

SUPPRESSION OF CHEMOTACTIC SINGULARITY VIA VISCOUS FLOW WITH LARGE BUOYANCY*

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Abstract. In this work, we study the Keller–Segel–Navier–Stokes equation with low Reynolds number and subject to large buoyancy force. We show that, for initial cell density with arbitrarily large mass (i.e., the L^1 norm), the solution remains regular for all times in the regime of sufficiently large buoyancy and viscosity. The major blowup suppression mechanism is a norm-stabilizing property possessed by a “static problem,” where the full problem can be seen as a perturbation of this quasi-stationary model.

Key words. chemotaxis, Keller–Segel equation, blowup suppression, global well-posedness, incompressible fluid, aggregation-diffusion equation

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1. Introduction. We consider the parabolic-elliptic Keller–Segel equation in a periodic channel $\Omega = \mathbb{T} \times [0, \pi]$ subject to the influence of the buoyancy-driven Navier–Stokes equation:

$$(1.1) \quad \begin{cases} \partial_t \rho + u \cdot \nabla \rho - \Delta \rho + \operatorname{div}(\rho \nabla (-\Delta_N)^{-1}(\rho - \rho_m)) = 0, \\ \rho_m = \frac{1}{|\Omega|} \int_{\Omega} \rho(t, x) dx, \\ \partial_t u + u \cdot \nabla u - \frac{1}{\operatorname{Re}} \Delta u + \nabla p = \operatorname{Ra} \rho(0, 1)^T, \\ \operatorname{div} u = 0. \end{cases}$$

We equip the system with initial data $\rho(0, x) = \rho_0(x)$, $u(0, x) = u_0(x)$, where ρ_0 is a nonnegative scalar function and u_0 is a divergence-free vector field. We also consider the following set of boundary conditions:

$$(1.2) \quad \partial_2 \rho = \nabla \rho \cdot n = 0, \quad u_2 = u \cdot n = 0, \quad \omega = \nabla^\perp \cdot u = 0 \quad \text{on } \partial\Omega.$$

Here, $\mathbb{T} := [-\pi, \pi)$ is the one-dimensional torus, and a function f defined on \mathbb{T} means that f assumes the periodic boundary condition with period 2π ; $n = (0, 1)^T$ denotes the unit normal derivative along $\partial\Omega = \mathbb{T} \times \{0, \pi\}$; ∇^\perp denotes the differential operator $(-\partial_2, \partial_1)$.

The first equation in (1.1) is the classical parabolic-elliptic Keller–Segel equation with advection. This equation characterizes a population of bacteria with the density ρ that moves in response to an attractive chemical that the bacteria themselves secrete. Specifically, the chemical-induced aggregation effect is modeled by the term $\operatorname{div}(\rho \nabla (-\Delta_N)^{-1}(\rho - \rho_m))$, indicating a scenario where chemicals homogenize much

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faster than the motion of micro-organisms. Here, $-\Delta_N$ denotes a homogeneous Neumann Laplacian that represents the classical condition of zero chemical flux across boundary $\partial\Omega$. Furthermore, chemotaxis usually takes place in ambient viscous fluids, which are classically modeled by Navier–Stokes equations with velocity u and pressure p . In nature, the micro-organisms and the fluid can interact through various means (see, e.g., [25]). The main interaction on which we focus is the coupling by buoyancy, which originates from the variation of bacterial density in the domain. Mathematically, such interactions appear in the fluid equation through the forcing term $\text{Ra}\rho(0,1)^T$, where Ra denotes the Rayleigh number measuring relative buoyancy strength due to density variation.

In this work, we focus on the study of chemotaxis–fluid interaction in a canonical domain $\mathbb{T} \times [0, \pi]$. We remark that previous studies classically consider domains without boundaries, e.g., $\mathbb{R}^2, \mathbb{T}^2, \mathbb{T} \times \mathbb{R}$, in order to study chemotaxis–fluid interaction in the bulk of fluid (see [2, 24, 40, 25]). Our work, in contrast, intends to investigate the buoyancy effect along the vertical direction. From this perspective, it is interesting to consider a domain with a top and bottom boundary and study the chemotaxis–fluid interaction via buoyancy when taking boundary effects into account. We also emphasize the boundary conditions that we insist for the rest of this work: On one hand, we assert the classical homogeneous Neumann boundary condition for the cell density. On the other hand, we impose the Lions boundary condition onto the fluid equation: $u \cdot n|_{\partial\Omega} = 0, \omega|_{\partial\Omega} = 0$. This particular boundary condition, as a special case of the more general Navier boundary condition, is prevalent in simulations of flows in the presence of rough boundaries, such as in hemodynamics. It was first introduced and rigorously studied by J.-L. Lions [42] and P.-L. Lions [43]. We refer the readers to works such as [39, 23, 12, 58] and references therein for a more thorough discussion regarding the Navier boundary condition.

When the ambient fluid is absent (i.e., when $\text{Ra} = 0$ and $u \equiv 0$ in (1.1)), we recover the classical parabolic–elliptic Keller–Segel equation:

$$(1.3) \quad \partial_t \rho - \Delta \rho + \text{div}(\rho \nabla (-\Delta_N)^{-1}(\rho - \rho_m)) = 0 \quad \text{in } \Omega.$$

First introduced by Patlak [49] and Keller and Segel [38], (1.3) has been classically studied in various settings. We refer the interested readers to the following list of works: [1, 3, 4, 5, 6, 8, 9, 28, 29, 36, 45, 46]. A remarkable feature enjoyed by (1.3) is that the solution can form singularity in finite time when dimension is greater than 1. In dimension 2, (1.3) is L^1 -critical. For any initial datum ρ_0 with finite second moment, the solution is globally regular if the initial mass $\|\rho_0\|_{L^1} < 8\pi$; see, e.g., [3, 6, 9, 36, 55]. If the initial mass is strictly greater than 8π , a finite-time singularity forms, as seen in [6, 8, 36, 45]. A more careful analysis of such blowup solutions is also carried out [13, 14, 52, 53, 54].

It is also curious to understand the behavior of the Keller–Segel equation under the influence of fluid advection, given the fact that most chemotactic processes take place in ambient fluid. The presence of fluid advection can bring complicated effects to chemotaxis. Among these effects, we would like to focus on the *regularization effects* induced by fluid advection. That is, we would like to understand how the transport term $u \cdot \nabla \rho$ could prevent potential singular behaviors in (1.3). For the past decade, much progress has been made to understand such regularization effects in the context of the Keller–Segel equation. In the regime of passive advection (i.e., the fluid velocity u is given and is not coupled to the Keller–Segel equation), Kiselev and Xu in [40] first demonstrate that, given any initial datum ρ_0 , there exists a relaxation-enhancing flow (see [17] for a precise definition) with sufficiently large amplitude that can suppress

the singularity formation. This result is later generalized by [35] to a larger class of passive flows and more general aggregation equations. Moreover, in [2, 24], the authors exploit the enhanced dissipation phenomenon induced by strong monotone shear flows. Such flows effectively reduce the dimensionality of the problem, where in two dimensions, the singularity can be suppressed. The fast-splitting scenario induced by hyperbolic flows is also explored in [26, 27].

There also have been numerous attempts to investigate the regularity properties of the Keller–Segel equation coupling to active fluid models, many of which address the global regularity of solutions to such coupled systems. We highlight that, among those results, either there is a smallness assumption on initial data (e.g., [10, 19, 44, 20]) or the global regularity of both the chemotaxis equation and fluid equation still hold if they are uncoupled (e.g., [56, 57]). We also note that the authors in [25, 59] study the blowup suppression mechanism of the Keller–Segel–Navier–Stokes equation near a strong Couette flow. These results are almost linear in a sense that the main driven mechanism is still brought by a dominating passive background flow.

Recently, in a series of works [33, 32] by the author joining with Kiselev and Yao, they analyze how buoyancy effects in fluid equations suppress chemotactic singularities in a genuinely nonlinear setting. In [33], the authors investigated the Keller–Segel equation evolving in ambient porous media under the influence of buoyancy. The authors demonstrated that a coupling with the porous media equation via an arbitrarily weak buoyancy constant suffices to arrest any potential chemotactic blowup. The key argument in [33] is a careful analysis of a potential energy and the coercive term $\|\partial_1 \rho\|_{H_0^{-1}}$ in its time derivative. The authors observed that this H_0^{-1} norm has to be small and induces an anisotropic mixing effect along x_1 -direction. This effect renders the system quasi-one-dimensional and therefore suppresses finite-time blowup. On the other hand, [32] studies how Stokes–Boussinesq flow with strong buoyancy suppresses blowup of a Keller–Segel equation equipped with zero Dirichlet boundary condition. A rather soft argument in [32] shows that the flow quenches the L^2 norm of cell density to be sufficiently small in the regime of large buoyancy.

The other motivation for studying the regularization by active fluid advection in the Keller–Segel equation is its connection to the global regularity of other equations that potentially have singular behaviors, such as some fluid equations. In fact, such a regularizing phenomenon is the main factor that competes with the main blowup mechanisms in some of the most classical fluid equations. For 3D Euler equations, a lack of/a weakened advection term is sufficient to induce finite-time singularities; see [15, 21]. The same is also observed in 3D Navier–Stokes equations; see [41, 30]. The crucial role played by advection stands out even more when one considers certain 1D models capturing Euler and Navier–Stokes dynamics; see [11, 37]. The Keller–Segel equation much resembles these fluid equations since it is also a critical equation that exhibits potential singularities. Hence, studying the regularization by fluid advection in the Keller–Segel equation contributes to a more substantial understanding in the problem of global existence for many fluid equations.

In this work, we plan to investigate (1.1) in the case where the fluid has large viscosity and exerts significant buoyancy onto the bacteria. We remark that not only do micro-organisms generally live in a viscous environment (see [50]), but intriguing biological phenomena also occur in a viscous fluid and in a buoyancy-dominating regime. For example, [51] studies an interesting phenomenon called the chemotactic Boycott effect in this regime, which describes the sedimentation process of both bacteria and fluid. In numerical experiments, the canonical choice of the coupling

fluid is the Navier–Stokes flow with parameters $Ra \sim 10^6$ and $Re^{-1} \sim 10^3$. Hence, while the chemotaxis model used in [51] is different from (1.1), the parameter regime $Ra \gg \frac{1}{Re} \gg 1$ is still highly relevant in the study of chemotaxis in viscous environment. Moreover, from an analytical perspective, this is also a first step to showing the blowup suppression by Navier–Stokes flow with only large buoyancy and moderate viscosity.

To make the problem more amenable to analysis, it is convenient to write $B = Re^{-1}$ and $g = Ra \cdot Re$. Another handy transformation is that, in the periodic channel Ω , one can reduce the Lions boundary condition in the following way: The no-flux boundary is equivalent to $u_2|_{\partial\Omega} = 0$. Since ∂_1 is the tangential derivative on $\partial\Omega$, we have the extra boundary constraint $\partial_1^k u_2|_{\partial\Omega} = 0$ for all $k \in \mathbb{N}$. Thus, the Lions boundary condition can be equivalently written as

$$u_2|_{\partial\Omega} = 0, \quad \omega|_{\partial\Omega} = \partial_2 u_1|_{\partial\Omega} = 0.$$

To sum up, we will work with the following equivalent, yet more convenient form of (1.1):

$$(1.4a) \quad \partial_t \rho + u \cdot \nabla \rho - \Delta \rho + \operatorname{div}(\rho \nabla(-\Delta_N)^{-1}(\rho - \rho_m)) = 0 \quad \text{in } \Omega,$$

$$(1.4b) \quad \partial_t u + u \cdot \nabla u - B \Delta u + \nabla p = B g \rho(0, 1)^T, \quad \operatorname{div} u = 0 \quad \text{in } \Omega,$$

$$(1.4c) \quad \partial_2 \rho|_{\partial\Omega} = 0, \quad u_2|_{\partial\Omega} = 0, \quad \partial_2 u_1|_{\partial\Omega} = 0,$$

$$(1.4d) \quad \rho(0, x) = \rho_0(x) \geq 0, \quad u(0, x) = u_0(x).$$

1.1. Main theorem. The main result of this work is that the *regular solution*¹ of (1.4) with arbitrary large mass is in fact globally regular given both parameters B and g sufficiently large, whose sizes only depend on initial data (ρ_0, u_0) . The precise statement of the main result is given as follows.

THEOREM 1.1. *Suppose initial data (ρ_0, u_0) with $\rho_0 \in H^1$ nonnegative and $u_0 \in V$, where V is the class of H^1 , divergence-free vector fields that satisfy the no-flux boundary condition.² There exists a couple $(g_1, B_1) = (g_1(\rho_m, \|\rho_0 - \rho_m\|_{L^2}), B_1(\rho_m, \|\rho_0 - \rho_m\|_{L^2}, \|u_0\|_1))$ such that, if $g \geq g_1$, $B \geq B_1 g^2 e^{g^2}$, (1.4) admits a unique, regular, and global-in-time solution.*

Remark 1.2. We compare this result with [32]. Firstly, the two works treat different scenarios: [32] investigates a Keller–Segel equation with zero Dirichlet boundary condition, while we study the Keller–Segel equation with classical zero Neumann boundary conditions. Secondly, the blowup-suppressing mechanism in the two works are different precisely due to the first difference above. In [32], the blowup is suppressed by an increased interaction between cells and the cold boundary induced by the strong buoyancy force. This mechanism forces the L^2 norm of cell density to drop below the mass threshold in short time. In our work, we instead use a dimension-reduction mechanism similar to that in [33]. It stabilizes the L^2 norm of cell density around a possibly large number.

1.2. Main strategy of proof. In this section, we would like to first emphasize the main difficulty of proving Theorem 1.1, after which we briefly sketch the idea of proving this main result.

¹Roughly speaking, the solution pair (ρ, u) is called a *regular solution* if it is C^∞ in space and for positive times. We will give a rigorous definition in section 3.

²The exact definition of V will be given in section 2.2.

The main obstacle is that, given the appearance of time derivative and advection term in the fluid equation (1.4b), the problem suffers from loss of an explicit velocity law, which directly connects density ρ and fluid velocity u . In general, such a loss prevents us from obtaining precise control of u in a way similar to [33]. In the parameter regime specified in Theorem 1.1, however, we will show that (1.4) is a suitable perturbation of a static problem (1.6), which actually has a velocity law that can be analyzed with tools developed in [33]. Thus, we arrange the proof into the following steps.

Step 1. Study of static problem (1.6). We first discuss the intuition of deriving the static problem (1.6). We start with dividing on both sides of (1.4b) by B . This yields

$$\frac{1}{B}(\partial_t u + u \cdot \nabla u) - \Delta u + \frac{1}{B} \nabla p = g\rho(0, 1)^T, \quad \operatorname{div} u = 0.$$

In the regime of large B , it is natural to treat the term $\frac{1}{B}(\partial_t u + u \cdot \nabla u)$ perturbatively. In fact, it is reasonable to guess that the following velocity law dominates the velocity evolution:

$$(1.5) \quad -\Delta u + \nabla P = g\rho(0, 1)^T, \quad \operatorname{div} u = 0,$$

where we write $P = B^{-1}p$. We remark that the scaling B^{-1} in front of the pressure is irrelevant due to incompressibility. This motivates us to first understand the following system, which we will later refer to as the **static problem**:

$$(1.6a) \quad \partial_t \rho + u \cdot \nabla \rho - \Delta \rho + \operatorname{div}(\rho \nabla (-\Delta_N)^{-1}(\rho - \rho_m)) = 0,$$

$$(1.6b) \quad -\Delta u + \nabla p = g\rho(0, 1)^T, \quad \operatorname{div} u = 0,$$

$$(1.6c) \quad \partial_2 \rho|_{\partial\Omega} = 0, \quad u_2|_{\partial\Omega} = 0, \quad \partial_2 u_1|_{\partial\Omega} = 0, \quad \rho(0, x) = \rho_0(x) \geq 0.$$

We remark that we do not specify the initial datum for u . Indeed, the problem (1.6) is closed by itself due to a remarkable property of (1.6b)–(1.6c) that one may reduce them to the compact form

$$(1.7) \quad u = -g\nabla^\perp(-\Delta_D)^{-2}\partial_1 \rho$$

thanks to Proposition 2.2. Here, $-\Delta_D$ denotes a homogeneous Dirichlet Laplacian, and $(-\Delta_D)^{-2} = (-\Delta_D)^{-1} \circ (-\Delta_D)^{-1}$. This renders no degree of freedom over choosing the initial datum for u in problem (1.6). We will work with the more convenient form (1.7) instead of the original formulation.

We also remark that the static problem (1.6) is reminiscent of the Keller–Segel equation coupling to Darcy’s law studied in [33], whose velocity law is given by

$$(1.8) \quad u = g\nabla^\perp(-\Delta_D)^{-1}\partial_1 \rho.$$

In [33], the authors exploited an anisotropic mixing effect that is sufficient to prevent chemotactic blowup when ρ achieves a large L^2 norm. However, the velocity law (1.7) yields a more regular fluid flow. The extra regularity induces a weaker mixing effect comparing to (1.8). To remedy this issue, we introduce a large g so that the flow has large amplitude to generate sufficient mixing effect. This treatment differs our work from [33]. In the large amplitude setting, we can achieve a crucial norm stabilizing property of (1.6). That is, the L^2 norm of ρ stabilizes around a level N_0 , whose size only depends on the initial datum ρ_0 in the regime of g sufficiently large. The abovementioned property is formalized in the statement below.

PROPOSITION 1.3 (norm-stabilizing property). *Consider problem (1.6) with non-negative initial datum $\rho_0 \in H^1$. Assume that $t_0 \geq 0$ is inside the lifespan of the unique regular local solution $\rho(t, x)$. Then, there exist $N_0 = N_0(\rho_m, \|\rho_0 - \rho_m\|_{L^2})$, a time $T_* = T_*(N_0, \rho_m)$, and $g_0 := g_0(N_0, \rho_m) > 0$ such that the following statement holds: If $\|\rho(t_0) - \rho_m\|_{L^2}^2 \leq \frac{N_0}{2}$, then, for all $g \geq g_0$, the regular solution $\rho(t, x)$ can be continued in interval $[t_0, t_0 + T_*]$ with estimate*

$$(1.9) \quad \sup_{t \in [t_0, t_0 + T_*]} \|\rho(t) - \rho_m\|_{L^2}^2 \leq N_0.$$

Moreover, there exists a time instance $T \in [t_0 + \frac{T_*}{2}, t_0 + T_*]$ such that

$$(1.10) \quad \|\rho(T) - \rho_m\|_{L^2}^2 \leq \frac{N_0}{8}.$$

We will devote section 4 to the proof of this key result. As a direct consequence, we will also see that the static problem (1.6) is globally regular.

