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A STOCHASTIC FLUID-STRUCTURE INTERACTION PROBLEM WITH THE NAVIER-SLIP BOUNDARY CONDITION*

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Abstract. We prove the existence of martingale solutions to a stochastic fluid-structure interaction problem involving a viscous, incompressible fluid flow, modeled by the 2D Navier–Stokes equations, through a deformable elastic tube modeled by the 1D shell/membrane equations. The fluid and the structure are nonlinearly coupled via the kinematic and dynamic coupling conditions at the fluid-structure interface. This article considers the case where the structure can have unrestricted displacement and explores the Navier-slip boundary condition imposed at the fluid-structure interface, displacement of which is not known a priori and is a part of the solution itself. The proof takes a time-discretization approach based on a Lie splitting scheme. The geometric nonlinearity stemming from the nonlinear coupling, the possibility of random fluid domain degeneracy, the potential jumps in the tangential components of the fluid and structure velocities at the moving interface, and the low regularity of the structure velocity require the development of new techniques that lead to the local-in-time existence of analytically weak martingale solutions.

Key words. stochastic moving boundary problems, fluid-structure interaction, martingale solutions, Navier-slip condition

MSC codes. 60H15, 35A01

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1. Introduction.

This paper introduces an approach for investigating solutions to a complex problem describing the interaction between a deformable (purely) elastic membrane and a 2D viscous, incompressible fluid flow, under the influence of multiplicative stochastic forces. The fluid flow is described by the 2D Navier–Stokes equations, while the membrane is characterized by shell equations. The fluid and the structure are fully coupled across the moving interface through a two-way coupling that ensures continuity of the normal components of their velocities and contact forces at the interface. There has been a lot of work done in the field of deterministic fluid-structure interaction (FSI) in the past two decades (see, e.g., [6, 17, 8, 23, 24, 18, 21] and the references therein); however, even though there is a lot of evidence pointing to the need for studying the stochastic perturbations of the benchmark FSI models, the mathematical theory of stochastic FSI or, more generally, of stochastic PDEs on randomly moving domains is completely undeveloped.

The main result of this paper is the establishment of the existence of local-in-time weak martingale solutions to this nonlinear stochastic fluid-structure interaction problem. To be precise, the solutions are weak both in the analytical and probabilistic sense and exist until an almost surely positive stopping time. The stopping time kicks in as the fluid domain approaches a degenerate configuration. In this existence proof, we employ a Lie operator splitting scheme which was first utilized in the context of deterministic FSI in [24] and later, for example, in [25, 4]. The recent articles [32, 31], which represent the only work addressing stochastic moving boundary problems, have

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demonstrated the existence of weak martingale solutions to FSI problems with scalar and unrestricted structural deformations, respectively. They also consider the no-slip boundary conditions imposed at the fluid-structure interface. While this is a common assumption in the blood flow literature (see, e.g., [28, 5]), the slip condition is considered to be a more realistic boundary condition in modeling near-contact dynamics, such as the closure of heart valves, as it allows for the possibility of collisions (see, e.g., [27, 26]). In terms of (deterministic) FSI literature involving slip boundary conditions, the authors of [14] analyze the motion of a rigid body in viscous incompressible fluid. See also [7] in the context of a rigid body immersed in a compressible fluid. However, the difficulty increases significantly when an elastic body instead of a rigid body is considered. *In this work, we give the first existence result for a stochastic moving boundary problem involving the Navier-slip boundary condition imposed at the interface of a fluid and an elastic body.* It also provides, for the first time, a compactness argument, in the context of FSI involving elastic structures, by constructing test functions that are allowed to have possible jumps in the tangential direction at the fluid-structure interface, which is a key feature of the slip condition. *Our compactness result thus generalizes the existing results while also revealing hidden regularities of the structure.*

The first mathematical issues that we come across are related to the facts that the fluid domain boundary is a random variable, not known a priori, which can possibly degenerate in a random fashion and that the incompressibility condition and the Navier-slip boundary condition lead to the dependence of the test functions on the randomly moving domains and thus require us to consider random test processes, which is highly unusual for typical stochastic PDEs on fixed domains. Due to the possibility of nonzero longitudinal structural displacement, extra care has to be taken in dealing with degenerate fluid domains, i.e., when the structure touches a part of the fluid domain boundary during deformation.

First, using the arbitrary Lagrangian–Eulerian (ALE) transformations, we map the fluid equations onto a fixed domain. ALE mappings have been extensively used in FSI and numerical simulations of moving boundary problems; see, e.g., [30, 11, 20, 28]. The use of these ALE maps and the analysis that follows is valid for as long as there is no loss of injectivity of the ALE transformation. To deal with this injectivity condition in the stochastic case we use a cut-off function and a stopping time argument. Furthermore, via the ALE maps, additional nonlinearities appear in the weak formulation of the problem that track several geometric quantities such as the fluid-structure interface tangent and normal.

The dependence of the test functions on the domain configurations (via the ALE maps) creates issues as we move to a new probability space in search of martingale solutions. Hence, we introduce a system that approximates the original system by augmenting it by a singular term that penalizes the divergence and the boundary behavior of the fluid velocity. However, addition of this penalty term and the low temporal regularity of the solutions create further difficulties in establishing compactness which we overcome by employing nonstandard compactness arguments. In establishing tightness of the laws of the approximate solutions, we also do not have the extra regularity for the structure velocity obtained from the fluid dissipation in the no-slip case. We finally show that the solutions to the approximate systems indeed converge to a desired martingale solution of the limiting equations.

Since the stochastic forcing appears not only in the structure equations but also in the fluid equations themselves, we come across additional difficulties, which are associated with the construction of the appropriate "test processes" on the approximate and limiting (time-dependent and random) fluid domains. Namely, along with the required divergence-free property on these domains, the test functions have to satisfy appropriate boundary and measurability conditions. We construct these approximate test functions by first constructing a Carathéodory function that gives the definition of a test function for the limiting equations and then by transforming this limiting test function in a way that preserves its desired properties on the approximate domains.

The paper is organized as follows: We describe the fluid and the structure problems along with the coupling conditions and the noise structure in section 2. This section also contains the definition of solutions and the main result of this paper. In section 3, we introduce the approximation scheme. Finally, in sections 4 and 5, we pass the time step and the structure velocity regularization parameter, which is added to deal with the corners of the fixed domain, to 0, respectively.

2. Problem setup.

We begin describing the problem by first considering the deterministic model.

2.1. The deterministic model and a weak formulation.

We consider the flow of an incompressible, viscous fluid in a 2D compliant cylinder $\mathcal{O} = (0,L) \times (0,1)$ with a deformable lateral boundary Γ. The left and the right boundary of the cylinder are the inlet and outlet for the time-dependent fluid flow. We assume "axial symmetry" of the data and of the flow, which allows us to consider the flow only in the upper half of the domain, with the bottom boundary fixed and equipped with the symmetry boundary conditions. Assume that the time-dependent fluid domain, whose displacement is not known a priori, is denoted by $\mathcal{O}_{\boldsymbol{\eta}}(t) = \boldsymbol{\varphi}(t,\mathcal{O})$, whereas its deformable interface is given by $\Gamma_{\boldsymbol{\eta}}(t) = \boldsymbol{\varphi}(t,\Gamma)$. Assume that $\boldsymbol{\varphi} : \mathcal{O} \to \mathcal{O}_{\boldsymbol{\eta}}$ is a C^1 diffeomorphism such that

$$\boldsymbol{\varphi}|_{\Gamma_{in},\Gamma_{out},\Gamma_b} = \text{id}, \quad \det\nabla\boldsymbol{\varphi}(t,\mathbf{x}) > 0,$$

where the inlet, outlet, and bottom boundaries of $\mathcal{O}$ are given by $\Gamma_{\text{in}} = \{0\} \times (0,1), \Gamma_{\text{out}} = \{L\} \times (0,1), \Gamma_b = (0,L) \times \{0\}$, respectively. The displacement of the elastic structure at the top lateral boundary Γ, which can be identified by $(0,L)$, will be given by $\boldsymbol{\eta}(t,z) = \boldsymbol{\varphi}(t,z) - (z,1)$ for $z \in (0,L)$ (see Figure 1). The mapping $\boldsymbol{\eta} : [0,L] \times [0,T] \to \mathbb{R}^2$ such that $\boldsymbol{\eta} = (\eta_z(z,t), \eta_r(z,t))$ is one of the unknowns in the problem.

The fluid subproblem. The fluid flow is modeled by the incompressible Navier–Stokes equations in the 2D time-dependent domain $\mathcal{O}_{\boldsymbol{\eta}}(t)$:

$$\begin{aligned} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} &= \nabla\cdot\sigma + F_u^{ext}, \\ \nabla\cdot\mathbf{u} &= 0, \end{aligned} \tag{2.1}$$

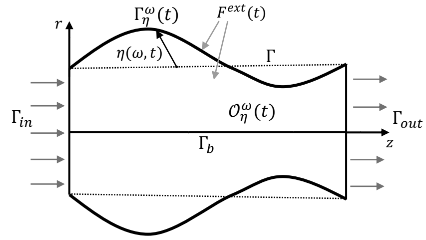

Fig. 1. *A snapshot of a realization of the fluid domain for some $\omega \in \Omega$ and $t \in [0,T]$.*

where $\mathbf{u}=(u_z,u_r)$ is the fluid velocity. The Cauchy stress tensor is $\sigma=-pI+2\nu\mathbf{D}(\mathbf{u})$, where p is the fluid pressure, ν is the kinematic viscosity coefficient, and $\mathbf{D}(\mathbf{u})=\frac{1}{2}(\nabla\mathbf{u}+(\nabla\mathbf{u})^T)$ is the symmetrized gradient. Here F_u^{ext} represents any external forcing impacting the fluid. In this work we will be assuming that this force is random, as we shall see below. The fluid flow is driven by dynamic pressure data given at the inlet and outlet boundaries, and we prescribe the symmetry boundary condition on the bottom boundary as follows:

$$p+\frac{1}{2}|\mathbf{u}|^2=P_{in/out}(t), \qquad u_r=0 \quad \text{on } \Gamma_{in/out}, \tag{2.2}$$

$$u_r=\partial_r u_z=0 \quad \text{on } \Gamma_b. \tag{2.3}$$

The structure subproblem. The elastodynamics problem for the displacement $\boldsymbol{\eta}=(\eta_z,\eta_r)$ of the structure with respect to Γ is as follows:

$$\partial_t^2\boldsymbol{\eta}+\mathscr{L}_e(\boldsymbol{\eta})=F_{\boldsymbol{\eta}} \quad \text{in } (0,L), \tag{2.4}$$

where $F_{\boldsymbol{\eta}}$ is the total force experienced by the structure and $\mathscr{L}_e$ is a continuous, self-adjoint, coercive, linear operator on $\mathbf{H}_0^2(0,L)$. This equation is supplemented with the following boundary conditions:

$$\boldsymbol{\eta}(0)=\boldsymbol{\eta}(L)=\partial_z\boldsymbol{\eta}(0)=\partial_z\boldsymbol{\eta}(L)=0. \tag{2.5}$$

The nonlinear fluid-structure coupling. The coupling between the fluid and the structure takes place across the moving fluid-structure interface.

- The kinematic coupling conditions in the *Navier-slip* case are

$$\partial_t\boldsymbol{\eta}(t,z)\cdot\mathbf{n}^{\boldsymbol{\eta}}=\mathbf{u}(\boldsymbol{\varphi}(t,z))\cdot\mathbf{n}^{\boldsymbol{\eta}}, \quad (t,z)\in[0,T]\times[0,L], \tag{2.6}$$

$$(\partial_t\boldsymbol{\eta}-\mathbf{u}(\boldsymbol{\varphi}(t,z)))\cdot\boldsymbol{\tau}^{\boldsymbol{\eta}}=\alpha\sigma(\boldsymbol{\varphi}(t,z))\mathbf{n}^{\boldsymbol{\eta}}\cdot\boldsymbol{\tau}^{\boldsymbol{\eta}}, \quad (t,z)\in[0,T]\times[0,L]. \tag{2.7}$$

- The dynamic coupling condition is

$$F_{\boldsymbol{\eta}}=-S_{\boldsymbol{\eta}}(t,z)(\sigma\mathbf{n}^{\boldsymbol{\eta}})|_{(t,z,\boldsymbol{\varphi}(t,z))}+F_{\boldsymbol{\eta}}^{ext}, \tag{2.8}$$

  where $\mathbf{n}^{\boldsymbol{\eta}}(t,z)$ is the unit outward normal to the top boundary at the point, $\boldsymbol{\tau}^{\boldsymbol{\eta}}$ is the tangent vector given by $\boldsymbol{\tau}^{\boldsymbol{\eta}}(t,z)=\partial_z\boldsymbol{\varphi}(t,z)$, and $S_{\boldsymbol{\eta}}(t,z)$ is the Jacobian of the transformation from Eulerian to Lagrangian coordinates. As earlier, $F_{\boldsymbol{\eta}}^{ext}$ denotes any external force impacting the structure.

This system is supplemented with the following initial conditions:

$$\mathbf{u}(t=0)=\mathbf{u}_0, \quad \boldsymbol{\eta}(t=0)=\boldsymbol{\eta}_0, \quad \partial_t\boldsymbol{\eta}(t=0)=\mathbf{v}_0. \tag{2.9}$$

Weak formulation on moving domain. Using the convention that boldface letters denote spaces containing vector-valued functions, we define the following relevant function spaces for the fluid velocity and the structure displacement:

$$\begin{aligned}
&\widetilde{\mathscr{V}}_F(t)=\{\mathbf{u}=(u_z,u_r)\in\mathbf{H}^1(\mathcal{O}_{\boldsymbol{\eta}}(t)):\nabla\cdot\mathbf{u}=0, \text{ and } u_r=0 \text{ on } \partial\mathcal{O}_{\boldsymbol{\eta}}\setminus\Gamma_{\boldsymbol{\eta}}(t)\},\\
&\widetilde{\mathscr{W}}_F(0,T)=L^\infty(0,T;\mathbf{L}^2(\mathcal{O}_{\boldsymbol{\eta}}(\cdot)))\cap L^2(0,T;\widetilde{\mathscr{V}}_F(\cdot)),\\
&\widetilde{\mathscr{W}}_S(0,T)=W^{1,\infty}(0,T;\mathbf{L}^2(0,L))\cap L^\infty(0,T;\mathbf{H}_0^2(0,L))\cap H^1(0,T;\mathbf{H}^1(0,L)),\\
&\widetilde{\mathscr{W}}(0,T)=\{(\mathbf{u},\boldsymbol{\eta})\in\widetilde{\mathscr{W}}_F(0,T)\times\widetilde{\mathscr{W}}_S(0,T):\mathbf{u}(\boldsymbol{\varphi}(t,z))\cdot\mathbf{n}^{\boldsymbol{\eta}}=\partial_t\boldsymbol{\eta}(t,z)\cdot\mathbf{n}^{\boldsymbol{\eta}},\\
&\qquad\qquad (t,z)\in(0,T)\times\Gamma\}.
\end{aligned}$$

Next, we derive a deterministic weak formulation of the problem on the moving domains. We consider $\mathbf{q} \in C^1([0,T];\widetilde{\mathscr{V}}_F(\cdot))$ such that $\mathbf{q}(\boldsymbol{\varphi}(t,z)) \cdot \mathbf{n}^{\boldsymbol{\eta}} = \boldsymbol{\psi}(t,z) \cdot \mathbf{n}^{\boldsymbol{\eta}}$ on $(0,T) \times \Gamma$ for some $\boldsymbol{\psi} \in C^1([0,T];\mathbf{H}_0^2(\Gamma))$. We multiply (2.1) by $\mathbf{q}$, integrate in time and space, and use Reynold's transport theorem to obtain

$$
\begin{aligned}
(\mathbf{u}(t),\mathbf{q}(t))_{\mathcal{O}_{\boldsymbol{\eta}}(t)} = (\mathbf{u}(0),\mathbf{q}(0))_{\mathcal{O}_{\boldsymbol{\eta}}(0)} &+ \int_0^t \int_{\mathcal{O}_{\boldsymbol{\eta}}(s)} \mathbf{u}(s) \cdot \partial_s \mathbf{q}(s) \\
&+ \int_0^t \int_{\Gamma_{\boldsymbol{\eta}}(s)} (\mathbf{u}(s)\cdot\mathbf{q}(s))(\mathbf{u}(s)\cdot\mathbf{n}^{\boldsymbol{\eta}}(s)) - \int_0^t \int_{\mathcal{O}_{\boldsymbol{\eta}}(s)} (\mathbf{u}(s)\cdot\nabla)\mathbf{u}(s)\mathbf{q}(s) \\
&- 2\nu \int_0^t \int_{\mathcal{O}_{\boldsymbol{\eta}}(s)} \mathbf{D}(\mathbf{u}(s))\cdot\mathbf{D}(\mathbf{q}(s))ds + \int_0^t \int_{\partial\mathcal{O}_{\boldsymbol{\eta}}(s)} (\sigma\mathbf{n}^{\boldsymbol{\eta}}(s))\cdot\mathbf{q}(s) \\
&+ \int_0^t \int_{\mathcal{O}_{\boldsymbol{\eta}}(s)} F_u^{ext}(s)\mathbf{q}(s).
\end{aligned}
$$

Set

$$
b(t,\mathbf{u},\mathbf{v},\mathbf{w}) := \frac{1}{2}\int_{\mathcal{O}_{\boldsymbol{\eta}}(t)} \left((\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w} - (\mathbf{u}\cdot\nabla)\mathbf{w}\cdot\mathbf{v}\right),
$$

and observe that

$$
\begin{aligned}
-((\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{q})_{\mathcal{O}_{\boldsymbol{\eta}}} &= -\frac{1}{2}((\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{q})_{\mathcal{O}_{\boldsymbol{\eta}}} + \frac{1}{2}((\mathbf{u}\cdot\nabla)\mathbf{q},\mathbf{u})_{\mathcal{O}_{\boldsymbol{\eta}}} - \frac{1}{2}\int_{\partial\mathcal{O}_{\boldsymbol{\eta}}} \mathbf{u}\cdot\mathbf{q}\mathbf{u}\cdot\mathbf{n}^{\boldsymbol{\eta}} \\
&= -b(s,\mathbf{u},\mathbf{u},\mathbf{q}) - \frac{1}{2}\int_{\Gamma_{\boldsymbol{\eta}}} \mathbf{u}\cdot\mathbf{q}\mathbf{u}\cdot\mathbf{n}^{\boldsymbol{\eta}} + \frac{1}{2}\int_{\Gamma_{in}} |u_z|^2 q_z - \frac{1}{2}\int_{\Gamma_{out}} |u_z|^2 q_z.
\end{aligned}
$$

Using the divergence-free property of fluid velocity $\mathbf{u}$ and the boundary conditions $u_r = 0$ on $\Gamma_{in/out}$, we have that $\partial_r u_r = -\partial_z u_z = 0$ on $\Gamma_{in/out}$. Hence,

$$
\int_{\Gamma_{in/out}} \sigma\mathbf{n}^{\boldsymbol{\eta}}\cdot\mathbf{q} = \int_{\Gamma_{in/out}} \pm p\, q_z = \int_{\Gamma_{in}} \left(P_{in} - \frac{1}{2}|\mathbf{u}|^2\right) q_z - \int_{\Gamma_{out}} \left(P_{out} - \frac{1}{2}|\mathbf{u}|^2\right) q_z,
$$

whereas $\int_{\Gamma_b} \sigma\mathbf{n}^{\boldsymbol{\eta}}\cdot\mathbf{q} = 0$. We also write $\int_{\Gamma_{\boldsymbol{\eta}}} \sigma\mathbf{n}^{\boldsymbol{\eta}}\cdot\mathbf{q}$ as

$$
\int_{\Gamma_{\boldsymbol{\eta}}} \sigma\mathbf{n}^{\boldsymbol{\eta}}\cdot((\mathbf{q}\cdot\mathbf{n}^{\boldsymbol{\eta}})\mathbf{n}^{\boldsymbol{\eta}} + (\mathbf{q}\cdot\boldsymbol{\tau}^{\boldsymbol{\eta}})\boldsymbol{\tau}^{\boldsymbol{\eta}}) = \int_{\Gamma_{\boldsymbol{\eta}}} \sigma\mathbf{n}^{\boldsymbol{\eta}}\cdot\mathbf{n}^{\boldsymbol{\eta}}(\boldsymbol{\psi}\cdot\mathbf{n}^{\boldsymbol{\eta}}) + \frac{1}{\alpha}(\partial_t\boldsymbol{\eta} - \mathbf{u})\cdot\boldsymbol{\tau}^{\boldsymbol{\eta}}(\mathbf{q}\cdot\boldsymbol{\tau}^{\boldsymbol{\eta}}).
$$

Next we multiply the structure equation (2.4) by $\boldsymbol{\psi}$ and integrate in time and space to obtain

$$
\begin{aligned}
(\partial_t\boldsymbol{\eta}(t),\boldsymbol{\psi}(t)) = (\mathbf{v}_0,\boldsymbol{\psi}(0)) &+ \int_0^t\int_0^L \partial_s\boldsymbol{\eta}\cdot\partial_s\boldsymbol{\psi}dzds - \int_0^t \langle\mathscr{L}_e(\boldsymbol{\eta}),\boldsymbol{\psi}\rangle ds \\
&- \int_0^t\int_0^L S_{\boldsymbol{\eta}}\sigma\mathbf{n}^{\boldsymbol{\eta}}\cdot\boldsymbol{\psi}dzds + \int_0^t\int_0^L F_{\boldsymbol{\eta}}^{ext}\cdot\boldsymbol{\psi}dzds.
\end{aligned}
$$

Observe that we can write

$$
\begin{aligned}
\int_0^t\int_0^L S_{\boldsymbol{\eta}}\sigma\mathbf{n}^{\boldsymbol{\eta}}\cdot\boldsymbol{\psi}dzds &= \int_0^t\int_0^L S_{\boldsymbol{\eta}}\sigma\mathbf{n}^{\boldsymbol{\eta}}\cdot((\boldsymbol{\psi}\cdot\mathbf{n}^{\boldsymbol{\eta}})\mathbf{n}^{\boldsymbol{\eta}} + (\boldsymbol{\psi}\cdot\tau^{\boldsymbol{\eta}})\tau^{\boldsymbol{\eta}})dzds \\
&= \int_0^t\int_0^L S_{\boldsymbol{\eta}}\left(\sigma\mathbf{n}^{\boldsymbol{\eta}}\cdot\mathbf{n}^{\boldsymbol{\eta}}(\boldsymbol{\psi}\cdot\mathbf{n}^{\boldsymbol{\eta}}) + (\boldsymbol{\psi}\cdot\tau^{\boldsymbol{\eta}})\frac{1}{\alpha}(\partial_t\boldsymbol{\eta} - \mathbf{u})\cdot\tau^{\boldsymbol{\eta}}\right)dzds.
\end{aligned}
$$

Hence, in conclusion, we look for $(\mathbf{u},\boldsymbol{\eta}) \in \widetilde{\mathscr{W}}(0,T)$, which satisfies the following equation for almost every $t \in [0,T]$ and for any test function $\mathbf{Q} = (\mathbf{q},\boldsymbol{\psi})$ described above:

(2.10)
$$\begin{aligned}&\int_{\mathcal{O}_{\boldsymbol{\eta}(t)}}\mathbf{u}(t)\mathbf{q}(t)d\mathbf{x}+\int_0^L\partial_t\boldsymbol{\eta}(t)\boldsymbol{\psi}(t)dz-\int_0^t\int_{\mathcal{O}_{\boldsymbol{\eta}(s)}}\mathbf{u}\cdot\partial_s\mathbf{q}d\mathbf{x}ds-\int_0^t\int_0^L\partial_s\boldsymbol{\eta}\partial_s\boldsymbol{\psi}dzds\\&\quad+\int_0^t b(s,\mathbf{u},\mathbf{u},\mathbf{q})ds-\frac{1}{2}\int_0^t\int_{\Gamma_{\boldsymbol{\eta}}}(\mathbf{u}\cdot\mathbf{q})(\mathbf{u}\cdot\mathbf{n}^{\boldsymbol{\eta}})dSds+2\nu\int_0^t\int_{\mathcal{O}_{\boldsymbol{\eta}(s)}}\mathbf{D}(\mathbf{u})\cdot\mathbf{D}(\mathbf{q})d\mathbf{x}ds\\&\quad+\frac{1}{\alpha}\int_0^t\int_0^L S_{\boldsymbol{\eta}}(\partial_t\boldsymbol{\eta}-\mathbf{u})\cdot\boldsymbol{\tau}^{\boldsymbol{\eta}}((\mathbf{q}-\boldsymbol{\psi})\cdot\boldsymbol{\tau}^{\boldsymbol{\eta}})dzds+\int_0^t\langle\mathscr{L}_e(\boldsymbol{\eta}),\boldsymbol{\psi}\rangle ds\\&=\int_{\mathcal{O}_{\eta_0}}\mathbf{u}_0\mathbf{q}(0)d\mathbf{x}+\int_0^L\mathbf{v}_0\boldsymbol{\psi}(0)dz+\int_0^t P_{in}\int_0^1 q_z\Big|_{z=0}drds-\int_0^t P_{out}\int_0^1 q_z\Big|_{z=1}drds\\&\quad+\int_0^t\int_{\mathcal{O}_{\boldsymbol{\eta}}(s)}\mathbf{q}\cdot F_u^{ext}\,d\mathbf{x}ds+\int_0^t\int_0^L\boldsymbol{\psi}\cdot F_{\boldsymbol{\eta}}^{ext}\,dzds.\end{aligned}$$

Here F_u^{ext} is the volumetric external force applied to the fluid and $F_{\boldsymbol{\eta}}^{ext}$ is the external force applied to the deformable boundary.

2.2. Stochastic framework on fixed domain. We will take $F_u^{ext}, F_{\boldsymbol{\eta}}^{ext}$ to be random forces. We consider a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ that satisfies the usual assumptions, i.e., $\mathcal{F}_0$ is complete and the filtration is right continuous; that is, $\mathcal{F}_t=\cap_{s\geq t}\mathcal{F}_s$ for all $t\geq 0$.

2.2.1. ALE mappings. To deal the geometric nonlinearity arising due to the motion of the fluid domain, we work with the arbitrary Lagrangian–Eulerian (ALE) mappings which are a family of diffeomorphisms from the fixed domain $\mathcal{O}=(0,L)\times(0,1)$ onto the moving domain $\mathcal{O}_{\boldsymbol{\eta}}(t)$. Notice that the presence of the stochastic forcing implies that the domains $\mathcal{O}_{\boldsymbol{\eta}}$ are themselves random and that we must define the ALE mappings *pathwise*. That is, for every $\omega\in\Omega$ we will consider the maps $A^\omega_{\boldsymbol{\eta}}(t):\mathcal{O}\to\mathcal{O}_{\boldsymbol{\eta}}(t,\omega)$ such that $A^\omega_{\boldsymbol{\eta}}(t)=\mathbf{id}+\boldsymbol{\eta}(t,\omega)$ on Γ and $A^\omega_{\boldsymbol{\eta}}(t)=\mathbf{id}$ on $\partial\mathcal{O}\setminus\Gamma$.

