Forum of Mathematics, Pi (2024), **Vol. 12**:e13 1–41
doi:10.1017/fmp.2024.10

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RESEARCH ARTICLE

Local parameters of supercuspidal representations

Wee Teck Gan[1], Michael Harris[2], Will Sawin[3] and Raphaël Beuzart-Plessis[4]

[1]Mathematics Department, National University of Singapore, Block S17, 10 Lower Kent Ridge Road Singapore 119076;
E-mail: matgwt@nus.edu.sg.
[2]Department of Mathematics, Columbia University, New York, NY 10027, USA;
E-mail: harris@math.columbia.edu (corresponding author).
[3]Department of Mathematics, Princeton University, Princeton, NJ 08540, USA; E-mail: wsawin@math.princeton.edu.
[4]Institut de Mathématiques de Marseille, Aix-Marseille Université, 13009 Marseille, France;
E-mail: raphael.beuzart-plessis@univ-amu.fr.

Received: 14 August 2023; **Revised:** 10 April 2024; **Accepted:** 14 May 2024

2020 Mathematics Subject Classification: *Primary* – 22E50, 11S37; *Secondary* – 11F80, 11F70

Abstract

For a connected reductive group G over a nonarchimedean local field F of positive characteristic, Genestier-Lafforgue and Fargues-Scholze have attached a semisimple parameter $\mathcal{L}^{ss}(\pi)$ to each irreducible representation π. Our first result shows that the Genestier-Lafforgue parameter of a tempered π can be uniquely refined to a tempered L-parameter $\mathcal{L}(\pi)$, thus giving the unique local Langlands correspondence which is compatible with the Genestier-Lafforgue construction. Our second result establishes ramification properties of $\mathcal{L}^{ss}(\pi)$ for unramified G and supercuspidal π constructed by induction from an open compact (modulo center) subgroup. If $\mathcal{L}^{ss}(\pi)$ is pure in an appropriate sense, we show that $\mathcal{L}^{ss}(\pi)$ is ramified (unless G is a torus). If the inducing subgroup is sufficiently small in a precise sense, we show $\mathcal{L}^{ss}(\pi)$ is wildly ramified. The proofs are via global arguments, involving the construction of Poincaré series with strict control on ramification when the base curve is $\mathbb{P}^1$ and a simple application of Deligne's Weil II.

Contents

1. Introduction

Let G be a connected reductive group over a nonarchimedean local field F and let π be an irreducible smooth representation of $G(F)$. The local Langlands conjecture (LLC) posits that to such a π can be attached an L-parameter

$$\mathcal{L}(\pi) : WD_F \to {}^L G(C),$$

which is an admissible homomorphism from the Weil-Deligne group $WD_F = W_F \times SL_2(C)$ with values in the Langlands L-group ${}^L G$ over an appropriate algebraically closed field C of characteristic 0. In what follows, we take $C = \overline{\mathbb{Q}}_\ell$ with ℓ different from $\mathrm{char}(F)$. One condition of being admissible is that $\mathcal{L}(\pi)$ should be *semisimple*, in the sense that if $\mathcal{L}(\pi)(WD_F) \cap \hat{G}(C)$ (where $\hat{G}(C)$ is the Langlands dual group of G) is contained in a parabolic subgroup $P \subset \hat{G}(C)$, then it is contained in a Levi subgroup of P. Such L-parameters can also be described in the language of semisimple Weil-Deligne parameters (ρ, N), as we recall in §3.

However, when F is of positive characteristic p, Genestier and Lafforgue [GLa] have defined a *semisimple* Weil parameter[1]

$$\mathcal{L}^{ss}(\pi) : W_F \to {}^L G(C).$$

This semisimple parameter $\mathcal{L}^{ss}(\pi)$ should be related to $\mathcal{L}(\pi)$ via

$$\mathcal{L}^{ss}(\pi)(w) = \mathcal{L}(\pi)\left(w, \begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{pmatrix}\right). \tag{1.1}$$

A natural question is then as follows: can the semisimple parameter $\mathcal{L}^{ss}(\pi)$ of Genestier-Lafforgue be enriched to give the true L-parameter $\mathcal{L}(\pi)$?

[1]Fargues and Scholze [FS] subsequently defined such a parameter, and Li Huerta has proved that the two parametrizations are equivalent; see the comment after Conjecture 11.7.

