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Article

Topological Edge Spectrum Along Curved Interfaces

Alexis Drouot and Xiaowen Zhu*

Department of Mathematics, University of Washington, C-138 Padelford Hall, Seattle, WA, 98105, USA
*Correspondence to be sent to: e-mail: xiaowenz@uw.edu
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We prove that if the boundary of a topological insulator divides the plane into two regions, each containing arbitrarily large balls, then it acts as a conductor. Conversely, we construct a counterexample to show that topological insulators that fit within strips do not need to admit conducting boundary modes. This constitutes a new setup where the bulk-edge correspondence is violated. Our proof relies on a seemingly paradoxical and underappreciated property of the bulk indices of topological insulators: they are global quantities that can be locally computed.

1 Introduction and Main Results

1.1 Introduction

Topological insulators are novel materials with striking properties. They are phases of matter insulating in their bulk (the Hamiltonian has a spectral gap), but turn into conductors when truncated to half-spaces (the spectral gap fills). The resulting edge conductance is equal to the difference of the bulk topological invariants across the cut, a principle known as the bulk-edge correspondence [3, 5, 12, 15, 16, 19, 20, 23, 27, 31]. Here, we consider truncations of topological insulators in regions more sophisticated than half-spaces (e.g., sectors or filled parabolas). We investigate how the shape of the resulting material affects the spectrum.

The main operator in our analysis is approximately equal to

$$H_e := \begin{cases} H_+ & \text{on} \quad \Omega, \\ H_- & \text{on} \quad \Omega^c = \mathbb{Z}^2 \setminus \Omega, \end{cases}$$

for two Hamiltonians $H_\pm$ on $\ell^2(\mathbb{Z}^2, \mathbb{C}^d)$ with distinct bulk invariants within a joint spectral gap $\mathcal{G}$ (potentially one of them representing the vacuum) and Ω a subset of $\mathbb{Z}^2$. We refer to §1.2 for precise definitions and assumptions. We ask for geometric conditions on Ω that guarantee that H_e has spectrum filling $\mathcal{G}$.

Our main result, Theorem 1, asserts that if both Ω and Ω^c contain arbitrarily large balls then H_e has spectrum filling $\mathcal{G}$ (referred to as edge spectrum). Hence, H_e behaves like a conductor near $\partial\Omega$. Examples of such domains Ω include half-planes, sectors, regions enclosed by hyperbolas, and so on. They exclude strips or half-strips; see Figure 1. In this last case, we actually show that there exist examples of distinct topological insulators $H_\pm$ such that H_e remains an insulator (Theorem 3). Therefore, topological materials fitting in strips can violate the bulk-edge correspondence: boundaries or interfaces are not systematic conductors. This violation was suggested in a question of Graf in an online talk by Thiang [21].

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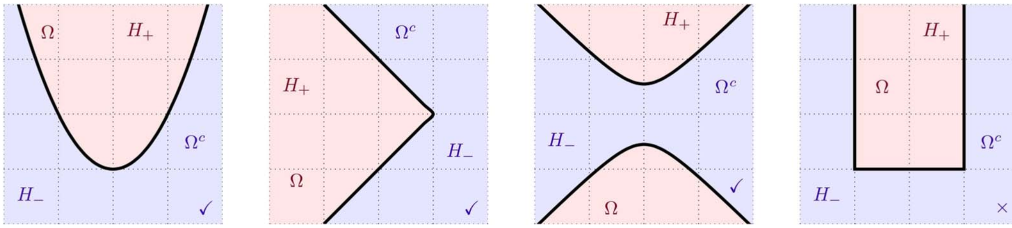

Fig. 1. The first three truncations systematically give rise to the edge spectrum. In the last case, we construct Haldane-type topological insulators H_-, H_+ such that H_e is also an insulator.

Our experience of the world is, by nature, an approximation of reality. Experiment samples (here, Ω) are always finite, and spectral measurements are valid only up to an uncertainty $\delta > 0$. Hence, in practice Ω never contains arbitrarily large balls—but neither can experiments assess with full certainty that spectral gaps completely fill: they can only measure the emergence of a δ-dense set of the spectrum. Theorem 2 is a quantitative formulation of Theorem 1 that relates to these observations. It predicts that there exist constants α, R_0 such that for all $R \geq R_0$, the following holds. Assume that both Ω, Ω^c contains a ball of radius R. Then the spectrum of H_e within $\mathcal{G}$ is $\alpha \ln(R)/R$-dense for some $\alpha > 0$:

$$\forall \lambda_* \in \mathcal{G}, \quad \exists \lambda \in \Sigma(H_e), \quad |\lambda - \lambda_*| \leq \frac{\alpha \ln(R)}{R}.$$

This justifies why topological insulators truncated to sufficiently large balls appear to conduct along their boundaries in experiments.

1.2 Topological insulators and interface operators

We briefly review standard facts from condensed matter physics. Electronic propagation through a given material is described via a self-adjoint operator H on a Hilbert space—here $\ell^2(\mathbb{Z}^2, \mathbb{C}^d)$. The spectrum $\Sigma(H)$ of H characterizes the electronic nature of the material: H is a conductor at energy λ if and only if $\lambda \in \Sigma(H)$ (sometimes the definition of conductor requires $\lambda \in \Sigma_{ac}(H)$, which is an additional condition on the spectral type at λ; see §1.6 for further discussion) and an insulator otherwise.

In the rest of this paper, $\nu \in (0, 1]$ **is a fixed parameter.**

We work here with short-range Hamiltonians: operators on $\ell^2(\mathbb{Z}^2, \mathbb{C}^d)$ whose kernels satisfy

$$\forall \mathbf{x}, \mathbf{y} \in \mathbb{Z}^2, \qquad \left|H(\mathbf{x}, \mathbf{y})\right| \leq \nu^{-1} e^{-2\nu|\mathbf{x}-\mathbf{y}|}, \qquad |\mathbf{x} - \mathbf{y}| := |x_1 - y_1| + |x_2 - y_2|. \tag{1.1}$$

Under (1.1), one can define the bulk conductance of H, denoted as $\sigma(H, \lambda)$ at an insulating energy λ. For $\lambda \notin \Sigma(H)$, let $P_\lambda(H) = \mathbb{1}_{(-\infty,\lambda)}(H)$ be the spectral projector below energy λ, and Λ_1 (respectively Λ_2) the indicator function of $\mathbb{N} \times \mathbb{Z}$ (respectively $\mathbb{Z} \times \mathbb{N}$). Then the operator $P_\lambda(H)[[P_\lambda(H), \Lambda_1], [P_\lambda(H), \Lambda_2]]$ is trace-class (see [16] and Remark 1 below) and

$$\sigma(H, \lambda) := -2\pi i \,\mathbf{Tr}\left(P_\lambda(H)[[P_\lambda(H), \Lambda_1], [P_\lambda(H), \Lambda_2]]\right)$$

is well defined. We comment that if $\mathcal{G}$ is a subinterval of $\Sigma(H)^c$ (referred to below as a spectral gap), then

$$\lambda, \lambda' \in \mathcal{G} \;\Rightarrow\; \sigma(H, \lambda) = \sigma(H, \lambda').$$

Therefore, there is no ambiguity in using the notation $\sigma(H, \mathcal{G})$ for $\sigma(H, \lambda)$, $\lambda \in \mathcal{G}$. It represents the bulk conductance for energies in $\mathcal{G}$ [4]. Under the gap condition $\lambda \notin \Sigma(H)$, $\sigma(H, \lambda)$ is an integer that measures topological aspects of the Hamiltonian H.

In this work, we ask under which conditions interfaces between two topologically distinct insulators (the bulk materials) carry currents. We make the following assumption on the bulk components:

Assumption 1. $H_\pm$ are two self-adjoint, short-range Hamiltonians on $\ell^2(\mathbb{Z}^2, \mathbb{C}^d)$, with a common spectral gap $\mathcal{G}$ (an interval contained in $\Sigma(H_+)^c \cap \Sigma(H_-)^c$) and distinct bulk conductances within $\mathcal{G}$:

$$\sigma(H_+, \mathcal{G}) \neq \sigma(H_-, \mathcal{G}).$$

Given a discrete domain $\Omega \in \mathbb{Z}^2$, we define its boundary following the idea of [24, Sec. 5.2] by

$$\partial\Omega := \{\mathbf{x} \in \Omega,\ B_1(\mathbf{x}) \not\subset \Omega\} \cup \{\mathbf{x} \in \Omega^c,\ B_1(\mathbf{x}) \not\subset \Omega^c\},$$

where $B_r(\boldsymbol{x})$ denotes the ℓ^1-ball of radius r centered at $\boldsymbol{x}$ in $\mathbb{Z}^2$. This definition is more commonly used for sets in $\mathbb{Z}^2$, in comparison to the boundaries defined for sets in $\mathbb{R}^2$. We will denote the distance from $\boldsymbol{x}$ to $\partial\Omega$ by $d(\mathbf{x}, \partial\Omega)$. We make the following assumptions on the interface operator:

Assumption 2. H_e is a self-adjoint, short-range Hamiltonian on $\ell^2(\mathbb{Z}^2, \mathbb{C}^d)$ satisfying the kernel condition:

$$\forall \mathbf{x}, \mathbf{y} \in \mathbb{Z}^2, \qquad \left|E(\mathbf{x}, \mathbf{y})\right| \le \nu^{-1} e^{-2\nu d(\mathbf{x}, \partial\Omega)}, \qquad E := H_e - \mathbb{1}_\Omega H_+ \mathbb{1}_\Omega - \mathbb{1}_{\Omega^c} H_- \mathbb{1}_{\Omega^c}. \tag{1.2}$$

The condition (1.2) means that H_e is equal to H_+ on Ω and H_- on Ω^c, up to corrections decaying exponentially away from $\partial\Omega$.

1.3 Main results

To formulate our main results, we will need the notion of filling radius for a subset Ω of $\mathbb{Z}^2$:

$$\mathbf{Fr}(\Omega) = \sup\{r :\ \exists\, \mathbf{x} \in \mathbb{Z}^2,\ B_r(\mathbf{x}) \cap \mathbb{Z}^2 \subset \Omega\}.$$

It measures the size of the largest ball contained in Ω: $\mathbf{Fr}(\Omega) \ge r$ if and only if Ω contains a ball of radius r.

