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Cyclic base change of cuspidal automorphic representations over function fields

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Abstract

Let G be a split semisimple group over a global function field K. Given a cuspidal automorphic representation Π of G satisfying a technical hypothesis, we prove that for almost all primes ℓ, there is a cyclic base change lifting of Π along any $\mathbb{Z}/\ell\mathbb{Z}$-extension of K. Our proof does not rely on any trace formulas; instead it is based on using modularity lifting theorems, together with a Smith theory argument, to obtain base change for residual representations. As an application, we also prove that for any split semisimple group G over a local function field F, and almost all primes ℓ, any irreducible admissible representation of $G(F)$ admits a base change along any $\mathbb{Z}/\ell\mathbb{Z}$-extension of F. Finally, we characterize local base change more explicitly for a class of toral representations considered in work of Chan and Oi.

Contents

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1. Introduction

A particular case of Langlands' principle of functoriality is *cyclic base change*, which has been established in significant generality over global fields of characteristic 0, but *not* global fields of characteristic p. The results towards cyclic base change functoriality in characteristic 0 were worked out in a long series of papers by many authors, including Saito, Shintani, Langlands, Arthur, Clozel, Labesse, and Harris (see the introduction to [Fen24] for references), all following a strategy proposed by Langlands that is based on studying the twisted trace formula. But this strategy encounters complications in characteristic p, for example because of the contribution of inseparable elements to the trace formula.

In this paper we establish (under technical assumptions) cyclic base change liftings over characteristic-p global fields, using a completely different strategy that is instead predicated upon the theory of automorphy lifting. Furthermore, we prove novel types of results regarding cyclic base change in the Genestier–Lafforgue correspondence. The results and proofs will be described in more detail in the rest of this introduction.

1.1 Global base change

Let X be a smooth, projective, geometrically connected curve over the finite field $\mathbb{F}_q$ with function field $K := \mathbb{F}_q(X)$, and let G be a split semisimple algebraic group over $\mathbb{F}_q$. Let Π be a cuspidal automorphic representation of $G(\mathbb{A}_K)$ defined over a number field.[1] For a field extension K'/K, a *(weak) base change lifting* of Π to K' is a cuspidal automorphic representation Π' of $G(\mathbb{A}_{K'})$ such that for almost all places v of K where Π is unramified, Π' is also unramified at all places of K' above v, and the Satake parameters of Π and of Π' over v are related by the *base change homomorphism* for spherical Hecke algebras. (Later in the paper we will define and prove stronger notions of base change lifting.)

Over number fields, the existence of such a base change lifting is proved in complete generality for cuspidal automorphic representations of GL_n [AC89], and it is proved under some technical

[1] Every cuspidal automorphic representation over a field of characteristic 0 admits a model over a number field, so this assumption incurs no loss of generality for the questions we will study.

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assumptions by Labesse [Lab99] for general groups, still over number fields, by a comparison of the stable trace formula for $G(\mathbb{A}_K)$ with the stable trace formula for $G(\mathbb{A}_{K'})$ twisted with respect to a generator of $\mathrm{Gal}(K'/K)$. It is expected that the stable trace formula will eventually provide weak base change over function fields as well, but the proof of such a formula faces a number of obstacles and even the first steps of the proof of the general trace formula are only now becoming available [LL21].

In this paper we take a different approach to establish the existence of weak base change liftings (under technical hypotheses), based on automorphy lifting arguments instead of the trace formula. In particular, we use the results of Vincent Lafforgue on the global Langlands correspondence [Laf18]. Let $\widehat{G}$ denote the dual group of G, considered as a split semisimple group scheme over $\mathbb{Z}$. For each prime ℓ we fix an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. A consequence of V. Lafforgue's work in [Laf18] is the construction, for each cuspidal automorphic representation Π of $G(\mathbb{A}_K)$ valued in $\overline{\mathbb{Q}}_\ell$, of a finite set of $\widehat{G}(\overline{\mathbb{Q}}_\ell)$-conjugacy classes of continuous Galois representations $\sigma : \mathrm{Gal}(K^{\mathrm{s}}/K) \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$, each of which has the property that its Frobenius eigenvalues at unramified places match the Satake parameters of Π (see § 4 for the precise formulation).

Suppose Π is a cuspidal automorphic representation defined over $\overline{\mathbb{Q}}$. Then for each $\ell \neq p$, we may choose an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$, and then apply Lafforgue's theory, obtaining as above a finite set of Galois representations. These will be referred to as the Galois representations 'attached to Π by Lafforgue's correspondence.' A consequence of our main result is the following theorem.

Theorem 1.1. *Let Π be a cuspidal automorphic representation of $G(\mathbb{A}_K)$ over $\overline{\mathbb{Q}}$. Suppose that for some (equivalently, any) prime $\ell' \neq p$, some (equivalently, each) of the Galois representations attached to Π by Lafforgue's correspondence has Zariski dense image in $\widehat{G}$. Then there exists a constant $c(\Pi)$ such that for all primes $\ell > c(\Pi)$, and all cyclic ℓ-extensions K'/K, there is a weak base change lifting of Π to a cuspidal automorphic representation Π' of $G(\mathbb{A}_{K'})$.*

This follows from a more precise and refined result, Theorem 6.3, which guarantees that Π' may be chosen to be a base change lifting in the strongest possible sense (including a compatibility at ramified places). However, the latter requires more discussion in order to formulate, so we postpone it for now.

Remark 1.2. In [ST21], the Ramanujan conjecture is proved for cuspidal automorphic representations satisfying certain types of local conditions, *conditionally* upon the existence of base change liftings (with compatibility at ramified places as well) for constant extensions of large enough degree. Our Theorem 6.3 provides this type of base change lifting. However, it is not hard to deduce the Ramanujan conjecture directly when our hypothesis is satisfied.

We sketch the approach of the proof. It is based on the automorphy lifting techniques pioneered by Taylor and Wiles. The first ingredient in this program is residual automorphy, and this is provided by [Fen24], which established the existence of base change for automorphic forms over $\overline{\mathbb{F}}_\ell$ in the special case where the extension K'/K is cyclic of order ℓ. (It was essential for the argument of [Fen24] that the characteristic ℓ of the automorphic forms coincides with the order ℓ of the extension.) To prove Theorem 1.1, we use *compatible systems* of Galois representations (whose existence is guaranteed by the Zariski density assumption) to bootstrap from this case and lift the automorphy to characteristic 0. In particular, we prove an automorphy lifting theorem for representations with arbitrary ramification, going beyond the everywhere unramified case which was established by four of the authors in [BHKT19].

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We briefly discuss some of the new issues that go into proving this ramified automorphy lifting result. First, we need to arrange local deformation rings that are well behaved, which is one reason for requiring $\ell > c(\Pi)$. We show that such a condition guarantees that the unrestricted local deformation condition is smooth (see § 5 for discussion of this). A second issue is that we need to know that the image of the associated ℓ-adic Galois representation is large enough to provide a sufficient supply of Taylor–Wiles primes, for all $\ell > c(\Pi)$. The condition used in [BHKT19] is that the image is $\widehat{G}$-*abundant*. For the 'potential automorphy' theorem of [BHKT19], it was enough to know that the image mod ℓ is abundant outside a set of ℓ having Dirichlet density 0. Since the ℓ-adic image is Zariski dense, this follows from a result of Larsen [Lar95]. In order to obtain base change liftings for $\ell > c(\Pi)$, we need the stronger result that the image mod ℓ is abundant for all but finitely many ℓ. For this we use a recent theorem proved by one of us with Gajda and Petersen [BGP19]; see also [Dri18, E.10].

1.2 Local base change

From Theorem 1.1 we draw some applications to the local Langlands correspondence. To state them, let F be a local function field of characteristic p, W_F the Weil group of F, and G a split semisimple group over F. Genestier and Lafforgue [GL17] have constructed a semisimplified local Langlands correspondence

$$\left\{\begin{matrix}\text{irreducible admissible representations}\\ \pi \text{ of } G(F) \text{ over } \overline{\mathbb{Q}}_\ell\end{matrix}\right\}/\sim \longrightarrow \left\{\begin{matrix}\text{semisimple } L\text{-parameters}\\ \sigma_\pi\colon W_F \to G(\overline{\mathbb{Q}}_\ell)\end{matrix}\right\}/\sim . \tag{1.1}$$

If F'/F is a field extension, we say that an irreducible admissible representation π' of $G(F')$ is a *base change lifting* of an irreducible admissible representation π of $G(F)$ if $\sigma_{\pi'} \cong \sigma_\pi|_{W_{F'}}$.

Using our global results, we deduce in § 7 the following theorem.

Theorem 1.3. *Let π be an irreducible admissible representation of $G(F)$ over $\overline{\mathbb{Q}}$. There exists a constant $c(\pi)$ such that for all primes $\ell > c(\pi)$, for any $\mathbb{Z}/\ell\mathbb{Z}$-extension F'/F there exists a base change lifting of $\pi \otimes_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}_\ell$ to $G(F')$.*

An analogous theorem was proved for mod ℓ representations, when F'/F is cyclic of degree ℓ, in [Fen24]. The strategy here is completely different. The idea is to globalize π to a cuspidal automorphic representation Π, and the extension F'/F to a global extension K'/K, to which we can apply Theorem 1.1. If we can do this, then we can extract the local component of Π' to obtain a local base change π'. However, we must take care to construct a globalization Π satisfying the hypotheses of Theorem 1.1. This is accomplished by setting up Π with specified local conditions at a finite number of auxiliary places, so that the corresponding Galois representation σ_Π has Zariski dense image. For example, we put a supercuspidal local component 'V_ϕ' at one auxiliary place whose Genestier–Lafforgue parameter σ_{V_ϕ} is already absolutely irreducible, in order to guarantee that σ_Π is absolutely irreducible. For this, we require knowledge of σ_{V_ϕ}. This is obtained by finding a *different* globalization Π^{aux} of V_ϕ whose corresponding global Galois representation $\sigma_{\Pi^{\mathrm{aux}}}$ can be calculated explicitly, from which we extract the Genestier–Lafforgue parameter of V_ϕ using local–global compatibility; the work of Heinloth, Ngô, and Yun [HNY13] provides a convenient such 'auxiliary globalization.'

In the final section § 8, we study local base change more explicitly for a class of supercuspidal representations singled out in [Kal19] under the name 'toral supercuspidal representations.' This is a fairly broad class of supercuspidal representations, encompassing examples of arbitrary depth, for which Kaletha has constructed in [Kal19, § 6] an explicit parametrization by L-parameters, satisfying good properties of the expected local Langlands correspondence. The 'Howe-unramified' toral supercuspidal representations have been studied further by

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Chan and Oi [CO23] (noting that the 'toral representations' of [Kal19] are described as '0-toral' in [CO23] (see, in particular, [CO23, § 8])—we follow the terminology of [CO23] in the body of this paper). For the representations studied in [CO23], we are able to explicitly characterize unramified base change along the Genestier–Lafforgue correspondence under some technical assumptions. A motivation behind the results of § 8 is consistency between Kaletha's parametrization and the Genestier–Lafforgue correspondence. Roughly speaking, we prove, under genericity conditions, that the base change of mod ℓ toral supercuspidal representations along unramified $\mathbb{Z}/\ell\mathbb{Z}$-extensions, under the Genestier–Lafforgue correspondence, behaves as would be predicted by Kaletha's parametrization. See § 8.4 for the precise formulations.

Next we comment on the proof. The mechanism for understanding base change of mod ℓ representations is a conjecture of Treumann and Venkatesh [TV16, Conjecture 6.6], which predicts that Tate cohomology should realize functoriality in the local Langlands correspondence for mod ℓ representations. This was proved in [Fen24, Theorem 1.2] for the Genestier–Lafforgue correspondence, thanks to which our task amounts to computing the Tate cohomology of toral supercuspidal representations. However, Tate cohomology is tricky to calculate in general; for a general supercuspidal representation presented via a Yu datum, as considered in [Kal19], it would be a challenge even to determine whether the Tate cohomology is non-zero. Crucial traction for this problem is provided by recent work of Chan and Ivanov [CI21] and of Chan and Oi [CO23] on geometric models for toral supercuspidal representations in terms of 'deep level Deligne–Lusztig induction'; we ultimately compute the Tate cohomology by applying equivariant localization tools to their deep level Deligne–Lusztig varieties.

2. Notation and terminology

We fix a finite field $\mathbb{F}_q$ of characteristic p. Let X be a smooth, geometrically connected, and projective curve over $\mathbb{F}_q$ and $K = \mathbb{F}_q(X)$, and let G be a split reductive group over K. The notation ℓ always denotes a prime not equal to p.

2.1 Notation related to global and local fields

We write K^{s} for a fixed choice of separable closure and $\Gamma_K := \mathrm{Gal}(K^{\mathrm{s}}/K)$ for the corresponding Galois group. For S a finite set of places of K, we write K_S for the maximal subextension of K^{s} unramified outside S, and $\Gamma_{K,S} := \mathrm{Gal}(K_S/K)$. If v is a place of K, then $\Gamma_{K_v} = \mathrm{Gal}(K_v^{\mathrm{s}}/K_v)$ will denote the decomposition group, and $\Gamma_{K_v} \to \Gamma_K$ the homomorphism corresponding to a fixed choice of K-embedding $K^{\mathrm{s}} \hookrightarrow K_v^{\mathrm{s}}$. If $v \notin S$, then $\mathrm{Frob}_v \in \Gamma_{K,S}$ denotes a choice of geometric Frobenius element at the place v. We will identify the set of places of K with the set of closed points of X. For a place $v \in |X|$ we write $q_v = \#k(v) = \#(\mathcal{O}_{K_v}/\varpi_v\mathcal{O}_{K_v})$ for the size of the residue field at v. We write $|\cdot|_v$ for the norm on K_v, normalized so that $|\varpi_v|_v = q_v^{-1}$; then the product formula holds. We write $\widehat{\mathcal{O}}_K = \prod_{v\in X} \mathcal{O}_{K_v}$. We will write W_{K_v} for the Weil group of the local field K_v.

2.2 Notation related to group schemes

If $G, H, \ldots$ are group schemes over a base S, then we use Gothic letters $\mathfrak{g}, \mathfrak{h}, \ldots$ to denote their Lie algebras, and $G_T, \mathfrak{g}_T, \ldots$ to denote the base changes of these objects relative to a scheme $T \to S$. If G acts on an S-scheme X and $x \in X(T)$, then we write $Z_G(x)$ or $Z_{G_T}(x)$ for the scheme-theoretic stabilizer of x; it is a group scheme over T. We denote the center of G by Z_G. We say that a group scheme G over S is *reductive* if G is smooth and affine with reductive (and therefore connected) geometric fibres.

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2.3 Notation related to coefficient rings

When doing deformation theory, we will generally fix a prime ℓ and an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. A finite extension $E/\mathbb{Q}_\ell$ inside $\overline{\mathbb{Q}}_\ell$ will be called a coefficient field; when such a field E has been fixed, we will write $\mathcal{O}$ or $\mathcal{O}_E$ for its ring of integers, k or k_E for its residue field, and ϖ or ϖ_E for a choice of uniformizer of $\mathcal{O}_E$. We write $\mathcal{C}_\mathcal{O}$ for the category of Artinian local $\mathcal{O}$-algebras with residue field k; if $A \in \mathcal{C}_\mathcal{O}$, then we write $\mathfrak{m}_A$ for its maximal ideal. Then A comes with the data of an isomorphism $k \cong A/\mathfrak{m}_A$.

2.4 Dual groups

In this paper, we will view the dual group $\widehat{G}$ of G as a split reductive group over $\mathbb{Z}$. Our definition of $\widehat{G}$ follows [BHKT19, § 2.1]. A prime ℓ is called a *very good characteristic* for $\widehat{G}$ if it satisfies the conditions in the table below for all the simple factors of $\widehat{G}$ (referring to the absolute root system types):

Condition	Types
$\ell \nmid n+1$	A_n
$\ell \neq 2$	B, C, D, E, F, G
$\ell \neq 3$	E, F, G
$\ell \neq 5$	E_8

3. Deformation problems

In this section we set up the results on Galois deformation theory that will be used later for automorphy lifting.

3.1 Setup for the general theory

Let $\widehat{G}$ be a split semisimple group over $\mathbb{Z}$. We fix a prime ℓ which is a very good characteristic for $\widehat{G}$, as well as a coefficient field $E \subset \overline{\mathbb{Q}}_\ell$. We also fix an absolutely irreducible representation $\overline{\sigma}$ of $\Gamma_{K,S}$. We recall the results of [BHKT19] on the deformation theory of representations of $\Gamma_{K,S}$ to $\widehat{G}$ with ℓ-adic coefficients.

Lemma 3.1 [BHKT19, Lemma 5.1]. *Let $\overline{\sigma} : \Gamma_{K,S} \to \widehat{G}(k)$ be an absolutely $\widehat{G}$-irreducible homomorphism. The scheme-theoretic centralizer of $\overline{\sigma}(\Gamma_{K,S})$ in $\widehat{G}^{\mathrm{ad}}_k$ is étale over k, and $H^0(\Gamma_{K,S}, \mathrm{ad}\,\overline{\sigma}) = 0$.*

Let $A \in \mathcal{C}_\mathcal{O}$. We define *liftings* and *deformations* of $\overline{\sigma}$ over A as in [BHKT19, § 5.1], and let $\mathrm{Def}_{\overline{\sigma}} : \mathcal{C}_\mathcal{O} \to \mathrm{Sets}$ be the functor that associates to $A \in \mathcal{C}_\mathcal{O}$ the set of deformations of $\overline{\sigma}$ over A. Then we have the following proposition.

Proposition 3.2 [BHKT19, Propositions 5.10]. *The functor $\mathrm{Def}_{\overline{\sigma}}$ is pro-represented by a complete Noetherian local $\mathcal{O}$-algebra $R_{\overline{\sigma}} = R_{\overline{\sigma},S}$. (We will include the subscript S in the notation when we want to let S vary.)*

3.2 Local deformation conditions

So far we have considered deformations of $\overline{\sigma}$ with no restriction on the points in S. We now introduce local deformation conditions, following the discussion in [CHT08, § 2.2], and for general groups in [Pat16, § 3.2], with notation as in [KT17, § 4]. For each $v \in S$ let $\Gamma_{K_v} \subset \Gamma_{K,S}$ be a decomposition group at v, and let $\overline{\sigma}_v$ denote the restriction of $\overline{\sigma}$ to Γ_{K_v}. Let $\mathcal{D}^{\square}_{\overline{\sigma},v}$ denote the

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functor on $\mathcal{C}_{\mathcal{O}}$ which to $A \in \mathcal{C}_{\mathcal{O}}$ associates the set of liftings of $\overline{\sigma}$ to A; then $\mathcal{D}^{\square}_{\overline{\sigma},v}$ is represented by a ring $R^{\square}_{\overline{\sigma},v}$. As in [KT17, Definition 4.1], we define a *local deformation problem for* $\overline{\sigma}_v$ to be a subfunctor $\mathcal{D}_v \subset \mathcal{D}^{\square}_{\overline{\sigma},v}$ such that:

- $\mathcal{D}_v$ is represented by a quotient of $R^{\square}_{\overline{\sigma},v}$; and
- for any $A \in \mathcal{O}$, if $\sigma \in \mathcal{D}_v(A)$ and $\alpha \in \ker[\widehat{G}(A) \to \widehat{G}(k)]$ then $\mathrm{ad}(\alpha)(\sigma) \in \mathcal{D}_v(A)$.

Definition 3.3. A *global deformation problem* for $\overline{\sigma}$ is a collection $\mathcal{D} := (S, \{\mathcal{D}_v\}_{v\in S})$ where for each $v \in S$, $\mathcal{D}_v$ is a local deformation problem for $\overline{\sigma}_v$.

For $A \in \mathcal{C}_{\mathcal{O}}$, a deformation $\sigma : \Gamma_{K,S} \to \widehat{G}(A)$ is of *type* $\mathcal{D}$ if for every $v \in S$, the restriction of σ to Γ_{K_v} belongs to $\mathcal{D}_v$.

The functor of deformations of $\overline{\sigma}$ of type $\mathcal{D}$ is denoted $\mathrm{Def}_{\overline{\sigma},\mathcal{D}} = \mathrm{Def}_{\overline{\sigma},S,\mathcal{D}}$. Again, we will emphasize the subscript S when we want to contemplate varying S.

To each local deformation problem $\mathcal{D}_v$ we attach a k-subspace $L_v \subset H^1(G_{K_v}, \mathrm{ad}(\overline{\sigma}))$ as in [Pat16, § 3.2].

Definition 3.4. The deformation problem $\mathcal{D}_v$ is *balanced* if $\dim L_v = \dim H^0(G_{K_v}, \mathrm{ad}(\overline{\sigma}))$.

Example. The minimal deformation problems considered in [CHT08, § 2.4.4] are balanced.

Example 3.5. The deformation problem $\mathcal{D}_v$ is *unrestricted* if $\mathcal{D}_v = \mathcal{D}^{\square}_{\overline{\sigma},v}$. In this case we have $L_v = H^1(\Gamma_{K_v}, \mathrm{ad}(\overline{\sigma}))$.

Recall that for all $v \in S$, Tate's local Euler characteristic formula for any $k[\Gamma_{K_v}]$-module M reduces to

$$h^0(\Gamma_{K_v}, M) - h^1(\Gamma_{K_v}, M) + h^2(\Gamma_{K_v}, M) = 0$$

because $\ell \neq p$. Hence, if $H^2(\Gamma_{K_v}, \mathrm{ad}(\overline{\sigma})) = 0$ then the unrestricted deformation problem $\mathcal{D}_v$ is balanced.

Note that local duality implies that

$$H^2(\Gamma_{K_v}, \mathrm{ad}(\overline{\sigma})) = 0 \Leftrightarrow H^0(\Gamma_{K_v}, \mathrm{ad}(\overline{\sigma})^\vee(1)) = 0. \tag{3.1}$$

3.3 Selmer groups

Given a global deformation problem $\mathcal{D}$, we define

$$H^1_{\mathcal{D}}(\Gamma_{K,S}, \mathrm{ad}(\overline{\sigma})) := \ker\left[H^1(\Gamma_{K,S}, \mathrm{ad}(\overline{\sigma})) \to \bigoplus_{v\in S} H^1(\Gamma_{K_v}, \mathrm{ad}(\overline{\sigma}))/L_v\right], \tag{3.2}$$

and let $h^1_{\mathcal{D}}(\Gamma_{K,S}, \mathrm{ad}(\overline{\sigma})) := \dim_k H^1_{\mathcal{D}}(\Gamma_{K,S}, \mathrm{ad}(\overline{\sigma}))$.

With L_v as above, let $L_v^\perp \subset H^1(\Gamma_{K_v}, \mathrm{ad}(\overline{\sigma})^\vee(1))$ denote its orthogonal complement under the local duality pairing. We define

$$H^1_{\mathcal{D}^\perp}(\Gamma_{K,S}, \mathrm{ad}(\overline{\sigma})^\vee(1)) := \ker\left[H^1(\Gamma_{K,S}, \mathrm{ad}(\overline{\sigma})^\vee(1)) \to \bigoplus_{v\in S} H^1(\Gamma_{K_v}, \mathrm{ad}(\overline{\sigma})^\vee(1))/L_v^\perp\right],$$

and let $h^1_{\mathcal{D}^\perp}(\Gamma_{K,S}, \mathrm{ad}(\overline{\sigma})^\vee(1)) := \dim_k H^1_{\mathcal{D}^\perp}(\Gamma_{K,S}, \mathrm{ad}(\overline{\sigma})^\vee(1))$.

Proposition 3.6. (i) *The functor* $\mathrm{Def}_{\overline{\sigma},\mathcal{D}}$ *is pro-represented by a complete Noetherian local* $\mathcal{O}$*-algebra* $R_{\overline{\sigma},\mathcal{D}} = R_{\overline{\sigma},S,\mathcal{D}}$.

(ii) *There is a surjection* $\mathcal{O}[\![X_1, \dots X_g]\!] \twoheadrightarrow R_{\overline{\sigma},S,\mathcal{D}}$, *where* $g = \dim_k H^1_{\mathcal{D}}(\Gamma_{K,S}, \mathrm{ad}\,\overline{\sigma})$.

Proof. For (i) the proof is the same as for [CHT08, Proposition 2.2.9], and for (ii) the proof is the same as for [CHT08, Corollary 2.2.12]. □

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3.4 Abundant subgroups

We recall two definitions from [BHKT19].

Definition 3.7. Let Γ be an abstract group, and let $\widehat{G}$ be a reductive group over k.

(i) A homomorphism $\sigma : \Gamma \to \widehat{G}(k)$ is said to be *absolutely $\widehat{G}$-completely reducible* (respectively, *absolutely $\widehat{G}$-irreducible*) if it is $\widehat{G}$-completely reducible (respectively, $\widehat{G}$-irreducible) after extension of scalars to an algebraic closure of k.

Suppose now that $\widehat{G}$ is semisimple, and let $\overline{k}$ be an algebraic closure of k.

(iii) A finite subgroup $H \subset \widehat{G}(\overline{k})$ is said to be *$\widehat{G}$-abundant* over k if it satisfies the following conditions:
  (a) The group H is contained in $\widehat{G}(k)$.
  (b) The groups $H^0(H, \widehat{\mathfrak{g}}_k)$, $H^0(H, \widehat{\mathfrak{g}}_k^\vee)$, $H^1(H, \widehat{\mathfrak{g}}_k^\vee)$, and $H^1(H, k)$ all vanish.
  (c) For each regular semisimple element $h \in H$, the torus $Z_{\widehat{G}}(h)^\circ \subset \widehat{G}$ is split over k.
  (d) For every simple $k[H]$-submodule $W \subset \widehat{\mathfrak{g}}^\vee$, there exists a regular semisimple element $h \in H$ such that $W^h \neq 0$ and $Z_{\widehat{G}}(h)$ is connected. (We recall that $Z_{\widehat{G}}(h)$ is always connected if $\widehat{G}$ is simply connected.)

Lemma 3.8. *Suppose $\widehat{G}$ is simple and split over k. Let $\widehat{G}^{\mathrm{sc}}$ denote the simply connected cover of $\widehat{G}$, and let $\widehat{G}(k)^+$ denote the image of $\widehat{G}^{\mathrm{sc}}(k)$ in $\widehat{G}(k)$. Let H be a group with $\widehat{G}^+(k) \subset H \subset \widehat{G}(k)$ and suppose that $(k$, type of $\widehat{G})$ is not in the following list:*

$$\{(\mathbb{F}_3, A_1), (\mathbb{F}_5, A_1)\} \cup \{(\mathbb{F}_q, C_n) \mid q \in \{3, 5, 9\}, n \geq 2\}.$$

Then there exists a finite field $k_1 \supset k$ such that for all finite fields $k' \supset k_1$ the group H is $\widehat{G}$-abundant over k'.