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By the properties of the Genestier-Lafforgue construction (recalled in §2), it suffices to address this question for (essentially) discrete series representations. For such a π, one desiderata of the LLC is that $\mathcal{L}(\pi)$ should be *irreducible* (or *elliptic*) – its image is contained in no proper parabolic P, and in particular is essentially tempered (or equivalently pure). It is not hard to see (cf. Lemma 3.5) that for any semisimple parameter $\mathcal{L}^{ss}(\pi)$, there is at most one essentially tempered L-parameter that could give rise to it via (1.1). The first result of this paper is the following theorem:

Theorem 1.2. *Let G be a connected reductive group over a local function field F of positive characteristic p. Let π be an irreducible discrete series representation of $G(F)$. Then there exists a (necessarily unique) essentially tempered L-parameter $\mathcal{L}(\pi)$ whose associated semisimple parameter (via (1.1)) is the Genestier-Lafforgue parameter $\mathcal{L}^{ss}(\pi)$.*

In particular, for a discrete series π, we have a unique candidate $\mathcal{L}(\pi)$ for its L-parameter. As a consequence, we have the following:

Corollary 1.3. *There is a well-defined*

$$\mathcal{L} : \mathcal{T}(G(F)) \longrightarrow \{\textit{essentially tempered L-parameters}\}$$

refining the construction of Genestier-Lafforgue, where $\mathcal{T}(G(F))$ is the set of equivalence classes of irreducible essentially tempered representations of $G(F)$.

Unfortunately, we are not quite able to show that the image of a discrete series representation under $\mathcal{L}$ is irreducible (or elliptic).

Assume now that G is an unramified reductive group. Our second result concerns the ramification property of the semisimple parameter $\mathcal{L}^{ss}(\pi)$ of a supercuspidal representation π. If the Frobenius eigenvalues of $\mathcal{L}^{ss}(\pi)$ are pure in an appropriate sense, then the LLC asserts that $\mathcal{L}^{ss}(\pi)$ itself is, in fact, *irreducible*. In particular, unless G is a torus, one expects that $\mathcal{L}^{ss}(\pi)$ is *ramified*. More precisely, we show the following:

Theorem 1.4. *Suppose F is the nonarchimedean local field $k((t))$, where $k = \mathbb{F}_q$ is a finite field of order $q = p^m > 5$ for some prime p. Let G be an unramified connected reductive group over F which is not a torus. Assume that π is an irreducible supercuspidal representation of $G(F)$ of the form*

$$\pi \xrightarrow{\sim} \mathrm{c-Ind}_U^{G(F)} \tau$$

for some (irreducible) smooth representation τ of a compact open (modulo center) subgroup U with coefficients in $\overline{\mathbb{Q}}_\ell$.

(i) *Suppose that $\mathcal{L}^{ss}(\pi)$ is pure: if $\mathrm{Frob}_F \in W_F$ is any Frobenius element, then the eigenvalues of $\mathcal{L}^{ss}(\pi)(\mathrm{Frob}_F)$ are all Weil q-numbers of the same weight (in which case we say that π is a **pure supercuspidal** representation). Then $\mathcal{L}^{ss}(\pi) = \mathcal{L}(\pi)$ is ramified: it is nontrivial on the inertia subgroup $I_F \subset W_F$.*

(ii) *Suppose that U is sufficiently small. Then $\mathcal{L}^{ss}(\pi)$ (and hence $\mathcal{L}(\pi)$) is wildly ramified: it is nontrivial on the wild inertia subgroup of W_F.*

Let us make a few remarks:

- Conjecturally, every π is compactly induced in the above sense. This property was recently proved by Fintzen to be true as long as p does not divide the order of the Weyl group of G [Fi]. Moreover, when p is odd and G is a classical group – that is, a unitary, symplectic, or special orthogonal group – Shaun Stevens proved in [St] that every supercuspidal representation is compactly induced. These cases can eventually be treated by comparison to GL(n), but the necessary methods, especially those involving the trace formula, are not yet available.
- It would be good to produce some examples of supercuspidal representations whose Genestier-Lafforgue parameter $\mathcal{L}^{ss}(\pi)$ is pure. This does not seem so simple to do. The Kloosterman

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representations used in §3.3 in the proof of Theorem 1.2 have irreducible parameters and are therefore pure. In §9, we describe a strategy to produce more examples of depth 0 generic supercuspidal representations.
- The precise definition of 'sufficiently small' in (ii) above is given at the beginning of §8.

As an immediate application of Theorem 1.4, we have the following corollary:

Theorem 1.5. *Let $F = k((t))$ and let σ be an irreducible representation of $G(F)$, with G unramified reductive over F. Suppose the Genestier-Lafforgue parameter $\mathcal{L}^{ss}(\sigma)$ is pure and unramified. Suppose every supercuspidal representation of every Levi factor of $G(F)$ contains an $\mathfrak{s}$-type. This is true in particular in the following cases:*

- *p does not divide the order of the Weyl group of G [Fi];*
- *p is odd and G is an unramified classical group (unitary, symplectic, or special orthogonal) [St].*

Then σ is an irreducible constituent of an unramified principal series representation of $G(F)$.

