directed bigon with b considered its right edge. As Σ' is diffeomorphic to Σ via a unique up to isotopy diffeomorphism, we identify $\mathcal{S}_n(\Sigma') = \mathcal{S}_n(\Sigma)$. The splitting homomorphism gives an algebra homomorphism

$$\Delta_b : \mathcal{S}_n(\Sigma) \to \mathcal{S}_n(\Sigma) \otimes \mathcal{O}_q(SL(n)). \tag{80}$$

The commutativity of splitting maps and the values of ϵ on horizontal stated arcs given by (71) imply that Δ_b is a right coaction of $\mathcal{O}_q(SL(n))$ on $\mathcal{S}_n(\Sigma)$. Moreover, the right coactions at different boundary edges commute. As Δ_b in Equation (80) is an algebra homomorphism, $\mathcal{S}_n(\Sigma)$ is a right *comodule-algebra* over $\mathcal{O}_q(SL(n))$, as defined in [33, section III.7].

If we split off a bigon (as above) and identify b with its left edge, we get a left $\mathcal{O}_q(SL(n))$-comodule structure on $\mathcal{S}_n(\Sigma)$.

The above construction of the $\mathcal{O}_q(SL(n))$-coactions on $\mathcal{S}_n(\Sigma)$ generalizes to $\mathcal{O}_q(SL(n))$-coactions on stated modules of marked 3-manifolds. Given a marking β of a marked 3-manifold $(M, \mathcal{N})$, consider its closed disk neighborhood D in ∂M, disjoint from the other markings of $(M, \mathcal{N})$. By pushing the interior of D inside M we get a new disk D' that is properly embedded in M. Splitting $(M, \mathcal{N})$ along D', we get a new marked 3-manifold $(M', \mathcal{N}')$ isomorphic to $(M, \mathcal{N})$, and another marked 3-manifold bounded by D and D'. The latter, after removing the common boundary of D and D', is isomorphic to the thickening of the bigon, with β considered its right face marking, as depicted in Figure 6a. Hence, this construction yields an R-linear splitting map

$$\Delta_\beta : \mathcal{S}_n(M, \mathcal{N}) \to \mathcal{S}_n(M, \mathcal{N}) \otimes \mathcal{O}_q(SL(n)).$$

As in the surface case, this is a right coaction of $\mathcal{O}_q(SL(n))$ on $\mathcal{S}_n(M, \mathcal{N})$, and the right coactions at different markings commute.

The completion $\widetilde{U_q(sl_n)}$ of $U_q(sl_n)$ of [61], see Subsection 3.9, has its integral version, $\widetilde{U^L}$, which contains the half-twist element X of Subsection 3.9, see [39, Comment 3.7]. Equivalently, this is a completion of the Lusztig integral version U^L of $U_q(sl_n)$ [48, section 1.3]. The Hopf algebras $\widetilde{U^L}$ and $\mathcal{O}_q(SL(n))$ are in Hopf duality over $\mathbb{Z}[v^{\pm 1}]$, which turns any right $\mathcal{O}_q(SL(n))$-comodule W to a left $\widetilde{U^L}$-module as follows: for $u \in \widetilde{U^L}$ and $x \in W$,

$$u * x = \sum x_{(1)} \langle f_{(2)}, u \rangle, \quad \text{where } \Delta(x) = \sum x_{(1)} \otimes f_{(2)}$$

is the $\mathcal{O}_q(SL(n))$-coaction map.

To make explicit the left action of $\widetilde{U^L}$ on $\mathcal{S}_n(M, \mathcal{N})$ coming to the right coaction Δ_β we extend the states of an n-web at marking β as follows. Suppose α is an n-web in $(M, \mathcal{N})$ with the sign sequence on β equal to $\boldsymbol{\eta} = (\eta_1, \dots, \eta_k) \in \{\pm\}^k$. The set $\{v_{\boldsymbol{i}} \mid \boldsymbol{i} \in \{1, \dots, n\}^k\}$ is the R-basis of the based module $V^{\boldsymbol{\eta}}$. Assume α is stated at all markings except β. For $x \in V^{\boldsymbol{\eta}}$ let $(\alpha, x) \in \mathcal{S}_n(M, \mathcal{N})$ be defined so that $(\alpha, v_{\boldsymbol{i}})$ is α with states $\boldsymbol{i}$ on β, and the map $x \to (\alpha, x)$ is R-linear. From the definition, we have

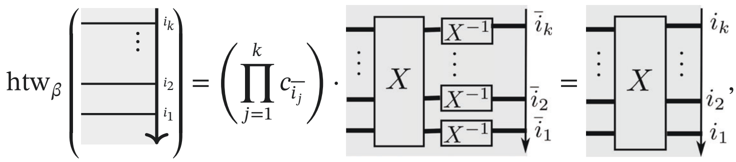
$$u * (\alpha, x) = (\alpha, ux) \tag{81}$$

Example 7.1. The action of the charmed element g on $\mathcal{S}_n(M, \mathcal{N})$ is exactly the map g_β of Subsection 4.10. In fact, g is a group-like element, that is, $\Delta^{[k]}(g) = g^{\otimes k}$ for $k = 1, 2, \dots$, and the actions of g on the based $U_q(sl_n)$-modules V and V^* are given by the same diagonal matrix with $g_1, \dots, g_n$ on the diagonal, see (42). That ensures that the action of g on $\mathcal{S}_n(M, \mathcal{N})$ coincides with the map g_β.

Example 7.2. The action of the half-ribbon element X is the half-twist homomorphism htw_β of Subsection 4.11. Indeed, for the positive half-twist H on k strands (with arbitrary orientations) by (40) we have

$$\mathsf{RT}(H) = \Delta^{[k]}(X) \circ (X^{-1})^{\otimes k} \circ \mathrm{rev}_k.$$

By applying this value of $\mathsf{RT}(H)$ to (62) we obtain

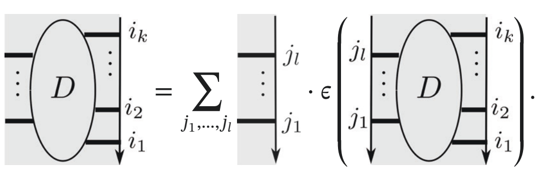


where the second identity follows from (41). This proves the statement.

Formula (81) makes it easy to study $\mathcal{S}_n(M,\mathcal{N})$ as an $\widetilde{U^L}$-module. For example, one can show that over the field $\mathbb{Q}(v)$ the $U_q(sl_n)$-module $\mathcal{S}_n(M,\mathcal{N}) \otimes_{\mathbb{Z}[v^{\pm 1}]} \mathbb{Q}(v)$ is integrable and is a direct sum of finite-dimensional simple $U_q(sl_n)$-modules. For the case $n = 2$, see [15].

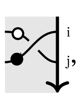
7.2 | Boundary relations revisited

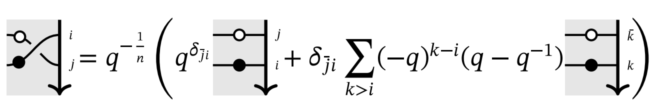
Let $\boldsymbol{D} = (D, \boldsymbol{i})$ be an n-web diagram D over the bigon $\mathfrak{B}$ right stated by $\boldsymbol{i} = (i_1, \dots, i_k)$. Assume that D has l left endpoints, which are not stated. Suppose further that α is a stated n-web in a marked 3-manifold $(M,\mathcal{N})$ and that in a cube Q that intersects $\mathcal{N}$ at a subinterval of a marking β the intersection $\alpha \cap Q$ has diagram equal to $\boldsymbol{D}$, as in the left side of (82). By the property of the counit of the coaction, we have



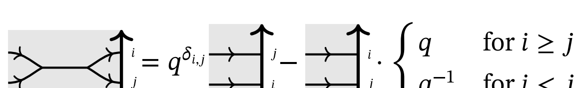
(82)

This identity provides a local relation in any stated skein module, called the $\boldsymbol{D}$-*relation*. By (75), the values of ϵ of stated n-webs are the entries of the matrix describing the Reshetikhin–Turaev operator $\mathsf{RT}(D)$ and are not difficult to calculate. All the boundary relations (47)-(50) are of this type. As Relations (43)–(50) are sufficient for defining the $\mathcal{O}_q(SL(n))$-coaction on $\mathcal{S}_n((M,\mathcal{N}))$, any $\boldsymbol{D}$-relation is a consequence of these relations.

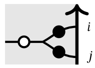
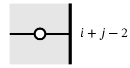
Example 7.3. For $\boldsymbol{D} =$ , the $\boldsymbol{D}$-relation is

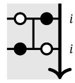
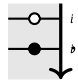


Example 7.4. The following relations for $n = 3$ will be useful later:



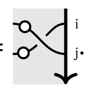
(83)

 $= \delta_{i\neq j}(-q)^{\delta_{i>j}} q^{-\frac{7}{6}}$  (84)

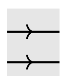
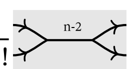
 $=$  , for $i = 1, 3$. (85)

7.3 | The last among the defining skein relations

Proposition 7.5. *If $[n-2]!$ is invertible in R then the last defining relation, (50), is a consequence of the other defining relations (43)–(49).*

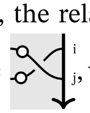
Proof. Note that Relation (50) is the $\boldsymbol{D}$-relation for $\boldsymbol{D} =$ .

As $[n-2]!$ is invertible, Identity (51) makes it possible to eliminate all crossings in every diagram,

 $= q^{\frac{n-1}{n}}$  $- (-1)^{\binom{n}{2}} \dfrac{q^{-\frac{1}{n}}}{[n-2]!}$  , (86)

For the purpose of this proof, let $\mathcal{S}'_n(M, \mathcal{N})$ be defined as $\mathcal{S}_n(M, \mathcal{N})$, only without Relation (50). The Splitting Theorem holds for $\mathcal{S}'_n(M, \mathcal{N})$, as using (86) we do not have to consider the invariance of the splitting homomorphism under moving a crossing through the splitting disk. Lemma 5.1 holds for $\mathcal{S}'_n(\mathfrak{B})$ because all crossings of diagrams on $\mathfrak{B}$ can be eliminated. Consequently, Lemma 6.5 holds as well and the proof of the isomorphism $\mathcal{S}_n(\mathfrak{B}) \simeq \mathcal{O}_q(sl_n; R)$ extends to an isomorphism

$$\mathcal{S}'_n(\mathfrak{B}) \xrightarrow{\simeq} \mathcal{S}_n(\mathfrak{B}) \xrightarrow{\simeq} \mathcal{O}_q(sl_n; R).$$

Furthermore, every marking β of $(M, \mathcal{N})$ defines a right coaction $\Delta'_\beta : \mathcal{S}'_n(M, \mathcal{N}) \to \mathcal{S}'_n(M, \mathcal{N}) \otimes_R \mathcal{O}_q(sl_n; R)$ as in previous subsection. Using the coaction, one sees that for every right stated n-web diagram $\boldsymbol{D} = (D, \boldsymbol{i})$ on $\mathfrak{B}$, the relation (82) is a consequence of the defining relations for $\mathcal{S}'_n(M, \mathcal{N})$. In particular, for $\boldsymbol{D} =$ , we get the statement of the proposition. □

8 | ALGEBRAIC STRUCTURE OF SKEIN ALGEBRAS

8.1 | Glueing over an ideal triangle

The standard ideal triangle $\mathfrak{T} \subset \mathbb{R}^2$ is the closed triangle with vertices $(-1, 0)$, $(1, 0)$ and (0,1) with these vertices removed. We will denote its sides by e_1, e_2, and $\partial_b \mathfrak{T}$ as in Figure 13. Suppose a_1, a_2 are two distinct boundary edges of a (possibly disconnected) pb surface Σ. Define

$$\Sigma_{a_1 \triangle a_2} = (\Sigma \sqcup \mathfrak{T})/(e_1 = a_1, e_2 = a_2),$$

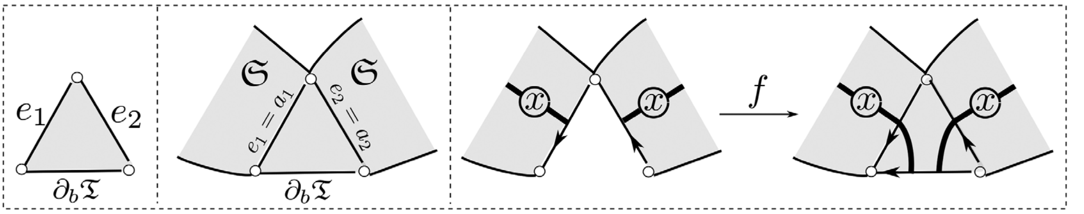


FIGURE 13 Left: The standard ideal triangle $\mathfrak{T}$. Middle: Glueing Σ and $\mathfrak{T}$ by $a_1 = e_1$ and $a_2 = e_2$ to get $\Sigma_{a_1 \triangle a_2}$. Right: Tangle diagram $x \in \mathcal{S}_n(\Sigma)$ and its image $\mathsf{glue}_{a_1,a_2}(x) \in \mathcal{S}_n(\Sigma_{a_1 \triangle a_2})$.

as in Figure 13. Define the R-linear homomorphism $\mathsf{glue}_{a_1,a_2} : \mathcal{S}_n(\Sigma) \to \mathcal{S}_n(\Sigma_{a_1\triangle a_2})$ so that if α is a stated n-web diagram over Σ with the negative height order on both a_1 and a_2 then $\mathsf{glue}_{a_1,a_2}(\alpha)$ is the result of continuing the strands of α with endpoints on a_1 and a_2 until they reach $\partial_b \mathfrak{T}$, as in Figure 13 (right). (As usual, the arrows indicate the height order.)

We are going to show that glue_{a_1,a_2} is a linear isomorphism and to construct its inverse, cut_{a_1,a_2}. Let

$$\varepsilon_{\mathfrak{T}} : \mathcal{S}_n(\mathfrak{T}) \to \mathcal{S}_n(\mathfrak{M}) = R, \quad \varepsilon_{\mathfrak{T}} = \varepsilon \circ \mathrm{fill}_*$$

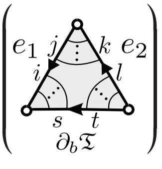
where fill embeds $\mathfrak{T}$ into a bigon by filling in the top vertex of $\mathfrak{T}$ and making the web ends at e_1 higher than those at e_2. Let us consider also the homomorphism

$$\mathsf{cut}_{a_1,a_2} : \mathcal{S}_n(\Sigma_{a_1\triangle a_2}) \to \mathcal{S}_n(\Sigma), \quad \mathsf{cut}_{a_1,a_2} = (\varepsilon_{\mathfrak{T}} \otimes id_{\mathcal{S}_n(\Sigma)}) \circ \Theta_{a_1} \circ \Theta_{a_2}.$$

Proposition 8.1. *The homomorphisms* cut_{a_1,a_2} *and* glue_{a_1,a_2} *are inverses of each other:*

$$\mathsf{cut}_{a_1,a_2} \circ \mathsf{glue}_{a_1,a_2} = id_{\mathcal{S}_n(\Sigma)} \text{ and } \mathsf{glue}_{a_1,a_2} \circ \mathsf{cut}_{a_1,a_2} = id_{\mathcal{S}_n(\Sigma_{a_1\triangle a_2})}$$

Proof. We follow Higgins' proof for $n = 3$. It is enough to check the above identities for the diagrams over Σ and over $\Sigma_{a_1\triangle a_2}$, respectively. We have



$$\varepsilon_{\mathfrak{T}}\left(\cdots \right) = \prod_{a=1}^{|\boldsymbol{j}|} \left(\delta_{j_a, \overline{k_a}} c_{j_a}^{-1} \right) \cdot \left(\prod_{b=1}^{|\mathbf{i}|} \delta_{i_b, s_b} \right) \cdot \left(\prod_{d=1}^{|\mathbf{l}|} \delta_{l_d, t_d} \right), \tag{87}$$

where the orientations of arcs in the triangle are arbitrary and $\boldsymbol{i}, \mathbf{s}$ and $\boldsymbol{j}, \mathbf{k}$ and $\mathbf{l}, \mathbf{t}$ are any three pairs are state sequences of equal length, $|\boldsymbol{i}| = |\mathbf{s}|$, $|\boldsymbol{j}| = |\mathbf{k}|$, and $|\mathbf{l}| = |\mathbf{t}|$.