Proof. We will prove the following claims for $(k, \text{type of } \widehat{G})$ not in the above list:

(i) The adjoint module $\widehat{\mathfrak{g}}_k$ is self-dual as a $k[H]$-representation,
(ii) The groups $H^0(H, \widehat{\mathfrak{g}}_k)$, $H^1(H, \widehat{\mathfrak{g}}_k)$, and $H^1(H, k)$ all vanish.
(iii) The action of H on $\widehat{\mathfrak{g}}_k$ is absolutely irreducible.
(iv) The group H contains strongly regular semisimple elements h, i.e. h such that $Z_{\widehat{G}}(h)$ is connected.

Let us first show that the claims suffice to prove the lemma. Indeed, condition (a) is obvious, condition (b) is the same as (ii) (given (i)), while condition (c) is automatic if k_1 is chosen to split all of the finitely many k-rational tori $Z_{\widehat{G}}(h)^\circ$ in $\widehat{G}$. Finally, given (iii), condition (d) comes down to the existence of strongly regular semisimple h, as in (iv), because $(\widehat{\mathfrak{g}}^\vee)^h$ always contains the Lie algebra of the centralizer of h.

It remains to establish the claim. Parts (i) to (iii) with many details can be found in [AB, § 3], which, however, largely builds on further references, which we now briefly recall. We first observe that $\widehat{\mathfrak{g}}_k$ is an absolutely irreducible $k[H]$-representation, essentially by results of Hiss [His84] and Hogeweij [Hog82]; see [AB, Proposition 3.22]. Now because $\widehat{\mathfrak{g}}_k$ is absolutely irreducible, to prove self-duality it suffices to see that $\widehat{\mathfrak{g}}_k$ and $\widehat{\mathfrak{g}}_k^\vee$ have the same weights. But this is clear since with any root α of the Lie algebra $\widehat{\mathfrak{g}}_k$, $-\alpha$ is also a root, and both have multiplicity 1. This settles (i) and (iii), and the first assertion of (ii).

Next, a largely classical result, completed by Tits, asserts that $\widehat{G}(k)^+$ is perfect, unless $(k, \text{type of } \widehat{G})$ is $(\mathbb{F}_3, A_1)$; see [MT11, Thm. 24.17]. Because $\widehat{G}$ is split and l is of very good characteristic for $\widehat{G}$, there is only a single exception. The latter also implies that $\widehat{G}(k)/\widehat{G}(k)^+$

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is of order prime to the characteristic l of k. Hence $H^1(H,k) = \mathrm{Hom}(H,k)$ vanishes, and this shows the third part of (ii); see [AB, Corollary 3.12].

We now turn to the remaining condition $H^1(\widehat{G}(k), \widehat{\mathfrak{g}}_k) = 0$ from (ii). The most complete vanishing results for $H^1(H, \widehat{\mathfrak{g}}_k)$ are due to Völklein with earlier work by Cline, Parshall, Scott in the A-D-E cases and by Hertzig (unpublished). We observe that $\mathrm{Lie}\,\widehat{G}$ has trivial center because l is very good for $\widehat{G}$. The latter also excludes the case $(k, \text{type of } \widehat{G}) = (\mathbb{F}_2, A_1)$. The cases $(\mathbb{F}_5, A_1)$ and $(\mathbb{F}_q, C_n)$, $q \in \{3,5,9\}$, $n \geq 2$, for $(k, \text{type of } \widehat{G})$ are ruled out by our list of exceptions. It follows from [Völ89, Theorem and Remarks] that $H^1(\widehat{G}(k), \widehat{\mathfrak{g}}_k) = 0$.

Finally, we prove (iv). If $\widehat{G}$ is simply connected, then by [SS70, II.3.9] the group $Z_{\widehat{G}}(h)$ is connected for all regular semisimple $h \in H$, and we are done. We now focus on the case where $\widehat{G}$ of adjoint type, where we find a suitable h using [FS88]. The last paragraph explains how this gives (iv) for $\widehat{G}$ of any other type.

Suppose that $\widehat{G}$ is of adjoint type. Then by [FS88, Theorem 3.1] there exists a maximal torus $T \subset \widehat{G}$ defined over k such that the following statements hold:

(1) The group $T^+(k) := \widehat{G}^+(k) \cap T(k)$ is cyclic. Let h be a generator.
(2) One has $Z_{\widehat{G}(k)}(h) = T(k)$, and the order of h is given in [FS88, Table III].
(3) If $\widehat{G}$ is of type A_n, then T is quotient of a totally anisotropic maximal torus of GL_{n+1} (over k) modulo its center, i.e. $T \cong \mathrm{Res}^k_{k^{(n+1)}} \mathbb{G}_m/\mathbb{G}_m$, the quotient of the Weil restriction of $\mathbb{G}_m$ from the unique degree-$(n+1)$ extension $k^{(n+1)}$ of k to k modulo $\mathbb{G}_m$.

Let $C := Z_{\widehat{G}}(h)$, and let $\pi_0(C) = C/C^0$ be its component group scheme. It follows from [Sei83, 2.9 and its proof] that the identity component C^0 is a (commuting) product of Chevalley groups over k and of the center of C^0, and so from (1) above the natural map $T \to C^0$ must be an isomorphism because any Chevalley group has k-points of order l. In particular, h is regular semisimple.

Next we gather results on $\pi_0(C)$ that will allow us to identify it with the trivial group. By [SS70, Corollary II.4.4] there is an injective morphism of group schemes $\pi_0(C) \to Z(\widehat{G}^{\mathrm{sc}})$ to the center $Z(\widehat{G}^{\mathrm{sc}})$ of $\widehat{G}^{\mathrm{sc}}$. Because l is good for $\widehat{G}$, the finite commutative group scheme $Z(\widehat{G}^{\mathrm{sc}})$ is étale over k. The centers are completely known, and a list can be found, for instance, in [AB, Table 1]. Using the Bruhat decomposition of $\widehat{G}$ over $\overline{k}$, and that h is regular semisimple, it is not difficult to show that C is a subgroup of the normalizer of T in $\widehat{G}$ (over $\overline{k}$), and hence $\pi_0(C)$ is a subgroup of the Weyl group of G. Moreover, by [SS70, I.2.11] one has a short exact sequence

$$0 \to T(k) \to C(k) \to \pi_0(C)(k) \to 0.$$

From (2) above we deduce that $T(k) \to C(k)$ is an isomorphism, and hence $\pi_0(C)(k)$ is trivial.

Now suppose the type is B_n, C_n, D_n or E_7. Then l cannot be 2, and so 2 divides the order of $k^\times$. It follows from the classification of the centers $Z(\widehat{G}^{\mathrm{sc}})$, that any non-trivial subgroup scheme will contain an element of exact order 2. But then $\pi_0(C)$ must be trivial, because we know already that $\pi_0(C)(k)$ is trivial.

The types E_8, F_4 and G_2 need not be considered, since here $Z(\widehat{G}^{\mathrm{sc}})$ is trivial and hence $\widehat{G}$ itself is simply connected. Next we consider type E_6, so that $Z(\widehat{G}^{\mathrm{sc}}) \cong \mu_3$. Let $q = \#k$. If $q \equiv 1 \pmod 3$, we can argue as in the previous paragraph to deduce that $\pi_0(C)$ is trivial. If, on the other hand, $q \equiv -1 \pmod 3$, then from [FS88, Table III] we find that

$$\#T(k) = (q^2+q+1)(q^4-q^2+1) \equiv -1 \pmod 3,$$

so that the order of h is not divisible by 3. It follows from [SS70, Corollary 4.6] that $\pi_0(C)$ is connected.

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It remains to discuss the type A_n. For $n = 1$, by (3) the element h is given as a diagonal matrix $\mathrm{diag}(x, x^q)$ in $\mathrm{GL}_2/\mathbb{G}_m$ for some $x \in (k^{(2)})^\times$ of order $q^2 - 1$, so that h has order $q + 1$. Because C is spanned by T and at most some Weyl group elements, we need to understand whether the unique non-trivial Weyl group element w of $\widehat{G} = \mathrm{PGL}_2$ lies in C. The element w exchanges the entries of h. If it would fix h, then $\mathrm{diag}(x, x^q) \equiv \mathrm{diag}(x^q, x) \pmod{\overline{k}^\times}$, so that there exists $a \in \overline{k}^\times$ with $ax = x^q$ and $ax^q = x$. This firstly implies $a \in \{\pm 1\}$, and then that the order of x divides $2(q-1)$. Because $q + 1$ divides the order of x, this could only happen for $q = 2$, which is forbidden, since l is assumed to be very good for $\widehat{G} = \mathrm{PGL}_2$.

Now suppose $n \geq 2$. Then h is given by $\mathrm{diag}(x, x^q, \ldots, x^{q^n}) \in \mathrm{GL}_{n+1}/\mathbb{G}_m(\overline{k})$ for some $x \in (k^{(n+1)})^\times$ of order $q^{n+1} - 1$. Let w be in the Weyl group of PGL_n, which we identify with S_{n+1}. Suppose that $h = w \circ h$, i.e. that w is the image of an element of C. Because $\pi_0(C) \subset Z(\mathrm{PGL}_{n+1}) \cong \mu_{n+1}$ we may assume that w is cyclic of order dividing $n + 1$. Let $r_1 \geq \cdots r_s \geq 2$ be the lengths in the cycle decomposition of w, and let $r = \gcd(r_1, \ldots r_s)$, which is now a divisor of $n + 1$. As in the case $n = 1$, we find $a \in \overline{k}^\times$ of order dividing r and such that $x^{q^{w(i)}} = ax^{q^i}$, or equivalently $x^{q^{w(i)} - q^i} = a$, for $i = 0, \ldots, n$.

To get further, let t be a large Zsygmondy prime for $(q, n + 1)$ in the sense of [FS88, Theorem 2.1], i.e. t is a prime that divides $q^{n+1} - 1$, that does not divide $q^m - 1$ for $1 \leq m \leq n$ and with either $t > n + 1$ or $t^2 | (q^{n+1} - 1)$. We deduce that t divides the order of $x^{q^{w(i)} - q^i}$ for any i with $w(i) \neq i$, and it follows that t divides r and in turn $n + 1$, unless w is trivial. But then modulo the prime t and using Fermat's little theorem we have

$$q^{n+1} - 1 = (q^t)^{(n+1)/t} - 1 \equiv q^{(n+1)/t} - 1 \pmod{t}.$$

This contradicts that t is a Zsygmondy prime for $(q, n + 1)$. Hence w must be trivial and $T = C$.

Finally, for a general $\widehat{G}$ consider the central isogeny $\widehat{G} \to \widehat{G}^{\mathrm{ad}}$ to the adjoint quotient $\widehat{G}^{\mathrm{ad}}$ of $\widehat{G}$. Let $H^{\mathrm{ad}} \subset \widehat{G}^{\mathrm{ad}}(k)$ be the image of H. Now the element h in $H^{ad,+}(k)$ from (1) above clearly lifts to some $h' \in \widehat{G}^+(k)$ with h' regular semisimple. Moreover, we have a left exact sequence $0 \to Z(\widehat{G}) \to Z_{\widehat{G}}(h') \to Z_{\widehat{G}^{\mathrm{ad}}}(h)$. In it, $Z_{\widehat{G}^{\mathrm{ad}}}(h)$ is a maximal torus, $Z_{\widehat{G}}(h')$ contains a maximal torus, and the center $Z(\widehat{G})$ lies in any maximal torus of $\widehat{G}$. Hence $Z_{\widehat{G}}(h')$ is a (maximal) torus, and thus connected. □

3.5 Taylor–Wiles primes

Following [BHKT19, Definition 5.16], we define a *Taylor–Wiles datum for* $\overline{\sigma}$ to be a pair $(Q, \{\varphi_v\}_{v \in Q})$, where the following statements hold:

- Q is a finite set of places v of K such that $\overline{\sigma}(\mathrm{Frob}_v)$ is regular semisimple and $q_v \equiv 1 \bmod \ell$
- For each $v \in Q$, $\varphi_v : \widehat{T}_k \cong Z_{\widehat{G}}(\overline{\sigma}(\mathrm{Frob}_v))$ is a choice of inner isomorphism. In particular, this forces $Z_{\widehat{G}}(\overline{\sigma}(\mathrm{Frob}_v))$ to be connected.

For a global deformation problem $\mathcal{D} = (S, \{\mathcal{D}_v\}_{v \in S})$ and a Taylor–Wiles datum $(Q, \{\varphi_v\}_{v \in Q})$ with $Q \cap S = \emptyset$, we define a new global deformation problem $\mathcal{D}_Q = (S \cup Q, (\mathcal{D}_Q)_v)$ with

$$(\mathcal{D}_Q)_v = \begin{cases} \mathcal{D}_v & v \in S, \\ \mathcal{D}^{\square}_{\overline{\sigma}, v} & v \in Q. \end{cases}$$

The proposition below guarantees the existence of sets of Taylor–Wiles primes suitable for patching, generalizing [BHKT19, Proposition 5.19]. Actually there are some typos in the statement and proof of [BHKT19, Proposition 5.19]; the argument below can be used to rectify them.

Proposition 3.9. *Suppose that the group* $\overline{\rho}(\Gamma_{K(\zeta_\ell)}) \subset \widehat{G}(k)$ *is* $\widehat{G}$*-abundant. Let* $\mathcal{D} := (S, \{\mathcal{D}_v\}_{v \in S})$ *be a global deformation problem such that for each* $v \in S$ *the local deformation*

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problem $\mathcal{D}_v$ is balanced, with corresponding subspace $L_v \subset H^1(\Gamma_{K_v}, \mathrm{ad}(\overline{\sigma}))$. Then for each $j \geq 1$, there is a Taylor–Wiles datum $(Q, \{\varphi_v\}_{v\in Q})$, satisfying the following conditions:

(i) *$Q \cap S = \emptyset$, and for each $v \in Q$, we have $q_v \equiv 1$ mod ℓ^j, and*
$$\#Q = h^1_{\mathcal{D}}(\Gamma_{K,S}, \mathrm{ad}(\overline{\sigma})) = h^1_{\mathcal{D}^\perp}(\Gamma_{K,S}, \mathrm{ad}(\overline{\sigma})^\vee(1)).$$
(ii) *We have $h^1_{\mathcal{D}_Q}(\Gamma_{K,S\cup Q}, \mathrm{ad}(\overline{\sigma})^\vee(1)) = 0$.*
(iii) *There exists a surjection $\mathcal{O}[\![X_1, \dots, X_g]\!] \twoheadrightarrow R_{\overline{\sigma},S\cup Q,\mathcal{D}}$ with $g = h^1_{\mathcal{D}}(\Gamma_{K,S}, \mathrm{ad}(\overline{\sigma})) + (r-1)\#Q$, where $r = \operatorname{rank} \widehat{G}$.*

Proof. If $S \cup Q = \emptyset$ then (i) and (ii) are vacuous and (iii) is an immediate consequence of Proposition 3.6(ii). For the rest of the proof, we assume that $S \cup Q$ is non-empty.

We apply [Čes15, Theorem 6.2], using that $H^0(\Gamma_{K,S\cup Q}, \mathrm{ad}(\overline{\sigma})(1)) = H^0(\Gamma_{K,S\cup Q}, \mathrm{ad}(\overline{\sigma})) = 0$ by the abundance assumption, and the local and global Euler characteristic formulas (note our assumption that $S \cup Q$ is non-empty) to deduce that
$$h^1_{\mathcal{D}_Q}(\Gamma_{K,S\cup Q}, \mathrm{ad}(\overline{\sigma})) - h^1_{\mathcal{D}^\perp_Q}(\Gamma_{K,S\cup Q}, \mathrm{ad}(\overline{\sigma})^\vee(1)) = \sum_{v\in S}(\dim \mathcal{L}_v - h^0(\Gamma_v, \mathrm{ad}(\overline{\sigma}))) + \sum_{v\in Q} r. \quad (3.3)$$
The assumption that $\mathcal{D}_v$ is balanced for all $v \in S$ implies that $\dim \mathcal{L}_v - h^0(G_{F_v}, \mathrm{ad}(\overline{\sigma})) = 0$ for all $v \in S$. We claim that once we can arrange (i) and (ii), then (iii) follows automatically. Indeed, if $h^1_{\mathcal{D}^\perp_Q}(\Gamma_{K,S\cup Q}, \mathrm{ad}(\overline{\sigma})^\vee(1))$ vanishes then (3.3) implies that
$$h^1_{\mathcal{D}_Q}(\Gamma_{K,S\cup Q}, \mathrm{ad}(\overline{\sigma})) = r\#Q = h^1_{\mathcal{D}}(\Gamma_{K,S}, \mathrm{ad}(\overline{\sigma})) + (r-1)\#Q,$$
from which (iii) follows by invoking Proposition 3.6.

It therefore remains to show that Q can be chosen so that
$$H^1_{\mathcal{D}^\perp_Q}(\Gamma_{K,S\cup Q}, \mathrm{ad}(\overline{\sigma})^\vee(1))$$
$$:= \ker\left(H^1(\Gamma_{K,S\cup Q}, \mathrm{ad}(\overline{\sigma})^\vee(1)) \to \left(\bigoplus_{v\in S} \frac{H^1(K_v, \mathrm{ad}(\overline{\sigma})^\vee(1))}{L_v} \oplus \bigoplus_{v\in Q} H^1(K_v, \mathrm{ad}(\overline{\sigma})^\vee(1))\right)\right)$$
vanishes. By a comparison of inflation-restriction exact sequences, the inflation map $H^1(\Gamma_{K,S}, \mathrm{ad}(\overline{\sigma})^\vee(1)) \to H^1(\Gamma_{K,S\cup Q}, \mathrm{ad}(\overline{\sigma})^\vee(1))$ takes
$$\ker\left(H^1(\Gamma_{K,S}, \mathrm{ad}(\overline{\sigma})^\vee(1)) \to \bigoplus_{v\in Q} H^1(k(v), \mathrm{ad}(\overline{\sigma})^\vee(1))\right)$$
$$\xrightarrow{\sim} \ker\left(H^1(\Gamma_{K,S\cup Q}, \mathrm{ad}(\overline{\sigma})^\vee(1)) \to \bigoplus_{v\in Q} H^1(K_v, \mathrm{ad}(\overline{\sigma})^\vee(1))\right).$$
Therefore it suffices by induction to show that for any j and any non-zero $[\psi] \in H^1(\Gamma_{K,S}, \widehat{\mathfrak{g}}^\vee_k(1))$, we can find infinitely many places $v \notin S$ of K such that $q_v \equiv 1 \bmod \ell^j$, $\overline{\rho}(\mathrm{Frob}_v)$ is regular semisimple with connected centralizer in $\widehat{G}_k$, and $\mathrm{res}_{K_v}[\psi] \neq 0$. The rest of the argument concludes exactly as in [BHKT19, Proof of Proposition 5.19]. □

4. Automorphy lifting

In this section we prove the automorphy lifting theorem that will be used to deduce cyclic base change. Such automorphy lifting theorems have been the subject of much study over number fields, and our proof employs similar techniques, but we are able to obtain results for more

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general groups thanks to the different numerical behavior of the Euler characteristics of function fields.

To put our results in context, we compare them to [BHKT19, Theorem 8.20], which handled everywhere unramified Galois representations. The main novelty of the present situation is that we need to implement automorphy lifting for arbitrarily ramified residual representations. In this section we use the Taylor–Wiles method to prove an automorphy lifting theorem (Theorem 4.6) whenever one knows that all the local deformation problems are well behaved (e.g. formally smooth). In the unramified case it was easy to see that the unramified local deformation problem is formally smooth; in the next section we give a suitable extension to unrestricted deformations for ramified representations, when ℓ is large enough.

4.1 Spaces of automorphic forms

We define integral spaces of automorphic forms as in [BHKT19, § 8.1]. For any open subgroup $U \subset G(\widehat{\mathcal{O}})$ and any $\mathbb{Z}[1/p]$-algebra R, we define:

- $C(U, R)$ to be the R-module of functions $f : G(K)\backslash G(\mathbb{A}_K)/U \to R$;
- $C_c(U, R) \subset C(U, R)$ to be the R-submodule of functions f which have finite support; and
- $C_{\mathrm{cusp}}(U, R)$ to be the R-submodule of functions f which are cuspidal, in the sense that for all proper parabolic subgroups $P \subset G$ and for all $g \in G(\mathbb{A}_K)$, the integral

$$\int_{n \in N(K)\backslash N(\mathbb{A}_K)} f(ng)\, dn$$

vanishes, where N is the unipotent radical of P.

This last integral is normalized by endowing $N(K)\backslash N(\mathbb{A}_K)$ with its probability Haar measure (which makes sense because we are assuming that p is a unit in R). We have $C_{\mathrm{cusp}}(U, R) \subset C_c(U, R)$ by [BHKT19, Proposition 8.2].

We define $C_{\mathrm{cusp}}(G, R) := \varinjlim_U C_{\mathrm{cusp}}(U, R)$.

Let $N = \sum_v n_v \cdot v \subset X$ be an effective divisor and let $U(N) := \ker(\prod_v G(\mathcal{O}_{K_v}) \to G(\mathcal{O}_N))$. The underlying set of places $|N| = \bigcup_{n_v > 0}\{v\}$ will play the role of the set S from § 3.

4.2 The excursion algebra and Lafforgue's parametrization

For the remainder of this section we choose a prime $\ell \neq p$ and coefficient field $E \subset \overline{\mathbb{Q}}_\ell$. For $R \in \{k, \mathcal{O}, E\}$ we denote by $\mathcal{B}(U(N), R)$ the R-subalgebra of $\mathrm{End}_R(C_{\mathrm{cusp}}(U(N), R))$ generated by V. Lafforgue's *excursion operators*, as in [BHKT19, § 8.4]. For any fixed N, this is a finite R-algebra. The Hecke operators $T_{V,v}$ as lying in $\mathcal{B}(U(N), R)$, for $v \notin |N|$ and V a representation of $\widehat{G}$, are defined as in [BHKT19].

The points of $\mathrm{Spec}\, \mathcal{B}(U(N), R)$ are naturally identified with semisimple L-parameters, in a manner that we will presently review. Combining [BHKT19, Corollaries 8.6, 8.11], we have the following (with notation as in those corollaries).

Theorem 4.1. *Let $R = E$ or k and $\mathfrak{m} \subset \mathcal{B}(U(N), R)$ be a maximal ideal. Fix an embedding $\mathcal{B}(U(N), R)/\mathfrak{m} \hookrightarrow \overline{R}$. Then the following statements hold:*

(i) *There exists a continuous, absolutely $\widehat{G}$-completely reducible representation $\sigma_{\mathfrak{m}} : \Gamma_K \to \widehat{G}(\overline{R})$ satisfying the following condition: for all excursion operators $S_{I,(\gamma_i)_{i\in I},f}$, we have*

$$f((\sigma_{\mathfrak{m}}(\gamma_i))_{i\in I}) = S_{I,(\gamma_i)_{i\in I},f} \bmod \mathfrak{m}. \tag{4.1}$$

(ii) *The representation $\sigma_{\mathfrak{m}}$ is uniquely determined up to $\widehat{G}(\overline{R})$-conjugacy by (4.1).*

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(iii) *The representation $\sigma_{\mathfrak{m}}$ is unramified outside $|N|$. If $v \notin |N|$, then it satisfies the expected local–global compatibility relation at v: for all irreducible representations V of $\widehat{G}_{\overline{\mathbb{Q}}_\ell}$, we have $T_{V,v} \in \mathcal{B}(U(N), R)$ and*

$$\chi_V(\sigma_{\mathfrak{m}}(\mathrm{Frob}_v)) = T_{V,v} \in \mathcal{B}(U(N), R)/\mathfrak{m}.$$

For $R \in \{k, E\}$ and any N, V. Lafforgue constructs a decomposition of $C_{\mathrm{cusp}}(U(N), \overline{R})$ into summands indexed by L-parameters $\sigma \in H^1(\Gamma_{K,N}, \widehat{G}(\overline{R}))$. More specifically, this decomposition comes from the generalized eigenspace decomposition for the action of $\mathcal{B}(U(N), R)$ on $C_{\mathrm{cusp}}(U(N), R)$. If $\Pi \subset C_{\mathrm{cusp}}(G, \overline{R})$ is a subspace such that $\Pi^{U(N)}$ is stable under the $\mathcal{B}(U(N), \overline{R})$-action and supported over a unique maximal ideal, we will denote this maximal ideal by $\mathfrak{m} = \mathfrak{m}_\Pi \subset \mathcal{B}(U(N), \overline{R})$. The corresponding L-parameter is then denoted $\sigma_\Pi := \sigma_{\mathfrak{m}}$, and we say that σ_Π is the L-parameter *attached* to Π. This assignment $\Pi \mapsto \sigma_\Pi$ is independent of the choice of N such that $\Pi^{U(N)} \neq 0$. This correspondence has the property that Π and σ_Π match under the local Langlands correspondence at all places where Π is unramified, although we caution that this property does not characterize it for general groups.

Remark 4.2 (A one-to-finite correspondence). We continue to assume $R \in \{k, E\}$. Not every irreducible $G(\mathbb{A})$-subspace Π of $C_{\mathrm{cusp}}(G, R)$ is attached to an L-parameter σ_Π in the sense of the previous paragraph. The notation σ_Π will only be used if Π is attached to σ_Π in the manner of the above paragraph. However, we do have the following construction which produces a finite set of L-parameters, each of which matches Π locally at all unramified places.

Let $C_{\mathrm{cusp}}(G, R)[\Pi]$ be the Π-isotypic subspace of $C_{\mathrm{cusp}}(G, R)$ for the $G(\mathbb{A}_K)$-action. Then the space $C_{\mathrm{cusp}}(G, R)[\Pi]^{U(N)}$ is stable under the action of $\mathcal{B}(U(N), R)$, and finite-dimensional for every N. We define $\underline{\rho}_\Pi$ to be the *finite set* of L-parameters corresponding to the maximal ideals of $\mathcal{B}(U(N), R)$ in the support of $C_{\mathrm{cusp}}(G, R)[\Pi]^{U(N)}$. Since each $\sigma \in \underline{\rho}_\Pi$ is associated to an automorphic representation isomorphic to Π at all local places, in particular the unramified ones, we have that each ρ matches Π at every unramified place under the (unramified) local Langlands correspondence.

We note that by [BHKT19, Proposition 6.4], if some $\sigma \in \underline{\rho}_\Pi$ has Zariski dense image in $\widehat{G}$, then in fact $\underline{\rho}_\Pi$ is a singleton, so $\underline{\rho}_\Pi = \{\sigma_\Pi\}$.

In § 5 we will need to work with collections of representations with $\overline{\mathbb{Q}}_\ell$ or $\overline{\mathbb{F}}_\ell$ coefficients, as ℓ varies. We may therefore write $\sigma_{\Pi,\ell}$ for σ_Π and $\overline{\sigma}_{\Pi,\ell}$ for $\overline{\sigma}_\Pi$ when it is necessary to specify the coefficients.