When $G = \mathrm{GL}(n)$, our three theorems are part of the local Langlands correspondence (LLC) proved by Laumon, Rapoport and Stuhler [LRS93]. When G is a split classical group, the results were proved in most cases by Ganapathy and Varma by comparison with Arthur's LLC over p-adic fields, via the method of close local fields [GV, A13]. Thus, the novelty of this paper is that, like the Genestier-Lafforgue result, Theorems 1.2 and 1.4 are valid uniformly for all (unramified) reductive groups, including exceptional groups.

Of course, Theorem 1.4(i) is a much weaker statement than the expected local Langlands correspondence, which not only asserts that $\mathcal{L}^{ss}(\pi)$ is irreducible when π is pure but gives a complete description of the *L-packet* of representations with the given local parameter. However, all proofs of the local Langlands correspondence for $\mathrm{GL}(n)$ begin with the observation that a supercuspidal representation π of $\mathrm{GL}(n, F)$ is not *incorrigible* [H19] – that it becomes a constituent of a principal series representation after a finite series of base changes corresponding to cyclic extensions F_{i+1}/F_i, $i = 0, \ldots, n$, where $F_0 = F$. Since one can choose the F_i so that the Galois parameter restricted to $W_{F_{n+1}}$ is unramified, this is obvious if we know that

(1) no supercuspidal representation of any Levi factor of $\mathrm{GL}(n, F)$ other than $\mathrm{GL}(1)^n$ has an unramified parameter, and
(2) the parametrization is compatible with cyclic stable base change.

In the proofs of the LLC in [LRS93, HT01, He00], point (1) is proved by reference to Henniart's numerical local correspondence [He88], whereas point (2) is proved by a global method. In Scholze's proof of the LLC for $\mathrm{GL}(n)$ [Sch13], point (1) is proved by a geometric argument using a study of nearby cycles in an integral model of the Lubin-Tate local moduli space.

Starting with point (1), the LLC is deduced by a study of the fibers of (stable) base change for cyclic extensions of p-adic fields. This is ultimately a consequence of deep properties – the 'advanced theory' – of the Arthur-Selberg trace formula that are established by Arthur and Clozel in [AC]. For $\mathrm{GL}(n)$ over a number field, the trace formula and its twisted analogue are automatically stable, but there is as yet no stable trace formula for general groups over function fields. If and when a stable trace formula is developed in this generality, it is likely that Theorem 1.4(i) will provide the starting point for an inductive proof of the LLC, at least for pure supercuspidal representations.

The proofs of both Theorems 1.2 and 1.4 proceed via global-to-local arguments and thus involve globalizing a given supercuspidal (or discrete series) representation. For a supercuspidal representation, this globalization is achieved by an adaptation of the method of [GLo] which involves a delicate construction of Poincaré series with precise control on ramification. Let us elaborate on the proof of Theorem 1.4(i) as an example.

Take the base curve to be $Y = \mathbb{P}^1$ and let K denote the global function field $k(Y) = k(t)$. By the method of Poincaré series used in [GLo], we construct a cuspidal automorphic representation Π of $G(\mathbf{A}_K)$ that is unramified outside the set of places of K corresponding to the set $Y(k)$ of k-rational points

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of Y. At the points $v = 0$ and $v = \infty$, we arrange for the local components of Π to be tamely ramified, with $\Pi_\infty^{I_-(p)} \neq 0$ and $\Pi_0^{I^+(p)} \neq 0$; here I_- and I^+ denote opposite Iwahori subgroups of $G(k(\frac{1}{t}))$ and $G(k(t))$, respectively, and $I_-(p) \subset I_-$ and $I^+(p) \subset I^+$ are the corresponding maximal pro-p subgroups. At the other rational points v, we assume Π_v to be isomorphic to the fixed supercuspidal π in the context of Theorem 1.4(i). Moreover, we assume Π_0 contains a sufficiently regular character α of the Iwahori subgroup I^+. V. Lafforgue has attached to such a Π a local system $\mathcal{L}(\Pi)$ over $Y \setminus \mathbb{P}^1(k)$. Supposing that $\mathcal{L}^{ss}(\Pi)$ is pure and unramified, it follows by Deligne [De80] that $\mathcal{L}(\Pi)$ extends to a local system over $\mathbb{G}_m$. Provided $q > 5$, we can then choose α to obtain a contradiction.

