THE X-RAY TRANSFORM ON ASYMPTOTICALLY CONIC SPACES

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Abstract. In this paper, partly based on Zachos' PhD thesis, we show that the geodesic X-ray transform is stably invertible near infinity on a class of asymptotically conic manifolds which includes perturbations of Euclidean space. In particular certain kinds of conjugate points are allowed. Further, under a global convex foliation condition, the transform is globally invertible.

The key analytic tool, beyond the approach introduced by Uhlmann and Vasy, is the introduction of a new pseudodifferential operator algebra, which we name the 1-cusp algebra, and its semiclassical version.

1. Introduction

The geodesic X-ray transform I on a Riemannian manifold (often with boundary) (M, g), of dimension $n \geq 2$, is a map from a class of functions, such as continuous functions on M, to a corresponding class functions on the space of geodesics: if γ is a geodesic, then

$$(If)(\gamma) = \int f(\gamma(s))\, ds.$$

Here one needs to make some assumption on the geometry and the function f so that the integral makes sense, for instance ensuring that one integrates over a finite interval or that f decays sufficiently fast along geodesics.

An important and well-studied question is whether the X-ray transform is (left-) invertible. In other words, if If is known, can f be determined? The answer, as one might expect, depends on (M, g) and the class of f to be considered. In addition, this problem, or its tensorial version, is the linearization of the boundary rigidity problem which asks whether the restriction of the distance function d_g to $\partial M \times \partial M$ determines g up to diffeomorphisms, or if g is in a fixed conformal class, $g = c^{-2} g_0$, with g_0 fixed, whether the same information determines the conformal factor c. (There are also some slightly different versions of these questions with some additional data.)

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A version of this problem was studied already over a century ago by Herglotz [6] and Wiechert and Zoeppritz [29], in the special case when M is a ball with a rotationally symmetric metric g and f is also rotationally invariant. These assumptions make the problem effectively one-dimensional, yet there is actual geometry involved: they proved the injectivity of I under an additional assumption which is the special case of the convex foliation assumption described below.

The 'standard' conjecture in the field is Michel's, namely that boundary rigidity holds on simple manifolds [14]. Recall that a Riemannian manifold with boundary (M, g) is simple if for any $p \in M$, the exponential map $\exp_p$ is a diffeomorphism from a neighborhood of the origin of T_pM and if ∂M is strictly convex with respect to g. There has been much work on this problem, primarily on compact manifolds. As we shall see below, there is a significant difference in the two vs. higher dimensional cases. Croke and Otal independently established boundary rigidity in the two-dimensional non-positively curved case [16], [3], before Pestov and Uhlmann proved Michel's conjecture in general in two-dimensions [19]. In the higher-dimensional setting, Stefanov and Uhlmann showed rigidity for metrics close to Euclidean ones [20]. Mukhometov showed rigidity for metrics conformal to the Euclidean metric [15]. There are also results under symmetry assumptions, while in [8] and [2] boundary rigidity is shown when one of the metrics is close to the Euclidean one, while [21] proves a generic result. Newer developments will be described below.

As a motivation for the current work we recall a result of Uhlmann and Vasy [25] concerning the local X-ray transform which introduced a new approach to this inverse problem. For an open set O in a manifold with boundary, the local transform is the X-ray transform restricted geodesic segments which are completely in O with endpoints on ∂M.

Theorem 1.1 (Uhlmann and Vasy [25])**.** *For compact Riemannian manifolds (M, g) with strictly convex boundary, the local geodesic X-ray transform is left-invertible on small enough collar neighborhoods of the boundary, and is globally left-invertible under a convex foliation assumption.*

Here the convex foliation assumption is a replacement for the simplicity condition; at this point the precise relationship between these is not completely clear. We recall the precise definition: the assumption is the existence of a C^∞ function x with non-vanishing differential which is strictly concave from the side of the super-level sets, i.e. for all geodesics γ,

$$\frac{d(x \circ \gamma)}{ds}(s_0) = 0 \Rightarrow \frac{d^2(x \circ \gamma)}{ds^2}(s_0) > 0.$$

This assumption is satisfied, for instance, on domains in simply connected negatively (non-positively) curved manifolds, with x being the distance from a point outside the domain, as well as on manifolds without conic points. A simple modification of the proof allows some singular level sets, like the radius function from the center of a ball, and then manifolds with non-negative curvature are also covered, as shown in [18]. Indeed, the setting of Herglotz [6] and Wiechert and Zoeppritz [29] becomes a special case of this setup.

The approach to this theorem was by adding an artificial boundary to create a collar neighborhood of the actual boundary and showing that the local geodesic X-ray transform on this collar neighborhood was an invertible operator in a particular operator class defined via microlocal analysis, as we explain below. These two authors, along with Stefanov, used this linear result to prove a nonlinear result about metric rigidity:

Theorem 1.2 (Stefanov, Uhlmann and Vasy [23], see also [22] in the conformal case)**.** *If (M, g) is an n-dimensional Riemannian manifold with boundary, where $n \geq 3$, with strictly convex boundary and a convex foliation, then if there is another Riemannian metric $\hat{g}$ on M such that ∂M is still strictly convex with respect to $\hat{g}$, and if g and $\hat{g}$ have identical boundary distance functions, then they are the same up to a diffeomorphism fixing the boundary pointwise.*

In this paper we extend the first result, Theorem 1.1, to a class of asymptotically conic manifolds. Recall that a conic metric, on a manifold $(0, \infty)_r \times Y$, with Y the cross section or link, which we always assume is compact and without boundary, is one of the form

$$g_\infty = dr^2 + r^2 h,$$

where h is a Riemannian metric on Y. An asymptotically conic metric is one on a manifold which outside a compact set is identified with $(r_0, \infty)_r \times Y$, with a metric that on this conic end tends to g_∞ as $r \to \infty$ in a specified way. To be concrete, for our purposes, it is useful to 'bring in' infinity, i.e. let $x = r^{-1}$, so $r \to \infty$ corresponds to $x \to 0$, and add a boundary $\{0\}_x \times Y$ to the manifold, thus compactifying it to $\overline{M}$. An asymptotically conic metric then, as introduced by Melrose [11], is a Riemannian metric on M which is of the form

$$g = \frac{dx^2}{x^4} + \frac{h}{x^2}$$

near $\partial\overline{M}$, where h is a smooth symmetric 2-cotensor on $\overline{M}$; g is thus asymptotic to g_∞ given by $h|_{x=0}$ on the cross section Y.

Theorem 1.3 (See Theorem 3.7 and Corollary 3.8)**.** *Suppose that M is a manifold of dimension ≥ 3, g is an asymptotically conic metric on M for which the cone's cross section (link) has no conjugate points within distance $\leq \pi/2$. Then on a collar neighborhood of infinity the geodesic X-ray transform is injective on the restriction to the collar neighborhood of sufficiently rapidly decaying exponential-power type weighted function spaces, i.e. for all $p > 0$ there is $C > 0$ such that injectivity on spaces such as $e^{-C/x^{2p}} L^2_g$ holds; here L^2_g is the L^2 space relative to the Riemannian density.*

Remark 1.4. While $\pi/2$ in the statement of the theorem might look peculiar, it is purely geometric, and is explained in Section 1.2.

In addition, the function x plays a dual role in the present discussion, as we explain below: one is connected to the asymptotically conic geometry, and is thus fixed, and other to the analysis of the X-ray transform inversion; the latter determines the exponential weight in the theorem. The arguments below will be given in detail for $p = 1$, in which case the decay assumption is sufficiently fast Gaussian decay, i.e. $e^{-C/x^2} L^2_g$, with $C > 0$ large. Working with general $p > 0$ only requires minor changes, and we will place these in remarks throughout the paper. See Remark 1.10.

Note that the assumption holds in particular on perturbations of asymptotically Euclidean metrics (for which the link has conjugate points at distance π), even though these typically have conjugate points, indeed this is necessarily the case if the metric keeps being asymptotic to Euclidean space but is not flat, as shown recently by Guillarmou, Mazzucchelli and Tzou [5]. This result thus partially strengthens the injectivity result of Guillarmou, Lassas and Tzou [4], in that that work requires the absence of conjugate points; however, this strengthening comes at the cost of imposing faster decay conditions in our case.

If $\overline{M}$ has a global convex foliation, our Theorem combined with the result of [25] immediately implies the full invertibility on sufficiently fast Gaussian decaying functions on M: first the restriction to a collar neighborhood of the boundary is determined, and thus if two functions have the same X-ray transforms, they are supported away from $x = 0$, so [25] applies.

To explain the context of these results, we note that it has been known for quite some time that under appropriate geometric assumptions, namely the absence of conjugate points, I^*I is an elliptic pseudodifferential operator. For our purposes it is best to consider I as a map from (say, continuous) functions on M to functions on the sphere bundle SM, or equivalently (via the Riemannian metric) the cosphere bundle S^*M, as

$$(If)(z, v) = \int f(\gamma_{z,v}(s))\, ds,$$

where $\gamma_{z,v}$ is the geodesic through $z \in M$ with tangent vector $v \in S_zM$. We then replace I^* by the map L from functions on SM to functions on M defined by

$$(Lw)(z) = \int_{S_zM} w(z, v)|d\sigma|,$$

where σ is a positive smooth density (e.g. the Riemannian one) on S_zM, smoothly dependent on z; LI is then an elliptic pseudodifferential operator of order -1. This gives that in the context of compact manifolds with boundary satisfying these geometric conditions, LI, thus I, has a finite, but potentially large, dimensional nullspace. The advance in the just mentioned papers was to exclude the possibility of such a nullspace as well as to localize the problem, thus eliminating the need for conjugate point assumptions. This was done by introducing an artificial boundary, and recovering f from If from information on geodesics that stay on one side of this boundary; moving the artificial boundary sufficiently close to the original boundary gave a small parameter in which asymptotic analysis techniques could be used. Technically, this involved a localizer $\tilde{\chi}$ on SM which becomes singular at the artificial boundary, localizing to geodesics that remain on the desired side via the consideration of $L\tilde{\chi}I$. Based on the precise nature of the singularity, one gets a different kind of an operator; with the particular choice made in these papers, the approach relied on Melrose's scattering algebra, associated to the new artificial boundary, effectively pushing it to infinity analytically; we describe this below.

In [28] a modified approach was introduced where the artificial boundary was replaced by a semiclassical scaling under somewhat more stringent geometric hypotheses; indeed, the two approaches could even be combined, thus eliminating the extra conditions and making sure that the only necessity for a combined approach is purely geometric (as opposed to analytic). This new semiclassical approach is more suited to our problem as otherwise it would be harder to keep track of the behavior of the combined pseudodifferential operator algebra when moving the artificial boundary in this case as had been done in [25]: one has both an artificial boundary, with the scattering algebra behavior, as well as a new algebra at infinity, called the 1-cusp algebra. The semiclassicalization of this joint algebra, on the other hand, easily gives the full invertibility (rather than mere ellipticity) results once the neighborhood of infinity is sufficiently small to control the geometry, allowing for fixed artificial boundary; this is the key tool in the proof of Theorem 1.3.

We prove new results of two different types. First, we develop a new operator algebra, called the 1-cusp algebra, which is related to the scattering pseudodifferential algebra but involves one more blow-up, and its semiclassical version. Next, we show, similarly to Uhlmann and Vasy [25], that the X-ray transform in the asymptotically conic setting can be modified to, via composition with other operators, an elliptic operator in this new algebra. In the remainder of the introduction in the two subsections we discuss each of these briefly.

1.1. **Analytic ingredients.** Before introducing the new 1-cusp algebra, we recall the scattering algebra with which it shares many similarities. The scattering pseudodifferential algebra was defined by Melrose in [11] in the general geometric setting, but his work had many predecessors. Indeed, this algebra actually can be locally reduced to a standard Hörmander algebra, which in turn was studied earlier by Parenti [17] and Shubin [24]. Concretely then, on $\mathbb{R}^n$, this algebra arises by the standard quantization,

$$(q_L(a)u)(z) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(z-z')\cdot\zeta} a(z,\zeta)u(z')\, d\zeta\, dz', \qquad u \in \mathcal{S}(\mathbb{R}^n), \tag{1.1}$$

of symbols which are separately symbolic, or symbolic of 'product type', in the position and momentum variables (z,ζ); symbols $a\in S^{m,l}$ of order (m,l) satisfy

$$|(D_z^\alpha D_\zeta^\beta a)(z,\zeta)|\leq C_{\alpha\beta}\langle z\rangle^{l-|\alpha|}\langle\zeta\rangle^{m-|\beta|} \tag{1.2}$$

for all multiindicies $\alpha,\beta\in\mathbb{N}^n$. Thus, the Schwartz kernel is the oscillatory integral (intepreted as a tempered distribution)

$$K_A(z,z')=(2\pi)^{-n}\int_{\mathbb{R}^n}e^{i(z-z')\cdot\zeta}a(z,\zeta)\,d\zeta$$

with respect to the density $|dz'|$. In Melrose's geometric version one works on a compact manifold with boundary; the correspondence is via the compactification $\overline{M}$ which we have already described in the general context of asymptotically conic spaces.

On a manifold with boundary M (we drop the bar over M when we discuss the general analytic structure on a manifold with boundary), scattering vector fields $V\in\mathcal{V}_{\mathrm{sc}}(M)$ are vector fields of the form $V=xV'$, where $V'\in\mathcal{V}_{\mathrm{b}}(M)$ is a vector field tangent to ∂M (by the definition of $\mathcal{V}_{\mathrm{b}}(M)$), and where x is a boundary defining function. This notion is independent of the choice of x since any two choices differ by a positive factor. In local coordinates near ∂M, with y coordinates on ∂M, scattering vector fields are thus of the form

$$a_0(x,y)x^2D_x+\sum_{j=1}^{n-1}a_j(x,y)xD_{y_j}.$$

Scattering differential operators are finite sums of products of such vector fields. Scattering pseudodifferential operators are a generalization of the latter, formally replacing polynomials in the vector fields by more general functions. More precisely, their Schwartz kernels K are locally given by oscillatory integrals of the form

$$(2\pi)^{-n}\int e^{i\left(\frac{x-x'}{x^2}\xi_{\mathrm{sc}}+\frac{y-y'}{x}\cdot\eta_{\mathrm{sc}}\right)}a(x,y,\xi_{\mathrm{sc}},\eta_{\mathrm{sc}})\,d\xi_{\mathrm{sc}}\,d\eta_{\mathrm{sc}}$$

with respect to the density $\frac{|dx'\,dy'|}{(x')^{n+1}}$, see [11]. Here the symbolic estimates of (1.2) become conormal estimates for a

$$|((x\partial_x)^j\partial_y^\alpha\partial_{\xi_{\mathrm{sc}}}^k\partial_{\eta_{\mathrm{sc}}}^\beta a)(x,y,\xi_{\mathrm{sc}},\eta_{\mathrm{sc}})|\leq C_{jk\alpha\beta}\langle\xi_{\mathrm{sc}},\eta_{\mathrm{sc}}\rangle^{m-k-|\beta|}x^{-l}.$$

The Schwartz kernel K can in turn be regarded as a well-behaved, namely conormal, distribution on a resolution (double blow-up) of the double space $M^2=M\times M$. That is, in variables $x,y,X=\frac{x-x'}{x^2},Y=\frac{y-y'}{x}$, K is conormal to the (lifted) diagonal $\{X=0,\ Y=0\}$. We discuss this resolution in some detail in Section 2.1. One advantage of the geometric approach, making the definition via a resolution, is that it is automatically invariantly defined, i.e. from the local perspective it is diffeomorphism invariant.

One reason that the scattering algebra is a useful object to work with is that it is not only a bi-filtered $*$-algebra, thus closed under composition, but the composition, to leading order, modulo $\Psi_{\mathrm{sc}}^{m-1,l-1}$, is symbolic, i.e. it can be expressed algebraically in terms of the principal symbols, meaning the class $[a]$ of a in (1.1) modulo $S^{m-1,l-1}$. Furthermore, the mapping properties are also very well behaved. Defining weighted Sobolev spaces $H^{s,r}$ by adding a weight, $H^{s,r} = \langle z\rangle^{-r}H^s$, where in the case of $\mathbb{R}^n$ H^s is the standard Sobolev space (and in general is transported to the asymptotically conic space via local coordinates), any scattering pseudodifferential operator $A \in \Psi_{\mathrm{sc}}^{m,\ell}$, maps weighted Sobolev spaces to weighted Sobolev spaces $A : H^{s,r} \to H^{s-m,r-\ell}$. The residual class $\Psi_{\mathrm{sc}}^{-\infty,-\infty}$ maps any $H^{s,r} \to H^{s',r'}$, so that they are all compact operators on any $H^{s,r}$.

The 1-cusp algebra shares many of these useful properties. In order to define the 1-cusp algebra, it is helpful to first consider the corresponding vector fields. Recall that cusp vector fields are defined on a manifold with boundary equipped with a boundary defining function x with a given differential at the boundary (so the boundary defining function is determined up to $O(x^2)$); $V \in \mathcal{V}_{\mathrm{cu}}(M)$ are smooth vector fields tangent to ∂M with the property that $Vx = O(x^2)$. In local coordinates thus they are of the form

$$a_0(x,y)x^2D_x + \sum a_j(x,y)D_{y_j}.$$

We then define the 1-cusp vector fields as $\mathcal{V}_{1\mathrm{c}}(M) = x\mathcal{V}_{\mathrm{cu}}(M)$, so in local coordinates thus they are of the form

$$a_0(x,y)x^3D_x + \sum a_j(x,y)xD_{y_j}.$$

These are thus also scattering vector fields, but with an additional order of vanishing in the D_x component.

When turning to the 1-cusp pseudodifferential operators, again defined on manifolds with a preferred boundary defining function x fixed up to $O(x^2)$, corresponding to this additional vanishing, we blow up the Schwartz kernel double space from the scattering coordinates (x,y,X,Y) to $(x,y,V=\frac{X}{x},Y)$. As $(x^2\partial_x, x\partial_y)$ and (X,Y) corresponded to $(\xi_{\mathrm{sc}},\eta_{\mathrm{sc}})$ in the scattering case, $(x^3\partial_x, x\partial_y)$ and (V,Y) correspond to $(\xi_{1\mathrm{c}},\eta_{1\mathrm{c}})$ in this new class when we write an oscillatory integral to define an operator using the symbol. We define our new class of symbols as smooth functions $a^{1\mathrm{c}}(x,y,\xi_{1\mathrm{c}},\eta_{1\mathrm{c}})$ which satisfy the inequalities

$$\left|(x\partial_x)^j\partial_y^\alpha\partial_{\xi_{1\mathrm{c}}}^k\partial_{\eta_{1\mathrm{c}}}^\beta a^{1\mathrm{c}}(x,y,\xi_{1\mathrm{c}},\eta_{1\mathrm{c}})\right| \le C_{ab}\langle\xi_{1\mathrm{c}},\eta_{1\mathrm{c}}\rangle^{m-k-|\beta|}x^{-\ell}.$$

While in coordinates this is the same definition as a scattering class symbol (conormal symbols), invariantly these are symbols (of the same type) on a different (scaled) cotangent bundle, and correspondingly we use a different quantization map to turn them into operators. Concretely, we use the resolved space, assuring that they specify conormal distributions with respect to $\{V=0,\ Y=0\}$:

$$K_A(x,y,V,Y) = \frac{1}{(2\pi)^n}\int a(x,y,\xi_{1c},\eta_{1c})e^{iV\xi_{1c}+Y\cdot\eta_{1c}}\,d\xi_{1c}\,d\eta_{1c}$$
$$= \frac{1}{(2\pi)^n}\int a(x,y,\xi_{1c},\eta_{1c})e^{i\left(\frac{x-x'}{x^3}\xi_{1c}+\frac{y-y'}{x}\cdot\eta_{1c}\right)}\,d\xi_{1c}\,d\eta_{1c},$$

with respect to the density $\frac{|dx'\,dy'|}{(x')^{n+2}}$.