The pathwise transformed gradient, symmetrized gradient, and divergence under this transformation will be denoted by

$$\nabla^{\boldsymbol{\eta}}f=\nabla f(\nabla A_{\boldsymbol{\eta}})^{-1},\quad \mathbf{D}^{\boldsymbol{\eta}}(\mathbf{u})=\frac{1}{2}(\nabla^{\boldsymbol{\eta}}\mathbf{u}+(\nabla^{\boldsymbol{\eta}})^T\mathbf{u}),\quad \text{and } \operatorname{div}^{\boldsymbol{\eta}}f=\operatorname{tr}(\nabla^{\boldsymbol{\eta}}f).$$

The Jacobian of the ALE mapping is given by $J^\omega_{\boldsymbol{\eta}}(t)=\det\nabla A^\omega_{\boldsymbol{\eta}}(t)$. Using $\mathbf{w}^{\boldsymbol{\eta}}$ to denote the ALE velocity $\mathbf{w}^{\boldsymbol{\eta}}=\frac{d}{dt}A_{\boldsymbol{\eta}}$, we note that $\partial_t J_{\boldsymbol{\eta}}=J_{\boldsymbol{\eta}}\nabla^{\boldsymbol{\eta}}\cdot\mathbf{w}^{\boldsymbol{\eta}}$. We also rewrite the advection term as follows:

$$b^\eta(\mathbf{u},\mathbf{w},\mathbf{q})=\frac{1}{2}\int_{\mathcal{O}}J_{\boldsymbol{\eta}}\left(((\mathbf{u}-\mathbf{w}^{\boldsymbol{\eta}})\cdot\nabla^{\boldsymbol{\eta}})\mathbf{u}\cdot\mathbf{q}-((\mathbf{u}-\mathbf{w}^{\boldsymbol{\eta}})\cdot\nabla^{\boldsymbol{\eta}})\mathbf{q}\cdot\mathbf{u}\right).$$

We will transform (2.10) using these ALE maps and give the definition of martingale solutions on the fixed domain $\mathcal{O}$.

We begin by describing the noise. We will assume that the external forces F_u^{ext} and $F_{\boldsymbol{\eta}}^{ext}$ are multiplicative stochastic forces and that we can then write the combined stochastic forcing F^{ext} as follows:

$$F^{ext}:=G(\mathbf{u},\boldsymbol{\eta})dW, \tag{2.11}$$

where W is a U-valued Wiener process with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, where U is a separable Hilbert space. We denote by Q the covariance operator of W, which is a positive, trace class operator on U, and define $U_0 := Q^{\frac{1}{2}}(U)$. Letting $\mathbf{L}^2 = \mathbf{L}^2(\mathcal{O}) \times \mathbf{L}^2(0,L)$, we now give assumptions on the noise coefficient G.

Assumption 2.1. The noise coefficient G is a function $G : \mathbf{L}^2(\mathcal{O}) \times \mathbf{H}_0^2(0,L) \to L_2(U_0;\mathbf{L}^2)$, such that for any $\frac{3}{2} \leq s < 2$ the following conditions hold true:

$$\begin{aligned} &\|G(\mathbf{u},\boldsymbol{\eta})\|_{L_2(U_0;\mathbf{L}^2)} \leq \|\mathbf{u}\|_{\mathbf{L}^2(\mathcal{O})} + \|\boldsymbol{\eta}\|_{\mathbf{H}_0^2(0,L)}, \\ &\|G(\mathbf{u}_1,\boldsymbol{\eta}_1) - G(\mathbf{u}_2,\boldsymbol{\eta}_2)\|_{L_2(U_0;\mathbf{L}^2)} \leq \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{L}^2(\mathcal{O})} + \|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_{\mathbf{L}^2(0,L)}. \end{aligned} \tag{2.12}$$

Here $L_2(X,Y)$ denotes the space of Hilbert–Schmidt operators from Hilbert spaces X to Y.

2.2.2. Definition of martingale solutions. We will now introduce the functional framework for the stochastic problem on the fixed reference domain $\mathcal{O} = (0,L) \times (0,1)$. The following are the functional spaces for the stochastic FSI problem defined on the fixed domain $\mathcal{O}$:

$$\begin{aligned} &V = \{\mathbf{u} = (u_z,u_r) \in \mathbf{H}^1(\mathcal{O}) : u_r = 0 \text{ on } \Gamma_{in/out/b}\}, \\ &\mathscr{W}_F = L^2(\Omega; L^\infty(0,T;\mathbf{L}^2(\mathcal{O}))) \cap L^2(\Omega; L^2(0,T;V)), \\ &\mathscr{W}_S = L^2(\Omega; W^{1,\infty}(0,T;\mathbf{L}^2(0,L)) \cap L^\infty(0,T;\mathbf{H}_0^2(0,L)) \cap H^1(0,T;\mathbf{H}_0^1(0,L))), \\ &\mathscr{W}(0,T) = \{(\mathbf{u},\boldsymbol{\eta}) \in \mathscr{W}_F \times \mathscr{W}_S : \mathbf{u}(t)|_\Gamma \cdot \mathbf{n}^{\boldsymbol{\eta}} = \partial_t\boldsymbol{\eta}(t) \cdot \mathbf{n}^{\boldsymbol{\eta}},\ \nabla^{\boldsymbol{\eta}} \cdot \mathbf{u} = 0 \ \mathbb{P}\text{-a.s.}\}. \end{aligned}$$

We also define the following spaces for test functions and fluid-structure velocities:

$$\mathscr{D} = V \times \mathbf{H}_0^2(0,L) \quad \text{and} \quad \mathscr{U} = V \times \mathbf{L}^2(0,L). \tag{2.13}$$

In what follows, the notation $\mathbf{v} = \partial_t\boldsymbol{\eta}$ will be used for denoting the structure velocity.

DEFINITION 2.1 (martingale solution). *Given compatible deterministic initial data,* $\mathbf{u}_0 \in \mathbf{L}^2(\mathcal{O})$, $\mathbf{v}_0 \in \mathbf{L}^2(0,L)$, *and initial structure displacement* $\boldsymbol{\eta}_0 \in \mathbf{H}_0^2(0,L)$ *that satisfies for some* $\delta_1, \delta_2 > 0$

$$\delta_1 < \inf_{\mathcal{O}} J_{\boldsymbol{\eta}_0} \quad \text{and} \quad \|\boldsymbol{\eta}_0\|_{\mathbf{H}_0^2(0,L)} < \frac{1}{\delta_2}, \tag{2.14}$$

we say that $(\mathscr{S},\mathbf{u},\boldsymbol{\eta},T^{\boldsymbol{\eta}})$ *is a martingale solution to the system* (2.1)–(2.9) *under the assumptions* (2.12) *if*

1. $\mathscr{S} = (\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P},W)$ *is a stochastic basis, that is,* $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ *is a filtered probability space satisfying the usual conditions and* W *is a* U*-valued Wiener process.*
2. $(\mathbf{u},\boldsymbol{\eta}) \in \mathscr{W}(0,T)$.
3. $T^{\boldsymbol{\eta}}$ *is a* $\mathbb{P}$*-a.s. strictly positive,* $\mathcal{F}_t$*-stopping time.*
4. $\mathbf{u}$ *and* $\boldsymbol{\eta}$ *are* $(\mathcal{F}_t)_{t\geq 0}$*-progressively measurable.*
5. *For every* $(\mathcal{F}_t)_{t\geq 0}$*-adapted, essentially bounded process* $\mathbf{Q} := (\mathbf{q},\boldsymbol{\psi})$ *with* C^1 *paths in* $\mathscr{D}$ *such that* $\nabla^{\boldsymbol{\eta}} \cdot \mathbf{q} = 0$ *and* $\mathbf{q}_\Gamma \cdot \mathbf{n}^{\boldsymbol{\eta}} = \boldsymbol{\psi} \cdot \mathbf{n}^{\boldsymbol{\eta}}$, *the equation*

(2.15)
$$\begin{aligned} \int_{\mathcal{O}} J_{\boldsymbol{\eta}}(t)\mathbf{u}(t)\mathbf{q}(t) + \int_0^L \partial_t\boldsymbol{\eta}(t)\boldsymbol{\psi}(t) &= \int_{\mathcal{O}} J_0\mathbf{u}_0\mathbf{q}(0) + \int_0^L \mathbf{v}_0\boldsymbol{\psi}(0) \\ &+ \int_0^t\int_{\mathcal{O}} J_{\boldsymbol{\eta}}\,\mathbf{u}\cdot\partial_t\mathbf{q} - \frac{1}{2}\int_0^t\int_{\mathcal{O}} J_{\boldsymbol{\eta}}(\mathbf{u}\cdot\nabla^{\boldsymbol{\eta}}\mathbf{u}\cdot\mathbf{q} - (\mathbf{u}-2\mathbf{w})\cdot\nabla^{\boldsymbol{\eta}}\mathbf{q}\cdot\mathbf{u}) \\ &- 2\nu\int_0^t\int_{\mathcal{O}} J_{\boldsymbol{\eta}}\mathbf{D}^{\boldsymbol{\eta}}(\mathbf{u})\cdot\mathbf{D}^{\boldsymbol{\eta}}(\mathbf{q}) - \frac{1}{\alpha}\int_0^t\int_\Gamma S_{\boldsymbol{\eta}}(\mathbf{u}-\partial_t\boldsymbol{\eta})\cdot\tau^{\boldsymbol{\eta}}((\mathbf{q}-\boldsymbol{\psi})\cdot\tau^{\boldsymbol{\eta}}) \end{aligned}$$

$$+\frac{1}{2}\int_0^t\int_\Gamma S_{\boldsymbol\eta}(\partial_t\boldsymbol\eta\cdot\mathbf{n}^{\boldsymbol\eta})(\mathbf{u}\cdot\mathbf{q})+\int_0^t\int_0^L\partial_t\boldsymbol\eta\partial_t\boldsymbol\psi-\int_0^t\langle\mathscr{L}_e(\boldsymbol\eta),\boldsymbol\psi\rangle$$
$$+\int_0^t\left(P_{in}\int_0^1 q_z\Big|_{z=0}dr-P_{out}\int_0^1 q_z\Big|_{z=1}dr\right)+\int_0^t(\mathbf{Q},G(\mathbf{u},\boldsymbol\eta)\,dW)$$

holds $\mathbb{P}$*-a.s. for almost every* $t\in[0,T^{\boldsymbol\eta})$.

We are now in a position to state the main result of this paper.

THEOREM 2.2. *Given compatible deterministic initial data* $\mathbf{u}_0\in\mathbf{L}^2(\mathcal{O})$, $\mathbf{v}_0\in\mathbf{L}^2(0,L)$, *and* $\boldsymbol\eta_0\in\mathbf{H}_0^2(0,L)$, *there exists at least one martingale solution to the FSI problem* (2.1)–(2.9) *in the sense of Definition* 2.1.

In what follows, we will present a proof for Theorem 2.2 based on the operator splitting scheme constructed in the following section.

3. Operator splitting scheme.

In this section we introduce a Lie operator splitting scheme that defines a sequence of approximate solutions to (2.15) by semidiscretizing the problem in time. Our aim is to show that up to a subsequence, approximate solutions converge in a certain sense to a martingale solution of the stochastic FSI problem.

3.1. Definition of the splitting scheme.

We discretize the problem in time and use an operator splitting to decouple the stochastic problem into two subproblems, viz., the structure and the fluid subproblems. We denote the time step by $\Delta t=\frac{T}{N}$ and use the notation $t^n=n\Delta t$ for $n=0,1,\dots,N$. Let $(\mathbf{u}^0,\boldsymbol\eta^0,\mathbf{v}^0)=(\mathbf{u}_0,\boldsymbol\eta_0,\mathbf{v}_0)$ be the initial data. Then at the ith time level, we update the vector $(\mathbf{u}^{n+\frac{i}{2}},\boldsymbol\eta^{n+\frac{i}{2}},\mathbf{v}^{n+\frac{i}{2}})$, for $i=1,2$ and $n=0,1,2,\dots,N-1$, as follows.

The structure subproblem. In this subproblem we update the structure displacement and the structure velocity while keeping the fluid velocity unchanged. That is, given $(\boldsymbol\eta^n,\mathbf{v}^n)\in\mathbf{H}_0^2(0,L)\times\mathbf{L}^2(0,L)$, we look for a pathwise solution $(\boldsymbol\eta^{n+\frac12},\mathbf{v}^{n+\frac12})\in\mathbf{H}_0^2(0,L)\times\mathbf{H}_0^2(0,L)$ to the following equations: For any $\boldsymbol\phi\in\mathbf{L}^2(0,L)$ and $\boldsymbol\psi\in\mathbf{H}_0^2(0,L)$,

$$\begin{aligned}&\mathbf{u}^{n+\frac12}=\mathbf{u}^n,\\ &\int_0^L(\boldsymbol\eta^{n+\frac12}-\boldsymbol\eta^n)\boldsymbol\phi dz=(\Delta t)\int_0^L\mathbf{v}^{n+\frac12}\boldsymbol\phi dz,\\ &\int_0^L\left(\mathbf{v}^{n+\frac12}-\mathbf{v}^n\right)\boldsymbol\psi dz+(\Delta t)\langle\mathscr{L}_e(\boldsymbol\eta^{n+\frac12}),\boldsymbol\phi\rangle+\varepsilon(\Delta t)\int_0^L\partial_z^2\mathbf{v}^{n+\frac12}\cdot\partial_z^2\boldsymbol\psi dz=0.\end{aligned}\tag{3.1}$$

For each $\omega\in\Omega$ and n, we define the ALE map associated with the structure variable $\boldsymbol\eta^n$ as the solution to

$$\begin{aligned}\Delta A^\omega_{\boldsymbol\eta^n}&=0\quad\text{ in }\mathcal{O},\\ A^\omega_{\boldsymbol\eta^n}=\mathbf{id}+\boldsymbol\eta^n(\omega)\text{ on }\Gamma,\quad&\text{and}\quad A^\omega_{\boldsymbol\eta^n}=\mathbf{id}\text{ on }\partial\mathcal{O}\setminus\Gamma.\end{aligned}\tag{3.2}$$

Note that we have added the last term in $(3.1)_3$ to regularize the structure velocity. This term provides the required regularity for the time derivative of the Jacobian of the ALE map in the construction of the fluid subproblem below (see (3.10)). Moreover, it is required to circumvent the issues associated with the "very weak" solutions to the Poisson equation on polygonal domains with corners. Our next result, Lemma 3.1 below, is an immediate consequence of this regularization term.

We will first pass $N \to \infty$ and then $\varepsilon \to 0$.

LEMMA 3.1. *Consider $v \in H_0^2(\Gamma)$ and $w \in H^2(\mathcal{O})$ that solve*

$$-\Delta w = 0 \text{ in } \mathcal{O}, \quad w = v \text{ on } \Gamma, \quad w = 0 \text{ on } \partial\mathcal{O} \setminus \Gamma. \tag{3.3}$$

Then,

$$\|w\|_{L^2(\mathcal{O})} \le \|v\|_{H^{-\frac{1}{2}}(\Gamma)}. \tag{3.4}$$

Proof. We introduce the following system:

$$-\Delta g = w \text{ in } \mathcal{O}, \quad g = 0 \text{ on } \partial\mathcal{O}.$$

Since $w, g \in H^2(\mathcal{O})$, thanks to Theorem 1.5.3.3. in [19] (compare with Remark 1.5.3.5 in [19]), the following holds:

$$(w,w) = -(\Delta g, w) = -(g, \Delta w) + \int_\Gamma v\, \partial_n g = \int_\Gamma v\, \partial_n g.$$

Hence, thanks to Theorem 1.5.2.1 in [19] we then obtain

$$\|w\|^2_{L^2(\mathcal{O})} \le \|v\|_{H^{-\frac{1}{2}}(\Gamma)} \|\partial_n g\|_{H^{\frac{1}{2}}(\Gamma)} \le \|v\|_{H^{-\frac{1}{2}}(\Gamma)} \|g\|_{H^2(\mathcal{O})} \le \|v\|_{H^{-\frac{1}{2}}(\Gamma)} \|w\|_{L^2(\mathcal{O})}. \qquad \square$$

The fluid subproblem. In this subproblem we update the fluid and structure velocities while keeping the structure displacement the same. As noted in the introduction, there are two major difficulties associated with constructing this half of the scheme. The *first difficulty* arises because the fluid domains can degenerate randomly. Hence, we introduce an "artificial structure displacement" random variable by the means of a cut-off function as follows: For $\delta = (\delta_1, \delta_2)$, let Θ_δ be the step function that satisfies $\Theta_\delta(x,y) = 1$ if $\delta_1 < x, y < \frac{1}{\delta_2}$, and $\Theta_\delta(x,y) = 0$ otherwise. For brevity we define

$$\theta_\delta(\boldsymbol{\eta}^n) := \min_{k \le n} \Theta_\delta\left(\inf_{\mathcal{O}} J^k, \|\boldsymbol{\eta}^k\|_{\mathbf{H}^s(\Gamma)}\right) \quad \text{for a fixed } s \in \left(\frac{3}{2}, 2\right), \tag{3.5}$$

where $J^k(\omega) = \det \nabla A^\omega_{\boldsymbol{\eta}^k}$ is the Jacobian of the map defined in (3.2). Note that θ_δ is a real-valued function which tracks all the structure displacements and is equal to 1 until the step for which the structure quantities leave the desired bounds given in terms of δ. Now we define the artificial structure displacement random variable as the following stopped process:

$$\boldsymbol{\eta}^n_*(z,\omega) = \boldsymbol{\eta}^{\max_{0 \le k \le n} \theta_\delta(\boldsymbol{\eta}^k) k}(z,\omega) \quad \text{for every } \omega \in \Omega,\ z \in [0,L]. \tag{3.6}$$

Observe that, for any $p > 2$ and $s \ge \frac{5}{2} - \frac{2}{p}$, we have the following regularity result for the harmonic extension of the boundary data associated with $\boldsymbol{\eta}^n_*$ on a square (see section 5 in [19]):

$$\|A^\omega_{\boldsymbol{\eta}^n_*} - \mathbf{id}\|_{\mathbf{W}^{2,p}(\mathcal{O})} \le C \|\boldsymbol{\eta}^n_*\|_{\mathbf{W}^{2-\frac{1}{p},p}(\Gamma)} \le C \|\boldsymbol{\eta}^n_*\|_{\mathbf{H}^s(\Gamma)}. \tag{3.7}$$

Then Morrey's inequality for some $p < 4$ gives us a constant $C_* > 0$ (see Theorem 7.26 in [15]) such that

$$\|\nabla(A^\omega_{\boldsymbol{\eta}^n_*} - \mathbf{id})\|_{\mathbf{C}^{0,\frac{1}{p}}(\bar{\mathcal{O}})} \le C \|\boldsymbol{\eta}^n_*\|_{\mathbf{H}^s(\Gamma)} \le \frac{C_*}{\delta_2}, \quad \frac{5}{2} - \frac{2}{p} \le s < 2. \tag{3.8}$$

Theorem 5.5-1 (B) of [9] then ensures that the map $A^\omega_{\boldsymbol{\eta}^n_*} \in \mathbf{C}^{1,\frac{1}{p}}(\bar{\mathcal{O}})$ is injective for any n if δ_2 satisfies

$$C_* < \delta_2. \tag{3.9}$$

Hence, such δ_2, the domain configurations corresponding to the artificial variables $\boldsymbol{\eta}^n_*$, are nondegenerate and their Jacobians have a deterministic lower bound of δ_1. These artificial domain configurations will be used to define the fluid subproblem.

The *second difficulty* in constructing the second subproblem is due to the dependence of the fluid test functions, through the transformed divergence-free condition and the kinematic coupling condition, on the structure displacement found in the previous subproblem. Hence to avoid dealing with random test functions we supplement the weak formulation in this subproblem by penalty terms, via the parameter $\varepsilon > 0$, that enforce the incompressibility condition and the continuity of velocities in the normal direction, only in the limit $\varepsilon \to 0$.

A penalized system on artificial domains. Let $\Delta_n W := W(t^{n+1}) - W(t^n)$. Then we look for $(\mathbf{u}^{n+1}, \mathbf{v}^{n+1}) \in \mathscr{U}$ that solves

(3.10)
$$\begin{aligned}
&\boldsymbol{\eta}^{n+1} := \boldsymbol{\eta}^{n+\frac{1}{2}},\\
&\int_{\mathcal{O}} J^n_* \left(\mathbf{u}^{n+1} - \mathbf{u}^{n+\frac{1}{2}}\right) \mathbf{q} d\mathbf{x} + \frac{1}{2}\int_{\mathcal{O}} \left(J^{n+1}_* - J^n_*\right) \mathbf{u}^{n+1} \cdot \mathbf{q} d\mathbf{x}\\
&\quad + \frac{1}{2}(\Delta t)\int_{\mathcal{O}} J^n_* ((\mathbf{u}^{n+1} - \mathbf{w}^n_*)\cdot \nabla^{\boldsymbol{\eta}^n_*} \mathbf{u}^{n+1} \cdot \mathbf{q} - (\mathbf{u}^{n+1} - \mathbf{w}^n_*)\cdot \nabla^{\boldsymbol{\eta}^n_*} \mathbf{q} \cdot \mathbf{u}^{n+1}) d\mathbf{x}\\
&\quad + 2\nu(\Delta t)\int_{\mathcal{O}} J^n_* \mathbf{D}^{\boldsymbol{\eta}^n_*}(\mathbf{u}^{n+1}) \cdot \mathbf{D}^{\boldsymbol{\eta}^n_*}(\mathbf{q}) d\mathbf{x}\\
&\quad + \frac{(\Delta t)}{\varepsilon}\int_{\mathcal{O}} \operatorname{div}^{\boldsymbol{\eta}^n_*} \mathbf{u}^{n+1} \operatorname{div}^{\boldsymbol{\eta}^n_*} \mathbf{q} d\mathbf{x} + \frac{(\Delta t)}{\varepsilon}\int_\Gamma (\mathbf{u}^{n+1} - \mathbf{v}^{n+1})\cdot \mathbf{n}^n_* (\mathbf{q} - \boldsymbol{\psi})\cdot \mathbf{n}^n_*\\
&\quad + \frac{(\Delta t)}{\alpha}\int_\Gamma S^n(\mathbf{u}^{n+1} - \mathbf{v}^{n+1})(\mathbf{q} - \boldsymbol{\psi}) dz + \int_0^L (\mathbf{v}^{n+1} - \mathbf{v}^{n+\frac{1}{2}})\boldsymbol{\psi} dz\\
&= \int_0^t \left(P^n_{in}\int_0^1 q_z\Big|_{z=0} dr - P^n_{out}\int_0^1 q_z\Big|_{z=1} dr\right) + (G(\mathbf{u}^n, \boldsymbol{\eta}^n_*)\Delta_n W, \mathbf{Q})_{\mathbf{L}^2}
\end{aligned}$$

for any $(\mathbf{q}, \boldsymbol{\psi}) \in \mathscr{U}$. Here we set $P^n_{in/out} := \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} P_{in/out}\, dt$. Moreover, the random variable $\mathbf{n}^n_*$ is the unit normal to $\Gamma_{\boldsymbol{\eta}^n_*}$ and

$$\mathbf{w}^n_* = \frac{1}{\Delta t}(A^\omega_{\boldsymbol{\eta}^{n+1}_*} - A^\omega_{\boldsymbol{\eta}^n_*}), \quad J^n_* = \det \nabla A^\omega_{\boldsymbol{\eta}^n_*}.$$

Remark 3.1. Note that, to obtain a stable scheme, we update $\boldsymbol{\eta}^n$ in the first subproblem using the data from the second subproblem and not $\boldsymbol{\eta}^n_*$. However, this means that after a certain random time, we will produce solutions that are meaningless. These discrepansies will be handled in the end in Lemma 5.7 by introducing an almost surely strictly positive stopping time until which the limiting solutions, corresponding to the approximations constructed in section 3.2 using $\boldsymbol{\eta}^n$'s and $\boldsymbol{\eta}^n_*$'s, are equal.

Now we introduce the following discrete energy and dissipation for $i = 0, 1$:

(3.11)
$$\begin{aligned}
E^{n+\frac{i}{2}} &= \frac{1}{2}\left(\int_{\mathcal{O}} J^n_* |\mathbf{u}^{n+\frac{i}{2}}|^2 d\mathbf{x} + \|\mathbf{v}^{n+\frac{i}{2}}\|^2_{\mathbf{L}^2(0,L)} + \|\boldsymbol{\eta}^{n+\frac{i}{2}}\|^2_{\mathbf{H}^2_0(0,L)}\right),\\
D^n_1 &= \varepsilon(\Delta t)\int_0^L |\partial_{zz}\mathbf{v}^{n+\frac{1}{2}}|^2,
\end{aligned}$$

$$D_2^n = \Delta t \int_{\mathcal{O}} \left(2\nu J_*^n |\mathbf{D}(\mathbf{u}^{n+1})|^2\right) d\mathbf{x} + \frac{\Delta t}{\alpha} \int_\Gamma |\mathbf{u}^{n+1} - \mathbf{v}^{n+1}|^2 + \frac{(\Delta t)}{\varepsilon} \int_{\mathcal{O}} |\text{div}^{\boldsymbol{\eta}_*^n} \mathbf{u}^{n+1}|^2 dx + \frac{\Delta t}{\varepsilon} \int_\Gamma |(\mathbf{u}^{n+1} - \mathbf{v}^{n+1}) \cdot \mathbf{n}_*^n|^2.$$

LEMMA 3.2 (existence for the structure subproblem). *Assume that $\boldsymbol{\eta}^n$ and $\mathbf{v}^n$ are $\mathbf{H}_0^2(0,L)$- and $\mathbf{L}^2(0,L)$-valued $\mathcal{F}_{t^n}$-measurable random variables, respectively. Then there exist $\mathbf{H}_0^2(0,L)$-valued $\mathcal{F}_{t^n}$-measurable random variables $\boldsymbol{\eta}^{n+\frac{1}{2}}, \mathbf{v}^{n+\frac{1}{2}}$ that solve* (3.1)*, and the following semidiscrete energy inequality holds:*

$$E^{n+\frac{1}{2}} + D_1^n + C_1^n = E^n, \tag{3.12}$$

where

$$C_1^n := \frac{1}{2}\|\mathbf{v}^{n+\frac{1}{2}} - \mathbf{v}^n\|_{\mathbf{L}^2(0,L)}^2 + \frac{1}{2}\|\boldsymbol{\eta}^{n+\frac{1}{2}} - \boldsymbol{\eta}^n\|_{\mathbf{H}_0^2(0,L)}^2$$

corresponds to numerical dissipation.