We conclude the introduction with a brief summary of the sections that follow. After recalling the results of V. Lafforgue and Genestier-Lafforgue in §2, and basic properties about Weil-Deligne parameters in §3, we complete the proof of Theorem 1.2 at the end of §3. We then lay the groundwork for the proof of Theorem 1.4(i). In §4, we establish some preparatory results about local systems on open subsets of $\mathbb{P}^1$, and in §5, we prove a useful result (Lemma 5.1) about parahoric subgroups of unramified groups. In §6, we construct some open compact subgroups of $G(\mathbb{A}_K)$ via building theoretic considerations and use these for the construction of appropriate Poincaré series. Using this Poincaré series construction and our earlier results about local systems on $\mathbb{P}^1$, we prove Theorem 1.4(i) in §7. The proof of Theorem 1.4(ii) is given in §8: this requires yet another Poincaré series construction. We then highlight some natural questions suggested by our results in §9. An example is whether one can show that a generic supercuspidal representation is pure. We show this for certain generic supercuspidal representations of depth 0. In §10, we explain the implications of our results for incorrigible representations and prove a weak version of the local Langlands correspondance for $GL(n)$: all supercuspidal representations of $GL(n, F)$ are either pure with ramified Galois parameters or have wildly ramified parameters. The purpose of this is to provide a sufficently general template that has the potential of being extended to general G. Finally, in §11, we propose the obvious conjecture linking the Fargues-Scholze parametrization of representations of groups over p-adic fields to the Genestier-Lafforgue parametrization, by means of the Deligne-Kazhdan method of close local fields.

2. Global and local Langlands-Lafforgue parameters

We begin by reviewing the global results of V. Lafforgue and the local results of Genestier-Lafforgue.

2.1. The global results of V. Lafforgue

Let Y be a smooth projective curve over k and $K = k(Y)$ be its global function field. Let G be a connected reductive algebraic group over K. Let $\hat{G}$ be the Langlands dual group of G with coefficients in $\overline{\mathbb{Q}}_\ell$ and let ${}^LG = \hat{G} \rtimes Gal(K^{sep}/K)$ be the Langlands L-group of G (in the Galois form).

Let $\mathcal{A}_0(G) = \mathcal{A}_0(G, Y)$ denote the set of cuspidal automorphic representations of G with central character of finite order. We let $\mathcal{G}^{ss}(G)$ denote the set of equivalence classes of compatible families of semisimple ℓ-adic homomorphisms, for $\ell \neq p$:

$$\rho_\ell : Gal(K^{sep}/K) \to {}^LG.$$

The term *semisimple* is understood to mean that if $\rho_\ell(Gal(K^{sep}/K)) \cap \hat{G}$ is contained in a parabolic subgroup $P \subset \hat{G}$, then it is contained in a Levi subgroup of P.

If $\nu : G \to \mathbb{G}_m$ is an algebraic character, with $\mathbb{G}_m$ here designating the split 1-dimensional torus over K, the theory of L-groups provides a dual character $\hat{\nu} : \mathbb{G}_m \to \hat{G} \subset {}^LG$. If Z is the center of G, and if $c : \mathbb{G}_m \to Z \subset G$ is a homomorphism, then the theory of L-groups provides an algebraic character ${}^Lc : {}^LG \to \mathbb{G}_m$.

We state Lafforgue's theorem for general reductive groups over general global function fields, but we will mainly be applying it to $K = k(\mathbb{P}^1)$. In what follows, a representation of G over a local field F is

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'unramified' if it contains a vector invariant under a hyperspecial maximal compact subgroup, in which case the group itself is assumed to be unramified over F.

Theorem 2.1. *Let K be any global function field and G a connected reductive algebraic group over K.*
(i) *[Laf18][Théorème 0.1] There is a map*

$$\mathcal{L} : \mathcal{A}_0(G) \to \mathcal{G}^{ss}(G)$$

with the following property: if v is a place of K and $\Pi \in \mathcal{A}_0(G)$ is a cuspidal automorphic representation such that Π_v is unramified, then $\mathcal{L}(\Pi)$ is unramified at v, and $\mathcal{L}^{ss}(\Pi)\,|_{W_{K_v}}$ is the Satake parameter of Π_v.[2]
(ii) *Suppose $\nu : G \to \mathbb{G}_m$ is an algebraic character and*

$$\chi : \mathbf{A}_K^\times / K^\times \to \overline{\mathbb{Q}}_\ell^{\,\times} = GL(1, \overline{\mathbb{Q}}_\ell)$$

is a continuous character of finite order. For any $\Pi \in \mathcal{G}_0(G)$, let $\Pi \otimes \chi \circ \nu$ denote the twist of Π by the character $\chi \circ \nu$ of $G(\mathbf{A})$. Then

$$\mathcal{L}(\Pi \otimes \chi \circ \nu) = \mathcal{L}(\Pi) \cdot \hat{\nu} \circ \hat{\chi},$$

where $\hat{\nu}$ is as above and where $\hat{\chi} : Gal(K^{sep}/K)^{ab} \to GL(1, \overline{\mathbb{Q}}_\ell)$ is the character corresponding to χ by class field theory.
(iii) *[GLa][Théorème 0.1] Let v be a place of K and let $F = K_v$. Then the semisimplification of the restriction of $\mathcal{L}(\Pi)$ to W_{K_v} depends only on F and the local component Π_v of Π, which is an irreducible admissible representation of $G(F)$, and not on the rest of the automorphic representation Π, nor on the global field K. We denote this parameter $\mathcal{L}^{ss}(\Pi_v)$.*