Theorem 1. Let $H_\pm, H_e$ satisfying Assumptions 1 and 2. If $\mathbf{Fr}(\Omega) = \mathbf{Fr}(\Omega^c) = \infty$, then $\mathcal{G} \subset \Sigma(H_e)$.

This means that if the boundary of a topological insulator divides the plane into two regions of infinite filling radius, then it is a conductor. In the context of translation-invariant bulk operators or Landau Hamiltonian on the hyperbolic plane, this result was already proved using coarse geometric methods by Ludewig–Thiang [26, 27]. We provide here a spectral approach that will rely on a novel quantitative version of Theorem 1. Given $\delta > 0$, we say that a set $S \subset \mathcal{G}$ is δ-dense within $\mathcal{G}$ if either $|\mathcal{G}| < \delta$ or if

$$\forall \lambda_* \in \mathcal{G},\ \exists \lambda \in S \text{ s.t.} |\lambda_* - \lambda| \le \delta.$$

Theorem 2. There exist constants $\alpha_\nu > 0$ and $R_\nu > 0$, depending on ν only, such that the following holds. For $H_\pm, H_e$ satisfying Assumptions 1 and 2 and any $R \ge R_\nu$:

$$\mathbf{Fr}(\Omega),\ \mathbf{Fr}(\Omega^c) \ge R \quad \Rightarrow \quad \Sigma(H_e) \cap \mathcal{G} \text{ is } \alpha_\nu \frac{\ln R}{R}\text{-dense within } \mathcal{G}.$$

Theorem 2 has the following physical interpretation. Assume that H_e represents the truncation of a topological insulator H_+ in the ball $B_R(0)$, and that we have a measurement procedure that can infer if energy is within δ of $\Sigma(H_e)$. Theorem 2 asserts that if $\ln(R)/R \ll \delta$, then experiments measure that the spectral gap of H_+ closes when truncating it to $B_R(0)$. This imperfect conclusion (H_e actually has a discrete spectrum when truncated to $B_R(0)$) is due to the limitation of the measuring procedure.

Theorem 2 implies Theorem 1: if $\mathbf{Fr}(\Omega) = \mathbf{Fr}(\Omega^c) = \infty$, then $\mathbf{Fr}(\Omega)$ and $\mathbf{Fr}(\Omega^c)$ are larger than R for any R, so $\Sigma(H_e) \cap \mathcal{G}$ is δ-dense within $\mathcal{G}$ for any $\delta > 0$. This means that $\Sigma(H_e) \cap \mathcal{G}$ is actually dense within $\mathcal{G}$; since it is a closed subset of $\mathcal{G}$ we conclude that $\Sigma(H_e) \cap \mathcal{G} = \mathcal{G}$, equivalently $\mathcal{G} \subset \Sigma(H_e)$.

A natural question is whether the conclusion of Theorem 1 fails for unbounded sets Ω with finite filling radius (see Figure 2).

Theorem 3. Fix $L > 0$ and $\Omega \subset [-L, L] \times \mathbb{Z}$. There exists operators $H_\pm$ satisfying Assumption 1 for a gap $\mathcal{G}$ containing 0; and H_e satisfying Assumption 2 such that $0 \notin \Sigma(H_e)$.

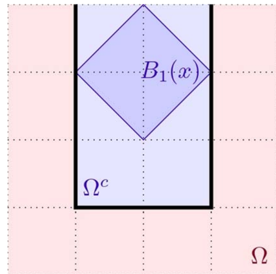

Fig. 2. A region Ω that does not satisfy the assumption of Theorem 1: the largest ball that fits in Ω^c has radius 1, so $\mathrm{Fr}(\Omega^c) = 1 < \infty$. There actually exist operators $H_\pm, H_e$ satisfying Assumptions 1, 2 such that $\mathcal{G} \not\subset \Sigma(H_e)$.

In other words, a topological insulator fitting in a strip does not have to be a conductor. This constitutes a new setup with a violation of the bulk-edge correspondence: an interface between two distinct topological phases does not have to support edge states, for instance, if the interface is the boundary of a half-strip. This adds up to a number of other violations; see, for example, [11, 22, 25, 32].

1.4 Sketch of proof

We explain here the main ideas leading to Theorem 1. Strictly speaking, the paper will focus on quantitative forms of these ideas to obtain Theorem 2, which (as explained above) implies Theorem 1.

Let H be a self-adjoint, short-range Hamiltonian on $\ell^2(\mathbb{Z}^2, \mathbb{C}^d)$ and $\lambda \notin \Sigma(H)$. Our main argument is the observation that $\sigma(H, \lambda)$ can be computed from the sole knowledge of H within a ball centered at any point $\mathbf{n}$ of $\mathbb{Z}^2$, as long as its radius R is sufficiently large (depending on ν, but independent of $\mathbf{n}$). While this may seem paradoxical, the global condition $\lambda \notin \Sigma(H)$ actually ensures that the result of the computation is independent of $\mathbf{n}$.

We proceed now by contradiction. Let $H_\pm, H_e$ satisfy the assumptions of Theorem 1, and assume that $\lambda \in \mathcal{G} \setminus \Sigma(H_e)$. We can then define $\sigma(H_e, \lambda)$, and, thanks to the above observations, compute it from sole knowledge of H_e on balls of the form $B_R(\mathbf{n})$.

Pick now $\mathbf{n}$ so that $B_R(\mathbf{n})$ lies deep in Ω, that is, in Ω and far from $\partial\Omega$; this is possible because under $\mathrm{Fr}(\Omega) = \infty$, Ω contains arbitrarily large balls. From Assumption 2, H_e is roughly H_+ there and we deduce $\sigma(H_e, \lambda) = \sigma(H_+, \lambda)$. Likewise, pick $\mathbf{n}$ so that $B_R(\mathbf{n})$ lies deep in Ω^c and deduce $\sigma(H_e, \lambda) = \sigma(H_-, \lambda)$. This contradicts the assumption $\sigma(H_+, \lambda) \neq \sigma(H_-, \lambda)$. Therefore, $\lambda \in \Sigma(H_e)$.

We comment that this proof also applies to materials made off three or more topological insulators, with at least two of them with different bulk invariant filling regions with infinite filling radius.

1.5 Relation to existing results

The question of how the shape of the truncation affects the edge spectrum has been considered before. The bulk-edge correspondence predicts the emergence of edge spectrum for half-space truncations: it gives the resulting interface conductance as a difference of Chern numbers [3, 5, 12, 15, 16, 19, 23, 27].

In [18], the authors focus on truncated quantum Hall Hamiltonians and derive a global analytic condition on Ω for the emergence of the edge spectrum. They verify that this condition holds for local perturbations of sectors. It is not evident how their condition relates to ours.

More recently the techniques have shifted to coarse geometry and K-theory. In [28, 33] the authors prove that magnetic Hamiltonians truncated to corners or sectors, and their local perturbations, have edge spectrum. The furthest-reaching works are due to Ludewig-Thiang [26, 27]. For translation-invariant bulk operators and Landau Hamiltonian on the hyperbolic plane, they produce a coarse-geometric condition equivalent to ours: if $d(\cdot, \partial\Omega)$ is unbounded on both Ω and Ω^c, then edge spectrum emerges between topologically distinct insulating phases.

Our proof uses spectral theory instead of coarse geometry. It has the benefit of being short and intuitive, and coming up with a quantitative form of the result. This version explains in what sense experimentalists observe edge spectrum in bounded samples. To the best of our knowledge, this is the first time such a result has been provided.

The shape of edge states matters in technological applications: they are the vectors of conduction along the edge. When the boundary is weakly curved – which corresponds to the adiabatic or semiclassical regime – several works constructed edge states as wave packets [6, 7, 13, 29]. The assumptions in the present work are significantly weaker (we only assume the existence of a spectral gap) but the result is also significantly weaker: we only prove the existence of an edge spectrum.

1.6 Open problems

An open problem is whether the bulk-edge correspondence generalizes to truncations to domains Ω satisfying $\mathrm{Fr}(\Omega) = \mathrm{Fr}(\Omega^c) = \infty$. We believe that this condition will need to be strengthened to something more quantitative for the bulk-edge correspondence to hold. There are already results that use the K-theoretic and coarse geometry framework [27]; it would be nice to provide a spectral proof.

Another open problem is the spectral type of the edge spectrum, which is widely expected to be absolutely continuous. In [9], the authors show that the edge spectrum is absolutely continuous when the edge is straight. They rely on (a) the bulk-edge correspondence for straight edges with a new form of edge conductance [17] and (b) a general result on the structure of unitary operators [2]. A follow-up to BEC for curved boundaries is to show that the edge spectrum for curved boundaries is absolutely continuous as well, extending the [9] result.

It has been shown that the bulk-edge correspondence holds when the gap condition ($\mathcal{G} \cap \Sigma(H_\pm)$ is empty) is replaced by a mobility gap condition ($H_\pm$ exhibits dynamical localization within $\mathcal{G}$); see [16]. At this point, we do not know if relaxing Assumption 1 to a mobility gap gives rise to an edge spectrum.

1.7 Notations

We will use the following notations:

- $\mathbf{x} = (x_1, x_2)$ denotes an element of $\mathbb{Z}^2$.
- $|\mathbf{x}| = |x_1| + |x_2|$ denotes the ℓ^1-norm on $\mathbb{Z}^2$.
- $B_r(\mathbf{x}) := \{\mathbf{y} \in \mathbb{Z}^2 : |\mathbf{y} - \mathbf{x}| \le r\}$ is the ball of radius $r \in \mathbb{R}^+$ centered at $\mathbf{x} \in \mathbb{Z}^2$.
- If $A \subset \mathbb{R}^2$ and $x \in \mathbb{R}^2$, $d(x, A)$ denotes the distance from x to A.
- Given an operator $H : \ell^2(\mathbb{Z}^2) \to \ell^2(\mathbb{Z}^2)$, we let $H(\mathbf{x}, \mathbf{y}) = \langle H\delta_{\mathbf{x}}, \delta_{\mathbf{y}}\rangle$ be the kernel of H; $\Sigma(H)$ denote the spectrum of H; $P_\lambda(H) := \mathbb{1}_{(-\infty,\lambda)}(H)$ denote the spectral projection below energy λ.
- In the whole paper, C_ν denotes a constant that can vary from line to line but depends only on the parameter ν from §1.2.
- Given $u \in \ell^2(\mathbb{Z}^2)$, denote its Fourier transform over $\mathbb{Z}^2$ by $\hat{u}(\xi) := \sum\limits_{n\in\mathbb{Z}^2} u_n e^{-i\langle n,\xi\rangle}$.