Step 2. Continuation argument. In fact, one may view Proposition 1.3 as a damping mechanism induced by the static problem (1.6). Keeping in mind that the full problem (1.4) is a perturbation of (1.6), we employ a bootstrap argument to show that the solution to the full problem is L^2 -close to the solution of the static problem for all times. Fixing initial condition ρ_0 and u_0 , we choose large N_0 as in Proposition 1.3. Define \mathcal{T}_0 to be the largest time such that the following bootstrap assumption holds:

$$(1.11) \quad \sup_{0 \leq t \leq \mathcal{T}_0} \|\rho(t) - \rho_m\|_{L^2}^2 \leq 9N_0.$$

Our goal is to prove the following improved estimate.

PROPOSITION 1.4. *Consider initial data $\rho_0 \in H^1$ that are nonnegative and $u_0 \in V$. There exists a couple $(g_1, B_1) = (g_1(\rho_m, \|\rho_0 - \rho_m\|_{L^2}), B_1(\rho_m, \|\rho_0 - \rho_m\|_{L^2}, \|u_0\|_1))$ such that, if $g \geq g_1$, $B \geq B_1 g^2 e^{g^2}$, the following improved estimate holds:*

$$(1.12) \quad \sup_{0 \leq t \leq \mathcal{T}_0} \|\rho(t) - \rho_m\|_{L^2}^2 \leq 5N_0.$$

The main ingredient of this step is based on various nonlinear estimates pertaining to the Navier–Stokes equation equipped with Lions boundary conditions. We devote section 5 to the proof of these estimates as well as that of Proposition 1.4.

Step 3. Closing the proof of Theorem 1.1. In the final step, we finish the proof of Theorem 1.1 using estimate (1.12).

Proof of Theorem 1.1. Using Proposition 1.4 and regularity criteria in Theorem 3.3, $[0, \mathcal{T}_0]$ is both open and closed. This implies that $\mathcal{T}_0 = \infty$, and we have obtained global existence of (1.4). \square

1.3. Organization of the paper. The paper is organized as follows. In section 2, we introduce notations and conventions used throughout this paper. We will also lay out a functional-analytic framework for studying the Stokes problem equipping the Lions boundary condition. In section 3, we state the local well-posedness of the full system (1.4) and the static problem (1.6); we also show corresponding L^2 -based regularity criteria in the same section. In section 4, we study the static problem (1.6) and prove the key Proposition 1.3, from which the global well-posedness of (1.6) follows. In section 5, we perform nonlinear estimates and prove the improved estimate (1.12). That is, we prove Proposition 1.4.

2. Preliminaries.

2.1. Notation and conventions. We will denote the usual L^p norm by $\|\cdot\|_{L^p}$ or $\|\cdot\|_{L^p(\cdot)}$. The second notation will be adopted if we would like to emphasize the domain. Similarly, we will denote the Sobolev H^s norm by $\|\cdot\|_s$ for $s \geq 1$. We define the space $H_0^1(\Omega)$ to be the completion of $C_c^\infty(\Omega)$ with respect to norm $\|\cdot\|_1$ and consider its dual space $H_0^{-1}(\Omega)$. Moreover, the following equivalence is standard:

$$\|f\|_{H_0^{-1}} = C \left(\int_{\Omega} f(-\Delta_D)^{-1} f dx \right)^{1/2},$$

where C is a constant that only depends on domain Ω . We also recall that Δ_D denotes the homogeneous Dirichlet Laplacian. We simply regard $C = 1$ because the exact value of this constant is inessential.

Recall the mean of ρ on Ω : $\rho_m(t) = \frac{1}{|\Omega|} \int_{\Omega} \rho(t, x) dx$. We remark that $\rho_m(t)$ is nonnegative and a conserved quantity along evolution given the initial datum ρ_0 being nonnegative due to a parabolic maximum principle and the divergence structure of the Keller–Segel equation (see Proposition 4.1 for the proof of this fact). Thus, we will omit the time dependence of the mean and write ρ_m .

We also consider the following decomposition to any function $\rho \in L^2(\Omega)$ that would play a crucial role in our analysis of the static problem: We write $\rho = \bar{\rho} + \tilde{\rho}$, where

$$\bar{\rho}(x_2) = \frac{1}{2\pi} \int_{\mathbb{T}} \rho(x_1, x_2) dx_1, \quad \tilde{\rho} = \rho - \bar{\rho}.$$

Note that $\bar{\rho}$ is exactly the projection onto the zeroth mode corresponding to direction x_1 , and $\tilde{\rho}$ is the orthogonal complement. We remark that $\bar{\rho} - \rho_m$ and $\tilde{\rho}$ are orthogonal in $L^2(\Omega)$:

$$\int_{\Omega} (\bar{\rho}(x_2) - \rho_m) \tilde{\rho}(x_1, x_2) dx_1 dx_2 = 0.$$

One can connect the 1D and 2D L^2 norms for $\bar{\rho} - \rho_m$ by a simple relation $\|\bar{\rho} - \rho_m\|_{L^2(\Omega)}^2 = 2\pi \|\bar{\rho} - \rho_m\|_{L^2([0, \pi])}^2$. In the rest of the paper, we will only use the 1D L^2 norm for $\bar{\rho}$. We always refer to the 1D L^2 norm when we write $\|\bar{\rho} - \rho_m\|_{L^2}$.

We will denote a universal constant in the upper bounds by uppercase letters and that in lower bounds by lowercase letters. Constants without subscripts (i.e., c, C, \mathcal{C}) are subject to change from line to line. Constants with subscripts (i.e., c_i, C_i, \mathcal{C}_i) are fixed once they are chosen. All such constants mentioned above only depend on domain Ω by default. We will also use the notation $C(X)$ to denote a constant C depending on quantity X . For two positive quantities X, Y , we write $X \lesssim Y$ to mean that $X \leq CY$ for some universal constant C that might only depend on domain Ω . $X \gtrsim Y$ is similarly defined. We will use I_i , $i \in \mathbb{N}$ to denote terms that will be estimated; this notation will be local to each lemma and will be reused in the proofs of other lemmas. Lastly, we also use summation convention. That is, the repeated indices are summed over.

2.2. Steady Stokes equation with Lions boundary condition. In this subsection, we introduce suitable functional-analytic settings for our study of the Stokes

equation equipped with the Lions boundary condition. We shall first study the following static Stokes equation and investigate its well-posedness:

$$(2.1) \quad \begin{cases} -\Delta u + \nabla p = f & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u_2|_{\partial\Omega} = 0, \partial_2 u_1|_{\partial\Omega} = 0. \end{cases}$$

In particular, we will derive suitable elliptic estimates similar to those satisfied by the classical Stokes operator that corresponds to the no-slip boundary condition for velocity (see [16]).

We introduce the following function spaces following the spirit of [39, 12]:

$$\begin{aligned} H &= \left\{ v \in L^2(\Omega) \times L^2(\Omega) : \operatorname{div} v = 0, \int_{\Omega} v_1(x) dx = 0 \right\}, \\ V &= \left\{ v \in H^1(\Omega) \times H^1(\Omega) : \operatorname{div} v = 0, v_2|_{\partial\Omega} = 0, \int_{\Omega} v_1(x) dx = 0 \right\}, \\ W &= \{ v \in V \cap (H^2(\Omega))^2 : \partial_2 v_1|_{\partial\Omega} = 0 \}, \end{aligned}$$

where we equip H with L^2 topology, V with topology induced by the inner product $(u, v)_V = \int_{\Omega} \partial_i u_j \partial_i v_j dx$, and W with H^2 topology. For $u \in V$, the following Poincaré inequality holds.

PROPOSITION 2.1. *Let $u \in V$. Then, the following estimate holds:*

$$(2.2) \quad \|u\|_{L^2} \leq C_P \|\nabla u\|_{L^2},$$

where C_P is a positive constant that only depends on the domain Ω .

Moreover, we have the following well-posedness and regularity result concerning (2.1).

PROPOSITION 2.2. *Assuming that $f \in H$, then (2.1) admits a unique solution $u \in W$ with estimate*

$$(2.3) \quad \|u\|_2 \leq C \|f\|_{L^2}.$$

In fact, u assumes the following explicit formula:

$$(2.4) \quad u = \nabla^\perp (-\Delta_D)^{-2} (\partial_2 f_1 - \partial_1 f_2),$$

where $\nabla^\perp := (-\partial_2, \partial_1)$. In addition, if $f \in H^s \cap H$, $s \geq 1$, we have the following improved regularity estimate:

$$(2.5) \quad \|u\|_{s+2} \leq C \|f\|_s.$$

We delay the proofs of the propositions above to the appendix.

3. Local well-posedness and regularity criteria. In this section, we show the local well-posedness for regular solutions to problems (1.4) and (1.6) and state L^2 regularity criteria for both problems. We start with giving rigorous definitions of a strong solution and a regular solution to both the full system (1.4) and the static problem (1.6).

DEFINITION 3.1.

1. Given initial data $\rho_0 \in H^1$, $u_0 \in V$, we say that the pair $(\rho(t, x), u(t, x))$ is a strong solution to (1.4) on $[0, T]$, $T > 0$ if

$$\rho \in C^0([0, T]; H^1) \cap L^2((0, T); H^2), \quad u \in C^0([0, T]; V) \cap L^2((0, T); W), \\ \partial_t \rho \in L^2([0, T]; L^2), \quad \partial_t u \in L^2([0, T]; H),$$

and (ρ, u) satisfies (1.4) in the sense of distribution. Moreover, a solution is regular if it is strong and, additionally,

$$\rho \in C^\infty((0, T] \times \Omega), \quad u \in C^\infty((0, T] \times \Omega).$$

2. Given initial datum $\rho_0 \in H^1$, we say that $\rho(t, x)$ is a strong solution to (1.6) on $[0, T]$, $T > 0$ if

$$\rho \in C^0([0, T]; H^1) \cap L^2((0, T); H^2), \quad \partial_t \rho \in L^2([0, T]; L^2),$$

and ρ is said to be regular if it is strong and $\rho \in C^\infty((0, T] \times \Omega)$.

With the definition above, we state the following local well-posedness result.

THEOREM 3.2. *Given initial datum $\rho_0 \in H^1$ (resp., $\rho_0 \in H^1, u_0 \in V$) such that $\rho_0 \geq 0$, there exists $T_{loc} = T_{loc}(\rho_0) > 0$ (resp., $\mathcal{T}_{loc} = \mathcal{T}_{loc}(\rho_0) > 0$) so that there exists a unique strong solution ρ (resp., (ρ, u)) to (1.6) (resp., (1.4)) on $[0, T_{loc})$ (resp., $[0, \mathcal{T}_{loc})$). Moreover, the strong solutions are regular.*

We conclude this section by stating the following regularity criteria for both (1.4) and (1.6).

THEOREM 3.3. *If the maximal lifespan T of the strong solution to (1.4) (resp., (1.6)) is finite, then necessarily,*

$$(3.1) \quad \lim_{t \rightarrow T} \int_0^t \|\rho(t) - \rho_m\|_{L^2}^2 dt = \infty.$$

We briefly comment on the proof of Theorem 3.2 and Theorem 3.3, as the details follow almost verbatim to section 2 of [32]. To prove the existence part of Theorem 3.2, we use the standard Galerkin's method to construct a solution to both (1.4) and (1.6) since Ω is a compact domain. Hence, it suffices to show L^2 -based a priori estimates as in [32]. As for the a priori estimates of (1.4), all estimates in section 2 of [32] carry over. The only extra estimate one has to make is on the advection term $u \cdot \nabla u$ because we study the Navier–Stokes equation in our case. The treatment of such terms, however, is standard in the study of well-posedness of 2D Navier–Stokes equations. We refer interested readers to the proof of [7, Theorem V.2.10] for related details. As for the a priori estimates of (1.6), we repeat the same proof as we do for (1.4) with the only change of applying Proposition 2.2 whenever we see $\|u\|_2$. Finally, uniqueness follows from the aforementioned a priori estimates that we have derived from the proof of existence in [32]. Higher regularity follows from standard parabolic smoothing. This ends the proof of Theorem 3.2.

The proof of Theorem 3.3 is the same as that of [32, Theorem 1.2] by studying the L^2 estimate of the ρ equation. The reason why the proofs of [32, Theorem 1.2] can be carried over is that [32, Theorem 1.2] only use $\operatorname{div} u = 0$, ρ nonnegative, and $\|\rho\|_{L^1}$ nonincreasing. These properties hold in both (1.4) and (1.6) (see Proposition 4.1).

4. Analysis of the static problem. In this section, we perform a detailed analysis of (1.6), which serves as a starting point prior to the study of the full problem (1.4). In section 4.1, we establish a series of a priori estimates concerning both cell density ρ and velocity field u of the static problem. We will further show a Nash-type inequality that elucidates an anisotropic mixing effect in the regime of large g . Finally, we prove Proposition 1.3 in section 4.2.

4.1. Key a priori estimates. As a first step, we establish several key energy estimates concerning the system (1.6). We first prove several a priori estimates for the Keller–Segel equation advected by a general incompressible flow. Then, we focus on showing estimates corresponding to the fluid equation (1.6b), which turn out to be crucial in establishing the norm-stabilizing property and global well-posedness of (1.6).

4.1.1. Estimates for a general Keller–Segel equation with advection. In this section, we derive appropriate estimates for the Keller–Segel equation advected by a general incompressible flow u , which may or may not depend on the cell density ρ itself:

$$(4.1) \quad \begin{aligned} \partial_t \rho + u \cdot \nabla \rho - \Delta \rho + \operatorname{div}(\rho \nabla (-\Delta_N)^{-1}(\rho - \rho_m)) &= 0, \\ \rho(0, x) &= \rho_0(x) \geq 0, \quad \partial_2 \rho|_{\partial\Omega} = 0. \end{aligned}$$

Here, u is a smooth, divergence-free vector field that satisfies no-flux boundary condition (i.e., $u_2|_{\partial\Omega} = 0$). We allow such a generality in (4.1) so that all results in this subsection hold for both the static problem (1.6) and the full problem (1.4). This is because an incompressible, viscous flow satisfying the Lions boundary condition necessarily satisfies all requirements prescribed in (4.1).

Our first result reveals that (4.1) conserves mass.

PROPOSITION 4.1. *Assume that $\rho(t, x)$ is a regular solution to (4.1) on $[0, T]$ with u being C^1 and divergence-free. Then, $\rho(t, x) \geq 0$ and $\|\rho(t, \cdot)\|_{L^1} = \|\rho_0\|_{L^1}$ for all $t \in [0, T]$.*

Proof. First, $\rho(t, x) \geq 0$ for all $t \in [0, T]$ comes from the facts that $\rho_0 \geq 0$ and the parabolic maximum principle. To prove the conservation of mass, we use nonnegativity of $\rho(t, x)$ to obtain that

$$\begin{aligned} \frac{d}{dt} \|\rho(t, \cdot)\|_{L^1} &= \frac{d}{dt} \int_{\Omega} \rho(t, x) dx \\ &= \int_{\Omega} (-u \cdot \nabla \rho + \Delta \rho - \operatorname{div}(\rho \nabla (-\Delta_N)^{-1}(\rho - \rho_m)))(t, x) dx = 0, \end{aligned}$$

where we integrated by parts and used that u is divergence-free, $u_2|_{\partial\Omega} = 0$, and $\partial_2 \rho|_{\partial\Omega} = 0$. \square

The next result, originally proved in [33], gives two variants of L^2 energy estimates of (4.1), which play a major role in establishing the norm-stabilizing property of (1.6). We point out that [33, Proposition 4.2 and Corollary 4.3] are proved where u is governed by Darcy's law (1.8). However, their proofs only used that u is divergence-free and satisfies the no-flux boundary condition. Hence, the proof of the result below follows verbatim from that of [33, Proposition 4.2 and Corollary 4.3].

PROPOSITION 4.2 (essentially Proposition 4.2, Corollary 4.3 of [33]). *Assume $\rho(t, x)$ to be a regular, nonnegative solution to (4.1) on $[t_0, t_0 + T]$ for some $t_0 \geq 0$ and*

$T > 0$. Then, for any $t \in (t_0, t_0 + T)$, there exists a constant C_0 depending only on the domain Ω such that

$$(4.2) \quad \frac{d}{dt} \|\rho - \rho_m\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 \leq C_0 \|\rho - \rho_m\|_{L^2}^4 + 2\rho_m \|\rho - \rho_m\|_{L^2}^2.$$

Furthermore, if $\|\rho(t, \cdot) - \rho_m\|_{L^2}^2 \geq \rho_m/4$ for all $t \in [t_0, t_0 + T]$, then there exists a positive constant C_1 depending only on the domain Ω such that

$$(4.3) \quad \frac{d}{dt} \|\rho - \rho_m\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 \leq C_1 \|\rho - \rho_m\|_{L^2}^4$$

and

$$(4.4) \quad \frac{d}{dt} \|\rho - \rho_m\|_{L^2}^2 \leq -\frac{1}{C_1 \rho_m^4} \|\bar{\rho} - \rho_m\|_{L^2}^6 + C_1 \|\bar{\rho}\|_{L^2}^4$$

for any $t \in [t_0, t_0 + T]$.

The next proposition shows that one can upgrade the $L_t^\infty L_x^2$ control of ρ to an $L_t^\infty L_x^\infty$ control.

PROPOSITION 4.3. *Let $\rho(t, x)$ be a regular solution to (4.1) on $[t_0, t_0 + T]$ for some $t_0 \geq 0$ and $N \geq \max(1, \rho_m)$ be an arbitrary number. If $\|\rho(t, \cdot) - \rho_m\|_{L^2} \leq 2N$ for all $t \in [t_0, t_0 + T]$, then we also have that $\sup_{t_0 \leq t \leq t_0 + T} \|\rho(t, \cdot) - \rho_m\|_{L^\infty} \leq CN^2$, where C is a domain dependent constant.*

Proof. The proof follows verbatim from that of [40, Proposition 9.1], where it is handled for \mathbb{T}^2 . We can do this since all integrations by parts do not introduce any boundary terms due to the Neumann boundary condition that ρ satisfies as well as the no-flux boundary condition satisfied by u . \square

Finally, we establish a corollary that shows a doubling time for the L^2 norm $\|\rho - \rho_m\|_{L^2}^2$.