Definition 4.3. Let Π be a cuspidal automorphic representation of $G(\mathbb{A}_K)$ defined over a number field L. We say that the *Mumford–Tate group of* Π *is* $\widehat{G}$ if for some (equivalently, every) prime $\ell \neq p$ and embedding $L \to \overline{\mathbb{Q}}_\ell$, the image of $\sigma_{\Pi,\ell}$ is Zariski dense in $\widehat{G}$. As explained in Remark 4.2, this notion does not depend on the choice of $\sigma_{\Pi,\ell} \in \underline{\rho}_{\Pi \otimes_L \overline{\mathbb{Q}}_\ell}$.

4.3 Automorphy lifting theorems

Let $\chi\colon \mathcal{B}(U(N), \mathcal{O}) \to \overline{\mathbb{Z}}_\ell$ be a homomorphism. After possibly enlarging E, we can assume that χ takes values in $\mathcal{O}$. Let $\mathfrak{m} \subset \mathcal{B}(U(N), \mathcal{O})$ be the maximal ideal which is the kernel of the composition of χ with $\mathcal{O} \to k$, and $\overline{\sigma}_{\mathfrak{m}}$ the Galois representation corresponding to $\mathfrak{m}$ under Theorem 4.1. After possibly further enlarging E, we can assume that $\overline{\sigma}_{\mathfrak{m}}$ takes values in $\widehat{G}(k)$.

With Definition 3.7 in mind, we make the following assumptions on $\overline{\sigma}_{\mathfrak{m}}$:

(i) $\ell \nmid \#W$. This implies, in particular, that ℓ is a very good characteristic for $\widehat{G}$.
(ii) The subgroup $Z_{\widehat{G}^{\mathrm{ad}}}(\overline{\sigma}_{\mathfrak{m}}(\Gamma_{K,|N|}))$ of $\widehat{G}_k$ is scheme-theoretically trivial.

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(iii) The representation $\overline{\sigma}_{\mathfrak{m}}$ is absolutely $\widehat{G}$-irreducible.
(iv) The subgroup $\overline{\sigma}_{\mathfrak{m}}(\Gamma_{K(\zeta_\ell),|N|})$ of $\widehat{G}(k)$ is $\widehat{G}$-abundant.

In addition, we choose a global deformation problem $\mathcal{D} := (|N|, \{\mathcal{D}_v\}_{v\in|N|})$ for $\overline{\sigma}_{\mathfrak{m}}$ such that all places of ramification for $\overline{\sigma}_{\mathfrak{m}}$ are contained in $|N|$, and:

(v) For each $v \in |N|$, the local deformation problem $\mathcal{D}_v$ is *balanced* in the sense of Definition 3.4 and unrestricted in the sense of Example 3.5. (So we require that $H^2(\Gamma_{K_v}, \operatorname{ad}\overline{\sigma}) = 0$ for $v \in |N|$.)

Remark 4.4. Points (i), (ii), and (iv) are copied from [BHKT19, § 8.4], where their roles are explained. Point (iii) is a weakening of the 'strongly $\widehat{G}$-irreducible' condition of [BHKT19, § 8.4], which we are able to avoid here by appeal to the results of [Xue20b]. Point (v) allows us to invoke Proposition 3.9 to produce Taylor–Wiles primes for patching.

Let $R_{\overline{\sigma}_{\mathfrak{m}},|N|,\mathcal{D}}$ be the corresponding global deformation ring (Proposition 3.6). From [BHKT19, Theorem 8.5] we have a pseudocharacter $\Theta_{U(N)} = (\Theta_{U(N),n})_{n\geq 1}$ valued in $\mathcal{B}(U(N),\mathcal{O})$, which factors through the quotient $\Gamma_K \to \Gamma_{K,|N|}$. Write $\Theta_{U(N),\mathfrak{m}}$ for the projection of the pseudocharacter $\Theta_{U(N)}$ to $\mathcal{B}(U(N),\mathcal{O})_{\mathfrak{m}}$, and $\sigma^{\mathrm{univ}}\colon \Gamma_K \to \widehat{G}(R_{\overline{\sigma}_{\mathfrak{m}},|N|,\mathcal{D}})$ for a representative of the universal deformation.

The following lemma is a variant of [BHKT19, Lemma 8.19].

Lemma 4.5. *There is a unique morphism $f_{\mathfrak{m}} : R_{\overline{\sigma}_{\mathfrak{m}},|N|,\mathcal{D}} \to \mathcal{B}(U(N),\mathcal{O})_{\mathfrak{m}}$ of $\mathcal{O}$-algebras such that $f_{\mathfrak{m},*}\operatorname{tr}\sigma^{\mathrm{univ}} = \Theta_{U(N),\mathfrak{m}}$. It is surjective.*

Proof. The existence and uniqueness of $f_{\mathfrak{m}}$ follow from [BHKT19, Theorem 4.10]. It is surjective because the ring $\mathcal{B}(U(N),\mathcal{O})_{\mathfrak{m}}$ is generated by the excursion operators $S_{I,(\gamma_i)_{i\in I},f}$, each of which is explicitly realized as the image of the element $f(\sigma^{\mathrm{univ}}(\gamma_i)_{i\in I}) \in R_{\overline{\sigma}_{\mathfrak{m}},|N|,\mathcal{D}}$. □

We can now prove the following generalization of [BHKT19, Theorem 8.20] that allows for ramification.

Theorem 4.6. *Suppose $\mathfrak{m}$ satisfies assumptions (i)–(v) as above. Then $C_{\mathrm{cusp}}(U(N),\mathcal{O})_{\mathfrak{m}}$ is a free $R_{\overline{\sigma}_{\mathfrak{m}},|N|,\mathcal{D}}$-module, $f_{\mathfrak{m}}$ is an isomorphism, and $R_{\overline{\sigma}_{\mathfrak{m}},|N|,\mathcal{D}}$ is a complete intersection $\mathcal{O}$-algebra.*

Proof. The proof is an implementation of the Taylor–Wiles method. The argument runs similarly to [BHKT19, Theorem 8.20], but we give a summarized version with the necessary changes for our situation, and only sketching the parts which are the same.

Lemma 4.7. *Let $U = \prod_v U_v$ be an open compact subgroup of $G(\widehat{\mathcal{O}}_K)$. Let $V = \prod_v V_v \subset U$ be an open normal subgroup such that U/V is abelian of ℓ-power order. Let v_0 be a place of K, and let ℓ^M denote the order of an ℓ-Sylow subgroup of $G(\mathbb{F}_{q_{v_0}})$. Let $V \leq W \leq U$ be a subgroup such that $(U/V)[\ell^M] \leq W/V$. Finally, $\mathfrak{m} \subset \mathcal{B}(W,\mathcal{O})$ be a maximal ideal such that $\overline{\sigma}_{\mathfrak{m}}$ is absolutely $\widehat{G}$-irreducible. Then $C_{\mathrm{cusp}}(W,\mathcal{O})_{\mathfrak{m}}$ is a finite free $\mathcal{O}[U/W]$-module.*

Proof. In the case that $\overline{\sigma}_{\mathfrak{m}}$ is strongly $\widehat{G}$-irreducible, this is exactly [BHKT19, Theorem 8.17]. Without this assumption, [BHKT19, Lemma 8.18] shows that $C_c(W,\mathcal{O})$ is a free $\mathcal{O}[U/W]$-module (though of infinite rank). Let u be a place of K such that $W_u = G(\mathcal{O}_{K_u})$, and let $\mathcal{H}_{G,u}$ denote the unramified Hecke algebra at u with $\mathcal{O}$-coefficients. Then $C_c(W,\mathcal{O})$ is a finite $\mathcal{H}_{G,u}$-module [Xue20b, Theorem 0.0.3]. According to [Xue20b, § 8], it is possible to define an $\mathcal{O}$-subalgebra $\mathcal{B}_c(W,\mathcal{O}) \leq \mathrm{End}_{\mathcal{O}}(C_c(W,\mathcal{O}))$ of excursion operators, leaving invariant $C_{\mathrm{cusp}}(W,\mathcal{O})$, and extending $\mathcal{B}(W,\mathcal{O})$. Moreover, this ring contains the image of $\mathcal{H}_{G,u}$ in $\mathrm{End}_{\mathcal{O}}(C_c(W,\mathcal{O}))$, showing that $\mathcal{B}_c(W,\mathcal{O})$ is a finite $\mathcal{H}_{G,u}$-algebra and that $C_c(W,\mathcal{O})$ is a finite $\mathcal{B}_c(W,\mathcal{O})$-module.

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Let us identify $\mathfrak{m}$ with its pre-image in $\mathcal{B}_c(W,\mathcal{O})$ under the canonical surjection $\mathcal{B}_c(W,\mathcal{O}) \to \mathcal{B}(W,\mathcal{O})$ (given by restricting excursion operators to $C_{\mathrm{cusp}}(W,\mathcal{O})$). To prove the lemma, it is enough to show that $C_{\mathrm{cusp}}(W,\mathcal{O})_{\mathfrak{m}}$ is a direct summand $\mathcal{B}_c(W,\mathcal{O})$-submodule of the free $\mathcal{O}[U/W]$-module $C_c(W,\mathcal{O})$. Using the constructions of [Xue20b], we see that there is a short exact sequence, respecting the action of excursion operators:

$$0 \to C_{\mathrm{cusp}}(W,\mathcal{O}) \to C_c(W,\mathcal{O}) \overset{\prod_{P\leq G} c_P}{\to} \prod_{P\leq G} C(U(\mathbb{A}_K)M(K)\backslash G(\mathbb{A}_K/W,\mathcal{O}),$$

where the sum ranges over a set of representatives for the conjugacy classes of proper parabolic subgroups $P = MU$ of G, and c_P denotes the corresponding constant term morphism. Let $\mathscr{C}$ denote the image of $C_c(W,\mathcal{O})$ under $\prod_{P\leq G} c_P$, a $\mathcal{B}_c(W,\mathcal{O})$-module. The proof will be complete if we can show that $\mathscr{C}_{\mathfrak{m}} = 0$. This follows from the absolute $\widehat{G}$-irreducibility of $\overline{\sigma}_{\mathfrak{m}}$. Indeed, if $\mathscr{C}_{\mathfrak{m}} \neq 0$, then the compatibility of excursion operators with constant term morphisms ([Xue20b, § 8.4]) shows, along exactly the same lines as in [Xue20a, § 4.2], that $\overline{\sigma}_{\mathfrak{m}}$ must factor through $\hat{M}(\overline{\mathbb{F}}_\ell)$ for some proper parabolic $P = MU$ of G. This is a contradiction to the absolute $\widehat{G}$-irreducibility of $\overline{\sigma}_{\mathfrak{m}}$. □

We begin by preparing the usual setup for patching. Recall that we have fixed a choice $T \subset B \subset G$ of split maximal torus and Borel subgroup of G. For a Taylor–Wiles datum $(Q, \{\varphi_v\}_{v\in Q})$, as in § 3.5, with Q disjoint from $|N|$, we introduce the following notation. Extend the global deformation problem $\mathcal{D} = (|N|, \{\mathcal{D}_v\}_{v\in|N|})$ to $\mathcal{D}_Q := (|N|\cup Q, \{\mathcal{D}_{Q,v}\}_{v\in|N|\cup Q})$ by setting $\mathcal{D}_{Q,v} = \mathcal{D}_v$ if $v \in |N|$ and $\mathcal{D}_{Q,v} = \mathcal{D}^{\square}_{\overline{\sigma},v}$ (the unrestricted deformation problem) if $v \in Q$. Define Δ_Q to be the maximal ℓ-power order quotient of the group $\prod_{v\in Q} T(k(v))$. Using local class field theory, the action of the universal deformation ring for the restriction of $\overline{\sigma}_{\mathfrak{m}}$ to the tame inertia groups at places in Q equips $R_{\overline{\sigma}_{\mathfrak{m}},|N|\cup Q,\mathcal{D}_Q}$ with a $\mathcal{O}[\Delta_Q]$-algebra structure. Writing $\mathfrak{a}_Q \subset \mathcal{O}[\Delta_Q]$ for the augmentation ideal, we have a canonical isomorphism $R_{\overline{\sigma}_{\mathfrak{m}},|N|\cup Q,\mathcal{D}_Q}/\mathfrak{a}_Q \cong R_{\overline{\sigma}_{\mathfrak{m}},|N|,\mathcal{D}}$.

We now define the relevant level structures for patching. We define open compact subgroups $U_1(Q) \subset U_0(Q) \subset U(N)$ as follows:

- $U_0(Q) = \prod_v U_0(Q)_v$, where $U_0(Q)_v = U(N)_v = \ker(G(\mathcal{O}_{K_v}) \to G(\mathcal{O}_{n_v\cdot v}))$ if $v \notin Q$, and $U_0(Q)_v$ is an Iwahori group if $v \in Q$;
- $U_1(Q) = \prod_v U_1(Q)_v$, where $U_1(Q)_v = U(N)_v$ if $v \notin Q$, and $U_1(Q)_v$ is the maximal pro-prime-to-ℓ subgroup of $U_0(Q)_v$ if $v \in Q$.

Thus $U_1(Q) \triangleleft U_0(Q)$ is a normal subgroup, and there is a canonical isomorphism $U_0/U_1 \cong \Delta_Q$.

We fix a place v_0 of K, and write ℓ^M for the order of the ℓ-Sylow subgroup of $G(\mathbb{F}_{q_{v_0}})$.

We now need to define auxiliary spaces of modular forms. We define $H'_0 = C_{\mathrm{cusp}}(U(N),\mathcal{O})_{\mathfrak{m}}$. There are surjective maps $\mathcal{B}(U_1(Q),\mathcal{O}) \twoheadrightarrow \mathcal{B}(U_0(Q),\mathcal{O}) \twoheadrightarrow \mathcal{B}(U(N),\mathcal{O})$, and we write $\mathfrak{m}$ as well for the pullback of $\mathfrak{m} \subset \mathcal{B}(U(N),\mathcal{O})$ to these two algebras. As in Lemma 4.5, we have surjective morphisms

$$R_{\overline{\sigma}_{\mathfrak{m}},|N|\cup Q,\mathcal{D}_Q} \twoheadrightarrow \mathcal{B}(U_1(Q),\mathcal{O})_{\mathfrak{m}} \twoheadrightarrow \mathcal{B}(U_0(Q),\mathcal{O})_{\mathfrak{m}}.$$

We define $H'_{Q,1} := C_{\mathrm{cusp}}(U_1(Q),\mathcal{O})_{\mathfrak{m}}$ and $H'_{Q,0} := C_{\mathrm{cusp}}(U_0(Q),\mathcal{O})_{\mathfrak{m}}$.

We discuss how to cut H'_0 out of $H'_{Q,0}$ as a direct summand, and an analogous construction for $H'_{Q,1}$. This will make use of the Hecke algebras $\mathcal{H}_{U_0(Q),v}$ at a place $v \in Q$ (where it is the Iwahori–Hecke algebra by our choice of level structure) and $\mathcal{H}_{U_1(Q),v}$. The 'translation part' of Bernstein's presentation of the Iwahori–Hecke algebra is an embedding $\mathcal{O}[X_*(T)] \to \mathcal{H}_{U_0(Q)_v}$,

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whose action on $H'_{Q,0}$ for $v \in Q$ induces an $R_{\overline{\sigma}_{\mathfrak{m}},|N|\cup Q,\mathcal{D}_Q}[\prod_{v\in Q} X_*(T)]$-module structure on $H'_{Q,0}$. We write $\mathfrak{n}_{Q,0} \subset \mathcal{O}[\prod_{v\in Q} X_*(T)]$ for the maximal ideal which is associated to the tuple of characters ($v \in Q$):

$$\varphi_v^{-1} \circ \overline{\sigma}_{\mathfrak{m}}|_{W_{K_v}} : W_{K_v} \to \widehat{T}(k). \tag{4.2}$$

Then $H'_{Q,0,\mathfrak{n}_{Q,0}}$ is a direct factor $R_{\overline{\sigma}_{\mathfrak{m}},|N|\cup Q,\mathcal{D}_Q}$-module of $H'_{Q,0}$, and there is a canonical isomorphism $H'_{Q,0,\mathfrak{n}_{Q,0}} \cong H'_0$ of $R_{\overline{\sigma}_{\mathfrak{m}},|N|\cup Q,\mathcal{D}_Q}$-modules.[2]

Similarly, for $v \in Q$ we write $T(K_v)_\ell$ for the quotient of $T(K_v)$ by its maximal pro-prime-to-ℓ subgroup. Then there is a structure of $R_{\overline{\sigma}_{\mathfrak{m}},|N|\cup Q,\mathcal{D}_Q}[\prod_{v\in Q} T(K_v)_\ell]$-module on $H'_{Q,1}$, where the copy of $T(K_v)_\ell$ corresponding to $v \in Q$ acts via the analogous embedding $\mathcal{O}[T(K_v)_\ell] \to \mathcal{H}_{U_1(Q)_v}$. We write $\mathfrak{n}_{Q,1} \subset \mathcal{O}[\prod_{v\in Q} T(K_v)_\ell]$ for the maximal ideal which is associated to the tuple of characters (4.2). Then $H'_{Q,1,\mathfrak{n}_{Q,1}}$ is a direct factor $R_{\overline{\sigma}_{\mathfrak{m}},|N|\cup Q,\mathcal{D}_Q}$-module of $H'_{Q,1}$, and the two structures of $\mathcal{O}[\Delta_Q]$-module on $H'_{Q,1,\mathfrak{n}_{Q,1}}$, one arising from the homomorphism $\mathcal{O}[\Delta_Q] \to R_{\overline{\sigma}_{\mathfrak{m}},|N|\cup Q,\mathcal{D}_Q}$ and the other from the homomorphism $\mathcal{O}[\Delta_Q] \to \mathcal{O}[\prod_{v\in Q} T(K_v)_\ell]$, are the same.

We need two key properties of the modules $H'_{Q,1}$ (with fixed j as in Proposition 3.9):

- The natural inclusion
$$C_{\mathrm{cusp}}(U_0(Q),\mathcal{O}) \subset C_{\mathrm{cusp}}(U_1(Q),\mathcal{O})$$
induces an identification $H'_{Q,0,\mathfrak{n}_{Q,0}} = (H'_{Q,1,\mathfrak{n}_{Q,1}})^{\Delta_Q}$.
- By Lemma 4.7, we then have that if $q_v \equiv 1 \bmod \ell^{M'}$ for each $v \in Q$ and for some $M' \geq M$, then $(H'_{Q,1})^{\ell^j\Delta_Q}$ is a free $\mathcal{O}[\Delta_Q/\ell^j\Delta_Q]$-module for each $0 \leq j \leq M' - M$. This property implies in turn that $(H'_{Q,1,\mathfrak{n}_{Q,1}})^{\ell^j\Delta_Q}$ is a free $\mathcal{O}[\Delta_Q/\ell^j\Delta_Q]$-module. Observe that $\Delta_Q/\ell^j\Delta_Q \cong (\mathbb{Z}/\ell^j\mathbb{Z})^{\oplus r\#Q}$, where $r = \operatorname{rank}\widehat{G}$.

For patching it is a bit more convenient to work with the modules $H_Q := \mathrm{Hom}_{\mathcal{O}}((H'_{Q,1,\mathfrak{n}_{Q,1}})^{\ell^j\Delta_Q},\mathcal{O})$ and $H_0 := \mathrm{Hom}_{\mathcal{O}}(H'_0,\mathcal{O})$. These are finite free $\mathcal{O}$-modules, which we endow with their natural structures of $R_{\overline{\sigma}_{\mathfrak{m}},|N|\cup Q,\mathcal{D}_Q} \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O}[\Delta_Q/\ell^j\Delta_Q]$-module and $R_{\overline{\sigma}_{\mathfrak{m}},|N|,\mathcal{D}_Q}$-module, respectively, by dualization. We can summarize the preceding discussion as follows:

- The module H_Q is a finite free $\mathcal{O}[\Delta_Q/\ell^j\Delta_Q]$-module, where $\mathcal{O}[\Delta_Q/\ell^j\Delta_Q]$ acts via the algebra homomorphism
$$\mathcal{O}[\Delta_Q/\ell^j\Delta_Q] \to R_{\overline{\sigma}_{\mathfrak{m}},|N|\cup Q,\mathcal{D}_Q} \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O}[\Delta_Q/\ell^j\Delta_Q].$$
- There is a natural surjective map $H_Q \twoheadrightarrow H_0$, which factors through an isomorphism $(H_Q)_{\Delta_Q} \xrightarrow{\sim} H_0$, and is compatible with the isomorphism $R_{\overline{\sigma}_{\mathfrak{m}},|N|\cup Q,\mathcal{D}_Q}/\mathfrak{a}_Q \cong R_{\overline{\sigma}_{\mathfrak{m}},|N|,\mathcal{D}}$.

Let $h := h^1_{\mathcal{D}}(\Gamma_{K,|N|}, \operatorname{ad}\overline{\sigma})$. By Proposition 3.9, we can find for each $j \geq 1$ a Taylor–Wiles datum $(Q_j, \{\varphi_v\}_{v\in Q_j})$ which satisfies the following conditions:

- Q_j is disjoint from $|N|$.
- For each $v \in Q_N$, we have $q_v \equiv 1 \bmod \ell^{j+M}$ and $\#Q_j = h$.
- There exists a surjection $\mathcal{O}[[x_1,\ldots,x_g]] \twoheadrightarrow R_{\overline{\sigma}_{\mathfrak{m}},|N|\cup Q_j,\mathcal{D}}$, where $g = hr$.

[2] This uses that for $v \in Q$, the stabilizer in the Weyl group of the regular semisimple element $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v) \in \widehat{G}(k)$ is trivial, which is equivalent to the condition that the centralizer in $\widehat{G}_k$ of $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)$ is connected, which is part of the definition of a Taylor–Wiles datum.

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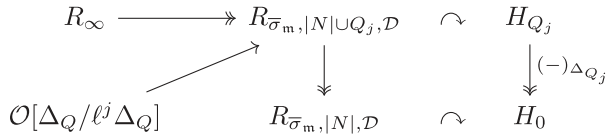
Define $R_\infty = \mathcal{O}[\![X_1, \dots, X_g]\!]$. The situation is summarized in the following diagram.



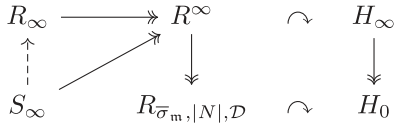
We now patch these objects together. This involves quotienting the objects in the diagram by open ideals to get diagrams of Artinian objects. Then because there are only finitely many isomorphism classes of such diagrams, one can pass to the inverse limit. The details of this process are the same as in [BHKT19, p. 48–49], so we skip to the conclusion. Define $\Delta_\infty := \mathbb{Z}_\ell^g$, $\Delta_j := \Delta_\infty/\ell^j\Delta_\infty$, $S_\infty := \mathcal{O}[\![\Delta_\infty]\!]$, $\mathfrak{b}_j := \ker(S_\infty \to \mathcal{O}[[\Delta_j]])$, and $\mathfrak{b}_0 := \ker(S_\infty \to \mathcal{O})$. By patching, we have the following objects:

- R^∞, a complete Noetherian local $\mathcal{O}$-algebra with residue field k, which is equipped with structures of S_∞-algebra and a surjective map $R_\infty \twoheadrightarrow R^\infty$;
- H_∞, a finite R^∞-module;
- α_∞, an isomorphism $R^\infty/\mathfrak{b}_0 \cong R_{\overline{\sigma}_\mathfrak{m},|N|,\mathcal{D}}$;
- β_∞, an isomorphism $H_\infty/\mathfrak{b}_0 \cong H_0$.

These objects have the following additional properties:

- H_∞ is free as an S_∞-module.
- The isomorphisms α_∞, β_∞ are compatible with the structure of $R_{\overline{\sigma}_\mathfrak{m},|N|,\mathcal{D}}$-module on H_0.

The situation is summarized in the following diagram.



Since S_∞ is topologically free, we may choose the dashed arrow so that the leftmost triangle commutes.

We find that

$$\dim R^\infty \geq \operatorname{depth}_{R^\infty} H_\infty \geq \operatorname{depth}_{S_\infty} H_\infty = \dim S_\infty = \dim R_\infty \geq \dim R^\infty,$$

and hence that these inequalities are equalities, $R_\infty \to R^\infty$ is an isomorphism, and (by the Auslander–Buchsbaum formula) H_∞ is also a free R^∞-module. It follows that $H_\infty/\mathfrak{b}_0 \cong H_0$ is a free $R^\infty/\mathfrak{b}_0 \cong R_{\overline{\sigma}_\mathfrak{m},|N|,\mathcal{D}}$-module, and that $R_{\overline{\sigma}_\mathfrak{m},|N|,\mathcal{D}}$ is an $\mathcal{O}$-flat complete intersection. This in turn implies that $C_{\mathrm{cusp}}(U,\mathcal{O})_\mathfrak{m} \cong \operatorname{Hom}_\mathcal{O}(H_0,\mathcal{O})$ is a free $R_{\overline{\sigma}_\mathfrak{m},|N|,\mathcal{D}}$-module (since complete intersections are Gorenstein). This completes the proof of the theorem. □

5. Formal smoothness of local deformations

This section is devoted to the proof of the following theorem, which guarantees that for all large enough ℓ, the unrestricted local condition will be balanced at a place of ramification for any given cuspidal automorphic representation.

THEOREM 5.1. *Let $\Pi \subset C_{\mathrm{cusp}}(G, \overline{\mathbb{Q}})$ be a cuspidal automorphic representation of $G(\mathbb{A}_K)$ with coefficients in $\overline{\mathbb{Q}}$. For any prime number $\ell \neq p$ let $\Pi_\ell = \Pi \otimes_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}_\ell$ denote its base change to $\overline{\mathbb{Q}}_\ell$. Suppose Π is unramified outside the finite set S, and $\sigma_\ell \in \underline{\rho}_{\Pi,\ell}$ (notation as in Remark 4.2) has Zariski dense image. (As explained in Remark 4.2, this implies* a posteriori *that $\sigma_{\Pi,\ell}$ exists, and*

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then $\sigma_\ell = \sigma_{\Pi,\ell}$.) Then, letting

$$\overline{\sigma}_\ell : \Gamma_{K,S} \to \widehat{G}(\overline{\mathbb{F}}_\ell)$$

be a reduction of σ_ℓ mod ℓ, there is an integer $c(\Pi)$, depending only on Π, such that, for all $v \in S$ and all $\ell > c(\Pi)$, we have

$$H^2(\Gamma_{K_v}, \mathrm{ad}(\overline{\sigma}_\ell)) = 0.$$

We concurrently found two very different proofs of Theorem 5.1, and include them both below, as they both establish general results along the way that may be of independent interest. The proof in § 5.1 is based on a lifting theorem (Theorem 5.6) for global Galois representations, which is analogous to [BGGT14, Theorem 4.3.1]; cf. also [KW09, Corollary 4.7]. The lifting theorem implies that if $H^2(\Gamma_{K_v}, \mathrm{ad}(\overline{\sigma}_\ell)) \neq 0$, then there is another lift of $\overline{\sigma}_\ell$ to characteristic 0 which is 'less ramified' at v than σ_ℓ. By the global Langlands correspondence for GL_n established by Lafforgue [Laf02], such a lift must come from another cuspidal automorphic representation—of which only finitely many contribute at any given level structure—congruent to Π modulo ℓ. The idea is then that there can only be finitely many such congruences, because otherwise strong multiplicity one for GL_n would be violated. We remark that a similar strategy, for an analogous situation over number fields, was used in the proof of [Gui20, Corollary 7.11].