Let us call diagrams over $\mathfrak{T}$ that are like in Equation (87) without the horizontal arcs, *vertical*. As for every $x \in \mathcal{S}_n(\Sigma_{a_1\triangle a_2})$, the skein $\Theta_{a_1} \circ \Theta_{a_2} \circ \mathsf{glue}_{a_1,a_2}(x)$ is a linear combination of diagrams that are vertical on $\mathfrak{T}$, the first identity follows.

To prove the second identity observe that each diagram in $\mathcal{S}_n(\Sigma_{a_1\triangle a_2})$ can be positioned so that it intersects $\mathfrak{T}$ in disjoint arcs only. By applying Relation (49), it can be presented as a linear combination of diagrams that are vertical on $\mathfrak{T}$. By Equation (87), for each of them we have

$$\mathsf{glue}_{a_1,a_2} \circ (\varepsilon_{\mathfrak{T}}(W) \otimes id_{\mathcal{S}_n(\Sigma)}) \circ \Theta_{a_1} \circ \Theta_{a_2}(W) = W.$$

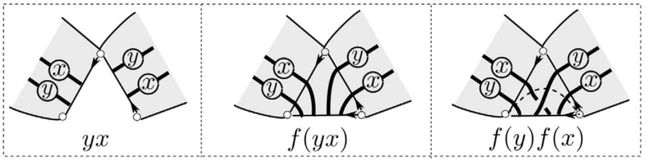


FIGURE 14 Diagrams of yx, $f(yx)$, and $f(y)f(x)$.

Hence, the right identity of Proposition 8.1 follows. □

Note that the bijective map glue_{a_1,a_2} is not an algebra isomorphism. In fact, $\mathsf{glue}_{a_1,a_2}(yx)$ and $\mathsf{glue}_{a_1,a_2}(y)\mathsf{glue}_{a_1,a_2}(x)$ are depicted in Figure 14, where $f = \mathsf{glue}_{a_1,a_2}$.

However, we are going to show that it is one with respect to the *self-braided tensor product* that we will define right now. The $n = 2$ version of this product was considered in [15].

There are two right $\mathcal{O}_q(SL(n))$-comodule algebra structures on $\mathcal{S}_n(\Sigma)$ given by

$$\Delta_i := \Delta_{a_i} : \mathcal{S}_n(\Sigma) \to \mathcal{S}_n(\Sigma) \otimes \mathcal{O}_q(SL(n)), \quad i = 1, 2,$$

which commute.

Define the R-linear map $\underline{\Delta} : \mathcal{S}_n(\Sigma) \to \mathcal{S}_n(\Sigma) \otimes \mathcal{O}_q(SL(n))$ by

$$\underline{\Delta}(x) = \sum x_{(1)} \otimes u_{(2)} u_{(3)},$$

in Sweedler's notation, where

$$(\Delta_1 \otimes \mathrm{Id}_{\mathcal{O}_q(SL(n))}) \circ \Delta_2(x) = \sum x_{(1)} \otimes u_{(2)} \otimes u_{(3)}.$$

For $x, y \in \mathcal{S}_n(\Sigma)$ define a new product by

$$y \underline{*} x = \sum y_{(1)} x_{(1)} \rho(u_{(2)} \otimes w_{(2)}), \tag{88}$$

where

$$\Delta_2(y) = \sum y_{(1)} \otimes u_{(2)}, \quad \Delta_1(x) = \sum x_{(1)} \otimes w_{(2)},$$

and ρ is the R-form.

It is proved in [15] that $\underline{\Delta}$ and $\underline{*}$ together give $\mathcal{S}_n(\Sigma)$ a right $\mathcal{O}_q(SL(n))$-comodule algebra structure for $n = 2$. That proof extends verbatim to all n. Denote by $\underline{\otimes}\mathcal{S}_n(\Sigma)$ the R-module $\mathcal{S}_n(\Sigma)$ with this $\mathcal{O}_q(SL(n))$-comodule algebra structure. On the other hand, $\mathcal{S}_n(\Sigma_{a_1 \triangle a_2})$ has a right $\mathcal{O}_q(SL(n))$-comodule algebra structure coming from the boundary edge $\partial_b \mathfrak{T}$. Here is a stronger version of Proposition 8.1.

Theorem 8.2. *The maps* $\mathsf{glue}_{a_1,a_2} : \underline{\otimes}\mathcal{S}_n(\Sigma) \to \mathcal{S}_n(\Sigma_{a_1 \triangle a_2})$, $\mathsf{cut}_{a_1,a_2} : \mathcal{S}_n(\Sigma_{a_1 \triangle a_2}) \to \underline{\otimes}\mathcal{S}_n(\Sigma)$ *are isomorphisms of right* $\mathcal{O}_q(SL(n))$*-comodule algebras.*

Proof. The geometric proof of [15] for $n = 2$ carries over to all n without modification. Here is a sketch. It is enough to show that $f = \mathsf{glue}_{a_1,a_2}$ is an algebra homomorphism. Let x, y be stated n-web diagrams.

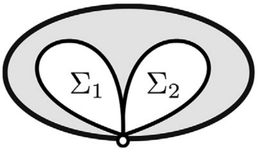
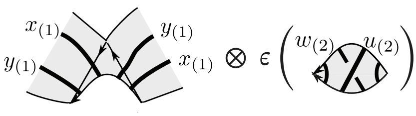


FIGURE 15 The operation $\Sigma_1 \triangle \Sigma_2$ on two surfaces with a single arc boundaries.

We present yx, $f(yx)$, $f(y)f(x)$ schematically as in Figure 14. By splitting along the dashed line in the picture of $f(y)f(x)$ and by using the counit property,

$$f(y)f(x) = \sum$$



where

$$\Delta_2(y) = \sum y_{(1)} \otimes u_{(2)}, \text{ and } \Delta_1(x) = \sum x_{(1)} \otimes w_{(2)}.$$

The above equals $\sum f(y_{(1)}x_{(1)})\rho(u_{(2)} \otimes w_{(2)})$, and, by (88), it reduces to $f(y \underline{*} x)$. Thus, f is an algebra homomorphism. □

A special case is when $\Sigma = \Sigma_1 \sqcup \Sigma_2$ and $a_i \subset \Sigma_i$ for $i = 1, 2$. In this case we say that $\Sigma_{a_1 \triangle a_2}$ is the result of gluing Σ_1 and Σ_2 over the triangle. Each $\mathcal{S}_n(\Sigma_i)$ is a right $\mathcal{O}_q(SL(n))$-comodule algebra via the coaction coming from the edge a_i. Then $\underline{\otimes}(\mathcal{S}_n(\Sigma))$ is the well-known *braided tensor product* $\mathcal{S}_n(\Sigma_1)$ and $\mathcal{S}_n(\Sigma_2)$ of the two $\mathcal{O}_q(SL(n))$-module algebras $\mathcal{S}_n(\Sigma_1)$ and $\mathcal{S}_n(\Sigma_2)$, defined in [52, Lemma 9.2.12].

Example 8.3 (Ideal triangle). Let $\Sigma_1 = \Sigma_2 = \mathfrak{B}$, where a_1 is the right edge of Σ_1 and a_2 is the left edge of Σ_2. Then $\Sigma_{a_1 \triangle a_2}$ is the triangle $\mathfrak{T}$. Hence, we have

$$\mathcal{S}_n(\mathfrak{T}) \cong \mathcal{O}_q(SL(n)) \underline{\otimes} \mathcal{O}_q(SL(n)),$$

where each copy of $\mathcal{O}_q(SL(n))$ is a right $\mathcal{O}_q(SL(n))$-comodule algebra via the coproduct. From here one can easily write down an explicit presentation of the algebra $\mathcal{S}_n(\mathfrak{T})$. Such presentation is used in the work [44] on the quantum trace for stated SL_n-skein algebras.

Let $\Sigma_{g,p+1}$ be a p-punctured genus g surface with a single loop boundary and let $\Sigma^*_{g,p}$ be $\Sigma_{g,p}$ with a boundary point removed.

Example 8.4. Let $\mathfrak{S} = \{\Sigma^*_{g,p}, g \geqslant 0, p \geqslant 1\}$ be the set of pb surfaces with a single arc boundary, considered up to a homeomorphism. For $\Sigma_1, \Sigma_2 \in \mathfrak{S}$, let $\Sigma_1 \triangle \Sigma_2$ be the result of gluing over a triangle along $a_1 = \partial\Sigma_1$ and $a_2 = \partial\Sigma_2$, as in Figure 15. Note that the $\triangle$ operation makes $\mathfrak{S}$ into a monoid with the identity $\mathfrak{M}$.

Theorem 8.2 implies that for any $\Sigma_1, \Sigma_2 \in \mathfrak{S}$, the algebra $\mathcal{S}_n(\Sigma_1 \triangle \mathcal{S}_n)$ is the braided tensor product $\mathcal{S}_n(\Sigma_1) \underline{\otimes} \mathcal{S}_n(\Sigma_2)$.

Therefore, $\mathcal{S}_n(\Sigma^*_{g,p})$ is the braided tensor product of $p-1$ copies of $\mathcal{S}_n(\Sigma^*_{0,2})$ and g copies of $\mathcal{S}_n(\Sigma^*_{1,1})$. We will analyze $\mathcal{S}_n(\Sigma^*_{0,2})$ and $\mathcal{S}_n(\Sigma^*_{1,1})$ in detail in Subsections 8.2 and 8.6 and we will see

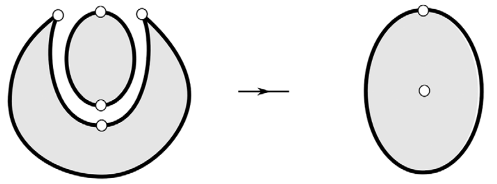


FIGURE 16 From the bigon to a punctured monogon by gluing over a triangle.

in particular that

$$\mathcal{S}_n(\Sigma_{0,2}^*) \simeq \mathcal{O}_q(SL(n)) \text{ and } \mathcal{S}_n(\Sigma_{1,1}^*) \simeq \mathcal{O}_q(SL(n))^{\otimes 2}$$

as R-modules. Consequently,

$$\mathcal{S}_n(\Sigma_{g,p}^*) \simeq \mathcal{O}_q(SL(n))^{\otimes(p-1+2g)}$$

as an R-module. (A version of this formula appeared in [7].) This statement will be generalized by Theorem 8.8.

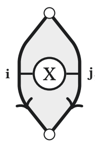
8.2 | Punctured monogon and Majid's transmutation

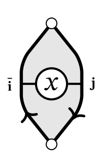
By attaching an ideal triangle to the bigon $\mathfrak{B}$ along its left and right edges e_l and e_r, as in Figure 16, we obtain the *once-punctured monogon*, $\Sigma_{0,2}^* = \mathfrak{B}_{a_1 \triangle a_2}$.

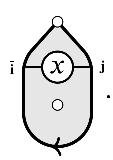
An algebraic description of the product on $\mathcal{S}_n(\Sigma_{0,2}^*)$ can be derived from that for $\mathcal{O}_q(SL(n))$ by the rule described in Equation (88). This allows to identify $\mathcal{S}_n(\Sigma_{0,2}^*)$ with Majid's transmutation of $\mathcal{O}_q(SL(n))$, as we explain now.

Let tm be the composition of the inverse of the half-twist around the left edge with the above triangle gluing map,

$$\mathsf{tm} = \mathsf{glue}_{e_l,e_r} \mathsf{htw}_{e_l}^{-1} : \mathcal{O}_q(SL(n)) = \mathcal{S}_n(\mathfrak{B}) \to \mathcal{S}_n(\Sigma_{0,2}^*). \tag{89}$$

This map can be visualized as follows: let a stated web x in $\mathfrak{B}$ be represented by a diagram 

, where the left horizontal line represents multiple horizontal edges (of possibly different directions) whose ends on the left are labeled by $\mathbf{i} = \begin{pmatrix} i_s \\ \vdots \\ i_1 \end{pmatrix}$ and, similarly, the right horizontal line represents multiple horizontal edges whose ends on the right are labeled by $\mathbf{j} = \begin{pmatrix} j_t \\ \vdots \\ j_1 \end{pmatrix}$. Then $\mathsf{htw}_{e_l}^{-1}(x) = \frac{1}{\prod c_{\mathbf{i}}}$ 

and $\mathsf{tm}(x) = \frac{1}{\prod c_{\mathbf{i}}}$ 

. By Propositions 4.11 and 8.1, tm is an R-linear isomorphism. We will prove that this map defines Magid's *transmutation* on $\mathcal{O}_q(SL(n))$. (That was the reason for denoting the above map by tm.)

Let us recall that notion first: for every Hopf algebra $\mathcal{H}$, Majid proved that

$$Ad : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}, \quad Ad(x) = \sum x_{(2)} \otimes (Sx_{(1)})x_{(3)}, \tag{90}$$

defines a coaction on $\mathcal{H}$ on itself [52, Example 1.6.14], called the adjoint $\mathcal{H}$-coaction and that there is an associative *braided product* (or, *covariantised*)

$$x \underline{\cdot} y = \sum x_{(2)} y_{(2)} \rho((Sx_{(1)})x_{(3)} \otimes Sy_{(1)}), \tag{91}$$

see [52, Example 9.4.10]. (This product should not be confused with the braided tensor product of Majid, which we discussed in Subsection 8.1.) Furthermore, he showed that $\mathcal{H}$ with the braided product and the adjoint $\mathcal{H}$-coaction is an $\mathcal{H}$-comodule coalgebra. With these structures, $\mathcal{H}$ is denoted by $\mathcal{T}H$ called the *transmutation* of $\mathcal{H}$.

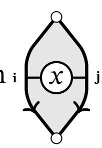
Proposition 8.5.

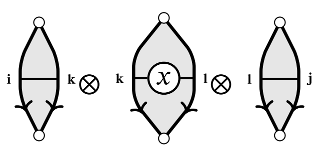
(1) tm *is an isomorphism between the* $\mathcal{O}_q(SL(n))$*-comodule algebra* $\mathcal{T}\mathcal{O}_q(SL(n))$ *and* $\mathcal{S}_n(\Sigma^*_{0,2})$. *Hence,* $(\mathsf{tm} \otimes I) \circ Ad = \Delta_{\partial \Delta^*_{0,2}} \circ \mathsf{tm}$.

(2) tm *is a ring isomorphism. Hence,* $\mathsf{tm}(x)\mathsf{tm}(y) = \mathsf{tm}(x \underline{\cdot} y)$.

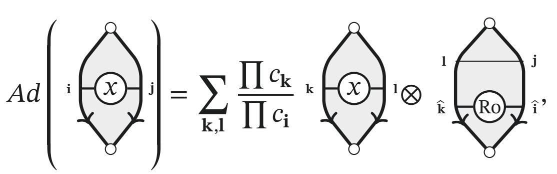
This statement was proved for $n = 2$ in [15].

Proof of Proposition 8.5.

(1) Let a stated web in $\mathfrak{B}$ be given by the diagram , as above. Then

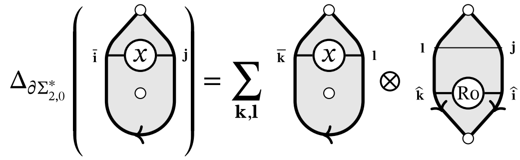
$\Delta^2(x) = \sum_{\mathbf{k},\mathbf{l}}$ 

where as above, horizontal arcs indicate multiple edges of possibly different directions. Consequently, by Equations (90) and (67),

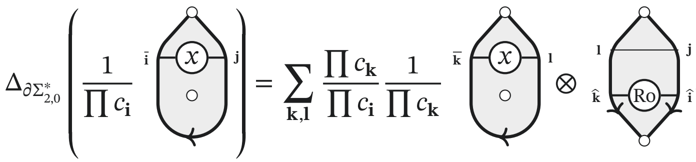
 (92)

where Ro denotes 180° rotation in plane, $\hat{\mathbf{i}}$ denotes conjugation of components and the inversion of their order: $\hat{\mathbf{i}} = \begin{pmatrix} \bar{i}_1 \\ \vdots \\ \bar{i}_s \end{pmatrix}$ for $\mathbf{i} = \begin{pmatrix} i_s \\ \vdots \\ i_1 \end{pmatrix}$, and $\prod c_{\mathbf{i}}$ denotes $\prod_{t=1}^{s} c_{i_t}$.