2.2. The local results of Genestier-Lafforgue

Now let F be a nonarchimedean local field of characteristic p, so that $F \xrightarrow{\sim} k'((t))$ for some finite extension k' of $\mathbb{F}_p$. Let $\mathcal{A}(G, F)$ denote the set of equivalence classes of irreducible admissible representations of $G(F)$ and let $\mathcal{G}^{ss}(G, F)$ denote the set of equivalence classes of semisimple ℓ-adic homomorphisms

$$\rho : W_F \to {}^L G.$$

By (iii) above, we thus obtain a (semisimple) parametrization of $\mathcal{A}(G, F)$:

$$\mathcal{L}^{ss} : \mathcal{A}(G, F) \to \mathcal{G}^{ss}(G, F). \tag{2.2}$$

Of course, the theorem quoted above only constructs $\mathcal{L}^{ss}(\pi)$ when π can be realized as the local component Π_v of a global cuspidal automorphic representation Π, but the statement of Théorème 0.1 of [GLa] includes the extension to a (semisimple) parametrization of all members of $\mathcal{A}(G, F)$. Theorem 2.1 continues:

Theorem 2.3 ([GLa],Théorème 0.1). *(iv) The local parametrization $\mathcal{L}^{ss}$ is compatible with parabolic induction in the following sense: Let $P \subset G$ be an F-rational parabolic subgroup with Levi factor M and let $\sigma \in \mathcal{A}(M, F)$. If $\pi \in \mathcal{A}(G, F)$ is an irreducible constituent of $Ind_{P(F)}^{G(F)} \sigma$ (normalized induction)*

[2]Here and below we will mainly refer to the restriction of a global Galois parameter to the local Weil group, rather than to the local Galois group, because the unramified Langlands correspondence relates spherical representations to unramified homomorphisms of the local Weil group to the L-group. But the difference is inessential.

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and $i_M : {}^L M \to {}^L G$ *is the inclusion of the L-group of M as a Levi factor of a parabolic subgroup in the L-group of G, then*

$$\mathcal{L}^{ss}(\pi) = i_M \circ \mathcal{L}^{ss}(\sigma).$$

(v) *In particular, suppose G is quasi-split over F with Borel subgroup B,* $T \subset B$ *is a maximal torus,* $I \subset G(F)$ *is an Iwahori subgroup, and* $I(p) \subset I$ *is its maximal pro-p subgroup. Suppose* $\pi^{I(p)} \neq 0$. *Then* $\mathcal{L}^{ss}(\pi)$ *takes values in the L-group of T.*

(vi) *The local and global parametrizations* $\mathcal{L}$ *and* $\mathcal{L}^{ss}$ *are compatible with homomorphisms*

$$\Upsilon : G_1 \to G_2$$

of reductive groups where the image of Υ *is a normal subgroup, in the following sense. Let* ${}^L\Upsilon : {}^L G_2 \to {}^L G_1$ *be the homomorphism of L-groups corresponding to* Υ. *Then for any cuspidal automorphic (resp. admissible irreducible) representation* π *of* G_2 *over K (resp. over F),*

$$\mathcal{L}(\pi \circ \Upsilon) = {}^L\Upsilon \circ \mathcal{L}(\pi)$$

(resp.

$$\mathcal{L}^{ss}(\pi \circ \Upsilon) = {}^L\Upsilon \circ \mathcal{L}^{ss}(\pi).$$

This applies in particular if Υ *is an isogeny of semisimple groups.*

Proof. In the situation of (v), the condition $\pi^{I(p)} \neq 0$ implies that π contains an eigenvector for I and thus is a constituent of a principal series representation. The assertion (v) then follows from (iv). Assertion (vi) in the local setting is the last assertion of [GLa, Théorème 0.1]; in the global setting, it is [Laf18, Proposition 12.5]. □

Corollary 2.4. *Let* $\Pi \in \mathcal{A}_0(G)$ *and suppose* $|X| \subset Y$ *is the largest closed subset such that* Π_v *is unramified for* $v \notin |X|$.