Then, we show that we can describe composition of operators in this new algebra symbolically, and that we can define ellipticity similarly and construct parametrices for elliptic operators in this class with residual errors. As in the scattering algebra, these residual errors are compact operators.

Proposition 1.5 ([30], see Proposition 2.7)**.** *If $A \in \Psi_{1c}^{m,\ell}$ with principal symbol, modulo $\Psi_{1c}^{m-1,\ell-1}$, $[a] \in S^{m,\ell}/S^{m-1,\ell-1}$ and $B \in \Psi_{1c}^{m',\ell'}$ with principal symbol $[b]$ then $A\circ B \in \Psi_{1c}^{m+m',\ell+\ell'}$ with principal symbol $[a][b]=[ab]$.*

As usual, this implies that there is a parametrix for elliptic operators:

Proposition 1.6 ([30], see Proposition 2.8)**.** *If $A \in \Psi_{1c}^{m,\ell}$ with principal symbol a is elliptic, i.e. for some $c>0$,*

$$|a(x,y,\xi_{1c},\eta_{1c})| \geq C\langle\xi_{1c},\eta_{1c}\rangle^m x^{-\ell} \text{ for } |(\xi_{1c},\eta_{1c})| \gg 1 \text{ or } x \ll 1$$

then there is a parametrix $B \in \Psi_{1c}^{-m,-\ell}$ with error $AB-I$, $BA-I$ in $\Psi_{1c}^{-\infty,-\infty}$.

As already alluded to earlier in the introduction, a semiclassical variant of the 1-cusp algebra plays a key role in this work. This is a foliation semiclassical algebra, associated to the full foliation $\mathcal{F}$ by the level sets of x. The foliation semiclassical algebra was described in [28] in both the standard (no boundary) and scattering (artificial boundary) settings; here we thus focus on the 1-cusp aspects. Near $x=0$, the foliation tangent 1c-vector fields are locally

$$\sum a_j(x,y)xD_{y_j};$$

the collection of these is denoted by $\mathcal{V}_{1c}(M;\mathcal{F})$. The semiclassical version of $\mathcal{V}_{1c}(M)$ is simply $\mathcal{V}_{1c,\hbar}(M) = h\mathcal{V}_{1c}(M)$; the semiclassical foliation version is

$$\mathcal{V}_{1c,\hbar,\mathcal{F}}(M;\mathcal{F}) = h\mathcal{V}_{1c}(M) + h^{1/2}\mathcal{V}_{1c}(M;\mathcal{F}).$$

Thus, the semiclassical foliation 1-cusp differential operators take the form

$$\sum_{\alpha+|\beta|\leq m} a_{\alpha\beta}(x,y,h)(hx^3D_x)^\alpha(h^{1/2}xD_y)^\beta.$$

The corresponding pseudodifferential operators $A \in \Psi_{1c,\hbar,\mathcal{F}}^{m,l}(M,\mathcal{F})$ again arise by a modified semiclassical quantization of standard semiclassical symbols a, i.e. ones satisfying (conormal in x) symbol estimates

$$|(xD_x)^\alpha D_y^\beta D_{\xi_{1c}}^\gamma D_{\eta_{1c}}^\delta a(x,y,\xi_{1c},\eta_{1c},h)| \leq C_{\alpha\beta\gamma\delta}\langle(\xi_{1c},\eta_{1c})\rangle^{m-\gamma-|\delta|}x^{-l},$$

namely

(1.3)
$$\begin{aligned}A_h u(x,y) &= Au(x,y,h)\\ &= (2\pi)^{-n} h^{-n/2-1/2} \int e^{i\left(\frac{x-x'}{x^3}\frac{\xi_{1c}}{h}+\frac{y-y'}{x}\frac{\eta_{1c}}{h^{1/2}}\right)}\\ &\qquad a(x,y,\xi_{1c},\eta_{1c})\, u(x',y')\,\frac{dx'\,dy'}{(x')^{n+2}}\,d\xi_{1c}\,d\eta_{1c}.\end{aligned}$$

Thus, in $x>0$, these are just the standard semiclassical foliation operators, in $h>0$ the standard 1-cusp pseudodifferential operators, with the combined behavior near $x=h=0$. In particular we have an elliptic theory as in the semiclassical foliation setting: if A is elliptic, meaning

$$|a(x,y,\xi_{1c},\eta_{1c})| \geq cx^{-l}\langle(\xi_{1c},\eta_{1c})\rangle^m, \qquad c>0,$$

then there is a parametrix $B\in\Psi^{-m,-l}_{1c,\hbar,\mathcal{F}}(M,\mathcal{F})$ with

$$AB-\mathrm{Id},\, BA-\mathrm{Id}\in h^\infty\Psi^{-\infty,-\infty}_{1c,\hbar,\mathcal{F}}(M,\mathcal{F}),$$

and there exists $h_0>0$ such that for $h<h_0$, $A\in\mathcal{L}(H^{s,r}_{1c,\hbar},H^{s-m,r-l}_{1c,\hbar})$ is invertible with uniform bounds. This is the real reason for the usefulness of the semiclassical setting: the errors of a parametrix are not only compact (or finite rank), but can be eliminated altogether.

1.2. **Conic geometry and inverse problems.** Bicharacteristics are integral curves of the Hamilton vector field of the dual metric function of g. For our asymptotically conic metrics the dual metric function is naturally a function on the same bundle ${}^{\mathrm{sc}}T^*M$, on which principal symbols of the scattering pseudodifferential operators live. When discussing the geometry, however, we will use (τ,μ) rather than $(\xi_{\mathrm{sc}},\eta_{\mathrm{sc}})$ as coordinates on the fibers of this bundle, i.e. we write covectors as

$$\tau\frac{dx}{x^2}+\mu\cdot\frac{dy}{x}.$$

This separate notation, in particular, serves to emphasize that these geometric objects will be unchanged even if one uses a different analytic scaling (when for analytic purposes x is replaced by x^p), cf. Remark 1.4. A computation of Melrose [11] gives that for asymptotically conic metrics g,

$$\frac{1}{2}H_g = x\Big(\tau(x\partial_x+\mu\cdot\partial_\mu)-|\mu|^2\partial_\tau+\frac{1}{2}H_h+xV\Big),$$

where V is a vector field tangent to the boundary $x=0$; this gives the arclength parametrization of geodesics. In view of the overall x factor, which makes this parameterization degenerate at the boundary, it is useful to work with ${}^{\mathrm{sc}}H_g = x^{-1}H_g$. While

$$\frac{1}{2}{}^{\mathrm{sc}}H_g = \tau(x\partial_x+\mu\cdot\partial_\mu)-|\mu|^2\partial_\tau+\frac{1}{2}H_h+xV$$

is non-vanishing at $x = 0$ in general, it still does vanish at $\{x = 0,\ \mu = 0\}$, which is called the radial set. When g is conic, and thus $V = 0$, within the unit level set of the dual metric function of g, $\mu = 0$ means $\tau = \pm 1$, i.e. at such a point ${}^{\mathrm{sc}}H_g$ is a (non-vanishing) multiple of the radial vector field $x\partial_x$, and thus the geodesic is radial (the Y component, y, is constant along it). As long as one stays microlocally away from these radial geodesics, as we do here, one can work with $\frac{1}{2|\mu|}{}^{\mathrm{sc}}H_g$ in place of ${}^{\mathrm{sc}}H_g$; we explain this in Section 3 in terms of a blow-up. This amounts to a reparameterization of the integral curves $c = c(s)$ of $\frac{1}{2}H_g$ via $\frac{dr}{ds} = x(c(s))|\mu(c(s))|$, i.e. if the reparameterized bicharacteristics are $\gamma = \gamma(r)$, then $\frac{ds}{dr} = x(\gamma(r))^{-1}|\mu(\gamma(r))|^{-1}$. Melrose and Zworski [12] computed these reparameterized bicharacteristics. Note that switching between $\frac{1}{2}{}^{\mathrm{sc}}H_g$ and $\frac{1}{2|\mu|}{}^{\mathrm{sc}}H_g$ is a smooth reparameterization away from $\mu = 0$, so one can equally well use either of these in that region. To leading order at $x = 0$, so globally for actually conic metrics, the interior, originally unit speed (prior to reparameterization), bicharacteristics can be written as follows:

$$\begin{aligned} &x = \frac{x_0}{\sin r_0}\sin(r + r_0),\ \tau = \cos(r + r_0),\ |\mu| = \sin(r + r_0),\\ &(y, \hat{\mu}) = \exp(rH_{\frac{1}{2}h})(y_0, \hat{\mu}_0),\ r \in (-r_0, -r_0 + \pi), \end{aligned} \tag{1.4}$$

with $(y, \hat{\mu})$ thus following a unit speed lifted geodesic of length π in Y. Note that the maximum of $x \circ \gamma$, which is the point of tangency to level sets of the function x, occurs halfway in the domain of γ, at (parameter) distance $\pi/2$ from either endpoint, at $r + r_0 = \pi/2$, and thus in terms of the boundary geodesic distance $\pi/2$ from either endpoint. In particular, near this point, where most of the action takes place for us, r and t (the parameterization for $\frac{1}{2}{}^{\mathrm{sc}}H_g$) can be used equally well. This also explains the $\pi/2$ in the statement of our main Theorem 1.3.

With these geometric preliminaries and with the properties of the new algebra established, we turn to the X-ray transform I and the operator L defined earlier as a replacement for I^*.

It turns out that for asymptotically conic metrics, if one uses a suitable localizer $\tilde{\chi}$, and conjugates $L\tilde{\chi}I$ by suitable exponential weights e^{Φ} to define the modified normal operator, the result is an element of our new algebra. Here $\tilde{\chi}$ localizes to (has support near) points in the sphere bundle which are almost tangent to level sets of the boundary function x. More precisely, the angle to the level sets goes to 0 as $x \to 0$ proportionally to x, i.e.

$$\tilde{\chi} = \tilde{\chi}(x, y, \lambda/x, \omega)$$

with compact support in the third slot, where we write tangent vectors as $\lambda(x\partial_x) + \omega \cdot \partial_y$ relative to a product decomposition near ∂M respecting the foliation, see Section 3.1 for detail. The exponential weight e^{Φ}, on the other hand, is Gaussian decaying, concretely $\Phi = -\frac{1}{2x^2}$. As

$$L\tilde{\chi}If = e^{\Phi}(e^{-\Phi}L\tilde{\chi}Ie^{\Phi})e^{-\Phi}f,$$

this means that the results we obtain are for $e^{-\Phi}f$, with its Gaussian growing weight, which means that on the one hand the estimates are strong at infinity, but on the other hand they only apply to Gaussian decaying functions f. The actual analytic result, with an ellipticity statement, is:

Theorem 1.7 (cf. [30], and see Theorem 3.1 for the full semiclassical version)**.** *For asymptotically conical metrics with cross sections without conjugate points within distance $\pi/2$ and for suitable localizers $\tilde\chi$, the modified normal operator of the X-ray transform is an elliptic operator (for sufficiently small x) in the 1-cusp algebra.*

Remark 1.8. If we replace x by x^p in the definition of the rescaled λ, i.e. take $\tilde\chi = \tilde\chi(x, y, \lambda/x^p, \omega)$, and similarly $\Phi = -\frac{1}{2px^{2p}}$, the conclusions remain valid, with the x^p-based 1-cusp algebra, i.e. the one defined in an analogous manner but with the boundary defining function x replaced by x^p, with a corresponding change of the smooth structure. Since we only need conormal behavior (as opposed to smoothness) of the coefficients of the algebra at x, the dependence of $\tilde\chi$ on x vs. x^p is immaterial. See Remark 3.3 for the key analytic reason.

As just described for the 1-cusp pseudodifferential algebra, this implies that there is a parametrix with a finite rank error acting on functions supported sufficiently close to infinity and with sufficiently fast decay (which arises from the modifications discussed below). In order to remove this error, we proceed by fixing an artificial boundary $x = c$, $c > 0$ sufficiently small, fixed by geometric considerations, namely the lack (in a precise sense discussed in Section 3.2) of conjugate points in $x < c$. Now we are on a manifold with boundary, with the two boundary hypersurfaces given by $x = 0$ and $x = c$, so can in particular consider the pseudodifferential algebra which is 1-cusp at $x = 0$ and scattering at $x = c$, corresponding to the artificial boundary there. The approach of [25] would be to allow c to become even smaller and use it as an asymptotic parameter. Instead, as already stated, we regard c as fixed, but introduce a semiclassical parameter h in the spirit of [28]. In fact, we need some additional information, namely we use the full foliation $\mathcal{F}$ by the level sets of x in $0 \le x \le c$; this allows one to define the semiclassical foliation version of the 1-cusp/scattering algebra.

The main technical result, Theorem 3.1, is that A, given by an exponential conjugate of $L\tilde\chi I$, is elliptic in the 1-cusp algebra, and indeed in the semiclassical foliation 1-cusp algebra. Here we use

$$\tilde\chi = \tilde\chi(x, y, \lambda/(h^{1/2}x), \omega)$$

and e^{Φ} with $\Phi = -\frac{1}{2hx^2}$.

Theorem 1.9 (See Theorem 3.1)**.** *For asymptotically conical metrics with cross sections without conjugate points within distance $\pi/2$ and for suitable localizers $\tilde\chi$, the modified normal operator of the X-ray transform is an elliptic operator (for sufficiently small h and x) with respect to the semiclassical foliation 1-cusp algebra.*

Remark 1.10. The semiclassical foliation version of Remark 1.8 is applicable; in this case $\tilde{\chi} = \tilde{\chi}(x, y, \lambda/(h^{1/2}x^p), \omega)$ and $\Phi = -\frac{1}{2phx^{2p}}$. See Remark 3.3 for the key analytic reason.

Thus, we can construct a parametrix, whose error is actually small for small h, implying invertibility. This immediately gives

Theorem 1.11 (See Corollaries 3.4 and 3.5)**.** *For manifolds as specified above, the original geodesic X-ray normal operator and thus the X-ray transform itself, acting on functions with Gaussian decay, will have a trivial nullspace supported in $x < \bar{x}$.*

The final ingredient of the proof of our main theorem, Theorem 1.3, is to eliminate the support condition by working with a combined scattering-1 cusp algebra; see Section 3.4.

2. The 1-cusp algebra and its semiclassical version

We proceed to create a new pseudodifferential algebra, the 1-cusp algebra, by performing blow-ups on the Schwartz kernel double space, and also discuss it in terms of explicit quantizations. This pseudodifferential operator algebra is local on the underlying manifold with boundary M, unlike say Melrose's b-algebra, or more relevantly, the cusp algebra, and can thus be described by explicit quantization and diffeomorphism invariance considerations, much as the case of the scattering algebra. We also discuss, as in the case of the scattering and cusp algebras, the connection to a class of pseudodifferential operators on $\mathbb{R}^n$. Yet an alternative approach to a description of certain (limited) aspects of this algebra would be to follow the work of Amman, Lauter, Nistor [1], which uses Lie algebroids.

2.1. The scattering double space and the scattering pseudodifferential algebra.

As a convenience for the reader, we restate some basic definitions and properties of the scattering algebra, which serves as a potential starting point for our new algebra, and shares some properties with it. For further details, refer to Melrose's original paper introducing the scattering algebra [11].

Melrose defined the scattering algebra on general manifolds with boundary; a motivation is that the Laplacian of an asymptotically conic (in particular an asymptotically Euclidean) Riemannian metric is an element of this algebra. Let x be a boundary defining function of M; this is determined up to a smooth positive factor. Then $\mathcal{V}_{\mathrm{sc}}(M)$ consists of C^∞ vector fields of the form xV', $V' \in \mathcal{V}_{\mathrm{b}}(M)$, i.e. V' is a smooth vector field tangent to ∂M. In local coordinates (x, y) near ∂M, with y local coordinates on ∂M, scattering vector fields $\mathcal{V}_{\mathrm{sc}}$ are those vector fields generated, over $C^\infty(M)$, by $\{x^2\partial_x, x\partial_{y_1}, \dots, x\partial_{y_{n-1}}\}$, i.e. are of the form

$$a_0(x, y)(x^2\partial_x) + \sum_{j=1}^{n-1} a_j(x, y)(x\partial_{y_j}).$$

These are thus all smooth sections of a vector bundle, the scattering tangent bundle, ${}^{\mathrm{sc}}TM$, whose elements at any point $p \in M$ can be written as $\lambda(x^2\partial_x)+\sum_{j=1}^{n-1}\omega_j(x\partial_{y_j})$, i.e. (λ,ω) are coordinates on the fibers of this vector bundle. The scattering cotangent bundle is then the dual vector bundle, and we can thus write scattering covectors as

$$\tau\,\frac{dx}{x^2}+\mu\cdot\frac{dy}{x},$$

i.e. (τ,μ) are local coordinates on the fibers of ${}^{\mathrm{sc}}T^*M$, and (x,y,τ,μ) on ${}^{\mathrm{sc}}T^*M$ itself. While we keep this notation for the asymptotically conic geometric discussion, in the analytic context we will use the notation $(\xi_{\mathrm{sc}},\eta_{\mathrm{sc}}) = (\tau,\mu)$, i.e. covectors are written as

$$\xi_{\mathrm{sc}}\,\frac{dx}{x^2}+\eta_{\mathrm{sc}}\cdot\frac{dy}{x}.$$

The Schwartz kernel of a scattering pseudodifferential operator is a conormal distribution on the scattering double space M^2_{sc}, which is a blow-up, or resolution, of the standard double space $M^2 = M\times M$; see Figure 1. We recall that the blow-up of a product-type, or p-, submanifold of a manifold with corners is a new manifold with corners in which different *normal* directions of approach to the submanifold being blown up are distinguished; this process is thus an invariant generalized version of the introduction of spherical coordinates around a submanifold, i.e. of cylindrical coordinates. (Melrose's paper [11] contains details of the blow-up process; see also [13], or indeed [27, Section 5] in a context that will play a role in Section 2.4.) The double space is constructed by taking $M^2 = M\times M$ and then first blowing up the corner $(\partial M)^2$ to get the b double space, a manifold with corners, where $(\partial M)^2$ has been blown up into the *b front face*. In the interior of the b front face, near the diagonal, we have coordinates

$$x,y,\frac{x-x'}{x},y',$$

with y local coordinates on ∂M. The lifted diagonal $\{\frac{x-x'}{x}=0,\ y=y'\}$ only meets the interior of this b front face, i.e. $\frac{x-x'}{x}$ is bounded away both from 1 and $-\infty$ along it. (Near the lift of $\partial M\times M$, i.e. $\{x=0\}$, and $M\times\partial M$, i.e. $\{x'=0\}$, we need to use somewhat different coordinates, but being near these faces amounts to $\frac{x'}{x}$ tending to $+\infty$, resp. 0.) Then, a second blow up, of the boundary $\{x=0,\ \frac{x-x'}{x}=0,\ y=y'\}$, of the lifted diagonal is performed with the new front face being the *scattering front-face*. We obtain coordinates

$$x,y,X=\frac{x-x'}{x^2},Y=\frac{y-y'}{x},$$

near the interior. (One can also replace x,y by x',y', and below we discuss another possibility.)