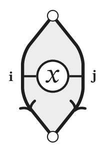
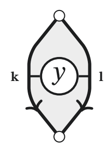
On the other hand, we have

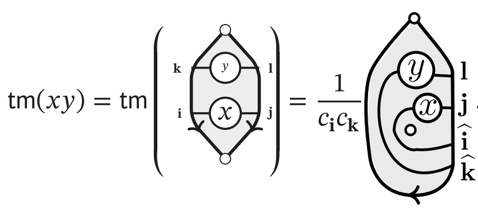


and, hence,



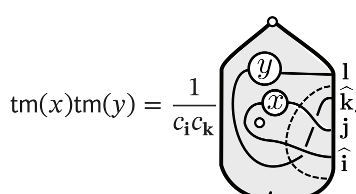
As this equality coincides with (92) after replacing x with $\mathsf{tm}(x)$ and Ad by $\Delta_{\partial\Sigma^*_{2,0}}$, the statement follows.

(2) Consider stated webs $x =$  and $y =$  in $\mathfrak{B}$. Then



(93)

On the other hand,



(94)

Denoting the stated web diagram on the right by $z \in \mathcal{S}_n(\Sigma^*_{0,2})$, we have

$$z = \sum z_{(1)}\varepsilon(z_{(2)}), \text{ where } \Delta(z) = \sum z_{(1)} \otimes z_{(2)} \in \mathcal{S}_n(\Sigma^*_{0,2}) \otimes \mathcal{O}_q(SL(n))$$

is the coaction along the dashed line and ε is the counit in $\mathcal{O}_q(SL(n))$. By applying this identify to (94), we obtain

$$\mathsf{tm}(x)\mathsf{tm}(y) = \frac{1}{c_{\mathbf{i}}c_{\mathbf{k}}} \sum \mathsf{tm}(x_{(2)}y_{(2)})T(x_{(1)}, x_{(3)}, y_{(1)}),$$

where $T(x_{(1)}, x_{(3)}, y_{(1)}) = \rho((Sx_{(1)})x_{(3)} \otimes Sy_{(1)})$, by Theorem 6.4. Consequently,

$$\mathrm{tm}(x)\mathrm{tm}(y) = \mathrm{tm}(x \underline{\cdot} y),$$

by (91). □

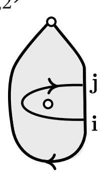
Consequently, our theory provides simple geometric proofs of the associativity of the (braided) product on $\mathcal{T}\mathcal{O}_q(SL(n))$ and of $\mathcal{T}\mathcal{O}_q(SL(n))$ being an $\mathcal{O}_q(SL(n))$-comodule algebra. (The proofs of these facts are quite technical and involved in [52].) Furthermore, our theory generalizes these statements to the boundary $\mathcal{O}_q(SL(n))$-coaction on the skein algebra of any essentially bordered punctured surface.

Let us discuss generators and relations of $\mathcal{T}\mathcal{O}_q(SL(n))$ now. A *reflection equation algebra* $A_q(M_n)$ is an R-algebra generated by formal variables x_{ij} for $i, j = 1, \ldots, n$ subject to the quadratic relations of the *reflection equation*:

$$X_2\hat{\mathcal{R}}X_2\hat{\mathcal{R}} = \hat{\mathcal{R}}X_2\hat{\mathcal{R}}X_2, \tag{95}$$

where $X = (x_{ij})_{i,j=1,\ldots,n}$, $X_2 = X \otimes Id$, and $\hat{\mathcal{R}}$ is the braiding matrix of Subsection 2.3. (These equations are written explicitly out in [18, section 3].)

It is proved in [36] that $\mathcal{T}\mathcal{O}_q(SL(n))$ is the quotient of the reflection equation algebra by the *braided determinant* which is the image of the quantum determinant under the linear isomorphism $\mathcal{T}\mathcal{O}_q(SL(n)) \simeq \mathcal{O}_q(SL(n))$ above. An explicit polynomial expression in x_{ij}'s for it appears in [31]. Consequently, that expression together with the relations (95) are a complete set of relations for $\mathcal{T}\mathcal{O}_q(SL(n))$.

Let us relate this discussion to $\mathcal{S}_n(\Sigma^*_{0,2})$ now. It is straightforward to verify that tm maps the generators $x_{ij} \in \mathcal{T}\mathcal{O}_q(SL(n))$ to the arcs 

which we will denote by $b_{i,j}$. (Then $\mathrm{tm}(a_{i,j}) = \frac{1}{c_{\bar{i}}} b_{\bar{i},j}$ for the generators a_{ij} for $\mathcal{S}_n(\mathfrak{B})$ of Subsection 6.2. Independently of the above considerations, it is easy to see that b_{ij}'s for $i, j = 1, \ldots, n$ generate $\mathcal{S}_n(\Sigma^*_{0,2})$, as any web in $\mathcal{S}_n(\Sigma^*_{0,2})$ can be pushed toward the boundary of $\partial\Sigma^*_{0,2}$ and simplified by the boundary relations to a polynomial expression in b_{ij}'s.)

Consequently, the above discussion provides a concrete finite presentation for $\mathcal{S}_n(\Sigma^*_{0,2})$.

8.3 | On injectivity of splitting homomorphism

Proposition 8.6. *Suppose Σ is an essentially bordered pb surface. Then for any interior ideal arc c of Σ, the splitting homomorphism $\Theta_c : \mathcal{S}_n(\Sigma) \to \mathcal{S}_n(\mathrm{cut}_c\Sigma)$ is injective.*

Proof. First assume that an endpoint of c is a boundary ideal point, which is an endpoint of a boundary edge $e \neq c$. In a small neighborhood of $e \cup c$, we can find an interior ideal arc c' such that e, c, c' cobound an ideal triangle; see Figure 17.

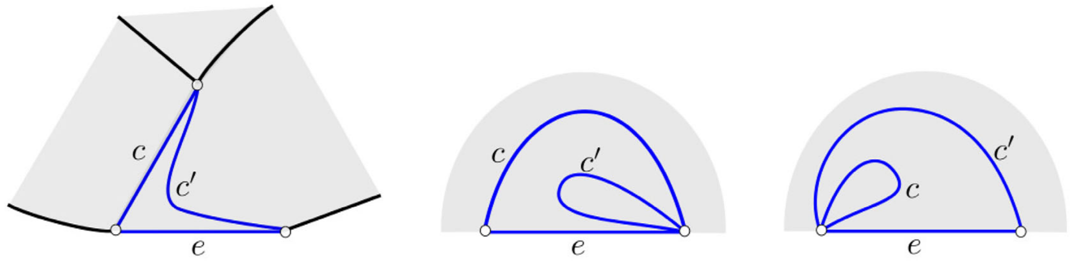


FIGURE 17 The curve c'. Left: General case. Middle: c and e form a bigon. Right: c cuts out a monogon.

Let $\mathring{\mathfrak{T}}$ be the interior of the triangle $\mathfrak{T}$ bounded by c, c', and e, and let $\Sigma' = \mathring{\Sigma} \setminus (\mathring{\mathfrak{T}} \cup e)$. Then $(\Sigma')_{c\Delta c'} = \Sigma$. By Theorem 8.2, the map $\mathrm{cut}_{c,c'} = (\varepsilon_{\mathfrak{T}} \otimes \mathrm{id}_{\mathcal{S}_n(\Sigma')}) \circ \Theta_{c'} \circ \Theta_c$ is bijective. It follows that Θ_c is injective.

Now assume that both endpoints of c are interior ideal points. As Σ is essentially bordered it contains an interior ideal arc d, disjoint from c, with one endpoint coinciding with an endpoint of c and the other endpoint being a boundary ideal point. By the above case, the splitting map $\Theta_d : \mathcal{S}_n(\Sigma) \to \mathcal{S}_n(\mathrm{cut}_d(\Sigma))$ is injective. As the interior ideal arc $c \subset \mathrm{cut}_d(\Sigma)$ has one endpoint on the boundary, $\Theta_c : \mathrm{cut}_d(\Sigma) \to \mathrm{cut}_{c,d}(\Sigma)$ is injective. From the commutativity $\Theta_c \circ \Theta_d = \Theta_d \circ \Theta_c$ we conclude that $\Theta_c : \mathcal{S}_n(\Sigma) \to \mathrm{cut}_c(\Sigma)$ is injective. □

Conjecture 8.7. *For any punctured bordered surface Σ and any interior ideal arc c the splitting homomorphism Θ_c is injective as well.*

The conjecture is true when $n = 2$ by [15] and for $n = 3$ by Higgins [28]. In both cases, explicit bases of $\mathcal{S}_n(\Sigma)$ were used. Proposition 8.6 shows the conjecture is true if Σ has nontrivial boundary. Furthermore, the argument of the proof reduces the conjecture to the empty boundary surfaces with a trivial ideal arc c, that is, an ideal arc bounding a disk in Σ. Corollary 9.2 will establish a weaker version of this conjecture for all pb surfaces.

8.4 | Skein algebras of surfaces with boundary

Let Σ be an essentially bordered pb surface. A collection $A = \{a_1, \dots, a_r\}$ of disjoint compact oriented arcs properly embedded into Σ is *saturated* if

(i) each connected component of $\Sigma \setminus \bigcup_{i=1}^{r} a_i$ contains exactly one ideal point (interior or boundary) of Σ, and
(ii) A is maximal with respect to the above condition.

Note that condition (i) does not imply (ii). For example, $A = \emptyset \subset \Sigma^*_{1,1}$ satisfies (i), but not (ii). Saturated A consists of two ideal arcs in this surface.

Let $U(a_1), \dots, U(a_r)$ be a collection of disjoint open tubular neighborhoods of $a_1, \dots, a_r$, respectively. Each $U(a_i)$ is homeomorphic with $a_i \times (-1, 1)$ (by an orientation preserving homeomorphism) and we require that $(\partial a_i) \times (-1, 1) \subset \partial \Sigma$.

Recall from Subsection 5.4 that any embedding of pb surfaces $\Sigma' \subset \Sigma$ together with an ordering on the boundary edges of Σ' in the boundary edges b of Σ, called b-orders, defines a linear

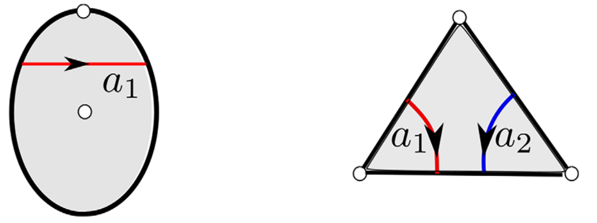


FIGURE 18 Examples of saturated systems. Left: $A = \{a_1\}$ in a punctured monogon, $\Sigma^*_{0,2}$. Right: $A = \{a_1, a_2\}$ in an ideal triangle $\mathfrak{T}$ with a_1 in blue and a_2 in red.

homomorphism $\mathcal{S}_n(\Sigma') \to \mathcal{S}_n(\Sigma)$. We will show that it is an isomorphism for a saturated system for arcs $a_1, \dots, a_r$ and $\Sigma' = U(A) = \bigcup_{i=1}^r U(a_i)$:

Theorem 8.8. *Assume Σ is an essentially bordered pb surface and $A = \{a_1, \dots, a_r\}$ is a saturated system of arcs.*

(1) *We have $r = r(\Sigma) := \#\partial\Sigma - \chi(\Sigma)$, where $\#\partial\Sigma$ is the number of boundary components of Σ and χ denotes the Euler characteristics.*
(2) *The embedding $U(A) \hookrightarrow \Sigma$ with negative b-orderings for all boundary edges b of Σ, induces an R-module isomorphism $f_A : \mathcal{S}_n(U(A)) \to \mathcal{S}_n(\Sigma)$.*

Note that each $U(a_i) = a_i \times (-1, 1)$ is naturally a directed bigon, with its sides $(\partial a_i) \times (-1, 1)$ oriented in the direction of $(-1, 1)$.

Example 8.9. The saturated systems of Figure 18 induce the linear isomorphisms

$$\mathsf{tm} : \mathcal{S}_n(\mathfrak{B}) \to \mathcal{S}_n(\Sigma^*_{0,2}) \quad \text{and} \quad \mathcal{S}_n(\mathfrak{B}) \otimes \mathcal{S}_n(\mathfrak{B}) \to \mathcal{S}_n(\mathfrak{T}) \tag{96}$$

of Equation (89) and Example 8.3.

By the above theorem, for any essentially bordered pb surface we have an R-linear isomorphism

$$\mathcal{O}_q(SL(n))^{\otimes r} \xrightarrow{\Psi^{\otimes r}} \mathcal{S}_n(U(A)) \xrightarrow{f_A} \mathcal{S}_n(\Sigma).$$

As $\mathcal{O}_q(SL(n))$ has a Kashiwara–Lusztig's canonical basis over $\mathbb{Z}[v^{\pm 1}]$, see [19, Proposition 5.1.1], we have

Corollary 8.10. *For any essentially bordered pb surface, $\mathcal{S}_n(\Sigma)$ is a free R-module with a basis given by the image of tensor product of Kashiwara–Lusztig's canonical bases on $\mathcal{O}_q(SL(n))^{\otimes r}$ under $f_A \Psi^{\otimes r}$.*

Remark 9.5 generalizes the above theorem and corollary to all nonclosed pb surfaces.

Remark 8.11. Part (1) implies that condition (ii) in the definition of a saturated system can be replaced by condition $|A| = r(\Sigma)$.

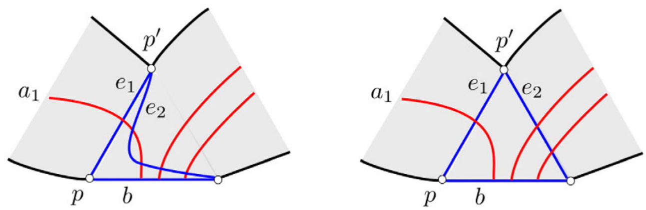
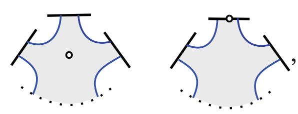


FIGURE 19 Left: The arcs in A are in red, the ideal arcs b, e_1, e_2 are in blue. Right: After pulling e_1, e_2 and arcs in A taut.

Proof of Theorem 8.8.

(1) By the maximality of A, its arcs cut Σ into pieces of the following two types:



whose Euler characteristics are, respectively, 0 and 1. Hence, the Euler characteristic of the result is the number of boundary ideal points, which is $\#\partial\Sigma$. On the other hand, each arc cut increases the Euler characteristic by 1. Hence, $\chi(\Sigma) + r = \#\partial\Sigma$, proving part (1).

(2) We prove it by induction on $\mathsf{tri}(\Sigma)$ which is the number of ideal triangles in an ideal triangulation of Σ, defined as follows. Let $\mathcal{E}$ be a maximal collection of nontrivial ideal arcs in Σ that are pairwise disjoint and pairwise nonisotopic. The ideal arcs in $\mathcal{E}$ not isotopic to boundary edges split Σ into pieces, each is either a monogon, a bigon, or a triangle. Then $\mathsf{tri}(\Sigma)$ is the number of triangles, which is known to be independent on the choice of $\mathcal{E}$. Note that $\mathsf{tri}(\Sigma) = 0$ if and only if Σ is a disjoint union of monogons and bigons, and the theorem is true for this case.

Suppose $\mathsf{tri}(\Sigma) > 0$. We can assume that Σ is connected.

Lemma 8.12. *There is a boundary edge of Σ containing at least two endpoints of arcs in A.*

Proof. As arcs of A are disjoint and simple, they have $2|A|$ endpoints and it is enough to prove that $2|A| > \#\partial\Sigma$. Assuming otherwise, $2|A| \leqslant \#\partial\Sigma$, and by part (1) we have $1 \leqslant \#\partial\Sigma \leqslant 2\chi(\Sigma)$. The positivity of the Euler characteristic implies that $\chi(\Sigma) = 1$ and Σ is a polygon. Then $\#\partial\Sigma \leqslant 2\chi(\Sigma) = 2$ implies Σ is a monogon or a bigon, contradicting the assumption $\mathsf{tri}(\Sigma) > 0$. □

Let b be a boundary edge containing at least two endpoints of A. Let p the ideal end point of b, following the positive direction of b. Among all arcs in A having endpoints in b assume a_1 has an endpoint closest to p. When Σ is cut by arcs in A, there are two pieces adjacent to a_1, one of them, denoted by P_1, contains the ideal point p. The other piece, denoted by P_2, contains an ideal point p'. Let e_1 be an ideal arc of Σ lying in the interior of $P_1 \cup P_2$ connecting p and p' and intersecting a_1 once. No other arcs in A intersects e_1. Because the geometric intersection of b with all arcs in A is at least 2, e_1 cannot be isotopic to b. Pushing the union $e_1 \cup b$ slightly into the interior of Σ yields an ideal arc e_2 such that e_1, e_2, b bounds an ideal triangle T, as in Figure 19. After an isotopy we can assume that A is taught with respect to e_1, e_2, b in the sense that for each $a_j \in A$ and each $e \in \{b, e_1, e_2\}$ the number $|a_j \cap e|$ is minimal when we replace a_j by any isotopic arc.