(i) *The parametrization of Theorem 2.1 defines a symmetric monoidal functor from the category* $Rep({}^L G)$ *of finite-dimensional algebraic representations of* ${}^L G$ *to the category of completely reducible local systems on* $Y \setminus |X|$: *to any* $\tau : {}^L G \to GL(N)$, *one attaches the N-dimensional local system attached to the representation*

$$\mathcal{V}(\Pi, \tau) := \tau \circ \mathcal{L}(\Pi)$$

of $\pi_1(Y \setminus |X|)$ *(any base point).*

(ii) *Similarly, let* $\sigma \in \mathcal{A}(G, F)$. *The local parametrization* $\mathcal{L}^{ss}$ *of (2.2) defines a symmetric monoidal functor from the category* $Rep({}^L G)$ *of finite-dimensional algebraic representations of* ${}^L G$ *to the category of completely reducible representations of* W_F:

$$\tau \mapsto \mathcal{L}^{ss}(\sigma, \tau) := \tau \circ \mathcal{L}^{ss}(\sigma).$$

(iii) *Suppose* Π *has the property that* $\mathcal{V}(\Pi, \tau)$ *is abelian and semisimple for some faithful representation* τ. *Then the image of* $\mathcal{L}(\Pi)$, *intersected with* $\hat{G}$, *lies in a maximal torus of* $\hat{G}$.

Proof. The first two assertions express the standard characterization of $\hat{G}$-local systems as symmetric monoidal functors (see, for example, the discussion on p. 52 of [Si]); the third assertion is obvious. □

We conclude this section with the following definition.

Definition 2.5. With notation as in Theorem 2.3, let $\sigma \in \mathcal{A}(G, F)$. We say σ is **pure** if, for some (equivalently, for all) faithful representations $\tau : {}^L G \to GL(N)$, $\mathcal{L}^{ss}(\sigma, \tau)$ has the property that, for

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any Frobenius element $Frob_F \in W_F$, the eigenvalues of $\mathcal{L}^{ss}(\sigma,\tau)(Frob)$ are all Weil q-numbers of the same weight.

3. Pure Weil-Deligne parameters

This section is devoted to the proof of Theorem 1.2. We shall begin with some background material on tempered L–parameters or their equivalent incarnation as pure Weil-Deligne parameters.

3.1. Tempered parameters and pure Weil-Deligne parameters

Let F be a nonarchimedean local field.

Definition 3.1. A Weil-Deligne parameter for F with values in ${}^LG(C)$ is a pair (ρ, N), where $\rho : W_F \to {}^LG(C)$ is a continuous homomorphism (for the discrete topology on C) and $N \in Lie(\hat{G})(C)$ is a nilpotent element such that $(Ad \circ \rho, N)$ is a homomorphism from the Weil-Deligne group of F to $Aut(Lie(\hat{G}))(C)$.

Without loss of generality, we may assume $C = \mathbb{C}$. There is a simple recipe for converting a Weil-Deligne parameter (ρ, N) for F with values in ${}^LG(\mathbb{C})$ to an L-parameter

$$\varphi = \varphi_{\rho,N} : W_F \times SL(2,\mathbb{C}) \to {}^LG(\mathbb{C}),$$

which is algebraic in the second factor, and vice versa. This recipe (and the proof that it gives an equivalence between the two notions) is given in [GR, §2.1 and Prop. 2.2].

We define a couple of properties of Weil-Deligne parameters.

Definition 3.2. The Weil-Deligne parameter (ρ, N) is *tempered* if the restriction of $\varphi_{\rho,N}$ to $W_F \times SU(2)$ has bounded image after projection to $\hat{G}(\mathbb{C})$. The parameter (ρ, N) is *essentially tempered* if its image in $\hat{G}^{ad}$ is bounded.

Definition 3.3.

(a) The Weil-Deligne representation (ρ, N) with values in $GL(V)$ is *pure* if there is a complex number t such that

(i) The eigenvalues of $\rho_t(Frob_F) := \rho(Frob_F)q^t$ are all q-numbers of integer weight.
(ii) The subspace $W_aV \subset V$ of eigenvectors for $\rho_t(Frob_F)$ with eigenvalues of weight $\leq a$ is invariant under (ρ, N);
(iii) Letting $gr_aV = W_aV/W_{a-1}V$, there is an integer w such that, for all $i \geq 0$, the map

$$N : gr_{w+i}V \to gr_{w-i}V$$

is an isomorphism.

The integer w in (iii) is then called the weight of (ρ, N) (twisted by t).

(b) Let G be a connected reductive group over F. The Weil-Deligne parameter (ρ, N) with values in ${}^LG(\mathbb{C})$ is pure if $(\sigma \circ \rho, N)$ is pure of some weight for some (equivalently every) faithful representation σ of ${}^LG$.

In particular, if $N = 0$, then the Weil-Deligne parameter (ρ, N) is pure if and only if the Weil group parameter ρ is pure (in the usual sense) of some weight (up to twist by a power of the norm). We distinguish the two notions of purity by referring to 'pure Weil-Deligne parameters' and 'pure Weil parameters', or 'pure semisimple parameters', respectively. In [TY], pure Weil parameters are called *strictly pure*.