Proof. The proof of existence and uniqueness of measurable solutions is straightforward and the reader is referred to [22] for details. This allows us to write

$$\mathbf{v}^{n+\frac{1}{2}} = \frac{\boldsymbol{\eta}^{n+\frac{1}{2}} - \boldsymbol{\eta}^n}{\Delta t}.$$

We now take $\boldsymbol{\psi} = \mathbf{v}^{n+\frac{1}{2}}$ in $(3.1)_3$ and using $a(a-b) = \frac{1}{2}(|a|^2 - |b|^2 + |a-b|^2)$, we obtain

$$\begin{aligned} &\|\mathbf{v}^{n+\frac{1}{2}}\|_{\mathbf{L}^2(0,L)}^2 + \|\mathbf{v}^{n+\frac{1}{2}} - \mathbf{v}^n\|_{\mathbf{L}^2(0,L)}^2 + \|\boldsymbol{\eta}^{n+\frac{1}{2}}\|_{\mathbf{H}_0^2(0,L)}^2 + \|\boldsymbol{\eta}^{n+\frac{1}{2}} - \boldsymbol{\eta}^n\|_{\mathbf{H}_0^2(0,L)}^2 \\ &\quad + \varepsilon(\Delta t)\|\partial_{zz}\mathbf{v}^{n+\frac{1}{2}}\|_{\mathbf{L}^2(0,L)}^2 = \|\mathbf{v}^n\|_{\mathbf{L}^2(0,L)}^2 + \|\boldsymbol{\eta}^n\|_{\mathbf{H}_0^2(0,L)}^2. \end{aligned} \tag{3.13}$$

Recalling that $\mathbf{u}^n = \mathbf{u}^{n+\frac{1}{2}}$ and adding the relevant terms on both sides of (3.13), we obtain (3.12). ∎

LEMMA 3.3 (existence for the fluid subproblem). *For a given $\delta = (\delta_1, \delta_2)$ satisfying* (3.9)*, and given $\mathcal{F}_{t^n}$-measurable random variables $(\mathbf{u}^{n+\frac{1}{2}}, \mathbf{v}^n)$ taking values in $\mathscr{U}$ and $\mathbf{v}^{n+\frac{1}{2}}$ taking values in $\mathbf{H}_0^2(0,L)$, there exists an $\mathcal{F}_{t^{n+1}}$-measurable random variable $\mathbf{U}^{n+1} = (\mathbf{u}^{n+1}, \mathbf{v}^{n+1})$ taking values in $\mathscr{U}$ that solves* (3.10)*, and the solution satisfies the following estimate:*

(3.14)

$$\begin{aligned} E^{n+1} + D_2^n + C_2^n \leq E^{n+\frac{1}{2}} &+ C\Delta t((P_{in}^n)^2 + (P_{out}^n)^2) \\ &+ C\|\Delta_n W\|_{U_0}^2 \|G(\mathbf{u}^n, \boldsymbol{\eta}_*^n)\|_{L_2(U_0;\mathbf{L}^2)}^2 \\ &+ |(G(\mathbf{u}^n, \boldsymbol{\eta}_*^n)\Delta_n W, (\mathbf{u}^n, \mathbf{v}^n))_{\mathbf{L}^2}| + \frac{1}{4}\int_0^L |\mathbf{v}^{n+\frac{1}{2}} - \mathbf{v}^n|^2 dz, \end{aligned}$$

where

$$C_2^n := \frac{1}{4}\int_{\mathcal{O}} \left(J_*^n |\mathbf{u}^{n+1} - \mathbf{u}^n|^2\right) d\mathbf{x} + \frac{1}{4}\int_0^L |\mathbf{v}^{n+1} - \mathbf{v}^{n+\frac{1}{2}}|^2 dz$$

is numerical dissipation, and $\boldsymbol{\eta}_*^n$ is as defined in (3.6).

Proof. The proof of existence and measurability of the solutions is given using Brouwer's fixed point theorem and the Kuratowski selection theorem in [32].

To obtain (3.13), we take $(\mathbf{q},\boldsymbol{\psi}) = (\mathbf{u}^{n+1},\mathbf{v}^{n+1})$ in (3.10) and use the identity $a(a-b) = \frac{1}{2}(|a|^2 - |b|^2 + |a-b|^2)$:

$$\begin{aligned}
&\frac{1}{2}\int_{\mathcal{O}} J_*^n \left(|\mathbf{u}^{n+1}|^2 - |\mathbf{u}^{n+\frac{1}{2}}|^2 + |\mathbf{u}^{n+1} - \mathbf{u}^{n+\frac{1}{2}}|^2\right) + \frac{1}{2}\int_{\mathcal{O}} \left(J_*^{n+1} - J_*^n\right)|\mathbf{u}^{n+1}|^2 d\mathbf{x}\\
&\quad + 2(\Delta t)\int_{\mathcal{O}} J_*^n |\mathbf{D}^{\boldsymbol{\eta}_*^n}(\mathbf{u}^{n+1})|^2 d\mathbf{x} + \frac{(\Delta t)}{\alpha}\int_{\Gamma} S^n|\mathbf{u}^{n+1} - \mathbf{v}^{n+1}|^2\\
&\quad + \frac{(\Delta t)}{\varepsilon}\int_{\mathcal{O}} |\mathrm{div}^{\boldsymbol{\eta}_*^n}\mathbf{u}^{n+1}|^2 d\mathbf{x} + \frac{(\Delta t)}{\varepsilon}\int_{\Gamma} |(\mathbf{u}^{n+1} - \mathbf{v}^{n+1})\cdot \mathbf{n}_*^n|^2 dz\\
&\quad + \frac{1}{2}\int_0^L |\mathbf{v}^{n+1}|^2 - |\mathbf{v}^{n+\frac{1}{2}}|^2 + |\mathbf{v}^{n+1} - \mathbf{v}^{n+\frac{1}{2}}|^2 dz\\
&= (\Delta t)\left(P_{in}^n \int_0^1 u_z^{n+1}\Big|_{z=0} dr - P_{out}^n \int_0^1 u_z^{n+1}\Big|_{z=1} dr\right)\\
&\quad + (G(\mathbf{u}^n,\boldsymbol{\eta}_*^n)\Delta_n W, (\mathbf{U}^{n+1} - \mathbf{U}^n)) + (G(\mathbf{u}^n,\boldsymbol{\eta}_*^n)\Delta_n W, \mathbf{U}^n).
\end{aligned}$$

Here we split the discrete stochastic integral into two terms. We estimate the first term by using the Cauchy–Schwarz inequality. For some $C(\delta) > 0$ independent of n, the following holds:

$$\begin{aligned}
&|(G(\mathbf{u}^n,\boldsymbol{\eta}_*^n)\Delta_n W, ((\mathbf{u}^{n+1},\mathbf{v}^{n+1}) - (\mathbf{u}^n,\mathbf{v}^n)))_{\mathbf{L}^2}| \le C\|\Delta_n W\|_{U_0}^2 \|G(\mathbf{u}^n,\boldsymbol{\eta}_*^n)\|_{L_2(U_0,\mathbf{L}^2)}^2\\
&\quad + \frac{1}{4}\int_{\mathcal{O}} J_*^n |\mathbf{u}^{n+1} - \mathbf{u}^n|^2 d\mathbf{x} + \frac{1}{8}\int_0^L |\mathbf{v}^{n+1} - \mathbf{v}^n|^2 dz\\
&\le C\|\Delta_n W\|_{U_0}^2 \|G(\mathbf{u}^n,\boldsymbol{\eta}_*^n)\|_{L_2(U_0;\mathbf{L}^2)}^2 + \frac{1}{4}\int_{\mathcal{O}} J_*^n |\mathbf{u}^{n+1} - \mathbf{u}^n|^2 d\mathbf{x}\\
&\quad + \frac{1}{4}\int_0^L |\mathbf{v}^{n+1} - \mathbf{v}^{n+\frac{1}{2}}|^2 dz + \frac{1}{4}\int_0^L |\mathbf{v}^{n+\frac{1}{2}} - \mathbf{v}^n|^2 dz.
\end{aligned}$$

We treat the terms with $P_{in/out}$ similarly to obtain (3.14). ∎

Next, we will obtain uniform estimates on the expectation of the kinetic and elastic energy and dissipation of the coupled problem.

THEOREM 3.4 (uniform estimates). *There exists a constant $C > 0$ that depends on the initial data, δ, T, and $P_{in/out}$, and is independent of N and ε such that*

1. $\mathbb{E}\left(\max_{1\le n\le N} E^n\right) < C$, $\mathbb{E}\left(\max_{0\le n\le N-1} E^{n+\frac{1}{2}}\right) < C$.
2. $\mathbb{E}\sum_{n=0}^{N-1} D^n < C$.
3. $\mathbb{E}\sum_{n=0}^{N-1} \int_{\mathcal{O}} J_*^n |\mathbf{u}^{n+1} - \mathbf{u}^n|^2 d\mathbf{x} + \|\mathbf{v}^{n+1} - \mathbf{v}^{n+\frac{1}{2}}\|_{\mathbf{L}^2(0,L)}^2 < C$.
4. $\mathbb{E}\sum_{n=0}^{N-1} \|\mathbf{v}^{n+\frac{1}{2}} - \mathbf{v}^n\|_{\mathbf{L}^2(0,L)}^2 + \|\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n\|_{\mathbf{H}^2(0,L)}^2 < C$,

where $D^n = D_1^n + D_2^n$ (see definitions (3.11)).

Proof. We first add (3.12) and (3.14) to obtain

$$\begin{aligned}
E^{n+1} + D^n + C_1^n + C_2^n &\le E^n + C\Delta t((P_{in}^n)^2 + (P_{out}^n)^2)\\
&\quad + C\|\Delta_n W\|_{U_0}^2 \|G(\mathbf{u}^n,\boldsymbol{\eta}_*^n)\|_{L_2(U_0;\mathbf{L}^2(\mathcal{O}))}^2 + |(G(\mathbf{u}^n,\boldsymbol{\eta}_*^n)\Delta_n W, \mathbf{U}^n)|.
\end{aligned} \tag{3.15}$$

Then for any $m \ge 1$, summing $0 \le n \le m-1$ gives us

$$\begin{aligned}
E^m + \sum_{n=0}^{m-1} D^n + \sum_{n=0}^{m-1} C_1^n + \sum_{n=0}^{m-1} C_2^n &\le E^0 + C\,\Delta t \sum_{n=0}^{m-1}\left((P_{in}^n)^2 + (P_{out}^n)^2\right)\\
&\quad + \sum_{n=0}^{m-1} |(G(\mathbf{u}^n,\boldsymbol{\eta}_*^n)\Delta_n W, \mathbf{U}^n)| + \sum_{n=0}^{m-1} \|G(\mathbf{u}^n,\boldsymbol{\eta}_*^n)\|_{L_2(U_0;\mathbf{L}^2)}^2 \|\Delta_n W\|_{U_0}^2.
\end{aligned} \tag{3.16}$$

We take supremum over $1 \le m \le N$ and then take expectation on both sides of (3.16). The discrete Burkholder–Davis–Gundy inequality, (3.6), and (3.8) give us for some $C(\delta) > 0$ that

$$\begin{aligned}
&\mathbb{E} \max_{1\le m\le N} \left| \sum_{n=0}^{m-1} (G(\mathbf{u}^n, \boldsymbol{\eta}_*^n)\Delta_n W, \mathbf{U}^n) \right| \\
&\le C\mathbb{E}\left[\Delta t \sum_{n=0}^{N-1} \|G(\mathbf{u}^n, \boldsymbol{\eta}_*^n)\|^2_{L_2(U_0,\mathbf{L}^2)} \left(\left\| (\sqrt{J_*^n})\mathbf{u}^n \right\|^2_{\mathbf{L}^2(\mathcal{O})} + \|\mathbf{v}^n\|^2_{\mathbf{L}^2(0,L)} \right) \right]^{\frac{1}{2}} \\
&\le C\mathbb{E}\Big[\max_{0\le m\le N} \left(\left\| (\sqrt{J_*^m})\mathbf{u}^m \right\|^2_{\mathbf{L}^2(\mathcal{O})} + \|\boldsymbol{\eta}_*^m\|^2_{\mathbf{H}_0^2(0,L)} \right) \\
&\quad \times \sum_{n=0}^{N-1} \Delta t \left(\|\sqrt{J_*^n}\mathbf{u}^n\|^2_{\mathbf{L}^2(\mathcal{O})} + \|\mathbf{v}^n\|^2_{\mathbf{L}^2(0,L)} \right) \Big]^{\frac{1}{2}} \\
&\le \frac{1}{2}E_0 + \frac{1}{2}\mathbb{E} \max_{1\le m\le N} \left[\left\| (\sqrt{J_*^m})\mathbf{u}^m \right\|^2_{\mathbf{L}^2(\mathcal{O})} + \|\boldsymbol{\eta}^m\|^2_{\mathbf{H}_0^2(0,L)} \right] \\
&\quad + C\Delta t \mathbb{E} \left(\sum_{n=0}^{N-1} \|(\sqrt{J_*^n})\mathbf{u}^n\|^2_{\mathbf{L}^2(\mathcal{O})} + \|\mathbf{v}^n\|^2_{\mathbf{L}^2(0,L)} \right).
\end{aligned}$$

We use the tower property and (2.12) for each $n = 0, \ldots, m-1$ to obtain

$$\begin{aligned}
\mathbb{E}[\|G(\mathbf{u}^n, \boldsymbol{\eta}_*^n)\|^2_{L_2(U_0,\mathbf{L}^2)} \|\Delta_n W\|^2_{U_0}] &= \mathbb{E}[\mathbb{E}[\|G(\mathbf{u}^n, \boldsymbol{\eta}_*^n)\|^2_{L_2(U_0,\mathbf{L}^2)} \|\Delta_n W\|^2_{U_0} | \mathcal{F}_n]] \\
&= \mathbb{E}[\|G(\mathbf{u}^n, \boldsymbol{\eta}_*^n)\|^2_{L_2(U_0,\mathbf{L}^2)} \mathbb{E}[\|\Delta_n W\|^2_{U_0} | \mathcal{F}_n]] \\
&= \Delta t (\mathrm{Tr}\mathbf{Q}) \mathbb{E} \|G(\mathbf{u}^n, \boldsymbol{\eta}_*^n)\|^2_{L_2(U_0,\mathbf{L}^2)} \\
&\le C\Delta t (\mathrm{Tr}\mathbf{Q}) \mathbb{E} \left(\|\sqrt{J_*^n}\mathbf{u}^n\|^2_{\mathbf{L}^2(\mathcal{O})} + \|\boldsymbol{\eta}_*^n\|^2_{\mathbf{H}_0^2(0,L)} \right). \qquad (3.17)
\end{aligned}$$

Thus, for some $C > 0$ depending on δ and on $\mathrm{Tr}\mathbf{Q}$, the following holds:

$$\begin{aligned}
&\mathbb{E} \max_{1\le n\le N} \left(\|(\sqrt{J_*^n})\mathbf{u}^n\|^2_{\mathbf{L}^2(\mathcal{O})} + \|\mathbf{v}^n\|^2_{\mathbf{L}^2(0,L)} \right) \le CE^0 + C\|P_{in/out}\|^2_{L^2(0,T)} \\
&\quad + C \sum_{n=1}^{N-1} \Delta t \mathbb{E} \max_{1\le m\le n} \left(\|(\sqrt{J_*^m})\mathbf{u}^m\|^2_{\mathbf{L}^2(\mathcal{O})} + \|\mathbf{v}^m\|^2_{\mathbf{L}^2(0,L)} \right).
\end{aligned}$$

The discrete Gronwall inequality finally gives us

$$\mathbb{E} \max_{1\le n\le N} \left(\|(\sqrt{J_*^n})\mathbf{u}^n\|^2_{\mathbf{L}^2(\mathcal{O})} + \|\mathbf{v}^n\|^2_{\mathbf{L}^2(0,L)} \right) \le Ce^T,$$

where C depends only on the given data and in particular δ.

Hence for E^n, D^n defined as in (3.11), we have

$$\mathbb{E} \max_{1\le n\le N} E^n + \mathbb{E} \sum_{n=0}^{N-1} (D^n + C_1^n + C_2^n) \le C(\delta)(E^0 + \|P_{in/out}\|^2_{L^2(0,T)} + Te^T). \qquad (3.18)$$ ◻

3.2. Approximate solutions. We use the solutions $(\mathbf{u}^{n+\frac{i}{2}}, \boldsymbol{\eta}^{n+\frac{i}{2}}, \mathbf{v}^{n+\frac{i}{2}})$, $i = 0, 1$, defined for every $N \in \mathbb{N} \setminus \{0\}$ at discrete times to define approximate solutions on the entire interval $(0, T)$. First we introduce approximate solutions that are piecewise constant on each subinterval $[n\Delta t, (n+1)\Delta t)$:

(3.19)

$$\mathbf{u}_N(t,\cdot) = \mathbf{u}^n, \quad \boldsymbol{\eta}_N(t,\cdot) = \boldsymbol{\eta}^n, \quad \boldsymbol{\eta}_N^*(t,\cdot) = \boldsymbol{\eta}_*^n, \quad \mathbf{v}_N(t,\cdot) = \mathbf{v}^n, \quad \mathbf{v}_N^{\#}(t,\cdot) = \mathbf{v}^{n+\frac{1}{2}}.$$

These processes are adapted to the given filtration $(\mathcal{F}_t)_{t\geq 0}$. The following are their time-shifted versions which are commonly used in deterministic settings:

$$\mathbf{u}_N^+(t,\cdot)=\mathbf{u}^{n+1}, \quad \boldsymbol{\eta}_N^+(t,\cdot)=\boldsymbol{\eta}^{n+1}, \quad \mathbf{v}_N^+(t,\cdot)=\mathbf{v}^{n+1}, \quad t\in(n\Delta t,(n+1)\Delta t].$$

We also define the corresponding piecewise linear interpolations for $t\in[t^n,t^{n+1}]$:

$$\begin{aligned}\widetilde{\mathbf{u}}_N(t,\cdot)&=\frac{t-t^n}{\Delta t}\mathbf{u}^{n+1}+\frac{t^{n+1}-t}{\Delta t}\mathbf{u}^n, \quad \widetilde{\mathbf{v}}_N(t,\cdot)=\frac{t-t^n}{\Delta t}\mathbf{v}^{n+1}+\frac{t^{n+1}-t}{\Delta t}\mathbf{v}^n,\\ \widetilde{\boldsymbol{\eta}}_N(t,\cdot)&=\frac{t-t^n}{\Delta t}\boldsymbol{\eta}^{n+1}+\frac{t^{n+1}-t}{\Delta t}\boldsymbol{\eta}^n, \quad \widetilde{\boldsymbol{\eta}}_N^*(t,\cdot)=\frac{t-t^n}{\Delta t}\boldsymbol{\eta}_*^{n+1}+\frac{t^{n+1}-t}{\Delta t}\boldsymbol{\eta}_*^n.\end{aligned} \tag{3.20}$$

Observe that

$$\frac{\partial\widetilde{\boldsymbol{\eta}}_N}{\partial t}=\mathbf{v}_N^{\#}, \quad \frac{\partial\widetilde{\boldsymbol{\eta}}_N^*}{\partial t}=\sum_{n=0}^{N-1}\theta_\delta(\boldsymbol{\eta}^{n+1})\mathbf{v}_N^{\#}\chi_{(t^n,t^{n+1})}:=\mathbf{v}_N^* \quad \text{a.e. on } (0,T), \tag{3.21}$$

where $\mathbf{v}_N^{\#}$ is introduced in (3.19). We also define piecewise constant interpolations of the ALE maps and Jacobians $A_{\boldsymbol{\eta}_N^*}, J_{\boldsymbol{\eta}_N^*}$ and their piecewise linear interpolation $\widetilde{A}_{\boldsymbol{\eta}_N^*}, \widetilde{J}_{\boldsymbol{\eta}_N^*}$. In the following lemma we summarize the results that are immediate consequences of the estimates obtained in Theorem 3.4.

LEMMA 3.5. *Given $\mathbf{u}_0\in\mathbf{L}^2(\mathcal{O})$, $\boldsymbol{\eta}_0\in\mathbf{H}_0^2(0,L)$, $\mathbf{v}_0\in\mathbf{L}^2(0,L)$, for a fixed $\delta=(\delta_1,\delta_2)$ satisfying* (3.9), *we have that*

1. $\{\boldsymbol{\eta}_N\},\{\boldsymbol{\eta}_N^+\},\{\boldsymbol{\eta}_N^*\}$ *and thus* $\{\widetilde{\boldsymbol{\eta}}_N\},\{\widetilde{\boldsymbol{\eta}}_N^*\}$ *are bounded independently of N, ε in* $L^2(\Omega;L^\infty(0,T;\mathbf{H}_0^2(0,L)))$.
2. $\{\mathbf{v}_N\},\{\mathbf{v}_N^+\},\{\mathbf{v}_N^{\#}\},\{\mathbf{v}_N^*\}$ *are bounded independently of N and ε in* $L^2(\Omega;L^\infty(0,T;\mathbf{L}^2(0,L)))$.
3. $\{\mathbf{u}_N\}$ *is bounded independently of N and ε in* $L^2(\Omega;L^\infty(0,T;\mathbf{L}^2(\mathcal{O})))\cap L^2(\Omega;L^2(0,T;\mathbf{L}^2(\mathcal{O})))$.
4. $\{\mathbf{u}_N^+\}$ *is bounded independently of N and ε in* $L^2(\Omega;L^2(0,T;\mathbf{H}^1(\mathcal{O})))$.
5. $\{\frac{1}{\sqrt{\varepsilon}}div^{\boldsymbol{\eta}_N^*}\mathbf{u}_N^+\}$ *is bounded independently of N and ε in* $L^2(\Omega;L^2(0,T;L^2(\mathcal{O})))$.
6. $\{\frac{1}{\sqrt{\varepsilon}}(\mathbf{u}_N^+-\mathbf{v}_N^+)\cdot\mathbf{n}_N^*\}$ *is bounded independently of N,ε in* $L^2(\Omega;L^2(0,T;L^2(\Gamma)))$.
7. $\{\sqrt{\varepsilon}\mathbf{v}_N^{\#}\}$ *is bounded independently of N and ε in* $L^2(\Omega;L^2(0,T;\mathbf{H}_0^2(0,L)))$.

Proof. Observe that for each $\omega\in\Omega$, $\nabla\mathbf{u}^{n+1}=\nabla^{\boldsymbol{\eta}_*^n}\mathbf{u}^{n+1}(\nabla A^\omega_{\boldsymbol{\eta}_*^n})$. Thus we have

$$\begin{aligned}\delta_1\mathbb{E}\int_{\mathcal{O}}|\nabla\mathbf{u}^{n+1}|^2d\mathbf{x}&\leq\mathbb{E}\int_{\mathcal{O}}(J_*^n)|\nabla\mathbf{u}^{n+1}|^2d\mathbf{x}=\mathbb{E}\int_{\mathcal{O}}(J_*^n)|\nabla^{\boldsymbol{\eta}_*^n}\mathbf{u}^{n+1}\cdot\nabla A^\omega_{\boldsymbol{\eta}_*^n}|^2d\mathbf{x}\\ &\leq C(\delta)\mathbb{E}\int_{\mathcal{O}}(J_*^n)|\nabla^{\boldsymbol{\eta}_*^n}\mathbf{u}^{n+1}|^2d\mathbf{x}\leq KC(\delta)\mathbb{E}\int_{\mathcal{O}}(J_*^n)|\mathbf{D}^{\boldsymbol{\eta}_*^n}\mathbf{u}^{n+1}|^2d\mathbf{x},\end{aligned}$$

where $K>0$ is the universal Korn constant that depends only on the reference domain $\mathcal{O}$. This result follows from Lemma 1 in [35] and because of uniform bounds (3.8) which imply that $\{A^\omega_{\boldsymbol{\eta}_N^*}(t);\omega\in\Omega, t\in[0,T]\}$ is compact in $\mathbf{W}^{1,\infty}(\mathcal{O})$. Thus, thanks to Theorem 3.4, there exists $C>0$, independent of N, such that

$$\mathbb{E}\int_0^T\int|\nabla\mathbf{u}_N^+|^2d\mathbf{x}ds=\mathbb{E}\sum_{n=0}^{N-1}\Delta t\int_{\mathcal{O}}|\nabla\mathbf{u}^{n+1}|^2d\mathbf{x}\leq C(\delta). \tag{3.22}$$

The proofs of the rest of the statements follow immediately from Theorem 3.4. ☐

4. Passing $N\to\infty$. In this section we will pass $N\to\infty$ by first establishing tightness of the laws of the approximate random variables defined in section 3.2.

4.1. Tightness results. Since we do not expect our candidate solutions to be differentiable in time, the tightness results, i.e., Lemmas 4.2 and 4.3 below, will rely on the following theorem (see [33, 29]).

LEMMA 4.1. *Let the translation in time by h of a function f be denoted by*

$$T_h f(t,\cdot) = f(t-h,\cdot), \quad h \in \mathbb{R}.$$

Assume that $\mathcal{Y}_0 \subset \mathcal{Y} \subset \mathcal{Y}_1$ are Banach spaces such that $\mathcal{Y}_0$ and $\mathcal{Y}_1$ are reflexive with compact embedding of $\mathcal{Y}_0$ in $\mathcal{Y}$; then for any $m>0$, the embedding

$$\left\{\mathbf{u} \in L^2(0,T;\mathcal{Y}_0) : \sup_{0<h<T} \frac{1}{h^m}\|T_h\mathbf{u}-\mathbf{u}\|_{L^2(h,T;\mathcal{Y}_1)} < \infty\right\} \hookrightarrow L^2(0,T;\mathcal{Y})$$

is compact.

We will now obtain our first tightness result for the fluid and structure velocities by bypassing the need for higher moment estimates (see also [16] in this context).

LEMMA 4.2. *The laws of $\mathbf{u}_N^+$ are tight in $L^2(0,T;\mathbf{H}^\alpha(\mathcal{O}))$, and those of $\mathbf{v}_N^+$ are tight in $L^2(0,T;\mathbf{H}^{-\beta}(0,L))$ for any $0 \le \alpha < 1$, $\beta > 0$.*

Proof. The aim of this proof is to apply Lemma 4.1 by obtaining bounds for

$$\begin{aligned}&\int_h^T \|T_h\mathbf{u}_N^+ - \mathbf{u}_N^+\|^2_{\mathbf{L}^2(\mathcal{O})} + \|T_h\mathbf{v}_N^+ - \mathbf{v}_N^+\|^2_{\mathbf{H}^{-\beta}(0,L)}\\ &\qquad = (\Delta t)\sum_{n=j}^N \|\mathbf{u}^n - \mathbf{u}^{n-j}\|^2_{\mathbf{L}^2(\mathcal{O})} + \|\mathbf{v}^n - \mathbf{v}^{n-j}\|^2_{\mathbf{H}^{-\beta}(0,L)}\end{aligned} \tag{4.1}$$

for any N and h. Here $1 \le j \le N$ such that $h = j\Delta t - l$ for some $l < \Delta t$. For simplicity we take $l = 0$.