2 Proof of Theorem 2

We proved Theorem 1 using Theorem 2 in §1.3. In this section, we prove Theorem 2 by assuming the key Proposition 2.1 below. This proposition essentially asserts that two insulators that coincide on a large enough ball (with a radius depending on ν but not on the center of the ball) must have the same bulk conductance.

Assumption 3. *H is a self-adjoint, short-range operator on $\ell^2(\mathbb{Z}^2; \mathbb{C}^d)$, such that for some $\lambda \in \mathbb{R}$ and $\delta \in (0, 1)$,*

$$(\lambda - \delta, \lambda + \delta) \cap \Sigma(H) = \emptyset.$$

Proposition 2.1. *There exists a constant $C_\nu \ge 1$ such that the following holds: For any $\epsilon > 0$, $r > 0$, $\mathbf{n} \in \mathbb{Z}^2$, and any H_1, H_2 satisfy Assumption 3 with*

$$|(H_1 - H_2)(\mathbf{x}, \mathbf{y})| \le \epsilon, \quad \mathbf{x}, \mathbf{y} \in B_{4r}(\mathbf{n}),$$

we have:

$$|\sigma(H_1, \lambda) - \sigma(H_2, \lambda)| \le \frac{C_\nu}{\delta^{12}} \left(e^{-\frac{\delta r}{2C_\nu}} + \epsilon^{1/2}\right).$$

Proof of Theorem 2 assuming Proposition 2.1. 1. We recall that ν is a fixed parameter. In this first step, we set the values of the constants α_ν and R_ν. (The constants chosen below are not be optimal but are good enough for the proof. Nevertheless, the order of δ-dense in Theorem 2, that is, $\mathcal{O}\left(\frac{\ln R}{R}\right)$ is most likely nearly sharp under the current form of Proposition 2.1.)

Let C_ν be given by Proposition 2.1; we set $\alpha_\nu = 600C_\nu$. Meanwhile, note that the quantity $\nu^{-1/2}R^{12}e^{-\nu R/2}$ goes to 0 as R goes to infinity. Therefore, there exists $R_\nu \geq 4$ such that for all $R \geq R_\nu$,

$$\nu^{-1/2}R^{12}e^{-\nu R/2} < \frac{1}{2}. \tag{2.1}$$

Fix now $R \geq R_\nu$ (in particular, (2.1) holds); and define

$$\delta = \alpha_\nu \frac{\ln R}{3R} = 200C_\nu \frac{\ln R}{R}. \tag{2.2}$$

We will prove that $\mathcal{G} \cap \Sigma(H_e)$ is 3δ-dense within $\mathcal{G}$, that is, $\alpha_\nu \frac{\ln R}{R}$-dense within $\Sigma(H_e)$.

2. Let us assume for now that for some $\lambda_* \in \mathcal{G}$, we have

$$\begin{aligned} (\lambda_* - \delta, \lambda_* + \delta) \cap \Sigma(H_e) &= \emptyset, \qquad (2.3)\\ (\lambda_* - \delta, \lambda_* + \delta) \cap \Sigma(H_+) &= \emptyset,\\ (\lambda_* - \delta, \lambda_* + \delta) \cap \Sigma(H_-) &= \emptyset, \end{aligned}$$

and let us aim for a contradiction. Note that these statements imply that $H_e, H_\pm$ satisfy Assumption 3.

Since Ω has filling radius at least R, there exists $\mathbf{n} \in \mathbb{Z}^2$ such that $B_{8r}(\mathbf{n}) \subset \Omega$, $r = R/8$. We now look at $(H_e - H_+)(\mathbf{x}, \mathbf{y})$ for $\mathbf{x}, \mathbf{y}$ in $B_{4r}(\mathbf{n})$. Because $\mathbf{x}, \mathbf{y} \in \Omega$, we have:

$$(H_e - H_+)(\mathbf{x}, \mathbf{y}) = (H_e - \mathbb{1}_\Omega H_+ \mathbb{1}_\Omega - \mathbb{1}_{\Omega^c} H_- \mathbb{1}_{\Omega^c})(\mathbf{x}, \mathbf{y}) = E(\mathbf{x}, \mathbf{y}),$$

where E is the operator defined in Assumption 2. Moreover,

$$d(\mathbf{x}, \partial\Omega) \geq d(\mathbf{n}, \partial\Omega) - |\mathbf{x} - \mathbf{n}| \geq 8r - 4r = 4r,$$

because $B_{8r}(\mathbf{n}) = B_R(\mathbf{n}) \subset \Omega$. It follows from (1.2) that

$$\left|(H_e - H_+)(\mathbf{x}, \mathbf{y})\right| \leq \nu^{-1}e^{-8\nu r}, \qquad \mathbf{x}, \mathbf{y} \in B_{4r}(\mathbf{n}).$$

Proposition 2.1 then yields

$$\left|\sigma(H_e, \lambda_*) - \sigma(H_+, \lambda_*)\right| \leq \frac{C_\nu}{\delta^{12}} e^{-\frac{\delta r}{2C_\nu}} + \frac{C_\nu \nu^{-1/2}}{\delta^{12}} e^{-4\nu r}.$$

We recall that δ has the value (2.2). Therefore, since $C_\nu \geq 1$ and $R \geq R_\nu \geq 4$,

$$\begin{aligned} \frac{C_\nu}{\delta^{12}} e^{-\frac{\delta r}{2C_\nu}} &= \frac{C_\nu R^{12}}{(200C_\nu \ln R)^{12}} e^{-\frac{200C_\nu \ln R}{R} \cdot \frac{R}{16C_\nu}}\\ &\leq R^{12}R^{-25/2} = R^{-1/2} \leq \frac{1}{2}. \end{aligned}$$

Likewise, because R satisfies (2.1),

$$\frac{C_\nu \nu^{-1/2}}{\delta^{12}} e^{-4\nu r} = \frac{C_\nu \nu^{-1/2} R^{12}}{(200C_\nu \ln R)^{12}} e^{-\frac{\nu R}{2}} \leq \nu^{-1/2}R^{12}e^{-\frac{\nu R}{2}} < \frac{1}{2}.$$

Going back to (2.3), we conclude that

$$\left|\sigma(H_e, \lambda_*) - \sigma(H_+, \lambda_*)\right| < 1.$$

Since bulk conductances are integers (see [16, Proposition 3] and Remark 1 below), we conclude that $\sigma(H_e, \lambda_*) = \sigma(H_+, \lambda_*)$.

Similarly, we conclude that $\sigma(H_e, \lambda_*) = \sigma(H_-, \lambda_*)$. This cannot be true, since $\sigma(H_+, \lambda_*) \neq \sigma(H_-, \lambda_*)$. We conclude that for each $\lambda_* \in \mathcal{G}$, one of the statements among (2.3) must fail. In other words, for all $\lambda_* \in \mathcal{G}$, there exists some $\lambda \in \Sigma(H_e) \cup \Sigma(H_+) \cup \Sigma(H_-)$ such that $|\lambda - \lambda_*| \le \delta$.

3. It remains to show that $\Sigma(H_e) \cap \mathcal{G}$ is 3δ-dense within $\mathcal{G}$. Write $\mathcal{G} = (a, b)$ with $b - a > 3\delta$ (otherwise any subset is 3δ-dense by definition). Let $\lambda_* \in (a + \delta, b - \delta)$ and $\lambda \in \Sigma(H_e) \cup \Sigma(H_+) \cup \Sigma(H_-)$ such that $|\lambda - \lambda_*| \le \delta$. In particular, $\lambda \in (a, b) = \mathcal{G}$, which is a spectral gap of $H_\pm$, so $\lambda \in \Sigma(H_e)$. Let now $\lambda_* \in (a, \mu_*)$, $\mu_* = a + 2\delta$; since $b - a > 3\delta$, $\mu_* \in (a + \delta, b - \delta)$ and by the previous step there exists $\mu \in \Sigma(H_e)$ such that $|\mu - \mu_*| \le \delta$. In particular, $|\mu - \lambda_*| \le 3\delta$. A similar argument works for $\lambda_* \in (b - 2\delta, b)$. We conclude that $\mathcal{G} \cap \Sigma(H_e)$ is 3δ-dense within $\mathcal{G}$. ■

3 Proof of the Key Proposition

We prove Proposition 2.1 in this section.

3.1 On short-range Hamiltonians

Throughout the proofs below, we will use the following estimates, proved in Appendix A: For $a \in (0, 1]$, $R > 0$, we have

$$\sum_{s \in \mathbb{Z}} e^{-2a|s|} \le \frac{2}{a}, \quad \sum_{\mathbf{x} \in \mathbb{Z}^2} e^{-2a|\mathbf{x}|} \le \frac{4}{a^2}, \quad \text{and} \quad \sum_{|\mathbf{x}| \ge R, \mathbf{x} \in \mathbb{Z}^2} e^{-2a|\mathbf{x}|} \le \frac{8}{a^2} e^{-aR}. \tag{3.1}$$

We make here a few observations on the self-adjoint, short-range Hamiltonians H on $\ell^2(\mathbb{Z}^2, \mathbb{C}^d)$. First, they are bounded in terms of the (fixed) parameter $\nu \in (0, 1]$ quantifying the short-range condition (1.1). Specifically, an application of Schur's test gives:

$$\|H\| \le \frac{4}{\nu^3}. \tag{3.2}$$

We refer to Appendix A for the proof.

As in [1, 16], we introduce

$$S_\alpha := \sup_{\mathbf{x} \in \mathbb{Z}^2} \sum_{\mathbf{y} \in \mathbb{Z}^2} |H(\mathbf{x}, \mathbf{y})| \left(e^{\alpha|\mathbf{x}-\mathbf{y}|} - 1\right).$$

We note that if H is short range under the definition (1.1), then for any $\alpha \in (0, 2\nu)$, $S_\alpha < +\infty$. Also, for later use, if $\alpha \in (0, \nu]$:

$$\frac{S_\alpha}{\alpha} \le \frac{S_\nu}{\nu} \le \frac{16}{\nu^4}, \tag{3.3}$$

Again, see Appendix A for the proof.