The proof in § 5.2 is of a more geometric nature. We reduce Theorem 5.1 to a torsion version of the weight-monodromy conjecture, formulated by Ito [Ito21], for the compactifications constructed by Laurent Lafforgue to prove the global Langlands correspondence for GL_n. Then we prove this torsion weight-monodromy conjecture for the intersection cohomology of any proper variety over an equal characteristic local field (generalizing work of Ito in the smooth proper case [Ito21]). A crucial tool is an integral version of the decomposition theorem recently proved by Cadoret and Zheng, which we use to reduce to the smooth and proper case established by Ito. This second argument actually yields a more general statement, Theorem 5.9.

Although neither argument gives an effective estimate on the constant $c(\Pi)$, they give different starting points for obtaining such an estimate. (That is one reason why it seems valuable to include both arguments.) In the first approach, what must be controlled is the set of primes at which cusp forms can have congruences to other cusp forms. In the second approach, what must be controlled has to do with the torsion in the integral intersection cohomology of L. Lafforgue's compactifications of moduli spaces of shtukas for GL_n.

5.1 Approach based on lifting theorems for Galois representations

First we show that Theorem 5.1 follows from the same statement when $G = \mathrm{GL}_n$. Pick a faithful irreducible representation of G on a finite projective $\mathbb{Z}$-module, say of rank $n > 1$. This induces an injection $\mathfrak{g} \hookrightarrow \mathfrak{gl}_n$, which is split over $\mathbb{Z}_\ell$ for all sufficiently large ℓ and therefore induces an injection $H^2(\Gamma_{K_v}, \mathrm{ad}(\overline{\sigma}_\ell)) \hookrightarrow H^2(\Gamma_{K_v}, \mathfrak{gl}_n \circ \overline{\sigma}_\ell)$. This reduces the claim to the case $\widehat{G} = \mathrm{GL}_n$, so for the rest of this subsection we focus on the case $G = \mathrm{GL}_n$.

5.1.1 *Some linear algebra.* We recall the following result of Deligne [Del80, Proposition 1.6.1].

Proposition 5.2. *Let k be a field and let V be a finite-dimensional k-vector space, equipped with a nilpotent endomorphism $N : V \to V$. Then there exists a unique increasing filtration $M_\bullet$ of V such that $NM_i \subset M_{i-2}$ and, for each $k \geq 0$, N^k induces isomorphisms $\mathrm{gr}_k M_\bullet \to \mathrm{gr}_{-k} M_\bullet$.*

If N is a nilpotent endomorphism of an n-dimensional vector space, we will write $\mathrm{Jord}(N)$ for the partition of n given by the sizes of the Jordan blocks of N. We recall that there is a

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partial order on partitions n which can be viewed as corresponding to the closure ordering of nilpotent orbits in the adjoint representation of GL_n.

Let v be a place of K, and suppose given a prime $\ell > n$ not dividing q. Let $E/\mathbb{Q}_\ell$ be a finite extension with residue field $k_E = \mathcal{O}_E/(\varpi_E)$, and suppose given a continuous representation $\rho : G_{K_v} \to \mathrm{GL}_n(\mathcal{O}_E)$. Fix a choice of (geometric) Frobenius lift $\phi_v \in G_{K_v}$, and $t_\ell \in G_{K_v}$ a generator for the ℓ-part of tame inertia. Let us say that ρ is *good* if the following conditions are satisfied:

- ρ is unipotently ramified. Let $N = \log(\rho(t_\ell)) \in M_n(\mathcal{O}_E)$; then $\phi_v N \phi_v^{-1} = q^{-1}N$.
- The eigenvalues of $\rho(\phi_v)$ all lie in E.
- ρ is pure of weight 0. This means that the eigenvalues of $\rho(\phi_v)$ on $V = E^n$ are q_v-Weil numbers and that, writing $V_i \subset V$ for the subspace where the eigenvalues are of weight i, N^k induces an isomorphism $V_k \to V_{-k}$ for each $k \geq 0$. (Note that the filtration associated to N by Proposition 5.2 is then $M_k = \bigoplus_{i \leq k} V_i$.)
- Let $f_i(X) = \det(X - \rho(\phi_v)|_{V_i}) \in \mathcal{O}_E[X]$. Then for each $i \neq j$, the reduced polynomials $\overline{f}_i(X), \overline{f}_j(X) \in k_E[X]$ are coprime.
- For each i, j, the polynomials $\overline{f}_i(X)$, $\overline{f}_j(q_v^{-1}X) \in k_E[X]$ fail to be coprime only if $j = i + 2$.

Let $L = \mathcal{O}_E^n$, so that $V = L \otimes_{\mathcal{O}_E} E$, and define $L_i = V_i \cap L$. Hensel's lemma implies that if ρ is good then $L = \bigoplus_{i \in \mathbb{Z}} L_i$.

LEMMA 5.3. *Suppose that ρ is good, and let $\overline{N} \in M_n(k_E)$ denote the reduction of N modulo ϖ_E. If $\mathrm{Jord}(\overline{N}) = \mathrm{Jord}(N)$, then $H^2(K_v, \mathrm{ad}\,\overline{\rho}) = 0$.*

Proof. The equality $\mathrm{Jord}(\overline{N}) = \mathrm{Jord}(N)$ is equivalent to the equalities $\dim_E \ker N^k = \dim_{k_E} \ker \overline{N}^k$ for each $k \geq 0$. Let $\overline{L} = L \otimes_{\mathcal{O}_E} k_E$ and $\overline{L}_i = L_i \otimes_{\mathcal{O}_E} k_E$. We first claim that for each $k \geq 0$, $\overline{N}^k$ induces an isomorphism $\overline{L}_k \to \overline{L}_{-k}$. Equivalently, the intersection $(\ker \overline{N}^k) \cap \overline{L}_k$ is 0.

The equality $\dim_E \ker N^k = \dim_{k_E} \ker \overline{N}^k$ implies that in fact $\ker \overline{N}^k = (\ker N^k \cap L) \otimes_{\mathcal{O}_E} k_E$. Since $\ker N^k$ is invariant under $\mathrm{Ad}\,\rho(\phi_v)$, we find that in fact

$$(\ker \overline{N}^k) \cap \overline{L}_k = (\ker N^k \cap L_k) \otimes_{\mathcal{O}_E} k_E = 0.$$

This establishes the claim. By linear algebra, we then have a decomposition

$$\overline{L} = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j=0}^{i} \overline{N}^j \overline{L}(i),$$

where $\overline{L}(i) = \overline{L}_i \cap (\ker \overline{N}^{i+1})$.

By Tate duality, we need to show that $H^0(K_v, \mathrm{ad}\,\overline{\rho}(1)) = \mathrm{Hom}_{G_{K_v}}(\overline{L}, \overline{L}(1)) = 0$. An element of this Hom space determines a linear map $F : \overline{L} \to \overline{L}$ which commutes with $\overline{N}$ and satisfies $F(\overline{L}_i) \subset \overline{L}_{i+2}$ for each $i \in \mathbb{Z}$. To show the Hom space vanishes, it is enough to show that F annihilates each $\overline{L}(i)$. However, if $x \in \overline{L}(i)$ then we find $F(x) \in \overline{L}_{i+2}$ and

$$\overline{N}^{i+1} F(x) = F(\overline{N}^{i+1} x) = 0.$$

The restriction of $\overline{N}^{i+1}$ to $\overline{L}_{i+2}$ is injective, so this forces $F(x) = 0$, hence $F = 0$, as required. □

5.1.2 *Local lifting rings.* Let $\ell > n$ be a prime not dividing q and let $E/\mathbb{Q}_\ell$ be a finite extension with residue field $k_E = \mathcal{O}_E/(\varpi_E)$. Let v be a place of K and fix a continuous homomorphism $\overline{\rho}_v : G_{K_v} \to \mathrm{GL}_n(k_E)$ and a continuous character $\chi : G_K \to \mathcal{O}_E^\times$ such that $\det \overline{\rho}_v = \chi|_{G_{K_v}}$. Let $\mathrm{CNL}_{\mathcal{O}_E}$ denote the category of complete Noetherian local $\mathcal{O}_E$-algebras with residue field k_E.

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Then the functor L_v^χ which associates to each $A \in \mathrm{CNL}_{\mathcal{O}_E}$ the set of lifts $\rho_{v,A} : G_{K_v} \to \mathrm{GL}_n(A)$ of $\overline{\rho}_v$ of determinant $\chi|_{G_{K_v}}$ is represented by an object $R_v^{\square,\chi} \in \mathrm{CNL}_{\mathcal{O}_E}$. This ring has a pleasant geometry.

Proposition 5.4. *Let assumptions be as above. Then the following statements hold:*

(i) *The ring $R_v^{\square,\chi}$ is a reduced local complete intersection.*
(ii) *For each minimal prime $\mathfrak{q} \subset R_v^{\square,\chi}$, the quotient $R_v^{\square,\chi}/\mathfrak{q}$ is a domain of Krull dimension n^2.*
(iii) *Let $\mathfrak{q} \subset R_v^{\square,\chi}$ be a minimal prime, and let $L_v^{\chi,\mathfrak{q}} \subset L_v^\chi$ be the corresponding subfunctor. Then $L_v^{\chi,\mathfrak{q}}$ is a local deformation problem, in the sense that for each $A \in \mathrm{CNL}_{\mathcal{O}_E}$, the subset $L_v^{\chi,\mathfrak{q}}(A) \subset L_v^\chi(A)$ is invariant under the conjugation action of the group $\ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(k_E))$.*

Proof. The first two points follow from [Yao, Proposition 2.13]. The third is established in [BGGT14, § 1.3] in the case where the determinant is not fixed and K_v is replaced by a finite extension of $\mathbb{Q}_p$ ($p \neq \ell$); the same proof applies here. □

We will use the following lemma.

Lemma 5.5. *With assumptions as above, suppose further that $\ell > n$ and that ρ_v is unipotently ramified. Then there exists a minimal prime $\mathfrak{q} \subset R_v^{\square,\chi}$ such that $R_v^{\square,\chi}/\mathfrak{q}$ is formally smooth over $\mathcal{O}_E$ and such that for every homomorphism $R_v^{\square,\chi}/\mathfrak{q} \to \overline{\mathbb{Q}}_\ell$, corresponding to a lifting $\rho_v : G_{K_v} \to \mathrm{GL}_n(\overline{\mathbb{Z}}_\ell)$ of $\overline{\rho}_v$ of determinant $\chi|_{G_{K_v}}$, the following conditions are satisfied:*

(i) *ρ_v is unipotently ramified.*
(ii) *If $t_\ell \in I_{K_v}(\ell)$ denotes a generator of the ℓ-part of tame inertia, then $\mathrm{Jord}(\log(\rho_v(t_\ell))) = \mathrm{Jord}(\log(\overline{\rho}_v(t_\ell)))$.*

Proof. We can take $R_v^{\square,\chi}/\mathfrak{q}$ to be the functor parametrizing liftings which are minimally ramified, in the sense of [CHT08, Definition 2.4.14], and of determinant $\chi|_{G_{K_v}}$. The claimed properties follow from [CHT08, Lemma 2.4.19]. □

5.1.3 *Existence of lifts.*

Theorem 5.6. *Let $\ell > n$ be a prime such that $(\ell, 2q) = 1$, and let $\overline{\rho} : G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_\ell)$ be a continuous representation such that $\overline{\rho}|_{G_{K \cdot \overline{\mathbb{F}}_q}}$ is irreducible. Let S be a finite set of places of K, containing the places at which $\overline{\rho}$ is ramified, and fix the following data:*

(i) *a finite extension $E/\mathbb{Q}_\ell$ inside $\overline{\mathbb{Q}}_\ell$;*
(ii) *a character $\chi : G_K \to \mathcal{O}_E^\times$ of finite order lifting $\det \overline{\rho}$, unramified outside S;*
(iii) *for each $v \in S$, a minimal prime $\mathfrak{q}$ of the universal lifting ring $R_v^{\square,\chi}$.*

Then we can find a continuous lifting $\rho : G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ of $\overline{\rho}$ satisfying the following conditions:

(i) *$\det \rho = \chi$.*
(ii) *For each place $v \notin S$ of K, $\rho|_{G_{K_v}}$ is unramified.*
(iii) *For each place $v \in S$ of K, $\rho|_{G_{K_v}}$ defines a homomorphism $R_v^{\square,\chi}/\mathfrak{q} \to \overline{\mathbb{Q}}_\ell$.*

Proof. Let $D : \mathrm{CNL}_{\mathcal{O}_E} \to \mathrm{Sets}$ denote the functor of deformations $\rho_A : G_K \to \mathrm{GL}_n(A)$ of $\overline{\rho}$ satisfying the following conditions:

(i) $\det \rho_A = \chi$.
(ii) If $v \in S$, then $\rho_A|_{G_{K_v}}$ defines a map $R_v^{\square,\chi}/\mathfrak{q} \to A$.
(iii) If $v \notin S$, then $\rho_A|_{G_{K_v}}$ is unramified.

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Then D is represented by an object $R \in \mathrm{CNL}_{\mathcal{O}_E}$, and a standard computation in Galois cohomology and using Proposition 5.4 (see [Yao, Corollary 5.4]) shows that R has Krull dimension $\dim R \geq 1$. On the other hand, R is a finite $\mathcal{O}_E$-algebra (see [Yao, Theorem 5.5], where this is deduced from work of de Jong and Gaitsgory, [deJ01, Gai07]). It follows that R is an $\mathcal{O}_E$-flat complete intersection and, in particular, that there exists a homomorphism $R \to \overline{\mathbb{Q}}_\ell$, which implies the existence of a lift $\rho : G_K \to \mathrm{GL}_n(\overline{\mathbb{Z}}_\ell)$ with the desired properties. □

5.1.4 *Minimality.*

Theorem 5.7. *Let π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_K)$ over $\overline{\mathbb{Q}}$ of finite central character and such that the associated compatible system $(r_\lambda(\pi))_\lambda$ is connected, in the sense of [BGP19]. Let S be a finite set of places of K, including the set of places at which π is ramified. Then there exists an integer $N > \max(2, q, n)$ such that for each prime number $\ell > N$ and for each place λ of $\overline{\mathbb{Q}}$ of residue characteristic ℓ, the following conditions are satisfied:*

(i) $\overline{\rho}_\lambda|_{G_{K \cdot \overline{\mathbb{F}}_q}}$ *is irreducible.*
(ii) *For each $v \in S$, $H^2(K_v, \mathrm{ad}\,\overline{\rho}_\lambda) = 0$.*

Proof. Let χ denote the central character of π, which we identify with a character $\chi : G_K \to \overline{\mathbb{Q}}^\times$. After replacing K by a finite Galois extension, we may assume that for each place $v \in S$, π_v is unipotently ramified. We can associate to each $v \in S$ a partition $\mathrm{Jord}(\pi_v)$ of n, namely the one given by the Jordan decomposition of the nilpotent part of the Weil–Deligne representation $\mathrm{rec}_{K_v}(\pi_v)$.

By [BGP19, Corollary 6.12], we can find an integer $N_0 > \max(2, q, n)$ such that for each prime number $\ell > N$ and for each place λ of $\overline{\mathbb{Q}}$ of residue characteristic ℓ, $\overline{\rho}_\lambda|_{G_{K\overline{\mathbb{F}}_q}}$ is irreducible. Since S is finite, it suffices to fix a single $w \in S$ and show that the number of primes $\ell > N_0$ such that there is a place $\lambda|\ell$ of $\overline{\mathbb{Q}}$ such that $H^2(K_w, \mathrm{ad}\,\overline{\rho}_\lambda) \neq 0$ is finite.

Let $L \subset \overline{\mathbb{Q}}$ be a number field such that the compatible system $(r_\lambda(\pi))_\lambda$ is L-rational and such that the roots of $f(X) = \det(X - \mathrm{rec}_{K_w}(\pi_w)(\phi_w))$ lie in L. By the Ramanujan conjecture (established in [Laf02, Theorem VI.10]), $\mathrm{rec}_{K_w}(\pi_w)$ is pure of weight 0, and we may factor $f(X) = \prod_i f_i(X)$, where the roots of $f_i(X)$ are q_w-Weil numbers of weight i. The polynomials $f(X)$, $f_i(X)$ lie in $\mathcal{O}_L[1/q, X]$. If $i \neq j$ then the polynomials $f_i(X), f_j(X)$ are coprime. If $i \neq j+2$ then the polynomials $f_i(X), f_{j+2}(q_w X)$ are coprime. After increasing N_0, we can therefore assume that for each prime $\ell > N_0$ and for each place $\lambda|\ell$ of $\overline{\mathbb{Q}}$, the representation $\rho_\lambda|_{G_{K_w}}$ (conjugated to take values in $\mathrm{GL}_n(\mathcal{O}_E)$ for some $E/\mathbb{Q}_\ell$) is good, in the sense of § 5.1.1.

Lemma 5.8. *For each prime number $\ell > N_0$ and place $\lambda|\ell$ of $\overline{\mathbb{Q}}$ such that $H^2(K_w, \mathrm{ad}\,\overline{\rho}_\lambda) \neq 0$, there exists a cuspidal automorphic representation π' of $\mathrm{GL}_n(\mathbb{A}_K)$ over $\overline{\mathbb{Q}}$ satisfying the following conditions:*

(i) *π' has central character χ and is unramified outside S.*
(ii) *For each place $v \in S$, π'_v is unipotently ramified.*
(iii) $\mathrm{Jord}(\pi'_w) < \mathrm{Jord}(\pi_w)$.
(iv) *There is an isomorphism $\overline{r_\lambda(\pi)} \cong \overline{r_\lambda(\pi')}$.*

Proof. We would like to apply Theorem 5.6. We need to specify a minimal prime of each lifting ring $R_v^{\square,\chi}$ ($v \in S$). If $v \neq w$ then we take any component containing the point corresponding to $r_\lambda(\pi)$. If $v = w$ then we take the minimal lifting ring described in Lemma 5.5. We then apply Theorem 5.6 with these choices to obtain a lift ρ of $\overline{r_\lambda(\pi)}$, which by the global Langlands correspondence for $\mathrm{GL}_n(\mathbb{A}_K)$ corresponds to a cuspidal automorphic representation

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π' of $\mathrm{GL}_n(\mathbb{A}_K)$. The only property that remains to be justified is that $\mathrm{Jord}(\pi'_w) < \mathrm{Jord}(\pi_w)$. By local–global compatibility, we have $\mathrm{Jord}(\pi'_w) = \mathrm{Jord}(\rho|_{G_{K_w}})$ (where we define the Jordan partition of a unipotently ramified representation to be that of the image of a generator of tame inertia). The definition of the minimally ramified deformation problem shows that $\mathrm{Jord}(\rho|_{G_{K_w}}) = \mathrm{Jord}(\overline{r_\lambda(\pi)}|_{G_{K_w}})$. Finally, Lemma 5.3 and our hypothesis that $H^2(K_w, \mathrm{ad}\,\overline{\rho}_\lambda) \neq 0$ show together that $\mathrm{Jord}(\overline{r_\lambda(\pi)}|_{G_{K_w}}) < \mathrm{Jord}(\pi_w)$. □

We now use the preceding lemma to finish the proof of the theorem. The set of cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_K)$ over $\overline{\mathbb{Q}}$ of central character χ, unramified outside S, and unipotently ramified inside S, is finite [Har74, Corollary 1.2.3]. Enlarge the number field L so that each such automorphic representation is in fact defined over L. If the number of places λ of L for which $H^2(K_w, \mathrm{ad}\,\overline{\rho}_\lambda) \neq 0$ is infinite, then by the pigeonhole principle we can find an automorphic representation π' in this set such that for infinitely many places λ of L there is an isomorphism $\overline{r_\lambda(\pi)} \cong \overline{r_\lambda(\pi')}$, and yet $\mathrm{Jord}(\pi'_w) < \mathrm{Jord}(\pi_w)$. In particular, $\pi \neq \pi'$.

If $v \notin S$ is a place of K then, in particular, we find $\det(X - r_\lambda(\pi)(\mathrm{Frob}_v)) \equiv \det(X - r_\lambda(\pi')(\mathrm{Frob}_v)) \bmod \lambda$ for infinitely many places λ, and therefore $\pi_v \cong \pi'_v$. The strong multiplicity-one theorem then implies that $\pi = \pi'$, contradicting the inequality $\mathrm{Jord}(\pi'_w) < \mathrm{Jord}(\pi_w)$. This concludes the proof. □

5.2 Geometric approach based on the torsion weight-monodromy conjecture

We will actually establish the following more general statement, which handles arbitrary compatible systems of irreducible representations (as opposed to specifically the adjoint representation).

Theorem 5.9. *Let $\{\sigma_\ell \colon \Gamma_{K,S} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)\}$ be any compatible system of irreducible n-dimensional representations. For each $v \in |X|$, Grothendieck's quasi-unipotence theorem implies that there is an open subgroup $J_v \subset I_{K_v}$ such that $\sigma_\ell(J_v)$ is unipotent for all $\ell \neq p$. Choose for each $v \in |X|$ a $t_v \in J_v$ projecting to a generator of the pro-ℓ part of tame inertia for all but finitely many $\ell \neq p$. Then all but finitely many primes ℓ have the properties that:*

- *$\overline{\sigma}_\ell$ is irreducible;*
- *for all v, $\sigma_\ell(t_v) - 1$ has torsion-free cokernel on some (equivalently any) lattice.*

5.2.1 *Some reductions.* By assumption G is semisimple; we may reduce to the case where G is simple since the statement of the theorem is compatible with taking finite direct sums.

The representation σ_ℓ is defined over a finite extension $E/\mathbb{Q}_\ell$. Let $\mathcal{O}_E$ be the ring of integers of E, and $\widetilde{\sigma}_\ell$ be a lattice in σ_ℓ, with mod-ℓ reduction $\overline{\sigma}_\ell$.

Lemma 5.10. *If the cokernel of $(\widetilde{\sigma}_\ell(t_v) - 1)^i$ is torsion-free for every i, then $H^2(K_v, \overline{\sigma}_\ell) = 0$.*

Proof. By [Ito21, Lemma 3.7], the torsion-freeness implies that $\mathrm{Jord}(\widetilde{\sigma}_\ell(t_v) - 1) = \mathrm{Jord}(\overline{\sigma}_\ell(t_v) - 1)$ in the notation of Lemma 5.3, and then applying the same proof as for Lemma 5.3 gives the conclusion. □

Lemma 5.11. *Let M be a finite flat $\mathbb{Z}_\ell$-module and γ a $\mathbb{Z}_\ell$-linear endomorphism of M. If $(M \otimes_{\mathbb{Z}_\ell} M)_{\Delta\gamma}$ is torsion-free, then M_γ is torsion-free. (Here $\Delta\gamma$ refers to the diagonal action of γ on the tensor product.)*

Proof. Suppose $m \in M$ represents a non-zero ℓ^k-torsion element of M_γ. Then $m \otimes m \in M \otimes_{\mathbb{Z}_\ell} M$ represents an ℓ^{2k}-torsion element of $(M \otimes_{\mathbb{Z}_\ell} M)_{\Delta\gamma}$. Furthermore, its image under

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$(M \otimes_{\mathbb{Z}_\ell} M)_{\Delta\gamma} \twoheadrightarrow M_\gamma \otimes_{\mathbb{Z}_\ell} M_{\Delta\gamma}$ is non-zero, so this shows that $(M \otimes_{\mathbb{Z}_\ell} M)_{\Delta\gamma}$ has a non-zero torsion element. □

Hence it suffices to show that for all sufficiently large ℓ, the cokernel of $(\widetilde{\sigma}_\ell(t_v) - 1)^i \otimes (\widetilde{\sigma}_\ell(t_v) - 1)^i$ is torsion-free for all i. The point of casting the problem this way is that this tensor product of Galois representations appears in the cohomology of shtukas, and through this we will be able to transform the question into a purely geometric one.

Let $n = \dim \mathfrak{g}$. Laurent Lafforgue constructed a proper Deligne–Mumford stack

$$\overline{\mathrm{Sht}}_N \to (X - N) \times (X - N)$$

(our $\overline{\mathrm{Sht}}_N$ refers to what is called in [Laf02] $\overline{\mathrm{Cht}_N^{n,\overline{P}\leq P}}'/a^{\mathbb{Z}}$ for sufficiently large Harder–Narasimhan polygon P, and appropriate a corresponding to the central character of Π) such that (because σ_ℓ is irreducible) $\Xi \boxtimes (\sigma_\ell \otimes \sigma_\ell)$ occurs, for some[3] Hecke module Ξ, as a Hecke–Galois stable summand of ${}^{\mathfrak{p}}H^0(\overline{\mathrm{Sht}}_N, \mathrm{IC}(E))$ [Laf02, Theorem VI.27], where the latter refers to the zeroth perverse cohomology of the geometric generic fiber of $\overline{\mathrm{Sht}}_N \to (X - N) \times (X - N)$.

Corollary 5.12. *For any $v \in |X|$, for all but finitely many ℓ, there exists N disjoint from v such that $\widetilde{\sigma}_\ell \otimes_{\mathcal{O}_E} \widetilde{\sigma}_\ell$ occurs as a Hecke stable[4] summand of ${}^{\mathfrak{p}}H^0(\overline{\mathrm{Sht}}_N, \mathrm{IC}(\mathcal{O}_E))$ where $\overline{\mathrm{Sht}}_N \to (X - N) \times (X - N)$ is a proper morphism of schemes.*

Proof. Fixing the Harder–Narasimhan polygon p, for all sufficiently large N [Var04, Proposition 2.16] implies that the Deligne–Mumford stack $\overline{\mathrm{Sht}}_N$ is represented by a scheme.

Since σ_ℓ is irreducible, the σ_ℓ-isotypic part of ${}^{\mathfrak{p}}H^0(\overline{\mathrm{Sht}}_N, \mathrm{IC}(\overline{\mathbb{Q}}_\ell))$ occurs as a summand of the finite-dimensional subspace $H^*_{\mathrm{ess}}(\overline{\mathbb{Q}}_\ell) \subset {}^{\mathfrak{p}}H^0(\overline{\mathrm{Sht}}_N, \mathrm{IC}(\overline{\mathbb{Q}}_\ell))$, where the 'essential part' is defined by the condition that its Hecke eigenvalues are cuspidal.