The scattering algebra consists of operators whose Schwartz kernels on this blown-up double space are well-behaved, meaning they are C^∞ in the

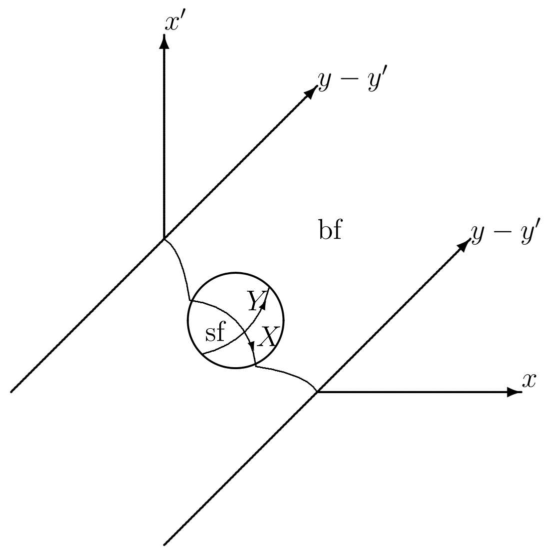

FIGURE 1. The scattering double space, with the b-front face, sf the scattering front face.

interior of M^2 away from the diagonal, are conormal to all boundary faces with infinite order vanishing at all of them except the scattering front face, and have a conormal singularity along the diagonal. Explicitly, the latter means that one can write the Schwartz kernels near the lifted diagonal intersecting the scattering front face, relative to the density $\frac{|dx'\,dy'|}{(x')^{n+1}}$, as

$$K_A(x,y,X,Y) = (2\pi)^{-n} \int e^{i(\xi_{\mathrm{sc}}X+\eta_{\mathrm{sc}}\cdot Y)} a(x,y,\xi_{\mathrm{sc}},\eta_{\mathrm{sc}})\, d\xi_{\mathrm{sc}}\, d\eta_{\mathrm{sc}}, \tag{2.1}$$

where $a \in S^{m,l}$ is a 'product type' symbol

$$\left|(x\partial_x)^j \partial_y^\alpha \partial_{\xi_{\mathrm{sc}}}^k \partial_{\eta_{\mathrm{sc}}}^\beta a(x,y,\xi_{\mathrm{sc}},\eta_{\mathrm{sc}})\right| \leq C_{jk\alpha\beta} \langle \xi_{\mathrm{sc}},\eta_{\mathrm{sc}}\rangle^{m-k-|\beta|} x^{-\ell}.$$

We emphasize that this description is a priori only valid in a neighborhood of the lifted diagonal, but in fact is also valid in a neighborhood of the scattering front face, though not globally. We give below a version that is global in charts $O \times O$, O open in M, via a reduction to $\overline{\mathbb{R}^n}$.

One very convenient feature of the scattering pseudodifferential algebra is that it can in fact be locally reduced to a standard Hörmander algebra [7] on $\mathbb{R}^n$, where 'locally' is understood on the radial compactification $\overline{\mathbb{R}^n}$, resp. the compact manifold with boundary M. Namely, taking 'product type' symbols $a \in S^{m,l}$ on $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$|\partial_z^\alpha \partial_\zeta^\beta a(z,\zeta)| \leq C_{\alpha\beta} \langle z\rangle^{l-|\alpha|} \langle \zeta\rangle^{m-|\beta|},$$

and defining the Schwartz kernel of the standard, say, left quantization,

$$K_A(z,z') = (2\pi)^{-n} \int e^{i(z-z')\cdot\zeta} a(z,\zeta)\, d\zeta \tag{2.2}$$

relative to the density $|dz'|$, the scattering algebra is obtained, modulo operators with a Schwartz class integral kernel on M^2, by identifying neighborhoods of points on M with corresponding neighborhoods on $\overline{\mathbb{R}^n}$, and pulling back the Schwartz kernel of an operator given by the just described left quantization. The principal symbol of A is defined as the equivalence class $[a]$ of a in $S^{m,l}/S^{m-1,l-1}$.

In order to connect the $\mathbb{R}^n$-based description to the geometric one, it is useful to use yet different coordinates near the scattering front face, namely

$$x,\ y,\ \tilde{X} = \frac{1}{x} - \frac{1}{x'} = \frac{x'-x}{xx'},\ \tilde{Y} = \frac{y}{x} - \frac{y'}{x'} = \frac{y-y'}{x} + \Big(\frac{1}{x} - \frac{1}{x'}\Big)y',$$

so

$$\tilde{X} = -\frac{x}{x'}X = -(1-xX)^{-1}X,$$
$$\tilde{Y} = Y - (1-xX)^{-1}Xy' = Y - (1-xX)^{-1}X(y-xY),$$

showing the smoothness of $\tilde{X}, \tilde{Y}$, and the reverse expressions are also similarly checked to be smooth. In particular, notice that $\tilde{X} = -X$, $\tilde{Y} = Y - Xy$ at $x = 0$. Hence (2.1) can be equally well-described as

$$K_A(x,y,X,Y) = (2\pi)^{-n} \int e^{i(\widetilde{\xi_{\rm sc}}\tilde{X} + \widetilde{\eta_{\rm sc}}\cdot\tilde{Y})} \tilde{a}(x,y,\widetilde{\xi_{\rm sc}},\widetilde{\eta_{\rm sc}})\, d\widetilde{\xi_{\rm sc}}\, d\widetilde{\eta_{\rm sc}}, \tag{2.3}$$

with $\tilde{a} \in S^{m,l}$ as well. To reduce this to the $\overline{\mathbb{R}^n}$ perspective, recall that where, say, $|z_n|$ is relatively large (and say z_n is positive), we can use z_n^{-1}, $\frac{z_j}{z_n}$ as coordinates on the radial compactification, with z_n^{-1} defining the boundary; the correspondence then is letting $x = z_n^{-1}$, $y_j = \frac{z_j}{z_n}$, $1 \leq j \leq n-1$, so $z_n = x^{-1}$, $z_j = y_j/x$, so (2.3) is in fact the same as (2.2), keeping in mind that $|dz'| = \frac{|dx'\,dy'|}{(x')^{n+1}}$. An advantage thus of (2.3) as well as (2.2) is that they are not restricted to the interior of the b-front face; they are also valid at the left and right faces, at least near the diagonal in M^2, i.e. on sets of the form $O \times O$, O a coordinate chart, thus eliminating dividing up treatments of the Schwartz kernels into several regions, such the interior of the scattering front face, the boundary of the scattering front face, etc.

One of the most significant features about the scattering algebra in contrast to its many relatives (such as the b-algebra [26] or the 0-algebra [10]) is that composition can be described algebraically in terms of symbols to leading order in every sense; this is immediate from the $\mathbb{R}^n$-based description above.

Proposition 2.1. *If $A \in \Psi_{\rm sc}^{m,\ell}$ with principal symbol, modulo $\Psi_{\rm sc}^{m-1,\ell-1}$, $[a] \in S^{m,\ell}/S^{m-1,\ell-1}$ and $B \in \Psi_{\rm sc}^{m',\ell'}$ with principal symbol $[b]$ then $A \circ B \in \Psi_{\rm sc}^{m+m',\ell+\ell'}$ with principal symbol $[a][b] = [ab]$.*

Proposition 2.2. *If $A \in \Psi_{\mathrm{sc}}^{m,\ell}$ with principal symbol a is elliptic, i.e. for some $c > 0$,*

$$|a(x, y, \xi_{\mathrm{sc}}, \eta_{\mathrm{sc}})| \geq C\langle \xi_{\mathrm{sc}}, \eta_{\mathrm{sc}}\rangle^m x^{-\ell} \text{ for } |(\xi_{\mathrm{sc}}, \eta_{\mathrm{sc}})| \gg 1 \text{ or } x \ll 1$$

then there is a parametrix $B \in \Psi_{\mathrm{sc}}^{-m,-\ell}$ with error $AB-I, BA-I \in \Psi_{\mathrm{sc}}^{-\infty,-\infty}$.

We define the usual weighted Sobolev spaces $H^{s,r}$ on $\mathbb{R}^n$ by imposing a weight: $H^{s,r} = \langle z\rangle^{-r} H^s$; these are then transported to the manifold via the just discussed identification to define the scattering Sobolev spaces $H_{\mathrm{sc}}^{s,r}$. Equivalently, for any real s, r, writing $\mathcal{S}'$ for the space of tempered distributions on M, i.e. the dual of C^∞ functions vanishing to infinite order at ∂M,

$$H_{\mathrm{sc}}^{s,r} = \{u \in \mathcal{S}' :\ \exists A \in \Psi_{\mathrm{sc}}^{s,r} \text{ elliptic and } Au \in L^2\},$$

where L^2 is with respect to a scattering density $\frac{|dx\, dy|}{x^{n+1}}$, which corresponds to $|dz|$ in the identification on $\mathbb{R}^n$.

These weighted Sobolev spaces can be used to describe the mapping properties of scattering pseudodifferential operators:

Proposition 2.3. *If $A \in \Psi_{\mathrm{sc}}^{m,\ell}$, then $A : H_{\mathrm{sc}}^{s,r} \to H_{\mathrm{sc}}^{s-m,r-\ell}$.*

Because the parametrix error is not only smoothing (order $-\infty$ in the differential sense) but includes a restriction on growth rates, the error is actually compact on any weighted Sobolev space and we can get desired Fredholm properties.

For example, $\sigma(\Delta+1) = \xi_{\mathrm{sc}}^2 + |\eta_{\mathrm{sc}}|^2 + 1$ is elliptic in the scattering algebra but $\sigma(\Delta - 1) = \xi_{\mathrm{sc}}^2 + |\eta_{\mathrm{sc}}|^2 - 1$ is not. Both of these operators are elliptic in the standard sense, but the scattering algebra explains why one operator has an infinite dimensional tempered distributional nullspace and the other does not.

2.2. **The cusp double space and algebra.** We now recall the definition of the cusp pseudodifferential algebra and its properties. It is defined on manifolds with boundary M with a boundary function x defined up to adding an element of $x^2 C^\infty(M)$, i.e. any other alternative boundary defining function for this structure is of the form $\tilde{x} = x + x^2\phi$, with $\phi \in C^\infty(M)$. In order to do so, we start by discussing the double space, as appears in the work of Mazzeo and Melrose [9]. Indeed, these authors provide a joint framework for the scattering and the cusp algebras within the class of 'fibred cusp' algebras. As a reference to the terminology of this paper we mention that in the scattering algebra case the corresponding boundary fibration is the identity map, while in the cusp algebra setting the boundary fibration is the map that sends every point on the boundary to a single point (so the fiber is the whole boundary).

The double space is obtained from $M^2 = M \times M$ by first performing the b-blow up, i.e. blowing up $(\partial M)^2$, as in the scattering setting, and then blowing up the lift of $x = x'$ at $x = 0$; see Figure 2. In valid coordinates on the b-double space near the lift of $x = x'$, thus in the interior of the b-face,

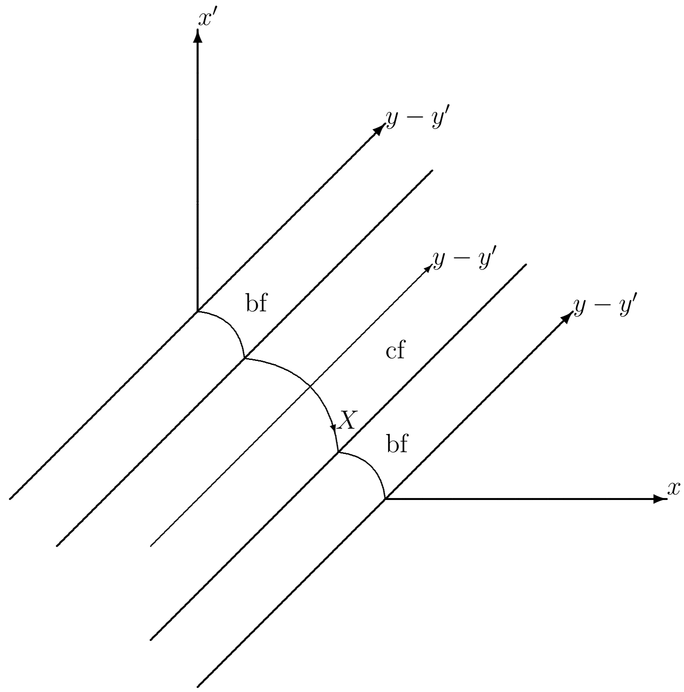

FIGURE 2. The cusp double space as a blow-up of the b-double space. The b font face is denoted by bf, while the cusp front face is denoted by cf.

coordinates are $x, y, \frac{x-x'}{x}, y'$, and the lift of $x = x'$ is this $\frac{x-x'}{x} = 0$. The result of this blow-up is to obtain coordinates

$$x, y, X = \frac{x-x'}{x^2}, y'$$

near the interior of the front face.

Here one needs to check that the submanifold being blown up is independent of the choice of x modulo $O(x^2)$. We now show that if we define a new boundary defining function $\tilde{x} = x + x^2\phi$ where ϕ is a smooth function, then this submanifold is unchanged. Pulling back $\tilde{x}$ from the left and right factors to the b-double space to get $\tilde{x}$ and $\tilde{x}'$,

$$\tilde{x} = x + x^2\phi(x, y) \qquad \tilde{x}' = x' + (x')^2\phi(x', y'),$$

so we can compute $\frac{x-x'}{x}$, the relevant b-front face coordinate in the new variables, to obtain

$$\begin{aligned}\frac{\tilde{x}-\tilde{x}'}{\tilde{x}} &= \frac{x-x'+x^2\phi(x,y)-(x')^2\phi(x',y')}{x(1+x\phi(x,y))}\\ &= \frac{x-x'}{x}(1+x\phi(x,y))^{-1}+x\frac{\phi(x,y)-\frac{(x')^2}{x^2}\phi(x',y')}{1+x\phi(x,y)}\\ &= \frac{x-x'}{x}(1+x\phi(x,y))^{-1}+x\frac{\phi(x,y)-(1-\frac{x-x'}{x})^2\phi(x',y')}{1+x\phi(x,y)}.\end{aligned}$$

Hence (since the blow-down maps are smooth, so x', y' are also smooth on the b-double space) $\frac{\tilde{x}-\tilde{x}'}{\tilde{x}}$ is smooth on the b-double space and its zero set at $x=0$ is exactly the same as that of $\frac{x-x'}{x}$. This means that the blow up creating the new double space produces the same space as we change from x to $\tilde{x}$ in the definition, and thus it is well-defined independent of such choices.

A cusp pseudodifferential operator of order m, ℓ then has a Schwartz kernel that is well-behaved on this double space in the sense that it is conormal to the new, cusp, front face away from the lifted diagonal, $\{X=0, y=y'\}$, vanishes to infinite order at all boundary faces except the cusp front face, is conormal up to the front face of order ℓ, and is conormal to the diagonal of order m. In particular, in a neighborhood of the diagonal it is given by an oscillatory integral

$$(2\pi)^{-n}\int e^{i(X\xi_{\mathrm{cu}}+(y-y')\eta_{\mathrm{cu}})}a(x,y,\xi_{\mathrm{cu}},\eta_{\mathrm{cu}})\,d\xi_{\mathrm{cu}}\,d\eta_{\mathrm{cu}}$$

relative to the density $\frac{|dx'\,dy'|}{(x')^2}$, where a is a symbol of order m, ℓ, i.e.

$$|(x\partial_x)^j\partial_y^\alpha\partial_{\xi_{\mathrm{cu}}}^k\partial_{\eta_{\mathrm{cu}}}^\beta a(x,y,\xi_{\mathrm{cu}},\eta_{\mathrm{cu}})| \le Cx^{-\ell}\langle(\xi_{\mathrm{cu}},\eta_{\mathrm{cu}})\rangle^{m-k-|\beta|}.$$

While, unlike the scattering algebra, the cusp algebra cannot be reduced modulo operators with Schwartz class integral kernels to a Hörmander algebra, in a somewhat weaker sense, that still captures the near diagonal behavior, it can. In this case the correspondence is with a different Hörmander algebra [7] on $\mathbb{R}^n$. Namely, taking symbols $a\in S_\infty^{m,l}$ on $\mathbb{R}^n\times\mathbb{R}^n$ such that

$$|\partial_z^\alpha\partial_\zeta^\beta a(z,\zeta)| \le C_{\alpha\beta}\langle z\rangle^l\langle z_n\rangle^{-\alpha_n}\langle\zeta\rangle^{m-|\beta|},$$

so the difference with the scattering case is that one only gains z_n decay upon differentiation in z_n (and no decay otherwise), and defining the Schwartz kernel of the standard, say, left quantization,

$$K_A(z,z') = (2\pi)^{-n}\int e^{i(z-z')\cdot\zeta}a(z,\zeta)\,d\zeta \tag{2.4}$$

relative to the density $|dz'|$, the cusp algebra is obtained in open sets of the form $O\times O\subset M^2$, where $O\subset M$ is identified with a similar open set (with compact closure, if desired) in $\mathbb{R}^{n-1}\times\overline{\mathbb{R}}$ via a diffeomorphism by pulling back the Schwartz kernel of an operator given by the just described left quantization. Notice that $\mathbb{R}^{n-1} \times \overline{\mathbb{R}}$ corresponds to a 'cylindrical end' perspective on $\mathbb{R}^n$, in which one coordinate, say the last one, is distinguished, and the remaining ones are required to stay in a bounded set. The reason this only captures the diagonal, thus differential order, behavior of the cusp algebra is that the cusp front face is global, i.e. includes points far from the diagonal in M^2. Indeed, we can remedy this by treating the off-diagonal behavior on M^2 by considering two disjoint open sets O, U in M, mapping them to disjoint open sets in $\mathbb{R}^{n-1} \times \overline{\mathbb{R}}$, and pulling back the Hörmander algebra Schwartz kernel from there. This perspective on the cusp algebra was explained in [26], and the equivalence is easily seen for $x = z_n^{-1}$ can be taken to be the coordinate near infinity in $\overline{\mathbb{R}}$, so with $y = (z_1, \dots, z_{n-1})$, $\eta_{1c} = (\zeta_1, \dots, \zeta_{n-1})$, the phase function in (2.4) can be written as

$$y \cdot \eta_{1c} + (x^{-1} - (x')^{-1})\zeta_n,$$

and

$$x^{-1} - (x')^{-1} = -\frac{x - x'}{xx'} = -\frac{x}{x'} X = -(1 - xX)^{-1} X,$$

which (or better yet, whose negative) could have equally well been used in the definition of the cusp algebra above. Note that the correspondence is $\zeta_n = -\xi_{1c}$, and the regularity of the amplitude a is in terms of $x\partial_x, \partial_y$, which equivalently means $z_n\partial_{z_n}, \partial_{z_j}, j = 1, \dots, n-1$, in the relevant region, $|z_j| < C$, $z_n > 1$.

