Quantum Entanglement Computation Using Canonical Quantization in a Waveguide and Scattering Theory

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***Abstract*—In this work, we propose two approaches of numerical study of quantum entanglement. In the first approach, canonical quantization with numerical mode decomposition is applied to inhomogeneous dispersive media. The initial quantum state is assumed to be a cat state, which is an entangled state with coherent states. Results in 2D modeling will be presented. In the second approach, we study the effect of scattering on momentum-entangled photon pair from spontaneous parametric down-conversion. Scattered two-photon wave function is calculated using computational electromagnetics. Schmidt number is calculated numerically to investigate the enhancement of entanglement after scattering compared to the incident state.**

I. Introduction

Quantum entanglement based on photons has important applications in quantum communication, quantum radar and quantum sensing. To describe entangled photons, formulations of quantum electromagnetics applicable to complex media are needed. In this work, we propose two approaches of investigating quantum entanglement.

In the first approach, canonical quantization with numerical mode decomposition is performed to rigorously quantize the Hamiltonian for finite-sized dispersive media [1]. A generalized Hermitian eigenvalue problem for electromagnetic fields coupled to Lorentz oscillators is developed. Eigenmodes in a 2D waveguide are obtained using computational electromagnetics methods. Hong-Ou-Mandel effect is reproduced as a validation. The cat state is then initialized to investigate entanglement.

In the second approach, we leverage electromagnetic scattering theory to study momentum entanglement of photon pair. A computational method is proposed to track the evolution of two-photon wave function from spontaneous parametric down-conversion (SPDC) after the photon pair hit the scatterer. Schmidt decomposition is performed to evaluate the degree of entanglement. Enhancement of entanglement after scattering is observed.

II. Canonical Quantization Approach

A. Formulation

When dispersive media are present in the system, auxiliary fields should be included. They act as Lorentz oscillators placed in dispersive media interacting with electromagnetic fields. Hence the conjugate pairs are defined accordingly [1]

$$\mathbf{q} = [\mathbf{A}, \Pi_\Phi, \Pi_P]^T, \quad \mathbf{p} = [\mathbf{\Pi}_{AP}, \Phi, -\mathbf{P}]^T \tag{1}$$

where $\mathbf{A}$, Φ and $\mathbf{P}$ are the vector potential, scalar potential and polarization current, and $\mathbf{\Pi}_{AP}$, Π_Φ and Π_P are their conjugate momenta, respectively. The reordering of conjugate pair eliminates the cross-coupling term, so that the Hamiltonian can be recast as

$$H = \frac{1}{2}\int_V \begin{bmatrix}\mathbf{q}\\ \mathbf{p}\end{bmatrix}^T \cdot \begin{bmatrix}\overline{\mathbf{K}} & 0\\ 0 & \overline{\mathbf{M}}\end{bmatrix} \cdot \begin{bmatrix}\mathbf{q}\\ \mathbf{p}\end{bmatrix} \tag{2}$$

where

$$\overline{\mathbf{K}} = \begin{bmatrix} \nabla\times\frac{1}{\mu_0}\nabla\times - \epsilon_0\nabla\frac{1}{\chi_0}\nabla\cdot\epsilon_0 & 0 & 0\\ 0 & -\frac{1}{\chi_0} & 0\\ 0 & 0 & \frac{\beta(\mathbf{r})}{\epsilon_0}\end{bmatrix} \tag{3}$$

$$\overline{\mathbf{M}} = \begin{bmatrix} \frac{1}{\epsilon_0} & 0 & \frac{1}{\epsilon_0}\\ 0 & \nabla\cdot\epsilon_0\nabla & -\nabla\cdot\\ \frac{1}{\epsilon_0} & \nabla & \frac{f(\mathbf{r})+1}{\epsilon_0}\end{bmatrix} \tag{4}$$

and $f(\mathbf{r}) = \omega_0^2(\mathbf{r})/\omega_p^2(\mathbf{r})$ and $\beta(\mathbf{r}) = 1/\omega_p^2(\mathbf{r})$, where $\omega_p^2(\mathbf{r})$ and $\omega_0^2(\mathbf{r})$ are the plasma and resonant frequencies of the local Lorentz oscillator. The elimination of cross coupling between $\mathbf{q}$ and $\mathbf{p}$ allows one to derive the decoupled equations of motion (EoMs). In frequency domain, the EoM for $\mathbf{q}$ is

$$\omega^2\,\overline{\mathbf{M}}^{-1}\cdot\mathbf{q} = \overline{\mathbf{K}}\cdot\mathbf{q} \tag{5}$$

which is an explicit eigenvalue problem. Besides, this reduction saves the number of unknowns by half.