The Hecke operators act on $H^*_{\mathrm{ess}}(\overline{\mathbb{Q}}_\ell)$, and the subalgebra of $\mathrm{End}(H^*_{\mathrm{ess}}(\overline{\mathbb{Q}}_\ell))$ generated by them forms a finite $\overline{\mathbb{Q}}_\ell$-algebra, so we may choose a finite number of Hecke operators $T_1, \ldots, T_k$ that form a generating set. Each of these finitely many operators on $H^*_{N,\mathrm{ess}}$ satisfies a characteristic polynomial which is defined over a number field, hence for all sufficiently large ℓ all their eigenvalues lie in $\overline{\mathbb{Z}}_\ell$. Since the projection to eigenspaces is given by universal polynomials in the eigenvalues, these projection operators are then defined over $\overline{\mathbb{Z}}_\ell$ for all sufficiently large ℓ, giving a decomposition which then further descends to $\mathcal{O}_E$. □

For $\overline{\mathrm{Sht}}_N$ as in Corollary 5.12, by Grothendieck's quasi-unipotence theorem we may choose t_v to act unipotently on all of $H^0((\overline{\mathrm{Sht}}_N)_{K_v^{\mathrm{s}}}, \mathrm{IC}(\mathbb{Z}_\ell))$. It then suffices to show that for almost all ℓ, the cokernel of $(t_v - 1)^i$ is torsion-free on $H^0(\overline{\mathrm{Sht}}_N, \mathrm{IC}(\mathbb{Z}_\ell))$ for all i. We will prove a much more general statement, for any smooth proper variety over K_v.

5.2.2 *Monodromy weight filtrations for torsion intersection cohomology.* Let Z be a proper variety over a characteristic-p local field F. Then we have an action of $\mathrm{Gal}(F^{\mathrm{s}}/F)$ on its (geometric) intersection cohomology $H^*(Z_{F^{\mathrm{s}}}, \mathrm{IC}_Z(\mathbb{Z}_\ell))$.

[3] Explicitly, Ξ is the $U(N)$ invariants on the cuspidal automorphic representation corresponding to σ_ℓ via L. Lafforgue's global Langlands correspondence for GL_n.

[4] The Hecke algebra does not act on all of ${}^{\mathfrak{p}}H^0(\overline{\mathrm{Sht}}_N, \mathrm{IC}(E))$ or ${}^{\mathfrak{p}}H^0(\overline{\mathrm{Sht}}_N, \mathrm{IC}(\mathcal{O}_E))$. There are Hecke correspondences on $\overline{\mathrm{Sht}}_N$ defined by normalization, which act on these cohomology groups, but not in a way satisfying the relations of the Hecke algebra. However, Lafforgue shows that on the 'essential' summand $H^*_{\mathrm{ess}}(\overline{\mathbb{Q}}_\ell) \subset {}^{\mathfrak{p}}H^0(\overline{\mathrm{Sht}}_N, \mathrm{IC}(\overline{\mathbb{Q}}_\ell))$, the Hecke operators satisfy the relations of the Hecke algebra. This is all that matters for us.

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There is an open subgroup $J \subset I_F$ such that its action is unipotent. Take $t \in J$ such that its projection to the ℓ-part of tame inertia is non-trivial for all but finitely many $\ell \neq p$.

Proposition 5.13. *For almost all $\ell \neq p$, the cokernel of $(t-1)^i \colon H^*(Z_{F^s}, \mathrm{IC}_Z(\mathbb{Z}_\ell)) \to H^*(Z_{F^s}, \mathrm{IC}_Z(\mathbb{Z}_\ell))$ is torsion-free for all i.*

In the special case where Z is smooth and proper, this is a result of Ito [Ito21], which ultimately relies on the weight-monodromy conjecture. Our proof will reduce to the smooth and proper case using the following proposition.

Proposition 5.14. *Let $f \colon Y \to Z$ be an alteration of varieties over a field L, with Y smooth over L. Then for all sufficiently large ℓ, $Rf_*\mathbb{Z}_\ell$ has $\mathrm{IC}_Z(\mathbb{Z}_\ell)$ as a direct summand.*

Proof of Proposition 5.13 assuming Proposition 5.14. According to de Jong [deJ96], we may find an alteration $f \colon Y \to Z$ with Y being smooth and proper. The cokernels in question are then direct summands of the analogous cokernels for Y, which by [Ito21, combination of Lemma 3.7, Proposition 3.9, and Theorem 3.6] are torsion-free for all sufficiently large ℓ. □

Remark 5.15. Our Proposition 5.14 also implies a torsion version of the weight-monodromy conjecture for the intersection cohomology of a proper variety Z with $\mathbb{F}_\ell$-coefficients, generalizing the case of smooth and proper Z treated in [Ito21].

The proof of Proposition 5.14 will be given at the end of this section. If we were considering $\mathbb{Q}_\ell$-coefficients instead, we would be able to deduce the analogous statement from the decomposition theorem (although it is not an entirely trivial deduction when F is not separably closed, since the splitting of the decomposition theorem over F^s is not canonical, and does not descend to F in general). For varieties over $\mathbf{C}$, the statement with $\mathbb{Z}_\ell$-coefficients would follow for all sufficiently large ℓ from the $\mathbb{Q}_\ell$-coefficient version plus the existence of a $\mathbb{Z}$-structure on $Rf_*\mathbb{Z}_\ell$ and $\mathrm{IC}(\mathbb{Z}_\ell)$. However, Proposition 5.14 in the stated generality seemed out of reach until we learned of the recent theorem of Cadoret and Zheng, which gives a version of the decomposition theorem with $\mathbb{Z}_\ell$-coefficients for all sufficiently large ℓ.

Theorem 5.16 (Cadoret and Zheng [CZ]). *Let $f \colon Y \to Z$ be a proper surjection of finite type schemes over a field L, with Y is smooth over L. Then for all sufficiently large ℓ, we have*

$$Rf_*\mathbb{Z}_\ell \cong \bigoplus_i {}^{\mathfrak{p}}\mathcal{H}^i(Rf_*\mathbb{Z}_\ell)[-i],$$

with each ${}^{\mathfrak{p}}\mathcal{H}^i(Rf_*\mathbb{Z}_\ell)$ being torsion-free as a perverse sheaf.

Furthermore, if L is separably closed, then for all sufficiently large ℓ, each ${}^{\mathfrak{p}}\mathcal{H}^i(Rf_*\mathbb{Z}_\ell)$ is isomorphic to a direct sum of intersection complexes of semisimple local systems.

5.2.3 *Perverse sheaves.* We would like to apply Theorem 5.16 to the situation of Proposition 5.14. A subtlety is that because the splitting provided by the decomposition theorem over L^s is not canonical, it does not in general descend to L. We must therefore pay careful attention to rationality issues. We will use a trick that we learned from Hansen's blog[5] (and which is credited there to Bhargav Bhatt, although we were later informed by Cadoret and Zheng that the argument essentially appears already around [BBD82, § 5.3.11]) in order to deduce the necessary results.

[5] Available at https://totallydisconnected.wordpress.com/2020/08/26/a-trick-and-the-decomposition-theorem/

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DEFINITION 5.17. Let $\mathcal{F}$ be a perverse sheaf on a variety Z. Let $j\colon U \hookrightarrow Z$ be the inclusion of the maximal (dense) open variety on which $\mathcal{F}$ is a (shifted) local system. We define the *generic part of* $\mathcal{F}$ to be the perverse sheaf $\mathcal{F}^{\mathrm{gen}} := j_{!*}j^*\mathcal{F}$.

We make no claim that there is a non-trivial map between $\mathcal{F}$ and $\mathcal{F}^{\mathrm{gen}}$ in general. However, we shall prove that in certain situations of interest, $\mathcal{F}^{\mathrm{gen}}$ can be realized as a direct summand of $\mathcal{F}$.

PROPOSITION 5.18. *Let $\mathcal{F}$ be a perverse sheaf on a variety Z over L. Suppose that $\mathcal{F}|_{Z_{L^s}}$ is isomorphic to a direct sum of IC sheaves. Then $\mathcal{F}^{\mathrm{gen}}$ is a direct summand of $\mathcal{F}$.*

Proof. Let $j\colon U \to Z$ be as in Definition 5.17 and B be the closed complement of U. The hypothesis implies that $\mathcal{F}|_{Z_{L^s}}$ is isomorphic to $\mathcal{F}^{\mathrm{gen}}_{Z_{L^s}} \oplus \mathcal{E}$ where $\mathcal{E}$ is a direct sum of IC sheaves supported on B_{L^s}.

Consider the maps $\alpha\colon ({}^{\mathfrak{p}}j_!)j^*\mathcal{F} \to \mathcal{F}$ and $\beta\colon \mathcal{F} \to ({}^{\mathfrak{p}}j_*)j^*\mathcal{F}$.

Let $\mathcal{G} := \mathrm{Im}(\alpha) \subseteq \mathcal{F}$. Then we have a tautological map $\mathcal{G} \twoheadrightarrow \mathcal{F}^{\mathrm{gen}}$. Since $({}^{\mathfrak{p}}j_!j^*\mathcal{F})_{L^s}$ has no proper quotients supported on B_{L^s} [Ach21, Exercise 3.1.6], the composition $\mathcal{G}_{L^s} \to \mathcal{F}|_{Z_{L^s}} \to \mathcal{E}$ vanishes, so that $\mathcal{G}_{L^s} \hookrightarrow \mathcal{F}^{\mathrm{gen}}|_{L^s}$. This implies that the given map $\mathcal{G} \to \mathcal{F}^{\mathrm{gen}}$ is an isomorphism, which gives an injection $\mathcal{F}^{\mathrm{gen}} \cong \mathcal{G} \hookrightarrow \mathcal{F}$. A dual argument shows that for $\mathcal{G}' := \mathrm{Im}(\beta)$, we have a surjection $\mathcal{F} \twoheadrightarrow \mathcal{G}' \cong \mathcal{F}^{\mathrm{gen}}$. We then check after base change to L^s that the composition $\mathcal{F}^{\mathrm{gen}} \hookrightarrow \mathcal{F} \twoheadrightarrow \mathcal{F}^{\mathrm{gen}}$ is the identity map. □

COROLLARY 5.19. *Let $f\colon Y \to Z$ be a proper map of L-varieties, with Y smooth over L. Then for all sufficiently large ℓ, ${}^{\mathfrak{p}}\mathcal{H}^0(Rf_*\,\mathrm{IC}_Y(\mathbb{Z}_\ell))^{\mathrm{gen}}$ is a direct summand of $Rf_*\,\mathrm{IC}_Y(\mathbb{Z}_\ell)$.*

Proof. By Theorem 5.16, the hypothesis of Proposition 5.18 holds, so that ${}^{\mathfrak{p}}\mathcal{H}^0(Rf_*\,\mathrm{IC}_Y(\mathbb{Z}_\ell))^{\mathrm{gen}}$ is a direct summand of ${}^{\mathfrak{p}}\mathcal{H}^0(Rf_*\,\mathrm{IC}_Y(\mathbb{Z}_\ell))$. Applying Theorem 5.16 again, ${}^{\mathfrak{p}}\mathcal{H}^0(Rf_*\,\mathrm{IC}_Y(\mathbb{Z}_\ell))$ is a direct summand of $Rf_*\,\mathrm{IC}_Y(\mathbb{Z}_\ell)$. □

Proof of Proposition 5.14. By the definition of alteration, there is an open dense subset $U \subset Z$ over which the map $f|_U$ is finite flat. By shrinking U further if necessary, we may assume that U is contained in the smooth locus of Z, so that $\mathrm{IC}_Z(\mathbb{Z}_\ell)$ is the intermediate extension of $\mathrm{IC}_Z(\mathbb{Z}_\ell)|_U \cong \mathbb{Z}_\ell[\dim U]$. Over U, the composition $\mathbb{Z}_\ell \to f_*\mathbb{Z}_\ell \xrightarrow{\mathrm{Tr}} \mathbb{Z}_\ell$ is multiplication by $\deg f|_U$, hence as long as $\ell > \deg f|_U$ it realizes the constant sheaf $(\mathbb{Z}_\ell)_U$ as a direct summand of $f_*((\mathbb{Z}_\ell)_{f^{-1}(U)}) \cong (Rf_*\mathbb{Z}_\ell)|_U$. Since restriction to open subsets is perverse t-exact, this says that $\mathrm{IC}_U(\mathbb{Z}_\ell)$ is a summand of ${}^{\mathfrak{p}}\mathcal{H}^0(Rf_*\,\mathrm{IC}_Y(\mathbb{Z}_\ell))|_U$, and then applying intermediate extension shows that $\mathrm{IC}_Z(\mathbb{Z}_\ell)$ is a direct summand of ${}^{\mathfrak{p}}\mathcal{H}^0(Rf_*\,\mathrm{IC}_Y(\mathbb{Z}_\ell))^{\mathrm{gen}}$. We then conclude using Corollary 5.19. □

6. Global cyclic base change

6.1 Automorphic Galois representations

Let $R = \overline{\mathbb{F}}_\ell$ or $\overline{E}_\lambda$. In [Fen24, § 2], the *abstract excursion algebra* $\mathrm{Exc}(\Gamma_K, \widehat{G}_R)$ is defined. This is an algebra over R such that the set of characters $\chi\colon \mathrm{Exc}(\Gamma_K, \widehat{G}_R) \to R$ is in bijection with semisimple (not necessarily continuous) representations $\sigma \in H^1(\Gamma_K, \widehat{G}(R))$. In [Xue20a, Xue20b], Cong Xue extends Lafforgue's methods to define an action of $\mathrm{Exc}(\Gamma_K, \widehat{G}_R)$ on $C_c^\infty(G(K)\setminus G(\mathbb{A}_K), R)$, preserving the cuspidal subspace. The same statement holds replacing Γ_K by $\Gamma_{K,N}$ and $G(K)\setminus G(\mathbb{A}_K)$ by $G(K)\setminus G(\mathbb{A}_K)/U(N)$. The algebra $\mathcal{B}(U(N), R)$ considered earlier in § 4.2 is the image of $\mathrm{Exc}(\Gamma_{K,N}, \widehat{G}_R)$ in $\mathrm{End}_R(C_{\mathrm{cusp}}(U(N), R))$.

Following [Fen24, Definition 5.5], we say that a Galois representation $\sigma\colon \Gamma_K \to \widehat{G}(R)$ is *automorphic* if the corresponding maximal ideal $\mathfrak{m}_\sigma$ appears in the support of the $\mathrm{Exc}(\Gamma_K, \widehat{G}_R)$-action

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on $C_c(U(N), R)$ for some N. *A priori* this does not imply that σ is related to a *cusp* form, but the following lemma shows that this is necessarily the case if σ is absolutely irreducible.

Lemma 6.1. *Let $\sigma\colon \Gamma_{K,S} \to \widehat{G}(R)$ be an absolutely irreducible Galois representation which, moreover, is automorphic. Let $\mathfrak{m}_\sigma$ the corresponding maximal ideal of $\mathrm{Exc}(\Gamma_K, \widehat{G}_R)$. If σ is irreducible, then $\mathfrak{m}_\sigma$ appears in the support of the $\mathrm{Exc}(\Gamma_K, \widehat{G}_R)$-action on $C_{\mathrm{cusp}}(U(N), R)$ for some N, and for this N there is an eigenvector $f \in C_{\mathrm{cusp}}(U(N), R)$ on which $\mathcal{B}$ acts through the character $\mathcal{B}/\mathfrak{m}_\sigma$.*

Proof. By [Xue20b, Proposition 7.6.2] and [Xue20a, § 6.2], for a parabolic subgroup $P \subset G$ with Levi M, the constant term map

$$\mathrm{CT}_G^P\colon C_c(G(K) \setminus G(\mathbb{A}_K), R) \to C_c(M(K) \setminus M(\mathbb{A}_K), R)$$

intertwines the action of $\mathrm{Exc}(\Gamma_K, \widehat{G}_R)$ on the source with the action of $\mathrm{Exc}(\Gamma_K, \widehat{M}_R)$ on the target, with respect to $\widehat{M}(R) \hookrightarrow \widehat{G}(R)$. Therefore, all Galois representations attached by Lafforgue's construction to the image of CT_G^P factor through $\widehat{M}(R)$. Hence if σ does not factor through such a parabolic, and $C_c(G, R)_{\mathfrak{m}_\sigma} \neq 0$, then we must have $C_{\mathrm{cusp}}(G, R)_{\mathfrak{m}_\sigma} \neq 0$.

For the last statement, we note that since $C_{\mathrm{cusp}}(G, R) = \varinjlim_N C_{\mathrm{cusp}}(U, R)$, we have $(C_{\mathrm{cusp}}(U, R))_{\mathfrak{m}_\sigma} \neq 0$ for some U. Since $C_{\mathrm{cusp}}(U, R)$ is a finite-dimensional vector space over R, it is the direct sum of its generalized eigenspaces for the $\mathrm{Exc}(\Gamma_K, \widehat{G}_R)$-action, and the assumption implies that the character $\mathrm{Exc}(\Gamma_K, \widehat{G}_R) \to \mathrm{Exc}(\Gamma_K, \widehat{G}_R)/\mathfrak{m}_\sigma$ appears among the systems of eigenvalues. □

6.2 Mod ℓ base change

We suppose ℓ is odd and is a *good prime* for the reductive group G. Explicitly, this means that we require $p > 2$ if $\widehat{G}$ has simple factors of type A, B, C or D; $p > 3$ if $\widehat{G}$ has simple factors of type G_2, F_4, E_6, E_7; and $p > 5$ if $\widehat{G}$ has simple factors of type E_8. Let K'/K be a cyclic extension of function fields of degree ℓ.

Theorem 6.2 [Fen24, Theorem 1.6]. *With notation as above, let $\overline{\sigma} : \Gamma_K \to \widehat{G}(\overline{\mathbb{F}}_\ell)$ be an automorphic Galois representation. Then the restriction $\overline{\sigma}|_{\Gamma_{K'}}$ is automorphic for $G_{K'}$.*

6.3 Existence of almost all cyclic base changes in characteristic 0

Theorem 6.3. *Let Π be a cuspidal automorphic representation of $G(\mathbb{A}_K)$ defined over a number field E.*[6] *Suppose the Mumford–Tate group of Π is $\widehat{G}$, as in Definition 4.3. For a place λ of E, let $\sigma_{\Pi,\lambda}$ be Lafforgue's parameter attached to Π (as noted in Remark 4.2, the hypothesis implies that $\underline{\rho}_{\Pi,\lambda}$ is a singleton, so that $\sigma_{\Pi,\lambda}$ exists).*

Then there is an integer $c(\Pi)$, depending only on Π, such that for any prime ℓ satisfying $\ell > c(\Pi)$, and any cyclic extension K'/K of degree ℓ, there is an automorphic representation Π' of $G(\mathbb{A}_{K'})$ attached to the L-parameter $\sigma_{\Pi,\lambda}|_{\mathrm{Gal}((K')^s/K')}$.

Proof. Let $\overline{\sigma}_{\Pi,\lambda}$ denote the reduction of $\sigma_{\Pi,\lambda}$ modulo λ; let k_λ denote the residue field of λ. Our hypothesis on the Mumford–Tate group of Π, together with the main theorem of [BGP19], implies that the image of $\overline{\sigma}_{\Pi,\lambda}(\Gamma_{K(\zeta_\ell)})$ contains $\widehat{G}(k_\lambda)^+$ for all λ of sufficiently large characteristic. It then follows from Lemma 3.8 that there is a constant $b_1(\Pi)$ such that for $\ell > b_1(\Pi)$ the image of $\overline{\sigma}_{\Pi,\lambda}(\Gamma_{K'(\zeta_\ell)})$ is abundant, and that $\overline{\sigma}_{\Pi,\lambda}$ is also absolutely irreducible.

[6] We emphasize again that every cuspidal automorphic representation of $G(\mathbb{A}_K)$ admits a model over the number field, so this is not a restriction.

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Now, it follows from Theorem 5.1 that there is a constant $b_2(\Pi)$ such that for all $\ell > b_2(\Pi)$, the restriction of $\overline{\sigma}_{\Pi,\lambda}$ to $\Gamma_{K'}$ satisfies condition (v) of Theorem 4.6. Taking $c(\Pi) := \max(b_1(\Pi), b_2(\Pi))$, we may then invoke Theorem 6.2 and Lemma 6.1 to apply Theorem 4.6. The freeneess guarantees the existence of an eigenform supported over a single point on the generic fiber of $R_{\overline{\sigma}_{\mathfrak{m}},|N|,\mathcal{D}}$, and we may take Π' to be an irreducible cuspidal subquotient of the automorphic representation generated by it. □

7. Local cyclic base change

Now let F be a local function field. In this section we will prove a *local* analog of Theorem 6.3, establishing existence of cyclic base change for irreducible smooth representations of $G(F)$, along any $\mathbb{Z}/\ell\mathbb{Z}$-extension of F for almost all ℓ.

7.1 The Genestier–Lafforgue correspondence

Let F be a local function field with ring of integers $\mathcal{O}_F$ and residue characteristic $\ell \neq p$. Let W_F be the Weil group of F. Let G be a reductive group over F. In [GL17, Théorèm 0.1], Genestier and Lafforgue construct a map

$$\left\{\begin{matrix}\text{irreducible admissible representations} \\ \pi \text{ of } G(F) \text{ over } \overline{\mathbb{Q}}_\ell\end{matrix}\right\}/\sim \longrightarrow \left\{\begin{matrix}\text{semisimple } L\text{-parameters} \\ \sigma_\pi \colon W_F \to \widehat{G}(\overline{\mathbb{Q}}_\ell)\end{matrix}\right\}/\sim .$$

We shall use the two properties of the correspondence $\pi \mapsto \sigma_\pi$ recalled below.

7.1.1 *Local–global compatibility.* For any automorphic representation $\Pi \cong \bigotimes_{x\in|X|}\Pi_x$ of $G(\mathbb{A}_K)$ that is associated to the L-parameter σ_Π by V. Lafforgue's global Langlands parametrization (§ 4.2), σ_{Π_x} is conjugate to the semisimplification of $\sigma|_{W_{K_x}}$ for any $\sigma \in \underline{\rho}_\Pi$ (notation as in Remark 4.2).

7.1.2 *Compatibility with parabolic induction.* If P is a parabolic subgroup of G with Levi quotient M, τ is an irreducible admissible representation of $M(F)$, and π is an irreducible subquotient of $\mathrm{Ind}_{P(F)}^{G(F)}\tau$ (with parabolic induction formed using the unitary normalization), then σ_π is conjugate to the composition $W_F \xrightarrow{\sigma_\tau} \widehat{M}(\overline{\mathbb{Q}}_\ell) \hookrightarrow \widehat{G}(\overline{\mathbb{Q}}_\ell)$.

7.2 Existence of almost all cyclic base changes in characteristic 0

DEFINITION 7.1. Let π be an irreducible admissible representation of $G(F)$ over $\overline{\mathbb{Q}}_\ell$. For a separable field extension F'/F, we say that an irreducible admissible representation π' of $G(F')$ over $\overline{\mathbb{Q}}_\ell$ is a *base change lifting of* π *to* $G(F')$ if $\sigma_{\pi'} \cong \sigma_\pi|_{W_{F'}}$.

This definition is an approximation to the notion of base change for L-packets. An L-packet for $G(F')$ should be said to be a base change lifting of an L-packet for $G(F)$ if the corresponding L-parameters are related by restriction on Weil groups. Since we lack a definition of L-packets for general groups and representations, we use the fibers of the Genestier–Lafforgue correspondence as a substitute for L-packets.

THEOREM 7.2. *Assume that p is good for G if G is not simply laced. Let π be an irreducible admissible representation of $G(F)$ over $\overline{\mathbb{Q}}$. There exists a constant $c(\pi)$ such that for all primes $\ell > c(\pi)$, for any $\mathbb{Z}/\ell\mathbb{Z}$-extension F'/F there exists a base change lifting of π to $G(F')$.*

Remark 7.3. In [Fen24, Theorem 1.1], a version of this result was established for mod ℓ representations, when the extension is cyclic of degree equal to the same prime ℓ, by completely different methods.

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Proposition 7.4. *Assume that p is good for G if G is not simply laced. Let π be an irreducible admissible supercuspidal representation of $G(F)$ over $\overline{\mathbb{Q}}$. Then there exist a global field K and a place $v \in |K|$ with $K_v \cong F$, and a cuspidal automorphic representation Π of $G(\mathbb{A}_K)$ with Mumford–Tate group $\widehat{G}$ such that $\Pi_v \cong \pi$.*

Proof of Theorem 7.2 assuming Proposition 7.4. First we reduce to the case where π is supercuspidal. Indeed, any π can be realized as an irreducible subquotient of a parabolic induction of supercuspidal representations, of the form $\mathrm{Ind}_{P(F)}^{G(F)} \tau$. By § 7.1.2, the parabolic induction from $M(F')$ to $G(F')$ of a base change of τ to $M(F')$ will have an L-parameter of the desired form. So it suffices to treat the supercuspidal case.

Hence we may and do assume for the rest of the argument that π is supercuspidal. Then we may apply Proposition 7.4 to embed π as the local component at v of cuspidal automorphic representation Π over a global field K with $K_v \cong F$. Using Remark 4.2, we can replace Π by an isomorphic $G(\mathbb{A}_K)$-representation (realized differently in the space of cuspidal functions on $G(K)\backslash G(\mathbb{A}_K)$) to assume that Π is attached to an L-parameter σ_Π. Then Theorem 6.3 applies, so let $c(\pi) := c(\Pi)$ be as in Theorem 6.3. For any $\mathbb{Z}/\ell\mathbb{Z}$-extension F'/F, we can find a $\mathbb{Z}/\ell\mathbb{Z}$-extension K'/K with a place v' lying over v such that $K_{v'} \cong F'$. By Theorem 6.3, there exists a cuspidal automorphic representation Π' of $G(\mathbb{A}_{K'})$ with L-parameter $\sigma_{\Pi'} \cong \sigma_\Pi|_{\Gamma_{K'}}$. If $\Pi'_{v'}$ is the local component of Π' at v', then the local–global compatibility of § 7.1.1 ensures that $\sigma_{\Pi'_{v'}} \cong \sigma_{\Pi'}|_{W_{K'_{v'}}}$, hence

$$\sigma_{\Pi'_{v'}} \cong \sigma_{\Pi'}|_{W_{K'_{v'}}} \cong \sigma_\Pi|_{W_{F'}} \cong \sigma_\pi|_{W_{F'}},$$

so $\Pi'_{v'}$ is the desired local base change. □

Remark 7.5. Since every supercuspidal representation of $G(F)$ admits a model over $\overline{\mathbb{Q}}$, at least after twisting by a central character (which is unnecessary for us because our G is semisimple), Theorem 7.2 applies to all supercuspidal representations. In fact, we may present any irreducible admissible representation in terms of parabolic inductions and twists of representations defined over $\overline{\mathbb{Q}}$, and then the compatibility of the Genestier–Lafforgue correspondence with parabolic induction and twisting would allow the formulation of a meaningful extension of Theorem 7.2 to all irreducible admissible representations.