Cusp pseudodifferential operators form a bi-filtered *-algebra as well under adjoints and composition, however, the principal symbol only captures the leading order behavior in the differential sense, thus is insufficient to capture compactness of operators on the corresponding cusp Sobolev spaces $H^{s,r}_{\mathrm{cu}}$. Indeed, these claims are immediate from the just-described connection with the Hörmander algebra, and were proved by Mazzeo and Melrose in [9] using geometric microlocal techniques.

Proposition 2.4. *If $A \in \Psi^{m,\ell}_{\mathrm{cu}}$ with principal symbol, modulo $\Psi^{m-1,\ell}_{\mathrm{cu}}$, $[a] \in S^{m,\ell}/S^{m-1,\ell}$ and $B \in \Psi^{m',\ell'}_{\mathrm{cu}}$ with principal symbol $[b]$ then $A \circ B \in \Psi^{m+m',\ell+\ell'}_{\mathrm{cu}}$ with principal symbol $[a][b] = [ab]$.*

Proposition 2.5. *If $A \in \Psi^{m,\ell}_{\mathrm{cu}}$ with principal symbol $[a]$ is elliptic, i.e. for some $c > 0$,*

$$|a(x, y, \xi_{\mathrm{cu}}, \eta_{\mathrm{cu}})| \geq c\langle \xi_{\mathrm{cu}}, \eta_{\mathrm{cu}} \rangle^m x^{-\ell} \text{ for } |(\xi_{\mathrm{cu}}, \eta_{\mathrm{cu}})| \gg 1$$

then there is a parametrix $B \in \Psi^{-m,-\ell}_{\mathrm{cu}}$ with error $AB - I, BA - I \in \Psi^{-\infty,0}_{\mathrm{cu}}$.

Proposition 2.6. *If $A \in \Psi^{m,\ell}_{\mathrm{cu}}$, then $A : H^{s,r}_{\mathrm{cu}} \to H^{s-m,r-\ell}_{\mathrm{cu}}$.*

Mazzeo and Melrose [9] define a normal operator to improve on this last result and thus obtain compact errors, but we shall not need this since our new algebra will have properties more akin to those of the scattering algebra.

2.3. **The 1-cusp double space and the cusp pseudodifferential operators.** The simplest way to obtain the 1-cusp double space is from the cusp one by blowing up the boundary of the lifted diagonal $\{X = 0, y - y' = 0\}$, i.e. $\{X = 0, y - y' = 0, x = 0\}$; see Figure 3. Notice that the lifted diagonal indeed only intersects the cusp front face (in particular does not intersect the b-front face), so local coordinates in the interior of the cusp front face can be used. Since this submanifold is purely geometric, it does not depend on any additional information beyond what went into the definition of the cusp double space, namely the boundary defining function defined up to $O(x^2)$ terms. Concretely, in a neighborhood of the interior of the front face x is relatively large, and we thus obtain coordinates

$$x, y, V = \frac{X}{x} = \frac{x - x'}{x^3}, Y = \frac{y - y'}{x}.$$

The Schwartz kernels of our new operators then are required to be well-behaved on the new double space in the sense that they are conormal to the new, 1-cusp, front face away from the lifted diagonal, $\{V = 0,\ Y = 0\}$, vanish to infinite order at all boundary faces, are conormal to the 1-cusp face of order ℓ and to the lifted diagonal of order m. In particular, in a neighborhood of the lifted diagonal they are given by an oscillatory integral

$$K_A(x, y, V, Y) = (2\pi)^{-n} \int e^{i(V\xi_{1c} + Y\eta_{1c})} a(x, y, \xi_{1c}, \eta_{1c})\, d\xi_{1c}\, d\eta_{1c} \tag{2.5}$$

relative to the density $\frac{|dx'\,dy'|}{(x')^{n+2}}$, which arises from Jacobian factors caused by the blow-ups and which we explain below, where a is a 'product type' symbol of order m, ℓ, i.e.

$$|(x\partial_x)^j \partial_y^\alpha \partial_{\xi_{1c}}^k \partial_{\eta_{1c}}^\beta a(x, y, \xi_{1c}, \eta_{1c})| \le C x^{-\ell} \langle(\xi_{1c}, \eta_{1c})\rangle^{m-k-|\beta|}. \tag{2.6}$$

On the other hand, away from the lifted diagonal but near the 1-cusp front face the Schwartz kernel satisfies estimates

$$|V^i Y^\gamma (x\partial_x)^j \partial_y^\alpha \partial_V^k \partial_Y^\beta K_A(x, y, V, Y)| \le C x^{-\ell}, \tag{2.7}$$

with C depending on the indices $i, j, k, \alpha, \beta, \gamma$, where $i, j, k \in \mathbb{N}$, $\alpha, \beta, \gamma \in \mathbb{N}^{n-1}$, which also encodes, via the powers of V, Y, the rapid decay to the cusp front face near the corner. Notice that as all the ingredients of the definition are diffeomorphism invariant, so is the class of 1-cusp pseudodifferential operators.

While strictly speaking, by the definition of conormal distributions, (2.5) is to be interpreted as a local oscillatory integral, i.e. valid with V, Y bounded, one *can* interpret it more globally. The reason is that by the basic properties of the Fourier transform, outside $V = 0$, $Y = 0$, it produces a Schwartz function in (V, Y) with values in conormal functions of (x, y), i.e. (2.7) holds for the right hand side of (2.5) regardless of m, ℓ.

We can shed some light on this algebra by also relating its double space to that of the scattering algebra; this relation is of some importance since the 1-cusp algebra itself arises for us in the setting of an asymptotically conic

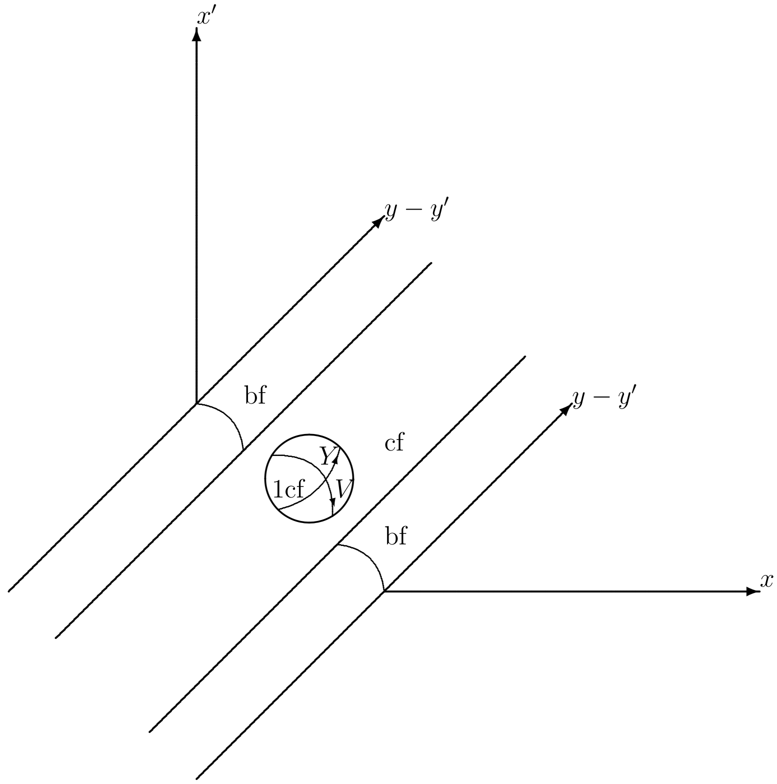

FIGURE 3. The 1-cusp double space as a blow-up of the cusp double space. The b font face is denoted by bf, the cusp front face is denoted by cf, while the 1-cusp front face by 1cf.

metric, which is naturally described by, and in particular has its bicharacteristics described by, the scattering geometry. Namely, for this perspective, within the scattering double space, one blows up the lift of $x = x'$, i.e. in local coordinates $\{x = 0, X = 0\}$, intersected with the scattering front face, $x = 0$. In the interior of the new front face this indeed produces local coordinates

$$x, y, V = \frac{X}{x}, Y,$$

matching those of the blow-up obtained from the cusp algebra, and establishing a natural local (in the region of validity of the two coordinates) diffeomorphism between the two spaces. A subtlety here, however, is that, unlike for the cusp approach above, the manifold we blow up intersects faces other than the scattering front face as well, namely the boundary of the scattering front face, so for a full discussion from this perspective valid coordinates must also be described and used in those regions. Another potential issue, which however is easily resolved using a straightforward modification of the above computation that the cusp algebra is well-defined, is that we need to check that the submanifold being blown up is well defined if x is only well-defined up to adding $O(x^2)$ terms. In any case, this well-definedness statement follows from the just established diffeomorphism, at least in the interior of the new front faces. The figure below represents this new double-space.

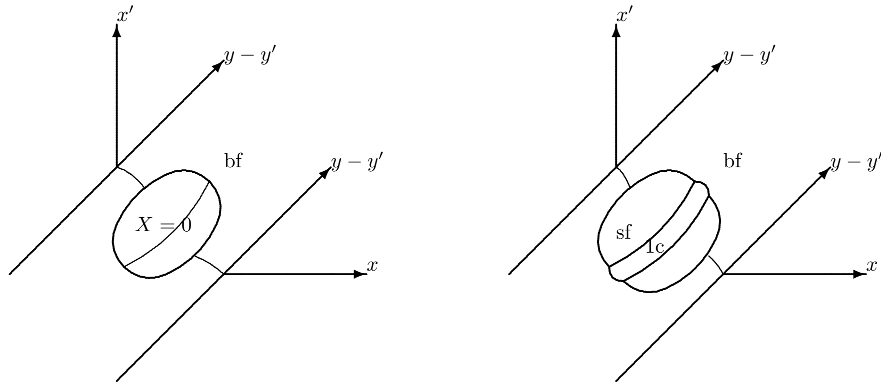

FIGURE 4. 1-cusp double space by a blow-up of the scattering double space: on the left the submanifold $X = 0$ of the scattering front face, sf, and on the right the resulting resolution, with the front face labelled by 1c. A neighborhood of the interior of the front face is naturally diffeomorphic to the front face 1cf shown in Figure 3 in the sense that the identity map between the interiors of M^2 smoothly extends up to these boundary faces, but the boundary of the two front faces is quite different.

The shaded portion in Figure 4 corresponds to the new front face, and V and Y are coordinates for this new front face (along with y, not shown in this picture).

While the identity map in the interior does not induce a global diffeomorphism between the space we just obtained and our 1-cusp double space, for instance due to intersection of the 1-cusp front face intersecting the b front face (unlike from the cusp approach), the space of conormal distributions, conormal to the diagonal and the front face, and vanishing to infinite order at every other face, is the same. Thus, we can consider the 1-cusp pseudodifferential operators both in relation to the cusp ones and to the scattering ones.

This definition is thus analogous to the geometric definition of scattering and cusp pseudodifferential operators earlier. As with scattering and cusp operators, smoothing operators (elements of $\Psi_{1c}^{-\infty,\ell}$) have Schwartz kernels without a conormal singularity along the diagonal, and residual operators (elements of $\Psi_{1c}^{-\infty,-\infty}$) have Schwartz kernels which additionally vanish to infinite order on the new front face. Therefore residual operators have Schwartz kernels which are also residual on the blown-down scattering or cusp space, so that they are residual scattering and cusp operators.

As already mentioned, if $A \in \Psi_{1c}^{-\infty,\ell}$, K_A satisfies (2.7), say near the interior of the cusp face, i.e. where $x|V|, x|Y| < C$, and if v is conormal of order r, i.e. $|(x\partial_x)^j \partial_y^\alpha v| \le C_{j\alpha} x^{-r}$, then

$$\begin{aligned}(Av)(x,y) &= \int K_A\Big(x, y, \frac{x-x'}{x^3}, \frac{y-y'}{x}\Big) v(x',y') \, \frac{dx'\, dy'}{(x')^{n+2}} \\ &= \int K_A(x,y,V,Y) v(x - x^3 V, y - xY) \, dV \, dY,\end{aligned}$$

which is immediately seen (by applying products of $x\partial_x$ and ∂_y, and using the rapid decay of K_A in V, Y) to be conormal of order $r + \ell$ (and indeed C^∞ if K_A and v are actually smooth), i.e. the orders add up; the Jacobian in the change of variables is the reason for the normalization of the density that we have adopted.

Just as in the scattering and cusp cases, there is a simple way to reduce this to an algebra in $\mathbb{R}^n$. In this case

$$x, y, \tilde{V} = \frac{1}{x^2} - \frac{1}{(x')^2}, \ \tilde{Y} = \frac{y}{x} - \frac{y'}{x'}$$

also give valid coordinates in the interior of the 1c-front face. Indeed,

$$\tilde{V} = \frac{(x'-x)(x'+x)}{x^2 (x')^2} = -V \frac{x(x'+x)}{(x')^2} = -V \frac{2 - x^2 V}{(1 - x^2 V)^2}$$

as $\frac{x'}{x} = 1 - x^2 V$, and

$$\tilde{Y} = \frac{y - y'}{x} + \Big(\frac{1}{x'} - \frac{1}{x}\Big) y' = Y + xV(1 - x^2 V)^{-1}(y - xY),$$

and indeed at $x = 0$, we have $\tilde{V} = -2V$, $\tilde{Y} = Y$; the converse direction is similar. Thus, we can equivalently write in place of (2.5)

$$K_A(x,y,\tilde{V},\tilde{Y}) = (2\pi)^{-n} \int e^{i\widetilde{\xi_{1c}}\tilde{V} + \widetilde{\eta_{1c}}\cdot\tilde{Y}} \tilde{a}(x, y, \widetilde{\xi_{1c}}, \widetilde{\eta_{1c}}) \, d\widetilde{\xi_{1c}} \, d\widetilde{\eta_{1c}}, \tag{2.8}$$

with $\tilde{a} \in S^{m,l}$, i.e. satisfying the same kinds of 'product type' symbol estimates (2.6). But then with $z_n = 1/x^2$, $z_j = y_j/x$ ($j = 1, \ldots, n-1$), so $x = z_n^{-1/2}$, $y_j = z_j / z_n^{1/2}$, in the region where (x, y_j) is bounded, i.e. z_n bounded away from 0 and $|z_j| < C z_n^{1/2}$, this amounts to exactly an oscillatory integral of the form

$$(2\pi)^{-n} \int e^{i(\zeta_n(z_n - z_n') + \sum_{j=1}^{n-1} \zeta_j'(z_j - z_j'))} \tilde{a}(z,\zeta) \, d\zeta, \tag{2.9}$$

relative to the density $|dz'| = 2\frac{|dx'\, dy'|}{(x')^{n+2}}$, with $\tilde{a}$ well-behaved (conormal) in terms of $z_n^{-1/2}$, $z_j/z_n^{1/2}$, i.e. in a parabolic compactification, with the relevant region being $z_n > 1$, $|z_j| < C z_n^{1/2}$, $j = 1, \ldots, n-1$. Concretely, as

$$x\partial_x = -2z_n\partial_{z_n} - \sum_{j=1}^{n-1} z_j\partial_{z_j} = -2z_n\partial_{z_n} - \sum_{j=1}^{n-1} \frac{z_j}{z_n^{1/2}} z_n^{1/2}\partial_{z_j}$$

and

$$\partial_{y_j} = z_n^{1/2}\partial_{z_j},$$

this means that, in this region, iterated regularity of $\tilde{a}$ with respect to $x\partial_x$ and ∂_{y_j} is equivalent to that with respect to $z_n\partial_{z_n}$ and $z_n^{1/2}\partial_{z_j}$. One can instead work globally (using $\langle z\rangle$ in place of z_n) with symbol estimates

$$|\partial_z^\alpha \partial_\zeta^\beta \tilde{a}(z,\zeta)| \leq C_{\alpha\beta}\langle\zeta\rangle^{m-|\beta|}\langle z\rangle^{\ell/2-|\alpha|/2-\alpha_n/2}. \tag{2.10}$$

Here the $\ell/2$ in the power of $\langle z\rangle$ arises from $z_n = x^{-2}$ in the relevant region, so this power is locally equivalent to $x^{-\ell}$. We define (2.10) as the parabolic symbol class $S^{m,\ell/2}_{\mathrm{para}}$, and write the corresponding pseudodifferential operators, via (2.9) as $\Psi^{m,\ell/2}_{\mathrm{para}}$. Notice that the basis for 1-cusp vector fields (over C^∞ functions of x, y, or equivalently of $z_n^{-1/2}$, $z_j/z_n^{1/2}$) is

$$x^3\partial_x = -2\partial_{z_n} - \sum_{j=1}^{n-1} \frac{z_j}{z_n^{1/2}} z_n^{-1/2}\partial_{z_j},\ x\partial_{y_j} = \partial_{z_j},\ j = 1, \ldots, n-1, \tag{2.11}$$

which is equivalent to $\partial_{z_1}, \ldots, \partial_{z_n}$.

2.4. **Algebraic properties.** Given the identification of $\Psi^{m,\ell}_{1\mathrm{c}}$ locally with the pseudodifferential operators $\Psi^{m,\ell/2}_{\mathrm{para}}$ on $\mathbb{R}^n$, the algebra properties of $\Psi_{1\mathrm{c}}$ follow immediately from those of Ψ_{para}. The latter in turn are immediate with the standard composition, etc, formulae on $\mathbb{R}^n$, applicable even in Hörmander's algebra Ψ_∞ (with just uniform z estimates, without decay on differentiation). Note that the principal symbol of $A \in \Psi^{m,\ell}_{1\mathrm{c}}$ needs to be understood modulo $S^{m-1,\ell-1}$, for this corresponds to the statement that for $\tilde{A} \in \Psi^{m,\ell/2}_{\mathrm{para}}$, the principal symbol is in $S^{m,\ell/2}/S^{m-1,\ell/2-1/2}$ in view of the defining estimate and the standard symbol expansion.

Proposition 2.7. *If $A \in \Psi^{m,\ell}_{1\mathrm{c}}$ with principal symbol, modulo $\Psi^{m-1,\ell-1}_{1\mathrm{c}}$, $[a] \in S^{m,\ell}/S^{m-1,\ell-1}$ and $B \in \Psi^{m',\ell'}_{1\mathrm{c}}$ with principal symbol $[b]$ then $A \circ B \in \Psi^{m+m',\ell+\ell'}_{1\mathrm{c}}$ with principal symbol $[a][b] = [ab]$.*

As usual, this implies that there is a parametrix for elliptic operators:

Proposition 2.8. *If $A \in \Psi^{m,\ell}_{1\mathrm{c}}$ with principal symbol a is elliptic, i.e. for some $c > 0$,*

$$|a(x,y,\xi_{1\mathrm{c}},\eta_{1\mathrm{c}})| \geq C\langle\xi_{1\mathrm{c}},\eta_{1\mathrm{c}}\rangle^m x^{-\ell} \text{ for } |(\xi_{1\mathrm{c}},\eta_{1\mathrm{c}})| \gg 1 \text{ or } x \ll 1$$

then there is a parametrix $B \in \Psi^{-m,-\ell}_{1\mathrm{c}}$ with error in $\Psi^{-\infty,-\infty}_{1\mathrm{c}}$.