We recall the Combes–Thomas inequality [10]:

Proposition 3.1. [1, Theorem 10.5] Let H be a selfadjoint, short-range operator on $\ell^2(\mathbb{Z}^2; \mathbb{C}^d)$. If $\alpha \in (0, 2\nu)$ and $z \in \mathbb{C}$ are such that $\Delta := d(z, \Sigma(H)) > S_\alpha$, then we have

$$|(H - z)^{-1}(\mathbf{x}, \mathbf{y})| \le \frac{1}{\Delta - S_\alpha} e^{-\alpha|\mathbf{x}-\mathbf{y}|}.$$

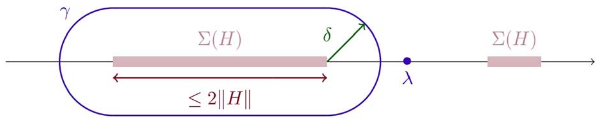

Fig. 3. The contour γ. We note that the spectrum of H is contained in $[-\|H\|, \|H\|]$, so it has a diameter of less than $2\|H\|$. Since $\delta \leq 1$, the two half-circles have a perimeter of less than 2π, and the length of γ is less than $4\|H\| + 2\pi$. Note that γ does not need to enclose λ.

3.2 Spectral projections

Recall that $P_\lambda(H) = \mathbb{1}_{(-\infty,\lambda)}(H)$ denotes the spectral projection below energy λ. An application of the Combes–Thomas inequality controls the kernel of spectral projections:

Lemma 3.1. There exists a constant C_ν, such that for any H, λ satisfying Assumption 3:

$$\left|P_\lambda(H)(\mathbf{x}, \mathbf{y})\right| \leq \frac{C_\nu}{\delta} e^{-\frac{\delta}{C_\nu}|\mathbf{x}-\mathbf{y}|}. \tag{3.4}$$

Proof. Set $\alpha = 2^{-5}\nu^4\delta$, so that $\alpha \leq \nu$ and $S_\alpha \leq \frac{16\alpha}{\nu^4} \leq \frac{\delta}{2}$, see (3.3). Let γ be a contour enclosing $\Sigma(H) \cap (-\infty, \lambda)$, at least δ-distant from $\Sigma(H)$. For $z \in \gamma$, we have:

$$\left|(H - z)^{-1}(\mathbf{x}, \mathbf{y})\right| \leq \frac{e^{-\alpha|\mathbf{x}-\mathbf{y}|}}{\Delta - S_\alpha} \leq \frac{e^{-\alpha|\mathbf{x}-\mathbf{y}|}}{\delta - \frac{\delta}{2}} \leq \frac{2}{\delta} e^{-\alpha|\mathbf{x}-\mathbf{y}|}. \tag{3.5}$$

Integrating this over γ, we have:

$$\left|P_\lambda(H)(\mathbf{x}, \mathbf{y})\right| = \left|\frac{1}{2\pi i} \oint_\gamma (H - z)^{-1}(\mathbf{x}, \mathbf{y}) dz\right| \leq \frac{|\gamma|}{\pi\delta} e^{-\alpha|\mathbf{x}-\mathbf{y}|}.$$

Note that we can always choose a tight loop γ rounded around $\Sigma(H) \cap (-\infty, \lambda)$, as shown in Figure 3, such that $|\gamma| \leq 2\pi + 4\|H\|$.

From the bound (3.2) on $\|H\|$, we deduce that

$$\frac{|\gamma|}{\pi} \leq 2 + \frac{16}{\pi\nu^3} \leq \frac{8}{\nu^3}. \tag{3.6}$$

We conclude that

$$\left|P_\lambda(H)(\mathbf{x}, \mathbf{y})\right| \leq \frac{8}{\nu^3\delta} e^{-\frac{\nu^4\delta}{32}|\mathbf{x}-\mathbf{y}|}.$$

This yields (3.4) (with for instance $C_\nu = 32\nu^{-4}$). ■

Remark 1. As a result, if H and λ satisfy Assumption 3, then the open bounded interval $(\lambda - \delta, \lambda + \delta)$ satisfies condition [16,(1.2)]; it follows that $\sigma(H, \lambda)$ is well defined and is an integer [16, Proposition 3].

Lemma 3.2. There exists a constant C_ν such that the following holds: let $r, \epsilon > 0$, and two triplets (H_1, λ, δ) and (H_2, λ, δ) satisfying Assumption 3, such that for $\mathbf{x}, \mathbf{y} \in B_{4r}(0)$,

$$|H_1(\mathbf{x}, \mathbf{y}) - H_2(\mathbf{x}, \mathbf{y})| \leq \epsilon. \tag{3.7}$$

Then for $(\mathbf{x}, \mathbf{y})$ in $B_{2r}(0)$,

$$|P_\lambda(H_1)(\mathbf{x}, \mathbf{y}) - P_\lambda(H_2)(\mathbf{x}, \mathbf{y})| \leq \frac{C_\nu}{\delta^6} \left(e^{-\frac{\delta r}{C_\nu}} + \epsilon\right). \tag{3.8}$$

Proof. Let α and γ be as in the proof of Lemma 3.1. We have:

$$\big(P_\lambda(H_1) - P_\lambda(H_2)\big)(\mathbf{x},\mathbf{y}) = \sum_{\mathbf{x}',\mathbf{y}'} \frac{1}{2\pi i} \oint_\gamma (H_2 - z)^{-1}(\mathbf{x},\mathbf{x}')\big(H_2(\mathbf{x}',\mathbf{y}') - H_1(\mathbf{x}',\mathbf{y}')\big)(H_1 - z)^{-1}(\mathbf{y}',\mathbf{y})dz.$$

From (3.5) and using the bound (3.6) for $|\gamma|$, we have

$$\begin{aligned}\big|\big(P_\lambda(H_1) - P_\lambda(H_2)\big)(\mathbf{x},\mathbf{y})\big| &\le \frac{2|\gamma|}{\pi\delta^2}\sum_{\mathbf{x}',\mathbf{y}'} e^{-\alpha|\mathbf{x}-\mathbf{x}'|-\alpha|\mathbf{y}-\mathbf{y}'|}\big|H_2(\mathbf{x}',\mathbf{y}') - H_1(\mathbf{x}',\mathbf{y}')\big| \\ &\le \frac{2^4}{\nu^3\delta^2}\sum_{\mathbf{x}',\mathbf{y}'} e^{-\alpha|\mathbf{x}-\mathbf{x}'|-\alpha|\mathbf{y}-\mathbf{y}'|}\big|H_2(\mathbf{x}',\mathbf{y}') - H_1(\mathbf{x}',\mathbf{y}')\big|.\end{aligned} \tag{3.9}$$

We split the RHS sum into two parts, depending on whether $\mathbf{x}', \mathbf{y}' \in B_{4r}(0)$ or not. When they do, we can use the bound (3.7) on the kernel of $H_1 - H_2$. It yields

$$\begin{aligned}&\sum_{\mathbf{x}',\mathbf{y}'\in B_{4r}(0)} e^{-\alpha|\mathbf{x}-\mathbf{x}'|-\alpha|\mathbf{y}-\mathbf{y}'|}\big|H_2(\mathbf{x}',\mathbf{y}') - H_1(\mathbf{x}',\mathbf{y}')\big| \\ \le\,& \epsilon \sum_{\mathbf{x}',\mathbf{y}'\in\mathbb{Z}^2} e^{-\alpha|\mathbf{x}-\mathbf{x}'|-\alpha|\mathbf{y}-\mathbf{y}'|} \\ \le\,& \epsilon \sum_{\mathbf{x}',\mathbf{y}'\in\mathbb{Z}^2} e^{-\alpha|\mathbf{x}'|-\alpha|\mathbf{y}'|} \le \epsilon\left(\frac{16}{\alpha^2}\right)^2,\end{aligned}$$

where in the last line we used (3.1).

When now restricting to $\mathbf{x}'$ or $\mathbf{y}' \in B_{4r}(0)^c$, we note that for $\mathbf{x},\mathbf{y} \in B_{2r}(0)$, either $|\mathbf{x}-\mathbf{x}'| \ge 2r$, or $|\mathbf{y}-\mathbf{y}'| \ge 2r$. Recall that by (1.1), $|H_i(\mathbf{x},\mathbf{y})| \le \nu^{-1}$ for any $\mathbf{x},\mathbf{y}$, $i = 1, 2$. Hence, we have

$$\begin{aligned}&\sum_{\mathbf{x}' \text{ or } \mathbf{y}'\in B_{4r}(0)^c} e^{-\alpha|\mathbf{x}-\mathbf{x}'|-\alpha|\mathbf{y}-\mathbf{y}'|}\big|H_2(\mathbf{x}',\mathbf{y}') - H_1(\mathbf{x}',\mathbf{y}')\big| \\ \le\,& \frac{2}{\nu}\sum_{\mathbf{x}' \text{ or } \mathbf{y}'\in B_{4r}(0)^c} e^{-\alpha|\mathbf{x}-\mathbf{x}'|-\alpha|\mathbf{y}-\mathbf{y}'|} \le \frac{4}{\nu}\sum_{\mathbf{x}'\in B_{2r}(0)^c,\mathbf{y}'\in\mathbb{Z}^2} e^{-\alpha|\mathbf{x}'|-\alpha|\mathbf{y}'|} \\ \le\,& \frac{4}{\nu}\left(\frac{32}{\alpha^2}e^{-\alpha r}\right)\left(\frac{16}{\alpha^2}\right) = \frac{8}{\nu}\left(\frac{2^8}{\alpha^4}\right)e^{-\alpha r}.\end{aligned}$$

In the last line, we applied (3.1). So, heading back to (3.9) and using the value $\alpha = 2^{-5}\nu^4\delta$ from the proof of Lemma 3.1, we obtain

$$\big|\big(P_\lambda(H_1) - P_\lambda(H_2)\big)(x,y)\big| \le \frac{2^7}{\nu^4\delta^2}\left(\frac{2^8}{\alpha^4}\right)(e^{-\alpha r} + \epsilon) = \frac{2^{35}}{\nu^{20}\delta^6}\left(e^{-\frac{\nu^4\delta}{32}r} + \epsilon\right).$$

This yields (3.8) (with $C_\nu = 2^{35}\nu^{-20}$; we made no attempts to minimize this constant). ■

3.3 Technical result

The key technical step in the proof of Proposition 2.1 is:

Proposition 3.2. Fix $\varepsilon > 0$, $C > 0$, $r > 0$, $\beta \in (0, 1]$. Let A_0, A_1, A_2 be three operators on $\ell^2(\mathbb{Z}^2, \mathbb{C}^d)$ with the following properties:

(i) For $j \in \{0, 1, 2\}$, $|A_j(\mathbf{x},\mathbf{y})| \le Ce^{-2\beta|x-y|}$.