Let Σ' be the result of removing b and the interior of T from Σ, and let A' be the collection $a_i' := a_i \cap \Sigma', i = 1, \dots, r$. As $|A'| = |A| = r(\Sigma) = r(\Sigma')$, the system A' is saturated for Σ'. As each a_i' is a shrinking of a_i, there is a natural isomorphism $f_{A\to A'} : \mathcal{S}_n(U(A)) \to \mathcal{S}_n(U(A'))$. As $\mathrm{tri}(\Sigma') = \mathrm{tri}(\Sigma) - 1$ the induction hypothesis applies to Σ'. From the definition, we see that f_A is the composition of

$$\mathcal{S}_n(U(A)) \xrightarrow{f_{A\to A'}} \mathcal{S}_n(U(A')) \xrightarrow{f_{A'}} \mathcal{S}_n(\Sigma') \xrightarrow{\mathsf{glue}_{e_1,e_2}} \mathcal{S}_n(\Sigma).$$

As each map in this composition is an R-linear isomorphism, so is f_A. □

We will show now that the assumption about the negativity of all b-orderings in Theorem 8.8(2) is unnecessary.

Let us enumerate the boundary edges of Σ by $b_1, \dots, b_s$ for bookkeeping purposes. Let $o_1, \dots, o_s$ be some b_1-,…, b_s-orderings of the boundary intervals of $U(A)$ in the boundary intervals of Σ and let $f_{A,o_1,\dots,o_s} : \mathcal{S}_n(U(A)) \to \mathcal{S}_n(\Sigma)$ be the homomorphism induced by that height ordered embedding.

To relate $f_{A,o_1,\dots,o_s}$ to f_A, note that each b-ordering o is obtained by a certain permutation σ of the negatively height ordered points $A \cap b$. Let us denote by $\sigma_1, \dots, \sigma_s$ the permutations corresponding to height orderings $o_1, \dots, o_s$. Then $f_{A,o_1,\dots,o_s}(x)$ is induced by the embedding of $U(A) \times (-1, 1)$ into $\Sigma \times (-1, 1)$ with the boundary intervals of $U(A)$ braided by $(\sigma_1)_+, \dots, (\sigma_s)_+$, see Figure 11. Let us elaborate on it more detail now.

Let us call skeins of the form $f_A(x_1 \otimes \dots \otimes x_r) \in \mathcal{S}_n(\Sigma)$ *pure*.

Lemma 8.13.

(i) *For any braids $\tau_1 \in B_{|A\cap b_1|}, \dots, \tau_s \in B_{|A\cap b_s|}$ there exists a unique linear transformation*

$$\mathsf{braid}_{A,\tau_1,\dots,\tau_s} : \mathcal{S}_n(\Sigma) \to \mathcal{S}_n(\Sigma)$$

which braids the endpoints of each pure skein in $\mathcal{S}_n(\Sigma)$ in b_i by τ_i, for $i = 1, \dots, s$. (All skeins are considered with negative b_i-orderings for $i = 1, \dots, s$.)

(ii) *Let $\sigma_1, \dots, \sigma_s$ be permutations corresponding to height orderings $o_1, \dots, o_s$ on $b_1, \dots, b_s$. Then for pure x,*

$$f_{A,o_1,\dots,o_s}(x) = \mathsf{braid}_{A,(\sigma_1)_+,\dots,(\sigma_s)_+} \circ f_A.$$

(iii) *$(\tau_1, \dots, \tau_s) \to \mathsf{braid}_{A,\tau_1,\dots,\tau_s}$ defines a group homomorphism from $B_{|A\cap b_1|} \times \dots \times B_{|A\cap b_s|}$ to the group of R-linear automorphisms of $\mathcal{S}_n(\Sigma)$. In particular, each $\mathsf{braid}_{A,\tau_1,\dots,\tau_s}$ is a linear isomorphism of $\mathcal{S}_n(\Sigma)$.*

Proof.

(i) For braids $\tau_1 \in B_{|A\cap b_1|}, \dots, \tau_s \in B_{|A\cap b_s|}$ consider the embedding

$$U(a_1) \cup \dots \cup U(a_n) \subset \Sigma \times (-1, 1)$$

modified by the braiding by τ_i of its components going toward $b_i \subset \Sigma$, for $i = 1, \dots, s$. This map is considered with the negative height order. It induces a linear map of skein algebras that we denote by $g_{A,\tau_1,\dots,\tau_s} : \mathcal{S}_n(U(A)) \to \mathcal{S}_n(\Sigma)$. Then for pure x, let

$$\mathsf{braid}_{A,\tau_1,\dots,\tau_s}(x) = g_{A,\tau_1,\dots,\tau_s} \circ f_A^{-1}.$$

By Theorem 8.8(2), pure skeins span $\mathcal{S}_n(\Sigma)$. Consequently, the condition of (i) determines $\mathsf{braid}_{A,\tau_1,\dots,\tau_s}$ completely.

(ii) Follows from the discussion above Lemma 8.13.

(iii) By definition,

$$\mathsf{braid}_{A,\tau_1,\dots,\tau_s} \circ \mathsf{braid}_{A,\tau'_1,\dots,\tau'_s} = \mathsf{braid}_{A,\tau_1\tau'_1,\dots,\tau_s\tau'_s},$$

for any $\tau_1, \tau'_1 \in B_{|A\cap b_1|}, \dots, \tau_s, \tau'_s \in B_{|A\cap b_s|}$. Consequently, each $\mathsf{braid}_{\tau_1,\dots,\tau_s}$ is a linear isomorphism of $\mathcal{S}_n(\Sigma)$. As $\mathsf{braid}_{A,id,\dots,id} = id$, the statement follows. □

By Theorem 8.8(2) and Lemma 8.13(2) and (3), we have:

Corollary 8.14. $f_{A,o_1,\dots,o_s} : \mathcal{S}_n(U(A)) \to \mathcal{S}_n(\Sigma)$ *is a linear isomorphism for every* $o_1, \dots, o_s$.

8.5 | Products on skein algebras of surfaces with boundary

In the previous subsection, we discussed R-module structures of skein algebras only. We will address the algebra products now.

Let $a_1, \dots, a_r$ be a saturated system of arcs in Σ as before. Note that the induced linear homomorphism $\mathcal{S}_n(U(a_i)) \to \mathcal{S}_n(\Sigma)$ is an algebra homomorphism if and only if a_i has its ends at different boundary intervals of Σ. (We have seen this already in Example 8.9, where the right map of Equation (96) is an algebra homomorphism on each of the components, $\mathcal{S}_n(\mathfrak{B})$, but the left map $tm : \mathcal{S}_n(\mathfrak{B}) \to \mathcal{S}_n(\Sigma^*_{0,2})$ is not an algebra homomorphism.)

Therefore, for the sake of studying algebra products on $\mathcal{S}_n(\Sigma)$ let us consider modified neighborhoods $U'(a_i) = U(a_i) \cup V$ for arcs a_i with both their ends in the same boundary interval, where V is a tubular neighborhood of the arc of $\partial\Sigma$ connecting the endpoints of a_i. We assume that V is small enough so that $U'(a_i)$ is homeomorphic to a punctured monogon. Note that the transmutation map is a linear isomorphism $\mathsf{tm} : \mathcal{S}_n(U(a_i)) \to \mathcal{S}_n(U'(a_i))$ by Proposition 8.5. We leave the chosen neighborhoods of the arcs with ends in different components of $\partial\Sigma$ unchanged, $U'(a_i) = U(a_i)$.

Let us consider the map

$$\mathsf{mult}_A : \mathcal{S}_n(U'(a_1)) \otimes \dots \otimes \mathcal{S}_n(U'(a_r)) \to \mathcal{S}_n(\Sigma), \quad \mathsf{mult}_A(x_1 \otimes \dots \otimes x_r) = x_1 \cdot \dots \cdot x_r.$$

Note that by the transmutation map for arcs in the same component of $\partial\Sigma$, mult_A coincides with $f_{A,o_1,\dots,o_s}$ for b_1-,…, b_s-orderings $o_1, \dots, o_s$, for which the boundary arcs of $U'(a_i)$ are higher than the boundary arcs of $U'(a_j)$ for $i > j$ in any boundary interval of $\partial\Sigma$.

Consequently, by the above discussion and by Corollary 8.14:

Corollary 8.15. $\mathsf{mult}_A : \mathcal{S}_n(U'(a_1)) \otimes \dots \otimes \mathcal{S}_n(U'(a_r)) \to \mathcal{S}_n(\Sigma)$ *is an R-linear isomorphism and an algebra homomorphism on*

$$\mathcal{S}_n(U'(a_i)) = R \otimes \dots \otimes R \otimes \mathcal{S}_n(U'(a_i)) \otimes R \otimes \dots \otimes R \subset \mathcal{S}_n(U'(a_1)) \otimes \dots \otimes \mathcal{S}_n(U'(a_r))$$

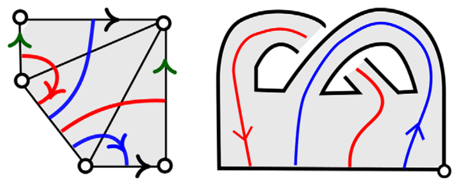


FIGURE 20 Left: A triangulation of $\Sigma^*_{1,1}$ (in black) with the horizontal edges identified and with the vertical edges identified. (Hence, all ideal vertices are identified.) The red and the blue arcs form a saturated arc collection. Right: Another presentation of $\Sigma^*_{1,1}$ with the corresponding red and blue arcs.

for every i (where each R is spanned by the appropriate identity element).

$\mathcal{S}_n(\Sigma)$ is not the tensor product of the algebras $\mathcal{S}_n(U'(a_1)), \dots, \mathcal{S}_n(U'(a_n))$ because elements of different component algebras do not necessarily commute in $\mathcal{S}_n(\Sigma)$.

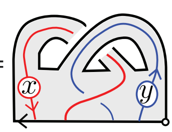
We have seen in Examples 8.3 and 8.9 already that the skein algebra of the ideal triangle, $\mathcal{S}_n(\mathfrak{T})$ is the braided tensor product $\mathcal{S}_n(\mathfrak{B})\underline{\otimes}\mathcal{S}_n(\mathfrak{B})$. We will see in the next subsection however that our stated skein algebras are not braided products of their component algebras, $\mathcal{S}_n(U'(a_1)), \dots, \mathcal{S}_n(U'(a_r))$ in general.

8.6 | Torus with an arc boundary

Let us apply the approach of the above section to analyze the skein algebra of the torus with an arc boundary, $\mathcal{S}_n(\Sigma^*_{1,1})$. Figure 20 shows a torus (in black) with a saturated arc collection: a_1 in red and a_2 in blue.

By Corollary 8.15,

$$\mathsf{mult}_A : \mathcal{S}_n(\Sigma^*_{0,2}) \otimes \mathcal{S}_n(\Sigma^*_{0,2}) \to \mathcal{S}_n(\Sigma^*_{1,1}), \quad \mathsf{mult}(x \otimes y) =$$



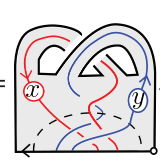
is an R-linear isomorphism and an algebra homomorphism on each of the components algebras.

We described a method of finding an algebraic presentation of $\mathcal{S}_n(\Sigma^*_{0,2})$ in Subsection 8.2. The above discussion allows for an algebraic description of the product on $\mathcal{S}_n(\Sigma^*_{0,2}) \otimes \mathcal{S}_n(\Sigma^*_{0,2})$ (induced from $\mathcal{S}_n(\Sigma^*_{1,1})$ by mult_A) as follows: by the construction of mult_A,

$$(x \otimes 1) \cdot (x' \otimes 1) = (x \cdot x') \otimes 1, \quad (1 \otimes y) \cdot (1 \otimes y') = 1 \otimes (y \cdot y'), \quad (x \otimes 1) \cdot (1 \otimes y) = x \otimes y$$

in $\mathcal{S}_n(\Sigma^*_{1,1})$ for $x \in \mathcal{S}_n(U'(a_1))$ and $y \in \mathcal{S}_n(U'(a_2))$. Therefore, to complete the algebraic description of the product in $\mathcal{S}_n(\Sigma^*_{1,1})$ it remains to consider

$$(1 \otimes y) \cdot (x \otimes 1) =$$



.

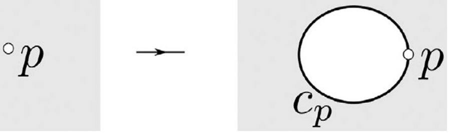


FIGURE 21 From Σ to Σ_p. Here p is an interior ideal point. The picture when p is a boundary ideal point is similar.

Denoting this diagram by z, and applying the identity (95), where Δ is our $\mathcal{O}_q(SL(n))$-coproduct taken with respect to the dashed line, we see that

$$(1 \otimes y) \cdot (x \otimes 1) = \sum x_{(2)} \cdot y_{(2)} \cdot T(x_{(1)}, x_{(3)}, y_{(1)}, y_{(3)}),$$

where $T(x_{(1)}, x_{(3)}, y_{(1)}, y_{(3)})$ is the counit value of the tangle in $\mathfrak{B}$ cut off from the diagram above by the dashed line.

The above formulae completely determine the multiplication in $\mathcal{S}_n(\Sigma^*_{1,1})$ and allow for writing a finite presentation of $\mathcal{S}_n(\Sigma^*_{1,1})$ in terms of generators and relators. Note that mult_A in this case is not a braided tensor product of the component algebras $\mathcal{S}_n(U'(a_1))$, for $i = 1, 2$, in the sense of [52].

For $R = \Bbbk(q)$, the skein algebra $\mathcal{S}_n(\Sigma_{1,1})$ is given by a semi-direct product $U_q(sl(n)) \ltimes \underline{\otimes}\mathcal{O}_q(SL(n))$ and is called the "elliptic double" of $U_q(sl(n))$, and also the "algebra of quantum differential operators on $SL(n, \Bbbk)$," see [7, section 6.4].

Remark 8.16. The finite presentations of $\mathcal{S}_n(\Sigma^*_{0,2})$ and $\mathcal{S}_n(\Sigma^*_{1,1})$ (discussed in Subsection 8.2 and above) induce finite presentations of algebras $\mathcal{S}_n(\Sigma^*_{g,p})$ for all $g \geqslant 0$, $p > 0$ by the method of Example 8.4.

Furthermore, Corollary 8.15 allows for a generalization of the above method to provide a finite presentation of $\mathcal{S}_n(\Sigma)$ in terms of generators and relators for every essentially bordered surface Σ.

9 | KERNEL AND IMAGE OF THE SPLITTING HOMOMORPHISM

9.1 | Kernel of the splitting homomorphism

Suppose Σ is a connected pb surface with an ideal point p and a trivial ideal arc c_p at p. Then $\mathsf{cut}_{c_p}\Sigma$ is the disjoint union of a monogon $\mathfrak{M}$ and of a new pb surface Σ_p that has c_p as its boundary edge, see Figure 21. Let $\mathcal{K}_p(\Sigma)$ be the kernel of the composition

$$\Theta_p : \mathcal{S}_n(\Sigma) \xrightarrow{\Theta_{c_p}} \mathcal{S}_n(\Sigma_p) \otimes_R \mathcal{S}_n(\mathfrak{M}) \xrightarrow{\cong} \mathcal{S}_n(\Sigma_p).$$

Explicitly Θ_p is given as follows. Any stated n-web α over Σ can be isotoped so that it is disjoint from c_p and, hence, lying in Σ_p. Then $\Theta_p(\alpha) = \alpha$ as elements of $\mathcal{S}_n(\Sigma_p)$.