The positive integer order 1-cusp Sobolev spaces can be defined via regularity with respect to the 1-cusp vector fields, i.e. $u \in H^s_{1c}$ if $u \in L^2$ (relative to a 1-cusp density, $\frac{|dx\,dy|}{x^{n+2}}$), and $V_1 \dots V_j u \in L^2$ as well if $j \leq s$ and $V_j \in \mathcal{V}_{1c}$; the negative integer ones then can be defined via duality. The weighted spaces $H^{s,r}_{1c}$ are $x^r H^s_{1c}$. Equivalently, we can say, for any real s, r, writing $\mathcal{S}'$ for the space of tempered distributions on M, i.e. the dual of C^∞ functions vanishing to infinite order at ∂M,

$$H^{s,r}_{1c} = \{u \in \mathcal{S}' : \exists A \in \Psi^{s,r}_{1c} \text{ elliptic and } Au \in L^2\}.$$

In view of the identification of the 1-cusp vector fields with those on $\mathbb{R}^n$, (2.11), respectively that of the pseudodifferential algebras, on $\mathbb{R}^n$ these Sobolev spaces correspond to the standard weighted Sobolev space $H^{s,r/2}$, and the following mapping result is immediate.

Proposition 2.9. *If* $A \in \Psi^{m,\ell}_{1c}$, *then* $A : H^{s,r}_{1c} \to H^{s-m,r-\ell}_{1c}$.

We comment on a different way of analyzing the 1-cusp algebra by relating it to the cusp algebra at a symbolic level. This relationship is exactly the same as that of the scattering (which is 1-b from this perspective) and b-algebras, as explained in the second microlocal discussion of [27, Section 5], thus we will be brief.

Concretely, the Schwartz kernels of the cusp operators are, near the diagonal, given by oscillatory integrals of the form

$$(2\pi)^{-n} \int e^{i(X\xi_{\mathrm{cu}} + (y-y')\eta_{\mathrm{cu}})} a^{\mathrm{cu}}(x, y, \xi_{\mathrm{cu}}, \eta_{\mathrm{cu}})\, d\xi_{\mathrm{cu}}\, d\eta_{\mathrm{cu}},$$

relative to the density $\frac{|dx'\,dy'|}{(x')^2}$, while those of the 1-cusp ones have the form

$$(2\pi)^{-n} \int e^{i(V\xi_{1c} + Y\eta_{1c})} a^{1c}(x, y, \xi_{1c}, \eta_{1c})\, d\xi_{1c}\, d\eta_{1c},$$

relative to the density $\frac{|dx'\,dy'|}{(x')^{n+2}}$. These are the same, however, if we write $\xi_{1c} = x\xi_{\mathrm{cu}}$, $\eta_{1c} = x\eta_{\mathrm{cu}}$, up to an overall factor of $(x/x')^n$ (which is irrelevant for the class, and is identically 1 on the cusp front face). This simply corresponds to the natural coordinates on the cotangent bundles:

$$\xi_{\mathrm{cu}} \frac{dx}{x^2} + \eta_{\mathrm{cu}} \cdot dy = \xi_{1c} \frac{dx}{x^3} + \eta_{1c} \cdot \frac{dy}{x}.$$

This means that geometrically one can (almost, as we explain) obtain the 1-cusp symbol space from the cusp one by blowing up the corner of the fiber-compactified cusp cotangent bundle; see Figure 5. Indeed, coordinates there, where say $|\xi_{\mathrm{cu}}|$ is large relative to $|\eta_{\mathrm{cu}}|$, are

$$x, y, |\xi_{\mathrm{cu}}|^{-1}, \frac{\eta_{\mathrm{cu}}}{|\xi_{\mathrm{cu}}|},$$

the corner is $x = 0, |\xi_{\mathrm{cu}}|^{-1} = 0$, so in the region where x is relatively large (relative to the other defining function, $|\xi_{\mathrm{cu}}|^{-1} = 0$, of the submanifold being blown up), on the blown up space the coordinates become

$$x, y, |\xi_{\mathrm{cu}}|^{-1}/x, \frac{\eta_{\mathrm{cu}}}{|\xi_{\mathrm{cu}}|},$$

which are exactly

$$x, y, |\xi_{1\mathrm{c}}|^{-1}, \frac{\eta_{1\mathrm{c}}}{|\xi_{1\mathrm{c}}|},$$

i.e. coordinates near the corner of the fiber-compactified 1-cusp cotangent bundle. The 'almost' in the identification refers to the region where $|\xi_{\mathrm{cu}}|^{-1}$ is relatively large, where valid coordinates are

$$x/|\xi_{\mathrm{cu}}|^{-1}, y, |\xi_{\mathrm{cu}}|^{-1}, \frac{\eta_{\mathrm{cu}}}{|\xi_{\mathrm{cu}}|},$$

which are

$$|\xi_{1\mathrm{c}}|, y, x|\xi_{1\mathrm{c}}|^{-1}, \frac{\eta_{1\mathrm{c}}}{|\xi_{1\mathrm{c}}|}.$$

But these are valid coordinates near the corner if one blows up the boundary of the zero section of the 1-cusp cotangent bundle. All remaining regions are handled similarly, cf. [27, Section 5].

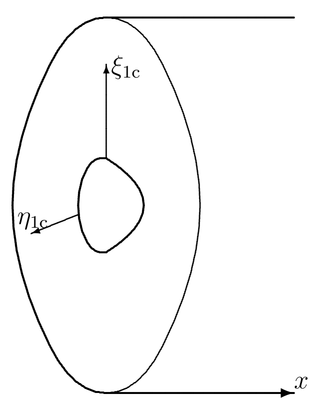

Figure 5. The resolution of the corner in the compactified cusp cotangent bundle, i.e. the symbol space. The curved boundary hypersurface in the center at $x = 0$ is the original boundary fiber of the cusp cotangent bundle. With the corner blown up, one obtains a new front face which is naturally diffeomorphic to the fiber of the 1-cusp cotangent bundle blown up at its zero section.

In particular, 1-cusp pseudodifferential symbols can be considered as cusp ones well-behaved on the corner-blown up cusp space. But blowing up the corner of a manifold with corners does not change the space of conormal functions. Thus, much as in the scattering-b relation of [27, Section 5], we can consider the 1-cusp operator symbols as conormal (non-classical) cusp operators symbols, and simply apply the results from the cusp algebra. Concretely,

$$\Psi_{1c}^{m,\ell} \subset \Psi_{cu}^{m,\max(\ell,\ell-m)} \cap \Psi_{cu}^{\max(m,0),\ell},$$

which in the 'base case' of $\Psi_{1c}^{0,0}$ is simply the statement $\Psi_{1c}^{0,0} \subset \Psi_{cu}^{0,0}$.

In fact, it is better yet to consider the 3-ordered second microlocal algebra with 3 orders from the resolved cusp cotangent bundle perspective: lifted fiber infinity, the lifted boundary and the front face. The resulting space is then a tri-filtered *-algebra by the cusp results, and $\Psi_{1c}^{m,\ell} \subset \Psi_{1c,cu}^{m,\ell,\ell}$.

Now, in order to deal with $\Psi_{1c}^{m,\ell}$ itself in this manner without the previous discussion, we need to 'blow down' the cusp face in the 2-microlocalized cusp cotangent bundle, i.e. show that the algebraic properties descend to the 'blown down space', namely the 1-cusp cotangent bundle, so that the composition of elements of $\Psi_{1c}^{m,\ell}$ and $\Psi_{1c}^{m',\ell'}$ is not merely in the 2-microlocal algebra, but in $\Psi_{1c}^{m+m',\ell+\ell'}$ itself. But modulo $\Psi_{1c}^{-\infty,\ell}$, we can write elements of $\Psi_{1c}^{m,\ell}$ as quantizations of symbols supported in $|(\xi_{1c}, \eta_{1c})| \geq 1$, and thus in the 2-microlocal space away from the cusp face. The cusp composition results imply that the composition of any two so-microlocalized elements of the cusp algebra in fact results in a similarly microlocalized element thus yielding an element of $\Psi_{1c}^{m+m',\ell+\ell'}$, and one then only needs to prove composition results for smoothing operators, from which we refrain here.

2.5. **The semiclassical version of the algebra.** In order to make the errors in the elliptic parametrix construction not just compact but small we also need a semiclassical version of the algebra, and indeed a semiclassical foliation algebra. In the standard and scattering settings the latter was introduced in [28] for the same reason. Here the foliation is given by level sets of the boundary defining function x, so now we regard x as fixed (not just up to adding $O(x^2)$ terms).

In the $\mathbb{R}^n$ version, the semiclassical foliation quantization takes the form

$$(2\pi)^{-n} h^{-n/2-1/2} \int e^{i(\zeta_n(z_n-z_n')/h+\sum_{j=1}^{n-1}\zeta_j'(z_j-z_j')/h^{1/2})} \tilde{a}(z,\zeta)\, d\zeta, \tag{2.12}$$

with symbols still satisfying

$$|\partial_z^\alpha \partial_\zeta^\beta \tilde{a}_h(z,\zeta)| \leq C_{\alpha\beta} \langle\zeta\rangle^{m-|\beta|} \langle z\rangle^{\ell/2-|\alpha|/2-\alpha_n/2}. \tag{2.13}$$

This could be regarded as a standard semiclassical quantization, i.e. where both powers of h in the exponent are 1 and the overall pre-factor is h^{-n}, with a worse behaved symbol, but the present version gives more precise algebraic properties. This is transferred over to the manifold M as in the non-semiclassical setting, ensuring that in M^2 away from the diagonal the Schwartz kernel not only vanishes to infinite order at the boundary, like in the 1-cusp case, but also to infinite order in h. With this definition the standard composition results on $\mathbb{R}^n$ yield the following results.

Proposition 2.10. *If* $A \in \Psi^{m,\ell}_{1c,\hbar,\mathcal{F}}$ *with principal symbol, modulo* $h^{1/2}\Psi^{m-1,\ell-1}_{1c,\hbar,\mathcal{F}}$, $[a] \in S^{m,\ell}/h^{1/2}S^{m-1,\ell-1}$ *and* $B \in \Psi^{m',\ell'}_{1c,\hbar,\mathcal{F}}$ *with principal symbol* $[b]$ *then* $A \circ B \in \Psi^{m+m',\ell+\ell'}_{1c,\hbar,\mathcal{F}}$ *with principal symbol* $[a][b] = [ab]$.

Here the gain of $h^{1/2}$ in the symbol algebra corresponds to the $h^{1/2}$ appearing with the foliation tangent variables, y or z_j, $j = 1, \dots, n-1$. As usual, this implies that there is a parametrix for elliptic operators:

Proposition 2.11. *If* $A \in \Psi^{m,\ell}_{1c,\hbar,\mathcal{F}}$ *with principal symbol* a *is elliptic, i.e. for some* $c > 0$,

$$|a(x, y, \xi_{1c}, \eta_{1c})| \geq C\langle \xi_{1c}, \eta_{1c}\rangle^m x^{-\ell} \text{ for } |(\xi_{1c}, \eta_{1c})| \gg 1 \text{ or } x \ll 1 \text{ or } h \ll 1$$

then there is a parametrix $B \in \Psi^{-m,-\ell}_{1c,\hbar,\mathcal{F}}$ *with error in* $h^\infty \Psi^{-\infty,-\infty}_{1c,\hbar,\mathcal{F}}$.

The triviality of the error, reflecting by the h^∞ factor, is what gives the smallness of the error for h sufficiently small, and thus the invertibility of A_h in that case.

The positive integer order semiclassical foliation 1-cusp Sobolev spaces are the same as the 1-cusp spaces, but with respect to an h-dependent norm. They can be defined via regularity with respect to the corresponding foliation semiclassical 1-cusp vector fields, $V \in \mathcal{V}_{1c,\hbar,\mathcal{F}}$, which are of the form

$$a_0(hx^3\partial_x) + \sum_{j=1}^{n-1} a_j(h^{1/2}x\partial_{y_j}).$$

Thus, the norm is locally (and in general by a partition of unity) given by

$$\|u\|^2_{H^s_{1c,\hbar,\mathcal{F}}} = \|u\|^2_{L^2} + \sum_{j+|\alpha|\leq s} \|(hx^3\partial_x)^j(h^{1/2}x\partial_{y_j})^\alpha u\|^2_{L^2};$$

the negative integer ones then can be defined via duality. The weighted norms $H^{s,r}_{1c,\hbar,\mathcal{F}}$ are those corresponding to $x^r H^s_{1c,\hbar,\mathcal{F}}$. Equivalently, we can say, for any real s, r, with $s, r \geq 0$, say,

$$\|u\|^2_{H^{s,r}_{1c,\hbar,\mathcal{F}}} = \|u\|^2_{L^2} + \|Au\|^2_{L^2},$$

where $A \in \Psi^{s,r}_{1c,\hbar,\mathcal{F}}$ is elliptic. This gives

Proposition 2.12. *If* $A \in \Psi^{m,\ell}_{1c,\hbar,\mathcal{F}}$, *then* $A : H^{s,r}_{1c,\hbar,\mathcal{F}} \to H^{s-m,r-\ell}_{1c,\hbar,\mathcal{F}}$.

3. Invertibility of the X-ray transform

Our main technical result concerns a modified normal operator for the X-ray transform, namely

$$A = e^{-\Phi/h} L\tilde{\chi} I e^{\Phi/h},$$

where

$$(Lw)(z) = \int_{S_z M} w(z, v)|d\sigma_z(v)|,$$

$\tilde{\chi} = \tilde{\chi}(z, \lambda/(h^{1/2}x), \omega)$, $\Phi = \frac{-1}{2x^2}$ is a Gaussian weight, and σ is a smooth positive measure on S_zM, i.e. in (λ, ω). Here we take a different normalization of L than [30], which introduces an additional x^{-1} factor in L, making the decay order 0 (rather than -1) for A in the following theorem.

Theorem 3.1. *The modified normal operator A of I is an operator in $h\Psi^{-1,-1}_{1\mathrm{c},\hbar,\mathcal{F}}$. Furthermore, for a suitable choice of $\tilde{\chi}$, its principal symbol is elliptic in a collar neighborhood of ∂M both in the sense of the 1-cusp algebra and semiclassically.*

The proof will take up Sections 3.1-3.2. Then in Section 3.3 we derive some immediate consequences, and then in Section 3.4 we introduce the artificial boundary method in this context to prove Theorem 3.7.

3.1. **Structure of the geodesics.**

3.1.1. *The reparameterization, with new parameter t, to make H_g nonvanishing at ∂M.* In order to get started, first we need to describe the geodesics in some detail. They satisfy Hamilton's equation of motion, i.e. have tangent vector given by the Hamilton vector field H_g of the dual metric function. The dual metric function is a symbol on ${}^{\mathrm{sc}}T^*M$, of order $(2, 0)$, which is elliptic away from the 0-section. Thus, H_g is of the form x times a b-vector field; this structure is generally the case of symbols of order $(2, 0)$ on ${}^{\mathrm{sc}}T^*M$, as follows from [11]. Indeed, as we already mentioned, explicitly

$$\frac{1}{2}H_g = x\Big(\tau(x\partial_x + \mu \cdot \partial_\mu) - |\mu|^2\partial_\tau + \frac{1}{2}H_h + xV\Big),$$

where $V \in \mathcal{V}_{\mathrm{b}}({}^{\mathrm{sc}}T^*M)$. Here and below, for convenience, we use a product decomposition of a neighborhood of ∂M respecting the preferred boundary defining function x; the concrete choice is irrelevant. Also, in this geometric discussion we write covectors as

$$\tau\frac{dx}{x^2} + \mu \cdot \frac{dy}{x}.$$

Correspondingly, it is useful to consider integral curves $\tilde{\gamma}$ of ${}^{\mathrm{sc}}H_g = x^{-1}H_g$. If the actual bicharacteristics are c, then the relationship is via the reparameterization of the integral curves via $\frac{dt}{ds} = x(c(s))$, i.e. $\frac{ds}{dt} = x(\tilde{\gamma}(t))^{-1}$; thus the X-ray transform can be rewritten in terms of $\tilde{\gamma}$, or its projection γ to the base manifold, as

$$If(\gamma) = \int f(\gamma(t))\, x(\gamma(t))^{-1}\, dt, \tag{3.1}$$

i.e. as a weighted X-ray transform. Later on we shall make a further change of parameterization to deal with the Schwartz kernel of the operator as $t \to \pm\infty$ (and thus $x(\gamma(t)) \to 0$).

3.1.2. *Finite t behavior of the reparameterized geodesics.* Along the integral curves $\tilde{\gamma} = \tilde{\gamma}(t)$ of $\frac{1}{2}{}^{\mathrm{sc}}H_g$, $\frac{dx}{dt} = \tau x + x^2 f_1$, f_1 smooth, and hence

$$\frac{d^2x}{dt^2} = (-|\mu|^2 + \tau^2)x + x^2 f_2,$$

with f_2 smooth, so along the unit level set of the dual metric function,

$$\frac{d^2x}{dt^2} = (-2|\mu|^2 + 1)x + x^2 f_2, \tag{3.2}$$

which is negative, with an upper bound given by x times a negative constant, where $|\mu| > 3/4$ say, provided x is sufficiently small. In particular, if $\frac{dx}{dt} = 0$, so $\tau = O(x)$, $\frac{d^2x}{dt^2} < 0$ showing the concavity of the level sets of x from the sublevel sets. In general we will work in a neighborhood $x < x_0$ of infinity in which this concavity statement holds.

In fact, one can have even stronger convexity by working with x^{-2} in place of x as $\frac{dx^{-2}}{dt} = -2x^{-2}(\tau + x f_1)$, hence

$$\frac{d^2x^{-2}}{dt^2} = 2x^{-2}(2\tau^2 + |\mu|^2 + x f_2),$$

which is positive, with an x^{-2} times a positive lower bound on the characteristic set for x small.