To achieve this goal, we will construct appropriate test functions for (3.1) and (3.10) that will result in the term on the right-hand side of (4.1). This is done by transforming $\mathbf{u}^k$ and $\mathbf{v}^k$ in such a way that they can be used as test functions for the equations for $\mathbf{u}^n$ and $\mathbf{v}^n$. This has to be done carefully since $\mathbf{u}^k$ and $\mathbf{u}^n$ are defined on different physical domains. Observe also that we cannot test $(3.1)_3$ directly with $\mathbf{v}^k$, as it does not have the required H_0^2-bounds independent of ε. Thus, we will use a space regularization of $\mathbf{v}^k$ to arrive at the desired test function for (3.1) and (3.10). First, for any $\mathbf{g} \in \mathbf{H}^1(\mathcal{O})$, we extend $\mathbf{g}$ by its boundary values at $\partial\mathcal{O}$ constantly in the normal direction outside of $\mathcal{O}$ and define it to be 0 elsewhere. Then, denoting the 1D bump function by ρ, we define its horizontally mollified version as follows:

$$\mathbf{g}_\sigma(z,r) = \int_{-1}^1 \mathbf{g}(z+\sigma y, r)\rho(y)dy.$$

Now let P_M denote the orthonormal projector in $L^2(\Gamma)$ onto the space $\text{span}_{1\le i\le M}\{\varphi_i\}$, where φ_i satisfies $-\Delta\varphi_i = \gamma_i\varphi_i$ and $\varphi_i(z) = 0$ when $z = 0, L$. For any $\mathbf{v} \in \mathbf{L}^2(\Gamma)$ we notate $\mathbf{v}_M = P_M\mathbf{v}$ and denote by $\mathbf{w}_M$ the harmonic extension of $\mathbf{v}_M$ in $\mathcal{O}$, such that $\mathbf{w}_M = 0$ on $\partial\mathcal{O}\setminus\Gamma$ (cf. (3.3)). Similarly, let $\mathbf{w}^k$ be the harmonic extension of $\mathbf{v}^k$ in $\mathcal{O}$ with 0 boundary values on $\partial\mathcal{O}\setminus\Gamma$. For the rest of the proof we fix $\sigma = \frac{L}{4}h^{\frac{1}{12}}$ and since $\gamma_M \sim \gamma_1 M^2$, we choose M such that $\gamma_M = ch^{-\frac{1}{6}}$.

Now we define the following function that will lead to a suitable test function for the fluid subproblem (see (4.8)):

$$
\begin{aligned}
\mathbf{u}^{k,n}_{\sigma,M} := {}& (J^n_*)^{-1}\nabla A_{\boldsymbol{\eta}^n_*}\left(J^k_*(\nabla A_{\boldsymbol{\eta}^k_*})^{-1}(\mathbf{u}^k-\mathbf{w}^k)\right)_\sigma + \mathbf{w}^k_M + \left(\lambda^k_\sigma - \lambda^{k,n}_M\right)\xi_0\chi \\
& - (J^n_*)^{-1}\nabla A_{\boldsymbol{\eta}^n_*}\mathcal{B}\Big(\mathrm{div}\Big((J^k_*(\nabla A_{\boldsymbol{\eta}^k_*})^{-1}\mathbf{w}^k)_\sigma - J^n_*(\nabla A_{\boldsymbol{\eta}^n_*})^{-1}\mathbf{w}^k_M \\
& - \left(\lambda^k_\sigma - \lambda^{k,n}_M\right)(J^n_*(\nabla A_{\boldsymbol{\eta}^n_*})^{-1}\xi_0\chi)\Big)\Big),
\end{aligned}
$$

where

- $\chi(r)$ is a smooth function on $\mathcal{O}$ such that $\chi(1)=1$ and $\chi(0)=0$;
- the $\mathbb{R}$-valued random variables $\lambda^k_\sigma = \int_0^L(\partial_z(\mathbf{id}+\boldsymbol{\eta}^k_*)\times\mathbf{v}^k)_\sigma$ and $\lambda^{k,n}_M = \int_0^L(\partial_z(\mathbf{id}+\boldsymbol{\eta}^n_*)\times\mathbf{v}^k_M)$;
- $\xi_0\in\mathbf{C}^\infty_0(0,L)$ that satisfies $\int_0^L(\partial_z(\mathbf{id}+\boldsymbol{\eta}^n_*)\times\xi_0)=1$ for any n;
- finally, $\mathcal{B}: L^2(\mathcal{O})\to\mathbf{H}^1_0(\mathcal{O})$ is Bogovski's operator (see [12]). $\mathcal{B}$ along with the constants in the previous point is used to correct the divergence of the extra terms appearing in the definition of $\mathbf{u}^{k,n}_{\sigma,M}$ due to the extension of structure velocities in the fluid domains.

Observe that to preserve 0 boundary conditions on $\partial\mathcal{O}\setminus\Gamma$ we must also "squeeze" the mollified function by $\sim 1+\sigma$ as in [32, 31]. However, we choose to leave it out of our discussion for a cleaner presentation as it does not change the following estimates and argument. Next, let

$$
\mathbf{v}^{k,n}_{\sigma,M} := \mathbf{v}^k_M - \left(\lambda^k_\sigma-\lambda^{k,n}_M\right)\xi_0.
$$

Observe that we used Piola transformations, which preserve divergence and nullity of normal components at the boundary, to define the fluid test functions $\mathbf{u}^{k,n}_{\sigma,M}$. Thanks to (3.8) and Theorem 1.7-1 in [9] we thus observe that

$$
J^n_*(\mathrm{div}^{\boldsymbol{\eta}^n_*}\mathbf{u}^{k,n}_{\sigma,M}) = \mathrm{div}\left(J^k_*(\nabla A_{\boldsymbol{\eta}^k_*})^{-1}\mathbf{u}^k\right)_\sigma = \left(\mathrm{div}\left(J^k_*(\nabla A_{\boldsymbol{\eta}^k_*})^{-1}\mathbf{u}^k\right)\right)_\sigma = (J^k_*\mathrm{div}^{\boldsymbol{\eta}^k_*}\mathbf{u}^k)_\sigma.
$$

Hence,

$$
\begin{aligned}
&\|\mathrm{div}^{\boldsymbol{\eta}^n_*}\mathbf{u}^{k,n}_{\sigma,M}\|_{L^2(\mathcal{O})} \le C(\delta)\|\mathrm{div}^{\boldsymbol{\eta}^k_*}\mathbf{u}^k\|_{L^2(\mathcal{O})},\\
&\|(\mathbf{u}^{k,n}_{\sigma,M}-\mathbf{v}^{k,n}_{\sigma,M})\cdot\mathbf{n}^n_*\|_{L^2(\Gamma)} \le C(\delta)\|(\mathbf{u}^k-\mathbf{v}^k)\cdot\mathbf{n}^k_*\|_{L^2(\Gamma)}.
\end{aligned}
\tag{4.2}
$$

Note that, to ensure that $(4.2)_2$ holds, we mollified only in the tangential direction, i.e., along $\Gamma_{\boldsymbol{\eta}^k_*}$ in the definition of $\mathbf{u}^{k,n}_{\sigma,M}$.

Observe that for any $\beta<\frac{1}{2}$, we have

$$
\|\mathbf{v}_M-\mathbf{v}\|_{\mathbf{H}^{-\beta}(0,L)} \le \gamma_M^{-\frac{\beta}{2}}\|\mathbf{v}\|_{\mathbf{L}^2(0,L)}, \quad \|\mathbf{v}_\sigma-\mathbf{v}\|_{\mathbf{H}^{-\beta}(0,L)} \le \sigma^\beta\|\mathbf{v}\|_{\mathbf{L}^2(0,L)}. \tag{4.3}
$$

Observe also that $\|T_h\boldsymbol{\eta}^*_N-\boldsymbol{\eta}^*_N\|_{L^\infty(0,T;\mathbf{H}^s(0,L))} \le \|\widetilde{\boldsymbol{\eta}}^*_N\|_{C^{0,\frac{1}{4}}(0,T;\mathbf{H}^s(0,L))}h^{\frac{1}{4}}$ for any $s>\frac{3}{2}$. Thanks to these observations and (3.4), for any $s\ge\frac{3}{2}$ and $\beta<\frac{1}{2}$, we calculate

$$
\begin{aligned}
|\lambda^k_\sigma-\lambda^{k,n}_M| \le {}& \int_0^L\int_{-1}^1\left(\int_z^{z+\sigma y}|\partial_{zz}\boldsymbol{\eta}^k_*(w)|dw\right)|\mathbf{v}^k(z+\sigma y)|dydz \\
& + \|\boldsymbol{\eta}^k_*\|_{\mathbf{H}^s(0,L)}\|\mathbf{v}^k_\sigma-\mathbf{v}^k\|_{\mathbf{H}^{-\frac{1}{2}}(0,L)} + \|\boldsymbol{\eta}^k_*-\boldsymbol{\eta}^n_*\|_{\mathbf{H}^s(0,L)}\|\mathbf{v}^k\|_{\mathbf{L}^2(0,L)} \\
& + \|\boldsymbol{\eta}^n_*\|_{\mathbf{H}^s(0,L)}\|\mathbf{v}^k_M-\mathbf{v}^k\|_{\mathbf{H}^{-\frac{1}{2}}(0,L)} \\
\le {}& C(\delta)\left(\|\boldsymbol{\eta}^n_*\|_{\mathbf{H}^2_0(0,L)}\sigma^{\frac{1}{2}}+\sigma^\beta+\gamma_M^{-\frac{\beta}{2}}+h^{\frac{1}{4}}\|\widetilde{\boldsymbol{\eta}}^*_N\|_{C^{0,\frac{1}{4}}(0,T;\mathbf{H}^s(0,L))}\right)\|\mathbf{v}^k\|_{\mathbf{L}^2(0,L)}.
\end{aligned}
\tag{4.4}
$$

Observe that due to the properties of mollification, $\partial_{zz}\mathbf{w}_\sigma = (\partial_{zz}\mathbf{w})_\sigma$ and $\partial_{rr}\mathbf{w}_\sigma = (\partial_{rr}\mathbf{w})_\sigma$. Hence $\mathbf{w}^k_\sigma$ is harmonic with 0 boundary values on $\partial\mathcal{O}\setminus\Gamma$ and such that $\mathbf{w}^k_\sigma(z,1) = \mathbf{v}^k_\sigma(z)$.

Moreover, thanks to Proposition 3.1 in [13] and the bounds (3.8) and (3.4), we obtain for $s>\frac{3}{2}$, $\beta<\frac{1}{2}$, and $n-j\le k\le n$ that

$$\begin{aligned}\|\mathbf{u}^{k,n}_{\sigma,M}-\mathbf{u}^k\|_{\mathbf{L}^2(\mathcal{O})} &\lesssim \|\nabla A_{\boldsymbol{\eta}^n_*}-\nabla A_{\boldsymbol{\eta}^k_*}\|_{\mathbf{L}^\infty(\mathcal{O})}\|\mathbf{u}^k\|_{\mathbf{L}^2(\mathcal{O})}+\|\mathbf{u}^k-\mathbf{u}^k_\sigma\|_{\mathbf{L}^2(\mathcal{O})}\\ &\quad+\|\mathbf{w}^k-\mathbf{w}^k_\sigma\|_{\mathbf{L}^2(\mathcal{O})}+\|\mathbf{w}^k-\mathbf{w}^k_M\|_{\mathbf{L}^2(\mathcal{O})}+|\lambda^k_\sigma-\lambda^{k,n}_M|\\ &\lesssim \|\boldsymbol{\eta}^n_*-\boldsymbol{\eta}^k_*\|_{\mathbf{H}^s(0,L)}\|\mathbf{u}^k\|_{\mathbf{L}^2(\mathcal{O})}+\sigma\|\mathbf{u}^k\|_{\mathbf{H}^1(\mathcal{O})}\\ &\quad+\|\mathbf{v}^k-\mathbf{v}^k_\sigma\|_{\mathbf{H}^{-1/2}(0,L)}+\|\mathbf{v}^k-\mathbf{v}^k_M\|_{\mathbf{H}^{-1/2}(0,L)}+|\lambda^k_\sigma-\lambda^{k,n}_M|\\ &\lesssim h^{\frac{1}{4}}\|\tilde{\boldsymbol{\eta}}^*_N\|_{C^{0,\frac{1}{4}}(0,T;\mathbf{H}^s(0,L))}(\|\mathbf{u}^k\|_{\mathbf{L}^2(\mathcal{O})}+\|\mathbf{v}^k\|_{\mathbf{L}^2(0,L)})+\sigma\|\mathbf{u}^k\|_{\mathbf{H}^1(\mathcal{O})}\\ &\quad+(\sigma^\beta+\gamma_M^{-\frac{\beta}{2}}+h^{\frac{1}{4}})\|\mathbf{v}^k\|_{\mathbf{L}^2(0,L)},\end{aligned}\tag{4.5}$$

where the hidden constants depend only on δ. Observe, due to (3.7), that

$$\begin{aligned}\|\mathbf{u}^{k,n}_{\sigma,M}\|_{\mathbf{H}^1(\mathcal{O})} &\le \|A_{\boldsymbol{\eta}^k_*}\|_{\mathbf{W}^{2,3}(\mathcal{O})}(\|\mathbf{u}^k\|_{\mathbf{H}^1(\mathcal{O})}+\|\mathbf{v}^k_\sigma\|_{\mathbf{H}^1(\Gamma)})+\|\mathbf{v}^k_M\|_{\mathbf{H}^1(\Gamma)}+|\lambda^k_\sigma-\lambda^{k,n}_M|\\ &\le C(\delta)(\|\mathbf{u}^k\|_{\mathbf{H}^1(\mathcal{O})}+\sigma^{-1}\|\mathbf{v}^k\|_{\mathbf{L}^2(\Gamma)})+\gamma_M^{\frac{1}{2}}\|\mathbf{v}^k\|_{\mathbf{L}^2(\Gamma)}+C(\delta)\|\mathbf{v}^k\|_{\mathbf{L}^2(\Gamma)}.\end{aligned}$$

Therefore, we obtain for some $C>0$ that depends only on T and δ that

$$\begin{aligned}&\mathbb{E}\left[(\Delta t)\sup_{0\le n\le N}\sum_{k=n-j+1}^{n}\|\mathbf{u}^{k,n}_{\sigma,M}\|^2_{\mathbf{H}^1(\mathcal{O})}\right]\\ &\quad\le C(\delta,T)\mathbb{E}\left((\Delta t)\sum_{n=0}^{N}\|\mathbf{u}^k\|^2_{\mathbf{H}^1(\mathcal{O})}+(\sigma^{-2}+\gamma_M)\sup_{0\le n\le N}\|\mathbf{v}^k\|^2_{\mathbf{L}^2(0,L)}\right)\\ &\quad\le C(\delta,T)(1+\sigma^{-2}+\gamma_M).\end{aligned}\tag{4.6}$$

Similarly (see also (4.4)),

$$\|\mathbf{v}^{k,n}_{\sigma,M}\|_{\mathbf{H}^2(0,L)}\le\gamma_M\|\mathbf{v}^k\|_{\mathbf{L}^2(0,L)}+C(\delta)\|\mathbf{v}^k\|_{\mathbf{L}^2(0,L)}.\tag{4.7}$$

Then, as in [17], for any $n\le N$ we "test" $(3.1)_3$ and $(3.10)_2$ with

$$\mathbf{Q}_n:=(\mathbf{q}_n,\boldsymbol{\psi}_n)=\left((\Delta t)\sum_{k=n-j+1}^{n}\mathbf{u}^{k,n}_{\sigma,M},\ (\Delta t)\sum_{k=n-j+1}^{n}\mathbf{v}^{k,n}_{\sigma,M}\right).\tag{4.8}$$

This gives us, for any $0\le n\le N$ and $N\in\mathbb{N}$, that

$$\begin{aligned}&-\int_{\mathcal{O}}\left((J^{n+1}_*)\mathbf{u}^{n+1}-(J^n_*)\mathbf{u}^n\right)\left(\Delta t\sum_{k=n-j+1}^{n}\mathbf{u}^{k,n}_{\sigma,M}\right)\\ &\quad-\int_0^L(\mathbf{v}^{n+1}-\mathbf{v}^n)\left(\Delta t\sum_{k=n-j+1}^{n}\mathbf{v}^{k,n}_{\sigma,M}\right)\\ &\quad=\frac{1}{2}\int_{\mathcal{O}}\left(J^{n+1}_*-J^n_*\right)\mathbf{u}^{n+1}\cdot\left(\Delta t\sum_{k=n-j+1}^{n}\mathbf{u}^{k,n}_{\sigma,M}\right)\\ &\quad\quad+(\Delta t)b^{\boldsymbol{\eta}^*_n}\left(\mathbf{u}^{n+1},\mathbf{w}^n_*,\left(\Delta t\sum_{k=n-j+1}^{n}\mathbf{u}^{k,n}_{\sigma,M}\right)\right)\end{aligned}\tag{4.9}$$

$$
\begin{aligned}
&+\frac{(\Delta t)}{\alpha}\int_\Gamma S_*^n(\mathbf{u}^{n+1}-\mathbf{v}^{n+1})\left((\Delta t)\sum_{k=n-j+1}^{n}\mathbf{u}^{k,n}_{\sigma,M}-\mathbf{v}^{k,n}_{\sigma,M}\right)\\
&+\frac{(\Delta t)}{\varepsilon}\int_{\mathcal{O}}\mathrm{div}^{\boldsymbol{\eta}_*^n}\mathbf{u}^{n+1}\mathrm{div}^{\boldsymbol{\eta}_*^n}\left(\Delta t\sum_{k=n-j+1}^{n}\mathbf{u}^{k,n}_{\sigma,M}\right)\\
&+\frac{(\Delta t)}{\varepsilon}\int_\Gamma(\mathbf{u}^{n+1}-\mathbf{v}^{n+1})\cdot\mathbf{n}_*^n\left((\Delta t)\sum_{k=n-j+1}^{n}\mathbf{u}^{k,n}_{\sigma,M}-\mathbf{v}^{k,n}_{\sigma,M}\right)\cdot\mathbf{n}_*^n\\
&+2\nu(\Delta t)\int_{\mathcal{O}}(J_*^n)\mathbf{D}^{\eta_*^n}(\mathbf{u}^{n+1})\cdot\mathbf{D}^{\eta_*^n}\left(\Delta t\sum_{k=n-j+1}^{n}\mathbf{u}^{k,n}_{\sigma,M}\right)\\
&+(\Delta t)\left(\mathscr{L}_e(\boldsymbol{\eta}^{n+\frac12}),\left(\Delta t\sum_{k=n-j+1}^{n}\mathbf{v}^{k,n}_{\sigma,M}\right)\right.\\
&\left.+\varepsilon\int_0^L\partial_{zz}\mathbf{v}^{n+\frac12}\cdot\partial_{zz}\left(\Delta t\sum_{k=n-j+1}^{n}\mathbf{v}^{k,n}_{\sigma,M}\right)\right)\\
&-(\Delta t)\left(P_{in}^n\int_0^1 q_z^n\Big|_{z=0}dr-P_{out}^n\int_0^1 q_z^n\Big|_{z=1}dr\right)-(G(\mathbf{u}^n,\boldsymbol{\eta}_*^n)\Delta_nW,\mathbf{Q}_n).
\end{aligned}
$$

Then we apply $\sum_{n=0}^N$ to this equation and then denote its right-hand side by $\sum_1^9 I_i$. To motivate the choice of this test function, we first apply a summation by parts formula for the two terms on the left-hand side of this equation. For the second term we obtain

$$
\begin{aligned}
-\sum_{n=0}^{N}\int_0^L(\mathbf{v}^{n+1}-\mathbf{v}^n)\left(\Delta t\sum_{k=n-j+1}^{n}\mathbf{v}^{k,n}_{\sigma,M}\right)&=\int_0^L\mathbf{v}^n(\mathbf{v}^{n,n}_{\sigma,M}-\mathbf{v}^{n-j,n}_{\sigma,M})\\
=\int_0^L\mathbf{v}^n\left(\mathbf{v}^n_M-(\lambda^n_\sigma-\lambda^{n,n}_M)\,\xi_0-\mathbf{v}^{n-j}_M+\left(\lambda^{n-j}_\sigma-\lambda^{n-j,n}_M\right)\xi_0\right)dz&\\
+\frac{(\Delta t)}{2}\sum_{n=0}^{N}\int_0^L|\mathbf{v}^n_M-\mathbf{v}^{n-j}_M|^2+(|\mathbf{v}^n_M|^2-|\mathbf{v}^{n-j}_M|^2)&\\
-\mathbf{v}^n(\lambda^n_\sigma-\lambda^{n-j}_\sigma+\lambda^{n-j,n}_M-\lambda^{n,n}_M)\xi_0&\\
:=I_0^1+I_0^2+I_0^3,&
\end{aligned}
$$

where we set $\mathbf{v}^n=0$ for $n<0$ and $n>N$. Observe that since

$$
\|\mathbf{v}^n-\mathbf{v}^n_M-\mathbf{v}^{n-j}-\mathbf{v}^{n-j}_M\|_{\mathbf{H}^{-\beta}(0,L)}\le\gamma_M^{-\frac{\beta}{2}}(\|\mathbf{v}^n\|_{\mathbf{L}^2(0,L)}+\|\mathbf{v}^{n-j}\|_{\mathbf{L}^2(0,L)}),
$$

for any $0\le\beta<\frac12$, the term I_0^1 will give us one of the desired terms in (4.1).

Similarly, using summation by parts and setting $\mathbf{u}^n=0$ for $n<0$ and $n>N$, we obtain the following for the first term on the left-hand side:

$$
\begin{aligned}
&-\sum_{n=0}^{N}\int_{\mathcal{O}}\left((J_*^{n+1})\mathbf{u}^{n+1}-(J_*^n)\mathbf{u}^n\right)\left(\Delta t\sum_{k=n-j+1}^{n}\mathbf{u}^{k,n}_{\sigma,M}\right)\\
&=(\Delta t)\sum_{n=1}^{N}\int_{\mathcal{O}}(J_*^n)\mathbf{u}^n\left(\sum_{k=n-j+1}^{n}\mathbf{u}^{k,n}_{\sigma,M}-\sum_{k=n-j}^{n-1}\mathbf{u}^{k,n-1}_{\sigma,M}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= (\Delta t)\sum_{n=1}^{N}\int_{\mathcal{O}}(J_*^n)\mathbf{u}^n\left(\mathbf{u}_{\sigma,M}^{n,n}-\mathbf{u}_{\sigma,M}^{n-j,n}+\sum_{k=n-j}^{n-1}\mathbf{u}_{\sigma,M}^{k,n}-\mathbf{u}_{\sigma,M}^{k,n-1}\right)\\
&= (\Delta t)\sum_{n=1}^{N}\left(\int_{\mathcal{O}}(J_*^n)\mathbf{u}^n(\mathbf{u}^n-\mathbf{u}^{n-j})+\int_{\mathcal{O}}(J_*^n)\mathbf{u}^n(\mathbf{u}_{\sigma,M}^{n,n}-\mathbf{u}^n-(\mathbf{u}_{\sigma,M}^{n-j,n}-\mathbf{u}^{n-j}))\right)\\
&\quad+(\Delta t)\sum_{n=1}^{N}\int_{\mathcal{O}}(J_*^n)\mathbf{u}^n\left(\sum_{k=n-j}^{n-1}\mathbf{u}_{\sigma,M}^{k,n}-\mathbf{u}_{\sigma,M}^{k,n-1}\right)\\
&= (\Delta t)\sum_{n=0}^{N}\frac{1}{2}\int_{\mathcal{O}}(J_*^n)|\mathbf{u}^n-\mathbf{u}^{n-j}|^2d\mathbf{x}+(\Delta t)\sum_{n=0}^{N}\left(\frac{1}{2}\int_{\mathcal{O}}(J_*^n)(|\mathbf{u}^n|^2-|\mathbf{u}^{n-j}|^2)d\mathbf{x}\right)\\
&\quad+(\Delta t)\sum_{n=0}^{N}\left(\int_{\mathcal{O}}(J_*^n)\mathbf{u}^n(\mathbf{u}_{\sigma,M}^{n,n}-\mathbf{u}^n-(\mathbf{u}_{\sigma,M}^{n-j,n}-\mathbf{u}^{n-j}))d\mathbf{x}\right)\\
&\quad+(\Delta t)\sum_{n=1}^{N}\int_{\mathcal{O}}(J_*^n)\mathbf{u}^n\left(\sum_{k=n-j}^{n-1}\mathbf{u}_{\sigma,M}^{k,n}-\mathbf{u}_{\sigma,M}^{k,n-1}\right)=:J_0^1+J_0^2+J_0^3+J_0^4.
\end{aligned}
$$

Observe again that J_0^1 is another one of the desired terms in (4.1). Now we will find appropriate bounds for the rest of the terms. We start with I_0^2:

$$
\mathbb{E}(\Delta t)\sum_{n=0}^{N}\int_0^L|\mathbf{v}_M^n|^2-|\mathbf{v}_M^{n-j}|^2=\mathbb{E}(\Delta t)\sum_{n=N-j+1}^{N}\int_\Gamma|\mathbf{v}_M^n|^2\le h\mathbb{E}\max_{0\le n\le N}\|\mathbf{v}^n\|^2_{\mathbf{L}^2(0,L)}.
$$

Next, thanks to (4.4), we see for any $0\le\beta<\frac{1}{2}$ that

$$
\begin{aligned}
\mathbb{E}|I_0^3|&:=\mathbb{E}\left|(\Delta t)\sum_{n=0}^{N}\int_0^L\mathbf{v}^n(\lambda_\sigma^{n,n}-\lambda_\sigma^{n-j,n}+\lambda_M^{n-j,n}-\lambda_M^{n,n})\xi_0\right|\\
&\le\mathbb{E}\Bigg(\left(\|\boldsymbol{\eta}_N^*\|_{L^\infty(0,T;\mathbf{H}_0^2(0,L))}\sigma^{\frac{1}{2}}+\sigma^\beta+\gamma_M^{-\frac{\beta}{2}}+h^{\frac{1}{4}}\|\widetilde{\boldsymbol{\eta}}_N^*\|_{C^{0,\frac{1}{4}}(0,T;\mathbf{H}^s(0,L))}\right)\\
&\qquad\times\|\mathbf{v}_N\|_{L^\infty(0,T;\mathbf{L}^2(0,L))}\Bigg)\le Ch^{\frac{\beta}{12}}.
\end{aligned}
$$