Theorem 9.1. *For any two ideal points p and p' of a connected punctured bordered surface Σ we have $\mathcal{K}_p(\Sigma) = \mathcal{K}_{p'}(\Sigma)$.*

Proof. Assume the two trivial ideal arcs c_p and $c_{p'}$ are disjoint. By splitting both c_p and $c_{p'}$ we get two monogons and a pb surface $\Sigma_{p,p'}$. Let $\Theta : \mathcal{S}_n(\Sigma) \to \mathcal{S}_n(\Sigma_{p,p'})$ be the composition of the two splittings, first along c_p and then along $c_{p'}$. As by Proposition 8.6, the second one is injective, we have $\ker \Theta_p = \ker \Theta$. By switching the order of the splitting, we have $\ker \Theta_{p'} = \ker \Theta$. Thus, $\mathcal{K}_p = \mathcal{K}_{p'}$. ☐

We denote this common ideal by $\mathcal{K}(\Sigma)$. The quotient $\bar{\mathcal{S}}_n(\Sigma) := \mathcal{S}_n(\Sigma)/\mathcal{K}(\Sigma)$ is called the *projected stated skein algebra of* Σ. By Proposition 8.6, $\mathcal{K}_p$ is trivial and $\bar{\mathcal{S}}_n(\Sigma) = \mathcal{S}_n(\Sigma)$ if $\partial\Sigma \neq \emptyset$,

Corollary 9.2. *For any ideal arc c, the splitting homomorphism descends to an injective algebra homomorphism*

$$\bar{\Theta}_c : \bar{\mathcal{S}}_n(\Sigma) \to \bar{\mathcal{S}}_n(\mathsf{cut}_c\,\Sigma) = \mathcal{S}_n(\mathsf{cut}_c\,\Sigma).$$

Proof. The proof is similar to that of Theorem 9.1. Assume c is disjoint from a trivial arc c_p. As the compositions $\Theta_c\Theta_{c_p}, \Theta_{c_p}\Theta_c : \mathcal{S}_n(\Sigma) \to \mathcal{S}_n(\mathsf{cut}_c(\Sigma_p))$ coincide and for both of them the second map is injective, $\ker \Theta_{c_p} = \ker \Theta_c$. ☐

Corollary 9.3. *Conjecture 8.7 is equivalent to the projection $\mathcal{S}_n(\Sigma) \to \bar{\mathcal{S}}_n(\Sigma)$ being an isomorphism. (And, hence, this projection is an isomorphism for $n = 2$ and 3.)*

In the next subsection, we will prove the following.

Theorem 9.4. *For any Σ, p and c_p as above, $\bar{\mathcal{S}}_n(\Sigma)$ coincides with the subalgebra of $\mathcal{S}_n(\Sigma_p)$ coinvariant under the coaction $\Delta_{c_p} : \mathcal{S}_n(\Sigma_p) \to \mathcal{S}_n(\Sigma_p) \otimes \mathcal{S}_n(\mathfrak{B})$ at c_p:*

$$\bar{\mathcal{S}}_n(\Sigma) = \{x \in \mathcal{S}_n(\Sigma_p) : \Delta_{c_p}(x) = x \otimes 1\}.$$

Remark 9.5. Let $\Sigma = \overline{\Sigma} - \mathcal{P}$, where $\mathcal{P}$ is a finite subset of compact surface $\overline{\Sigma}$, as in Subsection 5.1. Generalizing the setup of Subsection 8.4, consider a collection A of disjoint, oriented, arcs in Σ, each with endpoints in $\partial\Sigma \cup \mathcal{P}$, satisfying conditions (i) and (ii) above. Theorem 8.8 and the discussion of the projected stated skein algebra implies that such A defines an identification of $\bar{\mathcal{S}}_n(\Sigma)$ with $\mathcal{O}_q(SL(n))^{\otimes r}$ and, hence, it determines a basis of $\bar{\mathcal{S}}_n(\Sigma)$.

9.2 | The image of the splitting homomorphism

Let c be an interior oriented ideal arc of a pb surface Σ. Denote the two copies of c in $\mathsf{cut}_c\,\Sigma$ by a_1 and a_2. We have the splitting R-algebra homomorphism

$$\Theta_c : \mathcal{S}_n(\Sigma) \to \mathcal{S}_n(\mathsf{cut}_c\,\Sigma).$$

and $\mathcal{S}_n(\mathsf{cut}_c\,\Sigma)$ is a $\mathcal{O}_q(SL(n))$-bi-comodule with the right and left coactions

$$\Delta_{a_1} : \mathcal{S}_n(\mathsf{cut}_c\,\Sigma) \to \mathcal{S}_n(\mathsf{cut}_c\,\Sigma) \otimes \mathcal{O}_q(SL(n))$$

$$_{a_2}\Delta : \mathcal{S}_n(\mathsf{cut}_c\,\Sigma) \to \mathcal{O}_q(SL(n)) \otimes \mathcal{S}_n(\mathsf{cut}_c\,\Sigma),$$

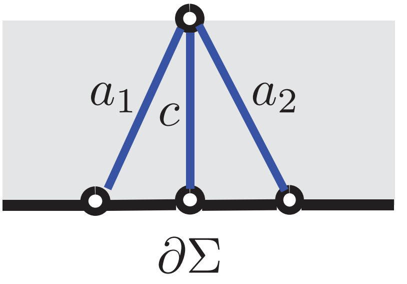


FIGURE 22 $(\mathrm{cut}_c\,\Sigma)_{a_1\triangle a_2}$.

respectively, where $\mathcal{O}_q(SL(n))$ is identified with the skein algebra of the bigon directed by the orientation of c. Recall that the Hochschild cohomology module is defined by

$$HH^0(\mathcal{S}_n(\mathrm{cut}_c\,\Sigma)) = \{x \in \mathcal{S}_n(\mathrm{cut}_c\,\Sigma) \mid \Delta_{a_1}(x) = \mathrm{fl}\circ_{a_2}\Delta(x)\},$$

where fl is the transposition

$$\mathrm{fl} : \mathcal{O}_q(SL(n)) \otimes \mathcal{S}_n(\Sigma) \to \mathcal{S}_n(\Sigma) \otimes \mathcal{O}_q(SL(n)), \quad \mathrm{fl}(x \otimes y) = y \otimes x.$$

Theorem 9.6 (See [15, 38] for $n = 2$ and [28] for $n = 3$.). *The image of* Θ_c *is equal to* $HH^0(\mathcal{S}_n(\mathrm{cut}_c\,\Sigma))$.

Proof. As the image of Θ_c is equal to the image of $\bar{\Theta}_c : \mathcal{S}_n(\Sigma) \to \mathcal{S}_n(\mathrm{cut}_c\,\Sigma)$, we can work with projected skein algebras. More specifically, we will assume that one end v of c is a boundary ideal point of Σ, as we can remove a disk from Σ, adjacent to v and disjoint from c, if necessary. We will present c, a_1, a_2 in $\mathrm{cut}_c\,\Sigma_{a_1\triangle a_2}$ as in Figure 22, with v in the bottom.

Let $(\mathrm{cut}_c\,\Sigma)_{a_1\wedge a_2}$ denote $(\mathrm{cut}_c\,\Sigma)_{a_1\triangle a_2} - \partial_0\mathfrak{T}$, for simplicity, where $\partial_0\mathfrak{T}$ is the bottom edge of $\mathfrak{T}$, as in Subsection 8.1. Note that we can identify the image of $\mathcal{S}_n((\mathrm{cut}_c\,\Sigma)_{a_1\wedge a_2}) \to \mathcal{S}_n((\mathrm{cut}_c\,\Sigma)_{a_1\triangle a_2})$ with $\overline{\mathcal{S}}_n(\Sigma)$. We will use this identification below.

Let

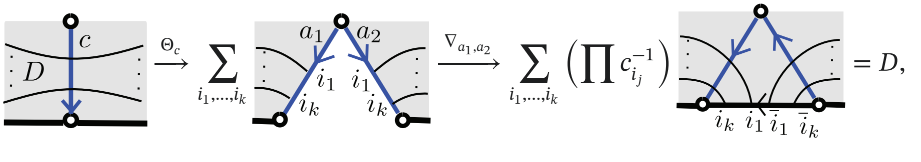
$$\nabla_{a_1,a_2} : \mathcal{S}_n(\mathrm{cut}_c\,\Sigma) \to \mathcal{S}_n((\mathrm{cut}_c\,\Sigma)_{a_1\triangle a_2}), \quad \nabla_{a_1,a_2} = \mathrm{glue}_{a_1,a_2}\circ \mathrm{htw}_{a_2}^{-1}.$$

It is an isomorphism by Propositions 4.11 and 8.1.

Lemma 9.7.

(1) $\nabla_{a_1,a_2}\Theta_c(\mathcal{S}_n(\Sigma)) = \mathcal{S}_n((\mathrm{cut}_c\,\Sigma)_{a_1\wedge a_2}) \simeq \overline{\mathcal{S}}_n(\Sigma)$.
(2) ∇_{a_1,a_2} *restricted to* $\mathrm{Im}\,\Theta_c$ *is the inverse to* Θ_c.

Proof. For every stated web diagram D on Σ, we have



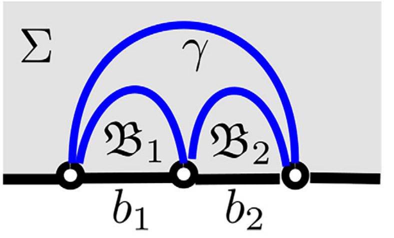


FIGURE 23 The curve γ.

with the last equality by Relation (49). Hence,

$$\nabla_{a_1,a_2}\Theta_c = \mathrm{id}_{\overline{S}_n(\Sigma)}.$$

As ∇_{a_1,a_2} restricted to $\mathrm{Im}\Theta_c$ is a bijection onto $\overline{S}_n(\Sigma)$, this identity implies (2). □

For any collection of boundary edges $X \subset \partial\Sigma$, let $\mathrm{Im}_\Sigma\, \mathcal{S}_n(\Sigma - X)$ denote the image $\iota_*(\mathcal{S}_n(\Sigma - X))$ in $\mathcal{S}_n(\Sigma)$ of the homomorphism induced by $\iota : \Sigma - X \hookrightarrow \Sigma$.

Lemma 9.8. *Let b_1, b_2 be two boundary components of Σ separated by a puncture. Then the embedding*

$$\mathrm{Im}_\Sigma\, \mathcal{S}_n(\Sigma - (b_1 \cup b_2)) \hookrightarrow \mathrm{Im}_\Sigma\, \mathcal{S}_n(\Sigma - b_1) \cap \mathrm{Im}_\Sigma\, \mathcal{S}_n(\Sigma - b_2)$$

is onto.

Proof. Consider an arc γ parallel to $b_1 \cup b_2$, as in Figure 23.

Then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Im}_\Sigma\, \mathcal{S}_n(\Sigma - (b_1 \cup b_2)) & \hookrightarrow & \mathrm{Im}_\Sigma\, \mathcal{S}_n(\Sigma - b_1) \cap \mathrm{Im}_\Sigma\, \mathcal{S}_n(\Sigma - b_2) \\ \downarrow \Theta_\gamma & & \downarrow \Theta_\gamma \\ \mathrm{Im}_{\mathrm{cut}_\gamma \Sigma}\, \mathcal{S}_n(\mathrm{cut}_\gamma \Sigma - (b_1 \cup b_2)) & \hookrightarrow & \mathrm{Im}_{\mathrm{cut}_\gamma \Sigma}\, \mathcal{S}_n(\mathrm{cut}_\gamma \Sigma - b_1) \cap \mathrm{Im}_{\mathrm{cut}_\gamma \Sigma}\, \mathcal{S}_n(\mathrm{cut}_\gamma \Sigma - b_2). \end{array}$$

By Proposition 8.6, both homomorphisms Θ_γ in the diagram are 1-1 and, hence, it is enough to show that the embedding in the bottom line is onto. As the skein algebra of a surface is the tensor product of its connected components, it is enough to show the statement of the lemma for the triangle $\mathfrak{T}$ bounded by b_1, b_2 and γ. By Proposition 8.2, $\mathcal{S}_n(\mathfrak{T})$ is isomorphic with $\mathcal{S}_n(\mathfrak{B}_1) \otimes \mathcal{S}_n(\mathfrak{B}_2)$, as an R-module, where $\mathfrak{B}_i$ for $i = 1$ are disjoint bigons in $\mathfrak{T}$ such that $b_i \subset \partial\mathfrak{B}_i$, as in Figure 23. Through that isomorphism the statement of the lemma reduces to

$$\mathcal{O}_q(SL(n)) \otimes R \cdot 1 \cap R \cdot 1 \otimes \mathcal{O}_q(SL(n)) = \mathcal{S}_n(\mathfrak{M}) = R,$$

where 1 is the identify in $\mathcal{O}_q(SL(n))$ and $\mathfrak{M}$ is the monogon $\mathfrak{T} - b_1 - b_2$. That follows from the fact that $R \cdot 1$ is a direct summand of $\mathcal{O}_q(SL(n))$, by [19, Proposition 5.1.1]. □

Let us continue with the proof of the theorem. To prove that the inclusion

$$\Theta_c(\mathcal{S}_n(\Sigma)) \subset Ker\,(\Delta_{a_1} - \mathrm{fl}\circ_{a_2}\Delta)$$

is an equality, we will show that for any $y \in Ker\,(\Delta_{a_1} - \mathrm{fl}\circ_{a_2}\Delta)$,

$$x = \nabla_{a_1,a_2}(y) \in \mathcal{S}_n((\mathrm{cut}_c\,\Sigma)_{a_1 \triangle a_2})$$

lies in $\mathcal{S}_n((\mathrm{cut}_c\,\Sigma)_{a_1 \wedge a_2})$. Then

$$\Theta_c(x) = y \tag{97}$$

by Proposition 9.7(2).

As mentioned above, it remains to be shown that $x \in \mathcal{S}_n((\mathrm{cut}_c\,\Sigma)_{a_1 \wedge a_2})$. Recall that Δ_{a_1} and $\mathrm{fl}_{a_2}\circ\Delta$ map y into $\mathcal{S}_n(\mathrm{cut}_c\,\Sigma) \otimes \mathcal{S}_n(\mathfrak{B})$, where the left and right edges of $\mathfrak{B}$ are denoted by e_l, e_r, respectively.

Let

$$z = \Delta_{a_1}(y) = \mathrm{fl}\circ_{a_2}\Delta(y).$$

By Proposition 9.7(2),

$$\nabla_{a_1,b_l}(z) = \nabla_{a_1,b_l}\Delta_{a_1}(y) = y,$$

where y at the end of the above equation is a skein in $\mathrm{cut}_c\,\Sigma \sqcup \mathfrak{B}/(a_1 = b_l)$ identified with $\mathrm{cut}_c\,\Sigma$. Then

$$\nabla_{b_r,a_2}\nabla_{a_1,b_l}(z) = \nabla_{b_r,a_2}(y) = \nabla_{b_r,a_2}\Theta_c(x) = x,$$

by (97).

By Proposition 9.7(1), $\nabla_{a_1,b_l}\Delta_{a_1}(y)$ belongs to $\mathcal{S}_n((\Sigma \sqcup \mathfrak{B})_{a_1 \wedge b_l})$. By applying ∇_{b_r,a_2} to it, we see that

$$x \in \mathcal{S}_n((\Sigma \sqcup \mathfrak{B})_{a_1 \wedge b_l, b_r \triangle a_2}).$$

As nablas for disjoint pairs of edges commute,

$$x = \nabla_{b_r,a_2}\nabla_{a_1,b_l}(z)$$

and by an analogous argument

$$x \in \mathcal{S}_n((\Sigma \sqcup \mathfrak{B})_{a_1 \triangle b_l, b_r \wedge a_2}).$$

Now the statement follows from Lemma 9.8. □

The construction of the inverse of the splitting map (Lemma 9.7) implies the following:

Corollary 9.9. *For any union C of ideal boundary arcs of Σ,*

$$\Theta_c(\mathcal{S}_n(\Sigma - C)) = \Theta_c(\mathcal{S}_n(\Sigma)) \cap \mathcal{S}_n(\mathrm{cut}_c\,\Sigma - C) \quad \textit{in } \mathcal{S}_n(\mathrm{cut}_c\Sigma).$$

Proof. The inclusion $\subset$ is obvious and the opposite inclusion $\supset$ is obtained by applying the inverse map to Θ_c of Lemma 9.7. □

Proof of Theorem 9.4.

$$\bar{\mathcal{S}}_n(\Sigma) \subset \{x \in \mathcal{S}_n(\Sigma) : \Theta_{c_p}(x) = x \otimes 1\}$$

is obvious. The opposite inclusion, $\supset$, is immediate for $\Sigma = \mathfrak{M}$: in that case p is the only vertex of $\mathfrak{M}$, $\mathfrak{M}_{c_p} = \mathfrak{B}$, where $c_p = \partial_r\mathfrak{B}$ and if $\Delta_{c_p} x = x \otimes 1$ for $x \in \mathfrak{M}_{c_p}$ then by applying $\varepsilon \otimes 1$ we obtain $x = \varepsilon(x)1 \in R$.