Now, ${}^{\mathrm{sc}}H_g$ being a b-vector field, it is a linear combination of $x\partial_x$, ∂_y, ∂_τ and ∂_μ. In view of this, it is useful to write the tangent vector to the projected bicharacteristic γ, which is thus the pushforward of ${}^{\mathrm{sc}}H_g$ to the base manifold, as a b-tangent vector,

$$\lambda x\partial_x + \omega\partial_y.$$

Notice that in fact λ is independent of the choice of the product decomposition respecting the preferred boundary defining function x, i.e. if (x', y') are other coordinates with $x' = x$, then

$$\lambda x\partial_x + \omega\partial_y = \lambda(x'\partial_{x'}) + \omega'\partial_{y'}.$$

Now writing the x, resp. y, component of γ as $\gamma^{(1)}$, resp. $\gamma^{(2)}$, as before, by the smoothness of the flow and as it is tangent to $x = 0$, so it preserves $x = 0$, we have

$$\gamma^{(1)}_{x,y,\lambda,\omega}(t) = x\tilde{\Gamma}^{(1)}(x, y, \lambda, \omega, t)$$

with $\tilde{\Gamma}^{(1)}$ smooth as

$$\gamma^{(1)}_{x,y,\lambda,\omega}(t) = x\int_0^1 \partial_1\gamma^{(1)}_{\sigma x,y,\lambda,\omega}(t)\,d\sigma,$$

with ∂_1 denoting derivative in the first subscript slot, and

$$\tilde{\Gamma}^{(1)}(x, y, \lambda, \omega, 0) = 1,\ \tilde{\Gamma}^{(1)}(0, y, \lambda, \omega, t) = \partial_1\gamma^{(1)}_{0,y,\lambda,\omega}(t).$$

Now, γ having tangent vector $\lambda x\partial_x+\omega\partial_y$ means that $\frac{d}{dt}\tilde\Gamma^{(1)}(x,y,\lambda,\omega,0)=\lambda$. Taylor expanding $\tilde\Gamma^{(1)}$ further in t, we have

$$\gamma^{(1)}_{x,y,\lambda,\omega}(t)=x(1+\lambda t+\alpha(x,y,\lambda,\omega)t^2+t^3\Gamma^{(1)}(x,y,\lambda,\omega,t)), \tag{3.3}$$

with $\Gamma^{(1)}$ smooth. Now, the sublevel sets $\{x<x_0\}$ of x (neighborhoods of infinity) are assumed to be geodesically concave, so bicharacteristics tangent to the level set $x=x_0$, i.e. with $\lambda=0$, satisfy $\frac{d^2}{dt^2}\gamma^{(1)}_{x,y,0,\omega}(t)<0$, so $\alpha<0$ when $\lambda=0$, cf. the discussion after (3.2) for seeing that for x sufficiently small, $\frac{d^2}{dt^2}\gamma^{(1)}_{x,y,0,\omega}(t)$ is bounded from above by x times a negative constant. Since the metric is Riemannian, it gives rise to an identification between ${}^{\mathrm{sc}}T^*M$ and ${}^{\mathrm{sc}}TM$; the pushforward of H_g to the base manifold is simply the sc-covector at which we are doing the pushforward so identified with a sc-tangent vector, and thus the pushforward of ${}^{\mathrm{sc}}H_g=x^{-1}H_g$ to the base is the sc-covector identified as a b-covector via division by x. In particular, for a warped product-type sc-metric we have $g=\tau^2+H(y,\mu)$, and then the pushforward of ${}^{\mathrm{sc}}H_g$ from (x,y,τ,μ) is $\tau(x\partial_x)+H(y)(\mu,.)$, i.e. $\lambda=\tau$, and ω is the standard identification of μ with a tangent vector on the cross section ∂X.

Thus, in summary,

$$\begin{aligned}\gamma_{x,y,\lambda,\omega}(t)&=(\gamma^{(1)}_{x,y,\lambda,\omega}(t),\gamma^{(2)}_{x,y,\lambda,\omega}(t))\\&=(x+x(\lambda t+\alpha t^2+t^3\Gamma^{(1)}(x,y,\lambda,\omega,t)),y+\omega t+t^2\Gamma^{(2)}(x,y,\lambda,\omega,t))\end{aligned}$$

with $\Gamma^{(1)},\Gamma^{(2)}$ smooth functions of x,y,λ,ω,t.

Notice that (3.3) implies that there are $T_0>0, C>0$ such that for $|t|<T_0$,

$$\gamma^{(1)}_{x,y,\lambda,\omega}(t)\le x(1+\lambda t-Ct^2)=x\Big(1-C\Big(t-\frac{\lambda}{2C}\Big)^2+\frac{\lambda^2}{4C}\Big).$$

So if $|\lambda|\le C_0\epsilon$, then $\gamma^{(1)}_{x,y,\lambda,\omega}(t)\le x(1+C_1\epsilon^2)$, hence

$$\gamma^{(1)}_{x,y,\lambda,\omega}(t)^{-2}\ge x^{-2}(1-C_2\epsilon^2),$$

hence

$$x^{-2}-\gamma^{(1)}_{x,y,\lambda,\omega}(t)^{-2}\le C_2x^{-2}\epsilon^2.$$

In particular, if $|\lambda|<C_3x$, then this quantity is bounded above by a constant. The weight we use is the exponential of this (once in the conjugation), and thus it is bounded. Indeed, this relationship between the dynamics and the weight motivates the choice of the latter: any larger weight would mean the operator is not in our pseudodifferential algebra and any smaller weight would mean that it is irrelevant, and we would not have an elliptic operator. In particular, this explains that if we instead had $|\lambda|<C_3x^p$, our weight would be of the form $e^{-1/x^{2p}}$, and similar results would apply.

A similar argument also implies that for $|t| > C_4|\lambda|$, but $|t| < T_0$, we have

$$x^{-2} - \gamma^{(1)}_{x,y,\lambda,\omega}(t)^{-2} \leq -C_5 x^{-2} t^2, \tag{3.4}$$

so in particular if t is bounded away from 0 this is bounded from above by a negative multiple of x^{-2}.

Due to the convexity of the level sets of x, $\gamma^{(1)}_{x,y,\lambda,\omega}(t)$ can have only one local maximum, which is necessarily near $t = 0$, within $C_4|\lambda|$ of it, so if $|\lambda| < C_3 x$ then within $C_4' x$ of it. Correspondingly, in fact for t bounded (3.4) automatically holds in $|t| > C_4|\lambda|$ (even for t with not necessarily $|t| < T_0$).

3.1.3. *Initial (non-uniform in t) no conjugate points requirement.* We will also need a no conjugate points requirement; we strenghten this below in Section 3.1.5. We will say that the metric satisfies the no-conjugate points assumptions if for all (x, y) with $x < x_0$ the smooth map

$$(t, \lambda, \omega) \mapsto (x^{-1}\gamma^{(1)}_{x,y,\lambda,\omega}(t), \gamma^{(2)}_{x,y,\lambda,\omega}(t)) = (\tilde{\Gamma}^{(1)}_{x,y,\lambda,\omega}(t), \gamma^{(2)}_{x,y,\lambda,\omega}(t))$$

has a full rank differential. For $x \neq 0$, this is directly equivalent to the usual statement (since the factor x^{-1} is irrelevant), but this is the uniform (in variables other than t) version we need. Note that taking into account the smooth dependence of the flow on the parameters, differentiating $\frac{d\gamma^{(1)}(t)}{dt} = \gamma^{(1)}(t)(\tau + \gamma^{(1)}(t) f_1)$ with respect to x and evaluating at $x = 0$ yields

$$\frac{d}{dt}\partial_1\gamma^{(1)}(t) = \tau\partial_1\gamma^{(1)}(t),$$

so $\partial_1\gamma^{(1)}(t)$ satisfies a first order homogeneous linear ODE which only depends on the asymptotic conic metric g_∞. Since the non-degeneracy condition for x sufficiently small follows from that for $x = 0$, we conclude that it suffices to check the non-degeneracy condition for g_∞, which we do below after a further reparameterization.

3.1.4. *Second reperameterization, to new parameter r, to deal with large |t| behavior.* In order to obtain uniformity as $|t| \to \infty$, it is very useful to change the parameterization again, keeping in mind the global nature of the Hamilton flow. The key point is that as the dual metric function at $x = 0$ is $g|_{x=0} = \tau^2 + h(y, \mu)$, as a b-vector field,

$${}^{\mathrm{sc}}H_g|_{x=0} = 2\tau(x\partial_x + \mu\partial_\mu) - 2|\mu|^2\partial_\tau + H_h,$$

as discussed earlier. Correspondingly, the points $x = 0$, $\mu = 0$ are 'radial points', where this vector field, considered as a b-vector field, is a multiple of $x\partial_x$. It is thus natural to blow these up in ${}^{\mathrm{sc}}T^*M$. Given our choice of cutoff, the integral curves of concern approach $x = 0$, $\mu = 0$ almost tangent to the boundary $x = 0$; in this region $\rho = \frac{x}{|\mu|}$, $|\mu|$, $\widehat{\mu} = \frac{\mu}{|\mu|}$, together with τ, y, are coordinates on the blown up space; $|\mu|$ defines the front face, ρ defines the lift of the original boundary. Then

$$\frac{1}{2}{}^{\mathrm{sc}}H_g = |\mu| V,$$

V a vector field tangent to the lift of the original boundary, $\rho = 0$, but transversal to the front face, and is indeed is $V|\mu|$ is ± 1 *at* the front face. Correspondingly, the integral curves of V,

$$\hat{\gamma} = \hat{\gamma}_{x,y,\lambda,\omega}(r), \text{ with } \frac{dr}{dt} = |\mu(\gamma(t))|,$$

intersect the front face in finite time, which enables standard flow arguments. Concretely, taking into account that $|\mu|$ is close to 1 at the initial point, so ρ and x can be used interchangeably there,

$$\rho = xF(x, y, \lambda, \omega, r),$$

with F smooth and positive, and $|\mu|$ also a smooth function of x, y, λ, ω, r, hence their product, $x(\hat{\gamma}(r))$, is also such a multiple of x, with the multiple going to 0 at the front face, and $y(\hat{\gamma}(r))$ is similarly a smooth function of x, y, λ, ω, r. Moreover,

$$|\mu(\hat{\gamma}(r))| = x(\hat{\gamma}(r))/\rho(\hat{\gamma}(r)) = x^{-1}x(\hat{\gamma}(r))F(x, y, \lambda, \omega, r)^{-1}. \tag{3.5}$$

Let $R(x, y, \lambda, \omega)$ be the value of r the front face is reached by the integral curve; thus, there are two values $R = R_{\pm}$ at the two ends of the integral curve but we suppress this in notation. Observe then that as $V|\mu|$ is ± 1 as the front face, $|\mu(\hat{\gamma}(r))| = a|r - R|$, where a is a smooth positive function of x, y, λ, ω, r, and is equal to 1 at $r = R$. Since, as sufficient for us, we assumed small $|\lambda|$ in our arguments (choosing projective coordinates at the front face), combining with (3.5), we have proved:

Lemma 3.2. *In $|\lambda| < \epsilon_0$, $\epsilon_0 > 0$ sufficiently small, the function $x(\hat{\gamma}(r)) = x(\hat{\gamma}_{x,y,\lambda,\omega}(r))$ is a smooth non-degenerate multiple of $x|r - R|$ near either endpoint of the integral curve $\hat{\gamma}(r)$), i.e. near $r = R_{\pm}$, and a smooth non-degenerate multiple of x away from these (where the $|r - R|$ factor is irrelevant).*

For the sake of completeness, let us also convert this into an estimate in the original t-parameterization, though below we always use the r parameterization near the ends of the bicharacteristics. As $\frac{dr}{dt} = |\mu|$, so $|r - R|\frac{dt}{dr} = a^{-1}$, t differs from $\pm \log |r - R_{\mp}|$ by a smooth function (of x, y, λ, ω, r), and in particular $|r - R| \sim e^{-|t|}$ (bounded above and below by positive multiples of this); recall that $R = R_{\pm}$. Hence

$$\gamma^{(1)}_{x,y,\lambda,\omega}(t) \sim xe^{-|t|},$$

and

$$\gamma^{(1)}_{x,y,\lambda,\omega}(t)^{-1} \le Cx^{-1}e^{|t|}. \tag{3.6}$$

We will also need an estimate of the exponential weight as we computed above in (3.4) for bounded t. Namely, we have

$$\begin{aligned}
&x^{-2}-\hat{\gamma}^{(1)}_{x,y,\lambda,\omega}(r)^{-2}\\
&=(\hat{\gamma}^{(1)}_{x,y,\lambda,\omega}(r)^2-x^2)x^{-2}(\hat{\gamma}^{(1)}_{x,y,\lambda,\omega}(r))^{-2}\\
&=-(\hat{\gamma}^{(1)}_{x,y,\lambda,\omega}(r))^{-2}W(x,y,\lambda,\omega,r),
\end{aligned}\tag{3.7}$$

with $W\to 1$ as $r\to R$. This can be combined with (3.6) if desired.

3.1.5. *Full no-conjugate point assumption.* We also need a no-conjugate point assumption which analogously to the finite t case means the non-degeneracy of the smooth function $(x^{-1}\hat{\gamma}^{(1)},\hat{\gamma}^{(2)})$ of (r,λ,ω); here in fact polynomial in $\hat{\gamma}^{(1)}$ degeneracy of the derivative is acceptable due to the exponential decay of the weight. Notice that for r away from the endpoints of the interval, this is equivalent to $(x^{-1}\gamma^{(1)},\gamma^{(2)})$ being non-degenerate as a function of (t,λ,ω) as $\frac{dr}{dt}=|\mu(\gamma(t))|\neq 0$ there, i.e. it reduces to the previous discussion. As discussed above for bounded t, using $\partial_1\hat{\gamma}=x^{-1}\hat{\gamma}$, the non-degeneracy for small x follows from non-degeneracy at $x=0$, which in turn follows from the corresponding property for the asymptotic metric g_∞. Hence, evaluating at $x=0$ and using the explicit g_∞-flow from (1.4), we have

$$(x^{-1}\hat{\gamma}^{(1)},\hat{\gamma}^{(2)})=\Big(\frac{\sin(r+r_0)}{\sin r_0},\exp(rH_{h/2})(y_0,\hat{\mu}_0)\Big)$$

with $\omega=h^{-1}(\hat{\mu}_0)$ and $\lambda=\cot r_0$, so the non-degeneracy is equivalent to $(x^{-1}\hat{\gamma}^{(1)},\hat{\gamma}^{(2)})$ being non-degenerate as a function of $(r,r_0,\hat{\mu}_0)$, which in turn is immediately seen as being equivalent to the absence of conjugate points under the boundary metric h within distance $\pi/2$.

3.2. **Invertibility of the geodesic X-ray transform on a collar neighborhood of infinity.** After these geometric preliminaries we work out the form of the Schwartz kernel of $A_h=e^{-\Phi/h}L\tilde{\chi}Ie^{\Phi/h}$. Relative to the density $|dz'|=|dx'\,dy'|$, this is, with $z=(x,y)$, $z'=(x',y')$,

$$\begin{aligned}
K_{A_h}(x,y,x',y')=\int e^{-\Phi(x)/h}e^{\Phi(x(\gamma_{x,y,\lambda,\omega}(t)))/h}\tilde{\chi}(x,y,\lambda/(h^{1/2}x),\omega)\\
\delta(z'-\gamma_{z,\lambda,\omega}(t))\gamma^{(1)}_{z,\lambda,\omega}(t)^{-1}\,dt\,|d\sigma|,
\end{aligned}\tag{3.8}$$

since that of I is

$$K_I(z,\lambda,\omega,z')=\int\delta(z'-\gamma_{z,\lambda,\omega}(t))\gamma^{(1)}_{z,\lambda,\omega}(t)^{-1}\,dt,$$

with $\gamma^{(1)}_{z,\lambda,\omega}(t)^{-1}$ corresponding to the Jacobian factor in (3.1). Hence,

(3.9)

$$\begin{aligned}K_{A_h}&(x,y,x',y')\\ &= (2\pi)^{-n}h^{-n/2-1/2}\int e^{-\Phi(x)/h}e^{\Phi(x(\gamma_{x,y,\lambda,\omega}(t)))/h}\tilde{\chi}(x,y,\lambda/(h^{1/2}x),\omega)\\ &\qquad e^{-i\xi'(x'-\gamma^{(1)}_{z,\lambda,\omega}(t))/h}e^{-i\eta'\cdot(y'-\gamma^{(2)}_{z,\lambda,\omega}(t))/h^{1/2}}\gamma^{(1)}_{z,\lambda,\omega}(t)^{-1}\,dt\,|d\sigma|\,|d\xi'|\,|d\eta'|,\end{aligned}$$

where we wrote the delta distribution as a semiclassical foliation Fourier transform of the constant function $(2\pi)^{-n}$, with

$$e^{-i\xi'(x'-\gamma^{(1)}_{z,\lambda,\omega}(t))/h}e^{-i\eta'\cdot(y'-\gamma^{(2)}_{z,\lambda,\omega}(t))/h^{1/2}}$$

being the kernel of the Fourier transform. The integrand of the $dt\,|d\sigma|$ integral can be considered as a semiclassical foliation Fourier transform $(\xi',\eta')\to(x',y')$ of

(3.10)

$$\begin{aligned}(2\pi)^{-n}&e^{-\Phi(x)/h}e^{\Phi(x(\gamma_{x,y,\lambda,\omega}(t)))/h}\tilde{\chi}(x,y,\lambda/(h^{1/2}x),\omega)\gamma^{(1)}_{z,\lambda,\omega}(t)^{-1}\\ &\qquad e^{i\xi'\gamma^{(1)}_{z,\lambda,\omega}(t)/h}e^{i\eta'\cdot\gamma^{(1)}_{z,\lambda,\omega}(t)/h^{1/2}},\end{aligned}$$

with the factors on the first line independent of the Fourier transform variables (ξ',η'). For the purposes below, it is useful to have the Schwartz kernel relative to the density $\frac{|dx'\,dy'|}{(x')^{n+2}}$, cf. our definition of the 1-cusp algebra. In view of the delta distribution in (3.8), this can be achieved by adding a factor $(\gamma^{(1)}_{z,\lambda,\omega}(t))^{n+2}$ to (3.9), and thus to (3.10).