B. Numerical Examples

Previous studies have shown numerical results in 1D, including Hong-Ou-Mandel experiment and nonlocal dispersion cancellation effect [1]. In this work, we consider a 2D geometry, as illustrated in Fig. 1. The TM modes in a 2D waveguide are calculated numerically. The waveguide is PEC-backed on one side, and is index-guided on the other side. A dielectric bump (ϵ_2) may be inserted as an imperfect beam splitter.

We first consider a Hong-Ou-Mandel experiment with a single photon initialized on each side of the beam splitter. Each photon is considered riding on a wavepacket and propagating toward the beam splitter. Second-order correlation function $g_2(\tau)$ will be calculated to observe quantum interference. We then consider each photon riding on a cat state $|\psi\rangle = |\alpha\rangle + |-\alpha\rangle$, where $|\alpha\rangle$ is a coherent state which, defined in the Fock state basis, has the form $|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$.

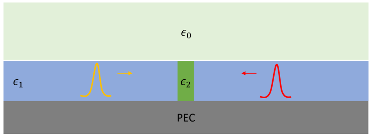

Fig. 1. Problem geometry of the 2D simulation to investigate quantum interference and entanglement.

III. Scattering of Momentum-Entangled Photon Pair

A. Formulation

We then consider the momentum entanglement as the photon pair propagate through an arbitrary scatterer. Assuming a strong, collimated pump beam propagating in z direction and degenerate down-conversion, the quantum state emerging from the nonlinear crystal due to SPDC is [2], [3]

$$|\Psi\rangle = \int d\mathbf{q}_1 d\mathbf{q}_2 \Phi(\mathbf{q}_1, \mathbf{q}_2)\hat{a}^\dagger(\mathbf{q}_1)\hat{a}^\dagger(\mathbf{q}_2)|0_1, 0_2\rangle. \tag{6}$$

The biphoton wave function in momentum space has the form of

$$\Phi(\mathbf{q}_1, \mathbf{q}_2) = \mathcal{N}\mathrm{sinc}\left(b^2|\mathbf{q}_1 - \mathbf{q}_2|^2\right) e^{-|\mathbf{q}_1+\mathbf{q}_2|^2/\sigma^2} \tag{7}$$

where b describes the phase matching condition and is related to the momentum of the pump photon through $b^2 = \frac{L}{4k_p}$. The transverse momentum of photon i is denoted by $\mathbf{q}_i = k_{x,i}\hat{\mathbf{k}}_{x,i} + k_{y,i}\hat{\mathbf{k}}_{y,i}$, and σ is the beam width in momentum space. $\mathcal{N}$ is the normalization constant. Note that $\Phi(\mathbf{q}_1, \mathbf{q}_2)$ should be interpreted as the probability amplitude.

Now we let the photon pair hit a scatterer, and use the two-photon wave function in (7) as the incident wave, as illustrated in Fig. 2(a). Considering far-field limit, the scattered biphoton wave function can be written as [4]

$$\Phi_s(\hat{\mathbf{k}}_1, \hat{\mathbf{k}}_2) = \int d\hat{\mathbf{k}}'_1 d\hat{\mathbf{k}}'_2 \langle \mathbf{k}_1|T|\mathbf{k}'_1\rangle\langle \mathbf{k}_2|T'|\mathbf{k}'_2\rangle\Phi_i(\hat{\mathbf{k}}'_1, \hat{\mathbf{k}}'_2). \tag{8}$$

The incident two-photon wave function $\Phi_i(\hat{\mathbf{k}}'_1, \hat{\mathbf{k}}'_2)$ is related to $\Phi(\mathbf{q}_1, \mathbf{q}_2)$ in (7) since the incident photons are assumbed to be monochromatic. The T-matrix in momentum space $\langle \mathbf{k}|T|\mathbf{k}'\rangle$ can be found by solving a scattering problem with an incident plane-wave in the $\mathbf{k}'$ direction, using integral equation. Observing the scattered field in the far-field in the $\mathbf{k}$ direction, after factoring out e^{ikr}/r, the remaining part is $\langle \mathbf{k}|T|\mathbf{k}'\rangle$. By evaluating the integral in (8) numerically, the scattered two-photon wave function can be found.