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Remark 3.4. The local parameters $\mathcal{L}^{ss}(\pi)$ attached by Genestier and Lafforgue are ℓ-adic. In order to define what it means for them to be pure (as Weil or Weil-Deligne parameters), one usually chooses an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$ (or more sensibly, an isomorphism between the algebraic closures of $\mathbb{Q}$ in the two fields) and declares them to be ι-pure if they are pure in the above sense after composing with ι. For this purpose, one would want to replace the complex number t in Definition 3.3 by an ℓ-adic number. In practice, the parameters that arise in the automorphic theory are ι-pure for every ι.

The following facts are well known and in any case are easy to verify.

Lemma 3.5.

(a) *The Weil-Deligne parameter (ρ, N) is pure if and only if $\varphi_{\rho,N}$ is essentially tempered.*

(b) *Let $\rho : W_F \to {}^L G(C)$ be a semisimple parameter. There is at most one way (up to equivalence) to complete ρ to a pure Weil-Deligne parameter (ρ, N). If it exists, we call it the pure completion of ρ. Equivalently, there is at most one essentially tempered L-parameter φ whose associated semisimple parameter (via (1.1)) is ρ.*

Proof. Both conditions in (a) are checked by composition with a faithful representation σ of ${}^L G$. This reduces the verification of (a) to the case of $GL(n)$, which is verified in [TY, Lemma 1.4(3)]. Part (b) for $GL(n)$ is [TY, Lemma 1.4(4)]. For general groups G, one can cite the argument on p. 10 of [A84]; Arthur is there classifying Arthur parameters, rather than Weil-Deligne parameters, of real groups, but the proof of uniqueness is the same. More precisely, suppose ρ can be completed to a pure Weil-Deligne parameter (ρ, N), and let $\varphi_{\rho,N} : W_F \times SL(2)$ be as above. Let φ be the restriction of $\varphi_{\rho,N}$ to the W_F factor. Since the image of φ is bounded and consists only of semisimple elements, its centralizer $C_\varphi \subset {}^L G$ is reductive. We need to show that the restriction of $\varphi_{\rho,N}$ to $SL(2)$, whose image lies in C_φ, is unique up to C_φ-conjugacy, but this is exactly what Arthur shows in his situation. □

3.2. An application of the purity of the monodromy weight filtration

The parametrization of Genestier-Lafforgue only attaches a semisimple parameter $\mathcal{L}^{ss}(\pi)$ to a local representation π. But when π is realized as a local component of a cuspidal automorphic representation Π of G over some global function field K – say $\pi = \Pi_x$ for some place x of K – then the restriction $\mathcal{L}(\Pi)_x$ to the decomposition group Γ_x at x need not be semisimple. We let

$$\mathcal{L}_\Pi(\pi) = (\mathcal{L}^{ss}(\pi), N_\Pi)$$

denote the Weil-Deligne parameter associated to the Galois parameter $\mathcal{L}(\Pi)_x$.

The following result of Sawin and Templier generalizes Deligne's theorem on the purity of the monodromy weight filtration to G-local systems:

Lemma 3.6. *[ST, Lemma 11.4] Let X be a smooth projective curve over a finite field k, with function field $K = k(X)$ and let $\rho : Gal(K^{sep}/K) \to {}^L G(C)$ be an irreducible homomorphism (with image not contained in a proper parabolic subgroup, where a parabolic subgroup is defined in the 'dynamic' way [CGP, Definition 2.2.1] as, for any homomorphism $\lambda\colon \mathbb{G}_m \to {}^L G$, the subgroup of $g \in {}^L G$ such that $\lim_{t\to 0} \lambda(t) g \lambda(t)^{-1}$ exists). Let z be any place of K and let (ρ_z^{ss}, N_z) be the Weil-Deligne parameter associated to the restriction ρ_z of ρ to a decomposition group at z. Then (ρ_z^{ss}, N_z) is a pure Weil-Deligne parameter.*

Proof. We explain how to deduce this from [ST, Lemma 11.4] since this result is not explicitly stated there. Under the irreducibility hypothesis, [ST, Lemma 11.4] states that the composition of ρ with any representation of ${}^L G(C)$ is pure of some weight w as a local system on an open subset of X. (The mixedness property assumed in [ST] is a consequence of [Laf02].) It then follows from [De80, Theorem 1.8.4] that the i'th associated graded of the monodromy filtration of the composition of ρ_z with the same representation is pure of weight $w + i$. In particular, all the Frobenius eigenvalues are q-numbers, and the sum of eigenspaces of Frobenius associated to eigenvalues of size $\leq q^{\frac{w+i}{2}}$ is the ith member of

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the monodromy filtration. Part (iii) of the definition of pure Weil-Deligne parameter then follows from the defining property [De80, Proposition 1.6.13]. □

Remark 3.7. The lemma is stated in [ST] when G is split semisimple, but the proof is valid for any reductive G, since it is easy to see that the limit $\lim_{t\to\infty} \alpha(t)g\alpha(t)^{-1}$ exists if and only if g preserves the filtration into weight spaces of α on a faithful representation.