In order to proceed, we recall from (1.3) that 1-cusp operators are given by the oscillatory integral

$$\begin{aligned}A_h u(x,y)&=Au(x,y,h)\\ &=(2\pi)^{-n}h^{-n/2-1/2}\int e^{i\left(\frac{x-x'}{x^3}\frac{\widetilde{\xi_{1c}}}{h}+\frac{y-y'}{x}\frac{\widetilde{\eta_{1c}}}{h^{1/2}}\right)}a_h(x,y,\widetilde{\xi_{1c}},\widetilde{\eta_{1c}})\,u(x',y')\,\frac{dx'\,dy'}{(x')^{n+2}}\,d\widetilde{\xi_{1c}}\,d\widetilde{\eta_{1c}},\end{aligned}$$

where a is a standard (conormal) symbol. Thus, the Schwartz kernel is

$$K_{A_h}(x,y,x',y')=(2\pi)^{-n}h^{-n/2-1/2}\int e^{i\left(\frac{x-x'}{x^3}\frac{\widetilde{\xi_{1c}}}{h}+\frac{y-y'}{x}\frac{\widetilde{\eta_{1c}}}{h^{1/2}}\right)}a_h(x,y,\widetilde{\xi_{1c}},\widetilde{\eta_{1c}})\,d\widetilde{\xi_{1c}}\,d\widetilde{\eta_{1c}},$$

relative to the density $\frac{dx'\,dy'}{(x')^{n+2}}$. , i.e. they are $(2\pi)^{-n}x^{n+2}$ (with the second factor due to the Jacobian in scaling the Fourier transform) times the semiclassical foliation Fourier transform in $(x^{-3}\widetilde{\xi_{1c}},x^{-1}\widetilde{\eta_{1c}})$ of

$$(x,y,\widetilde{\xi_{1c}},\widetilde{\eta_{1c}})\mapsto e^{i(x^{-2}\widetilde{\xi_{1c}}/h+x^{-1}y\cdot\widetilde{\eta_{1c}}/h^{1/2})}a(x,y,\widetilde{\xi_{1c}},\widetilde{\eta_{1c}}).$$

Inverting this Fourier transform, evaluating at $(x^{-3}\xi_{1c}, x^{-1}\eta_{1c})$,

$$\begin{aligned}&a_h(x,y,\xi_{1c},\eta_{1c})\\ &= (2\pi)^n x^{-n-2} e^{i(-x^{-2}\xi_{1c}/h - x^{-1}y\cdot\eta_{1c}/h^{1/2})}\\ &\qquad (\mathcal{F}^{-1}_{h,\mathcal{F}})_{(x',y')\to(x^{-3}\xi_{1c},x^{-1}\eta_{1c})} K_{A_h}(x,y,x',y').\end{aligned}$$

Proceeding from (3.9), taking into account the Fourier transform statement following it, we obtain

(3.11)

$$\begin{aligned}&a_h(x,y,\xi_{1c},\eta_{1c})\\ &= x^{-n-2} e^{-ix^{-3}\xi_{1c}x/h} e^{-ix^{-1}\eta_{1c}\cdot y/h^{1/2}} \int e^{-\Phi(x)/h} e^{\Phi(x(\gamma_{x,y,\lambda,\omega}(t)))/h} \tilde{\chi}(x,y,\lambda/(h^{1/2}x),\omega)\\ &\qquad e^{ix^{-3}\xi_{1c}\gamma^{(1)}_{z,\lambda,\omega}(t)/h} e^{ix^{-1}\eta_{1c}\cdot\gamma^{(2)}_{z,\lambda,\omega}(t)/h^{1/2}} \gamma^{(1)}_{z,\lambda,\omega}(t)^{n+1}\,dt\,|d\sigma|\\ &= \int e^{-\Phi(x)/h} e^{\Phi(x(\gamma_{x,y,\lambda,\omega}(t)))/h} \tilde{\chi}(x,y,\lambda/(h^{1/2}x),\omega)\\ &\qquad e^{ix^{-3}\xi_{1c}(\gamma^{(1)}_{z,\lambda,\omega}(t)-x)/h} e^{ix^{-1}\eta_{1c}\cdot(\gamma^{(2)}_{z,\lambda,\omega}(t)-y)/h^{1/2}} x^{-n-2}\gamma^{(1)}_{z,\lambda,\omega}(t)^{n+1}\,dt\,|d\sigma|.\end{aligned}$$

Note that this corresponds to [28, Equation (3.8)], taking into account the factor in (3.1), and that we write covectors as $\xi_{1c}\frac{dx}{x^3} + \eta_{1c}\frac{dy}{x}$, and thus $\xi = x^{-3}\xi_{1c}$, $\eta = x^{-1}\eta_{1c}$ in [28, Equation (3.8)]. Recall that $\Phi(x) = -\frac{1}{2x^2}$ here, and $\gamma^{(1)}_{z,\lambda,\omega}(t)^{-1}$ is bounded by $Cx^{-1}e^{|t|}$ by (3.6), while the combined Φ exponent is tending to $-\infty$ like $-\gamma^{(1)}_{z,\lambda,\omega}(t)^{-2}$ by (3.7), resulting in a superexponential suppression of the ends of the bicharacteristics γ. Thus, it remains to show that the right hand side of (3.11) is h times a symbol of order $-1,-1$, i.e. hx times a symbol of order $-1,0$.

We remark that (3.11) uses local coordinates. In general, for the Schwartz kernel $K_A(z,z')$ of our semiclassical operators we should be considering both the possibilities that z and z' are in the same chart, and also that they are away from each other, in different charts. In the latter case $|t|$ is necessarily bounded from below by a positive constant, and the argument below, discussed in the notation of the same chart, applies directly and shows that the Schwartz kernel is Schwartz and is $O(h^\infty)$. Indeed, in the argument given when discussing that region in t in the oscillatory integral the unprimed variables can be regarded as fixed, so for many purposes there is not even a need to consider a coordinate chart explicitly, and in any case the device described for the cusp pseudodifferential operators of taking two disjoint open sets O, U and identifying them with different open sets of $\mathbb{R}^n$, now in the parabolic sense, would be applicable. In the direct treatment one can describe the Schwartz kernel directly (as opposed to through the symbol, which requires a Fourier transform) which is residual in these regions in terms of (3.9); our computations then directly show that prior to the $|d\xi'|\,|d\eta'|$ integrals one already has rapid decay and smoothness in all variables (both x,h and ξ',η').

We break up our analysis into four regions by the use of a partition of unity (which we suppress in notation): $|t| < C_1 h^{1/2} x$, $C_2 h^{1/2} x < |t| < T_0$ (with T_0 corresponding to both flow and coordinate considerations, so it is sufficiently small and positive), $0 < C_3 < |t| < C_4$ (C_3, C_4 arbitrary positive) and $|t|$ near infinity, though the third and fourth regions can be combined.

We first analyze the pseudodifferential, i.e. near diagonal, behavior of the Schwartz kernel. For this, we may take z, z' in the same chart and $|t|$ bounded by a constant $T_0 > 0$. In order to proceed, we change the variables of integration to $\hat{t} = t/(h^{1/2}x)$ and $\hat{\lambda} = \lambda/(h^{1/2}x)$, so the $\hat{\lambda}$ integration is over a fixed interval. The phase is

(3.12)
$$\begin{aligned}
&x^{-3}\xi_{1c}\cdot(\gamma^{(1)}_{z,\lambda,\omega}(t)-x)/h + x^{-1}\eta_{1c}\cdot(\gamma^{(2)}_{z,\lambda,\omega}(t)-y)/h^{1/2}\\
&\quad= \xi_{1c}(\hat{\lambda}\hat{t} + \alpha(x,y,xh^{1/2}\hat{\lambda},\omega)\hat{t}^2 + xh^{1/2}\hat{t}^3\Gamma^{(1)}(x,y,xh^{1/2}\hat{\lambda},\omega,xh^{1/2}\hat{t}))\\
&\qquad\qquad + \eta_{1c}\cdot(\omega\hat{t} + xh^{1/2}\hat{t}^2\Gamma^{(2)}(x,y,xh^{1/2}\hat{\lambda},\omega,xh^{1/2}\hat{t})),
\end{aligned}$$

while the exponential damping factor (which we regard as a Schwartz function, part of the amplitude, when one regards $\hat{t}$ as a variable on $\mathbb{R}$) is (recall that $\alpha < 0$!)

(3.13)
$$\begin{aligned}
&1/(2hx^2) - 1/(2h\gamma^{(1)}_{x,y,\lambda,\omega}(t)^2)\\
&= \frac{1}{2}h^{-1}(\gamma^{(1)}_{x,y,\lambda,\omega}(t)^2 - x^2)x^{-2}(\gamma^{(1)}_{x,y,\lambda,\omega}(t))^{-2}\\
&= \frac{1}{2}h^{-1}x(\lambda t + \alpha(x,y,xh^{1/2}\hat{\lambda},\omega)t^2 + t^3\Gamma^{(1)}(x,y,\lambda,\omega,t))\\
&\qquad\qquad x(2 + \lambda t + \alpha(x,y,xh^{1/2}\hat{\lambda},\omega)t^2 + t^3\Gamma^{(1)}(x,y,\lambda,\omega,t))\\
&\qquad\qquad x^{-2}x^{-2}(1 + \lambda t + \alpha(x,y,xh^{1/2}\hat{\lambda},\omega)t^2 + t^3\Gamma^{(1)}(x,y,\lambda,\omega,t))^{-2}\\
&= \hat{\lambda}\hat{t} + \alpha(x,y,xh^{1/2}\hat{\lambda},\omega)\hat{t}^2 + \hat{t}^3 xh^{1/2}\hat{\Gamma}^{(1)}(x,y,xh^{1/2}\hat{\lambda},\omega,xh^{1/2}\hat{t}),
\end{aligned}$$

with $\hat{\Gamma}^{(1)}$ a smooth function. Thus, as we explain below in more detail, for ξ_{1c}, η_{1c} in a bounded region we conclude that a_h is a C^∞ function of all variables, including $h^{1/2}$. Furthermore, we observe that with (ξ_{1c}, η_{1c}) in place of (ξ, η), and in the new integration variables $\hat{t}$ and $\hat{\lambda}$, (3.11) has the same form as [28, Equation (3.8)], so identical stationary phase arguments are applicable.

Remark 3.3. If we used the scaling $\hat{\lambda} = \lambda/(h^{1/2}x^p)$ in the definition of $\tilde{\chi}$, and replaced x by x^p in the definition of Φ, with more precisely $\Phi = -\frac{1}{2px^{2p}}$, we would obtain essentially the same rescaled result for the phase and the exponential weight. Namely, write $\widetilde{\xi_{1c}}, \widetilde{\eta_{1c}}$ for the (slightly modified) x^p-based 1-cusp dual variables, i.e. write covectors as

$$\widetilde{\xi_{1c}}\frac{dx^p}{x^{3p}} + \widetilde{\eta_{1c}}\frac{dy}{x^p} = p\widetilde{\xi_{1c}}\frac{dx}{x^{2p+1}} + \widetilde{\eta_{1c}}\frac{dy}{x^p}.$$

Then the right hand side of the phase, (3.12), becomes, and $\hat{t} = t/(h^{1/2}x^p)$,

$$p\widetilde{\xi_{1c}}(\hat{\lambda}\hat{t} + \alpha(x, y, x^p h^{1/2}\hat{\lambda}, \omega)\hat{t}^2 + x^p h^{1/2}\hat{t}^3\Gamma^{(1)}(x, y, x^p h^{1/2}\hat{\lambda}, \omega, x^p h^{1/2}\hat{t}))$$
$$+ \widetilde{\eta_{1c}} \cdot (\omega\hat{t} + x^p h^{1/2}\hat{t}^2\Gamma^{(2)}(x, y, x^p h^{1/2}\hat{\lambda}, \omega, x^p h^{1/2}\hat{t})),$$

and similarly the analogue of the right hand side of (3.13) becomes

$$\hat{\lambda}\hat{t} + \alpha(x, y, x^p h^{1/2}\hat{\lambda}, \omega)\hat{t}^2 + \hat{t}^3 x^p h^{1/2}\hat{\Gamma}^{(1)}(x, y, x^p h^{1/2}\hat{\lambda}, \omega, x^p h^{1/2}\hat{t}).$$

These allow all the arguments below to proceed without any significant change; even the ellipticity computation is unaffected apart from scaling $\widetilde{\xi_{1c}}$ by an irrelevant factor of p.

Concretely, upon the rescaling t and λ to $\hat{t}$ and $\hat{\lambda}$, which introduces a factor of hx^2 from the Jacobian, the integrand of (3.11) is xh times a smooth function of all variables, integrated in a compact region except in $\hat{t}$. However, the Gaussian decay, in view of (3.4) and (3.13), of the exponential damping factor

$$e^{-\Phi(x)/h}e^{\Phi(x(\gamma_{z,\lambda,\omega}(t)))/h}$$

means that this non-compactness is not an issue, and (3.11) itself is hx times a smooth function of all variables, in accordance to the desired hx times a symbol of order $-1, 0$ conclusion, namely showing the smoothness part, but not yet the estimates as $|(\xi_{1c}, \eta_{1c})| \to \infty$.

We now consider the $|(\xi_{1c}, \eta_{1c})| \to \infty$ behavior. We use the stationary phase lemma, but with the slight complication that the $\hat{t}$ integration interval is non-compact. In order to deal with this, we divide the integration region into one in which $|\hat{t}|$ bounded, resp. one in which $|\hat{t}| \geq 1$. In the the former one can use the standard parameter dependent version of the stationary phase lemma, while in the latter the phase is non-stationary and one can use a direct integration by parts argument.

Starting with the former, at $h^{1/2} = 0$, the phase is

$$\xi_{1c}(\hat{\lambda}\hat{t} + \alpha(x, y, 0, \omega)\hat{t}^2) + \eta_{1c} \cdot \omega\hat{t}.$$

Consider first $\xi_{1c} \neq 0$: taking the $\hat{\lambda}$ derivative shows that $\hat{t} = 0$ at the critical set, and thus taking the $\hat{t}$ derivative shows that $\xi_{1c}\hat{\lambda} + \eta_{1c} \cdot \omega = 0$, i.e. $\hat{\lambda} = -\xi_{1c}^{-1}\eta_{1c} \cdot \omega$; this is actually critical with respect to the full set $(\hat{t}, \hat{\lambda}, \omega)$. Moreover, this set remains critical for $h^{1/2}$ non-zero due to the $\hat{t}^2$ vanishing factors in other terms of the phase. At $h^{1/2} = 0$ the Hessian of the phase in $(\hat{t}, \hat{\lambda})$ at the critical set is

$$\begin{pmatrix} 2\xi_{1c}\alpha(x, y, 0, \omega) & \xi_{1c} \\ \xi_{1c} & 0 \end{pmatrix},$$

which is invertible, with determinant $-\xi_{1c}^2$, hence remains so for small h. Thus, regarding ω as a parameter, the stationary phase lemma applies and yields that in this region a_h is xh times (due to the Jacobian factor discussed above!) a symbol of order -1 (from the reciprocal of the square root of the Hessian determinant). On the other hand, when $\eta_{1c} \neq 0$ (for the behavior as $|(\xi_{1c}, \eta_{1c})| \to \infty$ we only need to consider when at least one of ξ_{1c} and η_{1c} is non-zero), decompose ω corresponding to η_{1c} into a parallel and an orthogonal component, writing $\omega^{\|} = \omega \cdot \widehat{\eta_{1c}}$, $\widehat{\eta_{1c}} = \frac{\eta_{1c}}{|\eta_{1c}|}$, so the phase at $h^{1/2} = 0$ becomes

$$|\eta_{1c}|\Big(\frac{\xi_{1c}}{|\eta_{1c}|}(\hat{\lambda}\hat{t} + \alpha(x, y, 0, \omega)\hat{t}^2) + \omega^{\|}\hat{t}\Big). \tag{3.14}$$

First, taking the $\hat{\lambda}$ derivative shows that either $\frac{\xi_{1c}}{|\eta_{1c}|} = 0$ or $\hat{t} = 0$ at the critical set, and in the former case taking the $\hat{t}$ derivative shows that $\omega^{\|} = 0$, while in the latter case the same $\hat{t}$ derivative shows that (as we already have $\hat{t} = 0$) $\frac{\xi_{1c}}{|\eta_{1c}|}\hat{\lambda} + \omega^{\|} = 0$, so in view of the boundedness of $\hat{\lambda}$ and as we may assume the smallness of $\frac{\xi_{1c}}{|\eta_{1c}|}$ in view of the already treated case, $|\omega^{\|}|$ is bounded away from 1, and thus $\omega^{\|}$ is a valid coordinate at the critical set, $\hat{t} = 0$, $\frac{\xi_{1c}}{|\eta_{1c}|}\hat{\lambda} + \omega^{\|} = 0$, which is indeed critical with respect to the full set $(\hat{t}, \hat{\lambda}, \omega^{\|}, \omega^{\perp})$ of parameters. Further, this remains also true for $h^{1/2}$ non-zero due to the $\hat{t}^2$ vanishing factors in the other terms of the phase. The Hessian with respect to $(\hat{t}, \omega^{\|})$ is

$$|\eta_{1c}| \begin{pmatrix} 2\frac{\xi_{1c}}{|\eta_{1c}|}\alpha(x, y, 0, \omega) & 1 \\ 1 & 0 \end{pmatrix},$$

which is again invertible, with determinant $-|\eta_{1c}|^2$, and thus remains so for $h^{1/2}$ small. Thus, regarding $\hat{\lambda}, \omega^{\perp}$ as parameters, the stationary phase lemma applies and yields that in this region as well a_h is xh times a symbol of order -1. This completes the proof of the symbolic behavior of the contribution of the oscillatory intergral (3.11) from $\hat{t}$ bounded.

In hindsight, as this will be useful for the symbolic computation, we can rewrite the phase by regarding $\theta = (\hat{\lambda}, \omega)$ and (ξ_{1c}, η_{1c}) jointly, writing the latter as $|(\xi_{1c}, \eta_{1c})|(\widehat{\eta_{1c}}, \widehat{\xi_{1c}})$:

$$|(\xi_{1c}, \eta_{1c})|(\widehat{\xi_{1c}}(\hat{\lambda}\hat{t} + \alpha(x, y, 0, \omega)\hat{t}^2) + \widehat{\eta_{1c}} \cdot \omega\hat{t}).$$

Decomposing θ into parallel and orthogonal components relative to $(\widehat{\eta_{1c}}, \widehat{\xi_{1c}})$, so $\theta^{\|} = (\widehat{\eta_{1c}}, \widehat{\xi_{1c}}) \cdot \theta$, the phase is the large parameter $|(\xi_{1c}, \eta_{1c})|$ times

$$\theta^{\|}\hat{t} + \widehat{\xi_{1c}}\alpha(x, y, 0, \omega)\hat{t}^2. \tag{3.15}$$

As we already know that at the critical set $\hat{t} = 0$, we deduce that it is given by $\theta^{\|} = 0$, and that the Hessian of the phase there with respect to $(\hat{t}, \theta^{\|})$ is

$$|(\xi_{1c}, \eta_{1c})| \begin{pmatrix} 2\widehat{\xi_{1c}}\alpha(x, y, 0, \omega) & 1 \\ 1 & 0 \end{pmatrix}.$$

We next analyze the $|t| \le T_0$ small, $|\hat{t}| \ge 1$ region. Here we use a direct integration by parts argument, utilizing that if the derivative of the phase with respect to one of the integration variables $(\hat{t}, \hat{\lambda}, \omega)$ is bounded below by a positive multiple of $|(\xi_{1c}, \eta_{1c})||\hat{t}|^{-k}$ for some k, integration by parts in this variable, taking into account the Gaussian exponential damping factor bounded by $e^{-\epsilon t^2/(hx^2)} = e^{-\epsilon \hat{t}^2}$ in $|t| > Ch^{1/2}x$ by (3.4), gives rapid decay of the integral with respect to the large parameter $|(\xi_{1c}, \eta_{1c})|$. Hence, it remains to check that in all regions one has such a lower bound for some derivative; again it suffices to check this at $h^{1/2} = 0$. If $\xi_{1c} \neq 0$, then the $\hat{\lambda}$ derivative of the phase is $\xi_{1c}\hat{t}$, giving the desired statement. If $\eta_{1c} \neq 0$, the form (3.14) of the phase shows that first of all the $\hat{\lambda}$ derivative has such a lower bound as soon as $\frac{|\xi_{1c}|}{|\eta_{1c}|}$ is bounded from below by $|\hat{t}|^{-2}$. Then as long as $\omega^{\|}$ is a valid coordinate, the derivative with respect to $\omega^{\|}$ is $|\eta_{1c}|\hat{t}$, giving the desired lower bound. The remaining case, when $\omega^{\|}$ is not a valid coordinate, i.e. when $|\omega^{\|}|$ is close to 1, and $\frac{|\xi_{1c}|}{|\eta_{1c}|} \leq |\hat{t}|^{-2}$. In this case the $\hat{t}$ derivative becomes $|\eta_{1c}|(\frac{\xi_{1c}}{|\eta_{1c}|}(\hat{\lambda} + 2\alpha\hat{t}) + \omega^{\|})$, which is now bounded away from 0, completing the proof of the direct integration by parts argument in the region $|t| \leq T_0$ small, $|\hat{t}| \geq 1$.