More generally, let Σ be a disjoint union of an essentially bordered Σ' and of $\mathfrak{M}$ (with a vertex p and an arc c_p as above). Let $\Delta_{c_p} x = x \otimes 1$ for $x \in \Sigma' \sqcup \mathfrak{M}_{c_p}$. Then by Theorem 8.8, x can be written as $x = \sum_{i=1}^N y_i \otimes z_i$, where $y_1, \dots, y_N \in \mathcal{S}_n(\Sigma')$ are linearly independent and $z_1, \dots, z_N \in \mathcal{S}_n(\mathfrak{B})$. Then $\Delta_{c_p} z_i = z_i \otimes 1$ for every i and, hence, $z_1, \dots, z_N \in R$. That concludes the proof of the inclusion $\supset$ in that case.

Let $\Theta_{c_p}(x) = x \otimes 1$ now for some arbitrary Σ, p, c_p (as above) and $x \in \mathcal{S}_n(\Sigma_p)$. We need to show that x lies in the image of $\mathcal{S}_n(\Sigma_p - c_p)$ in $\mathcal{S}_n(\Sigma_p)$.

Let c'_p be an arc in Σ_p parallel to c_p, splitting Σ_p into Σ'_p and a bigon bounded by c_p and c'_p. Then

$$\Theta_{c_p}\Theta_{c'_p}(x) = \Theta_{c'_p}\Theta_{c_p}(x) = \Theta_{c'_p}(x) \otimes 1.$$

As $\Theta_{c'_p}(x) \in \mathcal{S}_n(\Sigma'_p \sqcup \mathfrak{M}_{c_p})$ the previous case implies that $\Theta_{c'_p}(x)$ is of the form $y \otimes 1 \in \mathcal{S}_n(\Sigma'_p) \otimes \mathcal{S}_n(\mathfrak{M}_{c_p})$, for some $y \in \mathcal{S}_n(\Sigma'_p)$. By Corollary 9.9 above for $C = c_p$, we have $\Theta_{c'_p}(x) \in \Theta_{c'_p}(\mathcal{S}_n(\Sigma_p - c_p))$. As $\Theta_{c'_p}$ is 1-1, x lies in $\mathcal{S}_n(\Sigma_p - c_p)$. $\square$

10 | RELATION TO FACTORIZATION HOMOLOGY, SKEIN CATEGORIES, AND LATTICE GAUGE THEORY

10.1 | Factorization homology

Factorization homology was introduced by Beilinson and Drinfeld [6] in the setting of conformal field theory and then in [1, 2, 47] in the topological context. Given an algebraic object $\mathcal{A}$ called an E_n-algebra, it associates to oriented n-dimensional manifolds (with boundary) M categories $\int_M \mathcal{A}$, which are linear over a certain ring of coefficients R.

For $n = 2$, the notion of E_2-algebra is equivalent to that of a braided tensor category. Important examples of such categories are the categories of finite-dimensional representations of quantum groups $U_q(\mathfrak{g})$. The factorization homology of surfaces for these categories was studied in [7], where the authors proved that if $\partial\Sigma = S^1$ then $\int_M \mathcal{A}$ is equivalent to the category of left modules over a certain algebra A_Σ (depending on the Lie algebra $\mathfrak{g}$).

The factorization homology of [7] and skein categories (discussed below) are theories parallel to ours. We show:

Theorem 10.1. *Let $R = \Bbbk(q)$ for a field $\Bbbk$ and let E_2 be the category of type 1 finite-dimensional representations of $U_q(sl(n))$ (over R). Then $A_{\Sigma_{g,p}}$ is isomorphic to $\mathcal{S}_n(\Sigma^*_{g,p})$ (as R-algebras) for every $g \geqslant 0, p \geqslant 1$.*

On the one hand, factorization homology of [7] is more general in that it is defined for all semi-simple Lie algebras and it can be viewed as a quantization of the entire moduli stacks of representations, rather than just the character varieties.

On the other hand, one may consider our theory more elementary because it does not involve higher category theory. More importantly, our stated skein modules are defined over nonfield rings of coefficients, for all 3-dimensional manifolds, and we worked out their theory for surfaces with multiple boundary components and multiple markings. Furthermore, unlike our stated skein algebras, the algebras A_Σ of [7] are defined up to an isomorphism only.

The existence of our stated skein algebras over $\mathbb{Z}[q^{\frac{1}{2n}}]$ allows to construct quantum trace homomorphisms of the stated skein algebras into quantum tori, over any ground ring. It was done for $n = 2$ by Bonahon–Wang, [12], and more generally in [15, 44]. The construction of quantum trace was generalized to all n in [45], where two versions of quantum trace maps, quantizing, respectively, the length coordinates and the shear coordinates trace formulae, were introduced.

Embeddings into quantum tori allow to study algebraic properties $\mathcal{S}_n(\Sigma)$ and their representations.

The above works relate our algebras to the theory of quantum cluster algebras, which provide alternative quantizations of character varieties. Further connections to quantum cluster algebras are through [13, 29, 59].

For completeness, let us summarize briefly the construction of the factorization homology of [7] (in dimension 2): it is based on the $(\infty, 1)$-category Mfld^2 whose objects are oriented surfaces (with boundary), morphisms are given by their embeddings, 2-morphisms are isotopies between embeddings and higher order morphisms are isotopies between them. This category has a monoidal structure given by disjoint embeddings and it has a full subcategory Disk^2 consisting of disks (with partial boundaries) and their embeddings. One can prove that any pivotal ribbon category defines a symmetric monoidal functor into a symmetric monoidal $(\infty, 1)$-category C whose objects are certain presentable categories and its monoidal structure is given by the categorical product. Then $\int_M \mathcal{A}$ is the left Kan extension

$$\begin{array}{ccc} \mathsf{Disk}^2 & \xrightarrow{F} & C \\ \downarrow & \nearrow_{\int_{-} \mathcal{A}} & \\ \mathsf{Mfld}^2 & & \end{array},$$

which, in more concrete terms, is a certain colimit in C over all possible embeddings of collections of disks into a given surface.

Proof of Theorem 10.1. Let $\mathfrak{g} = sl(n)$. Then the algebra $\mathfrak{F}_A$ of [7] is isomorphic with $\mathcal{S}_n(\Sigma^*_{0,2})$, see [7, section 6.1] Furthermore, one can see that $A_{\Sigma_{1,1}}$ coincides with $\mathcal{S}_n(\Sigma^*_{1,1})$, by comparing the "gluing pattern" of $\Sigma_{1,1}$ in [7, Theorem 5.11] with ours in Figure 20 (right) or by [7, Corollary 6.8].

By [7, Theorem 5.11], $A_{\Sigma_{g,p}}$ is the braided tensor product of $p - 1$ copies of $A_{\Sigma_{0,2}} = \mathcal{S}_n(\Sigma^*_{0,2})$ and g copies of $A_{\Sigma_{1,1}} = \mathcal{S}_n(\Sigma^*_{1,1})$. Now the statement follows from Example 8.4. □

10.2 | Skein categories

Skein categories are categorical analogous of skein algebras introduced by Walker and Johnson-Freyd [66, p. 70], [30, section 9]. A framing of a point p on surface Σ is a choice of a nonzero vector

$v \in T_p\Sigma$. Let $\mathcal{V}$ be a ribbon category, linear over R. The *ribbon category* $\mathsf{Rib}_{\mathcal{V}}(\Sigma)$ has objects given by finite sets of framed, signed disjoint points of Σ. Its morphisms are R-linear combinations of ribbon graphs in $\Sigma \times [0,1]$ (in the sense of Reshetikhin–Turaev) whose edges are decorated with objects of $\mathcal{V}$ and coupons are decorated with intertwiners. The ends of a ribbon graph Γ in $\Sigma \times \{0\}$ (respectively, in $\Sigma \times \{1\}$) determine the source (and, respectively, the target) of the morphism Γ.

For an oriented arc C, the Reshetikhin–Turaev construction defines a functor RT : $\mathsf{Rib}_{\mathcal{V}}(C \times [0,1]) \to \mathcal{V}$. (This functor was denoted by RT_0 for the ribbon category $\mathcal{C}_n$ of Subsection 3.1.)

The *skein category*, $\mathsf{Sk}_{\mathcal{V}}(\Sigma)$ is $\mathsf{Rib}_{\mathcal{V}}(\Sigma)$ modulo the relation on morphisms $\sum c_i\Gamma_i \sim 0$, whenever a restriction of $\sum c_i\Gamma_i$ to a certain cube $C \times [0,1] \times [0,1]$ is in the kernel of Reshetikhin–Turaev evaluation, RT.

Cooke proved that for the category $\mathcal{V}$ of finite-dimensional representations of a quantum group $U_q(\mathfrak{g})$, the skein category of any surface coincides with its factorization homology $\int_\Sigma \mathcal{V}$, [14]. Furthermore, [21, 27, 43] proved that for $\mathfrak{g} = sl(2)$ and R a field, the skein category of Σ with $\partial\Sigma = S^1$ is equivalent to the category of left modules over $\mathcal{S}_2(\Sigma)$. By Theorem 10.1, we obtain

Corollary 10.2. *For $\mathfrak{g} = sl(n)$ (for any n) and R a field, the skein category of Σ with $\partial\Sigma = S^1$ is equivalent to the category of left modules over $\mathcal{S}_n(\Sigma)$.*

Note that this equivalence is quite nonintuitive, as skein categories are built of unstated ribbon graphs with ends in $\Sigma \times \{0,1\}$ rather than of stated webs with ends in $\partial\Sigma \times (-1,1)$ considered in stated skein algebras.

Corollary 10.2 asserts that the $sl(2)$-skein category of Σ with $\partial\Sigma = S^1$ has an internal algebra object isomorphic to $\mathcal{S}_2(\Sigma)$. In fact, by [27, Theorem 1.1] and [21, Theorem 5.3] this internal algebra object is isomorphic to $\mathcal{S}_2(\Sigma)$ as a $\mathcal{O}q(SL(2))$-comodule algebra. We expect that this statement generalizes to $sl(n)$ for all n.

10.3 | Lattice gauge theory, quantum moduli spaces

A ciliated graph Γ is a finite graph with additional data specifying for each vertex of Γ a linear order of half-edges adjacent to it. Each ciliated graph Γ is ribbon and, hence, defines a surface that contracts onto Γ. Inspired by an earlier Fock–Rosly's work [24], Alekseev–Grosse–Schomerus and Buffenoir–Roche quantized moduli spaces of flat connections on such surfaces in [3, 4, 8, 9]. (See also [10].) Specifically, for each ciliated graph Γ and a quantized coordinate Hopf algebra $O_q(G)$ they have defined an $O_q(G)$-comodule $\mathcal{L}(\Gamma)$, called quantum moduli space, quantizing a (properly defined) algebra of functions on the space of flat G-connections on Γ.

Let $\Sigma(\Gamma)$ be a surface without boundary realizing the ribbon structure on Γ and let $\Sigma^0(\Gamma)$ be $\Sigma(\Gamma)$ with one of its punctures blown up into a disk, as in Figure 21. (Hence, $\Sigma^0(\Gamma) = \Sigma(\Gamma)_p$ for some puncture p, in the notation of Subsection 9.1. Note that $\Sigma^0(\Gamma)$ and $\Sigma(\Gamma)$ are uniquely determined up to a homeomorphism.) Then, as observed in [7], the defining equations for $\mathcal{L}(\Gamma)$ coincide with those induced by the gluing patterns of [7]. In other words, quantum moduli spaces are determined by the factorization homology of [7] and, consequently, for $G = SL(n)$,

$$\mathcal{L}(\Gamma) \simeq \mathcal{S}_n(\Sigma^0(\Gamma))$$

as $O_q(G)$-comodule algebras. This result was observed independently by the first author and proved by [37] for $n = 2$.

By Theorem 9.4, the coinvariant subalgebra $\mathcal{L}(\Gamma)^{\mathcal{O}_q(SL(n))}$ is isomorphic with our projected skein algebra $\bar{S}_n(\Sigma(\Gamma))$. This result generalizes the results of [10, 37] for $n = 2$.

11 | RELATION TO OTHER KNOWN CASES

11.1 | Compatibility with stated Kauffman bracket skein modules of 3-manifolds

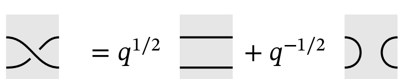
The stated Kauffman bracket skein algebras (of surfaces) of the first author [41] were generalized to stated skein modules of marked 3-manifolds in [11] (cf. also [43]). We are going to prove that these modules are isomorphic with our $SL(2)$-skein modules, $S_2(M, \mathcal{N})$.

To relate these modules to ours, let us replace the variable q of [41] with $q^{1/2}$ and denote the resulted stated Kauffman bracket skein module by $\mathscr{S}(M, \mathcal{N})_{q^{1/2}}$. Let a *framed link* in $(M, \mathcal{N})$ be a nonoriented 2-web without sinks nor sources, stated by signs $\pm$. By definition $\mathscr{S}(M, \mathcal{N})_{q^{1/2}}$ is the R-module freely spanned by isotopy classes of framed links subject to Relations (98)–(101).

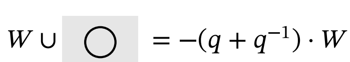
Theorem 11.1. *Suppose $(M, \mathcal{N})$ is a marked 3-manifold.*

(1) *There is a unique R-linear isomorphism $\Lambda : \mathscr{S}(M, \mathcal{N})_{q^{1/2}} \to S_2(M, \mathcal{N})$ that maps framed links α to stated 2-webs by assigning arbitrary orientations to them, and changing the minus state to 1 and the plus state to 2.*
(2) *The splitting homomorphism of [11, 41] coincides with ours through Λ.*

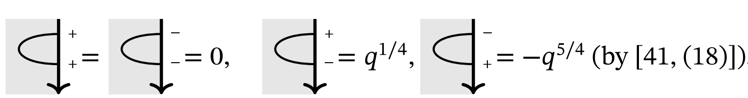
Proof of Theorem 11.1. Let $\mathcal{L}(M, \mathcal{N})$ be the set of all stated framed links in $(M, \mathcal{N})$. First let us record the defining relations for $\mathscr{S}(M, \mathcal{N})_{q^{1/2}}$:



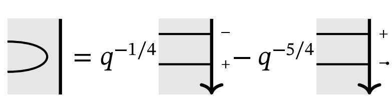
(98)



(99)



(100)

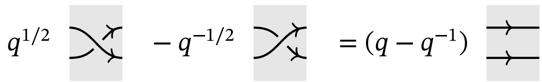


(101)

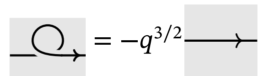
(This last equality is a consequence of applying a half-twist to [41, (13)].)

For convenience, we draw diagrams with the arrow down, rather than up as in [11, 41], to make them compatible with the skein relations of our $S_2(M, \mathcal{N})$. As the half-twist is an invertible operation, they form an alternative set of defining skein relations of the stated skein module of [11, 41].

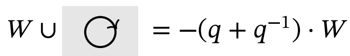
On the other hand, for $n = 2$ our skein relations are:



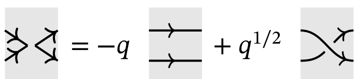
(102)



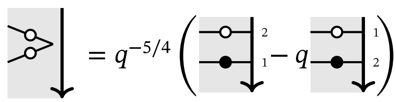
(103)



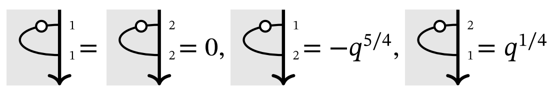
(104)



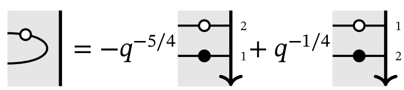
(105)



(106)



(107)

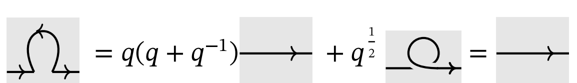


(108)

(By Proposition 7.5, Relation (50) is redundant.)