(ii) There exists $k \in \{0, 1, 2\}$ such that if $\mathbf{x},\mathbf{y} \in B_{2r}(0)$,then $|A_k(\mathbf{x},\mathbf{y})| \le C\varepsilon$.

Then $B := A_0[A_1, \Lambda_1][A_2, \Lambda_2]$ is trace-class and

$$\left|\operatorname{Tr}(B)\right| \le C^3 \frac{2^{16}}{\beta^6}\left(e^{-\beta r} + \varepsilon^{1/2}\right).$$

Let us start with a simple result:

Lemma 3.3. Let A be a bounded operator on $\ell^2(\mathbb{Z}^2, \mathbb{C}^d)$ with $|A(\mathbf{x}, \mathbf{y})| \le e^{-2\beta|\mathbf{x}-\mathbf{y}|}$. Then,

$$\left|[A, \Lambda_1](\mathbf{x}, \mathbf{y})\right| \le e^{-2\beta|x_1|-2\beta|y_1|-2\beta|x_2-y_2|}. \tag{3.10}$$

If moreover $|A(\mathbf{x}, \mathbf{y})| \le \varepsilon$ for $(\mathbf{x}, \mathbf{y}) \in B_{2r}(0)$, then

$$\left|[A, \Lambda_1](\mathbf{x}, \mathbf{y})\right| \le \varepsilon^{1/2} e^{-\beta|x_1|-\beta|y_1|-\beta|x_2-y_2|}, \qquad \mathbf{x}, \mathbf{y} \in B_{2r}(0). \tag{3.11}$$

Proof. The kernel of $[A, \Lambda_1]$ is

$$[A, \Lambda_1](\mathbf{x}, \mathbf{y}) = A(\mathbf{x}, \mathbf{y})(\Lambda_1(\mathbf{y}) - \Lambda_1(\mathbf{x})).$$

We note that $|\Lambda_1(\mathbf{y}) - \Lambda_1(\mathbf{x})| = 0$ if x_1 and y_1 are both positive or both negative; and it is at most 1 otherwise, that is, if $x_1 y_1 \le 0$. Therefore, we have the bound

$$\left|[A, \Lambda_1](\mathbf{x}, \mathbf{y})\right| \le C e^{-2\beta|\mathbf{x}-\mathbf{y}|} \mathbb{1}_{x_1 y_1 \le 0}.$$

Whenever $x_1 y_1 \le 0$, we have

$$|\mathbf{x} - \mathbf{y}| = |x_1 - y_1| + |x_2 - y_2| = |x_1| + |y_1| + |x_2 - y_2|.$$

It follows that

$$\left|[A, \Lambda_1](\mathbf{x}, \mathbf{y})\right| \le e^{-2\beta|x_2-y_2|-2\beta|x_1|-2\beta|y_1|}.$$

This completes the proof of (3.10). To prove (3.11), we recall that $|\Lambda_1(\mathbf{y}) - \Lambda_1(\mathbf{x})| \le 1$; which implies that $\left|[A, \Lambda_1](\mathbf{x}, \mathbf{y})\right| \le \varepsilon$. It suffices to interpolate this bound with (3.10). ■

For the proof of Proposition 3.2, we will use the following inequality: for $\beta \in (0, 1]$, $\mathbf{x}, \boldsymbol{w} \in \mathbb{Z}^2$,

$$\sum_{\mathbf{y},\mathbf{z}\in\mathbb{Z}^2} e^{-2\beta|\mathbf{x}-\mathbf{y}|-2\beta|y_2-z_2|-2\beta|y_1|-2\beta|z_1|-2\beta|z_1-w_1|-2\beta|z_2|-2\beta|w_2|} \le \left(\frac{4}{\beta}\right)^4 e^{-\beta|\mathbf{x}|-\beta|\boldsymbol{w}|}. \tag{3.12}$$

We refer to Appendix A for a proof.

Proof of Proposition 3.2. 1. By a scaling argument, we can assume that $C = 1$. We first control the kernel of B:

$$|B(\mathbf{x}, \boldsymbol{w})| = \left|\sum_{y,z\in\mathbb{Z}^2} A_0(\mathbf{x}, \mathbf{y}) B_1(\mathbf{y}, \mathbf{z}) B_2(\mathbf{z}, \mathbf{x})\right|,$$

where $B_j = [A_j, \Lambda_j]$. We control the kernels of A_0, B_1, B_2 using assumption (i) and (3.10). It yields:

$$\begin{aligned}|B(\mathbf{x},\boldsymbol{w})| &= \left|\sum_{\mathbf{y},\mathbf{z}\in\mathbb{Z}^2} A_0(\mathbf{x},\mathbf{y})B_1(\mathbf{y},\mathbf{z})B_2(\mathbf{z},\boldsymbol{w})\right| \\ &\leq \sum_{\mathbf{y},\mathbf{z}\in\mathbb{Z}^2} e^{-2\beta|\mathbf{x}-\mathbf{y}|-2\beta|y_2-z_2|-2\beta|y_1|-2\beta|z_1|-2\beta|z_1-w_1|-2\beta|z_2|-2\beta|w_2|} \\ &\leq \left(\frac{4}{\beta}\right)^4 e^{-\beta|\mathbf{x}|-\beta|\boldsymbol{w}|}.\end{aligned}$$

Thus, $|B(\mathbf{x},\boldsymbol{w})|$ decays exponentially, hence B is trace-class; moreover,

$$\begin{aligned}\mathbf{Tr}(B) \leq \sum_{\mathbf{x}\in\mathbb{Z}^2}|B(\mathbf{x},\mathbf{x})| &\leq \sum_{\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbb{Z}^2} e^{-2\beta|\mathbf{x}-\mathbf{y}|-2\beta|y_2-z_2|-2\beta|y_1|-2\beta|z_1|-2\beta|z_1-x_1|-2\beta|z_2|-2\beta|x_2|} \\ &:= \sum_{\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbb{Z}^2} f(2\beta,\mathbf{x},\mathbf{y},\mathbf{z}).\end{aligned} \tag{3.13}$$

2. We now split the sum in (3.13) into two pieces: $|\mathbf{x}| \geq r$ and $|\mathbf{x}| \leq r$. Thanks to (3.12) and (3.1), we have

$$\sum_{|\mathbf{x}|\geq r} f(2\beta,\mathbf{x},\mathbf{y},\mathbf{z}) \leq \left(\frac{4}{\beta}\right)^4 \sum_{|\mathbf{x}|\geq r} e^{-2\beta|\mathbf{x}|} \leq \left(\frac{4}{\beta}\right)^4 \frac{8}{\beta^2}e^{-\beta r} = \frac{2^{11}}{\beta^6}e^{-\beta r}.$$

We focus below on $|\mathbf{x}| \leq r$.

3. If $k = 0$ in (ii), then we split the sum in (3.13) according to $|\mathbf{y}| \geq 2r$ and $|\mathbf{y}| \leq 2r$. In the former case, $|\mathbf{x}-\mathbf{y}| \geq r$. Therefore, when $|\mathbf{x}| \leq r,\ |\mathbf{y}| \geq 2r$, we deduce that

$$\begin{aligned}\left|A_0(\mathbf{x},\mathbf{y})\right| &\leq e^{-2\beta|\mathbf{x}-\mathbf{y}|} \leq e^{-\beta r-\beta|\mathbf{x}-\mathbf{y}|}, \\ \left|A_0(\mathbf{x},\mathbf{y})B_1(\mathbf{y},\mathbf{z})B_2(\mathbf{z},\mathbf{x})\right| &\leq e^{-\beta r} f(\beta,\mathbf{x},\mathbf{y},\mathbf{z}).\end{aligned} \tag{3.14}$$

If now $|\mathbf{y}| \leq 2r$ (and $|\mathbf{x}| \leq r \leq 2r$), then we can use (ii). Interpolating with (i) gives, for $|\mathbf{x}| \leq r,\ |\mathbf{y}| \leq 2r$,

$$\begin{aligned}\left|A_0(\mathbf{x},\mathbf{y})\right| &\leq \varepsilon^{1/2} e^{-\beta|\mathbf{x}-\mathbf{y}|}, \\ \left|A_0(\mathbf{x},\mathbf{y})B_1(\mathbf{y},\mathbf{z})B_2(\mathbf{z},\mathbf{x})\right| &\leq \varepsilon^{1/2} f(\beta,\mathbf{x},\mathbf{y},\mathbf{z}).\end{aligned} \tag{3.15}$$

Summing the bounds (3.14) and (3.15) produces:

$$\begin{aligned}\sum_{\substack{|\mathbf{x}|\leq r,\\ \mathbf{y},\mathbf{z}\in\mathbb{Z}^2}} \left|A_0(\mathbf{x},\mathbf{y})B_1(\mathbf{y},\mathbf{z})B_2(\mathbf{z},\mathbf{x})\right| &\leq e^{-\beta r} \sum_{\substack{|\mathbf{x}|\leq r,\\ |\mathbf{y}|\geq 2r,\mathbf{z}\in\mathbb{Z}^2}} f(\beta,\mathbf{x},\mathbf{y},\mathbf{z}) + \varepsilon^{1/2} \sum_{\substack{|\mathbf{x}|\leq r,\\ |\mathbf{y}|\leq 2r,\mathbf{z}\in\mathbb{Z}^2}} f(\beta,\mathbf{x},\mathbf{y},\mathbf{z}) \\ &\leq \left(e^{-\beta r}+\varepsilon^{1/2}\right) \sum_{\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbb{Z}^2} f(\beta,\mathbf{x},\mathbf{y},\mathbf{z}) \\ &\leq \frac{2^{16}}{\beta^6}\left(e^{-\beta r}+\varepsilon^{1/2}\right),\end{aligned} \tag{3.16}$$

where we used (3.12) and (3.1) to get

$$\sum_{\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbb{Z}^2} f(\beta,\mathbf{x},\mathbf{y},\mathbf{z}) \leq \left(\frac{8}{\beta}\right)^4 \sum_{\mathbf{x}\in\mathbb{Z}^2} e^{-\beta|\mathbf{x}|} \leq \left(\frac{8}{\beta}\right)^4 \frac{16}{\beta^2} = \frac{2^{16}}{\beta^6}.$$

4. We now work on $k = 1$. We split the sum into two cases: $\mathbf{y}, \mathbf{z} \in B_{2r}(0)$, and $\mathbf{y}$ or $\mathbf{z}$ outside $B_{2r}(0)$. In the latter case, either $|\mathbf{x}-\mathbf{y}| \geq r$ or $|\mathbf{z}-\mathbf{x}| \geq r$. So either

$$\left|A_0(\mathbf{x},\mathbf{y})\right| \leq e^{-\beta r} e^{-\beta|\mathbf{x}-\mathbf{y}|} \text{ or } \left|B_2(\mathbf{z},\mathbf{x})\right| \leq e^{-\beta r} e^{-\beta|\mathbf{z}-\mathbf{x}|}.$$

In either case, we recover the bound (3.14). In the first case, we use (3.11) to recover (3.15). Since both (3.14) and (3.15) lead to (3.16), we obtain the desired bound.