B. Enhancement of Entanglement by Scattering

As a proof of principle, we first consider a spherical dielectric scatterer with $n = 1.5$, where the T-matrix can be obtained analytically [4]. To characterize the degree of entanglement of the entangled photon pair, we perform the Schmidt decomposition on both the incident and scattered two-photon wave function

$$\Phi(\hat{\mathbf{k}}_1, \hat{\mathbf{k}}_2) = \sum_n \sqrt{\lambda_n}\phi_1(\hat{\mathbf{k}}_1)\phi_2(\hat{\mathbf{k}}_2) \tag{9}$$

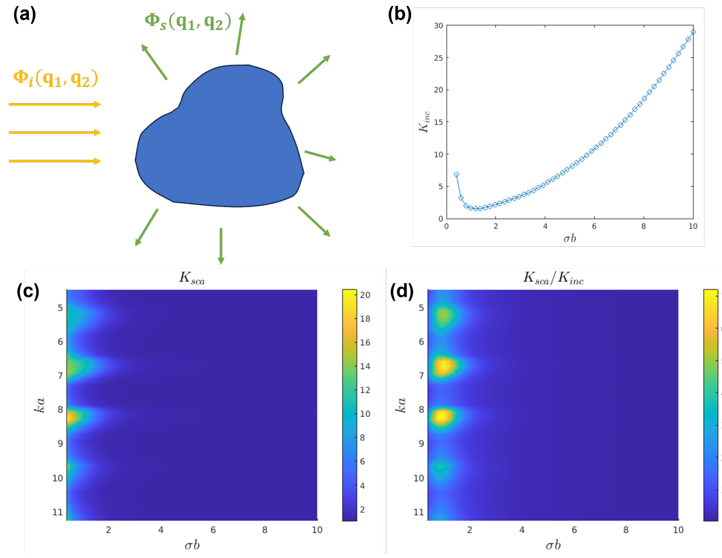

Fig. 2. (a) Problem setup of an entangled photon pair hitting a scatterer. (b) Schmidt number of the incident state as a function of σb. (c) Schmidt number of the scattered state with spherical scatterers of different sizes. (d) Enhancement of Schmidt number after scattering.

where $\sum_n \lambda_n = 1$. When there are morn than one nonzero λ_n, Schmidt number $K = (\sum_n \lambda_n^2)^{-1}$ is a general criterion to evaluate the degree of entanglement. Larger K indicates higher non-separability, i.e., higher degree of entanglement.

The Schmidt number of the scattered two-photon wave function is shown in Fig. 2(c), for different σb products and radii of scatterer. Dividing it by the incident Schmidt number (Fig. 2(b)), the enhancement of entanglement can be calculated. As shown in Fig. 2(d), a maximum of seven-fold enhancement is observed. For a general scatterer, integral equation will be used to obtain the T-matrix numerically and the numerical results will be presented.

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References

[1] D.-Y. Na, J. Zhu, and W. C. Chew, "Diagonalization of the hamiltonian for finite-sized dispersive media: Canonical quantization with numerical mode decomposition," *Physical Review A*, vol. 103, no. 6, p. 063707, 2021.

[2] J. Schneeloch and J. C. Howell, "Introduction to the transverse spatial correlations in spontaneous parametric down-conversion through the biphoton birth zone," *Journal of Optics*, vol. 18, no. 5, p. 053501, 2016.

[3] C. Law and J. Eberly, "Analysis and interpretation of high transverse entanglement in optical parametric down conversion," *Physical review letters*, vol. 92, no. 12, p. 127903, 2004.

[4] J. C. Schotland, A. Cazé, and T. B. Norris, "Scattering of entangled two-photon states," *Optics Letters*, vol. 41, no. 3, pp. 444–447, 2016.