7.3 Globalization of supercuspidal representations

This subsection is devoted to the proof of Proposition 7.4. We will use an argument due to Beuzart-Plessis, based on the Deligne–Kazhdan simple trace formula, to construct a globalization Π of π with specified local components at several auxiliary places that automatically force the image of Lafforgue's corresponding parameter to be Zariski dense.

7.3.1 *Genestier–Lafforgue parameters of simple supercuspidals.* The notion of 'simple supercuspidal representations' was singled out by Gross and Reeder [GR10]. Let us recall the definition.

Definition 7.6. Let G be a split semisimple group over a non-archimedean local field F of residue characteristic $p \neq \ell$. A *simple supercuspidal representation* is a representation 'V_ϕ' of $G(F)$ that arises in the following way. Let $B \subset G$ be a Borel subgroup, with unipotent radical U. Let $I \subset G(\mathcal{O}_F)$ be the corresponding Iwahori subgroup, and $I(1) \subset I$ its pro-unipotent radical. For an affine generic character $\phi\colon I(1) \to \mathbb{Q}_\ell(\mu_p)$, we define $V_\phi := \text{c-Ind}_{I(1)\times Z(G)}^{G(F)}(\phi \otimes \mathbb{1})$.

Gross and Reeder showed that simple supercuspidal representations are irreducible and supercuspidal. They anticipated the shape of the associated L-parameters. To explain this, let

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$\mathcal{J} \subset W_F$ be the inertia subgroup. Let us call *simple supercuspidal parameter* a discrete[7] Langlands parameter $\sigma\colon W_F \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$ such that the adjoint representation $\mathrm{Ad}\circ\sigma$ has $\widehat{\mathfrak{g}}^{\mathcal{J}} = 0$ and Swan conductor equal to the rank of $\widehat{G}$. We emphasize that a discrete Langlands parameter, a fortiori a simple supercuspidal parameter, is already semisimple; this will be used when invoking the local–global compatibility of § 7.1.1.

Proposition 7.7. *Assume that p is good for G if G is not simply laced. Then the Genestier–Lafforgue parameter of V_ϕ is a simple supercuspidal parameter.*

Proof. The idea is to find a globalization of V_ϕ to a cuspidal automorphic representation on $\mathbf{P}^1$, whose associated global L-parameter can be computed explicitly. A particularly convenient such globalization was studied by Heinloth, Ngô, and Yun [HNY13] and we only need to collect the relevant consequences of their work.

Let $K = \mathbb{F}_q(\mathbf{P}^1) = \mathbb{F}_q(t)$. Let I_0 be the Iwahori subgroup of $G(\mathbb{F}_q((t)))$, and $I(1)_\infty$ be the pro-unipotent radical of the Iwahori subgroup of $G(\mathbb{F}_q((1/t)))$. As a special case of [Yun16, Proposition 2.7], there is a unique automorphic representation $\Pi \cong \bigotimes'_{x\in|\mathbf{P}^1|}\Pi_x$ of $G(\mathbb{A}_K)$ such that:

- Π_x is unramified if $x \neq 0, \infty$;
- Π_0 has a vector invariant under I_0;
- Π_∞ has an eigenvector under $I(1)_\infty$, on which the action is given by ϕ.

Moreover, this Π is cuspidal and appears with multiplicity one in the automorphic spectrum, and

$$\dim C_c^\infty\left(G(K)\backslash G(\mathbb{A}_K)/\prod_{x\neq 0,\infty} G(\mathcal{O}_x), \overline{\mathbb{Q}}\right)^{I_0\times(I(1)_\infty,\phi)} = 1.$$

Since the excursion algebra acts on this one-dimensional $\overline{\mathbb{Q}}_\ell$-vector space, its action is automatically through a character, which determines the L-parameter σ_Π associated to Π by Lafforgue's correspondence.

In this case, Heinloth, Ngô, and Yun construct a Hecke eigensheaf A_ϕ on a moduli stack $\mathrm{Bun}_{G(0,2)}$ of G-bundles on $\mathbf{P}^1$ with level structure at 0 and ∞, whose associated Frobenius trace function is a non-zero $f_\phi \in C_c^\infty\big(G(K)\backslash G(\mathbb{A}_K)/\prod_{x\neq 0,\infty} G(\mathcal{O}_x), \overline{\mathbb{Q}}\big)^{I_0\times(I(1)_\infty,\phi)}$. They further prove that A_ϕ is a Hecke eigensheaf, with corresponding local system the (generalized) Kloosterman local system $\mathrm{Kl}_{\widehat{G}}(\phi)\colon \pi_1(\mathbb{P}^1 - \{0,\infty\}, \widehat{G}(\overline{\mathbb{Q}}_\ell))$. By [HNY13, Theorem 2, Corollary 2.15], letting $\mathcal{J}_\infty \subset W_{K_\infty}$ be the inertia group at ∞, the local monodromy representation $\mathrm{Kl}_{\widehat{G}}(\phi)|_{\mathcal{J}_\infty}\colon \mathcal{J}_\infty \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$ at ∞ is irreducible and it is the restriction to $\mathcal{J}_\infty$ of a simple supercuspidal parameter.

By [BHKT19, Proposition 6.4], the parameter associated to f_ϕ by Lafforgue's correspondence must coincide with $\mathrm{Kl}_{\widehat{G}}(\phi)$. By local–global compatibility, the Genestier–Lafforgue parameter of V_ϕ therefore agrees upon restriction to $\mathcal{J}_\infty$ with $\mathrm{Kl}_{\widehat{G}}(\phi)|_{\mathcal{J}_\infty}$. □

7.3.2 *Globalization.* The following lemma is proved by Beuzart-Plessis in the appendix to [GHS21], using the Deligne–Kazhdan simple trace formula.

Lemma 7.8. *Let π be a supercuspidal representation of $G(F)$ over $\overline{\mathbb{Q}}$. There exists a global curve X, with function field $K = \mathbb{F}_q(X)$ and places $v, w, w', w'' \in |X|$, and*

[7] Recall that a parameter $\sigma : W_F \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$ is called *discrete* if its image is not contained in a proper parabolic subgroup of $\widehat{G}$.

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a cuspidal representation $\Pi \cong \bigotimes'_{x\in|X|}\Pi_x$ *of* $G(\mathbb{A}_K)$ *such that the following statements hold:*

(i) $K_v \cong F$ *and* $\Pi_v \cong \pi$ *as* $G(F)$*-representations.*
(ii) Π_w *is a simple supercuspidal representation of* $G(K_w)$*.*
(iii) $\Pi_{w'}$ *has non-zero trace against the pseudo-coefficient for the Steinberg representation given by the Euler-Poincaré function [Lau96, Theorem 8.2.1].*
(iv) $\Pi_{w''}$ *is an unramified representation parametrized by a regular element of* $\widehat{T}//W$*. Moreover, the parameter can be chosen to avoid any finite union of proper subtori of* $\widehat{T}$*.*

We will need a characterization of irreducible subgroups of $\widehat{G}$ containing a principal unipotent element. The following result is probably known but we were unable to find a reference.

Lemma 7.9. *Let* $\mathfrak{g}$ *be a semisimple Lie algebra and* $\mathfrak{h}$ *a proper semisimple subalgebra containing a regular nilpotent element of* $\mathfrak{g}$*. Then the rank of* $\mathfrak{h}$ *is strictly less than that of* $\mathfrak{g}$*.*

Proof. Let E be the regular nilpotent of $\mathfrak{g}$ which lies in $\mathfrak{h}$. We first claim that E is also regular in $\mathfrak{h}$. We can complete E to an $\mathfrak{sl}_2$-triple (E, H, F) in $\mathfrak{h}$. The element H is regular semisimple in $\mathfrak{g}$, hence in $\mathfrak{h}$, which implies by [Bou05, VIII. 11.4, Proposition 7] that E is regular nilpotent in $\mathfrak{h}$.

Let $r(\mathfrak{g})$ and $\ell(\mathfrak{g})$ denote respectively the ranks of $\mathfrak{g}$ and its length as a representation of the three-dimensional subalgebra spanned by (E, H, F); define $r(\mathfrak{h})$ and $\ell(\mathfrak{h})$ analogously. The proposition just cited from [Bou05] implies that

$$r(\mathfrak{g}) = \ell(\mathfrak{g}); \quad r(\mathfrak{h}) = \ell(\mathfrak{h}).$$

Thus $r(\mathfrak{g}) = r(\mathfrak{h})$ if and only if $\mathfrak{g}$ and its subspace $\mathfrak{h}$ have the same length as (E, H, F)-modules; but this is only possible if $\mathfrak{g} = \mathfrak{h}$. □

Corollary 7.10. *Let* $\mathcal{H}$ *denote the set of conjugacy classes of proper semisimple subgroups of* $\widehat{G}$ *containing a principal unipotent element, and for each* $H \in \mathcal{H}$*, let* $\widehat{T}_H \subset \widehat{T}//W$ *denote the image of a maximal torus of* H *(which is independent of the choice of representative). Then the complement of the union of the* $\widehat{T}_H$ *is Zariski dense in* $\widehat{T}//W$*.*

Proof. A classification of $\mathcal{H}$ in all types appears in [SS97, Theorem A, Theorem B] (in characteristic 0, the result has been credited to earlier work of Dynkin). It is finite. On the other hand, it follows from Lemma 7.9 that each $\widehat{T}_H$ is of codimension at least 1 in $\widehat{T}//W$ (this is also clear from inspection of $\mathcal{H}$). Therefore, $\bigcup_{H\in\mathcal{H}}\widehat{T}_H$ is a finite union of positive-codimension subvarieties in $\widehat{T}//W$. □

7.3.3 *Determination of global monodromy.* Let Π be as in Lemma 7.8. Let $\underline{\rho}_\Pi$ be the set of Remark 4.2 and $\sigma \in \underline{\rho}_\Pi$. We claim that σ has Zariski dense image. This will complete the proof of Proposition 7.4.

So it only remains to prove the claim. By local–global compatibility, the semisimplification of $\sigma|_{K_x}$ corresponds to Π_x under the Genestier–Lafforgue correspondence for all $x \in |X|$. By Proposition 7.7 and local–global compatibility, the semisimplification of $\sigma_\Pi|_{K_w}$ is already absolutely irreducible, so σ_Π is absolutely irreducible. By [ST21, Lemma 11.4], for any representation V of $\widehat{G}$ the corresponding local system is $V \circ \sigma_\Pi$ is pure of weight 0. By condition (iii) and compatibility of the Genestier–Lafforgue correspondence with parabolic induction, $\sigma|_{W_{K_{w'}}}$ has the same semisimplification as the Steinberg representation. It follows as in [HNY13, § 4.3] from purity of the weight-monodromy filtration [Del80, Theorem I.8.4] that the image of $\sigma|_{K_{w'}}$ contains a principal unipotent element.

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By [Del80, Corollaire I.3.9] and the property that $V \circ \sigma_\Pi$ is pure of weight 0 for every V, the neutral component of the Zariski closure of the image of σ in $\widehat{G}$ is semisimple. Furthermore, we have just seen that it does not lie in any proper parabolic subgroup, and also that it contains a principal unipotent element. By Lemma 7.9, proper semisimple subgroups of $\widehat{G}$ have smaller rank. But this is ruled out by (iv), where the proper subtori are those in the statement of Corollary 7.10. □

8. Cyclic base change for toral supercuspidal representations

In this section we will investigate cyclic base change more explicitly for the class of toral supercuspidal representations studied in [CO23]. The strategy is to first explicate cyclic base change for the mod ℓ reductions of these representations, using a Conjecture of Treumann-Venkatesh (established in [Fen24] for the Genestier–Lafforgue correspondence) that 'base change functoriality is realized by Tate cohomology.'

The main new work is in calculating the Tate cohomology of toral supercuspidal representations, and what facilitates this calculation is a geometric model for these representations established by Chan and Oi [CO23], as compact inductions from parahoric subgroups of 'generalized Deligne–Lusztig inductions' studied by Chan and Ivanov. The generalized Deligne–Lusztig representations are produced from the cohomology of 'Deligne–Lusztig type varieties' built out of group schemes coming from the Moy–Prasad filtration, analogously to the way in which Deligne–Lusztig varieties are built from reductive groups over finite fields. Hence our computation naturally divides into two steps: (1) studying Tate cohomology of Deligne–Lusztig type varieties, which we do in § 8.2; and (2) studying Tate cohomology of compact inductions, which we do in § 8.3.

Kaletha constructs an explicit local Langlands parametrization of the toral supercuspidal representations in [Kal19]. On the other hand, we identify base change relations among the Genestier–Lafforgue parameters of toral supercuspidal representations. This gives some evidence for consistency between the Genestier–Lafforgue correspondence [GL17] and Kaletha's construction of L-packets of regular supercuspidal representations.

We emphasize that in this section, F is a non-archimedean local field having residue characteristic p, but we allow F to have characteristic 0 in all results up to Theorem 8.18.

8.1 Work of Chan and Ivanov

We briefly recall the generalized Deligne–Lusztig representations appearing in [CI21].

Let G be a reductive group over a non-archimedean local field F. Let $T \hookrightarrow G$ be an unramified maximal torus and $x \in \mathcal{B}(G/F)$ be a point of the Bruhat–Tits building of G that lies in the apartment of T. If F'/F is a tamely ramified extension, then we may also regard x as a point of $\mathcal{B}(G_{F'}/F')$ using the identification $\mathcal{B}(G/F) = \mathcal{B}(G_{F'}/F')^{\mathrm{Gal}(F'/F)}$. Corresponding to x we have by Bruhat–Tits theory a parahoric group scheme $\mathcal{G}/\mathcal{O}_F$, whose generic fiber is G/F.

Let $\mathbb{F}_q$ be the residue field of F. By assumption, T splits over $\breve{F} := W(\overline{\mathbb{F}}_q)$. Let U be the unipotent radical of an $\breve{F}$-rational Borel subgroup of $G_{\breve{F}}$ containing $T_{\breve{F}}$.

For $r \in \mathbb{Z}_{\geq 0}$, we have group schemes $\mathbb{G}_r, \mathbb{T}_r, \mathbb{U}_r$ over $\mathbb{F}_q$ as in [CI21, § 2.5, 2.6] corresponding to subquotients of the Moy–Prasad filtration at x, such that

$$\mathbb{G}_r(\mathbb{F}_q) = G_{x,0:r+} := G_{x,0}/G_{x,r+}, \quad \mathbb{T}_r(\mathbb{F}_q) = T_{0:r+} := T_{x,0}/T_{x,r+}$$

and $\mathbb{U}_r \subset (\mathbb{G}_r)_{\overline{\mathbb{F}}_q}$.

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8.1.1 *Deep level Deligne–Lusztig varieties.* We recall certain schemes constructed in [CI21, § 3], generalizing Deligne–Lusztig varieties. Let

$$S_{\mathbb{T}_r,\mathbb{U}_r} := \{x \in \mathbb{G}_r \colon x^{-1}\operatorname{Fr}_q(x) \in \mathbb{U}_r\}.$$

(The variety $S_{\mathbb{T}_r,\mathbb{U}_r}$ is called X_r in [CO23].) It is a separated, smooth, finite type scheme over $\overline{\mathbb{F}}_q$, with an action of $G_{x,0:r+} \times T_{x,0:r+}$ by multiplication on the left and right, and also a free action of $\mathbb{U}_r \cap \operatorname{Fr}_q^{-1}(\mathbb{U}_r)$ by right translation.

It is actually more convenient for us to work with

$$Y_{\mathbb{T}_r,\mathbb{U}_r} := S_{\mathbb{T}_r,\mathbb{U}_r}/\mathbb{U}_r \cap \operatorname{Fr}_q^{-1}(\mathbb{U}_r).$$

Since the natural map $S_{\mathbb{T}_r,\mathbb{U}_r} \to Y_{\mathbb{T}_r,\mathbb{U}_r}$ is a bundle in affine spaces, the compactly supported (geometric) cohomology groups of source and target are identified, equivariantly for the $G_{x,0:r+} \times T_{x,0:r+}$-action, up to Tate twist and an (even) shift of cohomological degrees, so they will lead to the same (virtual) representations. The $Y_{\mathbb{T}_r,\mathbb{U}_r}$ are called 'deep level Deligne–Lusztig varieties.'

Example. When $r = 0$, the definition of $Y_{\mathbb{T}_r,\mathbb{U}_r}$ specializes to that of a classical Deligne–Lusztig variety.

Definition 8.1 (Generalized Deligne–Lusztig induction). Let $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Q}}_\ell, k, \mathcal{O}\}$ be an ℓ-adic coefficient ring and $\theta \colon \mathbb{T}_r(\mathbb{F}_q) \to \Lambda^\times$ be a character. We denote

$$H_c^*(Y_{\mathbb{T}_r,\mathbb{U}_r};\Lambda)_\theta := H_c^*(Y_{\mathbb{T}_r,\mathbb{U}_r};\Lambda)_\theta \otimes_{\Lambda[T_{x,0:r+}]} \theta,$$

which is a graded representation of $G_{x,0:r+}$. We define the virtual representation of $G_{x,0:r+x}$,

$$R^{\mathbb{G}_r}_{\mathbb{T}_r,\mathbb{U}_r}(\theta) := \sum_i (-1)^i [H_c^i(Y_{\mathbb{T}_r,\mathbb{U}_r};\Lambda)_\theta] \in K_0(G_{x,0:r+};\Lambda).$$

The version of this definition with $\Lambda = \overline{\mathbb{Q}}_\ell$ is considered in [CI21, Definition 3.4], while we will also be interested in $\Lambda = \overline{\mathbb{F}}_\ell$.

Remark 8.2. By inflation, we may view θ as a character of $T_{x,0}$ that is trivial on $T_{x,r+}$, and $R^{\mathbb{G}_r}_{\mathbb{T}_r,\mathbb{U}_r}(\theta) \in K_0(G_{x,0};\Lambda)$. In practice, θ will come by restriction from a character of $T(F)$.

Definition 8.3. Following [CI21, § 2.10] in the case of $\overline{\mathbb{Q}}_\ell$-coefficients, we say that $\theta \colon \mathbb{T}_r(\mathbb{F}_q) \to k^\times$ is *regular* if for any absolute root α and any $d \geq 1$ such that $\operatorname{Fr}_q^d(\alpha) = \alpha$, the restriction of

$$\mathbb{T}_r(\mathbb{F}_{q^d}) \xrightarrow{\mathrm{Nm}} \mathbb{T}_r(\mathbb{F}_q) \xrightarrow{\theta} \overline{\mathbb{F}}_\ell^\times$$

to the subgroup (defined in [CI21, § 2.6]) $\mathbb{T}_r^{\alpha,r} \subset \mathbb{T}_r(\mathbb{F}_{q^d})$ is non-trivial. Here Nm is the norm map sending $t \mapsto t \cdot \sigma(t) \cdots \sigma^{\ell-1}(t)$.

For a finite group Γ and a virtual representation $V \in K_0(\Gamma;\Lambda)$ which is a representation up to sign, we define $|V|$ to be the underlying representation and $(-1)^V$ to be the sign of V, so that $V = (-1)^V|V|$.

Lemma 8.4. *Let* $\Lambda \in \{\overline{\mathbb{Q}}_\ell, \overline{\mathbb{F}}_\ell\}$.

(i) *If* θ *is regular, then* $R^{\mathbb{G}_r}_{\mathbb{T}_r,\mathbb{U}_r}(\theta) \in K_0(\mathbb{G}_r(\mathbb{F}_q);\Lambda)$ *is independent of* $\mathbb{U}_r$.
(ii) *If* θ *is regular and the stabilizer of* θ *in the Weyl group of the special fiber of* $\mathcal{G}$ *is trivial, then* $\pm R^\theta_{\mathbb{T}_r,\mathbb{U}_r}$ *is a (non-virtual) representation of* $G_{x,0:r+}$ *and* $|R^\theta_{\mathbb{T}_r,\mathbb{U}_r}|$ *is irreducible.*

Proof. Both (i) and (ii) are established in [CI21] when $\Lambda = \overline{\mathbb{Q}}_\ell$, so we shall simply explain the reduction from this case to $\Lambda = \overline{\mathbb{F}}_\ell$.

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(i) Any $\theta\colon T_{x,0:r+} \to \overline{\mathbb{F}}_\ell^\times$ lifts canonically to $\widetilde{\theta}\colon T_{x,0:r+} \to \overline{\mathbb{Z}}_\ell^\times \subset \overline{\mathbb{Q}}_\ell^\times$ via the Teichmüller map. Moreover, $R^{\mathbb{G}_r}_{\mathbb{T}_r,\mathbb{U}_r}(\theta)$ is the image of $R^{\mathbb{G}_r}_{\mathbb{T}_r,\mathbb{U}_r}(\widetilde{\theta})$, the characteristic-zero Deligne–Lusztig induction studied in [CI21, § 3.1], under the reduction map $K_0(G_{x,0:r+};\overline{\mathbb{Q}}_\ell) \to K_0(G_{x,0:r+};\overline{\mathbb{F}}_\ell)$. If θ is regular then $\widetilde{\theta}$ is regular in the sense of [CI21, § 2.10], so by [CI21, Theorem 1.1(i)] $R^{\mathbb{G}_r}_{\mathbb{T}_r,\mathbb{U}_r}(\widetilde{\theta})$ is independent of U.

(ii) This follows from lifting to characteristic 0 and a similar argument to that in (i), using [CI21, Theorem 1.1(i)] for the analogous statement in characteristic 0. We note that $|R^{\theta}_{\mathbb{T}_r,\mathbb{U}_r}|$ is non-zero by [CI21, Corollary 4.3]. □

8.2 Tate cohomology of some generalized Deligne–Lusztig inductions

8.2.1 *Recollections on Tate cohomology.* Let σ be an order-ℓ endomorphism of an abelian group V. Write $N := 1 + \sigma + \cdots + \sigma^{\ell-1} \in \mathbb{Z}[\langle\sigma\rangle]$. The *Tate cohomology* groups of V (with respect to the σ-action) are defined as

$$\mathrm{T}^0(\sigma, V) = \mathrm{T}^0(V) := \frac{\ker(1-\sigma\colon V \to V)}{N \cdot V},$$
$$\mathrm{T}^1(\sigma, V) = \mathrm{T}^1(V) := \frac{\ker(N\colon V \to V)}{(1-\sigma)\cdot V}.$$

It is sometimes convenient to extend the definition of $T^i(V)$ 2-periodically to all $i \in \mathbb{Z}$, so that $T^i(-) = T^{i+2}(-)$.

Given a short exact sequence

$$0 \to V' \to V \to V'' \to 0,$$

there is a (periodic) long exact sequence on Tate cohomology

$$\cdots \to \mathrm{T}^0(V') \to \mathrm{T}^0(V) \to \mathrm{T}^0(V'') \to \mathrm{T}^1(V') \to \mathrm{T}^1(V) \to \mathrm{T}^1(V'') \to \mathrm{T}^2(V') \to \cdots. \tag{8.1}$$

In [Fen24, § 3.4], we defined the notion of Tate cohomology for a scheme Y with an admissible action of $\mathbb{Z}/\ell\mathbb{Z} \cong \langle\sigma\rangle$. Admissibility automatically holds if Y is a quasiprojective variety over a field, and we will always be in this situation when invoking this theory, so let us assume Y is such. Another useful description of the (compactly supported) *Tate cohomology group* $\mathrm{T}^i(\sigma, Y; \Lambda) = \mathrm{T}^i(Y;\Lambda)$ *with coefficients in* Λ is the ith cohomology of the totalization of

$$\begin{array}{ccccccc} \cdots & & \deg -1 & & \deg 0 & & \deg 1 & & \cdots \\ \cdots & \xrightarrow{1-\sigma} & R\Gamma_c(Y;\Lambda) & \xrightarrow{N} & R\Gamma_c(Y;\Lambda) & \xrightarrow{1-\sigma} & R\Gamma_c(Y;\Lambda) & \xrightarrow{N} & \cdots \end{array} \tag{8.2}$$

It is immediate from this definition that $\mathrm{T}^i(-) \cong \mathrm{T}^{i+2}(-)$, so we sometimes treat T^i as being defined for $i \in \mathbb{Z}/2\mathbb{Z}$. We shall be interested in coefficients such as $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell, k, \mathcal{O}, \mathcal{O}/\ell^n\}$.

The double complex (8.2) leads to two spectral sequences abutting to $\mathrm{T}^*(Y;\overline{\mathbb{F}}_\ell)$:

- The 'vertical then horizontal' spectral sequence has first page

$$E_1^{ij} = H_c^j(Y;\Lambda).$$

Its jth row is the complex computing $\mathrm{T}^i(H_c^j(Y;\Lambda))$, namely,

$$\cdots \xrightarrow{N} H_c^j(Y;\Lambda) \xrightarrow{1-\sigma} H_c^j(Y;\Lambda) \xrightarrow{N} H_c^j(Y;\Lambda) \xrightarrow{1-\sigma} \cdots.$$

Therefore, the second page is

$$E_2^{ij} = \mathrm{T}^i(\sigma, H_c^j(Y;\Lambda)).$$

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- The 'horizontal then vertical' spectral sequence has
$$E_2^{ij} = H_c^j(Y^\sigma; \Lambda)$$
and, moreover, is degenerate starting from E_2 [TV16, Theorem 4.4].

8.2.2 *Tate cohomology of representations.* Let V be a representation over k of a finite group Γ. Suppose $\mathbb{Z}/\ell\mathbb{Z} \cong \langle\sigma\rangle$ acts on Γ. Then $V \mapsto V \circ \sigma$ defines an action of σ on isomorphism classes of representations of Γ over $\overline{\mathbb{F}}_\ell$.

Lemma 8.5. *If V is irreducible and $V \cong V \circ \sigma$ as Γ-representations, then there is a unique extension of the Γ-action on V to an action of $\Gamma \rtimes \langle\sigma\rangle$.*

Proof. If $A\colon V \xrightarrow{\sim} V \circ \sigma$ as Γ-representations, then Schur's lemma implies that the composition $A^\ell\colon V \xrightarrow{\sim} V \circ \sigma^\ell = V$ is multiplication by a scalar, say λ. Then defining σ to act as $\lambda^{-1/\ell}A$ gives an extension of the desired form. Schur's lemma also implies that A is unique up to scalar, and scaling A evidently results in the same extension. □

For a $\mathbb{Z}/\ell\mathbb{Z} \cong \langle\sigma\rangle$-module V, we have defined Tate cohomology groups $\mathrm{T}^0(V)$ and $\mathrm{T}^1(V)$. If the σ-action on V extends to an action of $\Gamma \rtimes \langle\sigma\rangle$, then $\mathrm{T}^0(V)$ and $\mathrm{T}^1(V)$ inherit an action of the subgroup of Γ fixed by σ, which we denote $Z_\Gamma(\sigma)$.