5. The case $k = 2$ follows the same path as $k = 0$. This completes the proof. ■

3.4 Comparison of bulk conductances

We will use the following result [16, Lemma 7(ii)], which essentially states that the bulk conductance is independent of Λ_1 and Λ_2:

Proposition 3.3. Let H be a short-range operator on $\ell^2(\mathbb{Z}^2, \mathbb{C}^d)$ and $\lambda \notin \Sigma(H)$. For any $\mathbf{n}$,

$$\sigma(H,\lambda) = -2\pi i \,\mathbf{Tr}\left(P_\lambda(H)[[P_\lambda(H), \Lambda_1(\cdot - n_1)], [P_\lambda(H), \Lambda_2(\cdot - n_2)]]\right).$$

We are now ready to prove Proposition 2.1.

Proof of Proposition 2.1. 1. For simplicity, use the notation $P_j = P_\lambda(H_j)$. Let T be the translation by $\mathbf{n}$: $Tu(\cdot) = u(\cdot - \mathbf{n})$. We have $\Lambda_j(\cdot - n_j) = T\Lambda_j(\cdot)T^*$ and $T^*P_\lambda(H)T = P_\lambda(T^*HT)$. Using these and Proposition 3.3, as well as the cyclicity of the trace, we obtain

$$\begin{aligned}
\sigma(H_j,\lambda) &= -2\pi i \,\mathbf{Tr}\left(P_j[[P_j, T\Lambda_1 T^*], [P_j, T\Lambda_2 T^*]]\right) \\
&= -2\pi i \,\mathbf{Tr}\left(P_j[T[T^*P_jT, \Lambda_1]T^*, T[T^*P_jT, \Lambda_2]T^*]\right) \\
&= -2\pi i \,\mathbf{Tr}\left(P_jT[[T^*P_jT, \Lambda_1], [T^*P_jT, \Lambda_2]]T^*\right) \\
&= -2\pi i \,\mathbf{Tr}\left(T^*P_jT[[T^*P_jT, \Lambda_1], [T^*P_jT, \Lambda_2]]\right) = \sigma(T^*H_jT, \lambda).
\end{aligned}$$

Therefore, by replacing H_j by T^*H_jT, we can simply assume that $\mathbf{n} = 0$.

2. Now we write

$$\sigma(H_1,\lambda) - \sigma(H_2,\lambda) = \mathcal{T}(P_1 - P_2, P_1, P_1) + \mathcal{T}(P_2, P_1 - P_2, P_1) + \mathcal{T}(P_2, P_2, P_1 - P_2), \tag{3.17}$$

where $\mathcal{T}(A_0, A_1, A_2)$ is the trilinear form

$$\mathcal{T}(A_0, A_1, A_2) = -2\pi i \,\mathbf{Tr}\left(A_0[A_1, \Lambda_1][A_2, \Lambda_2]\right) + 2\pi i \,\mathbf{Tr}\left(A_0[A_2, \Lambda_2][A_1, \Lambda_1]\right).$$

From Lemmas 3.1 and 3.2, we have (for a constant C_ν depending on ν only):

$$\begin{aligned}
\left|P_j(\mathbf{x},\mathbf{y})\right| &\leq \frac{C_\nu}{\delta} e^{-\delta|\mathbf{x}-\mathbf{y}|/C_\nu}, \qquad j = 1, 2; \\
\left|P_1(\mathbf{x},\mathbf{y}) - P_2(\mathbf{x},\mathbf{y})\right| &\leq \frac{C_\nu}{\delta^6}\left(e^{-\delta|\mathbf{x}-\mathbf{y}|/C_\nu} + \epsilon\right), \qquad \mathbf{x}, \mathbf{y} \in B_{2r}(\mathbf{n}).
\end{aligned}$$

Therefore, the triplets (A_0, A_1, A_2) involved in (3.17) satisfy the assumptions of Proposition 3.2, with constants

$$C = \frac{C_\nu}{\delta}, \quad \beta = \frac{\delta}{2C_\nu}, \quad \varepsilon = \frac{1}{\delta^5}\left(e^{-\delta r/C_\nu} + \epsilon\right).$$

So, we deduce that

$$\left|\sigma(H_1,\lambda) - \sigma(H_2,\lambda)\right| \leq \frac{C_\nu}{\delta^{3+6}}\left(e^{\frac{\delta r}{2C_\nu}} + \frac{1}{\delta^{5/2}}(e^{-\frac{\delta r}{C_\nu}} + \epsilon)^{1/2}\right) \leq \frac{C_\nu}{\delta^{12}}\left(e^{-\frac{\delta r}{2C_\nu}} + \epsilon^{1/2}\right).$$

This completes the proof of Proposition 2.1. ■

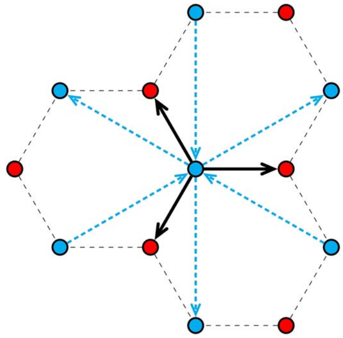

Fig. 4. The black arrows represent tunneling to the three nearest neighbors; the dashed blue arrows represent complex coupling to six second-nearest neighbors in the Haldane model [14].

4 Violation of the Bulk-Edge Correspondence in Strips

In this section, we show that topological insulators lying within strips do not necessarily support edge states along their boundary; this means that geometrically, Ω needs to be unbounded in all directions for the bulk spectral gaps to be systematically filled.

Specifically, for any $L > 0$, we construct an edge operator H_e satisfying Assumption 2 with:

- The bulk operators $H_\pm$ are insulating at energy 0, with bulk conductance ∓ 1;
- $\Omega \subset \mathbb{Z} \times [-L, L]$, in particular $\mathbf{Fr}(\Omega) \le L$; and
- $0 \notin \Sigma(H_e)$: the bulk gap did not fully close.

Hence, although the bulk operators $H_\pm$ represent topologically distinct topological phases, the interface $\partial\Omega$ does not support conducting states for H_e. In particular, a material made of topologically distinct insulators across $\partial\Omega$, $\Omega = \mathbb{N} \times [-L, L]$, violates the bulk-edge correspondence. This was suspected by G.M. Graf, but the problem was left open in an online talk by Thiang [21].

4.1 Haldane model

Our bulk operators are based on Haldane's model [14], which we review briefly.

A honeycomb lattice is generated by the parallel translation of the two nearest vertices, denoted by red and blue dots in Figure 4. Wave functions on the honeycomb lattice are denoted by $\psi = (\psi^A, \psi^B)^T$, where ψ^A and ψ^B denote wave functions on red and blue sites, respectively. The Haldane Hamiltonian models tunneling to the three nearest neighbors (called the Wallace model [34], denoted by H_0 below) and *complex* coupling to the six second-nearest neighbors (denoted by S below); see Figure 4. We will use a version based on the $\mathbb{Z}^2$-lattice (which only differs from the standard honeycomb version by a linear change of variable):

$$H_\pm = H_0 \pm S$$

where H_0 and S are self-adjoint, short-range Hamiltonians on $\psi = (\psi^A, \psi^B)^T \in \ell^2(\mathbb{Z}^2, \mathbb{C}^2)$ given by

$$(H_0\psi)_n = \begin{bmatrix} \psi^B_n + \psi^B_{n-e_1} + \psi^B_{n-e_2} \\ \psi^A_n + \psi^A_{n+e_1} + \psi^A_{n+e_2} \end{bmatrix},$$

$$(S\psi)_n = \mathrm{i}s \begin{bmatrix} \psi^A_{n+e_1} - \psi^A_{n-e_1} + \psi^A_{n-e_2} - \psi^A_{n+e_2} + \psi^A_{n+e_2-e_1} - \psi^A_{n+e_1-e_2} \\ -\psi^B_{n+e_1} + \psi^B_{n-e_1} - \psi^B_{n-e_2} + \psi^B_{n+e_2} - \psi^B_{n+e_2-e_1} + \psi^B_{n+e_1-e_2} \end{bmatrix}.$$

The parameter s above quantifies the ratio between first and second nearest neighbor coupling. We restrict it to $(0, 1]$ here.

As a result, the discrete Fourier transform w.r.t. $\mathbb{Z}^2$ is

$$\hat{H}_\pm(\xi) = \begin{bmatrix} \pm 2s\eta(\xi) & \overline{\omega(\xi)} \\ \omega(\xi) & \mp 2s\eta(\xi) \end{bmatrix}, \qquad \xi \in [-\pi, \pi]^2,$$

$$\omega(\xi) := 1 + e^{\mathrm{i}\xi_1} + e^{\mathrm{i}\xi_2},$$

$$\eta(\xi) := \sin(\xi_1) - \sin(\xi_2) + \sin(\xi_2 - \xi_1).$$

The eigenvalues of $\hat{H}_\pm(\xi)$ are $\pm\lambda(\xi)$ with

$$\lambda(\xi) = \sqrt{(2s\eta(\xi))^2 + |\omega(\xi)|^2}. \tag{4.1}$$

The functions η and ω do not vanish simultaneously; therefore, λ never vanishes and the Hamiltonians $H_\pm$ are insulating at energy 0: $0 \notin \Sigma(H_\pm)$, or equivalently, $H_\pm$ are invertible. Because of translational invariance, it is known their bulk conductance equals the Chern number of their low energy eigenbundle (see, e.g., [30, Equations (13) and Corollary 8.4.4]), and we have $\sigma(H_\pm; 0) = \mp 1$, see [8,§8]. In particular, the insulators described by $H_\pm$ are topologically distinct, and $H_\pm$ satisfy Assumption 1.