COROLLARY 4.4. *Suppose that $\rho(t, x)$ is a regular solution to (4.1) on $[t_0, t_0 + T]$ for some $t_0 \geq 0$ with $T > 0$ and $N > 0$ being an arbitrary number. If $\|\rho(t_0) - \rho_m\|_{L^2}^2 \leq N$, then there exists a time $T_*(N, \rho_m) = \frac{\min(N^{-1}, \rho_m^{-1})}{8(1+C_0)} =: c_0 \min(N^{-1}, \rho_m^{-1})$ such that*

$$(4.5) \quad \sup_{t_0 \leq t \leq t_0 + T_*(N, \rho_m)} \|\rho(t) - \rho_m\|_{L^2}^2 \leq 2N$$

and

$$(4.6) \quad \int_{t_0}^{t_0 + T_*(N, \rho_m)} \|\nabla \rho(t)\|_{L^2}^2 dt \leq 2N.$$

Here, C_0 is fixed as in Proposition 4.2.

Proof. To show (4.5), we employ a standard barrier argument: Suppose that there exists a time $\tau \in [t_0, t_0 + T_*(N, \rho_m)]$ such that $\|\rho(\tau) - \rho_m\|_{L^2}^2 = 2N$ and $\|\rho(t) - \rho_m\|_{L^2}^2 < 2N$ for any $t \in [t_0, \tau)$. Integrating (4.2) from t_0 to τ , we have that

$$\begin{aligned} 2N = \|\rho(\tau) - \rho_m\|_{L^2}^2 &\leq N + 2\rho_m \int_{t_0}^{\tau} \|\rho(s) - \rho_m\|_{L^2}^2 dx + C_0 \int_{t_0}^{\tau} \|\rho(s) - \rho_m\|_{L^2}^4 ds \\ &\leq N + 4\rho_m NT_* + 4N^2 C_0 T_* < 2N, \end{aligned}$$

where we used the definition of T_* in the final inequality. However, this yields a contradiction.

To show (4.6), we integrate (4.2) from t_0 to $t_0 + T_*(N, \rho_m)$, which yields

$$\begin{aligned} & \|\rho(t_0 + T_*(N, \rho_m)) - \rho_m\|_{L^2}^2 + \int_{t_0}^{t_0 + T_*(N, \rho_m)} \|\nabla \rho(t)\|_{L^2}^2 dt \\ & \leq N + 2\rho_m \int_{t_0}^{t_0 + T_*(N, \rho_m)} \|\rho(t) - \rho_m\|_{L^2}^2 dt + C_0 \int_{t_0}^{t_0 + T_*(N, \rho_m)} \|\rho(t) - \rho_0\|_{L^2}^4 dt \\ & \leq N + 2\rho_m(2N)T_* + C_0(4N^2)T_* < N + \frac{N}{2(1+C_0)} + \frac{C_0N}{2(1+C_0)} < 2N. \end{aligned}$$

The proof is thus complete. \square

4.1.2. Estimates for static problem (1.6). Now, we turn to some estimates concerning the system (1.6), which crucially exploit the structure of the Biot–Savart-type relation (1.7). As our first step, we prove the following key estimate, which displays the smallness of a mixing-type norm in the regime of large parameter g .

PROPOSITION 4.5. *Let $\rho(t, x)$ be a regular solution to (1.6) on $[t_0, t_0 + T]$ for some $t_0 \geq 0$ and $T > 0$. Let $N > 0$ be an arbitrary number. If $\|\rho(t_0) - \rho_m\|_{L^2}^2 \leq N$, the following bound holds:*

$$(4.7) \quad \int_{t_0}^{t_0 + T_*(N, \rho_m)} \|(-\Delta_D)^{-1} \partial_1 \rho\|_{L^2}^2 dt \leq C(\rho_m) g^{-1}.$$

Proof. On one hand, we compute that

$$\begin{aligned} \int_{t_0}^{t_0 + T_*(N, \rho_m)} \|(-\Delta_D)^{-1} \partial_1 \rho\|_{L^2}^2 dt &= \int_{t_0}^{t_0 + T_*(N, \rho_m)} \int_{\Omega} (-\Delta_D)^{-1} \partial_1 \rho (-\Delta_D)^{-1} \partial_1 \rho dx dt \\ &= \int_{t_0}^{t_0 + T_*(N, \rho_m)} \int_{\Omega} \partial_1 \rho (-\Delta_D)^{-2} \partial_1 \rho dx dt \\ &= - \int_{t_0}^{t_0 + T_*(N, \rho_m)} \int_{\Omega} \rho \partial_1 (-\Delta_D)^{-2} \partial_1 \rho dx dt \\ (4.8) \quad &= \frac{1}{g} \int_{t_0}^{t_0 + T_*(N, \rho_m)} \int_{\Omega} \rho u_2 dx dt. \end{aligned}$$

Note that we have used the self-adjointness of $(-\Delta_D)^{-1}$ in the second equality, integration by parts in the third equality, and (1.7) in the last equality. On the other hand, we test the ρ equation of (1.6) by x_2 , which yields

$$\begin{aligned} (4.9) \quad \int_{\Omega} \rho u_2 dx &= - \int_{\Omega} x_2 (u \cdot \nabla \rho) dx = \int_{\Omega} x_2 (\partial_t \rho - \Delta \rho + \operatorname{div}(\rho \nabla (-\Delta_N)^{-1} (\rho - \rho_m))) dx \\ &=: \sum_{i=1}^3 I_i, \end{aligned}$$

where the first equality follows from integration by parts and the no-flux boundary condition satisfied by u . Integrating in time from t_0 to $t_0 + T_*(N, \rho_m)$ and using Proposition 4.1 as well as the bound (4.6), we estimate time integrals of I_1, I_2 by

$$\begin{aligned}
\left| \int_{t_0}^{t_0+T_*(N,\rho_m)} I_1 dt \right| &= \left| \int_{\Omega} x_2(\rho(t_0+T_*(N,\rho_m)) - \rho(t_0)) dx \right| \leq C \|\rho_0\|_{L^1} \leq C \rho_m, \\
\left| \int_{t_0}^{t_0+T_*(N,\rho_m)} I_2 dt \right| &= \left| \int_{t_0}^{t_0+T_*(N,\rho_m)} \int_{\Omega} \partial_2 \rho dx dt \right| \\
&\leq CT_*(N,\rho_m)^{1/2} \left(\int_{t_0}^{t_0+T_*(N,\rho_m)} \|\nabla \rho(t)\|_{L^2}^2 dt \right)^{1/2} \leq C.
\end{aligned}$$

Here, we used the definition of $T_*(N, \rho_m)$ and (4.6) in the final inequality of the I_2 estimate. To estimate the term I_3 in (4.9), we use Sobolev embedding, elliptic estimates, and interpolation inequality to obtain that

$$\begin{aligned}
I_3 &= \int_{\Omega} \rho \partial_2 (-\Delta_N)^{-1} (\rho - \rho_m) \leq \|\rho\|_{L^2} \|\partial_2 (-\Delta_N)^{-1} (\rho - \rho_m)\|_{L^2} \\
&\leq (\|\rho - \rho_m\|_{L^2} + \sqrt{2\pi} \rho_m) \|\partial_2 (-\Delta_N)^{-1} (\rho - \rho_m)\|_{W^{1,6/5}} \\
&\leq C(\|\rho - \rho_m\|_{L^2} + \sqrt{2\pi} \rho_m) \|\rho - \rho_m\|_{L^{6/5}} \\
&\leq C \rho_m^{2/3} \|\rho - \rho_m\|_{L^2}^{1/3} (\|\rho - \rho_m\|_{L^2} + \rho_m) \\
&= C(\rho_m^{5/3} N^{1/6} + \rho_m^{2/3} N^{2/3}).
\end{aligned}$$

Then, we may estimate the time integral of A_3 by

$$\begin{aligned}
\int_{t_0}^{t_0+T_*(N,\rho_m)} I_3 dt &\leq C \min(N^{-1}, \rho_m^{-1}) (\rho_m^{5/3} N^{1/6} + \rho_m^{2/3} N^{2/3}) \\
&\leq \begin{cases} \rho_m^{2/3} + \rho_m^{-1/3}, & N \leq 1, \\ \rho_m^{5/3} N^{-5/6} + \rho_m^{2/3} N^{-1/3}, & N > 1 \end{cases} \leq C(\rho_m).
\end{aligned}$$

Collecting the estimates above yields

$$(4.10) \quad \left| \int_{t_0}^{t_0+T_*(N,\rho_m)} \int_{\Omega} \rho u_2 dx dt \right| \leq C(\rho_m),$$

Therefore, by (4.8),

$$\int_{t_0}^{t_0+T_*(N,\rho_m)} \|(-\Delta_D)^{-1} \partial_1 \rho\|_{L^2}^2 dt \leq C(\rho_m) g^{-1},$$

which concludes the proof. \square

Note that the norm $\|(-\Delta_D)^{-1} \partial_1 \rho\|_{L^2}$ could be viewed as a quantity that encodes information on mixing. This is because, formally, this quantity has the same scaling as the classical mix norm $\|\rho - \rho_m\|_{H_0^{-1}}$ (see [34], for example). Since, generally, the smallness of $\|\rho - \rho_m\|_{H^{-1}}$ induces enhanced dissipation by creating large gradients, it is tempting to uncover a similar effect by exploiting the smallness of $\|(-\Delta_D)^{-1} \partial_1 \rho\|_{L^2}$ in our setting. In order to achieve this goal, we follow the approach taken in [33], which studies a similar quantity $\|\partial_1 \rho\|_{H_0^{-1}}$. The technical ingredient that we need is a Nash's inequality with improved coefficient; let us recall the following result from [33], which is valid for any function $\rho \in L^2([s, r], H^1(\Omega))$, $0 \leq s < r \leq \infty$.

PROPOSITION 4.6 (Corollary 6.2 of [33]). Assume that, for some $K > 0$, there is

$$(4.11) \quad \int_s^r \|\partial_1 \rho\|_{H_0^{-1}}^2 dt \leq K^{-1} \int_s^r \|\tilde{\rho}\|_{L^2}^2 dt.$$

Then, there exists a constant $C > 0$ depending only on Ω such that

$$(4.12) \quad \int_s^r \|\tilde{\rho}\|_{L^2}^2 dt \leq CK^{-1/4} \left(\int_s^r \|\tilde{\rho}\|_{L^1}^2 dt \right)^{1/2} \left(\int_s^r \|\nabla \tilde{\rho}\|_{L^2}^2 dt \right)^{1/2}.$$

Note that the original statement in Corollary 6.2 of [33] only considers the case when $K \geq 1$. However, one could extend the result for any $K > 0$ in a rather trivial way: For $K \in (0, 1)$, we simply invoke the original Nash's inequality for $\tilde{\rho}$ without using the assumption (4.11). The fact that $K^{-1/4} > 1$ immediately yields the result.

With the proposition above, we may conclude the following crucial lemma, which precisely characterizes the creation of large gradients when the norm $\|(-\Delta_D)^{-1} \partial_1 \rho\|_{L^2}$ is small in an appropriate sense.

LEMMA 4.7. Fix $g > 0$. Assume that there is some constant $C > 0$ so that

$$(4.13) \quad \int_s^r \|(-\Delta_D)^{-1} \partial_1 \rho\|_{L^2}^2 dt \leq Cg^{-9/10} \left(\int_s^r \|\tilde{\rho}\|_{L^2}^2 dt \right)^2.$$

Then, the following estimate holds:

$$(4.14) \quad \int_s^r \|\nabla \rho\|_{L^2}^2 dt \geq \tilde{C}g^{9/50} \left(\int_s^r \|\rho - \rho_m\|_{L^1}^2 dt \right)^{-4/5} \left(\int_s^r \|\tilde{\rho}\|_{L^2}^2 dt \right)^{8/5}$$

for some constant $\tilde{C} > 0$ that only depends on the domain Ω and constant C appearing in (4.13).

Proof. Using the definition of the H_0^{-1} norm and the Cauchy–Schwarz inequality, we have that

$$\begin{aligned} \|\partial_1 \tilde{\rho}\|_{H_0^{-1}}^2 &= \int_{\Omega} \partial_1 \tilde{\rho} (-\Delta_D)^{-1} \partial_1 \tilde{\rho} dx \leq \|\partial_1 \tilde{\rho}\|_{L^2} \|(-\Delta_D)^{-1} \partial_1 \rho\|_{L^2} \\ &\leq \|\nabla \rho\|_{L^2} \|(-\Delta_D)^{-1} \partial_1 \rho\|_{L^2}, \end{aligned}$$

where we used the fact that $\partial_1 \rho = \partial_1 \tilde{\rho}$. Integrating the above inequality from s to r and applying the Cauchy–Schwarz inequality again, we have that

$$\begin{aligned} \int_s^r \|\partial_1 \tilde{\rho}\|_{H_0^{-1}}^2 dt &\leq \left(\int_s^r \|(-\Delta_D)^{-1} \partial_1 \rho\|_{L^2}^2 dt \right)^{1/2} \left(\int_s^r \|\nabla \rho\|_{L^2}^2 dt \right)^{1/2} \\ &\leq C^{1/2} g^{-9/20} \left(\int_s^r \|\nabla \rho\|_{L^2}^2 dt \right)^{1/2} \int_s^r \|\tilde{\rho}\|_{L^2}^2 dt. \end{aligned}$$

Then, an application of Proposition 4.6 via replacing N by $C^{-1/2} g^{9/20} (\int_s^r \|\nabla \rho\|_{L^2}^2 dt)^{-1/2} > 0$ implies (4.14) after some rearrangements of the inequality. \square

4.2. Norm-stabilizing property of the static problem. With the a priori estimates set up in previous sections, we are ready to prove the norm-stabilizing property (i.e., Proposition 1.3), from which the global existence of the static problem (1.6) follows.

Proof of Proposition 1.3. We first fix our choice of N_0 and T_* as follows: Choose N_0 so that

$$(4.15) \quad N_0 \geq \max\{1, 2\rho_m, 2\|\rho_0 - \rho_m\|_{L^2}^2\},$$

$$(4.16) \quad N_0 \geq \frac{32^3 C_1 \rho_m^4 (15/8 + 2C_1 c_0)}{4c_0},$$

and set $T_* = c_0 \min(2N_0^{-1}, \rho_m) = 2c_0 N_0^{-1}$. Recall that C_0, C_1 are universal constants fixed in Proposition 4.2; c_0 is also a universal constant fixed in Corollary 4.4. Moreover, we fix $g \geq g_0(N_0, \rho_m)$, where the threshold g_0 will be chosen sufficiently large in the course of the proof.

We first observe that, by Corollary 4.4, given $\|\rho(t_0) - \rho_m\|_{L^2}^2 = N_0/2$, we have that

$$(4.17) \quad \int_{t_0}^{t_0+T_*} \|\nabla \rho\|_{L^2}^2 dt \leq N_0,$$

and in particular, (1.9) holds.

We thus focus on the proof of (1.10). Assume, for the sake of contradiction, that $\|\rho(t) - \rho_m\|_{L^2}^2 > N_0/8$ for any $t \in [t_0 + \frac{T_*}{2}, t_0 + T_*]$. For $g \geq g_0$, we discuss the following dichotomy.

Case 1: Small $\|\tilde{\rho}\|_{L^2}$.

$$(4.18) \quad \int_{t_0+T_*/2}^{t_0+T_*} \|\tilde{\rho}\|_{L^2}^2 dt \leq c_0 g^{-1/20}.$$

In this case, we first observe that, due to the contradiction assumption, there is a lower bound

$$\int_{t_0+T_*/2}^{t_0+T_*} \|\rho - \rho_m\|_{L^2}^2 dt \geq \frac{T_* N_0}{16} = \frac{c_0}{16}.$$

Combining with (4.18) and using orthogonality between $\bar{\rho}$ and $\tilde{\rho}$, we have that

$$(4.19) \quad \int_{t_0+T_*/2}^{t_0+T_*} \|\bar{\rho} - \rho_m\|_{L^2}^2 dt \geq \left(\frac{1}{16} - g^{-1/20}\right) c_0 \geq \frac{c_0}{32}$$

if we choose

$$(4.20) \quad g_0 \geq 32^{20}.$$

On the other hand, an application of Hölder's inequality implies that

$$\int_{t_0+T_*/2}^{t_0+T_*} \|\bar{\rho} - \rho_m\|_{L^2}^2 dt \leq \left(\frac{T_*}{2}\right)^{2/3} \left(\int_{t_0+T_*/2}^{t_0+T_*} \|\bar{\rho} - \rho_m\|_{L^2}^6 dt\right)^{1/3}.$$

Combining the inequality above with (4.19), we have that

$$(4.21) \quad \int_{t_0+T_*/2}^{t_0+T_*} \|\bar{\rho} - \rho_m\|_{L^2}^6 dt \geq \frac{4}{T_*^2} \left(\int_{t_0+T_*/2}^{t_0+T_*} \|\bar{\rho} - \rho_m\|_{L^2}^2 dt\right)^3 \geq \frac{4c_0^3}{32^3 T_*^2} = \frac{4c_0 N_0^2}{32^3}.$$

Now, we shall invoke energy estimate (4.4) and integrate the differential inequality from $T_*/2$ to T_* . Note that we can do this since $\|\rho(t) - \rho_m\|_{L^2}^2 > N_0/8 \geq \rho_m/4$ for any

$t \in [t_0 + \frac{T_*}{2}, t_0 + T_*]$ by our contradiction assumption and our choice of N_0 in (4.15). Therefore, we arrive at

$$\begin{aligned} & \|\rho(t_0 + T_*) - \rho_m\|_{L^2}^2 \\ & \leq \|\rho(t_0 + T_*/2) - \rho_m\|_{L^2}^2 - \frac{1}{C_1 \rho_m^4} \int_{t_0+T_*/2}^{t_0+T_*} \|\bar{\rho} - \rho_m\|_{L^2}^6 dt + C_1 \int_{t_0+T_*/2}^{t_0+T_*} \|\bar{\rho}\|_{L^2}^4 dt \\ & \leq 2N_0 - \frac{4c_0}{32^3 C_1 \rho_m^4} N_0^2 + C_1 \sup_{t_0+T_*/2 \leq t \leq t_0+T_*/2} \|\rho - \rho_m\|_{L^2}^2 \int_{t_0+T_*/2}^{t_0+T_*} \|\bar{\rho}\|_{L^2}^2 dt \\ & \leq 2N_0 - \frac{4c_0}{32^3 C_1 \rho_m^4} N_0^2 + 2C_1 c_0 N_0 g^{-1/20} \leq 2N_0 - \frac{4c_0}{32^3 C_1 \rho_m^4} N_0^2 + 2C_1 c_0 N_0. \end{aligned}$$

Here, we used (4.21) in the second inequality, (4.18) in the third inequality, and $g \geq g_0 > 1$ in the last inequality. Then, by (4.16), a straightforward computation yields

$$\|\rho(t_0 + T_*) - \rho_m\|_{L^2}^2 \leq 2N_0 - \frac{4c_0}{32^3 C_1 \rho_m^4} N_0^2 + 2C_1 c_0 N_0 < \frac{N_0}{8},$$

which is a contradiction to our assumption.