8.2.3 *Torsion in integral cohomology.* We write
$$H_c^i(Y_{\mathbb{T}_r,\mathbb{U}_r}; \overline{\mathbb{Z}}_\ell) \cong H_c^i(Y_{\mathbb{T}_r,\mathbb{U}_r}; \overline{\mathbb{Z}}_\ell)_{\mathrm{tf}} \oplus H_c^i(Y_{\mathbb{T}_r,\mathbb{U}_r}; \overline{\mathbb{Z}}_\ell)_{\mathrm{tors}}$$
for the decomposition into the torsion-free and torsion summands, respectively. For a character $\widetilde{\theta}\colon \mathbb{T}_r(\mathbb{F}_q) \to \overline{\mathbb{Z}}_\ell^\times$, we write
$$H_c^i(Y_{\mathbb{T}_r,\mathbb{U}_r}; \overline{\mathbb{Z}}_\ell)_{\mathrm{tf},\widetilde{\theta}} := H_c^i(Y_{\mathbb{T}_r,\mathbb{U}_r}; \overline{\mathbb{Z}}_\ell)_{\mathrm{tf}} \otimes_{\overline{\mathbb{Z}}_\ell[\mathbb{T}_r(\mathbb{F}_q)]} \widetilde{\theta}.$$
Thus $H_c^i(Y_{\mathbb{T}_r,\mathbb{U}_r}; \overline{\mathbb{Z}}_\ell)_{\mathrm{tf},\widetilde{\theta}}$ is a lattice in $H_c^i(Y_{\mathbb{T}_r,\mathbb{U}_r}; \overline{\mathbb{Q}}_\ell)_{\widetilde{\theta}}$.

Write $\theta := \widetilde{\theta} \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell\colon \mathbb{T}_r(\mathbb{F}_q) \to \overline{\mathbb{F}}_\ell^\times$. We consider the hypothesis that
$$H_c^*(Y_{\mathbb{T}_r,\mathbb{U}_r}; \overline{\mathbb{F}}_\ell)_\theta \text{ is non-zero in only one degree.} \tag{8.3}$$
We will then be interested in studying $\mathrm{T}^i(\sigma, H_c^*(Y_{\mathbb{T}_r,\mathbb{U}_r}; \overline{\mathbb{F}}_\ell)_\theta)$ as a representation of $G_{x,0:r+}$, where σ is a Galois automorphism (see § 8.2.4 below). For regular θ, it is expected that $H_c^*(Y_{\mathbb{T}_r,\mathbb{U}_r}; \overline{\mathbb{Q}}_\ell)_\theta$ is non-zero in only one degree. Therefore, if θ is regular and q is sufficiently large, then one would expect (8.3) to hold for all large enough ℓ, as a case of the more general expectation that for any (finite type) variety over a separably closed field, the étale cohomology with coefficients in $\mathbb{Z}_\ell$ should be torsion-free for all sufficiently large ℓ; however, this does not appear to be known in general. When $r = 0$, in which case the $Y_{\mathbb{T}_r,\mathbb{U}_r}$ are the usual Deligne–Lusztig varieties of [DL76], this concentration of degree and torsion-freeness for non-singular θ is proved in [Bro90, Lemma 3.5] applied to the compactification of Deligne–Lusztig varieties, which already furnishes many interesting examples where (8.3) is known. Based on this, it seems natural to make the following conjecture.

Conjecture 8.6. If θ is a 0-toral character in the sense of [CO23, Definition 3.7], then (8.3) holds.

Later we will prove theorems concerning 0-toral θ under the assumption that (8.3) holds. It would be interesting to make these unconditional by proving Conjecture 8.6. As discussed, we already have many unconditional depth-zero examples.

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LEMMA 8.7. *Suppose* $\theta\colon \mathbb{T}_r(\mathbb{F}_q) \to \overline{\mathbb{F}}_\ell^\times$ *satisfies* (8.3) *and let* $\widetilde{\theta}\colon \mathbb{T}_r(\mathbb{F}_q) \to \overline{\mathbb{Z}}_\ell^\times$ *be the composition of* θ *with the Teichmüller lift. Let* i *be the unique degree in which* $H_c^i(Y_{\mathbb{T}_r,\mathbb{U}_r};\overline{\mathbb{F}}_\ell)_\theta$ *is non-zero. Then* $H_c^i(Y_{\mathbb{T}_r,\mathbb{U}_r};\overline{\mathbb{Z}}_\ell)_{\widetilde{\theta}}$ *is a lattice in* $H_c^i(Y_{\mathbb{T}_r,\mathbb{U}_r};\overline{\mathbb{Q}}_\ell)_{\widetilde{\theta}}$.

Proof. We must show that $H_c^i(Y_{\mathbb{T}_r,\mathbb{U}_r};\overline{\mathbb{Z}}_\ell)_{\widetilde{\theta}}$ is torsion-free. The universal coefficient theorem implies that $\big(H_c^i(Y_{\mathbb{T}_r,\mathbb{U}_r};\overline{\mathbb{Z}}_\ell)_{\widetilde{\theta}}\big)_{\mathrm{tors}} \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell$ injects into $H_c^{i+1}(Y_{\mathbb{T}_r,\mathbb{U}_r};\overline{\mathbb{F}}_\ell)_\theta = 0$, so $\big(H_c^i(Y_{\mathbb{T}_r,\mathbb{U}_r};\overline{\mathbb{Z}}_\ell)_{\widetilde{\theta}}\big)_{\mathrm{tors}} = 0$ by Nakayama's lemma. □

8.2.4 *Base change.* Recall that F_ℓ is the degree-ℓ unramified extension of F. Let $T' = \mathrm{Res}_{F_\ell/F}(T_{F_\ell})$, $G' = \mathrm{Res}_{F_\ell/F}(G_{F_\ell})$. We may view x as a point in the apartment of T', in the building of G'/F.

Given $U \subset G_{\breve{F}}$, we have several possible choices for $U' \subset G'_{\breve{F}}$. We will make a particular choice that is advantageous for the upcoming computation. A choice of generator $\sigma \in \mathrm{Gal}(F_\ell/F)$ induces an isomorphism

$$F_\ell \otimes_F \breve{F} \cong \underbrace{\breve{F} \times \cdots \times \breve{F}}_{\ell \text{ times}} =: \breve{F}^\ell$$

sending $x \otimes y \mapsto (xy, \sigma(x)y, \ldots, \sigma^{\ell-1}(x)y)$. This in turn induces

$$G'_{\breve{F}} \cong G_{\breve{F}} \times \cdots \times G_{\breve{F}}. \tag{8.4}$$

The action of σ on the left-hand side transports to the cyclic permutation on the right-hand side. We define $U' \subset G'_{\breve{F}}$ to be the image of $U \times \cdots \times U \subset G^\ell_{\breve{F}}$ under the isomorphism (8.4). Then U' is stable under the σ-action on $G'_{\breve{F}}$. The resulting deep level Deligne–Lusztig variety $Y_{\mathbb{T}'_r,\mathbb{U}'_r}$ therefore carries an action of σ, such that for $g \in G'_{x,0:r+} = \mathbb{G}'_r(\mathbb{F}_q)$ and $y \in Y_{\mathbb{T}'_r,\mathbb{U}'_r}(R)$ (for some ring R),

$$\sigma \cdot (g \cdot y) = \sigma(g) \cdot \sigma(y)$$

and similarly for $t \in T'_{x,0:r+} = \mathbb{T}'_r(\mathbb{F}_q)$. This induces a σ-action on $H_c^i(Y_{\mathbb{T}'_r,\mathbb{U}'_r};\overline{\mathbb{F}}_\ell)$ that is compatible in the same manner with the $\mathbb{G}'_r(\mathbb{F}_q) \times \mathbb{T}'_r(\mathbb{F}_q)$-action. We will consider Tate cohomology with respect to this action.

Recall that for a representation $V/\overline{\mathbb{F}}_\ell$ of a finite group Γ, the *Frobenius twist* of V is the representation

$$V^{(\ell)} := V \otimes_{\overline{\mathbb{F}}_\ell, \mathrm{Frob}_\ell} \overline{\mathbb{F}}_\ell.$$

Now note that $H_c^i(Y_{\mathbb{T}_r,\mathbb{U}_r};\overline{\mathbb{F}}_\ell)$ has an $\mathbb{F}_\ell$-structure induced by cohomology with coefficients in $\mathbb{F}_\ell$, and therefore has an action of $\mathrm{Aut}(\overline{\mathbb{F}}_\ell)$. Let $\theta\colon T_{x,0:r+} = \mathbb{T}_r(\mathbb{F}_q) \to \overline{\mathbb{F}}_\ell^\times$ be a character and $\theta'\colon \mathbb{T}'_r(\mathbb{F}_q) \to \overline{\mathbb{F}}_\ell^\times$ be the composition of θ with the norm map $\mathrm{Nm}\colon \mathbb{T}'_r(\mathbb{F}_q) \to \mathbb{T}_r(\mathbb{F}_q)$. Applying $\mathrm{Frob}_\ell \in \mathrm{Aut}(\overline{\mathbb{F}}_\ell)$ induces an isomorphism of $\mathbb{G}_r(\mathbb{F}_q)$-representations

$$\big(H_c^i(Y_{\mathbb{T}_r,\mathbb{U}_r};\overline{\mathbb{F}}_\ell) \otimes_{\overline{\mathbb{F}}_\ell[\mathbb{T}_r(\mathbb{F}_q)]} \theta\big)^{(\ell)} \xrightarrow{\sim} H_c^i(Y_{\mathbb{T}_r,\mathbb{U}_r};\overline{\mathbb{F}}_\ell) \otimes_{\overline{\mathbb{F}}_\ell[\mathbb{T}_r(\mathbb{F}_q)]} \theta^{\otimes \ell}. \tag{8.5}$$

THEOREM 8.8. *Let* $\theta\colon \mathbb{T}_r(\mathbb{F}_q) \to \overline{\mathbb{F}}_\ell^\times$. *Assume that* $\ell \nmid \#\mathbb{T}_0(\mathbb{F}_q)$.

(i) *If* $H_c^*(Y_{\mathbb{T}_r,\mathbb{U}_r};\overline{\mathbb{F}}_\ell)_\theta \neq 0$, *then* $\mathrm{T}^i(\sigma, H_c^{i'}(Y_{\mathbb{T}'_r,\mathbb{U}'_r};\overline{\mathbb{F}}_\ell)_{\theta'}) \neq 0$ *for some* $i \in \{0,1\}$ *and some* i'.

(ii) *Suppose that* θ *and* θ' *both satisfy* (8.3), *and are non-zero in degrees* j, j', *respectively. Then for each* $i \in \mathbb{Z}/2\mathbb{Z}$ *we have*

$$\mathrm{T}^i\big(\sigma, H_c^{j'}(Y_{\mathbb{T}'_r,\mathbb{U}'_r};\overline{\mathbb{F}}_\ell)_{\theta'} \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell\big) \cong (H_c^j(Y_{\mathbb{T}_r,\mathbb{U}_r,F};\overline{\mathbb{F}}_\ell)_\theta)^{(\ell)}$$

as representations of $\mathbb{G}_r(\mathbb{F}_q)$. *In particular, the left-hand side is independent of* i.

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Proof. We begin with some analysis of the relevant varieties. For a scheme Y over $\mathbb{F}_q$ we will write $\overline{Y}$ for its base change to $\overline{\mathbb{F}}_q$.

Parallel to (8.4), the choice of generator $\sigma \in \mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q)$ identifies

$$\overline{\mathbb{G}}'_r \xrightarrow{\sim} \overline{\mathbb{G}}_r \times \cdots \times \overline{\mathbb{G}}_r =: \overline{\mathbb{G}}_r^\ell. \tag{8.6}$$

The isomorphism (8.6) transports:

- the automorphism σ on the left-hand side to the cyclic rotation action on the right-hand side;
- the action of Fr_q on the left-hand side (coming from the $\mathbb{F}_q$-rational structure $\mathbb{G}'_r$ for $\overline{\mathbb{G}}'_r$) to the composition of the cyclic rotation with $(\mathrm{Fr}_q, \dots, \mathrm{Fr}_q)$ on the right-hand side, where each factor of Fr_q comes from the $\mathbb{F}_q$-rational structure $\mathbb{G}_r$ on $\overline{\mathbb{G}}_r$. Let us denote this endomorphism of $\overline{\mathbb{G}}_r^\ell$ by $F := \sigma \circ (\mathrm{Fr}_q, \dots, \mathrm{Fr}_q)$.

Similar remarks apply to $\mathbb{T}'_r \cong \mathrm{Res}_{\mathbb{F}_{q^\ell}/\mathbb{F}_q}(\mathbb{T}_{r,\mathbb{F}_{q^\ell}})$.

Under (8.6), the variety $Y_{\mathbb{T}'_r,\mathbb{U}'_r}$ has the presentation

$$Y_{\mathbb{T}'_r,\mathbb{U}'_r} \cong \{(g_1, \dots, g_\ell) \in \overline{\mathbb{G}}_r^\ell \colon (g_1, \dots, g_\ell)^{-1} F(g_1 \dots, g_\ell) \in \mathbb{U}_r^\ell\}/\mathbb{U}_r^\ell \cap F^{-1}\mathbb{U}_r^\ell.$$

The action of σ on $Y_{\mathbb{T}'_r,\mathbb{U}'_r}$ transports to the cyclic rotation on factors in the presentation on the right-hand side.

Lemma 8.9. *The diagonal map $Y_{\mathbb{T}_r,\mathbb{U}_r} \to Y_{\mathbb{T}'_r,\mathbb{U}'_r}$ identifies with the inclusion of the σ-fixed points on $Y_{\mathbb{T}'_r,\mathbb{U}'_r}$, equivariantly for the action of $\mathbb{G}_r(\mathbb{F}_q) \times \mathbb{T}_r(\mathbb{F}_q)$ by left and right translation.*

Proof. Immediate upon writing down the definitions. □

Corollary 8.10. *For each $i \in \mathbb{Z}$, restriction induces an isomorphism*

$$\mathrm{T}^i(Y_{\mathbb{T}'_r,\mathbb{U}'_r}; \overline{\mathbb{F}}_\ell) \cong \mathrm{T}^i(Y_{\mathbb{T}_r,\mathbb{U}_r}; \overline{\mathbb{F}}_\ell)$$

which is equivariant with respect to the action of $Z_{\mathbb{G}'_r(\mathbb{F}_q)\times\mathbb{T}'_r(\mathbb{F}_q)}(\sigma) = \mathbb{G}_r(\mathbb{F}_q) \times \mathbb{T}_r(\mathbb{F}_q)$, induced by left and right translation on $Y_{\mathbb{T}'_r,\mathbb{U}'_r}$ and $Y_{\mathbb{T}_r,\mathbb{U}_r}$.

Proof. This follows from Lemma 8.9 and the equivariant localization theorem of [Fen24, § 3.4.5]. □

With these preparations, we are now ready for the proofs of parts (i) and (ii) of the theorem.

Proof of (i). Restricting θ' along $\mathbb{T}_r(\mathbb{F}_q) \hookrightarrow \mathbb{T}'_r(\mathbb{F}_q)$ yields $\theta'|_{\mathbb{T}_r(\mathbb{F}_q)} = \theta^{\otimes \ell}$, so Corollary 8.10 induces an isomorphism of $\mathbb{G}_r(\mathbb{F}_q)$-representations

$$\mathrm{T}^*(Y_{\mathbb{T}'_r,\mathbb{U}'_r}; \overline{\mathbb{F}}_\ell) \otimes_{\overline{\mathbb{F}}_\ell[\mathbb{T}_r(\mathbb{F}_q)]} (\theta'|_{\mathbb{T}_r(\mathbb{F}_q)}) \cong \mathrm{T}^*(Y_{\mathbb{T}_r,\mathbb{U}_r}; \overline{\mathbb{F}}_\ell) \otimes_{\overline{\mathbb{F}}_\ell[\mathbb{T}_r(\mathbb{F}_q)]} \theta^{\otimes \ell}. \tag{8.7}$$

Similarly to (8.5), we have

$$\left(\mathrm{T}^*(Y_{\mathbb{T}_r,\mathbb{U}_r}; \overline{\mathbb{F}}_\ell) \otimes_{\overline{\mathbb{F}}_\ell[\mathbb{T}_r(\mathbb{F}_q)]} \theta\right)^{(\ell)} \xrightarrow{\sim} \mathrm{T}^*(Y_{\mathbb{T}_r,\mathbb{U}_r}; \overline{\mathbb{F}}_\ell) \otimes_{\overline{\mathbb{F}}_\ell[\mathbb{T}_r(\mathbb{F}_q)]} \theta^{\otimes \ell}.$$

Note that Frobenius twisting shows that $H_c^*(Y_{\mathbb{T}_r,\mathbb{U}_r}; \overline{\mathbb{F}}_\ell)_\theta \neq 0 \iff H_c^*(Y_{\mathbb{T}_r,\mathbb{U}_r}; \overline{\mathbb{F}}_\ell)_{\theta^{\otimes \ell}} \neq 0$.

As explained in § 8.2.2, there are two spectral sequences abutting to $\mathrm{T}^i(Y_{\mathbb{T}'_r,\mathbb{U}'_r}; \overline{\mathbb{F}}_\ell)$. One degenerates at E_2 and has $E_2^{ij} = H_c^j((Y_{\mathbb{T}'_r,\mathbb{U}'_r})^\sigma; \overline{\mathbb{F}}_\ell)$, which by Lemma 8.9 is $\mathbb{G}_r(\mathbb{F}_q) \times \mathbb{T}_r(\mathbb{F}_q)$-equivariantly isomorphic to $H_c^j(Y_{\mathbb{T}_r,\mathbb{U}_r}; \overline{\mathbb{F}}_\ell)$, and the other has $E_2^{ij} = \mathrm{T}^i(\sigma, H_c^j(Y_{\mathbb{T}'_r,\mathbb{U}'_r}; \overline{\mathbb{F}}_\ell))$.

Since $\ker(\mathbb{T}_r(\mathbb{F}_q) \to \mathbb{T}_0(\mathbb{F}_q))$ is pro-p, the assumption that $\ell \nmid \mathbb{T}_0(\mathbb{F}_q)$ implies that $\ell \nmid \mathbb{T}_r(\mathbb{F}_q)$. Therefore, $- \otimes_{\overline{\mathbb{F}}_\ell[\mathbb{T}_r(\mathbb{F}_q)]} \theta$ implements projection to a summand. By the preceding paragraph,

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the Jordan–Hölder factors (as $\mathbb{G}_r(\mathbb{F}_q)$-representations) of $H_c^j(Y_{\mathbb{T}_r,\mathbb{U}_r};\overline{\mathbb{F}}_\ell)_{\theta^{\otimes\ell}}$ are also subquotients of some $\mathrm{T}^i(H_c^{j'}(Y_{\mathbb{T}'_r,\mathbb{U}'_r};\overline{\mathbb{F}}_\ell))\otimes_{\overline{\mathbb{F}}_\ell[\mathbb{T}_r(\mathbb{F}_q)]}\theta^{\otimes\ell}$. Therefore the proof of (i) will be concluded by the following lemma, which allows us to 'commute' the formation of Tate cohomology with the projection to an isotypic component.

Lemma 8.11. *We have that*

$$\mathrm{T}^i(H_c^{j'}(Y_{\mathbb{T}'_r,\mathbb{U}'_r};\overline{\mathbb{F}}_\ell))\otimes_{\overline{\mathbb{F}}_\ell[\mathbb{T}_r(\mathbb{F}_q)]}\theta^{\otimes\ell}\cong\mathrm{T}^i(H_c^{j'}(Y_{\mathbb{T}'_r,\mathbb{U}'_r};\overline{\mathbb{F}}_\ell)_{\theta'})$$

as $\mathbb{G}_r(\mathbb{F}_q)$-representations.

Proof. Since $\#\mathbb{T}_r(\mathbb{F}_q)$ is coprime to ℓ, we have that

$$\mathrm{T}^i(H_c^{j'}(Y_{\mathbb{T}'_r,\mathbb{U}'_r};\overline{\mathbb{F}}_\ell))\otimes_{\overline{\mathbb{F}}_\ell[\mathbb{T}_r(\mathbb{F}_q)]}\theta^{\otimes\ell}\cong\mathrm{T}^i\big(H_c^{j'}(Y_{\mathbb{T}'_r,\mathbb{U}'_r};\overline{\mathbb{F}}_\ell)\otimes_{\overline{\mathbb{F}}_\ell[\mathbb{T}_r(\mathbb{F}_q)]}\theta^{\otimes\ell}\big)$$

as $\mathbb{G}_r(\mathbb{F}_q)$-representations. Here on the right-hand side we have projected to a $\mathbb{T}_r(\mathbb{F}_q)$-isotypic component before forming Tate cohomology, and we need to show that the same answer is computed if we instead project to a particular $\mathbb{T}'_r(\mathbb{F}_q)$-isotypic component.

Let $\{\vartheta_l\}$ be the set of characters of $\mathbb{T}'_r(\mathbb{F}_q)$ that extend $\theta^{\otimes\ell}$ on $\mathbb{T}_r(\mathbb{F}_q)\subset\mathbb{T}'_r(\mathbb{F}_q)$, indexed so that $\vartheta_1=\theta'$. Then

$$\mathrm{T}^i\big(H_c^{j'}(Y_{\mathbb{T}'_r,\mathbb{U}'_r};\overline{\mathbb{F}}_\ell)\otimes_{\overline{\mathbb{F}}_\ell[\mathbb{T}_r(\mathbb{F}_q)]}\theta^{\otimes\ell}\big)\cong\mathrm{T}^i\bigg(\bigoplus_l H_c^{j'}(Y_{\mathbb{T}'_r,\mathbb{U}'_r};\overline{\mathbb{F}}_\ell)\otimes_{\overline{\mathbb{F}}_\ell[\mathbb{T}'_r(\mathbb{F}_q)]}\vartheta_l\bigg).\tag{8.8}$$

We claim that θ' is the only σ-equivariant extension of $\theta^{\otimes\ell}$ to $\mathbb{T}'_r(\mathbb{F}_q)$. Indeed, any σ-equivariant character on $\mathbb{T}'_r(\mathbb{F}_q)$ factors through the norm map[8] $\mathbb{T}'_r(\mathbb{F}_q)\xrightarrow{\mathrm{Nm}}\mathbb{T}_r(\mathbb{F}_q)$, and the composition $\mathbb{T}_r(\mathbb{F}_q)\hookrightarrow\mathbb{T}'_r(\mathbb{F}_q)\xrightarrow{\mathrm{Nm}}\mathbb{T}_r(\mathbb{F}_q)$ is raising to the ℓth power. Since $\overline{\mathbb{F}}_\ell^\times$ is ℓ-torsion-free, there is only one character $\mathbb{T}_r(\mathbb{F}_q)\to\overline{\mathbb{F}}_\ell^\times$ that inflates to $\theta^{\otimes\ell}$ under raising to the ℓth power, namely θ.

Therefore, σ permutes the set $\{\vartheta_l\colon l>1\}$ without any fixed points, necessarily grouping them into free orbits, and it therefore freely permutes the summands of

$$\bigoplus_{l>1}H_c^{j'}(Y_{\mathbb{T}'_r,\mathbb{U}'_r};\overline{\mathbb{F}}_\ell)\otimes_{\overline{\mathbb{F}}_\ell[\mathbb{T}'_r(\mathbb{F}_q)]}\vartheta_l.\tag{8.9}$$

Since Tate cohomology of a free σ-module vanishes,

$$\mathrm{T}^i\bigg(\bigoplus_{l>1}H_c^{j'}(Y_{\mathbb{T}'_r,\mathbb{U}'_r};\overline{\mathbb{F}}_\ell)\otimes_{\overline{\mathbb{F}}_\ell[\mathbb{T}'_r(\mathbb{F}_q)]}\vartheta_l\bigg)=0.$$

Therefore, (8.8) is $\mathbb{G}_r(\mathbb{F}_q)$-equivariantly isomorphic to $\mathrm{T}^i(H_c^{j'}(Y_{\mathbb{T}'_r,\mathbb{U}'_r};\overline{\mathbb{F}}_\ell)_{\theta'})$. □

Proof of (ii). We consider the Tate spectral sequence, as in part (i). The additional assumptions imply that:

- $R\Gamma_c(Y_{\mathbb{T}'_r,\mathbb{U}'_r};\overline{\mathbb{F}}_\ell)_{\theta'}$ is concentrated in a single degree j', and so is quasi-isomorphic to $H_c^{j'}(Y_{\mathbb{T}'_r,\mathbb{U}'_r};\overline{\mathbb{F}}_\ell)_{\theta'}$;
- $R\Gamma_c(Y_{\mathbb{T}_r,\mathbb{U}_r};\overline{\mathbb{F}}_\ell)_{\theta}$ is concentrated in a single degree j, and so is quasi-isomorphic to $H_c^{j}(Y_{\mathbb{T}_r,\mathbb{U}_r};\overline{\mathbb{F}}_\ell)_{\theta}$.

[8] At the referee's suggestion, we add justification for this assertion. Obviously any character factoring through the norm map is σ-equivariant. Since ℓ is assumed coprime to $\#\mathbb{T}_r(\mathbb{F}_q)$, raising to the ℓth power is an automorphism of $\mathbb{T}_r(\mathbb{F}_q)$. Thus, if θ' is a σ-equivariant character, its factorization through the norm map can be reconstructed by restricting to $\mathbb{T}_r(\mathbb{F}_q)\subset\mathbb{T}'_r(\mathbb{F}_q)$ and then extracting ℓth roots.

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Corollary 8.10 then implies that

$$\mathrm{T}^i\big(\sigma, H_c^{j'}(Y_{\mathbb{T}'_r,\mathbb{U}'_r};\overline{\mathbb{F}}_\ell)\big) \cong H_c^j(Y_{\mathbb{T}_r,\mathbb{U}_r};\overline{\mathbb{F}}_\ell) \tag{8.10}$$

as $\mathbb{G}_r(\mathbb{F}_q) \times \mathbb{T}_r(\mathbb{F}_q)$-representations. Using Lemma 8.11, we find that

$$\mathrm{T}^i\big(\sigma, H_c^{j'}(Y_{\mathbb{T}'_r,\mathbb{U}'_r};\overline{\mathbb{F}}_\ell)_{\theta'}\big) \cong \mathrm{T}^{i+j'}(Y_{\mathbb{T}'_r,\mathbb{U}'_r};\overline{\mathbb{F}}_\ell) \otimes_{\overline{\mathbb{F}}_\ell[\mathbb{T}_r(\mathbb{F}_q)]} \theta^{\otimes \ell} \tag{8.11}$$

as $\mathbb{G}_r(\mathbb{F}_q)$-representations. Projecting (8.10) to the $\theta^{\otimes\ell}$-isotypic component and then using (8.5) to relate the Frobenius twist of the θ-isotypic component with the $\theta^{\otimes\ell}$-isotypic component, the proof is concluded. □

8.3 Tate cohomology of compact inductions

We study the relationship between compact induction and Tate cohomology. In this section, we let G'/F be a reductive group and $\mathsf{G}' = G'(F)$ with an action of $\mathbb{Z}/\ell\mathbb{Z} \cong \langle\sigma\rangle$, $\mathsf{H}' \subset \mathsf{G}'$ a σ-invariant open subgroup, $\mathsf{G} = (\mathsf{G}')^\sigma$ and $\mathsf{H} = (\mathsf{H}')^\sigma$.