4.2 Edge operator

Fix $L > 0$ and $\Omega \subset [-L, L] \times \mathbb{N}$. We define the edge operator by

$$H_e := H_+ - 2\mathbb{1}_{\Omega^c} S \mathbb{1}_{\Omega^c}.$$

Then formally, H_e is H_+ in Ω and H_- in Ω^c. We prove Theorem 3, formulated here as:

Proposition 4.1. H_e satisfies Assumption 2 at energy 0; however, there exists a numerical constant $\rho_0 > 0$ such that if $0 < s \le \rho_0 L^{-1}$, then $0 \notin \Sigma(H_e)$.

This implies that an interface lying in a strip between two topologically distinct insulating phases does not necessarily fill the bulk spectral gap.

Remark 2. Since the proof is perturbative, it can be adapted to show that a small neighborhood U of 0 stays in the gap $\Sigma(H_e)^c$. Meanwhile, Theorem 2 implies that $\mathcal{G} \cap \Sigma(H_e)$ is $\mathcal{O}\left(\frac{\ln R}{R}\right)$-dense in $\mathcal{G}$. However, we do not know if this provides a complete picture of $\mathcal{G} \cap \Sigma(H_e)$, that is, for instance, if $\mathcal{G} \cap \Sigma(H_e)$ contains intervals.

Remark 3. A general argument implies that the edge conductance of H_e across the x_2-axis is 0. Indeed, this conductance is stable under perturbations within strips orthogonal to the x_2-axis, such as $2\mathbb{1}_{\Omega^c} S \mathbb{1}_{\Omega^c}$; so it is equal to that of $H_e + 2\mathbb{1}_{\Omega^c} S \mathbb{1}_{\Omega^c} = H_+$, which is 0.
To the best of our knowledge, there is no general argument that implies that H_e has edge conductance across the x_1-axis equal to 0. For $0 < s \le \rho_0 L^{-1}$, it is a consequence of Proposition 4.1: H_e has no states with energy near 0. For $\Omega = \mathbb{N} \times [-L, L]$, this implies that no quantum particle may travel from one end of $\partial\Omega$ to the other with high probability.

Proposition 4.1 is a consequence of the uncertainty principle: a function localized in frequency may not be localized in position. For s small, the Fourier transforms of $H_\pm$ have eigenvalues of order 1, unless ξ is near the zeros $\xi_\pm^* = \pm 2\pi/3(1, -1)$ of $-\omega(\xi)$in which case they are of order s. Therefore, a $O(s)$-perturbation (such as S) may not close the gap unless it generates states for H_e that are concentrated in frequency near $\xi_\pm^*$. By (a tailored version of) the uncertainty principle, such states may not be localized within a strip (such as Ω).

4.3 Proof

We prove Proposition 4.1 here. We will need the following lemmas:

Lemma 4.1.

1) There exists $\lambda_0 > 0$, such that

$$\lambda(\xi) \ge \lambda_0 \cdot s, \quad \text{for all} \quad \xi \in [-\pi, \pi]^2, \quad s \in (0, 1].$$

2) There exists $\mu_0 > 0$ such that

$$|\omega(\xi)| \ge \mu_0 \cdot d(\xi, \{\xi_\pm^*\}), \quad \text{for all} \quad \xi \in [-\pi, \pi]^2.$$

Remark 4. At the physical level, these are well-known bounds; we tailor them here to our needs. Part (1) means that $H_\pm$ has a spectral gap at energy 0; part (2) means that the Wallace Hamiltonian H_0 has a Dirac cone.

Proof of Lemma 4.1. (1) From (4.1) and $s \in (0,1]$, we have

$$\lambda(\xi) = \sqrt{(2s\eta(\xi))^2 + |\omega(\xi)|^2} \geq s\sqrt{(2\eta(\xi))^2 + |\omega(\xi)|^2}. \tag{4.2}$$

Moreover,

$$\omega(\xi) = 0 \quad \Leftrightarrow \quad \xi = \xi_\pm^* = \pm\frac{2\pi}{3}(1,-1) \quad \Rightarrow \quad \eta(\xi_\pm^*) = \pm\frac{3\sqrt{3}}{2}.$$

Thus, $\omega(\xi)$ and $\eta(\xi)$ cannot vanish simultaneously and $\sqrt{(2\eta(\xi))^2 + |\omega(\xi)|^2}$ never vanishes. By continuity, $\sqrt{(2\eta(\xi))^2 + |\omega(\xi)|^2} \geq \lambda_0$ for some $\lambda_0 > 0$. This proves (1) by going back to (4.2).

(2) We first write down ω as a function valued in $\mathbb{R}^2$ instead of $\mathbb{C}$:

$$\omega(\xi) = (1 + \cos(\xi_1) + \cos(\xi_2), \sin(\xi_1) + \sin(\xi_2)).$$

With this notation,

$$\nabla(\omega(\xi)) = \begin{bmatrix} -\sin(\xi_1) & -\sin(\xi_2) \\ \cos(\xi_1) & \cos(\xi_2) \end{bmatrix}, \qquad \nabla(\omega(\xi_\pm^*)) = \frac{1}{2}\begin{bmatrix} \mp\sqrt{3} & \pm\sqrt{3} \\ -1 & -1 \end{bmatrix}.$$

As a result, for any $u = (u_1, u_2)^T$,

$$|\nabla\omega(\xi_\pm^*)u|^2 = \frac{1}{4}[3(u_1 - u_2)^2 + (u_1 + u_2)^2] = \frac{1}{4}(4u_1^2 + 4u_2^2 - 4u_1u_2) \geq \frac{1}{2}|u|^2.$$

Assume for any n, there is $\xi_n \neq \xi_\pm^* \in [-\pi,\pi]^2$ such that

$$|\omega(\xi_n)| \leq \frac{d(\xi_n, \{\xi_\pm^*\})}{n}. \tag{4.3}$$

By compactness of $[-\pi,\pi]^2$, there exists a subsequence of ξ_n that converges to some ξ_∞. From (4.3) and $d(\xi_n, \{\xi_\pm^*\}) \leq 4\pi$, we deduce $|\omega(\xi_\infty)| = 0$ hence ξ_∞ is either ξ_+^* or ξ_-^*. As a result, as $n \to \infty$,

$$\begin{aligned} \frac{1}{n} \geq \frac{|\omega(\xi_n) - \omega(\xi_\infty)|}{|\xi_n - \xi_\infty|} &= \frac{|\nabla\omega(\xi_\infty)(\xi_n - \xi_\infty)| + O(|\xi_n - \xi_\infty|^2)}{|\xi_n - \xi_\infty|} \\ &\geq \frac{1}{\sqrt{2}} + O(|\xi_n - \xi_\infty|) \to \frac{1}{\sqrt{2}}. \end{aligned}$$

We get a contradiction. Thus, there is some $\mu_0 > 0$ such that $|\omega(\xi)| \geq \mu_0|\xi - \xi_\pm^*|$. ∎

Lemma 4.2. There exists $C_0 > 0$ such that for all $s \in (0,1]$, $L > 0$ and $u \in \ell^2(\mathbb{Z}^2, \mathbb{C}^2)$:

$$\operatorname{Supp} u \subset \mathbb{Z} \times [-L, L] \qquad \Rightarrow \qquad \|H_+^{-1}u\|_{\ell^2} \leq C_0 L^{1/3} s^{-2/3} \|u\|_{\ell^2}.$$

Proof of Lemma 4.2. Recall that $\omega^{-1}(0) = \{\xi_+^*, \xi_-^*\}$. Since eigenvalues of $\hat{H}_+^{-1}(\xi)$ are $\pm\frac{1}{\lambda(\xi)}$, for any $\delta > 0$, we have

$$\begin{aligned}
\|H_+^{-1}u\|^2_{\ell^2(\mathbb{Z}^2)} &= \|\hat{H}_+^{-1}(\xi)\hat{u}(\xi)\|^2_{L^2([-\pi,\pi]^2)} \\
&\leq \int_{[-\pi,\pi]^2} \frac{1}{(\lambda(\xi))^2}|\hat{u}(\xi)|^2 d\xi \\
&\leq \int_{B_\delta(\xi_\pm^*)} \frac{1}{\lambda_0^2 s^2}|\hat{u}(\xi)|^2 d\xi + \int_{(B_\delta(\xi_\pm^*))^c} \frac{1}{\mu_0^2\delta^2}|\hat{u}(\xi)|^2 d\xi \\
&\leq \int_{B_\delta(\xi_\pm^*)} \frac{1}{\lambda_0^2 s^2}|\hat{u}(\xi)|^2 d\xi + \frac{(2\pi)^2}{\mu_0^2\delta^2}\|u\|^2_{\ell^2}
\end{aligned}$$

thanks to Plancherel's formula $\|\hat{u}\|_{L^2([-\pi,\pi]^2)} = 2\pi\|u\|_{\ell^2(\mathbb{Z}^2)}$. If $\mathbf{Supp}(u) \subset \mathbb{Z} \times [-L, L]$, then

$$\hat{u}(\xi_1, n_2) = \sum_{n_1} e^{-in_1\xi_1}u(n_1, n_2) = 0, \text{ if } n_2 \notin [-L, L].$$

Here we only did the Fourier transformation on n_1 but we abuse the notation and still use $\hat{u}$ to refer to it. Since $(a_1 + \cdots + a_n)^2 \leq n(a_1^2 + \cdots + a_n^2)$, we have

$$\begin{aligned}
\int_{B_\delta(\xi_\pm^*)} \frac{1}{\lambda_0^2 s^2}|\hat{u}(\xi)|^2 d\xi &\leq \frac{1}{\lambda_0^2 s^2}\int_{B_\delta(\xi_\pm^*)} \left|\sum_{n_2\in[-L,L]} e^{-in_2\xi_2}\hat{u}(\xi_1, n_2)\right|^2 d\xi \\
&\leq \frac{1}{\lambda_0^2 s^2}\int_{B_\delta(\xi_\pm^*)} 2L \sum_{n_2\in[-L,L]} |\hat{u}(\xi_1, n_2)|^2 d\xi \\
&\leq \frac{2L}{\lambda_0^2 s^2}\int_{|\xi_2-(\xi_\pm^*)_2|\leq\delta}\int_{-\pi}^{\pi}\sum_{n_2}|\hat{u}(\xi_1, n_2)|^2 d\xi_1 d\xi_2 \\
&= \frac{2L\cdot 4\delta}{\lambda_0^2 s^2}\int_{-\pi}^{\pi}\sum_{n_2}|\hat{u}(\xi_1, n_2)|^2 d\xi_1 = \frac{8\delta L\cdot 2\pi}{\lambda_0^2 s^2}\|u\|_{\ell^2}
\end{aligned}$$

where we use Plancherel's formula on n_1-coordinates only for the last line. Combining with the earlier estimates, we get

$$\|H_+^{-1}u\|^2_{\ell^2(\mathbb{Z}^2)} \leq \left(\frac{16\pi\delta L}{\lambda_0^2 s^2} + \frac{4\pi^2}{\mu_0^2\delta^2}\right)\|u\|^2_{\ell^2(\mathbb{Z}^2)}.$$