Case 2: Large $\|\bar{\rho}\|_{L^2}$.

$$(4.22) \quad \int_{t_0+T_*/2}^{t_0+T_*} \|\bar{\rho}\|_{L^2}^2 dt > c_0 g^{-1/20}.$$

In this case, we first use estimate (4.7) to obtain that

$$\int_{t_0+T_*/2}^{t_0+T_*} \|(-\Delta_D)^{-1} \partial_1 \rho\|_{L^2}^2 dt \leq C(\rho_m) g^{-1} < C(\rho_m) c_0^{-2} g^{-9/10} \left(\int_{t_0+T_*/2}^{t_0+T_*} \|\bar{\rho}\|_{L^2}^2 dt \right)^2.$$

Then, we can invoke Lemma 4.7 with $C = C(\rho_m) c_0^{-2}$ to conclude that

$$\begin{aligned} (4.23) \quad & \int_{t_0+T_*/2}^{t_0+T_*} \|\nabla \rho\|_{L^2}^2 dt \geq C(\rho_m) g^{9/50} \left(\int_{t_0+T_*/2}^{t_0+T_*} \|\bar{\rho}\|_{L^1}^2 dt \right)^{-4/5} \left(\int_{t_0+T_*/2}^{t_0+T_*} \|\bar{\rho}\|_{L^2}^2 dt \right)^{8/5} \\ & \geq C(\rho_m) T_*^{-4/5} g^{9/50} g^{-2/25} \geq C(\rho_m) N_0^{4/5} g^{1/10}. \end{aligned}$$

Here, we applied Proposition 4.1 in the first inequality, applied (4.22) in the second inequality, and used the definition of T_* in the final inequality. Now, choose g_0 sufficiently large that

$$(4.24) \quad C(\rho_m) N_0^{4/5} g^{1/10} > N_0.$$

Then, together with (4.23), we would arrive at $\int_{t_0+T_*/2}^{t_0+T_*} \|\nabla \rho\|_{L^2}^2 dt > N_0$. However, this contradicts the a priori estimate (4.17). Up until this point, both cases in the dichotomy lead to a contradiction, and we have completed the proof. \square

Remark 4.8.

1. The specific choice of parameters N_0, T_* , and g_0 are summarized as follows: We first fix N_0 according to (4.15) and (4.16), and we set $T_* = 2c_0 N_0^{-1}$. After N_0, T_* are fixed, we choose g_0 sufficiently large that (4.20) and (4.24) hold.

2. We also remark that the choice of $N_0 \geq 2\|\rho_0 - \rho_m\|_{L^2}^2$ does not play a role in the proof of Proposition 1.3. However, it is a rather technical choice that will play a role in the proofs of Corollary 4.9 and Proposition 1.4, which rely on a barrier argument.

Note that we can conclude the global well-posedness of system (2.1) from Proposition 1.3.

COROLLARY 4.9. *For arbitrary nonnegative initial data $\rho_0 \in H^1$, there exists $g_0 = g_0(\rho_m) > 0$ sufficiently large that the solution of the static problem (1.6) is globally regular for all $g \geq g_0$.*

Proof. Let g_0, N_0 be chosen as in Proposition 1.3 (and therefore, T_* is fixed). Let $t_0 \geq 0$ be the first time such that $\|\rho - \rho_m\|_{L^2}^2 = N_0$. If $t_0 = \infty$, then this means that $\|\rho(t) - \rho_m\|_{L^2}^2 \leq N_0$ for all $t \geq 0$ because $N_0 \geq \|\rho_0 - \rho_m\|_{L^2}^2$. Hence, by the regularity criterion stated in Theorem 3.3, we are already done. We may then suppose that $t_0 < \infty$. By Proposition 1.3, there exists $T_1 \in [t_0 + T_*/2, t_0 + T_*]$ such that $\|\rho(t_0 + T_1) - \rho_m\|_{L^2}^2 = N_0/8 < N_0$. Moreover, we know that $\sup_{0 \leq t \leq t_0 + T_*} \|\rho(t) - \rho_m\|_{L^2}^2 \leq 2N_0$. Then, let t_1 to be the first time after $t_0 + T_1$ such that $\|\rho - \rho_m\|_{L^2}^2 = N_0$. Repeating the argument above, there exists $T_2 \in [t_1 + T_*/2, t_1 + T_*]$ such that $\|\rho(t_1 + T_2) - \rho_m\|_{L^2}^2 = N_0/8$ and $\sup_{t_0 + T_1 \leq t \leq t_1 + T_*} \|\rho(t) - \rho_m\|_{L^2}^2 \leq 2N_0$. Repeating such steps indefinitely and using $T_* > 0$,

$$\sup_{t \geq 0} \|\rho(t) - \rho_m\|_{L^2}^2 \leq 2N_0 < \infty,$$

which implies global well-posedness by the L^2 criterion in Theorem 3.3. \square

5. Analysis of the full Keller–Segel–Navier–Stokes problem. In this section, we study the full Keller–Segel–Navier–Stokes system (1.4), and we prove Proposition 1.4. For the readers' convenience, let us recall the system (1.4) here:

$$(5.1a) \quad \partial_t \rho + u \cdot \nabla \rho - \Delta \rho + \operatorname{div}(\rho \nabla (-\Delta_N)^{-1}(\rho - \rho_m)) = 0,$$

$$(5.1b) \quad \partial_t u + u \cdot \nabla u - B \Delta u + \nabla p = Bg\rho(0, 1)^T, \quad \operatorname{div} u = 0,$$

$$(5.1c) \quad \partial_2 \rho|_{\partial\Omega} = 0, \quad u_2|_{\partial\Omega} = 0, \quad \partial_2 u_1|_{\partial\Omega} = 0,$$

$$(5.1d) \quad \rho(0, x) = \rho_0(x) \geq 0, \quad u(0, x) = u_0(x).$$

As pointed out in section 1.2, we will prove Proposition 1.4 with bootstrap assumption (1.11). We will do so by appropriately setting up a comparison scheme between the full problem (5.1) and the static problem (1.6). By examining Proposition 1.3 more closely, we observe that the damping effect induced by the static problem (1.6) becomes significant only when the L^2 norm of the cell density becomes sufficiently large, i.e., N_0 as specified in the statement of Proposition 1.3. Therefore, in the case of the full problem (5.1), we decompose the full solution (ρ, u) into the static problem dynamics (ρ_s, u_s) and the remainder term (r, v) as the L^2 norm of ρ turns to the level of N_0 . We then show suitable estimates for the remainder term (r, v) and show that they can be dominated by the damping effect introduced by the static part.

For now, given initial data (ρ_0, u_0) , we fix N_0 and T_* as in Proposition 1.3. Namely, we choose N_0 so large that (4.15) and (4.16) hold, and then set $T_* = 2c_0 N_0^{-1}$. Moreover, the thresholds $g_1, B_1 \geq 1$ are chosen to be sufficiently large in the course of our proof, whose exact values will be summarized in Remark 5.8. For the rest of this section, we assume $g \geq g_1$ and $B \geq B_1$ by default.

We organize this section in the following way: In section 5.1, we prove preliminary estimates of various derivatives to both ρ and u given bootstrap assumption (1.11). In section 5.2 and section 5.3, we prove suitable estimates for $u \cdot \nabla u$ and $\partial_t u$, respectively, which are crucial in closing the bootstrap argument. In section 5.4, we set up a comparison scheme in between the full problem (5.1) and the static problem (1.6) so as to close the bootstrap argument.

5.1. Preliminary estimates. For the rest of this section, we will consider a solution (ρ, u) to system (1.4) with initial data (ρ_0, u_0) . We begin with several preliminary estimates for ρ , u and some of their derivatives. Such estimates will be instrumental in later estimates of $\partial_t u$ and $u \cdot \nabla u$ under the bootstrap assumption. First, we introduce a time-integrated bound for $\nabla \rho$ on any time interval of scale T_* .

LEMMA 5.1. *For arbitrary $t_0 \in [0, T_0]$ and given bootstrap assumption (1.11), we have that*

$$(5.2) \quad \int_{t_0}^{t_0+T_*/9} \|\nabla \rho\|_{L^2}^2 dt \leq 18N_0.$$

Here, we recall that $[0, T_0]$ is the bootstrap horizon.

Proof. This lemma follows directly from applying Corollary 4.4 with $M = 9N_0$. \square

We then give an estimate of u that is uniform over the time interval $[0, T_0 + T_*/9]$, which is slightly longer than the bootstrap horizon. In the meantime, we also show a bound on ∇u on an interval of scale T_* .

LEMMA 5.2. *Given bootstrap assumption (1.11), the following estimate holds:*

$$(5.3) \quad \sup_{0 \leq t \leq T_0+T_*/9} \|u(t)\|_{L^2}^2 \leq C(\rho_m, \|u_0\|_{L^2})g^2N_0.$$

Moreover, for arbitrary $t_0 \in [0, T_0]$, we have the following gradient bound:

$$(5.4) \quad \int_{t_0}^{t_0+T_*/9} \|\nabla u(s)\|_{L^2}^2 ds \leq C(\rho_m, \|u_0\|_{L^2})g^2.$$

Proof. Fix $t \in [0, T_0 + T_*/9]$. Testing (5.1b) by u and using the Cauchy–Schwarz inequality, we obtain that

$$(5.5) \quad \frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + \int_{\Omega} u_j \partial_j u_i u_i dx + B \int_{\Omega} u \Delta u dx = Bg \int_{\Omega} \rho u_2 dx \leq Bg \|\rho\|_{L^2} \|u\|_{L^2}.$$

To treat the second and third terms on the left-hand side above, we use $\operatorname{div} u = 0$ and $u_2|_{\partial\Omega} = 0$ to deduce that

$$\int_{\Omega} u_j \partial_j u_i u_i dx = \frac{1}{2} \int_{\Omega} u_j \partial_j |u|^2 dx = -\frac{1}{2} \int_{\Omega} \operatorname{div} u |u|^2 dx + \int_{\mathbb{T}} u_2 |u|^2 \Big|_{x_2=0}^{x_2=\pi} dx_1 = 0.$$

Similarly,

$$\begin{aligned} \int_{\Omega} u \Delta u dx &= \int_{\mathbb{T}} \int_0^{\pi} u_j \partial_i \partial_j u_i dx_2 dx_1 = -\|\nabla u\|_{L^2}^2 + \int_{\mathbb{T}} u_j \partial_2 u_j \Big|_{x_2=0}^{x_2=\pi} dx_1 \\ &= -\|\nabla u\|_{L^2}^2 + \int_{\mathbb{T}} (u_1 \partial_2 u_1 + u_2 \partial_2 u_2) \Big|_{x_2=0}^{x_2=\pi} dx_1 = -\|\nabla u\|_{L^2}^2, \end{aligned}$$

where we used the Lions boundary condition in the final equality. Combining the two identities above with (5.5), we have that

$$(5.6) \quad \frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + B \|\nabla u\|_{L^2}^2 \leq Bg \|\rho\|_{L^2} \|u\|_{L^2}.$$

On the other hand, we apply the Poincaré inequality (2.2) and the Cauchy-Schwarz inequality to (5.6), which yields the following differential inequality:

$$\frac{d}{dt} \|u\|_{L^2}^2 \leq -\frac{B}{C} \|u\|_{L^2}^2 + 4CBg^2 \|\rho\|_{L^2}^2.$$

Furthermore, invoking Duhamel's formula, we deduce from the above estimate that

$$(5.7) \quad \|u(t)\|_{L^2}^2 \leq e^{-\frac{B}{C}t} \|u_0\|_{L^2}^2 + 4CBg^2 \int_0^t e^{-\frac{B}{C}(t-s)} \|\rho(s)\|_{L^2}^2 ds$$

for $t \in [0, T_0 + T_*/9]$. To estimate (5.7), we first use bootstrap assumption (1.11) and Corollary 4.4 to conclude that

$$\sup_{0 \leq t \leq T_0 + T_*/9} \|\rho(t) - \rho_m\|_{L^2}^2 \leq 18N_0.$$

Then, we deduce from (5.7) that

$$\begin{aligned} \|u(t)\|_{L^2}^2 &\leq \|u_0\|_{L^2}^2 + 4CBg^2 \left(\sup_{0 \leq t \leq T_0 + T_*/9} \|\rho(t) - \rho_m\|_{L^2}^2 + 2\pi^2 \rho_m^2 \right) \int_0^t e^{-\frac{B}{C}(t-s)} ds \\ &\leq \|u_0\|_{L^2}^2 + 4CBg^2 (18N_0 + 2\pi^2 \rho_m^2) \cdot B^{-1} (1 - e^{-\frac{Bt}{C}}) \\ &\leq \|u_0\|_{L^2}^2 + C(\rho_m) g^2 N_0 \leq C(\rho_m, \|u_0\|_{L^2}) g^2 N_0, \end{aligned}$$

where the third inequality follows from $\rho_m^2 \leq \frac{1}{2} \rho_m N_0$ due to our choice of $N_0 \geq 2\rho_m$. We also used $g, N_0 \geq 1$ in the final inequality. This concludes the proof of (5.3).

To show (5.4), we start with (5.6) to deduce that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + B \|\nabla u\|_{L^2}^2 &\leq C(\rho_m, \|u_0\|_{L^2}) Bg (18N_0 + 2\pi^2 \rho_m^2)^{1/2} g N_0^{1/2} \\ &\leq C(\rho_m, \|u_0\|_{L^2}) Bg^2 N_0, \end{aligned}$$

where we invoked the bootstrap assumption and (5.3) in the second inequality. Fixing $t_0 \in [0, T_0]$ and integrating the above from t_0 to $t_0 + T_*/9$, we obtain that

$$\begin{aligned} B \int_{t_0}^{t_0 + T_*/9} \|\nabla u(t)\|_{L^2}^2 dt &\leq \|u_0\|_{L^2}^2 + C(\rho_m, \|u_0\|_{L^2}) Bg^2 N_0 \cdot c_0 N_0^{-1} \\ &\leq C(\rho_m, \|u_0\|_{L^2}) Bg^2, \end{aligned}$$

where the last inequality holds as $g, B \geq 1$. Now, the desired estimate follows from dividing on both sides of the above inequality by B . \square

We conclude this subsection by an estimate on the time derivative of cell density ρ .

LEMMA 5.3. *Assume the bootstrap assumption (1.11). For any $t_0 \in [0, T_0]$, we have that*

$$(5.8) \quad \int_{t_0}^{t_0 + T_*/9} \|\partial_t \rho\|_{H_0^{-1}}^2 dt \leq C(\rho_m, \|u_0\|_{L^2}) g^2 N_0^2.$$

Proof. Picking arbitrary $\varphi \in H_0^1$ and using (5.1a), we have that

$$\begin{aligned} \left| \int_{\Omega} \partial_t \rho \varphi dx \right| &\leq \left| \int_{\Omega} u \cdot \nabla \rho \varphi dx \right| + \left| \int_{\Omega} \Delta \rho \varphi dx \right| + \left| \int_{\Omega} \varphi \operatorname{div}(\rho \nabla (-\Delta_N)^{-1}(\rho - \rho_m)) dx \right| \\ &=: \sum_{i=1}^3 I_i. \end{aligned}$$

Using the incompressibility of u and integrating by parts, we first see that

$$I_1 = \left| \int_{\Omega} u \cdot \nabla \varphi (\rho - \rho_m) dx \right| \leq \|\rho - \rho_m\|_{L^\infty} \|u\|_{L^2} \|\nabla \varphi\|_{L^2}.$$

Applying integration by parts in a similar fashion and using Sobolev embeddings as well as Hölder's inequality, we also have that

$$\begin{aligned} I_2 &= \left| \int_{\Omega} \nabla \rho \cdot \nabla \varphi dx \right| \leq \|\nabla \rho\|_{L^2} \|\nabla \varphi\|_{L^2}, \\ I_3 &= \left| \int_{\Omega} \rho \nabla \varphi \cdot \nabla (-\Delta_N)^{-1}(\rho - \rho_m) dx \right| \leq \|\rho\|_{L^4} \|\nabla (-\Delta_N)^{-1}(\rho - \rho_m)\|_{L^4} \|\nabla \varphi\|_{L^2} \\ &\leq \|\rho\|_{L^4} \|\rho - \rho_m\|_{L^2} \|\nabla \varphi\|_{L^2} \leq \|\rho\|_{L^\infty}^{1/2} \|\rho - \rho_m\|_{L^2}^{3/2} \|\nabla \varphi\|_{L^2}. \end{aligned}$$

Combining the estimates of I_i above and using duality, we conclude that, for any $t \in [t_0, t_0 + T_*/9]$,

$$\begin{aligned} \|\partial_t \rho\|_{H_0^{-1}}^2 &\lesssim \|\rho - \rho_m\|_{L^\infty}^2 \|u\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 + \|\rho\|_{L^\infty} \|\rho - \rho_m\|_{L^2}^3 \\ &\leq C(\rho_m, \|u_0\|_{L^2}) (g^2 N_0^3 + N_0^{5/2}) + \|\nabla \rho\|_{L^2}^2 \\ &\leq C(\rho_m, \|u_0\|_{L^2}) g^2 N_0^3 + \|\nabla \rho\|_{L^2}^2, \end{aligned}$$

where we used the bootstrap assumption (1.11), Proposition 4.3, and Lemma 5.2 in the second inequality. Finally, we integrate over $[t_0, t_0 + T_*/9]$ and invoke Lemma 5.1 to conclude that

$$\int_{t_0}^{t_0 + T_*/9} \|\partial_t \rho\|_{H_0^{-1}}^2 dt \leq \frac{c_0 N_0^{-1}}{9} \cdot C(\rho_m, \|u_0\|_{L^2}) g^2 N_0^3 + 18 N_0 \leq C(\rho_m, \|u_0\|_{L^2}) g^2 N_0^2$$

since $g \geq 1$ and $N_0 \geq 1$. \square

5.2. Control of the advection term. In this subsection, we estimate the advection term $u \cdot \nabla u$ on any time interval of length T in the bootstrap horizon.