Proposition 8.12. *Let π be a finite-dimensional representation of H. If $\mathsf{G}/\mathsf{H} \xrightarrow{\sim} (\mathsf{G}'/\mathsf{H}')^\sigma$, then*

$$\mathrm{T}^i\big(\text{c-Ind}_{\mathsf{H}'}^{\mathsf{G}'}\pi\big) \cong \text{c-Ind}_{\mathsf{H}}^{\mathsf{G}}(\mathrm{T}^i\pi)$$

as G'-representations.

Proof. A special case appears in [Ron16, Proposition 14], which already contains the main ideas of the proof. We use Bernstein and Zelevinsky's perspective of l-sheaves on l-spaces. There is an equivalence of categories between G'-equivariant sheaves on G'/H' and representations of H', which we denote $\mathcal{F}_\pi \leftrightarrow \pi$. Furthermore, the Tate cohomology of $\mathcal{F}_\pi$ as a sheaf transports to the Tate cohomology of π as a representation.

Under this equivalence and the analogous equivalence between G'-equivariant sheaves on a point and G'-representations, the G'-representation $\text{c-Ind}_{\mathsf{H}'}^{\mathsf{G}'}(\pi)$ corresponds to the G'-equivariant sheaf $\mathrm{pr}_!\,\mathcal{F}_\pi$ on pt, where $\mathrm{pr}\colon \mathsf{G}'/\mathsf{H}' \to \mathrm{pt}$, under the functor of taking global sections for G'-equivariant sheaves on pt. By [TV16, § 3.3], the restriction map on sections induces

$$\mathrm{T}^i\big(\text{c-Ind}_{\mathsf{H}'}^{\mathsf{G}'}\pi\big) \xrightarrow{\sim} \text{c-Ind}_{\mathsf{H}}^{\mathsf{G}}(\mathrm{T}^i\pi),$$

which completes the proof. □

Next we work out some situations where the hypothesis of Proposition 8.12 is satisfied. We have the long exact sequence

$$0 \to \mathsf{H} \to \mathsf{G} \to \mathsf{G}/\mathsf{H} \to H^1(\langle\sigma\rangle, \mathsf{H}') \to \cdots$$

so $\mathsf{G}/\mathsf{H} \xrightarrow{\sim} (\mathsf{G}'/\mathsf{H}')^\sigma$ if $H^1(\langle\sigma\rangle, \mathsf{H}') = 0$.

Lemma 8.13. *Let F_ℓ/F be the unramified $\mathbb{Z}/\ell\mathbb{Z}$-extension of local fields of characteristic $p \neq \ell$. Let H be a connected algebraic group over $\mathcal{O}_F$ and $\mathsf{H}' = H(\mathcal{O}_{F_\ell})$, with the action of $\mathrm{Gal}(F_\ell/F) \cong \langle\sigma\rangle$ by Galois conjugation on points. Then $H^1(\sigma, \mathsf{H}') = 0$.*

Proof. Let H^+ be the kernel of the reduction map $\mathsf{H}' = H(\mathcal{O}_{F_\ell}) \to H(\mathbb{F}_{q^\ell})$. From the long exact sequence of cohomology, we have the exact sequence

$$\cdots \to H^1(\sigma, \mathsf{H}'_+) \to H^1(\sigma, \mathsf{H}') \to H^1(\sigma, H(\mathbb{F}_{q^\ell})) \to \cdots.$$

Since H'_+ is pro-p, we have $H^1(\sigma, \mathsf{H}'_+) = 0$. By Lang's theorem, $H^1(\sigma, H(\mathbb{F}_{q^\ell})) = 0$. Therefore, $H^1(\sigma, \mathsf{H}') = 0$. □

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We shall be particularly interested in the case where $\mathsf{G}' = G(F_\ell)$ and $\mathsf{H}' = Z(\mathsf{G}')G(F_\ell)_{x,0}$ for $x \in \mathcal{B}(G/F_\ell)$ fixed by $\mathrm{Gal}(F_\ell/F)$, which guarantees that H' is stable under $\mathrm{Gal}(F_\ell/F)$. By unramified descent in Bruhat–Tits theory, $G(F_\ell)_{x,0}^{\mathrm{Gal}(F_\ell/F)} = G(F)_{x,0}$, so we have $\mathsf{G} := (\mathsf{G}')^{\mathrm{Gal}(F_\ell/F)} = G(F)$ and $\mathsf{H} := (\mathsf{H}')^{\mathrm{Gal}(F_\ell/F)} = Z(\mathsf{G})G(F)_{x,0}$.

Let $Z \subset G$ be the maximal central torus. Then Z_{F_ℓ} is the maximal central torus of G_{F_ℓ}.

LEMMA 8.14. *Assume that:*

(i) *the action of* $\mathrm{Gal}(F_\ell/F)$ *on the cocharacter group* $X^*(Z_{F_\ell})$ *is trivial;*
(ii) *the component group* $Z(\mathsf{G}')/Z$ *has order coprime to* ℓ.

Let $H' = Z(\mathsf{G}')G(F_\ell)_{x,0}$. *Then* $H^1(\sigma, \mathsf{H}') = 0$.

Example. Assumption (i) is satisfied, for example, if G is split reductive, or whenever G is (not necessarily split and) semisimple. For any given G, assumption (ii) is satisfied for all large enough ℓ.

Proof. Since G is assumed to be split over an unramified extension of F, Z is an unramified torus, hence has a canonical integral model $\mathcal{Z}/\mathcal{O}_F$ (the 'connected Néron model'), which has the property that $\mathcal{Z}(\mathcal{O}_F)$ is the maximal bounded subgroup of $Z(F)$. We have a split short exact sequence

$$0 \to \mathcal{Z}(\mathcal{O}_{F_\ell}) \to Z(F_\ell) \to Z(F_\ell)/\mathcal{Z}(\mathcal{O}_{F_\ell}) \to 0$$

where $Z(F_\ell)/\mathcal{Z}(\mathcal{O}_{F_\ell}) \cong X_*(Z_{F_\ell})$ is torsion-free, and $\mathcal{Z}(\mathcal{O}_{F_\ell}) = Z(F_\ell) \cap G(F_\ell)_{x,0} \subset G(F_\ell)_{x,0}$. Therefore, $Z(F_\ell)G(F_\ell)_{x,0} \cong X_*(Z_{F_\ell}) \times G(F_\ell)_{x,0}$ so

$$H^1(\sigma, Z(F_\ell)G(F_\ell)_{x,0}) \cong H^1(\sigma, X_*(Z_{F_\ell})) \times H^1(\sigma, G(F_\ell)_{x,0}).$$

Now, $H^1(\sigma, X_*(Z_{F_\ell})) = 0$ because the assumed condition (i) implies $H^1(\sigma, X_*(Z_{F_\ell})) \cong \mathrm{Hom}(\langle\sigma\rangle, X_*(Z_{F_\ell})) = 0$. Since $G(F_\ell)_{x,0}$ is the group of $\mathcal{O}_{F_\ell}$-points of a connected Bruhat–Tits group scheme, the group $H^1(\sigma, G(F_\ell)_{x,0})$ vanishes by Lemma 8.13. Therefore, $H^1(\sigma, Z(F_\ell)G(F_\ell)_{x,0}) = 0$.

Finally, assumption (ii) implies that the index of $Z(F_\ell)G(F_\ell)_{x,0}$ is a normal subgroup of $Z(\mathsf{G}')G(F_\ell)_{x,0}$ with finite index coprime to ℓ, so the long exact sequence implies that $H^1(\sigma, Z(\mathsf{G}')G(F_\ell)_{x,0}) = 0$. □

8.4 Toral supercuspidal representations

We will prove base change results for a class of supercuspidal representations studied by Chan and Oi in [CO23], which should correspond (in the language of [CO23]) to 0-toral Howe-unramified supercuspidal L-parameters.

8.4.1 *Assumptions.* We impose the same assumptions as in [CO23, § 7]. In particular, G is a reductive group over F, p is odd and not bad for G, $p \nmid \pi_1(G_{\mathrm{der}})$, and $p \nmid \pi_1(\widehat{G}_{\mathrm{der}})$. The maximal torus $T \subset G$ is unramified elliptic, and $\widetilde{\theta}\colon T(F) \to \overline{\mathbb{Z}}_\ell^\times$ is 0-toral and of depth $r > 0$, i.e. trivial on $T_{x,r+}$ for some $x \in \mathcal{B}(G/F)$. We write $\theta\colon T \to k^\times$ for the reduction of $\widetilde{\theta}$ modulo ℓ (where we recall that k is an algebraic closure of $\mathbb{F}_\ell$). We assume that $R^{\mathbb{G}_r}_{\mathbb{T}_r,\mathbb{U}_r}(\widetilde{\theta})$ is non-zero, which is automatic under a mild regularity hypothesis (see the proof of Lemma 8.4(ii)).

8.4.2 *Work of Chan and Oi.* The assumptions imply that $R^{\mathbb{G}_r}_{\mathbb{T}_r,\mathbb{U}_r}(\widetilde{\theta}) \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{Q}}_\ell$ is irreducible and independent of the choice of $\mathbb{U}_r$, hence we simply abbreviate it as $R^{\mathbb{G}_r}_{\mathbb{T}_r}(\widetilde{\theta})_{\overline{\mathbb{Q}}_\ell}$.

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Furthermore, by [CO23, Theorem 7.2] the $\overline{\mathbb{Q}}_\ell$-representation $\text{c-Ind}_{T(F)G(F)_{x,0}}^{G}(|R_{\mathbb{T}_r}^{\mathbb{G}_r}(\widetilde{\theta})_{\overline{\mathbb{Q}}_\ell}|)$ is irreducible supercuspidal.

Given $\widetilde{\theta}$, there is defined in [CO23, Definition 4.8] a certain sign character $\epsilon^{\text{ram}}[\widetilde{\theta}]\colon T(F) \to \overline{\mathbb{Q}}_\ell^\times$. We define

$$\pi_{T,\widetilde{\theta}\cdot\epsilon^{\text{ram}}[\widetilde{\theta}]} := \text{c-Ind}_{T(F)G(F)_{x,0}}^{G(F)} |R_{\mathbb{T}_r,\mathbb{U}_r}(\widetilde{\theta})_{\overline{\mathbb{Q}}_\ell}|.$$

The twisting by $\epsilon^{\text{ram}}[\widetilde{\theta}]$ is for consistency with Kaletha's indexing of regular supercuspidal representations [Kal19]; $\pi_{T,\widetilde{\theta}\cdot\epsilon^{\text{ram}}[\widetilde{\theta}]}$ is a regular supercuspidal representation by [CO23, Theorem 7.2]. In particular, it is irreducible.

Since $\epsilon^{\text{ram}}[\widetilde{\theta}]$ takes values in $\{\pm 1\}$, we may regard it also as a character valued in $\overline{\mathbb{F}}_\ell^\times$ by reduction modulo ℓ. We may define

$$\pi_{T,\theta\cdot\epsilon^{\text{ram}}[\theta]} := \text{c-Ind}_{T(F)G(F)_{x,0}}^{G(F)} R_{\mathbb{T}_r,\mathbb{U}_r}(\theta) \in K_0(G(F);\overline{\mathbb{F}}_\ell).$$

A priori this is only a class in $K_0(G(F);\overline{\mathbb{F}}_\ell)$, but there are two circumstances in which we can lift it to an honest representation, which will be denoted $|\pi_{T,\theta\cdot\epsilon^{\text{ram}}[\theta]}|$:

- If ℓ is banal, in which case we may lift $\pm R_{\mathbb{T}_r,\mathbb{U}_r}(\theta)$ to a representation $|R_{\mathbb{T}_r,\mathbb{U}_r}(\theta)|$.
- If $H_c^*(Y_{\mathbb{T}_r,\mathbb{U}_r};E)_{\widetilde{\theta}}$ concentrates in a single degree (where we recall from § 2.3 that E denotes some sufficiently large extension of $\mathbb{Q}_\ell$, containing the values of $\widetilde{\theta}$), in which case we define

$$|\pi_{T,\theta\cdot\epsilon^{\text{ram}}[\theta]}| := \text{c-Ind}_{T(F)G(F)_{x,0}}^{G(F)}(H_c^*(Y_{\mathbb{T}_r,\mathbb{U}_r};\mathcal{O})_{\text{tf},\widetilde{\theta}} \otimes_{\mathcal{O}} k).$$

  In this case $|\pi_{T,\theta\cdot\epsilon^{\text{ram}}[\theta]}|$ is the mod ℓ reduction of a $\overline{\mathbb{Z}}_\ell$-lattice in $\pi_{T,\widetilde{\theta}\cdot\epsilon^{\text{ram}}[\widetilde{\theta}]}$, namely the one induced by the lattice $H_c^*(Y_{\mathbb{T}_r,\mathbb{U}_r};\mathcal{O})_{\text{tf},\widetilde{\theta}} \subset |R_{\mathbb{T}_r,\mathbb{U}_r}(\widetilde{\theta})|$. Hence, if ℓ is banal, then this definition coincides with the one in the previous bullet point, so that $|\pi_{T,\theta\cdot\epsilon^{\text{ram}}[\theta]}|$ is unambiguously defined. Furthermore, if (8.3) is satisfied, then by Lemma 8.7 we have

$$|\pi_{T,\theta\cdot\epsilon^{\text{ram}}[\theta]}| \cong \text{c-Ind}_{T(F)G(F)_{x,0}}^{G(F)}(H_c^*(Y_{\mathbb{T}_r,\mathbb{U}_r};k)_\theta).$$

Proposition 8.15. *If ℓ is banal for G, then (under our running hypotheses) the representation $\pi_{T,\theta\cdot\epsilon^{\text{ram}}[\theta]}$ is irreducible and cuspidal.*

Proof. Indeed, [CO23, Theorem 7.2] identifies $\pi_{T,\widetilde{\theta}\cdot\epsilon^{\text{ram}}[\widetilde{\theta}]}$ with the compact induction of Yu's representation, denoted ${}^\circ\tau_d$ in [CO23]. Since ℓ is banal the mod ℓ reduction of ${}^\circ\tau_d$ is again irreducible and obtained from Yu's construction. Its compact induction is identified with $\pi_{T,\theta\cdot\epsilon^{\text{ram}}[\theta]}$. The result then follows from [Fin22, Theorem 6.1]. □

Remark 8.16. Marie-France Vignéras has indicated another proof of Proposition 8.15. Let

$$\tau = \left|R_{\mathbb{T}_r}^{\mathbb{G}_r}(\widetilde{\theta})_{\overline{\mathbb{Q}}_\ell}\right|$$

and let $[\tau]$ denote the reduction mod ℓ of any $\overline{\mathbb{Z}}_\ell$-lattice in τ; this is independent of the choice because ℓ is banal. Write $U = T(F)G(F)_{x,0}$. Lemma 3.2 of her article [Vig01] proves a simple criterion for irreducibility of the compact induction of $[\tau]$, which we adapt to our present notation:

(a) $\text{End}_{\overline{\mathbb{F}}_\ell[G]}(\text{c-Ind}_U^G[\tau]) = \overline{\mathbb{F}}_\ell$.

(b) Let π be any irreducible $\overline{\mathbb{F}}_\ell$-representation of G. If $[\tau]$ is contained in the restriction of π to U then $[\tau]$ is also a quotient of $\pi|_U$.

Now (b) is automatic because ℓ is banal and therefore prime to the pro-order of U. On the other hand, it follows from the supercuspidality of $\text{c-Ind}_U^G \tau$ that for any $g \in G \setminus U$ the restrictions of τ

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and $g(\tau)$ to the intersection $U \cap gUg^{-1}$ are disjoint. Again, since ℓ is banal, the same holds for the restrictions of $[\tau]$ and $g([\tau])$. Point (a) then follows, and this implies irreducibility. Cuspidality is then a consequence of [Vig96, Theorem II.2.7], because the matrix coefficients of the compactly induced representation are compact modulo center.

8.4.3 *Base change.* Let G be a reductive group over F and F'/F a $\mathbb{Z}/\ell\mathbb{Z}$-extension. Let π' be an irreducible (admissible) representation of $G(F')$ over $\overline{\mathbb{F}}_\ell$. Choose a generator σ of $\mathrm{Gal}(F'/F) \cong \mathbb{Z}/\ell\mathbb{Z}$. Then σ acts on $G(F')$ through its Galois action on E. We say that π' is *σ-fixed* if $\pi' \cong \pi' \circ \sigma$ as $G(F')$-representations.

LEMMA 8.17 [TV16, Proposition 6.1]. *If π' is σ-fixed, then the action of $G(F')$ extends uniquely to a $G(F') \rtimes \langle\sigma\rangle$-action.*

The *Tate cohomology* groups of π', with respect to the σ-action, are

$$\mathrm{T}^0(\pi') := \frac{\ker(1-\sigma\colon \pi' \to \pi')}{(1+\sigma+\cdots+\sigma^{\ell-1})\cdot\pi'}, \quad \mathrm{T}^1(\pi') := \frac{\ker(1+\sigma+\cdots+\sigma^{\ell-1}\colon \pi' \to \pi')}{(1-\sigma)\cdot\pi'}.$$

The $G(F')$-action on π' induces an action of $G(F)$ on $\mathrm{T}^i(\pi')$.

Let π be an irreducible admissible representation of $G(F)$ over $\overline{\mathbb{F}}_\ell$, and π' be an irreducible admissible representation of $G(F')$ over $\overline{\mathbb{F}}_\ell$. Recall that in Definition 7.1 we defined what it means for π' to be a *base change* of π. In this situation we say that π is a *base change descent* of π'. Addressing [TV16, Conjecture 6.5], it was proved in [Fen24, Theorem 1.2] that if F has characteristic $p \neq \ell$, and ℓ is odd and good for $\widehat{G}$, then any irreducible $G(F)$-subquotient π of $\mathrm{T}^i(\pi')$ base-changes to $(\pi')^{(\ell)}$, the Frobenius twist of π'. The theorem below computes Tate cohomology of mod ℓ toral supercuspidal representations for $F' = F_\ell$ the unramified extension of order ℓ.

THEOREM 8.18. *Let G, T, θ be as in § 8.4.1. Let G', T', θ' be as in § 8.2.4. Assume that $T(F_\ell)$ is elliptic.*

(i) *Assume that $\ell \nmid \#\mathbb{T}_0(\mathbb{F}_q)$ and G satisfies the assumptions of Lemma 8.14. If $H_c^*(Y_{\mathbb{T}_r,\mathbb{U}_r}, \overline{\mathbb{F}}_\ell)_\theta \neq 0$, then there exist $i \in \{0,1\}$, i' such that*

$$\mathrm{T}^i\Big(\mathrm{c\text{-}Ind}_{T(F_\ell)G(F_\ell)_{x,0}}^{G(F_\ell)} H_c^{i'}(Y_{\mathbb{T}'_r,\mathbb{U}'_r}; \overline{\mathbb{F}}_\ell)_{\theta'}\Big) \neq 0.$$

(ii) *In addition to the assumptions from (i), suppose further that θ and θ' both satisfy (8.3). Then we have an isomorphism of $G(F)$-representations for each $i \in \mathbb{Z}/2\mathbb{Z}$,*

$$\mathrm{T}^i(\pi_{T',\theta'\cdot\epsilon^{\mathrm{ram}}[\theta']}) \cong \pi^{(\ell)}_{T,\theta\cdot\epsilon^{\mathrm{ram}}[\theta]}.$$

In particular, the left-hand side is independent of i.

Proof. (i) Since $T(F_\ell)$ is elliptic, we have $T(F_\ell)G(F_\ell)_{x,0} = Z(G(F_\ell))G(F_\ell)_{x,0}$. By Lemma 8.14, we may then apply Proposition 8.12 to deduce that

$$\mathrm{T}^i\Big(\mathrm{c\text{-}Ind}_{T(F_\ell)G(F_\ell)_{x,0}}^{G(F_\ell)} H_c^{i'}(Y_{\mathbb{T}'_r,\mathbb{U}'_r}; \overline{\mathbb{F}}_\ell)_{\theta'}\Big) = \mathrm{c\text{-}Ind}_{T(F_\ell)G(F_\ell)_{x,0}}^{G(F_\ell)} \mathrm{T}^i\big(H_c^{i'}(Y_{\mathbb{T}'_r,\mathbb{U}'_r}; \overline{\mathbb{F}}_\ell)_{\theta'}\big).$$

By Theorem 8.8, there exist i and i' for which $\mathrm{T}^i(H_c^{i'}(Y_{\mathbb{T}'_r,\mathbb{U}'_r}; \overline{\mathbb{F}}_\ell)_{\theta'} \neq 0$.

(ii) Let j and j' be the non-vanishing degrees of $H_c^*(Y_{\mathbb{T}_r,\mathbb{U}_r}; \overline{\mathbb{F}}_\ell)_\theta$ and $H_c^*(Y_{\mathbb{T}'_r,\mathbb{U}'_r}; \overline{\mathbb{F}}_\ell)_{\theta'}$, respectively. By Lemma 8.7, the assumptions imply that:

- $H_c^*(Y_{\mathbb{T}'_r,\mathbb{U}'_r}; \overline{\mathbb{Z}}_\ell)_{\mathrm{tf},\widetilde{\theta}'} \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell \cong H_c^*(Y_{\mathbb{T}'_r,\mathbb{U}'_r}; \overline{\mathbb{F}}_\ell)_{\theta'}$ as representations of $T(F_\ell)G(F_\ell)_{x,0}$, where $T(F_\ell)$ acts through θ' and $G(F_\ell)_{x,0}$ acts through inflation from $G(F_\ell)_{x,0:r+}$;

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- $H_c^*(Y_{\mathbb{T}_r,\mathbb{U}_r};\overline{\mathbb{Z}}_\ell)_{\mathrm{tf},\widetilde{\theta}} \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell \cong H_c^*(Y_{\mathbb{T}_r,\mathbb{U}_r};\overline{\mathbb{F}}_\ell)_\theta$ as representations of $T(F)G(F)_{x,0}$, where $T(F)$ acts through θ and $G(F)_{x,0}$ acts through inflation from $G(F)_{x,0:r+}$.

By Lemma 8.14, we may apply Proposition 8.12 to deduce that

$$\mathrm{T}^i\Big(\text{c-Ind}_{T(F_\ell)G(F_\ell)_{x,0}}^{G(F_\ell)} H_c^{j'}(Y_{\mathbb{T}'_r,\mathbb{U}'_r};\overline{\mathbb{F}}_\ell)_{\theta'}\Big) \cong \text{c-Ind}_{T(F)G(F)_{x,0}}^{G(F)} \mathrm{T}^i\big(H_c^{j'}(Y_{\mathbb{T}'_r,\mathbb{U}'_r};\overline{\mathbb{F}}_\ell)_{\theta'}\big) \tag{8.12}$$

as $G(F)$-representations. Next, the assumptions that θ and θ' satisfy (8.3) allow us to apply Theorem 8.8(ii) to deduce that

$$\mathrm{T}^i\big(H_c^{j'}(Y_{\mathbb{T}'_r,\mathbb{U}'_r};\overline{\mathbb{F}}_\ell)_{\theta'}\big) \cong (H_c^j(Y_{\mathbb{T}_r,\mathbb{U}_r};\overline{\mathbb{F}}_\ell)_\theta)^{(\ell)}$$

as $T(F)G(F)_{x,0}$-representations. We then conclude by noting that Frobenius twist commutes with compact induction. □

Remark 8.19. For a σ-fixed irreducible representation π' of $G(F_\ell)$, it is typically not obvious that $\mathrm{T}^i(\pi') \neq 0$. (Thanks to [Fen24, Theorem 1.2], this would already imply that π has a base change descent to $G(F)$.) We emphasize that our traction on the toral supercuspidal representations $\pi_{T,\theta\cdot\epsilon^{\mathrm{ram}}[\theta]}$ comes from the *geometric* description of these representations developed in [CI21]. Another advantage of the geometric description, which was observed in [CO23], is that it naturally incorporates the twisting character $\epsilon^{\mathrm{ram}}[\theta]$.

Corollary 8.20. *Let assumptions be as in Theorem 8.18(i),(ii) and suppose ℓ is odd and banal for G. Then $\pi_{T,\theta\cdot\epsilon^{\mathrm{ram}}[\theta]}$ base-changes to $\pi_{T',\theta'\cdot\epsilon^{\mathrm{ram}}[\theta']}$.*

Proof. Under our assumptions, Proposition 8.15 implies that $\pi_{T',\theta'\cdot\epsilon^{\mathrm{ram}}[\theta']}$ and $\pi_{T,\theta\cdot\epsilon^{\mathrm{ram}}[\theta]}$ are irreducible and cuspidal (noting that $\ell \neq p$ is banal for $G(F)$ if and only if ℓ is banal for $G(F_\ell)$, since the residue field cardinalities of F_ℓ and of F are congruent modulo ℓ). The claim then follows from [Fen24, Theorem 1.2], using Theorem 8.18(ii) to calculate the Tate cohomology of $\pi_{T',\theta'\cdot\epsilon^{\mathrm{ram}}[\theta']}$. □

Remark 8.21. One would expect compatibility between the Genestier–Lafforgue correspondence and Kaletha's correspondence for regular supercuspidal representations [Kal19]. No results towards such compatibility are known at present for general groups. In fact, Kaletha's work has not yet been extended to function fields or to mod ℓ representations, but Corollary 8.20 appears to be in accordance with what one would expect from such an extension. Namely, if we instead let F be a local field of characteristic *zero* and residue characteristic $p \neq \ell$ sufficiently large relative to G, and let $\widetilde{\theta}\colon T(F) \to \overline{\mathbb{Q}}_\ell^\times$ be 0-toral, then it is computed in [CO23, § 8] that the L-parameter of $\pi_{T,\widetilde{\theta}\cdot\epsilon^{\mathrm{ram}}[\widetilde{\theta}]}$ (according to Kaletha's correspondence for regular supercuspidal representations) is

$$W_F \xrightarrow{\varphi_{\widetilde{\theta}}} {}^L T(\overline{\mathbb{Q}}_\ell) \xrightarrow{{}^L j} {}^L G(\overline{\mathbb{Q}}_\ell) \tag{8.13}$$

where $\varphi_{\widetilde{\theta}}$ corresponds to $\widetilde{\theta}$ under local class field theory, and ${}^L j$ is determined by $T \hookrightarrow G$. In particular, this implies that $\pi_{T,\widetilde{\theta}\cdot\epsilon^{\mathrm{ram}}[\widetilde{\theta}]}$ base-changes to $\pi_{T',\widetilde{\theta}'\cdot\epsilon^{\mathrm{ram}}[\widetilde{\theta}']}$ under Kaletha's correspondence for regular supercuspidal representations. One would then expect the same of the mod ℓ reductions and local function fields F, which suggests the statement of Corollary 8.20.

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Conflicts of interest
None.

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