Hence, using the Chebyshev inequality we obtain

$$
\mathbb{P}(|I_0^2+I_0^3|>R)\le\frac{Ch^{\frac{\beta}{12}}}{R}.
$$

To treat the term J_0^2, we recall that for two matrices A and B, the derivative of the determinant $D(\det)(A)B=\det(A)\mathrm{tr}(BA^{-1})$. Hence applying the mean value theorem to $\det(\nabla A_{\boldsymbol{\eta}_*^n})-\det(\nabla A_{\boldsymbol{\eta}_*^{n+j}})$, using (3.8) and that $\det(A)\le(\max A_{ij})^2$, we obtain, for some $\beta\in[0,1]$, that (for more details see (73) in [26])

$$
\begin{aligned}
\|J_*^{n+j}-J_*^n\|_{L^\infty(\mathcal{O})}&=\|\det(\nabla A^{n,\beta})\nabla^{n,\beta}\cdot(A_{\boldsymbol{\eta}_*^{n+j}}-A_{\boldsymbol{\eta}_*^n})\|_{L^\infty(\mathcal{O})}\\
&\le C(\delta)\|A_{\boldsymbol{\eta}_*^{n+j}}-A_{\boldsymbol{\eta}_*^n}\|_{\mathbf{C}^1(\mathcal{O})},
\end{aligned}
\tag{4.10}
$$

where $\nabla^{n,\beta}=\nabla^{\boldsymbol{\eta}_*^n}+\beta(\nabla^{\boldsymbol{\eta}_*^{n+j}}-\nabla^{\boldsymbol{\eta}_*^n})$ and $\nabla A^{n,\beta}=\nabla A_{\boldsymbol{\eta}_*^n}+\beta(\nabla A_{\boldsymbol{\eta}_*^{n+j}}-\nabla A_{\boldsymbol{\eta}_*^n})$. Using (3.8) again, we find the following bounds for any $s>\frac{3}{2}$:

$$
\begin{aligned}
J_0^2 &:= (\Delta t)\sum_{n=0}^{N}\left(\int_{\mathcal{O}}(J_*^n)(|\mathbf{u}^n|^2-|\mathbf{u}^{n-j}|^2)d\mathbf{x}\right)\\
&= (\Delta t)\left(\sum_{n=N-j+1}^{N}\int_{\mathcal{O}}(J_*^n)|\mathbf{u}^n|^2d\mathbf{x}+\sum_{n=0}^{N-j}\int_{\mathcal{O}}(J_*^n-J_*^{n+j})|\mathbf{u}^n|^2d\mathbf{x}\right)\\
&\le (\Delta t)\sum_{n=N-j+1}^{N}\int_{\mathcal{O}}(J_*^n)|\mathbf{u}^n|^2d\mathbf{x}+(\Delta t)\sum_{n=0}^{N-j}\|A_{\boldsymbol{\eta}_*^{n+j}}-A_{\boldsymbol{\eta}_*^n}\|_{\mathbf{C}^1(\mathcal{O})}\|\mathbf{u}^n\|^2_{\mathbf{L}^2(\mathcal{O})}\\
&\le \left(h\sup_{1\le n\le N}\int_{\mathcal{O}}(J_*^n)|\mathbf{u}^n|^2d\mathbf{x}+\sup_{0\le n\le N-1}\|\mathbf{u}^n\|^2_{\mathbf{L}^2(\mathcal{O})}h^{\frac14}\|\boldsymbol{\eta}_*^{n+j}-\boldsymbol{\eta}_*^n\|_{\mathbf{H}^s(0,L)}\right)\\
&\le h^{\frac14}\sup_{0\le n\le N-1}\|\mathbf{u}^n\|^2_{\mathbf{L}^2(\mathcal{O})}\left(1+\|\widetilde{\boldsymbol{\eta}}_N^*\|_{C^{0,\frac14}(0,T;\mathbf{H}^s(0,L))}\right).
\end{aligned}
$$

In what follows, we will repeatedly make use of the fact that for any two random variables X and Y, we have

$$
\begin{aligned}
\mathbb{P}(|X|+|Y|>R)&\le \mathbb{P}\left(|X|>\frac{R}{2}\right)+\mathbb{P}\left(|Y|>\frac{R}{2}\right),\\
\mathbb{P}(|XY|>R)&\le \mathbb{P}(|X|>\sqrt{R})+\mathbb{P}(|Y|>\sqrt{R}).
\end{aligned}
$$

The embedding $W^{1,\infty}(0,T;L^2(\Gamma))\cap L^2(0,T;H_0^2(\Gamma))\hookrightarrow C^{0,\frac14}(0,T;H^{\frac32}(\Gamma))$ then gives us for any $s\ge\frac32$ that

$$
\begin{aligned}
\mathbb{P}\left(|J_0^2|>R\right)&\le \mathbb{P}\left(h^{\frac18}\sup_{1\le n\le N}\|\mathbf{u}^n\|^2_{\mathbf{L}^2(\mathcal{O})}\ge\sqrt{R}\right)+\mathbb{P}\left(h^{\frac18}\|\widetilde{\boldsymbol{\eta}}_N^*\|_{C^{0,\frac14}(0,T;\mathbf{H}^s(\Gamma))}\ge\sqrt{R}\right)\\
&\le \frac{h^{\frac18}}{\sqrt{R}}\mathbb{E}\left(\sup_{1\le n\le N}\|\mathbf{u}^n\|^2_{\mathbf{L}^2(\mathcal{O})}+\|\widetilde{\boldsymbol{\eta}}_N^*\|_{C^{0,\frac14}(0,T;\mathbf{H}^s(\Gamma))}\right)\\
&\le \frac{h^{\frac18}}{\sqrt{R}}\mathbb{E}\left(\sup_{1\le n\le N}\|\mathbf{u}^n\|^2_{\mathbf{L}^2(\mathcal{O})}+\|\widetilde{\boldsymbol{\eta}}_N^*\|_{W^{1,\infty}(0,T;\mathbf{L}^2(\Gamma))\cap L^2(0,T;\mathbf{H}_0^2(\Gamma))}\right)\le C\frac{h^{\frac18}}{\sqrt{R}}.
\end{aligned}
$$

Next, thanks to (4.5), we see that

$$
\begin{aligned}
J_0^3 &= \left|(\Delta t)\sum_{n=0}^{N}\left(\int_{\mathcal{O}}(J_*^n)\mathbf{u}^n(\mathbf{u}_{\sigma,M}^{n,n}-\mathbf{u}^n-(\mathbf{u}_{\sigma,M}^{n-j,n}-\mathbf{u}^{n-j}))d\mathbf{x}\right)\right|\\
&\le C(\delta)(\Delta t)\sum_{n=0}^{N}\|\mathbf{u}^n\|_{\mathbf{L}^2(\mathcal{O})}(\|\mathbf{u}_{\sigma,M}^{n,n}-\mathbf{u}^n\|_{\mathbf{L}^2(\mathcal{O})}+\|\mathbf{u}_{\sigma,M}^{n-j,n}-\mathbf{u}^{n-j}\|_{\mathbf{L}^2(\mathcal{O})})\\
&\lesssim \sup_{1\le n\le N}\|\mathbf{u}^n\|_{\mathbf{L}^2(\mathcal{O})}(\Delta t)\sum_{k=0}^{N}\Big(h^{\frac14}\|\widetilde{\boldsymbol{\eta}}_N^*\|_{C^{0,\frac14}(0,T;\mathbf{H}^s(0,L))}(\|\mathbf{u}^k\|_{\mathbf{L}^2(\mathcal{O})}+\|\mathbf{v}^k\|_{\mathbf{L}^2(0,L)})\\
&\quad+\sigma\|\mathbf{u}^k\|_{\mathbf{H}^1(\mathcal{O})}+(\sigma^\beta+\gamma_M^{-\frac{\beta}{2}}+h^{\frac14})\|\mathbf{v}^k\|_{\mathbf{L}^2(0,L)}\Big).
\end{aligned}
$$

Thus, we obtain that $\mathbb{P}(|J_0^3|>R)\le\frac{C}{R}(h^{\frac14}+\sigma^\beta+\gamma_M^{-\frac{\beta}{2}})\le\frac{C}{R}h^{\frac{\beta}{12}}$. For the term J_0^4 we find, using the properties of the Bogovski operator $\mathcal{B}$, that

$$
\begin{aligned}
|J_4^0| &= \left|(\Delta t)\sum_{n=0}^{N}\int_{\mathcal{O}}(J_*^n)\mathbf{u}^n\left(\sum_{k=n-j}^{n-1}\mathbf{u}_{\sigma,M}^{k,n}-\mathbf{u}_{\sigma,M}^{k,n-1}\right)\right| \\
&\leq C(\delta)(\Delta t)\sum_{n=0}^{N}\Bigg(\|\mathbf{u}^n\|_{\mathbf{L}^6(\mathcal{O})}\|\frac{(J_*^n)^{-1}\nabla A_{\boldsymbol{\eta}_*^n}-(J_*^{n-1})^{-1}\nabla A_{\boldsymbol{\eta}_*^{n-1}}}{\Delta t}\|_{L^2(\mathcal{O})} \\
&\quad \cdot\sum_{k=n-j}^{n-1}(\Delta t)\|(\mathbf{u}^k-\mathbf{w}^k)_\sigma\|_{\mathbf{L}^3(\mathcal{O})}\Bigg)+(\Delta t)\sum_{n=0}^{N}\|\mathbf{u}^n\|_{\mathbf{L}^2(\mathcal{O})}(\Delta t)\sum_{k=n-j}^{n-1}\frac{|\lambda_M^{k,n}-\lambda_M^{k,n-1}|}{\Delta t} \\
&\leq h\sigma^{-\frac{1}{2}}C(\delta)\left((\Delta t)\sum_{n=0}^{N}\|\mathbf{u}^n\|^2_{\mathbf{H}^1(\mathcal{O})}\right)^{\frac{1}{2}}\left((\Delta t)\sum_{n=0}^{N}\|\mathbf{v}_*^n\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}\right)^{\frac{1}{2}} \\
&\quad \cdot\max_{0\leq n\leq N}(\|\mathbf{u}^n\|_{\mathbf{L}^2(\mathcal{O})}+\|\mathbf{v}^n\|_{\mathbf{L}^2(\Gamma)}) \\
&\quad +h\gamma_M^{\frac{1}{2}}C(\delta)\max_{0\leq n\leq N}(\|\mathbf{u}^n\|_{\mathbf{L}^2(\mathcal{O})}\|\mathbf{v}_*^n\|_{\mathbf{L}^2(\Gamma)}\|\mathbf{v}^k\|_{\mathbf{L}^2(\Gamma)}).
\end{aligned}
$$

Here we also used the calculations in (4.10) that gave us

$$
\left\|\frac{J_*^{n+1}-J_*^n}{\Delta t}\right\|_{L^2(\mathcal{O})}\leq C(\delta)\|\mathbf{w}_*^{n+1}\|_{\mathbf{H}^1(\mathcal{O})}\leq C(\delta)\|\mathbf{v}_*^{n+1}\|_{\mathbf{H}^{\frac{1}{2}}(0,L)}. \tag{4.11}
$$

Hence, we conclude, for some constant C that depends on ε, that

$$
\mathbb{P}(|J_0^4|\geq R)\leq C(\varepsilon)\frac{h^{\frac{11}{12}}}{\sqrt{R}}. \tag{4.12}
$$

Now we use (4.11) again to bound I_1 on the right-hand side of (4.9):

$$
\begin{aligned}
I_1 &:= \left|\sum_{n=0}^{N}\int_{\mathcal{O}}\left(J_*^{n+1}-J_*^n\right)\mathbf{u}^{n+1}\cdot\left(\Delta t\sum_{k=n-j+1}^{n}\mathbf{u}_{\sigma,M}^{k,n}\right)d\mathbf{x}\right| \\
&\leq (\Delta t)\sum_{n=0}^{N}\|\mathbf{v}^{n+\frac{1}{2}}\|_{\mathbf{H}^{\frac{1}{2}}(0,L)}\|\mathbf{u}^{n+1}\|_{\mathbf{L}^6(\mathcal{O})}\left((\Delta t)\sum_{k=n-j+1}^{n}\|\mathbf{u}_{\sigma,M}^{k,n}\|^2_{\mathbf{L}^3(\mathcal{O})}\right)^{\frac{1}{2}}\sqrt{h} \\
&\leq C\sqrt{h}\left((\Delta t)\sum_{n=0}^{N}\|\mathbf{v}^{n+\frac{1}{2}}\|^2_{\mathbf{H}^{\frac{1}{2}}(0,L)}\right)^{\frac{1}{2}}\left((\Delta t)\sum_{n=0}^{N}\|\mathbf{u}^n\|^2_{\mathbf{H}^1(\mathcal{O})}\right)^{\frac{1}{2}} \\
&\quad \times\left((\Delta t)\sup_{0\leq n\leq N}\sum_{k=n-j+1}^{n}\|\mathbf{u}_{\sigma,M}^{k,n}\|^2_{\mathbf{H}^1(\mathcal{O})}\right)^{\frac{1}{2}}.
\end{aligned}
$$

This implies, thanks to (4.6), that

$$
\mathbb{P}\left(|I_1|\geq R\right)\leq\frac{C}{\varepsilon}\frac{h^{\frac{1}{3}}}{R^{\frac{2}{3}}}+\frac{C(\delta)h^{\frac{1}{3}}}{R^{\frac{2}{3}}}(1+\sigma^{-2}+\gamma_M)\leq C(\varepsilon)\frac{h^{\frac{1}{6}}}{R^{\frac{2}{3}}}. \tag{4.13}
$$

Note that this estimate depends on ε. Next, we treat the nonlinear term in Navier–Stokes equations. Using the embedding $H^{\frac{1}{2}}(\mathcal{O})\hookrightarrow L^4(\mathcal{O})$ we obtain, for some $C>0$ which depends only on δ, that

$$\begin{aligned}
I_2 &:= \left|(\Delta t)\sum_{n=0}^N b^{\boldsymbol{\eta}_*^n}\left(\mathbf{u}^{n+1},\mathbf{w}_*^n,\left(\Delta t\sum_{k=n-j+1}^n \mathbf{u}_{\sigma,M}^{k,n}\right)\right)\right|\\
&\le \sqrt{h}(\Delta t)\sum_{n=0}^N \|\mathbf{u}^{n+1}-\mathbf{w}_*^n\|_{\mathbf{L}^4(\mathcal{O})}\|\mathbf{u}^{n+1}\|_{\mathbf{L}^4(\mathcal{O})}\left(\Delta t\sum_{k=n-j+1}^n \|\mathbf{u}_{\sigma,M}^{k,n}\|^2_{\mathbf{H}^1(\mathcal{O})}\right)^{\frac12}\\
&\le \sqrt{h}(\Delta t)\sum_{n=0}^N\left(\|\mathbf{u}^{n+1}\|_{\mathbf{H}^1(\mathcal{O})}+\|\mathbf{w}_*^n\|_{\mathbf{H}^{\frac12}(0,L)}\right)\|\mathbf{u}^{n+1}\|_{\mathbf{H}^1(\mathcal{O})}\\
&\quad\times\left(\Delta t\sum_{k=n-j+1}^n \|\mathbf{u}_{\sigma,M}^{k,n}\|^2_{\mathbf{H}^1(\mathcal{O})}\right)^{\frac12}.
\end{aligned}$$

Again, thanks to (4.6), for any $R>0$, we have

$$\begin{aligned}
\mathbb{P}(|I_2|>R) &\le \frac{h^{\frac14}}{\sqrt{R}}\mathbb{E}\left[(\Delta t)\sum_{n=0}^N\|\mathbf{u}^{n+1}\|^2_{\mathbf{H}^1(\mathcal{O})}\right]+\frac{h^{\frac14}}{\sqrt{R}}\mathbb{E}\left[\sup_{0\le n\le N-1}\|\mathbf{v}^{n+\frac12}\|^2_{\mathbf{L}^2(0,L)}\right]\\
&\quad+\frac{h^{\frac12}}{R}\mathbb{E}\left[(\Delta t)^2\sum_{n=0}^N\sum_{k=n-j+1}^n\|\mathbf{u}_{\sigma,M}^{k,n}\|^2_{\mathbf{H}^1(\mathcal{O})}\right]\\
&\le \frac{C(\delta)h^{\frac14}}{\sqrt{R}}+\frac{C(\delta)h^{\frac12}}{R}(1+\sigma^{-2}+\gamma_M)\le\frac{C(\delta)}{\sqrt{R}}h^{\frac14}.
\end{aligned}$$

The term I_3 is easier to handle and is treated similarly. For the penalty term, thanks to $(4.2)_1$, we have

$$\begin{aligned}
\mathbb{E}[I_4] &:= \mathbb{E}\left|\frac{(\Delta t)}{\varepsilon}\sum_{n=0}^N\int_{\mathcal{O}}\operatorname{div}^{\boldsymbol{\eta}_*^n}\mathbf{u}^{n+1}\left(\Delta t\sum_{k=n-j+1}^n\operatorname{div}^{\boldsymbol{\eta}_*^n}(\mathbf{u}_{\sigma,\lambda}^{k,n})\right)d\mathbf{x}\right|\\
&\le \mathbb{E}\frac{(\Delta t)}{\varepsilon}\sum_{n=0}^N\|\operatorname{div}^{\eta_*^n}\mathbf{u}^{n+1}\|_{L^2(\mathcal{O})}\left((\Delta t)\sum_{k=n-j+1}^n\|\operatorname{div}^{\boldsymbol{\eta}_*^n}(\mathbf{u}_{\sigma,\lambda}^{k,n})\|^2_{L^2(\mathcal{O})}\right)^{\frac12}\sqrt{h}\\
&\le C\sqrt{hT}\mathbb{E}\left(\frac{(\Delta t)}{\varepsilon}\sum_{n=0}^N\|\operatorname{div}^{\boldsymbol{\eta}_*^n}(\mathbf{u}^{n+1})\|^2_{L^2(\mathcal{O})}\right)\le Ch^{\frac12}.
\end{aligned}$$

Notice that, due to Lemma 3.5(5), the constant C in the estimate above does not depend on ε. The other penalty term I_5 is handled identically thanks to $(4.2)_2$ and Lemma 3.5(6). Similarly, due to (4.6), we have

$$\begin{aligned}
\mathbb{E}[I_6] &= \mathbb{E}\left|(\Delta t)\sum_{n=0}^N\int_{\mathcal{O}}\mathbf{D}^{\eta_*^n}\mathbf{u}^{n+1}\left(\Delta t\sum_{k=n-j+1}^n\mathbf{D}^{\eta_*^n}(\mathbf{u}_{\sigma,M}^{k,n})\right)d\mathbf{x}\right|\\
&\le \sqrt{h}\mathbb{E}\left((\Delta t)\sum_{n=0}^N\|\mathbf{u}^{n+1}\|_{\mathbf{H}^1(\mathcal{O})}\left((\Delta t)\sum_{k=n-j+1}^n\|(\mathbf{u}_{\sigma,M}^{k,n})\|^2_{\mathbf{H}^1(\mathcal{O})}\right)^{\frac12}\right)\\
&\le \sqrt{h}\mathbb{E}\sum_{n=0}^N(\Delta t)\|\mathbf{u}^{n+1}\|^2_{\mathbf{H}^1(\mathcal{O})}+\sqrt{h}\mathbb{E}\left((\Delta t)\sup_{0\le n\le N}\sum_{k=0}^N\|(\mathbf{u}_{\sigma,M}^{k,n})\|^2_{\mathbf{H}^1(\mathcal{O})}\right)\\
&\le C(\delta)h^{\frac14}(1+\sigma^{-2}+\gamma_M)\le Ch^{\frac{1}{12}}.
\end{aligned}$$

Next, using (4.7), we obtain that

$$\begin{aligned}\mathbb{E}[I_8] &:= \mathbb{E}\left|(\Delta t)\sum_{n=0}^{N}\left(\mathscr{L}_e(\boldsymbol{\eta}^{n+1}), \left(\Delta t\sum_{k=n-j+1}^{n}\mathbf{v}^k_{\sigma,M}\right)\right)\right| \\ &\leq C(\delta)(\Delta t)^2\mathbb{E}\sum_{n=0}^{N}\left(\|\partial_{zz}\boldsymbol{\eta}^n\|_{\mathbf{L}^2(0,L)}\sum_{k=n-j-1}^{n}\|\partial_{zz}\mathbf{v}^{k,n}_{\sigma,M}\|_{\mathbf{L}^2(0,L)}\right) \\ &\leq \mathbb{E}(\Delta t)^2\sum_{n=0}^{N}\left(\|\partial_{zz}\boldsymbol{\eta}^n\|_{\mathbf{L}^2(0,L)}\sum_{k=n-j-1}^{n}\gamma_M\|\mathbf{v}^k\|_{\mathbf{L}^2(0,L)}\right) \\ &\leq CTh\gamma_M\mathbb{E}\max_{0\leq n\leq N}\left(\|\boldsymbol{\eta}^n\|^2_{\mathbf{H}^2(0,L)} + \|\mathbf{v}^n\|^2_{\mathbf{L}^2(0,L)}\right) \leq Ch^{\frac{5}{6}}.\end{aligned}$$

The term I_7 is handled similarly. Finally, we treat the stochastic term (see also (3.17)) using Young's inequality:

$$\begin{aligned}\mathbb{E}\left(\sum_{n=0}^{N}|(G(\mathbf{u}^n,\boldsymbol{\eta}^n_*)\Delta_n W, \mathbf{Q}_n)|\right) &\leq \mathbb{E}\left(\sum_{n=0}^{N}\|G(\mathbf{u}^n,\boldsymbol{\eta}^n_*)\|_{L_2(U_0;\mathbf{L}^2)}\|\Delta_n W\|_{U_0}\right. \\ &\quad\left.\times\left((\Delta t)\sum_{k=n-j+1}^{n}\|\mathbf{u}^{k,n}_{\sigma,M}\|^2_{\mathbf{L}^2(\mathcal{O})} + \|\mathbf{v}^k_{\sigma,M}\|^2_{\mathbf{L}^2(0,L)}\right)^{\frac{1}{2}} h^{\frac{1}{2}}\right) \\ &\leq h^{\frac{1}{2}}\mathbb{E}\sum_{n=0}^{N}\|G(\mathbf{u}^n,\boldsymbol{\eta}^n_*)\|^2_{L_2(U_0;\mathbf{L}^2)}\|\Delta_n W\|^2_{U_0} \\ &\quad + \mathbb{E}\left((\Delta t)\sum_{k=n-j+1}^{n}\|\mathbf{u}^{k,n}_{\sigma,M}\|^2_{\mathbf{L}^2(\mathcal{O})} + \|\mathbf{v}^k_{\sigma,M}\|^2_{\mathbf{L}^2(0,L)}\right) \leq Ch^{\frac{1}{2}}.\end{aligned}$$

Now to show that the laws of the random variables mentioned in the statement of the theorem are tight, we will consider the following sets for $0 \leq \alpha < 1$ and $0 < \beta < \frac{1}{2}$ and any $R > 0$:

$$\begin{aligned}\mathcal{B}_R := \Bigg\{&(\mathbf{u},\mathbf{v}) \in L^2(0,T;\mathbf{H}^1(\mathcal{O})) \times L^2(0,T;\mathbf{L}^2(0,L)) : \|\mathbf{u}\|^2_{L^2(0,T;\mathbf{H}^1(\mathcal{O}))} \\ &+ \|\mathbf{v}\|^2_{L^2(0,T;\mathbf{L}^2(0,L))} + \sup_{0<h<1} h^{-\frac{\beta}{12}}\int_h^T \left(\|T_h\mathbf{u}-\mathbf{u}\|^2_{\mathbf{L}^2(\mathcal{O})} + \|T_h\mathbf{v}-\mathbf{v}\|^2_{\mathbf{H}^{-\beta}(0,L)}\right) \leq R\Bigg\}.\end{aligned}$$

Thanks to Lemma 4.1, $\mathcal{B}_R$ is compact in $L^2(0,T;\mathbf{H}^\alpha(\mathcal{O})) \times L^2(0,T;\mathbf{H}^{-\beta}(0,L))$ for each $R > 0$, $0 \leq \alpha < 1$, and $0 < \beta < \frac{1}{2}$. Now we apply Chebyshev's inequality to obtain the desired result:

$$\begin{aligned}\mathbb{P}((\mathbf{u}^+_N,\mathbf{v}^+_N) \notin \mathcal{B}_R) &\leq \mathbb{P}\left(\|\mathbf{u}^+_N\|^2_{L^2(0,T;\mathbf{H}^1(\mathcal{O}))} + \|\mathbf{v}^+_N\|^2_{L^2(0,T;\mathbf{L}^2(0,L))} > \frac{R}{2}\right) \\ &+ \mathbb{P}\left(\sup_{0<h<1} h^{-\frac{\beta}{12}}\int_h^T\left(\|T_h\mathbf{u}^+_N - \mathbf{u}^+_N\|^2_{\mathbf{L}^2(\mathcal{O})} + \|T_h\mathbf{v}^+_N - \mathbf{v}^+_N\|^2_{\mathbf{H}^{-\beta}(0,L)}\right) > \frac{R}{2}\right) \leq \frac{C}{\sqrt{R}},\end{aligned}$$

where $C > 0$ depends only on δ, TrQ, ε, and the given data and is independent of the parameter N. ∎

Next we give the tightness results for the structure displacements and velocities.

LEMMA 4.3. *For fixed δ, the following statements hold:*

1. *The laws of $\{\widetilde{\boldsymbol{\eta}}_N\}_{N\in\mathbb{N}}$ and those of $\{\widetilde{\boldsymbol{\eta}}_N^*\}_{N\in\mathbb{N}}$ are tight in $C([0,T],\mathbf{H}^s(0,L))$ for any $s<2$.*
2. *The laws of $\{\|\mathbf{u}_N^+\|_{L^2(0,T;V)}\}_{N\in\mathbb{N}}$ are tight in $\mathbb{R}$.*
3. *For a fixed ε, the laws of $\{\|\mathbf{v}_N^*\|_{L^2(0,T;\mathbf{H}_0^1(0,L))}\}_{N\in\mathbb{N}}$ are tight in $\mathbb{R}$.*

Proof. The Aubin–Lions theorem gives us the following: For $0<s<2$,

$$L^\infty(0,T;\mathbf{H}_0^2(0,L))\cap W^{1,\infty}(0,T;\mathbf{L}^2(0,L))\subset\subset C([0,T];\mathbf{H}^s(0,L)).$$

Hence for any $R>0$ we consider

$$\begin{aligned}\mathcal{K}_R:=\{\boldsymbol{\eta}\in L^\infty(0,T;\mathbf{H}_0^2(0,L))\cap W^{1,\infty}(0,T;\mathbf{L}^2(0,L)):\\ \|\boldsymbol{\eta}\|^2_{L^\infty(0,T;\mathbf{H}_0^2(0,L))}+\|\boldsymbol{\eta}\|^2_{W^{1,\infty}(0,T;\mathbf{L}^2(0,L))}\le R\}.\end{aligned}$$

Using the Chebyshev inequality and Lemma 3.5 we obtain for some $C>0$ independent of N and ε that the following holds:

$$\begin{aligned}\mathbb{P}\left[\widetilde{\boldsymbol{\eta}}_N\notin\mathcal{K}_R\right]&\le\mathbb{P}\left[\|\widetilde{\boldsymbol{\eta}}_N\|^2_{L^\infty(0,T;\mathbf{H}_0^2(0,L))}\ge\frac{R}{2}\right]+\mathbb{P}\left[\|\widetilde{\boldsymbol{\eta}}_N\|^2_{W^{1,\infty}(0,T;\mathbf{L}^2(0,L))}\ge\frac{R}{2}\right]\\&\le\frac{4}{R^2}\mathbb{E}\left[\|\widetilde{\boldsymbol{\eta}}_N\|^2_{L^\infty(0,T;\mathbf{H}_0^2(0,L))}+\|\widetilde{\boldsymbol{\eta}}_N\|^2_{W^{1,\infty}(0,T;\mathbf{L}^2(0,L))}\right]\le\frac{C}{R^2}.\end{aligned}$$

The proof of statements (2) and (3) follow by an identical application of the Chebyshev inequality and the bounds obtained in Lemma 3.5. For any $R>0$,

$$\mathbb{P}[\|\mathbf{v}_N^*\|_{L^2(0,T;\mathbf{H}_0^1(0,L))}>R]\le\frac{1}{R^2}\mathbb{E}[\|\mathbf{v}_N^*\|^2_{L^2(0,T;\mathbf{H}_0^1(0,L))}]\le\frac{C(\varepsilon)}{R^2}.$$

This completes the proof of Lemma 4.3. ∎

To obtain almost sure convergence of the rest of the random variables we will use the following lemma which is a consequence of the bounds on numerical dissipation.