LEMMA 5.4. *Assume that the bootstrap assumption (1.11) holds, and fix arbitrary $T \in (0, \mathcal{T}_0 + T_*/9)$. There exists a constant $C_0 = C_0(\rho_m, \|u_0\|_{L^2})$ such that, for $B \geq C_0 g N_0^{1/2}$, the following estimate holds:*

$$(5.9) \quad \int_{t_0}^{t_0 + T} \|u \cdot \nabla u(t)\|_{L^2}^2 dt \leq C(\rho_m, \|u_0\|_1) g^2 N_0 (g^2 N_0 T + 1)$$

for any $t_0 \in [0, \mathcal{T}_0 + T_*/9 - T]$.

Proof. First, using Hölder's inequality and Ladyzhenskaya's inequality in two dimensions, we observe that

$$\begin{aligned}\|u \cdot \nabla u\|_{L^2} &\leq \|u\|_{L^4} \|\nabla u\|_{L^4} \lesssim \|u\|_{L^2}^{1/2} \|u\|_1 \|\nabla u\|_1^{1/2} \\ &\leq \|u\|_{L^2}^{1/2} (\|u\|_{L^2} + \|\nabla u\|_{L^2}) (\|\nabla u\|_{L^2}^{1/2} + \|\nabla^2 u\|_{L^2}^{1/2}) \\ &= \|u\|_{L^2}^{3/2} \|\nabla u\|_{L^2}^{1/2} + \|u\|_{L^2}^{3/2} \|\nabla^2 u\|_{L^2}^{1/2} + \|u\|_{L^2}^{1/2} \|\nabla u\|_{L^2}^{3/2} \\ &\quad + \|u\|_{L^2}^{1/2} \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2}^{1/2}.\end{aligned}$$

We show how to estimate the most singular term $\|u\|_{L^2}^{1/2} \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2}^{1/2}$, and the rest of the terms follow from a similar argument. Using the interpolation inequality $\|\nabla u\|_{L^2}^2 \leq \|u\|_{L^2} \|\Delta u\|_{L^2}$ for $u \in W$ and the calculus Lemma A.3, we have that

$$(5.10) \quad \|u\|_{L^2} \|\nabla u\|_{L^2}^2 \|\nabla^2 u\|_{L^2} \leq \|u\|_{L^2}^2 \|\Delta u\|_{L^2}^2.$$

Integrating (5.10) in time from t_0 to $t_0 + T$ and using (5.3), we have that

$$\begin{aligned}\int_{t_0}^{t_0+T} \|u\|_{L^2}^2 \|\Delta u\|_{L^2}^2 dt &\leq \sup_{0 \leq t \leq T_0+T_*/9} \|u(t)\|_{L^2}^2 \int_{t_0}^{t_0+T} \|\Delta u(t)\|_{L^2}^2 dt \\ &\leq C(\rho_m, \|u_0\|_{L^2}) g^2 N_0 \int_{t_0}^{t_0+T} \|\Delta u(t)\|_{L^2}^2 dt,\end{aligned}$$

we used (5.3) in the second inequality. Proceeding similarly with other terms, we can conclude the following bound:

$$(5.11) \quad \int_{t_0}^{t_0+T} \|u \cdot \nabla u\|_{L^2}^2 dt \leq C(\rho_m, \|u_0\|_{L^2}) g^2 N_0 \int_{t_0}^{t_0+T} \|\Delta u\|_{L^2}^2 dt.$$

Thus, to close the estimate, we need to obtain a control of Δu over an interval of length T . Testing (5.1b) by $-\Delta u$, we obtain that, for any $t \in (t_0, t_0 + T)$,

$$\begin{aligned}(5.12) \quad &\frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + B \|\Delta u\|_{L^2}^2 \\ &= \int_{\Omega} (\Delta u)(u \cdot \nabla u) dx + Bg \int_{\Omega} \rho \Delta u_2 dx \\ &\leq \|\Delta u\|_{L^2} \|u \cdot \nabla u\|_{L^2} + \frac{B}{4} \|\Delta u\|_{L^2}^2 + Bg^2 \|\rho\|_{L^2}^2 \leq C \|u\|_{L^2} \|\Delta u\|_{L^2}^2 \\ &\quad + \frac{B}{4} \|\Delta u\|_{L^2}^2 + Bg^2 \|\rho\|_{L^2}^2 \\ &\leq C(\rho_m, \|u_0\|_{L^2}) g N_0^{1/2} \|\Delta u\|_{L^2}^2 + \frac{B}{4} \|\Delta u\|_{L^2}^2 + Bg^2 \|\rho\|_{L^2}^2,\end{aligned}$$

where we used (5.10) in the second inequality and (5.3) in the last inequality above. Choose

$$(5.13) \quad B \geq 4C(\rho_m, \|u_0\|_{L^2}) g N_0^{1/2} =: C_0 g N_0^{1/2},$$

where constant $C(\rho_m, \|u_0\|_{L^2})$ is the one appearing in the last inequality of (5.12). Plugging (5.13) into (5.12) and rearranging, we have that

$$\frac{d}{dt} \|\nabla u\|_{L^2}^2 + B \|\Delta u\|_{L^2}^2 \leq 2Bg^2 \|\rho\|_{L^2}^2 \leq 2Bg^2 (18N_0 + 2\pi^2 \rho_m^2)$$

for $t \in (t_0, t_0 + T)$. Here, we used the bootstrap assumption (1.11) and Corollary 4.4.

Now, integrating from t_0 to $t_0 + T$, we obtain that

$$B \int_{t_0}^{t_0+T} \|\Delta u(t)\|_{L^2}^2 dt \leq \|\nabla u_0\|_{L^2}^2 + C(\rho_m) B g^2 N_0 T \leq C(\rho_m, \|u_0\|_1) (B g^2 N_0 T + 1).$$

After dividing by B ,

$$\int_{t_0}^{t_0+T} \|\Delta u(t)\|_{L^2}^2 dt \leq C(\rho_m, \|u_0\|_1) (g^2 N_0 T + 1),$$

where where we used $B \geq 1$ in the inequality. Combining with (5.11), we obtain the desired result. \square

5.3. Control of $\partial_t u$. In this section, our ultimate goal is to show both a pointwise-in-time and a time-integrated bound on the time derivative $\partial_t u$. As the first step, we show an estimate of $\partial_t u$ near time zero, which is presented in Lemma 5.5. The fundamental reason that we prove this lemma is rather technical; we do not assume appropriate compatibility condition on initial datum u_0 . For such a general datum, one generally cannot obtain a pointwise-in-time control of $\|\partial_t u(0, \cdot)\|_{L^2}$ by naively taking the limit of $t \rightarrow 0$ using the equation. (We refer to [22] for a more detailed discussion on compatible data for parabolic problems.) While control of $\partial_t u$ exactly at time zero is not available in general, we may instead control $\|\partial_t u\|_{L^2}$ at some later time due to parabolic smoothing. More precisely, we have Lemma 5.5.

LEMMA 5.5. *Define a time instance $s_0 := c_0 \min(\|\rho_0 - \rho_m\|_{L^2}^{-2}, \rho_m^{-1})$, where c_0 is fixed as in Corollary 4.4. Then, for $B \geq g \max(C_0 N_0^{1/2}, N_0)$, where C_0 is defined in Lemma 5.4, we have that*

$$(5.14) \quad \int_0^{s_0} \|\partial_t u(t)\|_{L^2}^2 dt \leq C(\rho_m, \|u_0\|_1) B^2 g^2.$$

In particular, there exists $\tau_0 \in (0, s_0)$ such that

$$(5.15) \quad \|\partial_t u(\tau_0)\|_{L^2}^2 \leq \frac{C(\rho_m, \|u_0\|_1) B^2 g^2}{s_0} =: \mathcal{C}_1(\rho_m, \|\rho_0 - \rho_m\|_{L^2}, \|u_0\|_1) B^2 g^2.$$

Proof. Note that, given (5.14), (5.15) directly follows from Chebyshev's inequality. Hence, we focus on showing (5.14). Multiplying $\partial_t u$ on both sides of (5.1b) and integrating in space, we have that

$$(5.16) \quad \|\partial_t u\|_{L^2}^2 + B \int_{\Omega} \partial_t u \cdot (-\Delta u) dx = B g \int_{\Omega} \rho \partial_t u_2 dx - \int_{\Omega} \partial_t u (u \cdot \nabla u) dx.$$

Since $u \in W$, we can further compute that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 &= \int_{\Omega} \partial_t \partial_i u_j \partial_i u_j dx = \int_{\Omega} \partial_t u \cdot (-\Delta u) dx + \int_{\partial\Omega} \partial_t u_j \partial_2 u_j dS \\ &= \int_{\Omega} \partial_t u \cdot (-\Delta u) dx + \int_{\partial\Omega} \partial_t u_1 \partial_2 u_1 dS + \int_{\partial\Omega} \partial_t u_2 \partial_2 u_2 dS \\ &= \int_{\Omega} \partial_t u \cdot (-\Delta u) dx, \end{aligned}$$

where we used the boundary conditions $u_2|_{\partial\Omega} = \partial_2 u_1|_{\partial\Omega} = 0$ in the third inequality above. Combining with (5.16) and using the Cauchy–Schwarz inequality, we obtain the following energy inequality:

$$\|\partial_t u\|_{L^2}^2 + \frac{B}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 \leq \frac{1}{2} \|\partial_t u\|_{L^2}^2 + \frac{3}{4} (B^2 g^2 \|\rho\|_{L^2}^2 + \|u \cdot \nabla u\|_{L^2}^2), \quad t \in [0, s_0].$$

Rearranging and integrating in time from 0 to s_0 , we have that

$$(5.17) \quad B \|\nabla u(s_0)\|_{L^2}^2 + \int_0^{s_0} \|\partial_t u(t)\|_{L^2}^2 dt \leq B \|\nabla u_0\|_{L^2}^2 + \frac{3}{2} \int_0^{s_0} (B^2 g^2 \|\rho(t)\|_{L^2}^2 + \|u \cdot \nabla u(t)\|_{L^2}^2) dt.$$

Invoking Lemma 5.4 with $t_0 = 0$, $T = s_0$, we arrive at

$$\int_0^{s_0} \|u \cdot \nabla u(t)\|_{L^2}^2 dt \leq C(\rho_m, \|u_0\|_1) g^2 N_0 (g^2 N_0 s_0 + 1).$$

Note that we can indeed invoke Lemma 5.4 here due to $s_0 < \mathcal{T}_0$ and our assumption on B . Combining this bound with (5.17), we conclude that

$$\begin{aligned} \int_0^{s_0} \|\partial_t u(t)\|_{L^2}^2 dt &\leq B \|\nabla u_0\|_{L^2}^2 + C(\rho_m, \|u_0\|_1) (B^2 g^2 \|\rho_0 - \rho_m\|_{L^2}^2 \cdot s_0 + g^4 N_0^2 s_0 + g^2 N_0) \\ &\leq C(\rho_m, \|u_0\|_1) (B + B^2 g^2 + g^4 N_0^2 s_0 + g^2 N_0) \leq C(\rho_m, \|u_0\|_1) B^2 g^2, \end{aligned}$$

where the last inequality follows from choosing $B^2 \geq g^2 N_0^2 \geq N_0$, as well as the fact that $s_0 \leq \rho_m^{-1}$. Hence, we have shown (5.14). \square

With the L^2 estimate (5.15) of $\partial_t u$ near time zero above, we may further derive the following crucial estimates of $\partial_t u$ up to \mathcal{T}_0 thanks to the bootstrap assumption (1.11). More precisely, see Lemma 5.6.

LEMMA 5.6. *Assume the bootstrap assumption (1.11) holds. There exists $\tau_0 < s_0$ and $B_0(\rho_m, \|u_0\|_1)$ large so that, for any $B \geq B_0 g^2$,*

$$(5.18) \quad \sup_{\tau_0 \leq t \leq \mathcal{T}_0} \|\partial_t u(t)\|_{L^2}^2 \leq 5\mathcal{C}_1(\rho_m, \|\rho_0 - \rho_m\|_{L^2}, \|u_0\|_1) B^2 g^2,$$

where \mathcal{C}_1 is defined in (5.15). Moreover, for any $t_0 \in [\tau_0, \mathcal{T}_0]$, we have that

$$(5.19) \quad \int_{t_0}^{t_0 + T_*/9} \|\partial_t u(t)\|_{L^2}^2 dt \leq C(\rho_m, \|u_0\|_1) B g^2.$$

Additionally, if $t_0 \in [\tau_0, \mathcal{T}_0 - \frac{8}{9} T_*]$, then

$$(5.20) \quad \int_{t_0}^{t_0 + T_*} \|\partial_t u(t)\|_{L^2}^2 dt \leq C(\rho_m, \|u_0\|_1) B g^2.$$

Remark 5.7. We remark that (5.19) does not follow directly from integrating (5.18) in time. In fact, doing so will only lead to an insufficient upper bound of size $B^2 g^2 N_0^{-1}$.

Proof. The proof for this lemma is rather lengthy, so we split the proof in several steps.

Step 1: Energy estimate for $\partial_t u$. We start with the evolution of $\partial_t u$. By differentiating (5.1b) in time, it is straightforward to check that $\partial_t u$ satisfies the following equation:

$$\partial_t^2 u + \partial_t u \cdot \nabla u + u \cdot \nabla \partial_t u - B \Delta \partial_t u + \nabla \partial_t p = Bg(\partial_t \rho)(0, 1)^T.$$

Testing by $\partial_t u$, using $\operatorname{div}(\partial_t u) = 0$ and the fact that $\partial_t u$ still satisfies the Lions boundary condition, we have, for arbitrary $\epsilon > 0$, that

$$\begin{aligned} (5.21) \quad & \frac{1}{2} \frac{d}{dt} \|\partial_t u\|_{L^2}^2 + B \|\nabla \partial_t u\|_{L^2}^2 \\ &= - \int_{\Omega} (\partial_t u_j)(\partial_j u_i)(\partial_t u_i) dx + Bg \int_{\Omega} \partial_t \rho \partial_t u_2 dx \\ &\leq - \int_{\Omega} (\partial_t u_j)(\partial_j u_i)(\partial_t u_i) dx + \epsilon B \|\partial_t u_2\|_{H_0^1}^2 + C(\epsilon) Bg^2 \|\partial_t \rho\|_{H_0^{-1}}^2 \\ &\leq - \int_{\Omega} (\partial_t u_j)(\partial_j u_i)(\partial_t u_i) dx + \epsilon C B \|\nabla \partial_t u\|_{L^2}^2 + C(\epsilon) Bg^2 \|\partial_t \rho\|_{H_0^{-1}}^2, \end{aligned}$$

where we used the H_0^1 – H_0^{-1} duality in the first inequality and the Poincaré inequality in the final inequality. The use of duality is indeed valid since the Lions boundary condition implies that u_2 , and thus $\partial_t u_2$, satisfies the zero Dirichlet boundary condition.

To estimate the first term on the right-hand side of the estimate above, an application of Ladyzhenskaya's inequality in two dimensions and the Cauchy–Schwarz inequality gives

$$\begin{aligned} (5.22) \quad & \int_{\Omega} (\partial_t u_j)(\partial_j u_i)(\partial_t u_i) dx \leq \|\partial_t u\|_{L^4}^2 \|\nabla u\|_{L^2} \leq C \|\partial_t u\|_{L^2} (\|\partial_t u\|_{L^2} + \|\nabla \partial_t u\|_{L^2}) \|\nabla u\|_{L^2} \\ & \leq \frac{B}{4} \|\nabla \partial_t u\|_{L^2}^2 + \frac{C}{B} \|\partial_t u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 + C \|\partial_t u\|_{L^2}^2 \|\nabla u\|_{L^2}. \end{aligned}$$

Now, choosing $\epsilon > 0$ sufficiently small in (5.21), combining with (5.22), and rearranging, we have the following differential inequality:

$$\begin{aligned} (5.23) \quad & \frac{d}{dt} \|\partial_t u\|_{L^2}^2 \leq -B \|\nabla \partial_t u\|_{L^2}^2 + \frac{C}{B} \|\partial_t u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 + C \|\partial_t u\|_{L^2}^2 \|\nabla u\|_{L^2} + C Bg^2 \|\partial_t \rho\|_{H_0^{-1}}^2 \\ & \leq \left(\frac{C}{B} \|\nabla u\|_{L^2}^2 + C \|\nabla u\|_{L^2} - \frac{B}{C_P} \right) \|\partial_t u\|_{L^2}^2 + C Bg^2 \|\partial_t \rho\|_{H_0^{-1}}^2, \end{aligned}$$

where we used the Poincaré inequality (2.2) in the second inequality.

Choose τ_0 as in Lemma 5.5. We write (5.23) in Duhamel form and use (5.15) to obtain that, for any $t \in [\tau_0, \tau_0 + T_*/9]$,

(5.24)

$$\begin{aligned}
& \|\partial_t u(t)\|_{L^2}^2 \\
& \leq C_1 B^2 g^2 \exp\left(-\frac{B(t-\tau_0)}{C_P} + C \int_{\tau_0}^t \left(\frac{1}{B} \|\nabla u(s)\|_{L^2}^2 + \|\nabla u(s)\|_{L^2}\right) ds\right) \\
& \quad + C B g^2 \int_{\tau_0}^t \exp\left(-\frac{B(t-s)}{C_P} + C \int_s^t \left(\frac{1}{B} \|\nabla u(\tau)\|_{L^2}^2 + \|\nabla u(\tau)\|_{L^2}\right) d\tau\right) \\
& \quad \times \|\partial_t \rho(s)\|_{H_0^{-1}}^2 ds \leq C_1 B^2 g^2 \exp\left(-\frac{B(t-\tau_0)}{C_P} + C(\rho_m, \|u_0\|_1) \left(\frac{g^2}{B} + g(t-\tau_0)^{1/2}\right)\right) \\
& \quad + C B g^2 \int_{\tau_0}^t \exp\left(-\frac{B(t-s)}{C_P} + C(\rho_m, \|u_0\|_1) \left(\frac{g^2}{B} + g(t-s)^{1/2}\right)\right) \|\partial_t \rho(s)\|_{H_0^{-1}}^2 ds.
\end{aligned}$$

Here, we used (5.4) in the second inequality above. We also utilized the following consequence of (5.4): For any $\tau_0 \leq a \leq b \leq a + T_*/9$,

$$\int_a^b \|\nabla u(s)\|_{L^2} ds \leq (b-a)^{1/2} \left(\int_a^{a+T_*/9} \|\nabla u(s)\|_{L^2}^2 ds \right)^{1/2} \leq C(\rho_m, \|u_0\|_1) g(b-a)^{1/2}.$$

Then, we choose B sufficiently large that

$$(5.25) \quad B \geq \frac{C_2}{\log 2} g^2,$$

where $C_2 := C(\rho_m, \|u_0\|_1)$ is the constant that appeared in (5.24). This reduces (5.24) to

(5.26)

$$\begin{aligned}
\|\partial_t u(t)\|_{L^2}^2 & \leq 2C_1 B^2 g^2 \exp\left(-\frac{B(t-\tau_0)}{C_P} + C_2 g(t-\tau_0)^{1/2}\right) \\
& \quad + C B g^2 \int_{\tau_0}^t \exp\left(-\frac{B(t-s)}{C_P} + C_2 g(t-s)^{1/2}\right) \|\partial_t \rho(s)\|_{H_0^{-1}}^2 ds =: I_1 + I_2.
\end{aligned}$$

This concludes the proof of energy estimates that we will use in later steps.