Consider now the region where t is bounded away from 0, but is bounded; in this case the exponential weight is bounded by $e^{-\epsilon/(x^2 h)}$, cf. (3.4), thus is rapidly decaying. Recall that the phase is

$$\begin{aligned}x^{-3}\xi_{1c} \cdot (\gamma^{(1)}_{z,\lambda,\omega}(t) - x)/h + x^{-1}\eta_{1c} \cdot (\gamma^{(2)}_{z,\lambda,\omega}(t) - y)/h^{1/2}\\ = x^{-1}h^{-1/2}\Big((x^{-1}\xi_{1c}/h^{1/2})(x^{-1}\gamma^{(1)}_{z,\lambda,\omega}(t) - 1) + \eta_{1c} \cdot (\gamma^{(2)}_{z,\lambda,\omega}(t) - y)\Big),\end{aligned}$$

and $\partial_t(x^{-1}\gamma^{(1)})$ is non-zero (bounded away from 0) in this region by the convexity properties of the foliation. Thus, for t away from 0 there is $C_0 > 0$ such that if $|\xi_{1c}|x^{-1}/h^{1/2} > C_0|\eta_{1c}|$ then the phase is non-stationary with respect to t, hence the integral is rapidly decaying in $|(\xi_{1c}, \eta_{1c})|/(h^{1/2}x)$. On the other hand, under the no conjugate points assumption, in the precise sense described in Section 3.1.3, letting $\widetilde{\xi_{1c}} = x^{-1}\xi_{1c}/h^{1/2}$, if $|\widetilde{\xi_{1c}}| = |\xi_{1c}|x^{-1}/h^{1/2} < 2C_0|\eta_{1c}|$, one has the standard no-conjugate points argument available as the phase is a standard homogeneous degree 1 phase in $(\widetilde{\xi_{1c}}, \eta_{1c})$ times $x^{-1}h^{-1/2}$. Here the no-conjugate points argument uses the non-degenerateness (full rank of the Jacobian) of the smooth function $(x^{-1}\gamma^{(1)}, \gamma^{(2)})$ as a function of (t, λ, ω), $t \neq 0$, as discussed in Section 3.1.3.

Thus, it remains to consider $|t| \to \infty$. Following Section 3.1.4 it is very useful to change the parameterization again, keeping in mind the global nature of the Hamilton flow, to r from t. Recall that here we write

$$\hat{\gamma} = \hat{\gamma}_{x,y,\lambda,\omega}(r), \text{ with } \frac{dr}{dt} = |\eta_{sc}(\gamma(t))|,$$

and the integral curves intersect the boundary in finite time; this enables standard integration by parts arguments. As follows from (3.7), the amplitude is exponentially decaying in $h^{-1}(\hat{\gamma}^{(1)}_{x,y,\lambda,\omega}(r))^{-2}$, thus together with the extra factor

$$x(\hat{\gamma}(r))^{-1}|\eta_{sc}(\hat{\gamma}(r))|^{-1} = x(\hat{\gamma}(r))^{-2}xF(x, y, \lambda, \omega, r)$$

it still has this property, suppressing the endpoint of the bicharacteristic. Now, if $|\xi_{1c}|x^{-1}/h^{1/2} > C_0|\eta_{1c}|$ with a sufficiently large C_0, then the phase is non-stationary with respect to r, giving the desired decay result; otherwise the no-conjugate points assumption achieves this. Again, as discussed in Section 3.1.5, for this the non-degeneracy of the smooth function $(x^{-1}\hat{\gamma}^{(1)}, \hat{\gamma}^{(2)})$ of (r, λ, ω) is used, where in fact polynomial in $\hat{\gamma}^{(1)}$ degeneracy of the derivative is acceptable due to the exponential decay of the weight. In combination this proves the claimed pseudodifferential property.

Finally, the ellipticity computation is as in [28, Equation (3.12)] with (ξ_{1c}, η_{1c}) in place of (ξ_{sc}, η_{sc}). Concretely, in order to compute the semiclassical principal symbol from (3.11), we may simply let $h^{1/2} = 0$ in the rescaled expression, apart from the overall prefactor, so

$$a_h(x, y, \xi_{1c}, \eta_{1c}) = xh \int e^{i(\xi_{1c}(\hat{\lambda}\hat{t} + \alpha(x,y,0,\omega)\hat{t}^2) + \eta_{1c}\cdot\omega\hat{t})} \\ e^{\hat{\lambda}\hat{t} + \alpha(x,y,0,\omega)\hat{t}^2} \tilde{\chi}(z, \hat{\lambda}, \omega)\, d\hat{t}\, d\hat{\lambda}\, d\omega,$$

up to errors gaining $O(xh^{1/2}\langle \xi_{1c}, \eta_{1c}\rangle^{-1})$ relative to the leading order $O(xh\langle \xi_{1c}, \eta_{1c}\rangle^{-1})$. In order to compute the principal symbol of this, i.e. the behavior as $|(\xi_{1c}, \eta_{1c})| \to \infty$, we recall from (3.15) that it is useful to regard $\theta = (\hat{\lambda}, \omega)$ as a joint variable, decomposed relative to $(\widehat{\eta_{1c}}, \widehat{\xi_{1c}})$, with critical set given by $\hat{t} = 0$, $\theta^{\|} = 0$. Thus, by the stationary phase lemma, the principal symbol of the semiclassical principal symbol is an elliptic multiple of

$$\int_{\mathbb{S}^{n-2}} \tilde{\chi}(z, \hat{\lambda}(\theta^{\perp}), \omega(\theta^{\perp}))\, d\theta^{\perp},$$

which is elliptic for $\chi \geq 0$ with $\chi(0, \cdot) > 0$ since the codimension one planes $\theta^{\|} = 0$ and $\hat{\lambda} = 0$ necessarily intersect in a line through the origin, and thus non-trivially intersect the sphere as $n \geq 2 + 1 = 3$. Hence it remains to compute the semiclassical principal symbol at finite points.

The computation at finite points is more difficult, but it simplifies greatly if we take χ to be a Gaussian (which is not technically allowed, but we will approximate it below). Namely, recalling that $\alpha < 0$, the semiclassical principal symbol is, for c a non-zero constant,

$$\begin{aligned} xh &\int e^{\alpha\hat{t}^2(1+i\xi_{1c}) + \hat{t}(\hat{\lambda}(1+i\xi_{1c}) + i\eta_{1c}\cdot\omega)} \tilde{\chi}(z, \hat{\lambda}, \omega)\, d\hat{t}\, d\hat{\lambda}\, d\omega \\ &= xh \int e^{\alpha(1+i\xi_{1c})(\hat{t} + \frac{\hat{\lambda}(1+i\xi_{1c}) + i\eta_{1c}\cdot\omega}{2\alpha(1+i\xi_{1c})})^2} e^{\frac{(\hat{\lambda}(1+i\xi_{1c}) + i\eta_{1c}\cdot\omega)^2}{4\alpha(1+i\xi_{1c})}} \tilde{\chi}(z, \hat{\lambda}, \omega)\, d\hat{t}\, d\hat{\lambda}\, d\omega \\ &= cxh \int |\alpha|^{-1/2}(1 + i\xi_{1c})^{-1/2} e^{-\frac{(\hat{\lambda}(1+i\xi_{1c}) + i\eta_{1c}\cdot\omega)^2}{4\alpha(1+i\xi_{1c})}} \tilde{\chi}(z, \hat{\lambda}, \omega)\, d\hat{\lambda}\, d\omega. \end{aligned}$$

Letting $\tilde{\chi}(z,\hat{\lambda},\omega) = e^{\hat{\lambda}^2/(2\alpha)}$ this can be rewritten as

$$\begin{aligned}
&cxh \int |\alpha|^{-1/2}(1+i\xi_{1c})^{-1/2} e^{\frac{\hat{\lambda}^2}{2\alpha}} e^{-\frac{\hat{\lambda}^2(1+i\xi_{1c})}{4\alpha}} e^{-i\frac{\hat{\lambda}\eta_{1c}\cdot\omega}{2\alpha}} e^{\frac{(\eta_{1c}\cdot\omega)^2}{4\alpha(1+i\xi_{1c})}}\, d\hat{\lambda}\, d\omega \\
&= cxh \int |\alpha|^{-1/2}(1+i\xi_{1c})^{-1/2} e^{\frac{\hat{\lambda}^2(1-i\xi_{1c})}{4\alpha}} e^{-i\frac{\hat{\lambda}\eta_{1c}\cdot\omega}{2\alpha}} e^{\frac{(\eta_{1c}\cdot\omega)^2}{4\alpha(1+i\xi_{1c})}}\, d\hat{\lambda}\, d\omega \\
&= cxh \int |\alpha|^{-1/2}(1+i\xi_{1c})^{-1/2} e^{\frac{1-i\xi_{1c}}{4\alpha}(\hat{\lambda}-i\frac{\eta_{1c}\cdot\omega}{1-i\xi_{1c}})^2} e^{\frac{(\eta_{1c}\cdot\omega)^2}{4\alpha(1-i\xi_{1c})}} e^{\frac{(\eta_{1c}\cdot\omega)^2}{4\alpha(1+i\xi_{1c})}}\, d\hat{\lambda}\, d\omega \\
&= c'xh \int |\alpha|^{-1}(1+\xi_{1c}^2)^{-1/2} e^{\frac{(\eta_{1c}\cdot\omega)^2}{2\alpha(1+\xi_{1c}^2)}}\, d\omega
\end{aligned}$$

with c' non-zero, and now the integral is positive since the integrand is such. Since we need χ to be compactly supported, we approximate the Gaussian in the space of Schwartz functions by compactly supported χ; for suitable approximation the same positivity property follows. This completes the proof of the ellipticity, and thus the proof of Theorem 3.1.

3.3. **Consequences of Theorem 3.1.** Having proved Theorem 3.1, we can apply the results from Section 2.5 to the modified normal operator A. This means that there is a parametrix B in this new operator class, where the errors $A \circ B - \mathrm{Id}$ and $B \circ A - \mathrm{Id}$ are residual operators in the semiclassical foliation 1-cusp algebra. However, this ellipticity only applied for $x_0 \le \bar{x}_0$, and so these residual errors are only residual on the operator over this domain. This can be used by viewing the operator A acting on functions with support in this region $x_0 \le \bar{x}_0$. For functions supported in this collar region we conclude:

Corollary 3.4. *The modified normal operator A, in a region where $x \le \bar{x}_0$, has a left-parametrix which in the region $x \le \bar{x}_0$ is in the class $h^{-1}\Psi^{1,1}_{1c,\hbar,\mathcal{F}}$ with error in $h^\infty\Psi^{-\infty,-\infty}_{1c,\hbar,\mathcal{F}}$, and therefore for any sufficiently small h, A is left invertible on functions supported in this region.*

Proof of Corollary. Let O be a collar neighborhood of $\partial\overline{M}$ on which A is elliptic, and let $K = \{x \le \bar{x}_0\} \subset O$. Let ϕ be a cutoff function, identically 1 on K, supported in O. Let O' be open with $\overline{O'} \subset O$ and $\operatorname{supp}\phi \subset O'$. Then ellipticity gives us that there is an operator $B \in h^{-1}\Psi^{1,1}_{1c,\hbar,\mathcal{F}}$, such that the errors $E_1 = \mathrm{Id} - A \circ B$ and $E_2 = \mathrm{Id} - B \circ A$, while globally only satisfy that $E_1, E_2 \in \Psi^{0,0}_{1c,\hbar,\mathcal{F}}$, but locally on O' these errors are residual, and thus $\phi E_i \phi \in h^\infty\Psi^{-\infty,-\infty}_{1c,\hbar,\mathcal{F}}$, $i = 1, 2$. Now, $\phi BA\phi = \phi^2 + \phi E_2\phi$, and for v supported in K, $\phi v = v$, so

$$\phi BAv = v + \phi E_2\phi v = (\mathrm{Id} + \phi E_2\phi)v.$$

Now, $\phi E_2 \phi$ is $O(h^\infty)$ as a bounded operator on any weighted Sobolev space, so for h sufficiently small $\mathrm{Id} + \phi E_2\phi$ is invertible, and hence

$$v = (\mathrm{Id} + \phi E_2\phi)^{-1}\phi BAv.$$

This completes the proof of the stated left invertibility. □

In view if the definition of A, taking a sufficiently small h, this immediately implies:

Corollary 3.5. *There is a collar neighborhood of the boundary such that the (local) geodesic X-ray I is injective on sufficiently fast Gaussian decaying functions supported in this neighborhood.*

3.4. **Artificial boundary.** In order to prove the main result, Theorem 1.3, we simply need to add an artificial boundary, $x = \bar{x}_0$. We then work on the domain $\overline{\Omega} = \{x \le \bar{x}_0\}$, which has two disjoint boundary hypersurfaces, $x = 0$ and $x = \bar{x}_0$. We work with a foliation semiclassical algebra corresponding to the level sets of x such that in addition at $x = 0$ the algebra is 1-cusp, while at $x = \bar{x}_0$ it is scattering. Since the two boundary hypersurfaces are disjoint, this joint algebra can simply defined by localization. Indeed, we have already discussed the semiclassical foliation 1-cusp algebra $\Psi_{1\mathrm{c},\hbar,\mathcal{F}}$, which gives the localized behavior near $x = 0$ (or more strongly away from $x = \bar{x}_0$). In addition in [28] the semiclassical foliation algebra has been defined; this is the model near $x = \bar{x}_0$ (and more strongly away from $x = 0$). In both algebras if ϕ, ψ are C^∞ (on the compact underlying manifold) with disjoint support, the Schwartz kernels $\phi A \psi$, where A is an element of the algebra, are C^∞ and rapidly decreasing both in h and at the boundary. Thus, one can define the joint algebra, $\Psi_{\mathrm{sc},1\mathrm{c},\hbar,\mathcal{F}}$ by:

Definition 3.6. The space $\Psi^{m,l_1,l_2}_{\mathrm{sc},1\mathrm{c},\hbar,\mathcal{F}}$ consists of operators A on $\dot{C}^\infty(\overline{\Omega})$ such that

(1) If $\phi, \psi \in C^\infty(\overline{\Omega})$ with support disjoint from $x = 0$, then $\phi A \psi \in \Psi^{m,l_1}_{\mathrm{sc},\hbar,\mathcal{F}}$.
(2) If $\phi, \psi \in C^\infty(\overline{\Omega})$ with support disjoint from $x = \bar{x}_0$, then $\phi A \psi \in \Psi^{m,l_2}_{1\mathrm{c},\hbar,\mathcal{F}}$.
(3) If $\phi, \psi \in C^\infty(\overline{\Omega})$ with disjoint support then $\phi A \psi$ has Schwartz kernel which is C^∞ with rapid vanishing in h as well as all boundary hypersurfaces of $\overline{\Omega} \times \overline{\Omega}$.

Indeed, notice that if $1 = \phi_{\mathrm{sc}} + \phi_0 + \phi_{1\mathrm{c}}$ is a partition of unity with

$$\operatorname{supp} \phi_{\mathrm{sc}} \cap \{x = 0\} = \emptyset,\ \operatorname{supp} \phi_{1\mathrm{c}} \cap \{x = \bar{x}_0\} = \emptyset,$$
$$\operatorname{supp} \phi_{\mathrm{sc}} \cap \operatorname{supp} \phi_{1\mathrm{c}} = \emptyset,\ \operatorname{supp} \phi_0 \subset \{0 < x < \bar{x}_0\}$$

then any two of $\phi_{\mathrm{sc}}, \phi_0, \phi_{1\mathrm{c}}$ pairwise satisfy one of these conditions, so e.g. $\phi_0 A \phi_{1\mathrm{c}} \in \Psi^{m,l_2}_{1\mathrm{c},\hbar,\mathcal{F}}$, etc.

It is straightforward to check that $\Psi^{\infty,\infty,\infty}_{\mathrm{sc},1\mathrm{c},\hbar,\mathcal{F}}$ is a tri-filtered $*$-algebra, inheriting the properties of the two individual algebras whose amalgamation it is.

For the X-ray problem then let both $\tilde{\chi}$ and Φ be a combination of all the various forms of $\tilde{\chi}, \Phi$ for the ingredients. Concretely let

$$\Phi = F \circ x,$$

with $F' > 0$, $F(x) = -\frac{1}{2x^2}$, for x near 0, $F(x) = \frac{1}{\bar{x}_0 - x}$, for x near $\bar{x}_0$ (but $x < \bar{x}_0$), so our exponential weight is $e^{\Phi/h}$. Also let

$$\tilde{\chi} = \chi(x^{1/2}\lambda\sqrt{\Phi'}/(h^{1/2}|\alpha|^{1/2})),$$

with χ compactly supported, non-negative, identically 1 near 0. Then for x near 0,

$$\tilde{\chi} = \chi(\lambda/(h^{1/2}x|\alpha|^{1/2}))$$

as considered in the previous section (with the $|\alpha|^{1/2}$ factor irrelevant here, but showing up in the Gaussian being approximated for ellipticity), while for x near $\bar{x}_0$,

$$\tilde{\chi} = \chi(x^{1/2}\lambda/(h^{1/2}(\bar{x}_0 - x)|\alpha|^{1/2})),$$

as in the scattering setting of [28], with the extra factor of $x^{1/2}$ really should be considered in the context of $(x\lambda)/(h^{1/2}(\bar{x}_0 - x)|x\alpha|^{1/2})$, in view of the definition of λ and α here, see (3.3), vs. in [28]; in the latter our leading factor of x in (3.3) would be incorporated into these. The modified normal operator is then

$$A = e^{-\Phi/h}L\tilde{\chi}Ie^{\Phi/h}.$$

A simple combination of the pseudodifferential computations of the earlier sections and [28] shows that this operator is in $h\Psi^{-1,-2,-1}_{\mathrm{sc},1\mathrm{c},\hbar,\mathcal{F}}(\overline{\Omega})$ provided that there are no conjugate points on the boundary within distance $\pi/2$ as well as that geodesics do not have conjugate points to the point of tangency to an x-level set, with the latter following from the former if $\bar{x}_0$ is sufficiently small. Further, for suitable $\tilde{\chi}$, given by approximating a Gaussian $e^{-|.|^2/2}$ on $\mathbb{R}$ in Schwartz functions by compactly supported functions χ, the same combination yields ellipticity. This proves the main theorem:

Theorem 3.7. *On a sufficiently small collar neighborhood of infinity, specified by a level set of x as the artificial boundary, on an asymptotically conic manifold with no conjugate points within distance $\pi/2$, the modified normal operator $e^{-\Phi/h}L\tilde{\chi}Ie^{\Phi/h} \in h\Psi^{-1,-2,-1}_{\mathrm{sc},1\mathrm{c},\hbar,\mathcal{F}}(\overline{\Omega})$ is elliptic in the sense of the standard (differential), the 1-cusp (at infinity) and scattering (at the artificial boundary) boundary as well as the semiclassical principal symbols. In particular, it is an invertible operator for h sufficiently small.*

As a consequence, we can determine functions from their X-ray transform *without* a support condition.

Corollary 3.8. *The geodesic X-ray I, restricted to geodesics that stay in $x \le \bar{x}_0$, is injective on the restrictions to $x \le \bar{x}_0$ of sufficiently fast Gaussian decaying functions.*

As explained after Theorem 1.3, this proves Theorem 1.3.

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