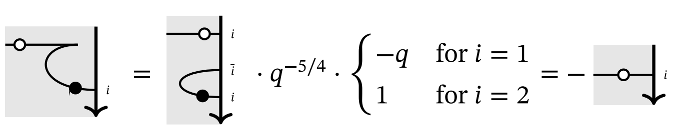
Lemma 11.2. *The value of any framed link T in $(M, \mathcal{N})$ considered as a 2-web in $\mathcal{S}_2(M, \mathcal{N})$ does not depend on the orientation of T.*

Proof. By (105), (103), and (104), we have



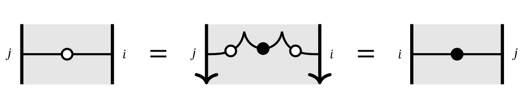
(109)

For any arc with a 2-vertex near its end, we have

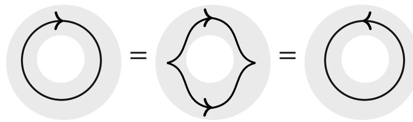


(110)

Hence, for any arc we have,



(Both ends may lie on the same marking in $\mathcal{N}$.) Similarly, by (109), for loops we have

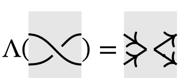


□

By Lemma 11.2, assigning arbitrary orientations of links defines a map $\Lambda : R\mathcal{L}(M, \mathcal{N}) \to \mathcal{S}_2(M, \mathcal{N})$ preserving Relations (99)–(101). To see that (98) is preserved as well, we start with the following combinatorial observation:

We say that a crossing or in an unoriented framed link L in $(M, \mathcal{N})$ is of -*type* if L with that smoothing can be oriented, so that it is a 2-web with only two 2-vertices, looking like . We define a crossing of -type analogously. It is straightforward to verify that every crossing is of one of these two types. (However, it can be of both types simultaneously, if the crossing involves an arc.)

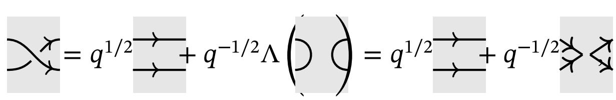
Lemma 11.3. *If a crossing or in L is of* -type *then for that smoothing of L we have*



Proof. There are two possibilities.

(1) The NE end of is connected to the NW or the SW end. Then the statement follows by introducing two 2-valent vertices as in (109).
(2) The NE end is connected to the marking. Then one of the SE or SW ends must be connected to a marking and the statement follows by applying (110) twice near the markings. □

Suppose that the crossing on the left side of (98) is of -type. Then by the above lemma, Λ maps that skein relation to



which coincides with (105) in $\mathcal{S}_2(M, \mathcal{N})$. The proof for a crossing of -type is analogous. Thus, the R-linear map $\Lambda : \mathcal{S}(M, \mathcal{N})_{q^{1/2}} \to \mathcal{S}_2(M, \mathcal{N})$ is well-defined.

We prove that Λ is an isomorphism by constructing its inverse: Consider first the map $R\mathcal{W}_2(M, \mathcal{N}) \to \mathcal{S}(M, \mathcal{N})_{q^{1/2}}$ sending webs α to $(-1)^{|V_2(\alpha)|}\bar{\alpha}$, where $\bar{\alpha}$ is the result of forgetting the orientation and of smoothing all the 2-valent vertices. It is immediate to see that it factors through Relations (102)–(108) into a homomorphism $\mathcal{S}_2(M, \mathcal{N}) \to \mathcal{S}(M, \mathcal{N})_{q^{1/2}}$. As $\mathcal{S}(M, \mathcal{N})_{q^{1/2}}$

and $\mathcal{S}_2(M,\mathcal{N})$ are spanned by links (i.e., webs with no sinks nor sources) and as Λ and the above map $\mathcal{S}_2(M,\mathcal{N}) \to \mathscr{S}(M,\mathcal{N})_{q^{1/2}}$ are inverses of each other on links, the statement follows.

The proof of part (2) is straightforward. □

11.2 | Compatibility with the SU(n)-skein modules

In this subsection, we are going to show that for any 3-manifold M and any n our skein module $\mathcal{S}_n(M,\emptyset)$ is isomorphic with the $SU(n)$-skein module introduced by the second author in [58]. That module is built of *based n-webs* in M that are defined as our n-webs in $(M,\emptyset)$, except that the half-edges incident to any of their n-valent vertices are linearly ordered. We denote the set of all such webs up to isotopy by $\mathcal{W}_n^b(M)$. Let $\mathcal{S}_n^b(M)$ be the quotient of the R-module freely generated by $\mathcal{W}_n^b(M)$ subject to Relations (16)–(19), which are the internal annihilators of the functor RT_0. For an invertible $u \in R$ let $\mathcal{S}_n^b(M;u)$ be an R-module defined as $\mathcal{S}_n^b(M)$, except that the right side of (19) is multiplied by u. From the definition, we see that $\mathcal{S}_n^b(M;u)$ is isomorphic to $\mathcal{S}_n^b(M)$ via the map $\alpha \to u^{\#sinks(\alpha)}\alpha$. The SU_n-skein module defined in [58] is actually $\mathcal{S}_n^b(M;(-q)^{n(n-1)/2})$.

Given a based n-web α, let α° denote the underlying n-web in $\mathcal{S}_n(M,\mathcal{N})$. Recall that every oriented 3-manifold has a spin structure.

Theorem 11.4. *Let M be an oriented 3-manifold.*

(a) *For n odd, the operation $\alpha \to \alpha^\circ$ on based n-webs extends to an isomorphism $\mathcal{S}_n^b(M) \cong \mathcal{S}_n(M,\emptyset)$.*
(b) *Every spin structure on M defines a function $s : \mathcal{W}_n^b(M) \to \{1,-1\}$ such that the map $\alpha \to f(\alpha) = (-1)^{s(\alpha)}\alpha^\circ$ induces a unique R-linear isomorphism $\mathcal{S}_n^b(M) \cong \mathcal{S}_n(M,\emptyset)$.*

Proof.

(a) For n odd, Relations (20) that are consequences of the defining relations, (16)–(19), show that a based n-web α, as an element of $\mathcal{S}_n^b(M)$, is determined by α°. Furthermore, the defining relations (16)–(19) coincide with the defining relations (25)–(28).
(b) Let n be even now. Fix a Riemannian metric on M and let UM be a principal $SO(3)$-bundle associated to the tangent bundle of M. A section at a point is the group $SO(3)$, which can be identified with the set of all ordered, positively oriented, orthonormal bases (v_1,v_2,v_3) of the tangent space at the point. Any such ordered orthonormal basis is totally determined by the first two vectors. A smooth embedding $a : [0,1] \to M$ equipped with a normal vector field defines a lift $\tilde{a} : [0,1] \to UM$ where the first and the second vectors are, respectively, the velocity vector and the framing vector, normalized to have length 1. For a based n-web α define $s(\alpha) \in \{0,1\}$ as follows: First isotope α so that the framing is normal everywhere, and at every n-valent vertex the n half edges have the same velocity vector. The latter condition implies the lift of the endpoints at all the half-edges at an n-valent vertex are the same. As n is even the lifts of all edges of α and of all its circle components form a $\mathbb{Z}/2$ one-cycle $\tilde{\alpha}$ of UM. Recall that a spin structure s of M can be identified with a cohomology class in $H^1(UM,\mathbb{Z}/2)$ which is nontrivial at the section at every point of M. Let $s(\alpha)$ be the evaluation of the spin structure, considered as an element of $H^1(UM,\mathbb{Z}/2)$ on $\tilde{\alpha}$. Clearly, $s(\alpha)$ depends only on the isotopy class of α. From the definition, $s(\alpha) = 1$ if α is the trivial loop. If α' is the result of adding a positive twist to an edge or loop of α then $s(\alpha') = -s(\alpha)$. Thus, f maps the defining relations

(16)–(19), respectively, to the defining relations (25)-(28). Hence, it extends to a well-defined R-linear map $f : \mathcal{S}_n^b(M) \to \mathcal{S}_n(M, \emptyset)$.

For the inverse, note that Relations (20) show that the map $R\mathcal{W}_n(M, \emptyset) \to \mathcal{S}_n^b(M)$ sending $\alpha^\circ \to (-1)^{s(\alpha)}\alpha$ for every based n-web α is well-defined. As $\bar{f}$ maps the defining relations (25)–(28) to the defining relations (16)–(19), it descends to a well-defined R-linear map from $\mathcal{S}_n(M, \emptyset)$ to $\mathcal{S}_n(M)$, which is the inverse of f. □

11.3 | Compatibility with Higgins' SL3 skein algebras

In his recent work [28], Higgins introduced his version of stated SL_3-skein algebras, denoted by $\mathcal{S}_q^{SL_3}(\Sigma)$, of punctured bordered surfaces Σ. His skein algebra is the R-module freely generated by 3-webs stated by $-1, 0, 1$, subject to his system of skein relations.

Let us identify Higgins's states $1, 0, -1$ of Higgins with our states 1,2,3, respectively.

Theorem 11.5. *For any punctured bordered surface Σ, there is an isomorphism $\phi : \mathcal{S}_q^{SL_3}(\Sigma) \to \mathcal{S}_3(\Sigma)$ sending every stated 3-web α to*

$$(-1)^{h_-(\alpha)+v_3(\alpha)} \cdot q^{(3v_3(\alpha)+S_{in}(\alpha)-S_{out}(\alpha))/2} \cdot \alpha, \tag{111}$$

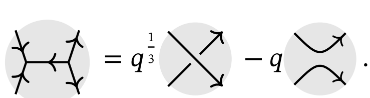
where

- *$v_3(\alpha)$ is the number of 3-valent sources and sinks of α,*
- *$h_-(\alpha)$ is the number of Higgins' -1 states in α, and*
- *$S_{in}(\alpha)$ and $S_{out}(\alpha)$ are sums of Higgins' states of all edges edges coming into and coming out of the boundary, respectively.*

We thank V. Higgins for suggesting the above formula to us.

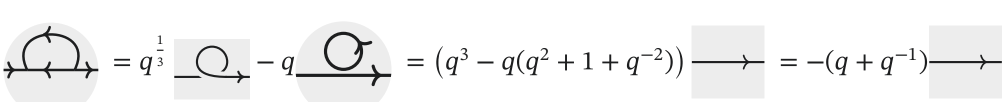
Furthermore, our theory recovers most of Higgins' for $n = 3$. Specifically, Higgins constructed splitting homomorphisms for his skein algebras and an isomorphism $\mathcal{S}_q^{SL_3}(\Sigma) \simeq \mathcal{O}_q(sl_3)$. It is straightforward to check that these maps coincide with ours through ϕ. Higgins also proved a version of Theorem 8.2 for $n = 3$. (However, additionally, he defined bases of his skein algebras $\mathcal{S}_q^{SL_3}(\Sigma)$ for all pb surfaces Σ. There appears no easy generalization of these bases to $n > 3$, as the relay on the confluence method of [60], which works for $n = 2$ and 3 only.)

Proof of Theorem 11.5. By Identity (51),



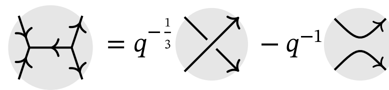
(112)

By capping the skeins of Equation (112) from the top, we get



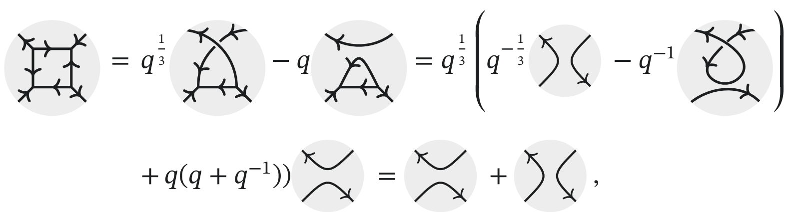
(113)

Taking the reflection of Equation (112),



(114)

Hence, we have



(115)

by Equation (113).

Let $\mathcal{W}_3^H(\Sigma)$ be the set of all isotopy classes of 3-webs over Σ with states $-1, 0, 1$. Consider an R-linear isomorphism $\tilde{\phi} : R\mathcal{W}_3^H(\Sigma) \to R\mathcal{W}_3(\Sigma)$ given on stated 3-webs α by (111).

By definition, the Higgins algebra $\mathcal{S}_q^{SL_3}(\Sigma)$ is $R\mathcal{W}_3^H(\Sigma)$ modulo his skein relations [28, (I1a)–(I4b),(B1)–(B4)]. It is easy to see that Higgins' internal relations are pullbacks under $\tilde{\phi}$ of Relations (112), (114), (115), (113), and (45), respectively.

Higgins' boundary relations (B1) and (B3) are pullbacks of Relation (54), (B4) is a pullback of (54), and, finally, (B2) is a pullback of (114) at the boundary combined with (58).

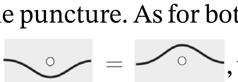
Hence, we showed that $\tilde{\phi}$ descends to an R-linear homomorphism $\phi : \mathcal{S}_q^{SL_3}(\Sigma) \to \mathcal{S}_3(\Sigma)$. The definition of ϕ suggests an obvious inverse homomorphism $\mathcal{S}_3(\Sigma) \to \mathcal{S}_q^{SL_3}(\Sigma)$ and, indeed, one can verify that it is well-defined. However, as checking that it respects our relation (46) requires a lengthy calculation, we enclose an alternative proof of ϕ being an isomorphism:

As it is clearly a surjective algebra homomorphism, it remains to show that ϕ is injective. From the definition it clear that ϕ commutes with the splitting homomorphism. As $\mathcal{S}_q^{SL_3}(\Sigma)$ satisfies the splitting homomorphism, $\mathcal{S}_q^{SL_3}(\Sigma) = \mathcal{O}_q(sl_3)$, and the gluing over a triangle is given by the same isomorphism as described in Theorem 8.2. Theorem 8.8 is also valid with $\mathcal{S}_3$ replaced by $\mathcal{S}_q^{SL_3}$. Part (2) of Theorem 8.2 shows that ϕ is an isomorphism when Σ is essentially bordered.

Suppose Σ is a connected, having empty boundary, and at least one puncture. Let c be an ideal arc of Σ. In the commutative diagram

$$\begin{array}{ccc} \mathcal{S}_q^{SL_3}(\Sigma) & \xrightarrow{\Theta_c} & \mathcal{S}_q^{SL_3}(\mathsf{cut}_c\Sigma) \\ \big\downarrow \phi & & \big\downarrow \cong \\ \mathcal{S}_3(\Sigma) & \xrightarrow{\Theta_c} & \mathcal{S}_3(\mathsf{cut}_c\Sigma) \end{array}$$

the upper Θ_c is injective by Higgins result, which forces ϕ to be injective.

Consider the remaining case when Σ is a closed surface without ideal point. Remove a point p from Σ to obtained a pb surface Σ' having one puncture. As for both $\mathcal{S} = \mathcal{S}_3$ and $\mathcal{S} = \mathcal{S}_q^{SL_3}$ we have $\mathcal{S}(\Sigma) = \mathcal{S}(\Sigma')/\mathrm{Rel}$, where Rel is the relation , we conclude that $\phi : \mathcal{S}_q^{SL_3}(\Sigma) \to \mathcal{S}_3(\Sigma)$ is an isomorphism. This completes the proof. □

11.4 | Relation to the Frohman–Sikora SU(3)-skein algebras

Frohman and the second author considered in [25] the "reduced $SU(3)$-skein algebra" of marked surfaces built of unstated 3-webs, subject to the $SU(3)$-skein relations of [40], extended by certain boundary skein relations, which depend on an invertible parameter $a \in R$. We denote that algebra by $\mathcal{S}_{FS}(\Sigma, B)$ for the value 1 of that parameter.

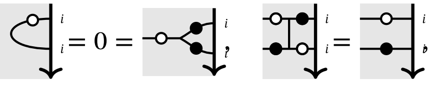
For an unstated 3-web α in Σ, let $\eta_+(\alpha)$ (respectively, $\eta_-(\alpha)$) denote α stated with 3s (respectively, 1's) at all its ends.

Theorem 11.6.