Step 2: Proof of (5.18). The plan of proving (5.18) is as follows: We first show a uniform bound similar to the desired control (5.18), yet on a shorter time interval, together with an improved bound at the endpoint of this interval. Namely, we will show that

$$(5.27) \quad \sup_{\tau_0 \leq t \leq \tau_0 + T_*/9} \|\partial_t u(t)\|_{L^2}^2 \leq 5C_1 B^2 g^2,$$

and additionally,

$$(5.28) \quad \|\partial_t u(\tau_0 + T_*/9)\|_{L^2}^2 \leq C_1 B^2 g^2 = \|\partial_t u(\tau_0)\|_{L^2}^2.$$

With (5.27) and (5.28), we can prove (5.18) using an iteration argument.

Step 2.1: Proof of (5.27) and (5.28). To prove the desired estimates (5.27) and (5.28), we study the inequality (5.26). The key point is that we have to estimate (5.26) in two different time scales. Roughly speaking, we will choose $B^{-1} \ll T_*$,

i.e., the dissipation time for u being much smaller than that for ρ . In this scenario, $\|\partial_t u\|_{L^2}$ may grow on time interval $[\tau_0, \tau_0 + B^{-1}]$, but it will not grow much due to the smallness of B^{-1} . Yet, on $[\tau_0 + B^{-1}, \tau_0 + T_*/9]$, the dissipation effect becomes significant and is sufficient to damp growth generated by nonlinear terms.

More precisely, let us choose B such that

$$(5.29) \quad \frac{100C_P}{B} < \frac{c_0}{9N_0} = \frac{T_*}{9}, \quad B \geq \frac{100C_P\mathcal{C}_2^2}{(\log 2)^2} g^2,$$

where C_P is the Poincaré constant that appeared in (5.26). From (5.29), we obtain

$$(5.30) \quad \exp\left(-\frac{B(t-a)}{C_P} + \mathcal{C}_2 g(t-a)^{1/2}\right) \leq \exp\left(-\frac{B(t-a)}{C_P}\right) \exp\left(\frac{10\sqrt{C_P}\mathcal{C}_2 g}{\sqrt{B}}\right) \leq 2$$

for any time a so that $0 \leq t-a \leq \frac{100C_P}{B}$.

Estimate on $[\tau_0, \tau_0 + \frac{100C_P}{B}]$. If $t \in [\tau_0, \tau_0 + \frac{100C_P}{B}]$, we may invoke (5.30) to estimate I_1 and I_2 by

$$I_1 \leq 4\mathcal{C}_1 B^2 g^2, \quad I_2 \leq CBg^2 \int_{\tau_0}^{\tau_0 + T_*/9} \|\partial_t \rho(s)\|_{H_0^{-1}}^2 ds \leq C(\rho_m, \|u_0\|_{L^2}) Bg^4 N_0^2,$$

where we used Lemma 5.3 in the last inequality. We choose B sufficiently large that

$$(5.31) \quad C(\rho_m, \|u_0\|_{L^2}) g^2 N_0^2 < \frac{\mathcal{C}_1 B}{2},$$

where $C(\rho_m, \|u_0\|_{L^2})$ is the constant appearing in the previous estimate. We then conclude that

$$(5.32) \quad \sup_{\tau_0 \leq t \leq \tau_0 + \frac{100C_P}{B}} \|\partial_t u(t)\|_{L^2}^2 < 5\mathcal{C}_1 B^2 g^2.$$

Estimate on $(\tau_0 + \frac{100C_P}{B}, \tau_0 + T_*/9]$. Suppose that $t \in (\tau_0 + \frac{100C_P}{B}, \tau_0 + T_*/9]$. In this case, we further choose B large so that

$$(5.33) \quad B \geq \frac{(2C_P\mathcal{C}_2)^2}{100C_P} g^2.$$

Such a choice guarantees that, for any time $a \in [\tau_0, t - \frac{100C_P}{B}]$, we have that

$$(5.34) \quad -\frac{B(t-a)}{C_P} + \mathcal{C}_2 g(t-a)^{1/2} \leq -\frac{B(t-a)}{2C_P} \leq -50.$$

Therefore, using (5.34), we can bound I_1 and I_2 in (5.26) by

$$(5.35) \quad I_1 \leq 2\mathcal{C}_1 B^2 g^2 \exp\left(-\frac{B(t-\tau_0)}{2C_P}\right) \leq e^{-50} \mathcal{C}_1 B^2 g^2 < \frac{1}{4} \mathcal{C}_1 B^2 g^2,$$

(5.36)

$$\begin{aligned} I_2 &= CBg^2 \left(\int_{\tau_0}^{t - \frac{100C_P}{B}} + \int_{t - \frac{100C_P}{B}}^t \right) \exp\left(-\frac{B(t-s)}{C_P} + \mathcal{C}_2 g(t-s)^{1/2}\right) \|\partial_t \rho(s)\|_{H_0^{-1}}^2 ds \\ &\leq CBg^2 (e^{-50} + 2) \int_{\tau_0}^{\tau_0 + T_*/9} \|\partial_t \rho(s)\|_{H_0^{-1}}^2 ds \leq C(\rho_m, \|u_0\|_{L^2}) Bg^4 N_0^2, \end{aligned}$$

where we used (5.34) to estimate the first integral and (5.30) to estimate the second integral in the first inequality above. Then, we choose B even larger that

$$(5.37) \quad C(\rho_m, \|u_0\|_{L^2})g^2N_0^2 \leq \frac{C_1}{2}B,$$

where $C(\rho_m, \|u_0\|_{L^2})$ is the constant appearing in the last line of (5.36). Combining (5.37) with (5.36), we conclude that I_2 in (5.26) can be bounded as follows:

$$(5.38) \quad I_2 \leq \frac{C_1}{2}B^2g^2.$$

Finally, we combine (5.35) and (5.38) to deduce that

$$(5.39) \quad \sup_{\tau_0 + \frac{100C_P}{B} \leq t \leq \tau_0 + T_*/9} \|\partial_t u(t)\|_{L^2}^2 < C_1 B^2 g^2.$$

Collecting both (5.32) and (5.39) directly yields (5.27) and (5.28).

Step 2.2: Iterative procedure. With (5.27) and (5.28), we may start to iterate Steps 2.1 and 2.2 for subsequent time intervals $[\tau_0 + iT_*/9, \tau_0 + (i+1)T_*/9]$, $i \geq 1$. Notice that all estimates involved in Steps 2.1 and 2.2 only used that $[\tau_0, \tau_0 + T_*/9] \subset [0, \mathcal{T}_0 + T_*/9]$ and $\|\partial_t u(\tau_0)\|_{L^2}^2 \leq C_1 B^2 g^2$. Hence, we may define i_* to be the largest natural number so that $[\tau_0 + i_* T_*/9, \tau_0 + (i_* + 1)T_*/9] \subset [0, \mathcal{T}_0 + T_*/9]$. The above arguments will conclude that

$$\sup_{\tau_0 \leq t \leq \tau_0 + (i_* + 1)T_*/9} \|\partial_t u(t)\|_{L^2}^2 \leq 5C_1 B^2 g^2.$$

We may, without loss of generality, consider $\mathcal{T}_0 < \infty$. In this case, by definition of i_* , we must have $\tau_0 + (i_* + 1)T_*/9 > \mathcal{T}_0$ since otherwise, $[\tau_0 + (i_* + 1)T_*/9, \tau_0 + (i_* + 2)T_*/9] \subset [0, \mathcal{T}_0 + T_*/9]$, contradicting the maximality of i_* . We may then conclude that

$$\sup_{\tau_0 \leq t \leq \mathcal{T}_0} \|\partial_t u(t)\|_{L^2}^2 \leq \sup_{\tau_0 \leq t \leq \tau_0 + (i_* + 1)T_*/9} \|\partial_t u(t)\|_{L^2}^2 \leq 5C_1 B^2 g^2.$$

This concludes the proof of (5.18) after we choose threshold B_0 so large that (5.25), (5.29), (5.31), (5.33), and (5.37) hold for all $B \geq B_0 g^2$. Note that we can indeed choose such B_0 since all conditions of choosing B assume the form $B \geq Cg^2$ with some C .

Step 3: Proof of (5.19). We write (5.26) in Duhamel form starting at time $t_0 \in [\tau_0, \mathcal{T}_0]$:

$$(5.40) \quad \begin{aligned} & \int_{t_0}^{t_0 + T_*/9} \|\partial_t u(t)\|_{L^2}^2 dt \\ & \leq 2\|\partial_t u(t_0)\|_{L^2}^2 \int_{t_0}^{t_0 + T_*/9} \exp\left(-\frac{B}{C_P}(t-t_0) + C_2 g(t-t_0)^{1/2}\right) dt \\ & \quad + CBg^2 \int_{t_0}^{t_0 + T_*/9} \int_{t_0}^t \exp\left(-\frac{B}{C_P}(t-s) + C_2 g(t-s)^{1/2}\right) \|\partial_t \rho(s)\|_{H_0^{-1}}^2 ds dt \\ & \leq 10C_1 B^2 g^2 \int_{t_0}^{t_0 + T_*/9} \exp\left(-\frac{B}{C_P}(t-t_0) + C_2 g(t-t_0)^{1/2}\right) dt \\ & \quad + CBg^2 \int_{t_0}^{t_0 + T_*/9} \int_{t_0}^t \exp\left(-\frac{B}{C_P}(t-s) + C_2 g(t-s)^{1/2}\right) \|\partial_t \rho(s)\|_{H_0^{-1}}^2 ds dt. \end{aligned}$$

Here, we used (5.18) in the second inequality. Since (5.29) and (5.33) hold, we may run a similar argument in deducing (5.30) and (5.34) to conclude that

$$(5.41) \quad \exp\left(-\frac{B}{C_P}(t-s) + C_2 g(t-s)^{1/2}\right) \leq \begin{cases} 2 \exp\left(-\frac{B(t-s)}{C_P}\right), & 0 \leq t-s \leq \frac{100C_P}{B}, \\ \exp\left(-\frac{B(t-s)}{2C_P}\right) & \frac{100C_P}{B} < t-s \leq \frac{T_*}{9} \end{cases} \\ \leq 2 \exp\left(-\frac{B(t-s)}{2C_P}\right)$$

for $0 \leq t-s \leq T_*/9$. Applying (5.41) to (5.40), we have that

$$\begin{aligned} \int_{t_0}^{t_0+T_*/9} \|\partial_t u(t)\|_{L^2}^2 dt &\leq C(\rho_m, \|u_0\|_1) B^2 g^2 \frac{C_P}{B} \left(1 - e^{-\frac{BT_*}{18C_P}}\right) \\ &\quad + CBg^2 \int_{t_0}^{t_0+T_*/9} \left(\int_{t_0}^s e^{-\frac{B(t-s)}{2C_P}} dt\right) \|\partial_t \rho(s)\|_{H_0^{-1}}^2 dt \\ &\leq C(\rho_m, \|u_0\|_1) Bg^2 + Cg^2 \int_{t_0}^{t_0+T_*/9} \|\partial_t \rho(s)\|_{H_0^{-1}}^2 dt \\ &\leq C(\rho_m, \|u_0\|_1) (Bg^2 + g^4 N_0^2) \leq C(\rho_m, \|u_0\|_1) Bg^2 \end{aligned}$$

if we choose B sufficiently large that

$$(5.42) \quad B \geq g^2 N_0^2.$$

Note that we used (5.18) and the Fubini theorem in the first inequality and Lemma 5.3 in the third inequality. This concludes the proof of (5.19) after we choose threshold B_0 so that (5.25), (5.29), (5.31), (5.33), (5.37), and (5.42) hold for all $B \geq B_0 g^2$.

Step 4: Proof of (5.20). Given $t_0 \in [\tau_0, \mathcal{T}_0 - \frac{8}{9}T_*]$, we note that (5.20) follows directly from (5.19):

$$\int_{t_0}^{t_0+T_*} \|\partial_t u(t)\|_{L^2}^2 dt = \sum_{i=0}^8 \int_{t_0+iT_*/9}^{t_0+(i+1)T_*/9} \|\partial_t u(t)\|_{L^2}^2 dt \leq C(\rho_m, \|u_0\|_1) Bg^2,$$

where we applied (5.19) for each term in the inequality above. \square

5.4. Closing the bootstrap. In this section, we show the improved estimate as stated in Proposition 1.4. Consider a time instance \mathcal{T}_1 to be the first time such that $\|\rho - \rho_m\|_{L^2}^2$ achieves N_0 , where we recall that N_0 was chosen to satisfy (4.15) and (4.16). Note that \mathcal{T}_1 verifies the following two properties:

1. $\mathcal{T}_1 > s_0$, where s_0 is chosen as in Lemma 5.5;
2. $\mathcal{T}_1 + T_* < \mathcal{T}_0$.

The first property is due to $N_0 > 2\|\rho_0 - \rho_m\|_{L^2}^2$ and Corollary 4.4; the second property is due to the choice that $T_* = 2c_0 N_0^{-1}$.

Now, we set up the following comparison scheme: Let ρ_s, u_s be the (global) regular solution to the following static problem:

$$(5.43) \quad \begin{cases} \partial_t \rho_s + u_s \cdot \nabla \rho_s - \Delta \rho_s + \operatorname{div}(\rho_s \nabla (-\Delta_N)^{-1}(\rho_s - (\rho_s)_m)) = 0, \\ -\Delta u_s + \nabla p_s = g \rho_s(0, 1)^T, \quad \operatorname{div} u_s = 0, \\ \partial_2 \rho_s|_{\partial\Omega} = 0, (u_s)_2|_{\partial\Omega} = \partial_2(u_s)_1|_{\partial\Omega} = 0, \rho_s(0, x) = \rho(\mathcal{T}_1, x) \geq 0. \end{cases}$$

Define $r(t, x) = \rho(t, x) - \rho_s(t - \mathcal{T}_1, x)$, $v(t, x) = u(t, x) - u_s(t - \mathcal{T}_1, x)$ for $t \in [\mathcal{T}_1, \mathcal{T}_1 + T_*]$. A straightforward computation yields the following equations for r and v :

(5.44a)

$$\partial_t r - \Delta r + v \cdot \nabla \rho + u_s \cdot \nabla r + \operatorname{div}(r \nabla (-\Delta_N)^{-1}(\rho - \rho_m)) + \operatorname{div}(\rho_s \nabla (-\Delta_N)^{-1} r) = 0, \quad (5.44b)$$

$$-\Delta v + \nabla(B^{-1}p - p_s) = -\frac{1}{B} \partial_t u - \frac{1}{B}(u \cdot \nabla u) + gr(0, 1)^T, \operatorname{div} v = 0, \quad (5.44c)$$

$$r(\mathcal{T}_1, x) = 0, \quad v(\mathcal{T}_1, x) = u(\mathcal{T}_1, x) - u_s(\mathcal{T}_1, x), \quad (5.44d)$$

$$\partial_2 r|_{\partial\Omega} = 0, \quad v_2|_{\partial\Omega} = \partial_2 v_1|_{\partial\Omega} = 0.$$

Notice that, in the derivation of (5.44a), we used the fact that

$$\rho(t, x) - \rho_m - (\rho_s(t - \mathcal{T}_1, x) - (\rho_s)_m) = \rho(t, x) - \rho_s(t - \mathcal{T}_1, x) = r(t, x),$$

thanks to the observation that $(\rho_s)_m = (\rho(\mathcal{T}_1, \cdot))_m = \rho_m$. Our plan is to establish appropriate controls of $\|r(t)\|_{L^2}$ and $\|\rho_s(t - \mathcal{T}_1) - \rho_m\|_{L^2}$ on time interval $[\mathcal{T}_1, \mathcal{T}_1 + T_*]$. **Step 1: Control of $\|r(t)\|_{L^2}$.** To control the remainder r , we study (5.44a) by testing this equation by r . By incompressibility of u_s , we deduce that, for any $t \in [\mathcal{T}_1, \mathcal{T}_1 + T_*]$,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|r\|_{L^2}^2 + \|\nabla r\|_{L^2}^2 &= - \int_{\Omega} r v \cdot \nabla \rho - \int_{\Omega} r \operatorname{div}(r \nabla (-\Delta_N)^{-1}(\rho - \rho_m)) \\ &\quad - \int_{\Omega} r \operatorname{div}(\rho_s \nabla (-\Delta_N)^{-1} r) \\ &= K_1 + K_2 + K_3. \end{aligned}$$

We first treat K_2, K_3 . Integrating by parts, we have that

(5.45)

$$K_2 = \frac{1}{2} \int_{\Omega} \nabla(r^2) \cdot \nabla(-\Delta_N)^{-1}(\rho - \rho_m) = \frac{1}{2} \int_{\Omega} r^2(\rho - \rho_m) dx \leq \frac{1}{2} \|\rho - \rho_m\|_{L^\infty} \|r\|_{L^2}^2.$$

To estimate K_3 , we integrate by parts and obtain that

$$\begin{aligned} K_3 &= \int_{\Omega} \rho_s \nabla r \cdot \nabla(-\Delta_N)^{-1} r dx \leq \|\nabla r\|_{L^2} \|\rho_s\|_{L^4} \|\nabla(-\Delta_N)^{-1} r\|_{L^4} \\ &\leq \frac{1}{4} \|\nabla r\|_{L^2}^2 + \frac{3}{4} \|\rho_s\|_{L^4}^2 \|\nabla(-\Delta_N)^{-1} r\|_{L^4}^2 \\ (5.46) \quad &\leq \frac{1}{4} \|\nabla r\|_{L^2}^2 + \frac{3}{4} \|\rho_s\|_{L^4}^2 \|r\|_{L^2}^2, \end{aligned}$$

where we used the Sobolev embedding $H^1 \subset L^4$ and elliptic estimates. Finally, we treat K_1 , which potentially contributes to the growth of $\|r\|_{L^2}$ due to the appearance of v in the term. In fact, we have that

$$\begin{aligned} K_1 &= - \int_{\Omega} r v \cdot \nabla \rho dx = \int_{\Omega} \operatorname{div}(r v) \rho dx - \int_{\partial\Omega} r v_2 \rho d\sigma \\ &= \int_{\Omega} \nabla r \cdot v \rho dx \leq \|\nabla r\|_{L^2} \|v\|_{L^\infty} \|\rho\|_{L^2} \leq \frac{1}{4} \|\nabla r\|_{L^2}^2 + \frac{3}{8} \|v\|_2^2 \|\rho\|_{L^2}^2, \end{aligned}$$

where we used $v_2|_{\partial\Omega} = 0$ in the third equality and Sobolev embedding $H^2 \subset L^\infty$ in the final inequality.