In particular, taking $\delta = \left(\frac{\pi\lambda_0^2 s^2}{4\mu_0 L}\right)^{\frac{1}{3}}$, we get

$$\|H_+^{-1}u\|^2_{\ell^2(\mathbb{Z}^2)} \leq C_0^2\left(\frac{L}{s^2}\right)^{\frac{2}{3}}\|u\|^2_{\ell^2}, \qquad C_0 = 2^{\frac{13}{6}}\pi^{\frac{2}{3}}\mu_0^{-\frac{2}{3}}\lambda_0^{-\frac{2}{3}}.$$

This completes the proof. ■

Proof of Proposition 4.1. (1) We have

$$\begin{aligned}
H_e &= \mathbb{1}_\Omega H_+ \mathbb{1}_\Omega + \mathbb{1}_{\Omega^c}(H_+ - 2S)\mathbb{1}_{\Omega^c} + \mathbb{1}_\Omega H_+ \mathbb{1}_{\Omega^c} + \mathbb{1}_{\Omega^c} H_+ \mathbb{1}_\Omega \\
&= \mathbb{1}_\Omega H_+ \mathbb{1}_\Omega + \mathbb{1}_{\Omega^c} H_- \mathbb{1}_{\Omega^c} + \mathbb{1}_\Omega H_+ \mathbb{1}_{\Omega^c} + \mathbb{1}_{\Omega^c} H_+ \mathbb{1}_\Omega.
\end{aligned} \tag{4.4}$$

By (4.4),

$$E = H_e - \mathbb{1}_\Omega H_+ \mathbb{1}_\Omega + \mathbb{1}_{\Omega^c} H_- \mathbb{1}_{\Omega^c} = \mathbb{1}_\Omega H_+ \mathbb{1}_{\Omega^c} + \mathbb{1}_{\Omega^c} H_+ \mathbb{1}_\Omega.$$

Since $\mathbb{1}_\Omega H_+ \mathbb{1}_{\Omega^c}(x,y) = \mathbb{1}_\Omega(x) H_+(x,y) \mathbb{1}_{\Omega^c}(y) = 0$ if $|x-y| > 2$, we have E satisfies (1.2); thus, H_e satisfies Assumption 2.

(2) Recall that since $\lambda(\xi)$ never vanishes, H_+ is invertible. Thus,

$$H_e = H_+ - 2\mathbb{1}_{\Omega^c} S \mathbb{1}_{\Omega^c} \quad \Leftrightarrow \quad H_+^{-1} H_e = \mathrm{Id} - 2H_+^{-1} \mathbb{1}_{\Omega^c} S \mathbb{1}_{\Omega^c}.$$

To show H_e is invertible, it is enough to show $\|2H_+^{-1}\mathbb{1}_{\Omega^c} S \mathbb{1}_{\Omega^c}\| < 1$. Since $\|S\| \le 6s$,

$$\begin{aligned} \|2H_+^{-1}\mathbb{1}_{\Omega^c} S \mathbb{1}_{\Omega^c} u\|_{\ell^2} &\le C_0 L^{1/3} s^{-2/3} \|\mathbb{1}_{\Omega^c} S \mathbb{1}_{\Omega^c} u\|_{\ell^2} \\ &\le 6 C_0 L^{\frac{1}{3}} s^{\frac{1}{3}} \|u\|_{\ell^2}. \end{aligned}$$

Thus, when $s < \rho_0 L^{-1}$, $\rho_0 = 6^{-3} C_0^{-3}$, we have $\|2H_+^{-1}\mathbb{1}_{\Omega^c} S \mathbb{1}_{\Omega^c}\| < 1$. Thus, H_e is invertible. ■

Remark 5. Numerics actually yield the values

$$\lambda_0 = 1, \qquad \mu_0 = \frac{3}{\pi\sqrt{26}} \simeq 0.18, \qquad C_0 \simeq 31, \qquad s < 1.5 \cdot 10^{-7} L^{-1}.$$

That is, if the second-nearest neighbor hopping is much smaller than the first-nearest neighbor hopping (depending on L), then a topological insulator fitting in a strip of width L may not have an edge spectrum.

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Appendix. Proof of Some Estimates

Proof of (3.1). Fix $a \in (0,1]$. Then:

$$\sum_{s\in\mathbb{Z}} e^{-2a|s|} = \frac{2}{1-e^{-2a}} - 1 = \frac{1+e^{-2a}}{1-e^{-2a}} = \frac{1}{\tanh(a)} \le \frac{2}{a},$$

where the last inequality follows from the fact that $\tanh(x)$ is concave when $x > 0$; thus, $\tanh(a) \ge \tanh(1)a \ge a/2$ for $a \in (0,1]$. This yields the first inequality in (3.1). The second inequality follows immediately since $e^{-2a|\mathbf{x}|} = e^{-2a|x_1|} e^{-2a|x_2|}$.

Now fix $r \ge 0$. If $|\mathbf{x}| \ge r$, then either $|x_1| \ge r/2$ or $|x_2| \ge r/2$. This induces a splitting into two mutually symmetric sums:

$$\sum_{|\mathbf{x}|\ge r} e^{-2a|\mathbf{x}|} \le 2 \sum_{|x_1|\ge r/2,\ x_2\in\mathbb{Z}} e^{-2a|\mathbf{x}|} \le 2\left(\frac{2}{a}\right)^2 e^{-ar}.$$

This completes the proof. ■

Proof of (3.2). We apply Schur's test. For a self-adjoint operator, it reads:

$$\|H\| \leq \sup_{\mathbf{x}\in\mathbb{Z}^2} \sum_{\mathbf{y}\in\mathbb{Z}^2} \left|H(\mathbf{x},\mathbf{y})\right| \leq \frac{1}{\nu} \sup_{\mathbf{x}\in\mathbb{Z}^2} \sum_{\mathbf{y}\in\mathbb{Z}^2} e^{-2\nu|\mathbf{x}-\mathbf{y}|}$$

$$= \frac{1}{\nu} \sum_{\mathbf{y}\in\mathbb{Z}^2} e^{-2\nu|\mathbf{y}|} = \frac{1}{\nu}\left(\sum_{y_1\in\mathbb{Z}} e^{-2\nu|y_1|}\right)^2 \leq \frac{4}{\nu^3}.$$

In the last inequality, we used the first inequality in (3.1), which is valid since $\nu \in (0,1]$. ∎

Proof of (3.3). By the slope inequality for convex functions $f(x) = e^{xs}$ with any $s \geq 0$, we have for $0 < \alpha \leq \nu$:

$$\frac{e^{\alpha s}-1}{\alpha} \leq \frac{e^{\nu s}-1}{\nu}.$$

By applying this inequality to $s = |\mathbf{x}-\mathbf{y}|$, we deduce that $\frac{S_\alpha}{\alpha} \leq \frac{S_\nu}{\nu}$. We now estimate S_ν. We have

$$S_\nu = \sup_{\mathbf{x}\in\mathbb{Z}^2} \sum_{\mathbf{y}\in\mathbb{Z}^2} |H(\mathbf{x},\mathbf{y})| \left(e^{\nu|\mathbf{x}-\mathbf{y}|}-1\right) \leq \frac{1}{\nu} \sup_{\mathbf{x}\in\mathbb{Z}^2} \sum_{\mathbf{y}\in\mathbb{Z}^2} e^{-2\nu|\mathbf{x}-\mathbf{y}|}\left(e^{\nu|\mathbf{x}-\mathbf{y}|}-1\right)$$

$$\leq \frac{1}{\nu} \sum_{\mathbf{y}\in\mathbb{Z}^2} e^{-\nu|\mathbf{y}|} \leq \frac{16}{\nu^3},$$

where we used (3.1) again. ∎

Proof of (3.12). 1. We first note that we have, by $|t-s|+|s| \geq |t|$ and (3.1):

$$\sum_{s\in\mathbb{Z}} e^{-2\beta|t-s|-2\beta|s|} \leq e^{-\beta|t|} \sum_{s\in\mathbb{Z}} e^{-\beta|t-s|-\beta|s|} \leq e^{-\beta|t|} \sum_{s\in\mathbb{Z}} e^{-\beta|s|} \leq \frac{4e^{-\beta|t|}}{\beta}. \tag{A.1}$$

2. We now control S, the sum in the LHS of (3.12). To this end, we apply (A.1) four times: first to $(t,s) = (x_1,y_1)$, then (w_1,z_1), then (y_2,z_2) and finally (x_2,y_2). This gives:

$$S \leq \frac{4}{\beta} e^{-\beta|x_1|-2\beta|w_2|} \sum_{y_2,\mathbf{z}} e^{-2\beta|x_2-y_2|-2\beta|y_2-z_2|-2\beta|z_1|-2\beta|z_1-w_1|-2\beta|z_2|}$$

$$\leq \left(\frac{4}{\beta}\right)^2 e^{-\beta|x_1|-\beta|w_1|-\beta|w_2|} \sum_{y_2,z_2} e^{-2\beta|x_2-y_2|-2\beta|y_2-z_2|-2\beta|z_2|}$$

$$\leq \left(\frac{4}{\beta}\right)^3 e^{-\beta|x_1|-\beta|\mathbf{w}|} \sum_{y_2} e^{-2\beta|x_2-y_2|-2\beta|y_2|} \leq \left(\frac{4}{\beta}\right)^4 e^{-\beta|\mathbf{x}|-\beta|\mathbf{w}|}.$$

This is (3.12). ∎

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