(1) *For any punctured bordered surface Σ, the above operations extent to R-linear homomorphisms*
$$\eta_\pm : \mathcal{S}_{FS}(\Sigma) \to \mathcal{S}_3(\Sigma).$$
(2) *$\eta_\pm$ are embeddings and $\eta_\pm(\mathcal{S}_{FS}(\Sigma))$ are direct summands of $\mathcal{S}_3(\Sigma)$.*

Proof.

(1) $\eta_\pm$ maps the internal relations of [25] to (45), (112)–(115), and the boundary relations (for $a_{FS} = 1$) to



for $i = 1, 3$, which are satisfied by (48), (84), and (85).

(2) To prove that we identify $\mathcal{S}_3(\Sigma)$ with Higgins' skein algebra, through Theorem 11.5. Now it is easy to see that $\eta_\pm$ map the basis of reduced nonelliptic webs without British highways of [25] 1-1 into the basis composed of irreducible webs of [28]. □

APPENDIX: PROOF OF PROPOSITION 3.13 (A CALCULATION OF MATRICES OF X)

We need to prove Identity (41). The generators The quantized enveloping algebra $U_q(sl_n)$ is generated by $E_i, F_i, K_i^{\pm 1}$ with relations given in [35]. Its action on $V = \mathbb{Q}(q)^n$ with the standard basis $e_1, \dots, e_n$ is given by
$$E_i e_j = \delta_{i,j} e_{i+1},\ F_i e_j = \delta_{i,j+1} e_i,\ K_i e_j = q^{\delta_{i,j+1} - \delta_{i,j}} e_j. \tag{A.1}$$
Note that e_n is the highest weight vector.

By definition [61], a half-ribbon element is an invertible element $X \in \widetilde{U_q(sl_n)}$ satisfying
$$\mathcal{R} = (X^{-1} \otimes X^{-1})\Delta(X),\ \text{and } X^2 = \vartheta, \text{the ribbon element.} \tag{A.2}$$
In [61, section 4], a half-ribbon element, denoted here by X_0, was constructed based on work of Kirillov–Reshetikhin [34] and Levendorskii–Soibelman [46].

To calculated the action of X_0, we use the following identities from [61, Lemma 3.10],
$$X_0 F_i X_0^{-1} = -E_{n-i}, \tag{A.3}$$
$$X_0 K_i X_0^{-1} = K_{n-i}^{-1}. \tag{A.4}$$
$$X(T_{w_0}^{-1}(e_n)) = t_0^{1/2} e_n, \tag{A.5}$$

where $T_{w_0} \in \widetilde{U_q(sl_n)}$ the quantum braid group element corresponding to the longest element w_0 of the symmetric group S_n whose exact definition is not needed here.

Let δ be the sum of all the fundamental weights. From (A.4), we get

$$X_0 K_\delta X_0^{-1} = K_\delta^{-1}.$$

It follows that $X := K_\delta^{-1} X_0$ also satisfies (A.2) and hence is a half-ribbon element. Actually, X is a half-ribbon element considered in [34, 46].

If $x \in V$ has weight λ, then $T_{w_0}(x)$ has weight $w_0(\lambda)$. As each weight subspace of V is 1-dimensional and T_{w_0} is invertible, we have

$$T_{w_0}^{-1}(e_n) = c e_1, \quad 0 \neq c \in \mathbb{Q}(v).$$

By [39, Proposition 5.9], there are positive integers $m_1, \dots, m_k$ and a sequence $i_1, \dots, i_k \in \{1, \dots, n-1\}$ such that

$$F_{i_1}^{(m_1)} \dots F_{i_k}^{(m_k)}(e_n) = T_{w_0}^{-1}(e_n) = c e_1, \quad \text{where } F_i^{(m)} = F_i^m/[m]!$$

As $F_i^2 = 0$ on V, all the m_j must be 1. As $F_i e_j$ is either 0 or another $e_{j'}$ by Equation (A.1), we must have $c = 1$. Hence, $T_{w_0}^{-1}(e_n) = e_1$, and Equation (A.5) becomes

$$X_0(e_1) = t_0^{1/2} e_n.$$

Applying F_{n-1} to the above equation and using (A.3), we get $X_0 e_2 = -t_0^{1/2} e_{n-1}$. Continue applying $F_{n-2}, F_{n-3}, \dots$ and using (A.3), we get

$$X_0 e_j = (-1)^{j-1} t_0^{1/2} e_{\bar{j}}.$$

As K_δ^{-1} acts on V by $K_\delta^{-1}(e_i) = q^{\frac{n+1}{2}-i} e_i$, we the matrix of the action of X on V

$$X_j^i = \delta_{i,\bar{j}} \, (-1)^{n-i} t_0^{1/2} q^{\frac{n+1}{2}-i} = \delta_{i,\bar{j}} \, c_i.$$

The action of X on a dual space is given by the antipode S. By [61, Proposition 4.3], we have $S(X) = gX$. From here one can easily calculate

$$X e^i = c_{\bar{i}} e^{\bar{i}} = f^i.$$

It follows that the action of X on V^* in the basis $\{f^1, \dots, f^n\}$ is given by $X_j^i = \delta_{i,\bar{j}} c_i$. This completes the proof of Identity (41).

ACKNOWLEDGMENTS

The authors would like to thank F. Costantino, D. Douglas, V. Higgins, J. Korinman, and T. Yu for helpful discussions. The first author is partially supported by National Science Foundation Grant DMS-1811114. Many results of the paper were proved and discussed during the visit of both authors to Toulouse in Spring 2017, supported by the Centre International de Mathématiques et d'Informatique's Excellence program during the Thematic Semester "Invariants in low-dimensional geometry and topology". The authors presented the results of this paper in the form of talks or mini-courses at many conferences, including at Noncommutative Geometry Seminar, IMPAN, March 2017 and the special session at the AMS meeting at Hunter College in May 2017 and would like to thank the organizers for the opportunities to present their work.

JOURNAL INFORMATION

The *Journal of Topology* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

ORCID

Adam S. Sikora https://orcid.org/0000-0001-9396-3903

REFERENCES

1. D. Ayala and J. Francis, *Factorization homology of topological manifolds*, J. Topol. **8** (2015), no. 4, 1045–1084, arXiv:1206.5522.
2. D. Ayala, J. Francis, and H. L. Tanaka, *Factorization homology of stratified spaces*, Selecta Math. **23** (2016), no. 1, 293–362, arXiv:1409.0848.
3. A. Alekseev, H. Grosse, and V. Schomerus, *Combinatorial quantization of the Hamiltonian Chern–Simons Theory II*, Comm. Math. Phys. **174** (1996), no. 3, 561–604.
4. A. Y. Alekseev and V. Schomerus, *Representation theory of Chern–Simons observables*, Duke Math. J. **85** (1996), no. 2, 447–510.
5. M. F. Atiyah and R. Bott, *Yang–Mills Equations over Riemann Surfaces*, Philos. Trans. Royal Soc. London Ser. A, Mathematical and Physical Sciences **308** (1983), no. 1505, 523–615.
6. A. Beilinson and V. Drinfeld, *Chiral algebras*, American Mathematical Society Colloquium Publications, vol. 51, Amer. Math. Soc., Providence, RI, 2004.
7. D. Ben-Zvi, A. Brochier, and D. Jordan, *Integrating quantum groups over surfaces*, J. Topol. **11** (2018), no. 4, 873–916, arXiv: 1501.04652.
8. E. Buffenoir and Ph. Roche, *Two-dimensional lattice gauge theory based on a quantum group*, Comm. Math. Phys. **170** (1995), no. 3, 669–698.
9. E. Buffenoir and Ph. Roche, *Link invariants and combinatorial quantization of Hamiltonian Chern–Simons theory*, Comm. Math. Phys. **181** (1996), no. 2, 331–365.
10. D. Bullock, C. Frohman, and J. Kania-Bartoszyńska, *Topological interpretations of lattice gauge field theory*, Comm. Math. Physics **198** (1998), 47–81, arXiv:q-alg/9710003.
11. W. Bloomquist and T. T. Q. Lê, *The Chebyshev–Frobenius homomorphism for skein module of marked 3-manifolds*, Math. Z. **301** (2022), no. 1, 1063–1105.
12. F. Bonahon and H. Wong, *Quantum traces for representations of surface groups in* $SL_2(\mathbb{C})$, Geom. Topol. **15** (2011), 1569–1615, arXiv:1003.5250.
13. L. O. Chekhov and M. Shapiro, *Darboux coordinates for symplectic groupoid and cluster algebras*, arxiv: 2003.07499.
14. J. Cooke, *Excision of skein categories and factorisation homology*, julietcooke.net, arXiv:1910.02630.
15. F. Costantino and T. T. Q. Lê, *Stated skein algebras of surfaces*, J. EMS **24** (2022), no. 12, 4063–4142.
16. V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge University Press, Cambridge, 1994. MR 1300632
17. S. Cautis, J. Kamnitzer, and S. Morrison, *Webs and quantum skew Howe duality*, Math. Ann. **360** (2014), no. 1–2, 351–390.
18. M. Domokos and T. H. Lenagan, *Quantized trace rings*, Quart. J. Math. **56** (2005), 507–523, arXiv:math/0407053.
19. J. Du, *q-Schur algebras, asymptotic forms, and quantum SLn*, J. Algebra **177** (1995), no. 2, 385–408.

20. P. Etingof and O. Schiffmann, *Lectures on quantum groups*, 2nd ed., Lectures in Mathematical Physics, International Press, Massachusetts, USA, 2002.
21. M. Faitg, *Holonomy and (stated) skein algebras in combinatorial quantization*, arXiv:2003.08992
22. V. V. Fock and A. B. Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, Publ. Math. Inst. Hautes Études Sci. **103** (2006), 1–211.
23. V. V. Fock and A. B. Goncharov, *Cluster ensembles, quantization and the dilogarithm*, Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), 865–930, arXiv:0311245.
24. V. V. Fock and A. A. Rosly, *Poisson structures on moduli of flat connections on Riemann surfaces and r-matrices*, Am. Math. Soc. Transl. **191** (1999), 67–86.
25. C. Frohman and A. S. Sikora, *SU(3)-skein algebras and webs on surfaces*, Math. Z. **300** (2022), 33–56, arXiv:2002.08151.
26. A. Gyoja, *A q-analogue of Young symmetrizer*, Osaka J. Math. **23** (1986), 841–852.
27. B. Haïoun, *Relating stated skein algebras and internal skein algebras*, aXiv:2104.13848.
28. V. Higgins, *Triangular decomposition of* SL_3 *skein algebras*, Quantum Topol. **14** (2023), 1–63, arXiv: 2008.09419.
29. D. Jordan, I. Le, G. Schrader, and A. Shapiro, *Quantum decorated character stacks*, arXiv:2102.12283.
30. T. Johnson-Freyd, *Heisenberg-picture quantum field theory*, Progress in Mathematics volume in honour of Kolya Reshetikhin, arXiv:1508.05908, 2020.
31. D. Jordan and N. White, *The center of the reflection equation algebra via quantum minors*, J. Algebra **542** (2020), 308–342.
32. M. Kashiwara, *Global crystal bases of quantum groups*, Duke Math. J. **69** (1993), no. 2, 455–485.
33. C. Kassel, *Quantum groups*, Graduate texts in mathematics, vol. 155, Springer, New York, 1995.
34. A. N. Kirillov and N. Reshetikhin, *q-Weyl group and a multiplicative formula for universal R-matrices*, Comm. Math. Phys. **134** (1990), no. 2, 421–431.
35. A. Klimyk and K. Schmüdgen, *Quantum groups and their representations*, Texts and Monographs in Physics, Springer, Berlin, 1997.
36. S. Kolb and J. Stokman, *Reflection equation algebras, coideal subalgebras and their centres*, Selecta Math. (N.S.) **15** (2009), no. 4, 621–664.
37. J. Korinman, *Finite presentations for stated skein algebras and lattice gauge field theory*, arXiv: 2012.03237.
38. J. Korinman and A. Quesney, *Classical shadows of stated skein representations at roots of unity*, arXiv:1905.03441, 2019.
39. J. Kamnitzer and P. Tingley, *The crystal commutor and Drinfeld's unitarized R-matrix*, J. Algebraic Combin. **29** (2009), no. 3, 315–335.
40. G. Kuperberg, *Spiders for rank 2 Lie algebras*, Comm. Math. Phys. **180** (1996), no. 1, 109–151.
41. T. T. Q. Lê, *Triangular decomposition of skein algebras*, Quantum Topol. **9** (2018), 591–632, arXiv: 1609.04987.
42. T. T. Q. Lê, *Quantum Teichmüller spaces and quantum trace map*, J. Inst. Math. Jussieu **18** (2019), no. 2, 249–291, arXiv:1511.06054.
43. T. T. Q. Lê and T. Yu, *Stated skein modules of marked 3-manifolds/surfaces, a survey*, Acta Math. Vietnam. **46** (2021), no. 2, 265–287, arXiv: 2005.14577.
44. T. T. Q. Lê and T. Yu, *Quantum traces and embeddings of stated skein algebras into quantum tori*, arXiv:2012.15272.
45. T. T. Q. Lê and T. Yu, *Quantum traces for* SL_n*-skein algebras*, arXiv:2303.08082.
46. S. Z. Levendorskii and Y. S. Soibelman, *The quantum Weyl group and a multiplicative formula for the R-matrix of a simple Lie algebra*, (Russian) Funktsional. Anal. i Prilozhen. **25** (1991), no. 2, 73–76; translation in Funct. Anal. Appl. **25** (1991), no. 2, 143–145.
47. J. Lurie, *Higher algebra*. https://www.math.ias.edu/~lurie/, 2017.
48. G. Lusztig, *Quatum groups at roots of 1*, Geom. Dedicata **35** (1990), 89–114.
49. G. Lusztig, *Canonical bases in tensor products*, Proc. Natl. Acad. Sci. USA **89** (1992), 8177–8179.
50. G. Lusztig, *Introduction to quantum groups*, Progress in Mathematics, vol. 110, Birkhäuser Boston Inc., Cambridge, MA, 1993.
51. G. Lusztig, *Study of a Z-form of the coordinate ring of a reductive group*, J. Amer. Math. Soc. **22** (2009), no. 3, 739–769, arXiv:0709.1286.
52. S. Majid, *Foundations of quantum group theory*, Cambridge University Press, Cambridge, 1995.
53. S. E. Morrison, *A diagrammatic category for the representation theory of* $U_q(sl_n)$, Ph.D. thesis, 2007.

54. A. Poudel, *A comparison between SL_n spider categories*, arXiv:2210.09289, 2022.
55. J. H. Przytycki, *Skein modules of 3-manifolds*, Bull. Polish Acad. Sci. Math. **39** (1991), no. 1–2, 91–100.
56. J. H. Przytycki and A. S. Sikora, *On skein algebras and $SL_2(C)$-character varieties*, Topology **39** (2000), no. 1, 115–148.
57. N. Y. Reshetikhin and V. G. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Comm. Math. Phys. **127** (1990), 1–26.
58. A. S. Sikora, *Skein theory for SU(n)-quantum invariants*, Alg. Geom. Topol. **5** (2005), 865–897, arXiv:math/0407299v5.
59. L. Shen, *Shen cluster nature of quantum groups*, arXiv:2209.06258.
60. A. S. Sikora and B. W. Westbury, *Confluence theory for graphs*, Alg. Geom. Topol. **7** (2007), 439–478, arXiv: math/0609832.
61. N. Snyder and P. Tingley, *The half-twist for $U_q(\mathfrak{g})$ representations*, Algebra Number Theory **3** (2009), no. 7, 809–834.
62. M. Takeuchi, *Some topics on $GL_q(n)$*, J. Algebra **147** (1992), no. 2, 379–410.
63. M. Takeuchi, *A short course on quantum matrices*, New directions in Hopf algebras, Math. Sci. Res. Inst. Publ., vol. 43, Cambridge University Press, Cambridge, 2002, pp. 383–435.
64. V. G. Turaev, *The Conway and Kauffman modules of a solid torus*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **167** (1988), no. Issled. Topol. 6, 190, 79–89.
65. V. G. Turaev, *Quantum invariants of knots and 3-manifolds*, De Gruyter Studies in Mathematics, vol. 18, 1994. https://www.degruyter.com/document/doi/10.1515/9783110883275/html?lang=en
66. K. Walker, TQFTs (early incomplete draft). http://canyon23.net/math/tc.pdf, 2006.
67. E. Witten, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. **121** (1989), no. 3, 351–399.