Now, we claim that $-B^{-1}\partial_t u(t, \cdot) - B^{-1}u \cdot \nabla u + gr(t, \cdot)(0, 1)^T \in H$. Indeed, because $u(t, \cdot) \in W$ by Theorem 3.2, we have that $\int_{\Omega} u_1(x) dx = 0$. This leads to

$$\int_{\Omega} (-B^{-1}\partial_t u + gr(0, 1)^T)_1 dx = \int_{\Omega} -B^{-1}\partial_t u_1 dx = -B^{-1}\partial_t \int_{\Omega} u_1(x) dx = 0.$$

Moreover, we observe that, using the Lions boundary condition satisfied by u ,

$$\int_{\Omega} (u \cdot \nabla u)_1 dx = \int_{\Omega} u_j \partial_j u_1 dx = - \int_{\Omega} \operatorname{div} u u_1 dx + \int_{\partial\Omega} u_2 u_1 dS(x) = 0.$$

The two computations above readily verify the claim. Hence, we are eligible to apply the elliptic estimate (2.3) and obtain that

$$\|v\|_2^2 \lesssim \frac{1}{B^2} (\|\partial_t u\|_{L^2}^2 + \|u \cdot \nabla u\|_{L^2}^2) + g^2 \|r\|_{L^2}^2.$$

We therefore estimate K_1 by

$$(5.47) \quad K_1 \leq \frac{1}{4} \|\nabla r\|_{L^2}^2 + C \left(\frac{1}{B^2} (\|\partial_t u\|_{L^2}^2 + \|u \cdot \nabla u\|_{L^2}^2) + g^2 \|r\|_{L^2}^2 \right) \|\rho\|_{L^2}^2.$$

Combining (5.47), (5.45), and (5.46), we have the following differential inequality: For any $t \in [\mathcal{T}_1, \mathcal{T}_1 + T_*]$,

$$\begin{aligned} \frac{d}{dt} \|r\|_{L^2}^2 + \|\nabla r\|_{L^2}^2 &\leq C (g^2 \|\rho\|_{L^2}^2 + \|\rho - \rho_m\|_{L^\infty} + \|\rho_s\|_{L^4}^2) \|r\|_{L^2}^2 \\ &\quad + \frac{C}{B^2} \|\rho\|_{L^2}^2 (\|\partial_t u\|_{L^2}^2 + \|u \cdot \nabla u\|_{L^2}^2) \\ &\leq F(t) \|r\|_{L^2}^2 + \frac{C}{B^2} (\|\rho - \rho_m\|_{L^2}^2 + 2\pi^2 \rho_m^2) (\|\partial_t u\|_{L^2}^2 + \|u \cdot \nabla u\|_{L^2}^2), \end{aligned}$$

where we set $F(t) = C(g^2 \|\rho(t)\|_{L^2}^2 + \|\rho(t) - \rho_m\|_{L^\infty} + \|\rho_s(t - \mathcal{T}_1)\|_{L^4}^2)$. Note that, by the bootstrap assumption, Proposition 4.3, Proposition 1.3, and $g \geq 1$, we have that

$$\int_{\mathcal{T}_1}^{\mathcal{T}_1 + T_*} F(s) ds \leq C(\rho_m, \|\rho_0\|_{L^2}) g^2.$$

Now, we would like to estimate $\|r(t)\|_{L^2}^2$ on $[\mathcal{T}_1, \mathcal{T}_1 + T_*]$. Since $r(\mathcal{T}_1, x) = 0$, an application of Grönwall inequality and bootstrap assumption (1.11) yields that, for any $t \in [\mathcal{T}_1, \mathcal{T}_1 + T_*]$,

$$\begin{aligned} (5.48) \quad \|r(t)\|_{L^2}^2 &\leq \frac{C}{B^2} \int_{\mathcal{T}_1}^t (\|\rho(s) - \rho_m\|_{L^2}^2 + 2\pi^2 \rho_m^2) (\|\partial_t u(s)\|_{L^2}^2 + \|u \cdot \nabla u(s)\|_{L^2}^2) ds \\ &\quad \times \left[\exp \left(\int_{\mathcal{T}_1}^t F(s) ds \right) \right] \\ &\leq C(\rho_m, \|\rho_0 - \rho_m\|_{L^2}) e^{g^2} \cdot \frac{1}{B^2} \int_{\mathcal{T}_1}^{\mathcal{T}_1 + T_*} (\|\partial_t u(s)\|_{L^2}^2 + \|u \cdot \nabla u(s)\|_{L^2}^2) ds. \end{aligned}$$

Choose

$$(5.49) \quad B \geq \max(B_0 g^2, C_0 g N_0^{1/2}),$$

where B_0 and C_0 are chosen as in Lemma 5.6 and Lemma 5.4, respectively. Then, we may invoke (5.20) (since $\mathcal{T}_1 + T_* < \mathcal{T}_0$ and $\mathcal{T}_1 > s_0 \geq \tau_0$) and Lemma 5.4 to estimate that

$$\begin{aligned} \int_{\mathcal{T}_1}^{\mathcal{T}_1+T_*} \|\partial_t u(s)\|_{L^2}^2 + \|u \cdot \nabla u(s)\|_{L^2}^2 ds &\leq C(\rho_m, \|u_0\|_1)(g^2 B + g^4 N_0^2 T_* + g^2 N_0) \\ &= C(\rho_m, \|u_0\|_1)(g^2 B + g^4 N_0 + g^2 N_0), \end{aligned}$$

where we used the fact that $T_* = 2c_0 N_0^{-1}$ in the last equality. Combining this estimate with (5.48), we obtain that

$$\begin{aligned} (5.50) \quad \sup_{\mathcal{T}_1 \leq t \leq \mathcal{T}_1+T_*} \|r(t)\|_{L^2}^2 &\leq C(\rho_m, \|\rho_0 - \rho_m\|_{L^2}, \|u_0\|_1) \frac{(g^2 B + g^4 N_0 + g^2 N_0)e^{g^2}}{B^2} \\ &\leq C(\rho_m, \|\rho_0 - \rho_m\|_{L^2}, \|u_0\|_1) \frac{g^2 e^{g^2}}{B}, \end{aligned}$$

where we used $g \geq 1$ and (5.49) in the last inequality above. Now, choosing

$$(5.51) \quad B \geq \frac{4C(\rho_m, \|\rho_0 - \rho_m\|_{L^2}, \|u_0\|_1)}{N_0} g^2 e^{g^2},$$

where $C(\rho_m, \|\rho_0 - \rho_m\|_{L^2}, \|u_0\|_1)$ is the constant appearing in the last line of (5.50), we deduce that

$$(5.52) \quad \sup_{\mathcal{T}_1 \leq t \leq \mathcal{T}_1+T_*} \|r(t)\|_{L^2}^2 \leq \frac{N_0}{4}.$$

Step 2: Control of $\|\rho_s(t - \mathcal{T}_1) - \rho_m\|_{L^2}$. Since ρ_s satisfies the static problem (5.43), we shall use Proposition 1.3 to demonstrate that ρ_s produces sufficient damping so as to close the bootstrap argument. However, there is a technical caveat: N_0 is chosen according to the initial datum ρ_0 , while ρ_s corresponds to the initial datum $\rho(\mathcal{T}_1, \cdot)$, which has a different L^2 norm from ρ_0 . Therefore, applying Proposition 1.3 to ρ_s will produce new thresholds, which we call \widetilde{N}_0 , \widetilde{T}_* , and \widetilde{g}_0 . Yet, we will demonstrate that one is still able to produce sufficient damping from ρ_s .

Recall that $(\rho_s)_m = \rho_m$, and $\|\rho_s(0, \cdot) - \rho_m\|_{L^2}^2 = N_0$ by the definition of \mathcal{T}_1 . Then, by Proposition 1.3 and setting

$$\widetilde{N}_0 = \max \left(1, 2\rho_m, 2N_0, \frac{32^3 C_1 \rho_m^4 (15/8 + 2C_1 c_0)}{4c_0} \right),$$

$\widetilde{T}_* = 2c_0 \widetilde{N}_0^{-1}$, there exists $\widetilde{g}_0 = \widetilde{g}_0(\widetilde{N}_0, \rho_m)$ such that, for any $g \geq \widetilde{g}_0$, we have that $\sup_{0 \leq \tau \leq \widetilde{T}_*} \|\rho_s(\tau) - \rho_m\|_{L^2}^2 \leq \widetilde{N}_0$, and there exists $\widetilde{T} \in (\widetilde{T}_*/2, \widetilde{T}_*)$ such that $\|\rho_s(\widetilde{T}) - \rho_m\|_{L^2}^2 \leq \frac{\widetilde{N}_0}{8}$. An elementary but crucial observation is that $\widetilde{N}_0 = 2N_0$; this is because we choose N_0 to satisfy (4.15) and (4.16), from which we have that

$$N_0 \geq \max \left(1, 2\rho_m, \frac{32^3 C_1 \rho_m^4 (15/8 + 2C_1 c_0)}{4c_0} \right),$$

and therefore, $\widetilde{N}_0 = 2N_0$. Thus, by definition, we also have that $\widetilde{T}_* = \frac{T_*}{2}$.

Now, we choose

$$(5.53) \quad g_1 = g_1(\|\rho_0 - \rho_m\|_{L^2}, \rho_m) := \widetilde{g}_0(\widetilde{N}_0, \rho_m).$$

With the discussions above, we conclude the following: Assuming that $g \geq g_1$, then there is

$$\|\rho_s(t) - \rho_m\|_{L^2}^2 \leq \widetilde{N}_0 = 2N_0$$

for $t \in [0, T_*/2]$. Moreover, there exists $S_1 := \widetilde{T} \in [T_*/4, T_*/2]$ such that

$$\|\rho_s(S_1) - \rho_m\|_{L^2}^2 \leq \frac{\widetilde{N}_0}{8} = \frac{N_0}{4}.$$

Step 3: Proof of improved estimate (1.12). By the triangle inequality and (5.52),

$$\begin{aligned} \sup_{\mathcal{T}_1 \leq t \leq \mathcal{T}_1 + T_*/2} \|\rho(t) - \rho_m\|_{L^2}^2 &\leq 2 \sup_{\mathcal{T}_1 \leq t \leq \mathcal{T}_1 + T_*/2} \|\rho_s(t - \mathcal{T}_1) - \rho_m\|_{L^2}^2 \\ &\quad + 2 \sup_{\mathcal{T}_1 \leq t \leq \mathcal{T}_1 + T_*/2} \|r(t)\|_{L^2}^2 \\ &\leq 4N_0 + \frac{N_0}{4} < 5N_0 \end{aligned}$$

and

$$\|\rho(\mathcal{T}_1 + S_1) - \rho_m\|_{L^2}^2 \leq \frac{N_0}{2} + \frac{N_0}{4} < N_0.$$

Now, we may iterate this argument in a similar spirit to the proof of (5.20) in Lemma 5.6. For any $i > 1$ with \mathcal{T}_{i-1} and $S_{i-1} \in [T_*/4, T_*/2]$ defined, we consider \mathcal{T}_i to be the first time instance after $\mathcal{T}_{i-1} + S_{i-1}$ such that $\|\rho - \rho_m\|_{L^2}^2 = N_0$. By the definition of \mathcal{T}_i , we have that $\mathcal{T}_i - \mathcal{T}_{i-1} \geq S_{i-1} \geq T_*/4$. Moreover, we note that \mathcal{T}_i still verifies that $\mathcal{T}_i > s_0$ and $\mathcal{T}_i + T_* < \mathcal{T}_0$. Indeed, the first inequality is due to the monotonicity of \mathcal{T}_i in i , and the second inequality holds true since T_* is the least doubling time of $\|\rho - \rho_m\|_{L^2}^2$ at level N_0 . So, we may repeat Steps 2 and 3 above on interval $[\mathcal{T}_i, \mathcal{T}_i + T_*]$:

$$\sup_{\mathcal{T}_i \leq t \leq \mathcal{T}_i + T_*/2} \|\rho(t) - \rho_m\|_{L^2}^2 \leq 5N_0$$

and

$$\|\rho(\mathcal{T}_i + S_i) - \rho_m\|_{L^2}^2 \leq N_0$$

for some $S_i \in [T_*/4, T_*/2]$. Iterating for $j_* := \lfloor \frac{4\mathcal{T}_0}{T_*} \rfloor + 1$ times, we reach the estimate that

$$\sup_{0 \leq t \leq \mathcal{T}_{j_*} + T_*/2} \|\rho(t) - \rho_m\|_{L^2}^2 \leq 5N_0.$$

However, we must have that $\mathcal{T}_{j_*} + T_*/2 \geq \mathcal{T}_0$ since $\mathcal{T}_{i+1} - \mathcal{T}_i \geq T_*/4$ for every i . From this fact, we conclude that

$$\sup_{0 \leq t \leq \mathcal{T}_0} \|\rho(t) - \rho_m\|_{L^2}^2 \leq \sup_{0 \leq t \leq \mathcal{T}_{j_*} + T_*/2} \|\rho(t) - \rho_m\|_{L^2}^2 \leq 5N_0,$$

which is the improved bound (1.12).

Remark 5.8. We summarize the choice of thresholds g_1 and B_1 as follows. We first choose $g_1 = g_1(\rho_m, \|\rho_0 - \rho_m\|_{L^2})$ as in (5.53). Fixing $g \geq g_1$, we then choose $B_1 = B_1(\rho_m, \|\rho_0 - \rho_m\|_{L^2}, \|u_0\|_1)$ large so that (5.49) and (5.51) hold for all $B \geq B_1 g^2 e^{g^2}$.

6. Discussion. In this section, we discuss extensions of Theorem 1.1 and some open problems.

1. By performing a similar but less laborious analysis, one could prove a similar result to Theorem 1.1 to the following Keller–Segel–Stokes system:

$$(6.1) \quad \begin{cases} \partial_t \rho + u \cdot \nabla \rho - \Delta \rho + \operatorname{div}(\rho \nabla (-\Delta_N)^{-1}(\rho - \rho_m)) = 0, \\ \rho_m = \frac{1}{|\Omega|} \int_{\Omega} \rho(t, x) dx, \\ \partial_t u - \frac{1}{\operatorname{Re}} \Delta u + \nabla p = \operatorname{Ra} \rho(0, 1)^T, \\ \operatorname{div} u = 0 \end{cases}$$

equipped with boundary conditions

$$(6.2) \quad \partial_2 \rho = 0, \quad u_2 = 0, \quad \partial_2 u_1 = 0 \quad \text{on } \partial\Omega.$$

The proof of this result was contained in an earlier arXiv version of this work, but we only presented the Keller–Segel–Navier–Stokes case here due to considerable redundancies in the proof of the Keller–Segel–Stokes case.

2. It is natural to attempt to generalize our main result to the Keller–Segel–Navier–Stokes system equipping the classical no-slip boundary condition on the Navier–Stokes equation. However, in this case, the existence of more singular boundary layers might complicate the analysis. Such effect is manifested in a different leading-order problem. Instead of considering the Biot–Savart law $u = -g \nabla^\perp (-\Delta_D)^{-2} \partial_1 \rho$ as we did in this work, one needs to use $u = \nabla^\perp \psi$, where ψ solves the following biharmonic equation (see [18, 48, 47] for a more detailed discussion):

$$(6.3) \quad \begin{cases} \Delta^2 \psi = \partial_1 \rho, \\ \psi|_{\partial\Omega} = \partial_2 \psi|_{\partial\Omega} = 0. \end{cases}$$

Appendix A.

A.1. Stokes operator equipped with Lions boundary condition. In this section, we first rigorously prove Proposition 2.1 and Proposition 2.2, which concern the Stokes operator corresponding to the Lions boundary condition. For the convenience of the readers, we reiterate the statements of the propositions as follows.

PROPOSITION A.1. *Let $u \in V$. Then, the following estimate holds:*

$$(A.1) \quad \|u\|_{L^2} \leq C_P \|\nabla u\|_{L^2},$$

where C_P is a positive constant that only depends on the domain Ω .

Proof. Since $\|u\|_{L^2}^2 = \|u_1\|_{L^2}^2 + \|u_2\|_{L^2}^2$, we will estimate each velocity component separately. Since $u \in V$, then, in particular, u_1 has zero spatial mean and $u_2 \in H_0^1(\Omega)$. Thus, (A.1) follows from applying the Poincaré inequality to u_1 and u_2 separately. \square

PROPOSITION A.2. *Assuming that $f \in H$, then (2.1) admits a unique solution $u \in W$ with estimate*

$$(A.2) \quad \|u\|_2 \leq C \|f\|_{L^2}.$$

In fact, u is given by the following explicit formula:

$$(A.3) \quad u = \nabla^\perp (-\Delta_D)^{-2} (\partial_2 f_1 - \partial_1 f_2).$$

More generally, if $f \in H^s \cap H$, $s \geq 1$, we have the following improved regularity estimate:

$$(A.4) \quad \|u\|_{s+2} \leq C\|f\|_s.$$

Proof. To show the well-posedness of problem (2.1), it is convenient to show on the level of stream function. Since we would find u such that $\int_{\Omega} u_1 dx = 0$ and $u_2|_{\partial\Omega} = 0$, there exists a unique stream function ψ such that $u = \nabla^{\perp}\psi$ by Hodge decomposition. Moreover, ψ satisfies the following:

$$\Delta\psi = \omega, \quad \psi|_{\partial\Omega} = 0.$$

Taking $\nabla^{\perp} \cdot$ on (2.1) and denoting $\omega = \nabla^{\perp} \cdot u$, we must have that

$$-\Delta\omega = \partial_1 f_2 - \partial_2 f_1, \quad \omega|_{\partial\Omega} = 0.$$

Therefore, the discussion above motivates us to study the following auxiliary coupled system:

$$(A.5) \quad \begin{cases} \Delta\psi = \omega, & \psi|_{\partial\Omega} = 0, \\ -\Delta\omega = \partial_1 f_2 - \partial_2 f_1, & \omega|_{\partial\Omega} = 0. \end{cases}$$

First, note that, since $f \in L^2$, we have that $\partial_1 f_2 - \partial_2 f_1 \in H_0^{-1}$ in the sense of distribution. Then, by standard elliptic theory, there exists a unique $\omega \in H_0^1$ such that the ω equation holds with estimate $\|\omega\|_1 \leq C(\Omega)\|f\|_{L^2}$. Applying standard elliptic theory again to the ψ equation, we have a unique solution $\psi \in H^3 \cap H_0^1$, with estimate $\|\psi\|_3 \leq C(\Omega)\|\omega\|_1$. Define $u = \nabla^{\perp}\psi$. Then, u strongly solves (2.1) with boundary conditions $u_2|_{\partial\Omega} = \partial_2 u_1|_{\partial\Omega} = 0$ satisfied in the trace sense. Moreover, we have the estimate

$$\|u\|_2 \leq \|\psi\|_3 \leq C(\Omega)\|f\|_{L^2}.$$

Moreover, from the discussion above, we obtain formula (A.3). Finally, to show the higher regularity estimates (A.4), we note that, given $f \in H^s \cap H$, $s \geq 1$, $\nabla f \in H^{s-1}$. Using the explicit formula (A.3) and standard elliptic estimate, we conclude (A.4). \square

Finally, we present the following calculus lemma, Lemma A.3.