LEMMA 4.4. *The following convergence results hold:*

1. $\lim_{N\to\infty}\mathbb{E}\int_0^T\|\mathbf{u}_N-\mathbf{u}_N^+\|^2_{\mathbf{L}^2(\mathcal{O})}dt=0$, $\lim_{N\to\infty}\mathbb{E}\int_0^T\|\mathbf{u}_N-\widetilde{\mathbf{u}}_N\|^2_{\mathbf{L}^2(\mathcal{O})}dt=0$.
2. $\lim_{N\to\infty}\mathbb{E}\int_0^T\|\mathbf{v}_N-\widetilde{\mathbf{v}}_N\|^2_{\mathbf{L}^2(0,L)}dt=0$, $\lim_{N\to\infty}\mathbb{E}\int_0^T\|\mathbf{v}_N-\mathbf{v}_N^{\#}\|^2_{\mathbf{L}^2(0,L)}dt=0$.
3. $\lim_{N\to\infty}\mathbb{E}\int_0^T\|\boldsymbol{\eta}_N-\widetilde{\boldsymbol{\eta}}_N\|^2_{\mathbf{H}_0^2(0,L)}dt=0$, $\lim_{N\to\infty}\mathbb{E}\int_0^T\|\boldsymbol{\eta}_N^+-\widetilde{\boldsymbol{\eta}}_N\|^2_{\mathbf{H}_0^2(0,L)}dt=0$.

Proof. Statement $(1)_1$ is true thanks to Theorem 3.4(3). We prove $(1)_2$ below exactly as in [32].

$$\begin{aligned}&\mathbb{E}\int_0^T\|\mathbf{u}_N-\widetilde{\mathbf{u}}_N\|^2_{\mathbf{L}^2(\mathcal{O})}dt\\&\quad=\mathbb{E}\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}}\frac{1}{\Delta t}\|(t-t^n)\mathbf{u}^{n+1}+(t^{n+1}-t-\Delta t)\mathbf{u}^n\|^2_{\mathbf{L}^2(\mathcal{O})}dt\\&\quad=\mathbb{E}\sum_{n=0}^{N-1}\|\mathbf{u}^{n+1}-\mathbf{u}^n\|^2_{\mathbf{L}^2(\mathcal{O})}\int_{t^n}^{t^{n+1}}\left(\frac{t-t^n}{\Delta t}\right)^2dt\le\frac{CT}{\delta_1 N}\to0\quad\text{as }N\to\infty.\end{aligned}$$

The rest follows identically from the uniform estimates stated in Theorem 3.4. ∎

To pass $N \to \infty$ in the weak formulation, we consider the following random variable:

$$\widetilde{\mathbf{U}}_N(t) := ((\widetilde{J}^*_N(t))\widetilde{\mathbf{u}}_N(t), \widetilde{\mathbf{v}}_N(t)) - E_N(t), \tag{4.14}$$

where E_N is an error term that appears due to discretizing the stochastic integral (see (4.26)) given by

$$E_N(t) = \sum_{m=0}^{N-1}\left(\frac{t-t^m}{\Delta t}G(\mathbf{u}^m,\boldsymbol{\eta}^m_*)\Delta_m W - \int_{t^m}^t G(\mathbf{u}^m,\boldsymbol{\eta}^m_*)dW\right)\chi_{[t^m,t^{m+1})}.$$

Let $\mathscr{U}_1 := \mathscr{U} \cap (\mathbf{H}^2(\mathcal{O}) \times \mathbf{H}^3_0(0,L))$. For any $\mathscr{V}_1 \subset\subset \mathscr{U}_1$, we denote by $\mu^{u,v}_N$ the probability measure of $\widetilde{\mathbf{U}}_N$:

$$\mu^{u,v}_N := \mathbb{P}\left(\widetilde{\mathbf{U}}_N \in \cdot\right) \in Pr(C([0,T];\mathscr{V}_1')),$$

where $Pr(S)$ denotes the set of probability measures on a metric space S. Then we have the following tightness result which is proved identically as Lemma 4.6 in [32].

LEMMA 4.5. *For a fixed $\varepsilon > 0$ and δ, the laws $\{\mu^{u,v}_N\}_N$ of the random variables $\{\widetilde{\mathbf{U}}_N\}_N$ are tight in $C([0,T];\mathscr{V}_1')$.*

We note here that this result is available only in the case of fixed $\varepsilon > 0$ and that we will not have this result in the next section. In the following lemma, we will show that the stochastic error term vanishes as $N \to \infty$.

LEMMA 4.6. *The numerical error E_N of the stochastic term has the following property:*

$$\mathbb{E}\int_0^T \|E_N(t)\|^2_{\mathbf{L}^2(\mathcal{O})\times\mathbf{L}^2(0,L)}dt \to 0 \qquad \text{as } N \to \infty.$$

Proof. For any N, we begin by writing $E_N(t) =: E^1_N + E^2_N$. Observe that E^1_N satisfies

$$\begin{aligned}
\mathbb{E}\int_0^T \|E^1_N(t)\|^2_{\mathbf{L}^2}dt &= \mathbb{E}\sum_{n=0}^{N-1}\|G(\mathbf{u}^n,\boldsymbol{\eta}^n_*)\Delta_n W\|^2_{\mathbf{L}^2}\int_{t^n}^{t^{n+1}}\left|\frac{t-t^m}{\Delta t}\right|^2 dt\\
&= \mathbb{E}\sum_{n=0}^{N-1}\|G(\mathbf{u}^n,\boldsymbol{\eta}^n_*)\Delta_n W\|^2_{\mathbf{L}^2}\frac{\Delta t}{3} \le \mathbb{E}\sum_{n=0}^{N-1}\|G(\mathbf{u}^n,\boldsymbol{\eta}^n_*)\|^2_{L_2(U_0;\mathbf{L}^2)}\|\Delta_n W\|^2_{U_0}\Delta t\\
&= (\Delta t)^2\mathbb{E}\sum_{n=0}^{N-1}\|G(\mathbf{u}^n,\boldsymbol{\eta}^n_*)\|^2_{L_2(U_0;\mathbf{L}^2)} \le C\Delta t,
\end{aligned}$$

where, as a consequence of Theorem 3.4, $C > 0$ is independent of N and ε.

To estimate E^2_N we use the Itô isometry as follows:

$$\begin{aligned}
\mathbb{E}\int_0^T \|E^2_N(t)\|^2_{\mathbf{L}^2}dt &= \mathbb{E}\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}}\left\|\int_{t^n}^t G(\mathbf{u}^n,\boldsymbol{\eta}^n_*)dW\right\|^2_{\mathbf{L}^2}dt\\
&= \mathbb{E}\sum_{n=0}^{N-1}\int_{t^n}^{t^{n+1}}\int_{t^n}^t \|G(\mathbf{u}^n,\boldsymbol{\eta}^n_*)\|^2_{L_2(U_0;\mathbf{L}^2)}dsdt\\
&= \frac{1}{2}\mathbb{E}\sum_{n=0}^{N-1}\|G(\mathbf{u}^n,\boldsymbol{\eta}^n_*)\|^2_{L_2(U_0;\mathbf{L}^2)}(\Delta t)^2 \le C\Delta t.
\end{aligned}$$

∎

4.2. Almost sure convergence. In this section we will obtain almost sure convergence results for our approximate solutions. To that end, let μ_N be the joint law of the random variable $\mathcal{U}_N = (\mathbf{u}_N^+, \mathbf{v}_N^+, \tilde{\boldsymbol{\eta}}_N^*, \tilde{\boldsymbol{\eta}}_N, \|\mathbf{u}_N^+\|_{L^2(0,T;V)}, \|\mathbf{v}_N^*\|_{L^2(0,T;\mathbf{H}_0^1(0,L))}, \tilde{\mathbf{U}}, W)$ taking values in the phase space

$$\Upsilon := [L^2(0,T;\mathbf{L}^2(\mathcal{O}))] \times [L^2(0,T;\mathbf{H}^{-\beta}(0,L))] \times [C([0,T],\mathbf{H}^s(0,L))]^2 \times \mathbb{R}^2 \times C([0,T];\mathscr{V}_1') \times C([0,T];U)$$

for some fixed $\frac{3}{2} < s < 2$ and $0 < \beta < \frac{1}{2}$.

Since $C([0,T];U)$ is separable and metrizable by a complete metric, the sequence of Borel probability measures, $\mu_N^W(\cdot) := \mathbb{P}(W \in \cdot)$, that are constantly equal to one element, is tight on $C([0,T];U)$. Thus, recalling Lemmas 4.2, 4.3, and 4.6 and using Tychonoff's theorem, it follows that the sequence of the probability measures μ_N of the approximating sequence $\mathcal{U}_N$ is tight on the Polish space Υ. Hence, by applying the Prohorov theorem and a variant of the Skorohod representation theorem (Theorem 1.10.4 in [34]), we obtain the following convergence result.

THEOREM 4.7. *There exist a probability space* $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$*, random variables* $\bar{\mathcal{U}}_N := (\bar{\mathbf{u}}_N^+, \bar{\mathbf{v}}_N^+, \bar{\tilde{\boldsymbol{\eta}}}_N^*, \bar{\tilde{\boldsymbol{\eta}}}_N, m_N, k_N, \bar{\tilde{\mathbf{U}}}_N, \bar{W}_N)$ *and* $\bar{\mathcal{U}} := (\bar{\mathbf{u}}, \bar{\mathbf{v}}, \bar{\boldsymbol{\eta}}^*, \bar{\boldsymbol{\eta}}, m, k, \bar{\tilde{\mathbf{U}}}, \bar{W})$ *defined on this new probability space, and measurable maps* $\phi_N : \bar{\Omega} \to \Omega$ *such that*

$$\bar{\mathcal{U}}_N(\bar{\omega}) = \mathcal{U}_N(\phi_N(\bar{\omega})) \quad \text{for } \bar{\omega} \in \bar{\Omega}, \quad \text{and} \quad \bar{\mathbb{P}} \circ \phi_N^{-1} = \mathbb{P}, \tag{4.15}$$

such that

$$\bar{\mathcal{U}}_N \to \bar{\mathcal{U}}, \qquad \bar{\mathbb{P}}\text{-a.s.} \quad \text{in the topology of } \Upsilon. \tag{4.16}$$

Now we define rest of the approximate random variables:

$$\bar{\mathbf{u}}_N = \mathbf{u}_N \circ \phi_N, \quad \bar{\tilde{\mathbf{u}}}_N = \tilde{\mathbf{u}}_N \circ \phi_N, \quad \bar{\mathbf{v}}_N = \mathbf{v}_N \circ \phi_N,$$
$$\bar{\boldsymbol{\eta}}_N = \boldsymbol{\eta}_N \circ \phi_N, \quad \bar{\boldsymbol{\eta}}_N^+ = \boldsymbol{\eta}_N^+ \circ \phi_N, \quad \bar{\boldsymbol{\eta}}_N^* = \boldsymbol{\eta}_N^* \circ \phi_N.$$

Then, from Lemma 4.4(1) and an application of the Borel–Cantelli lemma, we obtain for $\frac{3}{2} < s < 2$ and $0 < \beta < \frac{1}{2}$ that

$$\bar{\mathbf{u}}_N \to \bar{\mathbf{u}}, \quad \bar{\tilde{\mathbf{u}}}_N \to \bar{\mathbf{u}}, \quad \bar{\mathbb{P}}\text{-a.s.} \quad \text{in } L^2(0,T;\mathbf{L}^2(\mathcal{O})), \tag{4.17}$$

$$\bar{\boldsymbol{\eta}}_N \to \bar{\boldsymbol{\eta}}, \quad \bar{\boldsymbol{\eta}}_N^+ \to \bar{\boldsymbol{\eta}}, \quad \bar{\boldsymbol{\eta}}_N^* \to \bar{\boldsymbol{\eta}}^*, \quad \bar{\mathbb{P}}\text{-a.s.} \quad \text{in } L^2(0,T;\mathbf{H}^s(0,L)), \tag{4.18}$$

$$\bar{\mathbf{v}}_N \to \bar{\mathbf{v}}, \quad \bar{\tilde{\mathbf{v}}}_N \to \bar{\mathbf{v}}, \quad \bar{\mathbb{P}}\text{-a.s.} \quad \text{in } L^2(0,T;\mathbf{H}^{-\beta}(0,L)). \tag{4.19}$$

Thanks to these explicit maps we can identify the real-valued random variables $k_N = \|\bar{\mathbf{v}}_N^*\|_{L^2(0,T;\mathbf{H}_0^1(0,L))}$. Thus, a.s. convergence of k_N implies that for a fixed $\varepsilon > 0$, $\|\bar{\mathbf{v}}_N^*\|_{L^2(0,T;\mathbf{H}_0^1(0,L))}$ is bounded a.s. and thus, up to a subsequence,

$$\bar{\mathbf{v}}_N^* \rightharpoonup \bar{\mathbf{v}}^* \quad \text{weakly in } L^2(0,T;\mathbf{H}_0^1(0,L)) \quad \bar{\mathbb{P}}\text{-a.s.} \tag{4.20}$$

Similarly $\bar{\mathbf{u}}_N^+ \rightharpoonup \bar{\mathbf{u}}$ weakly in $L^2(0,T;V)$ a.s.

Observe also that the bounds obtained in Lemma 3.5 hold for the new random variables $\bar{\mathcal{U}}_N$ as well. Particularly, thanks to (4.16) and (4.15), we have the same deterministic bounds $\|\bar{\boldsymbol{\eta}}^*\|_{C(0,T;\mathbf{H}^s(0,L))} \le \frac{1}{\delta_2}$ for $\frac{3}{2} < s < 2$. We also have the following convergence results for the displacements. Namely, notice that Theorem 4.7 implies that for given $\frac{3}{2} < s < 2$ (see [24, Lemma 3]),

$$\bar{\boldsymbol{\eta}}_N \to \bar{\boldsymbol{\eta}} \text{ and } \bar{\boldsymbol{\eta}}_N^* \to \bar{\boldsymbol{\eta}}^* \quad \text{in } L^\infty(0,T;\mathbf{H}^s(0,L)) \text{ a.s.} \tag{4.21}$$

and thus,

$$\bar{\boldsymbol{\eta}}_N \to \bar{\boldsymbol{\eta}} \text{ and } \bar{\boldsymbol{\eta}}^*_N \to \bar{\boldsymbol{\eta}}^* \quad \text{in } L^\infty(0,T;\mathbf{C}^1[0,L]) \text{ a.s.} \tag{4.22}$$

Next we define piecewise constant interpolations of the ALE maps and Jacobians $A_{\bar{\boldsymbol{\eta}}^*_N}, J_{\bar{\boldsymbol{\eta}}^*_N}$ and their piecewise linear interpolation $\widetilde{A}_{\bar{\boldsymbol{\eta}}^*_N}, \widetilde{J}_{\bar{\boldsymbol{\eta}}^*_N}$. Then (4.22), (3.8), and (3.7) yield

$$\begin{aligned}
&\nabla A_{\bar{\boldsymbol{\eta}}^*_N} \to \nabla A_{\bar{\boldsymbol{\eta}}^*} \text{ and } (\nabla A_{\bar{\boldsymbol{\eta}}^*_N})^{-1} \to (\nabla A_{\bar{\boldsymbol{\eta}}^*})^{-1} \quad \text{in } L^\infty(0,T;\mathbf{C}(\mathcal{O})) \text{ a.s.},\\
&J_{\bar{\boldsymbol{\eta}}^*_N} \to J_{\bar{\boldsymbol{\eta}}^*} = \det(\nabla A_{\bar{\boldsymbol{\eta}}^*}) \quad \text{in } L^\infty(0,T;C(\bar{\mathcal{O}})) \text{ a.s.},\\
&S_{\bar{\boldsymbol{\eta}}_N} \to S_{\bar{\boldsymbol{\eta}}} \text{ and } S_{\bar{\boldsymbol{\eta}}^*_N} \to S_{\bar{\boldsymbol{\eta}}^*} \quad \text{in } L^\infty(0,T;C(\bar{\Gamma})) \text{ a.s.}
\end{aligned} \tag{4.23}$$

Furthermore, $A_{\bar{\boldsymbol{\eta}}^*} \in L^\infty(0,T;\mathbf{W}^{2,p}(\mathcal{O}))$ for $p<4$ and is the solution to (3.2) corresponding to the boundary data $\mathbf{id}+\bar{\boldsymbol{\eta}}^*$ on Γ. Next let $\bar{\mathbf{w}}^*_N = \partial_t \widetilde{A}_{\bar{\boldsymbol{\eta}}^*_N} = \sum_{n=0}^N \frac{1}{\Delta t}(A^\omega_{\bar{\boldsymbol{\eta}}^n_*} - A^\omega_{\bar{\boldsymbol{\eta}}^{n-1}_*})\chi_{(t_n,t_{n+1})}$. Note that for every $\omega \in \bar{\Omega}$, we have (see [19])

$$\|\bar{\mathbf{w}}^*_N\|_{L^2(0,T;\mathbf{H}^{k+\frac{1}{2}}(\mathcal{O}))} \le C\|\bar{\mathbf{v}}^*_N\|_{L^2(0,T;\mathbf{H}^k(0,L))} \quad \text{for any } 0 \le k \le 1,$$

where C depends only on k. Thus using (4.20), for a fixed $\varepsilon > 0$, we obtain that

$$\bar{\mathbf{w}}^*_N \rightharpoonup \bar{\mathbf{w}}^* \quad \text{weakly in } L^2(0,T;\mathbf{H}^1(\mathcal{O})), \quad \bar{\mathbb{P}}\text{-a.s.}, \tag{4.24}$$

where $\bar{\mathbf{w}}^*$ satisfies (3.3) with values $\bar{\mathbf{v}}^*$ on Γ. Similarly, (4.11) gives us

$$\partial_t \widetilde{J}_{\bar{\boldsymbol{\eta}}^*_N} \rightharpoonup \partial_t J_{\bar{\boldsymbol{\eta}}^*} \quad \text{weakly in } L^2(0,T;\mathbf{L}^2(\mathcal{O})), \quad \bar{\mathbb{P}}\text{-a.s.}$$

Finally, we give the definition of the required filtrations. we denote by $\mathcal{F}'_t$ the σ-field generated by the random variables $(\bar{\mathbf{u}}(s), \bar{\mathbf{v}}(s)), \bar{\boldsymbol{\eta}}(s), \bar{W}(s)$ for all $s \le t$. Then we define

$$\mathcal{N} := \{\mathcal{A} \in \bar{\mathcal{F}} \mid \bar{\mathbb{P}}(\mathcal{A}) = 0\}, \qquad \bar{\mathcal{F}}^0_t := \sigma(\mathcal{F}'_t \cup \mathcal{N}), \qquad \bar{\mathcal{F}}_t := \bigcap_{s \ge t} \bar{\mathcal{F}}^0_s. \tag{4.25}$$

We note here that the stochastic processes $(J_{\bar{\boldsymbol{\eta}}^*}\bar{\mathbf{u}}, \bar{\boldsymbol{\eta}})$ are $(\bar{\mathcal{F}}_t)_{t\ge 0}$-progressively measurable. For each N we also define a filtration $(\bar{\mathcal{F}}^N_t)_{t\ge 0}$ on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ the same way. Moreover, using usual arguments we can see that $\bar{W}_N$ is an $\bar{\mathcal{F}}^N_t$-Wiener process (see, e.g., [2]). Next, relative to the new stochastic basis $(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}^N_t)_{t\ge0}, \bar{\mathbb{P}}, \bar{W}_N)$, thanks to (4.15) we can see that for each N, the following equation holds:

$$\begin{aligned}
&(\widetilde{J}_{\bar{\boldsymbol{\eta}}^*_N}(t), \mathbf{q}) + (\bar{\tilde{\mathbf{v}}}_N(t), \boldsymbol{\psi}) = (J_0\mathbf{u}_0, \mathbf{q}) + (\mathbf{v}_0, \boldsymbol{\psi}) - \int_0^t \langle \mathscr{L}_e(\bar{\boldsymbol{\eta}}^+_N), \boldsymbol{\psi}\rangle\\
&\quad + \int_0^t \int_{\mathcal{O}} \frac{\partial \widetilde{J}_{\bar{\boldsymbol{\eta}}^*_N}}{\partial t}(2\bar{\tilde{\mathbf{u}}}_N - \bar{\mathbf{u}}_N)\cdot\mathbf{q} - \frac{1}{2}\frac{\partial \widetilde{J}_{\bar{\boldsymbol{\eta}}^*_N}}{\partial t}\bar{\mathbf{u}}^+_N\cdot\mathbf{q} - \varepsilon\int_0^t\int_0^L \partial_{zz}\bar{\mathbf{v}}^\#_N \cdot \partial_{zz}\boldsymbol{\psi}\\
&\quad - \frac{1}{2}\int_0^t\int_{\mathcal{O}} (J_{\bar{\boldsymbol{\eta}}^*_N})((\bar{\mathbf{u}}^+_N - \bar{\mathbf{w}}^*_N)\cdot\nabla^{\bar{\boldsymbol{\eta}}^*_N}\bar{\mathbf{u}}^+_N\cdot\mathbf{q} - (\bar{\mathbf{u}}^+_N - \bar{\mathbf{w}}^*_N)\cdot\nabla^{\bar{\boldsymbol{\eta}}^*_N}\mathbf{q}\cdot\bar{\mathbf{u}}^+_N)\\
&\quad - 2\nu\int_0^t\int_{\mathcal{O}} (J_{\bar{\boldsymbol{\eta}}^*_N})\mathbf{D}^{\bar{\boldsymbol{\eta}}^*_N}(\bar{\mathbf{u}}^+_N)\cdot\mathbf{D}^{\bar{\boldsymbol{\eta}}^*_N}(\mathbf{q}) - \frac{1}{\alpha}\int_0^t\int_\Gamma S_{\bar{\boldsymbol{\eta}}_N}(\bar{\mathbf{u}}^+_N - \bar{\mathbf{v}}^+_N)(\mathbf{q}-\boldsymbol{\psi})\\
&\quad - \frac{1}{\varepsilon}\int_0^t\int_{\mathcal{O}} \mathrm{div}^{\bar{\boldsymbol{\eta}}^*_N}\bar{\mathbf{u}}^+_N \mathrm{div}^{\bar{\boldsymbol{\eta}}^*_N}\mathbf{q}\,d\mathbf{x} - \frac{1}{\varepsilon}\int_0^t\int_\Gamma (\bar{\mathbf{u}}^+_N - \bar{\mathbf{v}}^+_N)\cdot\bar{\mathbf{n}}^*_N(\mathbf{q}-\boldsymbol{\psi})\cdot\bar{\mathbf{n}}^*_N\\
&\quad \pm \int_0^t \left(P_{in/out}\int_0^1 q_z\Big|_{z=0/1}\right) + \int_0^t (G(\bar{\mathbf{u}}_N, \bar{\boldsymbol{\eta}}^*_N)d\bar{W}_N, \mathbf{Q}) + (E_N(t), \mathbf{Q}),
\end{aligned} \tag{4.26}$$

$\bar{\mathbb{P}}$-a.s. for every $t \in [0,T]$ and any $\mathbf{Q} \in \mathscr{D} \cap \mathscr{U}_1$.

Using the convergence results stated in Theorem 4.7 we can then pass $N \to \infty$ in the deterministic terms in (4.26). For the stochastic integral see Lemma 5.6. We mention here how we treat the convective term which is the only term that needs an explanation. By integrating by parts we obtain

$$\begin{aligned}\int_{\mathcal{O}} J_{\bar{\boldsymbol\eta}^*_N} \bar{\mathbf{w}}^*_N \cdot \nabla^{\bar{\boldsymbol\eta}^*_N} \bar{\mathbf{u}}^+_N \cdot \mathbf{q} = &- \int_{\mathcal{O}} J_{\bar{\boldsymbol\eta}^*_N} \nabla^{\bar{\boldsymbol\eta}^*_N} \cdot \bar{\mathbf{w}}^*_N \bar{\mathbf{u}}^+_N \cdot \mathbf{q} - \int_{\mathcal{O}} J_{\bar{\boldsymbol\eta}^*_N} \bar{\mathbf{w}}^*_N \cdot \nabla^{\bar{\boldsymbol\eta}^*_N} \mathbf{q} \cdot \bar{\mathbf{u}}^+_N \\ &+ \int_{\Gamma} S_{\bar{\boldsymbol\eta}^*_N} (\bar{\mathbf{v}}^*_N \cdot \mathbf{n}^*_N)(\bar{\mathbf{u}}^+_N \cdot \mathbf{q}),\end{aligned} \tag{4.27}$$

where $S_{\bar{\boldsymbol\eta}^*_N}$ is the Jacobian of the transformation from Eulerian to Lagrangian coordinates. Thus, using the weak and strong convergence results in Theorem 4.7, (4.20), (4.24), and (4.22) we can pass $N \to \infty$ in (4.27). Notice that the addition of the viscous regularization for a fixed $\varepsilon > 0$ in the structure subproblem (3.1) allowed for discretization of the time derivative of the Jacobian in the fluid subproblem (3.10) and thus it also makes the limiting term $\int_0^t \int_{\mathcal{O}} \partial_t J_{\bar{\boldsymbol\eta}^*} \bar{\mathbf{u}} \cdot \mathbf{q}$ well-defined at this stage. However, to give sense to this term in the vanishing ε regime, we use the fact that

$$\int_0^t \int_{\mathcal{O}} \partial_t J_{\bar{\boldsymbol\eta}^*} \bar{\mathbf{u}} \cdot \mathbf{q} = \int_0^t \int_{\mathcal{O}} J_{\bar{\boldsymbol\eta}^*} \nabla^{\boldsymbol\eta^*} \cdot \mathbf{w}^* \bar{\mathbf{u}} \cdot \mathbf{q} \tag{4.28}$$

to arrive at the following approximate weak formulation.