LEMMA A.3. $\|\nabla^2 u\|_{L^2} = \|\Delta u\|_{L^2}$ for $u \in W \cap C^\infty(\Omega)$.

Proof. Integrating by parts twice, we obtain that

$$\begin{aligned} \|\Delta u\|_{L^2}^2 &= \int_{\Omega} \partial_i \partial_i u_j \partial_k \partial_k u_j dx \\ &= \int_{\Omega} \partial_i \partial_k u_j \partial_i \partial_k u_j dx + \int_{\partial\Omega} \Delta u_j \partial_2 u_j dx_1 - \int_{\partial\Omega} \partial_k \partial_2 u_j \partial_k u_j dx_1 \\ &=: \|\nabla^2 u\|_{L^2}^2 + BT_1 + BT_2. \end{aligned}$$

We moreover notice that

$$BT_1 = \int_{\partial\Omega} \Delta u_1 \partial_2 u_1 dx_1 + \Delta u_2 \partial_2 u_2 dx_1 = 0.$$

Now, we observe that, for any $x \in \partial\Omega$, since u is smooth and belongs to the class W , we have that

$$\Delta u_2 = \partial_1^2 u_2 + \partial_2^2 u_2 = \partial_2^2 u_2 = -\partial_1 \partial_2 u_1 = 0,$$

where we used $u_2|_{\partial\Omega} = 0$ in the second equality, $\operatorname{div} u = 0$ in the third equality, and $\partial_2 u_1|_{\partial\Omega} = 0$ in the last equality. Using this fact and $\partial_2 u_1 = 0$ for $x \in \partial\Omega$, we expand the sum and obtain

$$\begin{aligned} BT_2 &= \int_{\partial\Omega} \partial_1 \partial_2 u_1 \partial_1 u_1 dx_1 + \int_{\partial\Omega} \partial_1 \partial_2 u_2 \partial_1 u_2 dx_1 \\ &\quad + \int_{\partial\Omega} \partial_2 \partial_2 u_1 \partial_2 u_1 dx_1 + \int_{\partial\Omega} \partial_2 \partial_2 u_2 \partial_2 u_2 dx_1 \\ &= \int_{\partial\Omega} \partial_2 \partial_2 u_2 \partial_2 u_2 dx_1, \end{aligned}$$

where we used the boundary conditions of u in the last equality. Since $\partial_2^2 u_2 = -\partial_1 \partial_2 u_1 = 0$ for $x \in \partial\Omega$, we finally conclude that $BT_2 = 0$, and thus, $\|\nabla^2 u\|_{L^2} = \|\Delta u\|_{L^2}$. \square

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REFERENCES

- [1] J. BEDROSSIAN, *Large mass global solutions for a class of L^1 -critical nonlocal aggregation equations and parabolic-elliptic Patlak–Keller–Segel models*, Comm. Partial Differential Equations, 40 (2015), pp. 1119–1136, <https://doi.org/10.1080/03605302.2014.999938>.
- [2] J. BEDROSSIAN AND S. HE, *Suppression of blow-up in Patlak–Keller–Segel via shear flows*, SIAM J. Math. Anal., 49 (2017), pp. 4722–4766, <https://doi.org/10.1137/16M1093380>.
- [3] J. BEDROSSIAN AND N. MASMOUDI, *Existence, uniqueness and Lipschitz dependence for Patlak–Keller–Segel and Navier–Stokes in \mathbb{R}^2 with measure-valued initial data*, Arch. Ration. Mech. Anal., 214 (2014), pp. 717–801, <https://doi.org/10.1007/s00205-014-0796-z>.
- [4] P. BILER, G. KARCH, P. LAURENÇOT, AND T. NADZIEJA, *The 8π -problem for radially symmetric solutions of a chemotaxis model in the plane*, Math. Methods Appl. Sci., 29 (2006), pp. 1563–1583, <https://doi.org/10.1002/mma.743>.
- [5] A. BLANCHET, J. A. CARRILLO, AND N. MASMOUDI, *Infinite time aggregation for the critical Patlak–Keller–Segel model in \mathbb{R}^2* , Comm. Pure Appl. Math., 61 (2008), pp. 1449–1481, <https://doi.org/10.1002/cpa.20225>.
- [6] A. BLANCHET, J. DOLBEAULT, AND B. PERTHAME, *Two-dimensional Keller–Segel model: Optimal critical mass and qualitative properties of the solutions*, Electron. J. Differential Equations, 2006 (2006), 44, <https://eudml.org/doc/127363>.
- [7] F. BOYER AND P. FABRIE, *Mathematical Tools for the Study of the Incompressible Navier–Stokes Equations and Related Models*, Springer, Cham, 2012.
- [8] V. CALVEZ AND L. CORRIAS, *The parabolic-parabolic Keller–Segel model in \mathbb{R}^2* , Commun. Math. Sci., 6 (2008), pp. 417–447, <http://dx.doi.org/10.4310/CMS.2008.v6.n2.a8>.
- [9] J. A. CARRILLO AND J. ROSADO, *Uniqueness of bounded solutions to aggregation equations by optimal transport methods*, in Proceedings of the 5th European Congress of Mathematics, European Mathematical Society, Zurich, 2010, pp. 3–16, <https://doi.org/10.4171/077-1/1>.
- [10] M. CHAE, K. KANG, AND J. LEE, *Global existence and temporal decay in Keller–Segel models coupled to fluid equations*, Comm. Partial Differential Equations, 39 (2014), pp. 1205–1235, <https://doi.org/10.1080/03605302.2013.852224>.
- [11] J. CHEN, *On the Regularity of the de Gregorio Model for the 3D Euler Equations*, preprint, arXiv:2107.04777, 2021.
- [12] T. CLOPEAU, A. MIKELIC, AND R. ROBERT, *On the vanishing viscosity limit for the 2D incompressible Navier–Stokes equations with the friction type boundary conditions*, Nonlinearity, 11 (1998), 1625–1636, <https://doi.org/10.1088/0951-7715/11/6/011>.
- [13] C. COLLOT, T.-E. GHOUL, N. MASMOUDI, AND V. T. NGUYEN, *Refined description and stability for singular solutions of the 2D Keller–Segel system*, Comm. Pure Appl. Math., 75 (2022), pp. 1419–1516, <https://doi.org/10.1002/cpa.21988>.

- [14] C. COLLOT, T.-E. GHOUL, N. MASMOUDI, AND V. T. NGUYEN, *Spectral analysis for singularity formation of the two dimensional Keller–Segel system*, Ann. PDE, 8 (2022), 5, <https://doi.org/10.1007/s40818-022-00118-5>.
- [15] P. CONSTANTIN, *Note on loss of regularity for solutions of the 3-D incompressible Euler and related equations*, Comm. Math. Phys., 104 (1986), pp. 311–326, <https://doi.org/10.1007/BF01211598>.
- [16] P. CONSTANTIN AND C. FOIAS, *Navier–Stokes Equations*, University of Chicago Press, Chicago, IL, 2020.
- [17] P. CONSTANTIN, A. KISELEV, L. RYZHIK, AND A. ZLATOŠ, *Diffusion and mixing in fluid flow*, Ann. of Math., 168 (2008), pp. 643–674, <https://doi.org/10.4007/annals.2008.168.643>.
- [18] A.-L. DALIBARD, J. GUILLOD, AND A. LEBLOND, *Long-Time Behavior of the Stokes-Transport System in a Channel*, preprint, arXiv:2306.00780, 2023.
- [19] M. DI FRANCESCO, A. LORZ, AND P. MARKOWICH, *Chemotaxis-fluid coupled model for swimming bacteria with nonlinear diffusion: Global existence and asymptotic behavior*, Discrete Contin. Dyn. Syst., 28 (2010), pp. 1437–1453, <https://doi.org/10.3934/dcds.2010.28.1437>.
- [20] R. DUAN, A. LORZ, AND P. MARKOWICH, *Global solutions to the coupled chemotaxis-fluid equations*, Comm. Partial Differential Equations, 35 (2010), pp. 1635–1673, <https://doi.org/10.1080/03605302.2010.497199>.
- [21] T. M. ELGINDI, *Finite-time singularity formation for $C^{1,\alpha}$ solutions to the incompressible Euler equations on \mathbb{R}^3* , Ann. of Math., 194 (2021), pp. 647–727, <https://doi.org/10.4007/annals.2021.194.3.2>.
- [22] L. C. EVANS, *Partial Differential Equations*, Grad. Stud. Math. 19, American Mathematical Society, Providence, RI, 2022.
- [23] M. C. L. FILHO, H. J. N. LOPES, AND G. PLANAS, *On the inviscid limit for two-dimensional incompressible flow with Navier friction condition*, SIAM J. Math. Anal., 36 (2005), pp. 1130–1141, <https://doi.org/10.1137/S0036141003432341>.
- [24] S. HE, *Suppression of blow-up in parabolic–parabolic Patlak–Keller–Segel via strictly monotone shear flows*, Nonlinearity, 31 (2018), pp. 3651–3688, <https://doi.org/10.1088/1361-6544/aac1ce>.
- [25] S. HE, *Enhanced dissipation and blow-up suppression in a chemotaxis-fluid system*, SIAM J. Math. Anal., 55 (2023), pp. 2615–2643, <https://doi.org/10.1137/22M1517159>.
- [26] S. HE AND E. TADMOR, *Suppressing chemotactic blow-up through a fast splitting scenario on the plane*, Arch. Ration. Mech. Anal., 232 (2019), pp. 951–986, <https://doi.org/10.1007/s00205-018-01336-7>.
- [27] S. HE, E. TADMOR, AND A. ZLATOŠ, *On the fast spreading scenario*, Comm. Amer. Math. Soc., 2 (2022), pp. 149–171, <https://doi.org/10.1090/cams/6>.
- [28] D. HORSTMANN, *From 1970 until present: The Keller–Segel model in chemotaxis and its consequences*, I. Jahresber. Deutsch. Math.-Verein., 3 (2003), pp. 103–165.
- [29] D. HORSTMANN, *From 1970 until present: The Keller–Segel model in chemotaxis and its consequences*, II. Jahresber. Deutsch. Math.-Verein., 2 (2004), pp. 51–69.
- [30] T. Y. HOU, C. LI, Z. SHI, S. WANG, AND X. YU, *On singularity formation of a nonlinear nonlocal system*, Arch. Ration. Mech. Anal., 199 (2011), pp. 117–144, <https://doi.org/10.1007/s00205-010-0319-5>.
- [31] C.-Y. HSIEH AND Y. YU, *Long-time dynamics of classical Keller–Segel equation*, Nonlinear Anal. Real World Appl., 76 (2024), 104026, <https://doi.org/10.1016/j.nonrwa.2023.104026>.
- [32] Z. HU AND A. KISELEV, *Suppression of chemotactic blowup by strong buoyancy in Stokes–Boussinesq flow with cold boundary*, J. Funct. Anal., 287 (2024) 110541, <https://doi.org/10.1016/j.jfa.2024.110541>.
- [33] Z. HU, A. KISELEV, AND Y. YAO, *Suppression of Chemotactic Singularity by Buoyancy*, preprint, arXiv:2305.01036, 2023.
- [34] G. IYER, A. KISELEV, AND X. XU, *Lower bounds on the mix norm of passive scalars advected by incompressible enstrophy-constrained flows*, Nonlinearity, 27 (2014), pp. 973–986, <https://doi.org/10.1088/0951-7715/27/5/973>.
- [35] G. IYER, X. XU, AND A. ZLATOŠ, *Convection-induced singularity suppression in the Keller–Segel and other non-linear PDEs*, Trans. Amer. Math. Soc., 374 (2021), pp. 6039–6058, <https://doi.org/10.1090/tran/8195>.
- [36] W. JÄGER AND S. LUCKHAUS, *On explosions of solutions to a system of partial differential equations modelling chemotaxis*, Trans. Amer. Math. Soc., 329 (1992), pp. 819–824, <https://doi.org/10.1090/S0002-9947-1992-1046835-6>.
- [37] H. JIA, S. STEWART, AND V. SVERAK, *On the de Gregorio modification of the Constantin–Lax–Majda model*, Arch. Ration. Mech. Anal., 231 (2019), pp. 1269–1304, <https://doi.org/10.1007/s00205-018-1298-1>.

- [38] E. F. KELLER AND L. A. SEGEL, *Model for chemotaxis*, J. Theoret. Biol., 30 (1971), pp. 225–234, [https://doi.org/10.1016/0022-5193\(71\)90050-6](https://doi.org/10.1016/0022-5193(71)90050-6).
- [39] J. P. KELLIHER, *Navier–Stokes equations with Navier boundary conditions for a bounded domain in the plane*, SIAM J. Math. Anal., 38 (2006), pp. 210–232, <https://doi.org/10.1137/040612336>.
- [40] A. KISELEV AND X. XU, *Suppression of chemotactic explosion by mixing*, Arch. Ration. Mech. Anal., 222 (2016), pp. 1077–1112, <https://doi.org/10.1007/s00205-016-1017-8>.
- [41] Z. LEI AND T. Y. HOU, *On the stabilizing effect of convection in three-dimensional incompressible flows*, Comm. Pure Appl. Math., 62 (2009), pp. 501–564, <https://doi.org/10.1002/cpa.20254>.
- [42] J.-L. LIONS, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, 1969.
- [43] P.-L. LIONS, *Mathematical Topics in Fluid Mechanics, Vol. 1: Incompressible Models*, Oxford Lecture Ser. Math. Appl., 1996.
- [44] A. LORZ, *A coupled Keller–Segel–Stokes model: Global existence for small initial data and blow-up delay*, Commun. Math. Sci., 10 (2012), pp. 555–574, <https://doi.org/10.4310/CMS.2012.v10.n2.a7>.
- [45] T. NAGAI, *Blow-up of radially symmetric solutions to a chemotaxis system*, Adv. Math. Sci. Appl., 5 (1995), 581–601.
- [46] T. NAGAI, T. SENBA, AND K. YOSHIDA, *Application of the Trudinger–Moser inequality to a parabolic system of chemotaxis*, Funkc. Ekvacioj, 40 (1997), pp. 411–433.
- [47] J. PARK, *Long Time Stability and Instability in the Two-Dimensional Boussinesq System with Kinematic Viscosity*, preprint, arXiv:2404.00985, 2024.
- [48] J. PARK, *Stability Analysis of the Incompressible Porous Media Equation and the Stokes Transport System Via Energy Structure*, preprint, arXiv:2403.14187, 2024.
- [49] C. S. PATLAK, *Random walk with persistence and external bias*, Bull. Math. Biophys., 15 (1953), pp. 311–338, <https://doi.org/10.1007/BF02476407>.
- [50] E. M. PURCELL, *Life at low Reynolds number*, Amer. J. Phys., 45 (1977), pp. 3–11, <https://doi.org/10.1119/1.10903>.
- [51] I. TUVAL, L. CISNEROS, C. DOMBROWSKI, C. W. WOLGEMUTH, J. O. KESSLER, AND R. E. GOLDSTEIN, *Bacterial swimming and oxygen transport near contact lines*, Proc. Natl. Acad. Sci. USA, 102 (2005), pp. 2277–2282, <https://doi.org/10.1073/pnas.0406724102>.
- [52] J. J. VELÁZQUEZ, *Stability of some mechanisms of chemotactic aggregation*, SIAM J. Math. Anal., 62 (2002), pp. 1581–1633, <https://doi.org/10.1137/S0036139900380049>.
- [53] J. J. L. VELÁZQUEZ, *Point dynamics in a singular limit of the Keller–Segel model 1: Motion of the concentration regions*, SIAM J. Math. Anal., 64 (2004), pp. 1198–1223, <https://doi.org/10.1137/S0036139903433888>.
- [54] J. J. L. VELÁZQUEZ, *Point dynamics in a singular limit of the Keller–Segel model 2: Formation of the concentration regions*, SIAM J. Math. Anal., 64 (2004), pp. 1224–1248, <https://doi.org/10.1137/S003613990343389X>.
- [55] D. WEI, *Global well-posedness and blow-up for the 2-D Patlak–Keller–Segel equation*, J. Funct. Anal., 274 (2018), pp. 388–401, <https://doi.org/10.1016/j.jfa.2017.10.019>.
- [56] M. WINKLER, *Global large-data solutions in a chemotaxis-(Navier–) Stokes system modeling cellular swimming in fluid drops*, Comm. Partial Differential Equations, 37 (2012), pp. 319–351, <https://doi.org/10.1080/03605302.2011.591865>.
- [57] M. WINKLER, *Suppressing blow-up by gradient-dependent flux limitation in a planar Keller–Segel–Navier–Stokes system*, Z. Angew. Math. Phys., 72 (2021), 72, <https://doi.org/10.1007/s00033-021-01508-8>.
- [58] Y. XIAO AND Z. XIN, *On the vanishing viscosity limit for the 3D Navier–Stokes equations with a slip boundary condition*, Comm. Pure Appl. Math., 60 (2007), pp. 1027–1055, <https://doi.org/10.1002/cpa.20187>.
- [59] L. ZENG, Z. ZHANG, AND R. ZI, *Suppression of blow-up in Patlak–Keller–Segel–Navier–Stokes system via the Couette flow*, J. Funct. Anal., 280 (2021), 108967, <https://doi.org/10.1016/j.jfa.2021.108967>.