THEOREM 4.8. *For the stochastic basis $(\bar\Omega, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t)_{t\geq 0}, \bar{\mathbb{P}}, \bar W)$ constructed in Theorem 4.7, given any fixed $\varepsilon > 0$ and $\delta = (\delta_1, \delta_2)$ satisfying (3.9), the processes $(\bar{\mathbf{u}}, \bar{\boldsymbol\eta}, \bar{\boldsymbol\eta}^*)$ obtained in Theorem 4.7 are such that $\bar{\boldsymbol\eta}^*$ and $(\bar J_{\boldsymbol\eta^*}\bar{\mathbf{u}}, \partial_t \bar{\boldsymbol\eta})$ are $(\bar{\mathcal{F}}_t)_{t\geq 0}$-progressively measurable with $\bar{\mathbb{P}}$-a.s. continuous paths in $\mathbf{H}^s(0,L)$, $s < 2$, and $\mathscr{V}_1'$, respectively, and the following weak formulation holds $\bar{\mathbb{P}}$-a.s. for every $t \in [0,T]$ and $\mathbf{Q} \in \mathscr{D}$:*

$$\begin{aligned}(J_{\bar{\boldsymbol\eta}^*}(t)\bar{\mathbf{u}}(t), \mathbf{q}) + (\partial_t \bar{\boldsymbol\eta}(t), \boldsymbol\psi) &= ((J_0)\mathbf{u}_0, \mathbf{q}) + (\mathbf{v}_0, \boldsymbol\psi) - \int_0^t \langle \mathscr{L}_e(\bar{\boldsymbol\eta}), \boldsymbol\psi \rangle \\ &- \frac{1}{2}\int_0^t \int_{\mathcal{O}} (J_{\bar{\boldsymbol\eta}^*})(\bar{\mathbf{u}} \cdot \nabla^{\bar{\boldsymbol\eta}^*} \bar{\mathbf{u}} \cdot \mathbf{q} - (\bar{\mathbf{u}} - 2\bar{\mathbf{w}}^*) \cdot \nabla^{\bar{\boldsymbol\eta}^*} \mathbf{q} \cdot \bar{\mathbf{u}}) \\ &+ \frac{1}{2}\int_0^t \int_{\Gamma} S_{\bar{\boldsymbol\eta}^*}(\partial_t \bar{\boldsymbol\eta}^* \cdot \mathbf{n}^*)(\bar{\mathbf{u}} \cdot \mathbf{q}) - \frac{1}{\alpha}\int_0^t \int_{\Gamma} S_{\bar{\boldsymbol\eta}}(\bar{\mathbf{u}} - \partial_t \bar{\boldsymbol\eta})(\mathbf{q} - \boldsymbol\psi) \\ &- 2\nu \int_0^t \int_{\mathcal{O}} J_{\bar{\boldsymbol\eta}^*} \mathbf{D}^{\bar{\boldsymbol\eta}^*}(\bar{\mathbf{u}}) \cdot \mathbf{D}^{\bar{\boldsymbol\eta}^*}(\mathbf{q}) d\mathbf{x} - \varepsilon \int_0^t \int_0^L \partial_{zz}\partial_t \bar{\boldsymbol\eta} \cdot \partial_{zz}\boldsymbol\psi dz \\ &- \frac{1}{\varepsilon}\int_0^t \int_{\mathcal{O}} \mathrm{div}^{\bar{\boldsymbol\eta}^*} \bar{\mathbf{u}} \mathrm{div}^{\bar{\boldsymbol\eta}^*} \mathbf{q} d\mathbf{x} - \frac{1}{\varepsilon}\int_0^t \int_{\Gamma} (\bar{\mathbf{u}} - \partial_t \bar{\boldsymbol\eta}) \cdot \mathbf{n}^* (\mathbf{q} - \boldsymbol\psi) \cdot \mathbf{n}^* \\ &+ \int_0^t \left(P_{in} \int_0^1 q_z\Big|_{z=0} dr - P_{out} \int_0^1 q_z\Big|_{z=1} dr \right) ds + \int_0^t (G(\bar{\mathbf{u}}, \bar{\boldsymbol\eta}^*) d\bar W, \mathbf{Q}).\end{aligned} \tag{4.29}$$

Next, we argue that

$$\bar{\boldsymbol\eta}^*(t) = \bar{\boldsymbol\eta}(t) \quad \text{for any } t < T^{\boldsymbol\eta}, \quad \bar{\mathbb{P}}\text{-a.s.}, \tag{4.30}$$

where for a given $\delta = (\delta_1, \delta_2)$,

$$T^{\boldsymbol\eta} := T \wedge \inf\left\{ t > 0 : \inf_{\mathcal{O}} (J_{\bar{\boldsymbol\eta}}(t)) \leq \delta_1 \text{ or } \|\bar{\boldsymbol\eta}(t)\|_{\mathbf{H}^s(0,L)} \geq \frac{1}{\delta_2} \right\}.$$

Indeed, to prove (4.30), we introduce the following stopping times. For $\frac{3}{2} < s < 2$ we define

$$T_N^{\boldsymbol{\eta}} := T \wedge \inf \left\{ t > 0 : \inf_{\mathcal{O}} (J_{\bar{\boldsymbol{\eta}}_N}(t)) \le \delta_1 \text{ or } \|\bar{\boldsymbol{\eta}}_N(t)\|_{\mathbf{H}^s(0,L)} \ge \frac{1}{\delta_2} \right\}.$$

Then (4.22) implies that $T^{\boldsymbol{\eta}} \le \liminf_{N\to\infty} T_N^{\boldsymbol{\eta}}$ a.s. Observe further that for almost any $\omega \in \bar{\Omega}$ and $t < T^{\boldsymbol{\eta}}$, and for any $\epsilon > 0$, there exists an N such that

$$\begin{aligned}\|\bar{\boldsymbol{\eta}}(t) - \bar{\boldsymbol{\eta}}^*(t)\|_{\mathbf{H}^s(0,L)} < \|\bar{\boldsymbol{\eta}}(t) - \bar{\boldsymbol{\eta}}_N(t)\|_{\mathbf{H}^s(0,L)} + \|\bar{\boldsymbol{\eta}}^*(t) - \bar{\boldsymbol{\eta}}_N^*(t)\|_{\mathbf{H}^s(0,L)} \\ + \|\bar{\boldsymbol{\eta}}_N^*(t) - \bar{\boldsymbol{\eta}}_N(t)\|_{\mathbf{H}^s(0,L)} < \epsilon.\end{aligned}$$

This is true because the uniform convergence (4.22) implies that for any $\epsilon > 0$ there exists an $N_1 \in \mathbb{N}$ such that the first two terms on the right-hand side of the above inequality are each bounded by $\frac{\epsilon}{2}$ for all $N \ge N_1$. Moreover, since $t < T_N^{\boldsymbol{\eta}}$ for infinitely many N's, the third term is equal to 0. This concludes the proof of (4.30).

5. Passing to the limit $\boldsymbol{\varepsilon \to 0}$. In what follows, to emphasize the dependence on the parameter $\varepsilon > 0$, we will use the notation $(\bar{\mathbf{u}}_\varepsilon, \bar{\mathbf{v}}_\varepsilon, \bar{\mathbf{v}}_\varepsilon^*, \bar{\boldsymbol{\eta}}_\varepsilon, \bar{\boldsymbol{\eta}}_\varepsilon^*, \bar{W}_\varepsilon)$ and $(\Omega^\varepsilon, \mathcal{F}^\varepsilon, (\mathcal{F}_t^\varepsilon)_{t\ge 0}, \mathbb{P}^\varepsilon)$ for the martingale solution found in the previous section. The aim of this section is to pass $\varepsilon \to 0$ in (4.29) by constructing appropriate test functions. Most of the results in the first half of this section can be proved as in the previous section, and so we will only summarize the important theorems without proof. Observe that, thanks to the weak lower-semicontinuity of norm, the uniform estimates obtained in the previous section still hold. As a consequence of Lemma 3.5 and Theorem 4.7, we thus have the following uniform bounds.

Lemma 5.1 (uniform boundedness). *For a fixed $\delta = (\delta_1, \delta_2)$ that satisfies* (3.9), *we have for some $C > 0$* independent of *ε that the following hold:*

1. $\mathbb{E}^\varepsilon \|\bar{\mathbf{u}}_\varepsilon\|^2_{L^\infty(0,T;\mathbf{L}^2(\mathcal{O}))\cap L^2(0,T;V)} < C$.
2. $\mathbb{E}^\varepsilon \|\bar{\mathbf{v}}_\varepsilon^*\|^2_{L^\infty(0,T;\mathbf{L}^2(0,L))} < C$.
3. $\mathbb{E}^\varepsilon \|\bar{\boldsymbol{\eta}}_\varepsilon\|^2_{L^\infty(0,T;\mathbf{H}_0^2(0,L)\cap W^{1,\infty}(0,T;\mathbf{L}^2(0,L)))} < C$.
4. $\mathbb{E}^\varepsilon \|\bar{\boldsymbol{\eta}}_\varepsilon^*\|^2_{L^\infty(0,T;\mathbf{H}_0^2(0,L)\cap W^{1,\infty}(0,T;\mathbf{L}^2(0,L)))} < C$.
5. $\mathbb{E}^\varepsilon \|\text{div}^{\bar{\boldsymbol{\eta}}_\varepsilon^*} \bar{\mathbf{u}}_\varepsilon\|^2_{L^2(0,T;L^2(\mathcal{O}))} < C\varepsilon$, $\mathbb{E}^\varepsilon \|(\bar{\mathbf{u}}_\varepsilon|_\Gamma - \bar{\mathbf{v}}_\varepsilon) \cdot \mathbf{n}_\varepsilon^*\|^2_{L^2(0,T;L^2(0,L))} < C\varepsilon$.
6. $\sqrt{\varepsilon}\mathbb{E}^\varepsilon \|\partial_{zz} \bar{\mathbf{v}}_\varepsilon\|^2_{L^2(0,T;\mathbf{L}^2(0,L))} < C$.
7. $\|\bar{\boldsymbol{\eta}}_\varepsilon^*\|_{C(0,T;\mathbf{H}^s(0,L))} \le \frac{1}{\delta_2}$ *for* $\frac{3}{2} < s < 2$, *for almost every* $\omega \in \Omega^\varepsilon$.

Next we have the following tightness results.

Lemma 5.2 (tightness of the laws).

1. *The sequences $\mathbb{P}^\varepsilon \circ (\bar{\mathbf{u}}_\varepsilon)^{-1}$ and $\mathbb{P}^\varepsilon \circ (\bar{\mathbf{v}}_\varepsilon)^{-1}$ are tight in $L^2(0,T;\mathbf{H}^\alpha(\mathcal{O}))$ and $L^2(0,T;\mathbf{H}^{-\beta}(0,L))$, respectively, for any $0 \le \alpha < 1$, $0 < \beta < \frac{1}{2}$.*
2. *The sequences $\mathbb{P}^\varepsilon \circ (\bar{\boldsymbol{\eta}}_\varepsilon)^{-1}$ and $\mathbb{P}^\varepsilon \circ (\bar{\boldsymbol{\eta}}_\varepsilon^*)^{-1}$ are tight in $C([0,T];\mathbf{H}^s(0,L))$ for $\frac{3}{2} < s < 2$.*
3. *The sequence $\mathbb{P}^\varepsilon \circ (\|\bar{\mathbf{u}}_\varepsilon\|_{L^2(0,T;V)})^{-1}$ is tight in $\mathbb{R}$.*
4. *The sequence $\mathbb{P}^\varepsilon \circ (\|\bar{\mathbf{v}}_\varepsilon^*\|_{L^2(0,T;\mathbf{L}^2(0,L))})^{-1}$ is tight in $\mathbb{R}$.*

Proof. We describe how to prove the first statement, which follows from the proof of Lemma 4.2 almost identically. Construction of suitable test functions $(\mathbf{q}_\varepsilon, \boldsymbol{\psi}_\varepsilon)$ is the same as (4.8) and we apply the variant of Itô's formula stated in Lemma 5.1 in [3], which justifies testing (4.29) with the continuous-in-time versions of the random test functions (4.8) (i.e., where $(\Delta t)\sum_{n-j+1}^{n}$ is replaced by $\int_{t-h}^{t} dt$). Recall that all the bounds obtained in the proof of Lemma 4.2, except in (4.12) and (4.13), do not depend on ε. However, for (4.13), observe due to integrating by parts in (4.27) and applying (4.28), that the weak formulation (4.29) now contains the boundary integral

$\int_0^t \int_\Gamma S_{\bar{\boldsymbol{\eta}}^*_\varepsilon}(\bar{\mathbf{v}}^*_\varepsilon \cdot \mathbf{n}^*_\varepsilon)(\bar{\mathbf{u}}_\varepsilon \cdot \mathbf{q}_\varepsilon)$ instead of the aforementioned term involving the derivatives of $\bar{\mathbf{w}}^*_\varepsilon$. Then, for the process $\mathbf{q}_\varepsilon$ taking values in $\mathbf{H}^1(\mathcal{O})$, described above and constructed as in (4.8), we can bound this boundary integral independently of ε, by using the fact that $\|\mathbf{n}^*_\varepsilon\|_{\mathbf{L}^\infty((0,T)\times(0,L))} < C(\delta)$ together with the bounds for the trace $\mathbb{E}^\varepsilon\|\bar{\mathbf{u}}_\varepsilon|_\Gamma\|^2_{L^2(0,T;\mathbf{H}^{\frac{1}{2}}(\Gamma))} \le C$ (see Theorem 1.5.2.1 in [19]) and $\mathbb{E}^\varepsilon\|\bar{\mathbf{v}}^*_\varepsilon\|^2_{L^2(0,T;\mathbf{L}^2(0,L))} \le C$ which are independent of ε. We similarly treat the term in (4.12) by integrating by parts (see (5.13)). ∎

Now for an infinite denumerable set of indices Λ, we denote by μ_ε the joint law of the random variable $\bar{\mathcal{U}}_\varepsilon := (\bar{\mathbf{u}}_\varepsilon, \bar{\mathbf{v}}_\varepsilon, \bar{\boldsymbol{\eta}}_\varepsilon, \bar{\boldsymbol{\eta}}^*_\varepsilon, \|\bar{\mathbf{u}}_\varepsilon\|_{L^2(0,T;V)}, \|\bar{\mathbf{v}}^*_\varepsilon\|_{L^2(0,T;\mathbf{L}^2(\Gamma))}, \bar{W}_\varepsilon)$ taking values in the phase space

$$\mathcal{S} = L^2(0,T;\mathbf{H}^{\frac{3}{4}}(\mathcal{O})) \times L^2(0,T;\mathbf{H}^{-\beta}(\Gamma)) \times [C([0,T],\mathbf{H}^s(\Gamma))]^2 \times \mathbb{R}^2 \times C([0,T];U)$$

for some $0<\beta<\frac{1}{2}$, $\frac{3}{2}<s<2$.

Then the tightness of $\mu_\varepsilon = \mathbb{P}^\varepsilon \circ (\bar{\mathcal{U}}_\varepsilon)^{-1}$ on $\mathcal{S}$ and an application of the Prohorov theorem and the almost sure representation in [34] give us the following result.

THEOREM 5.3. *There exist a probability space* $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ *and random variables* $\hat{\mathcal{U}}_\varepsilon = (\hat{\mathbf{u}}_\varepsilon, \hat{\mathbf{v}}_\varepsilon, \hat{\boldsymbol{\eta}}_\varepsilon, \hat{\boldsymbol{\eta}}^*_\varepsilon, m_\varepsilon, k_\varepsilon, \hat{W}_\varepsilon)$ *and* $\hat{\mathcal{U}} = (\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\eta}}^*, m, k, \hat{W})$ *such that the following hold:*

1. $\hat{\mathcal{U}}_\varepsilon =^d \mu_\varepsilon$ *for every* $\varepsilon \in \Lambda$.
2. $\hat{\mathcal{U}}_\varepsilon \to \hat{\mathcal{U}}$ $\hat{\mathbb{P}}$*-a.s. in the topology of* $\mathcal{S}$ *as* $\varepsilon \to 0$.
3. $\partial_t \hat{\boldsymbol{\eta}} = \hat{\mathbf{v}}$ *and* $\partial_t \hat{\boldsymbol{\eta}}^* = \hat{\mathbf{v}}^*$ *in the sense of distributions, a.s.*

We recall again that Theorem 1.10.4 in [34] tells us that the random variables $\hat{\mathcal{U}}_\varepsilon$ can be chosen such that for every $\varepsilon \in \Lambda$,

$$\hat{\mathcal{U}}_\varepsilon(\omega) = \bar{\mathcal{U}}_\varepsilon(\phi_\varepsilon(\omega)), \quad \omega \in \hat{\Omega}, \tag{5.1}$$

and $\hat{\mathbb{P}} \circ \phi_\varepsilon^{-1} = \mathbb{P}^\varepsilon$, where $\phi_\varepsilon : \hat{\Omega} \to \Omega^\varepsilon$ is measurable.

Thanks to these explicit maps we identify the real-valued random variables m_ε as $m_\varepsilon = \|\hat{\mathbf{u}}_\varepsilon\|_{L^2(0,T;V)}$ and notice that m_ε converge almost surely due to Theorem 5.3. Hence as in Theorem 4.7, we obtain, up to a subsequence, that

$$\hat{\mathbf{u}}_\varepsilon \rightharpoonup \hat{\mathbf{u}} \quad \text{weakly in } L^2(0,T;V), \quad \hat{\mathbb{P}}\text{-a.s.} \tag{5.2}$$

Similarly,

$$\hat{\mathbf{v}}^*_\varepsilon \rightharpoonup \hat{\mathbf{v}}^* \quad \text{weakly in } L^2(0,T;\mathbf{L}^2(0,L)), \quad \hat{\mathbb{P}}\text{-a.s.} \tag{5.3}$$

As in the previous section, we also have that

$$\hat{\boldsymbol{\eta}}_\varepsilon \to \hat{\boldsymbol{\eta}} \text{ and } \hat{\boldsymbol{\eta}}^*_\varepsilon \to \hat{\boldsymbol{\eta}}^* \quad \text{in } L^\infty(0,T;\mathbf{C}^1[0,L]) \text{ a.s.,} \tag{5.4}$$

and that

$$\begin{aligned}
&A_{\hat{\boldsymbol{\eta}}^*_\varepsilon} \to A_{\hat{\boldsymbol{\eta}}^*}, (A_{\hat{\boldsymbol{\eta}}^*_\varepsilon})^{-1} \to (A_{\hat{\boldsymbol{\eta}}^*})^{-1} \text{ in } L^\infty(0,T;\mathbf{W}^{2,p}(\mathcal{O})) \text{ for any } p<4 \text{ a.s.,}\\
&J_{\hat{\boldsymbol{\eta}}^*_\varepsilon} \to J_{\hat{\boldsymbol{\eta}}^*} = \det(\nabla A_{\hat{\boldsymbol{\eta}}^*}) \quad \text{in } L^\infty(0,T;C(\bar{\mathcal{O}})), \quad \hat{\mathbb{P}}\text{-a.s.,}\\
&S_{\hat{\boldsymbol{\eta}}_\varepsilon} \to S_{\hat{\boldsymbol{\eta}}} \quad \text{in } L^\infty(0,T;C(\bar{\Gamma})), \quad \hat{\mathbb{P}}\text{-a.s.,}\\
&\hat{\mathbf{w}}^*_\varepsilon \rightharpoonup \hat{\mathbf{w}}^* \quad \text{weakly in } L^2(0,T;\mathbf{H}^{\frac{1}{2}}(\mathcal{O})), \quad \hat{\mathbb{P}}\text{-a.s.,}\\
&\hat{\mathbf{n}}^*_\varepsilon \to \hat{\mathbf{n}}^* \quad \text{in } L^\infty(0,T;C(\bar{\Gamma})), \quad \hat{\mathbb{P}}\text{-a.s.}
\end{aligned} \tag{5.5}$$

Due to the lack of the equivalent of Lemma 4.5, we have one more obstacle to deal with. Namely, that the candidate solution for fluid velocity, $\hat{\mathbf{u}}$, does not have the desired temporal regularity to be a stochastic process in the classical sense. Hence, we construct an appropriate filtration as follows: first define the σ-fields

$$\sigma_t(\hat{\mathbf{u}}) := \bigcap_{s\geq t} \sigma\left(\bigcup_{\mathbf{Q}\in C_0^\infty((0,s);\mathscr{D})} \{(\hat{\mathbf{u}},\mathbf{q})<1\} \cup \mathcal{N} \right), \quad \mathcal{N} = \{\mathcal{A}\in\hat{\mathcal{F}} \mid \hat{\mathbb{P}}(\mathcal{A})=0\}.$$

Let $\hat{\mathcal{F}}'_t$ be the σ-field generated by the random variables $\hat{\boldsymbol{\eta}}(s), \hat{W}(s)$ for all $0\leq s\leq t$. Then we define the history of the random distributions $\hat{\mathbf{u}}$, $\hat{\mathcal{F}}_t$, as follows:

$$\hat{\mathcal{F}}_t^0 := \bigcap_{s\geq t} \sigma(\hat{\mathcal{F}}'_s \cup \mathcal{N}), \qquad \hat{\mathcal{F}}_t := \sigma(\sigma_t(\hat{\mathbf{u}}) \cup \hat{\mathcal{F}}_t^0). \tag{5.6}$$

This gives a complete, right-continuous filtration $(\hat{\mathcal{F}}_t)_{t\geq 0}$, on the probability space $(\hat{\Omega},\hat{\mathcal{F}},\hat{\mathbb{P}})$, to which the noise and the candidate solution $\hat{\mathbf{u}}$ are adapted. Now we state the following result from [2].

LEMMA 5.4. *There exists a stochastic process taking values in* $L^2(0,T;\mathbf{L}^2(\mathcal{O}))$ *a.s. which is an* $(\hat{\mathcal{F}}_t)_{t\geq 0}$*-progressively measurable representative of* $\hat{\mathbf{u}}$.

THEOREM 5.5. *For any fixed* $\delta=(\delta_1,\delta_2)$ *that satisfies* (3.9)*, the random variables* $(\hat{\mathbf{u}},\hat{\boldsymbol{\eta}},\hat{\boldsymbol{\eta}}^*)$ *constructed in Theorem* 5.3 *satisfy the following:*

$$\begin{aligned}
&(J_{\hat{\boldsymbol{\eta}}^*}(t)\hat{\mathbf{u}}(t),\mathbf{q}(t)) + (\partial_t\hat{\boldsymbol{\eta}}(t),\boldsymbol{\psi}(t)) = ((J_0)\mathbf{u}_0,\mathbf{q}(0)) + (\mathbf{v}_0,\boldsymbol{\psi}(0))\\
&\quad + \int_0^t\int_{\mathcal{O}} J_{\hat{\boldsymbol{\eta}}^*}\hat{\mathbf{u}}\cdot\partial_t\mathbf{q} + \int_0^t\int_0^L \partial_t\hat{\boldsymbol{\eta}}\,\partial_t\boldsymbol{\psi} - \int_0^t \langle \mathscr{L}_e(\hat{\boldsymbol{\eta}}),\boldsymbol{\psi}\rangle\\
&\quad - \frac{1}{2}\int_0^t\int_{\mathcal{O}} J_{\hat{\boldsymbol{\eta}}^*}(\hat{\mathbf{u}}\cdot\nabla^{\hat{\boldsymbol{\eta}}^*}\hat{\mathbf{u}}\cdot\mathbf{q} - (\hat{\mathbf{u}}-2\hat{\mathbf{w}}^*)\cdot\nabla^{\hat{\boldsymbol{\eta}}^*}\mathbf{q}\cdot\hat{\mathbf{u}}) + \frac{1}{2}\int_0^t\int_\Gamma S_{\hat{\boldsymbol{\eta}}^*}(\hat{\mathbf{v}}^*\cdot\hat{\mathbf{n}}^*)(\hat{\mathbf{u}}\cdot\mathbf{q})\\
&\quad - 2\nu\int_0^t\int_{\mathcal{O}} J_{\hat{\boldsymbol{\eta}}^*}\mathbf{D}^{\hat{\boldsymbol{\eta}}^*}(\hat{\mathbf{u}})\cdot\mathbf{D}^{\hat{\boldsymbol{\eta}}^*}(\mathbf{q}) - \frac{1}{\alpha}\int_0^t\int_\Gamma S_{\hat{\boldsymbol{\eta}}}(\hat{\mathbf{u}}-\partial_t\hat{\boldsymbol{\eta}})\cdot\tau^{\hat{\boldsymbol{\eta}}}(\mathbf{q}-\boldsymbol{\psi})\cdot\tau^{\hat{\boldsymbol{\eta}}}\\
&\quad + \int_0^t\left(P_{in}\int_0^1 q_z\Big|_{z=0}dr - P_{out}\int_0^1 q_z\Big|_{z=1}dr\right)ds + \int_0^t (G(\hat{\mathbf{u}},\hat{\boldsymbol{\eta}}^*)d\hat{W},\mathbf{Q}),
\end{aligned} \tag{5.7}$$

$\hat{\mathbb{P}}$*-a.s. for almost every* $t\in[0,T]$ *and for any* $(\hat{\mathcal{F}}_t)_{t\geq 0}$*-adapted process* $\mathbf{Q}=(\mathbf{q},\boldsymbol{\psi})$ *with* C^1*-paths in* $\mathscr{D}$ *such that* $\nabla^{\hat{\boldsymbol{\eta}}^*}\cdot\mathbf{q}=0$ *and* $\mathbf{q}|_\Gamma\cdot\mathbf{n}^{\hat{\boldsymbol{\eta}}^*}=\boldsymbol{\psi}\cdot\mathbf{n}^{\hat{\boldsymbol{\eta}}^*}$ *a.s. Moreover,* $\nabla^{\hat{\boldsymbol{\eta}}^*}\cdot\hat{\mathbf{u}}=0$.

Proof of Theorem 5.5. First we must construct $\mathscr{D}$-valued test processes $(\mathbf{q}_\varepsilon,\boldsymbol{\psi}_\varepsilon)$, satisfying the kinematic coupling condition and such that $\mathbf{q}_\varepsilon$ satisfies the transformed divergence-free condition. This is required so that the two penalty terms in the approximate weak formulation drop out.

We first construct an appropriate test functions for the limiting equation (5.7) as follows: Recall that the maximal domain $\mathcal{O}_\delta=(0,L)\times(0,R_\delta)$ is a rectangular domain comprising of all the moving domains $\mathcal{O}_{\hat{\boldsymbol{\eta}}_\varepsilon^*}$. Consider a smooth $(\hat{\mathcal{F}}_t)_{t\geq 0}$-adapted process $\mathbf{g}=(g_z,g_r)$ on $\bar{\mathcal{O}}_\delta$ such that $\nabla\cdot\mathbf{g}=0$ and such that $\mathbf{g}$ satisfies the required boundary conditions $g_r=0$ on $z=0,L,r=0$, and $\partial_r g_z=0$ on Γ_b. Assume also that, on the top lateral boundary of the moving domain associated with $\hat{\boldsymbol{\eta}}^*$, $\Gamma_{\hat{\boldsymbol{\eta}}^*}$, the function $\mathbf{g}$ satisfies $\mathbf{g}(t)|_{\Gamma_{\hat{\boldsymbol{\eta}}^*(t)}}\cdot\hat{\mathbf{n}}^*(t)=\boldsymbol{\psi}(t)\cdot\hat{\mathbf{n}}^*(t)$ for some smooth $(\hat{\mathcal{F}}_t)_{t\geq 0}$-adapted process $\boldsymbol{\psi}=(\psi_z,\psi_r)$. We define

$$\mathbf{q}(t,z,r,\omega)=\mathbf{g}(t,\omega)\circ A^\omega_{\hat{\boldsymbol{\eta}}^*}(t)(z,r).$$

To observe that $\mathbf{q}$ is a suitable test function we consider, for any $t \in [0,T]$ and given process $\mathbf{g}$, the map $\mathcal{C}_{\mathbf{g}} : \hat{\Omega} \times \mathbf{C}([0,L]) \to \mathbf{C}^1(\bar{\mathcal{O}})$,

$$\mathcal{C}_{\mathbf{g}}(\omega, \boldsymbol{\eta}) = F_{\boldsymbol{\eta}}(\mathbf{g}(t,\omega)),$$

where $F_{\boldsymbol{\eta}}(\mathbf{f}) := \mathbf{f} \circ A^{\omega}_{\boldsymbol{\eta}}$ is a well-defined map from $\mathbf{C}(\bar{\mathcal{O}}_{\boldsymbol{\eta}})$ to $\mathbf{C}(\bar{\mathcal{O}})$ for any $\boldsymbol{\eta} \in \mathbf{C}([0,L])$. Due to the continuity of the composition operator $F_{\boldsymbol{\eta}}$, the assumption that $\mathbf{g}(t)$ is $\hat{\mathcal{F}}_t$-measurable implies for any $\boldsymbol{\eta}$ that the $\mathbf{C}^1(\bar{\mathcal{O}})$-valued map $\omega \mapsto \mathcal{C}_{\mathbf{g}}(\omega, \boldsymbol{\eta})$ is $\hat{\mathcal{F}}_t$-measurable (where $\mathbf{C}^1(\bar{\mathcal{O}})$ is endowed with Borel σ-algebra). Note also that for a fixed ω, the map $\boldsymbol{\eta} \mapsto \mathcal{C}_{\mathbf{g}}(\omega, \boldsymbol{\eta})$ is continuous. Hence, we infer that $\mathcal{C}_{\mathbf{g}}$ is a Carathéodory function. Recall also that $\hat{\boldsymbol{\eta}}^*$ is $(\hat{\mathcal{F}}_t)_{t \geq 0}$-adapted. Therefore, we deduce that the $\mathbf{C}^1(\bar{\mathcal{O}})$-valued process $\mathbf{q}(t,\omega) = \mathcal{C}_{\mathbf{g}}(\omega, \hat{\boldsymbol{\eta}}^*(t,\omega))$ is $(\hat{\mathcal{F}}_t)_{t \geq 0}$-adapted as well.

In summary, we have $\{\hat{\mathcal{F}}_t\}_{t \geq 0}$-adapted processes $(\mathbf{q}, \boldsymbol{\psi})$ with continuous paths in $\mathscr{D}$ such that

$$\nabla^{\hat{\boldsymbol{\eta}}^*} \cdot \mathbf{q} = 0 \quad \text{ and } \quad \mathbf{q}\Big|_{\Gamma} \cdot \mathbf{n}^{\hat{\boldsymbol{\eta}}^*} = \boldsymbol{\psi} \cdot \mathbf{n}^{\hat{\boldsymbol{\eta}}^*}.$$

Moreover, for any $\omega \in \hat{\Omega}$ we have that $\mathbf{q} \in L^\infty(0,T; \mathbf{H}^{2+k}(\mathcal{O})) \cap H^1(0,T; \mathbf{H}^k(\mathcal{O}))$ for any $k \leq \frac{1}{2}$. Now we define the approximate test functions $(\mathbf{q}_\varepsilon, \boldsymbol{\psi}_\varepsilon)$, with the aid of the Piola transformation as done in the proof of Lemma 4.2:

$$\begin{aligned}\mathbf{q}_\varepsilon = {} & J^{-1}_{\hat{\boldsymbol{\eta}}^*_\varepsilon} \nabla A_{\hat{\boldsymbol{\eta}}^*_\varepsilon} J_{\hat{\boldsymbol{\eta}}^*} \nabla A^{-1}_{\hat{\boldsymbol{\eta}}^*} (\mathbf{q} - \boldsymbol{\psi}\chi) + \boldsymbol{\psi}\chi - \left(\lambda^{\hat{\boldsymbol{\eta}}^*_\varepsilon} - \lambda^{\hat{\boldsymbol{\eta}}^*}\right)(\xi_0 \chi) \\ & - J^{-1}_{\hat{\boldsymbol{\eta}}^*_\varepsilon} \nabla A_{\boldsymbol{\eta}^n_*} \mathcal{B}\Big(\operatorname{div}\Big((J_{\hat{\boldsymbol{\eta}}^*}(\nabla A_{\hat{\boldsymbol{\eta}}^*})^{-1} - J_{\hat{\boldsymbol{\eta}}^*_\varepsilon}(\nabla A_{\hat{\boldsymbol{\eta}}^*_\varepsilon})^{-1})\boldsymbol{\psi}\chi \\ & - \left(\lambda^{\hat{\boldsymbol{\eta}}^*_\varepsilon} - \lambda^{\hat{\boldsymbol{\eta}}^*}\right) J_{\hat{\boldsymbol{\eta}}^*_\varepsilon}(\nabla A_{\hat{\boldsymbol{\eta}}^*_\varepsilon})^{-1} \xi_0 \chi\Big)\Big).\end{aligned}$$

And for the structure test function we let

$$\boldsymbol{\psi}_\varepsilon = \boldsymbol{\psi} - (\lambda^{\hat{\boldsymbol{\eta}}^*_\varepsilon} - \lambda^{\hat{\boldsymbol{\eta}}^*})\xi_0,$$

where we pick an appropriate $\xi_0 \in \mathbf{C}^\infty_0((0,T) \times \Gamma)$ such that $-\int_\Gamma (\mathbf{id} + \hat{\boldsymbol{\eta}}^*_\varepsilon(t)) \times \partial_z \xi_0(t)\, dz = 1$ for every $\varepsilon > 0$ and $t \in [0,T]$. We also define the real-valued corrector functions,

$$\lambda^{\hat{\boldsymbol{\eta}}^*_\varepsilon}(t) = -\int_\Gamma (\mathbf{id} + \hat{\boldsymbol{\eta}}^*_\varepsilon(t)) \times \partial_z \boldsymbol{\psi}(t) dz, \quad \lambda^{\hat{\boldsymbol{\eta}}^*}(t) = -\int_\Gamma (\mathbf{id} + \hat{\boldsymbol{\eta}}^*(t)) \times \partial_z \boldsymbol{\psi}(t) dz.$$

As earlier, $\chi(r)$ is a smooth function on $\mathcal{O}$ such that $\chi(1) = 1$ and $\chi(0) = 0$. Observe that the properties of the Piola transformation (see, e.g., Theorem 1.7 in [9]) imply that

$$\nabla^{\hat{\boldsymbol{\eta}}^*_\varepsilon} \cdot \mathbf{q}_\varepsilon = J_{\hat{\boldsymbol{\eta}}^*} J^{-1}_{\hat{\boldsymbol{\eta}}^*_\varepsilon} \nabla^{\hat{\boldsymbol{\eta}}^*} \cdot \mathbf{q} = 0 \qquad \text{and} \qquad \mathbf{q}_\varepsilon|_\Gamma \cdot \mathbf{n}^*_\varepsilon = \boldsymbol{\psi}_\varepsilon \cdot \mathbf{n}^*_\varepsilon.$$

Furthermore, we have

$$|\frac{d}{dt}\lambda^{\hat{\boldsymbol{\eta}}^*_\varepsilon}| \leq \|\hat{\mathbf{v}}^*_\varepsilon\|_{\mathbf{L}^2(0,L)} \|\boldsymbol{\psi}\|_{\mathbf{H}^1_0(0,L)} + \|\hat{\boldsymbol{\eta}}^*_\varepsilon\|_{\mathbf{L}^\infty(0,L)} \|\partial_t \partial_z \boldsymbol{\psi}\|_{\mathbf{L}^\infty(0,L)}.$$

Hence $\lambda^{\hat{\boldsymbol{\eta}}^*_\varepsilon} \to \lambda^{\hat{\boldsymbol{\eta}}^*}$ strongly in $L^\infty(0,T)$ and weakly in $H^1(0,T)$ a.s. Additionally, thanks to (5.5) we obtain that

$$\begin{aligned}& \|\mathbf{q}_\varepsilon - \mathbf{q}\|_{L^\infty(0,T;\mathbf{H}^1(\mathcal{O}))} \\ & \quad \leq \|A_{\hat{\boldsymbol{\eta}}^*_\varepsilon} - A_{\hat{\boldsymbol{\eta}}^*}\|_{L^\infty(0,T;\mathbf{W}^{2,3}(\mathcal{O}))} \left(\|\mathbf{q}\|_{L^\infty(0,T;\mathbf{H}^1(\mathcal{O}))} + \|\boldsymbol{\psi}\|_{L^\infty(0,T;\mathbf{H}^1_0(0,L))}\right) \\ & \qquad + \|\lambda^{\hat{\boldsymbol{\eta}}^*_\varepsilon} - \lambda^{\hat{\boldsymbol{\eta}}^*}\|_{L^\infty(0,T)} \to 0, \qquad \hat{\mathbb{P}}\text{-a.s.}\end{aligned} \tag{5.8}$$

Similarly, for any k we have

$$\begin{aligned} \boldsymbol{\psi}_\varepsilon \to \boldsymbol{\psi} \quad &\text{in } L^\infty(0,T;\mathbf{C}^k(\bar\Gamma)), \quad \hat{\mathbb{P}}-a.s., \\ \partial_t\boldsymbol{\psi}_\varepsilon \rightharpoonup \partial_t\boldsymbol{\psi} \quad &\text{weakly in } L^2(0,T;\mathbf{C}^k(\Gamma)), \quad \hat{\mathbb{P}}-a.s. \end{aligned} \tag{5.9}$$

Now we test (4.29) with $(\mathbf{q}_\varepsilon, \boldsymbol{\psi}_\varepsilon)$ for which we invoke the variant of Itô's formula derived in Lemma 5.1 in [3]. We can now pass $\varepsilon \to 0$ starting with the stochastic integral.

LEMMA 5.6. *The processes* $(\int_0^t (G(\hat{\mathbf{u}}_\varepsilon(s), \hat{\boldsymbol{\eta}}^*_\varepsilon(s))d\hat W_\varepsilon(s), \mathbf{Q}_\varepsilon(s)))_{t\in[0,T]}$ *converge to* $(\int_0^t (G(\hat{\mathbf{u}}(s), \hat{\boldsymbol{\eta}}^*(s))d\hat W(s), \mathbf{Q}(s)))_{t\in[0,T]}$ *in* $L^1(\hat\Omega; L^1(0,T;\mathbb{R}))$ *as* $\varepsilon \to 0$.

Proof. Under the assumptions (2.12) we observe that

$$\begin{aligned} &\int_0^T \|(G(\hat{\mathbf{u}}_\varepsilon, \hat{\boldsymbol{\eta}}^*_\varepsilon), \mathbf{Q}_\varepsilon) - (G(\hat{\mathbf{u}}, \hat{\boldsymbol{\eta}}^*), \mathbf{Q})\|^2_{L_2(U_0,\mathbb{R})} ds \\ &\quad \le \int_0^T \|(G(\hat{\mathbf{u}}_\varepsilon, \hat{\boldsymbol{\eta}}^*_\varepsilon) - G(\hat{\mathbf{u}}, \hat{\boldsymbol{\eta}}^*), \mathbf{Q}_\varepsilon)\|^2_{L_2(U_0;\mathbb{R})} + \int_0^T \|(G(\hat{\mathbf{u}}, \hat{\boldsymbol{\eta}}^*), \mathbf{Q}_\varepsilon - \mathbf{Q})\|^2_{L_2(U_0;\mathbb{R})} \\ &\quad \le \int_0^T \left(\|\hat{\boldsymbol{\eta}}^*_\varepsilon - \hat{\boldsymbol{\eta}}^*\|^2_{\mathbf{L}^2(0,L)} + \|\hat{\mathbf{u}}_\varepsilon - \hat{\mathbf{u}}\|^2_{\mathbf{L}^2(\mathcal{O})} \right) \|\mathbf{Q}_\varepsilon\|^2_{\mathbf{L}^2} \\ &\qquad + \int_0^T \left(\|\hat{\boldsymbol{\eta}}^*\|^2_{\mathbf{H}^2_0(0,L)} + \|\hat{\mathbf{u}}\|^2_{\mathbf{L}^2(\mathcal{O})} \right) \|\mathbf{Q}_\varepsilon - \mathbf{Q}\|^2_{\mathbf{L}^2}. \end{aligned}$$

Then thanks to Theorem 4.7, (5.8), and $(5.9)_1$, the right-hand side of the inequality above converges to 0, $\hat{\mathbb{P}}$-a.s. as $\varepsilon \to 0$. That is,

$$(G(\hat{\mathbf{u}}_\varepsilon, \hat{\boldsymbol{\eta}}^*_\varepsilon), \mathbf{Q}_\varepsilon) \to (G(\hat{\mathbf{u}}, \hat{\boldsymbol{\eta}}^*), \mathbf{Q}), \qquad \hat{\mathbb{P}}\text{-a.s.} \qquad \text{in } L^2(0,T; L_2(U_0,\mathbb{R})). \tag{5.10}$$

Now using classical ideas from [1] (see Lemma 2.1 of [10]), we obtain from (5.10) that

$$\int_0^t (G(\hat{\mathbf{u}}_\varepsilon, \hat{\boldsymbol{\eta}}^*_\varepsilon)d\hat W_\varepsilon, \mathbf{Q}_\varepsilon) \to \int_0^t (G(\hat{\mathbf{u}}, \hat{\boldsymbol{\eta}}^*)d\hat W, \mathbf{Q}) \ \text{ in probability in } L^2(0,T;\mathbb{R}). \tag{5.11}$$

Furthermore, for some $C > 0$ independent of ε we have the following bounds that follow from Itô's isometry:

$$\begin{aligned} \hat{\mathbb{E}} \int_0^T \left| \int_0^t (G(\hat{\mathbf{u}}_\varepsilon, \hat{\boldsymbol{\eta}}^*_\varepsilon)d\hat W_\varepsilon(s), \mathbf{Q}_\varepsilon) \right|^2 dt &= \int_0^T \hat{\mathbb{E}} \int_0^t \|(G(\hat{\mathbf{u}}_\varepsilon, \hat{\boldsymbol{\eta}}^*_\varepsilon), \mathbf{Q}_\varepsilon)\|^2_{L_2(U_0,\mathbb{R})} ds dt \\ &\le T\hat{\mathbb{E}} \int_0^T \left(\|\hat{\boldsymbol{\eta}}^*_\varepsilon\|^2_{\mathbf{H}^2_0(0,L)} + \|\hat{\mathbf{u}}_\varepsilon\|^2_{\mathbf{L}^2(\mathcal{O})} \right) \|\mathbf{Q}_\varepsilon\|^2_{\mathbf{L}^2} ds \\ &\le C. \end{aligned} \tag{5.12}$$

Here we also used the a.s. bounds $\|\mathbf{q}_\varepsilon\|_{L^\infty(0,T;\mathbf{L}^2(\mathcal{O}))} \le C(\delta)(\|\mathbf{q}\|_{L^\infty(0,T;\mathbf{L}^2(\mathcal{O}))} + \|\boldsymbol{\psi}\|_{L^\infty(0,T;\mathbf{L}^2(0,L))})$. Combining (5.11), (5.12) with the Vitali convergence theorem, we conclude the proof of Lemma 5.6. ∎

The rest of the convergence results follows as in [32]. One of the terms that requires further explanation is the boundary integral $\int_0^t \int_\Gamma S_{\hat{\boldsymbol{\eta}}^*_\varepsilon}(\hat{\mathbf{v}}^*_\varepsilon \cdot \hat{\mathbf{n}}^*_\varepsilon)(\hat{\mathbf{u}}_\varepsilon \cdot \mathbf{q}_\varepsilon)$. Observe that, due to the embedding $H^{\frac14}(\Gamma) \hookrightarrow L^4(\Gamma)$, Theorem 5.3, and (5.8), $\hat{\mathbf{u}}_\varepsilon \cdot \mathbf{q}_\varepsilon$ converges to $\hat{\mathbf{u}} \cdot \mathbf{q}$ in $L^2(0,T;L^2(\Gamma))$. Combining this with (5.3) and $(5.5)_5$, we obtain the convergence of $\int_0^t \int_\Gamma S_{\hat{\boldsymbol{\eta}}^*_\varepsilon}(\hat{\mathbf{v}}^*_\varepsilon \cdot \hat{\mathbf{n}}^*_\varepsilon)(\hat{\mathbf{u}}_\varepsilon \cdot \mathbf{q}_\varepsilon)$ to $\int_0^t \int_\Gamma S_{\hat{\boldsymbol{\eta}}^*}(\hat{\mathbf{v}}^* \cdot \hat{\mathbf{n}}^*)(\hat{\mathbf{u}} \cdot \mathbf{q})$ a.s.

Finally we comment on the term $\int_0^T\int_{\mathcal{O}} J_{\hat{\boldsymbol\eta}^*_\varepsilon}\hat{\mathbf{u}}_\varepsilon\cdot\partial_t\mathbf{q}_\varepsilon$. Observe that by integrating by parts we can realize that it is indeed a well-defined term. For example, we observe that the crucial term in the expansion of $\partial_t\mathbf{q}_\varepsilon$ can be written as

(5.13)
$$\begin{aligned}\int_0^t\int_{\mathcal{O}} J_{\hat{\boldsymbol\eta}^*_\varepsilon}\hat{\mathbf{u}}_\varepsilon\cdot\left(J^{-1}_{\hat{\boldsymbol\eta}^*_\varepsilon}J_{\hat{\boldsymbol\eta}^*}(\partial_t\nabla A_{\hat{\boldsymbol\eta}^*_\varepsilon})\nabla A^{-1}_{\hat{\boldsymbol\eta}^*}\mathbf{q}\right) &= \int_0^t\int_{\mathcal{O}}\hat{\mathbf{u}}_\varepsilon\cdot\left(J_{\hat{\boldsymbol\eta}^*}(\partial_t\nabla A_{\hat{\boldsymbol\eta}^*_\varepsilon})\nabla A^{-1}_{\hat{\boldsymbol\eta}^*}\mathbf{q}\right)\\ = -\int_0^t\int_{\mathcal{O}}\hat{\mathbf{w}}^*_\varepsilon\cdot\operatorname{div}\left(\hat{\mathbf{u}}_\varepsilon\otimes\left(J_{\hat{\boldsymbol\eta}^*}\nabla A^{-1}_{\hat{\boldsymbol\eta}^*}\mathbf{q}\right)\right) &+ \int_0^t\int_\Gamma(\hat{\mathbf{u}}_\varepsilon\cdot\hat{\mathbf{v}}^*_\varepsilon)(J_{\hat{\boldsymbol\eta}^*}\nabla A^{-1}_{\hat{\boldsymbol\eta}^*}\mathbf{q}\cdot\mathbf{n}),\end{aligned}$$

where the right-hand side converges to

$$-\int_0^t\int_{\mathcal{O}}\hat{\mathbf{w}}^*\cdot\operatorname{div}\left(\hat{\mathbf{u}}\otimes\left(J_{\hat{\boldsymbol\eta}^*}\nabla A^{-1}_{\hat{\boldsymbol\eta}^*}\mathbf{q}\right)\right)+\int_0^t\int_\Gamma(\hat{\mathbf{u}}\cdot\hat{\mathbf{v}}^*)(J_{\hat{\boldsymbol\eta}^*}\nabla A^{-1}_{\hat{\boldsymbol\eta}^*}\mathbf{q}\cdot\mathbf{n}),\ dt\otimes\hat{\mathbb{P}}\text{-a.e.}$$

Similarly writing $\partial_t J_{\hat{\boldsymbol\eta}^*} = -(J_{\hat{\boldsymbol\eta}^*})^{-2}\mathrm{tr}((\mathrm{cof}A_{\hat{\boldsymbol\eta}^*})^T\nabla\mathbf{w}^*)$, we can treat the rest of the terms by integration by parts, identically. This completes the proof of Theorem 5.5. ☐

Notice that the weak formulation in Theorem 5.5 still contains $\hat{\boldsymbol\eta}^*(t)$ in several terms. We will now show that in fact $\hat{\boldsymbol\eta}^*(t)$ can be replaced by the stochastic process $\hat{\boldsymbol\eta}(t)$ to obtain the desired weak formulation until some strictly positive stopping time $T^{\boldsymbol\eta}$.

Lemma 5.7 (almost surely positive stopping time). *Let the deterministic initial data $\boldsymbol\eta_0$ satisfy the assumptions* (2.14). *Then, for any $\delta=(\delta_1,\delta_2)$ satisfying* (3.9), *there exists an almost surely positive stopping time $T^{\boldsymbol\eta}$, given by*

$$T^{\boldsymbol\eta} := T\wedge\inf\left\{t>0:\inf_{\mathcal{O}} J_{\hat{\boldsymbol\eta}}(t)\le\delta_1\right\}\wedge\inf\left\{t>0:\|\hat{\boldsymbol\eta}(t)\|_{\mathbf{H}^s(0,L)}\ge\frac{1}{\delta_2}\right\},\tag{5.14}$$

such that

$$\hat{\boldsymbol\eta}^*(t)=\hat{\boldsymbol\eta}(t)\quad \textit{for } t<T^{\boldsymbol\eta}.\tag{5.15}$$

Proof. We write the stopping time as

$$T^{\boldsymbol\eta}=T\wedge\inf\left\{t>0:\inf_{\mathcal{O}} J_{\hat{\boldsymbol\eta}}(t)\le\delta_1\right\}\wedge\inf\left\{t>0:\|\hat{\boldsymbol\eta}(t)\|_{\mathbf{H}^s(0,L)}\ge\frac{1}{\delta_2}\right\}=:T\wedge T_1^{\boldsymbol\eta}+T_2^{\boldsymbol\eta}.$$

Observe that using the triangle inequality, for any $\delta_0>\delta_2$, we obtain for $T_2^{\boldsymbol\eta}$ that

$$\begin{aligned}\hat{\mathbb{P}}\Big[T_2^{\boldsymbol\eta}=0,\|\boldsymbol\eta_0\|_{\mathbf{H}^2(0,L)}<\frac{1}{\delta_0}\Big] &= \lim_{\epsilon\to0^+}\hat{\mathbb{P}}\left[T_2^{\boldsymbol\eta}<\epsilon,\|\boldsymbol\eta_0\|_{\mathbf{H}^2(0,L)}<\frac{1}{\delta_0}\right]\\ &\le\limsup_{\epsilon\to0^+}\hat{\mathbb{P}}\left[\sup_{t\in[0,\epsilon)}\|\hat{\boldsymbol\eta}(t)\|_{\mathbf{H}^s(0,L)}>\frac{1}{\delta_2},\|\boldsymbol\eta_0\|_{\mathbf{H}^2(0,L)}<\frac{1}{\delta_0}\right]\\ &\le\limsup_{\epsilon\to0^+}\hat{\mathbb{P}}\left[\sup_{t\in[0,\epsilon)}\|\hat{\boldsymbol\eta}(t)-\boldsymbol\eta_0\|_{\mathbf{H}^s(0,L)}>\frac{1}{\delta_2}-\frac{1}{\delta_0}\right]\\ &\le\frac{1}{(\frac{1}{\delta_2}-\frac{1}{\delta_0})}\limsup_{\epsilon\to0}\hat{\mathbb{E}}\left[\sup_{t\in[0,\epsilon)}\|\hat{\boldsymbol\eta}(t)-\boldsymbol\eta_0\|_{\mathbf{H}^s(0,L)}\right]\\ &\le\frac{1}{(\frac{1}{\delta_2}-\frac{1}{\delta_0})}\limsup_{\epsilon\to0}\hat{\mathbb{E}}\left[\sup_{t\in[0,\epsilon)}\|\hat{\boldsymbol\eta}(t)-\boldsymbol\eta_0\|^{1-\frac{s}{2}}_{\mathbf{L}^2(0,L)}\|\hat{\boldsymbol\eta}(t)-\boldsymbol\eta_0\|^{\frac{s}{2}}_{\mathbf{H}^2(0,L)}\right]\end{aligned}$$

$$\le \frac{1}{(\frac{1}{\delta_2}-\frac{1}{\delta_0})}\limsup_{\epsilon\to 0}\hat{\mathbb{E}}\left[\sup_{t\in[0,\epsilon)}\epsilon\|\hat{\mathbf{v}}(t)\|^{1-\frac{s}{2}}_{\mathbf{L}^2(0,L)}\|\hat{\boldsymbol{\eta}}(t)-\boldsymbol{\eta}_0\|^{\frac{s}{2}}_{\mathbf{H}^2(0,L)}\right]$$
$$\le \limsup_{\epsilon\to 0}\frac{\epsilon}{(\frac{1}{\delta_2}-\frac{1}{\delta_0})}\left(\hat{\mathbb{E}}\left[\sup_{t\in[0,\epsilon)}\|\hat{\mathbf{v}}(t)\|^2_{\mathbf{L}^2(0,L)}\right]\right)^{\frac{2-s}{4}}$$
$$\times\left(\hat{\mathbb{E}}\left[\sup_{t\in(0,\epsilon)}\|\hat{\boldsymbol{\eta}}(t)-\boldsymbol{\eta}_0\|^2_{\mathbf{H}^2(0,L)}\right]\right)^{\frac{s}{4}}=0.$$

Hence, by continuity from below, we infer that for any $\delta_2>0$,

$$\hat{\mathbb{P}}\left[T_2^{\boldsymbol{\eta}}=0,\|\boldsymbol{\eta}_0\|_{\mathbf{H}^2(0,L)}<\frac{1}{\delta_2}\right]=0. \tag{5.16}$$

We estimate $T_1^{\boldsymbol{\eta}}$ similarly, by observing that for any $t\in[0,T]$, we have $\inf_{\mathcal{O}} J_{\hat{\boldsymbol{\eta}}}(t)\ge \inf_{\mathcal{O}}\hat{J}_0-\|J_{\hat{\boldsymbol{\eta}}}(t)-J_0\|_{C(\mathcal{O})}$. Hence, for any $\delta_0>\delta_1$ we write

$$\hat{\mathbb{P}}[T_1^{\boldsymbol{\eta}}=0,\inf_{\mathcal{O}} J_0>\delta_0]\le\limsup_{\epsilon\to 0^+}\hat{\mathbb{P}}\left[\inf_{t\in[0,\epsilon)}\inf_{\hat{\mathcal{O}}} J_{\hat{\boldsymbol{\eta}}}(t)<\delta_1,\inf_{\mathcal{O}}\hat{J}_0>\delta_0\right]$$
$$\le\limsup_{\epsilon\to 0^+}\hat{\mathbb{P}}\left[\sup_{t\in[0,\epsilon)}\|J_{\hat{\boldsymbol{\eta}}}(t)-J_0\|_{C(\mathcal{O})}>\delta_0-\delta_1\right]$$
$$\le\frac{1}{(\delta_0-\delta_1)^2}\limsup_{\epsilon\to 0}\hat{\mathbb{E}}\left[\sup_{t\in[0,\epsilon)}\|J_{\hat{\boldsymbol{\eta}}}(t)-J_0\|^2_{C(\mathcal{O})}\right]=0.$$

Thus, for given $\delta_1>0$,

$$\hat{\mathbb{P}}\left[T_1^{\boldsymbol{\eta}}=0,\inf_{\mathcal{O}} J_0>\delta_1\right]=0. \tag{5.17}$$

In conclusion we have

$$\hat{\mathbb{P}}\left[T^{\boldsymbol{\eta}}=0,\inf_{\mathcal{O}} J_0>\delta_1,\|\boldsymbol{\eta}_0\|_{\mathbf{H}^2(0,L)}<\frac{1}{\delta_2}\right]=0. \tag{5.18}$$

□

Finally, by combining Theorem 5.5, Lemma 5.7, and (5.15), we conclude the proof of our main result Theorem 2.2.

Concluding remarks. We thus conclude that for any given $\delta=(\delta_1,\delta_2)$ satisfying (3.9), if the deterministic initial data $\boldsymbol{\eta}_0$ satisfies (2.14), then the stochastic processes $(\hat{\mathbf{u}},\hat{\boldsymbol{\eta}},T^{\boldsymbol{\eta}})$ along with the stochastic basis constructed in Theorem 5.3 determine a martingale solution in the sense of Definition 2.1 of the stochastic FSI problem (2.1)–(2.9). Note that even though we have proved this result in the case of 2D-1D fluid-structure interaction, our method is robust to include the 3D-2D case as well, given that the structure displacement is Lipschitz continuous in space. This Lipschitz condition in the 3D-2D case can be achieved, for example, by considering a sixth order regularization term in the elastic equations (2.4). We finally remark that the uniqueness of the solution, which is intimately related to the existence of a pathwise solution on a preordained stochastic basis, remains largely unanswered even in the deterministic case due to the nonlinearities in the problem.

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