Corollary 3.8. *Let π be an irreducible admissible representation of $G(F)$. Suppose*

- *X is a smooth projective curve over a finite field k*
- *z is a place of the function field $K = k(X)$ such that $K_z \xrightarrow{\sim} F$.*
- *there is a cuspidal automorphic representation Π of G_K, with $\Pi_z \xrightarrow{\sim} \pi$, such that the global parameter $\mathcal{L}(\Pi)$ is irreducible.*

Then the Weil-Deligne parameter $\mathcal{L}_\Pi(\pi)$ is pure. Moreover, it is the unique pure completion of the semisimple parameter $\mathcal{L}^{ss}(\pi)$.

Not every semisimple parameter admits a completion to a pure Weil-Deligne parameter. Corollary 3.8 asserts that $\mathcal{L}^{ss}(\pi)$ does admit a pure completion provided π can be realized as a local component of a cuspidal automorphic representation whose global parameter is irreducible in the indicated sense.

3.3. *Proof of Theorem 1.2*

We are now ready to prove Theorem 1.2. In view of Corollary 3.8, it suffices to construct cuspidal globalizations of any essentially discrete series representation π whose global parameter is irreducible. The following proposition achieves this when π is supercuspidal.

Proposition 3.9. *Let G be a reductive group over F. Let π be a supercuspidal representation of $G(F)$. Let X be a smooth projective curve over a finite field k and let z be a place of the function field $K = k(X)$ such that $K_z \xrightarrow{\sim} F$. Then there exist infinitely many cuspidal automorphic representations Π of G_K, such that*

- *$\Pi_z \xrightarrow{\sim} \pi$;*
- *the global parameter $\mathcal{L}(\Pi)$ is irreducible.*

In particular, the semisimple parameter $\mathcal{L}^{ss}(\pi)$ admits a completion to a pure Weil-Deligne parameter that is realized as the specialization at z of the global parameter $\mathcal{L}(\Pi)$, for any such Π.

Proof. We shall use the results of [GLo] to produce the desired globalizations. Let $x \neq z$ be any other place of K such that $G(K_x)$ is a split reductive group. There are infinitely many such x – just take any x that splits completely in a finite cover of X over which G splits. Thus, it suffices to show

(i) There is a supercuspidal representation π' of $G(K_x)$ such that $\mathcal{L}^{ss}(\pi')$ is irreducible, and
(ii) There is an automorphic representation Π of G_K with $\Pi_x \xrightarrow{\sim} \pi'$ and $\Pi_z \xrightarrow{\sim} \pi$, and with Π_y tamely ramified for all $y \neq x, z$.

Once (i) is given, the existence of Π as in (ii) is proved in [GLo] by a Poincaré series construction. It thus remains to produce a supercuspidal representation π' as in (i).

We let $k' = k(x)$ denote the residue field at x and construct the Kloosterman sheaf $\mathrm{Kl}_{\hat{G}}(\phi, \chi)$ of [HNY] over $\mathbb{P}^1_{k'}$. Let π' be the local component at ∞ of the corresponding automorphic representation $\pi(\phi, \chi)$. Theorem 2 of [HNY] implies that the restriction to the inertia group at ∞ of the local monodromy representation of $\mathrm{Kl}_{\hat{G}}(\phi, \chi)$ is irreducible. The statement in [HNY] includes slightly restrictive hypotheses on the prime p for non-simply laced groups, but these hypotheses have since been removed by Xu and Zhu in [XZ, Corollary 4.5.8, Remark 4.5.10], under the hypothesis that $G(K_x)$ is split, which we are assuming. However, it follows from construction that $\mathrm{Kl}_{\hat{G}}(\phi, \chi)$ is the global parameter attached by [Laf18] to $\pi(\phi, \chi)$. More precisely, [HNY, XZ] computes the local monodromy

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of $\mathrm{Kl}_{\hat{G}}(\phi,\chi)$ at unramified places and identifies it with that obtained from the Satake parameters of $\pi(\phi,\chi)$. (The construction of $\pi(\phi,\chi)$ is made explicit in [Y16] but its existence is implicit in [HNY].) It then follows that $\mathcal{L}^{ss}(\pi')$ is irreducible, as desired. □

Remark 3.10. The paper [HNY] was written before [Laf18], but the ${}^LG$-valued parameters attached to the automorphic representations constructed in [HNY] coincide at all unramified places with those obtained by Lafforgue, because both are given by the Satake correspondence. It follows by Chebotarev density that the semisimple ℓ-adic local systems obtained by composing the ${}^LG$-valued parameters of [HNY] and [Laf18] with any representation of ${}^LG$ are identical. In particular, the irreducibility property proved by [HNY, XZ] implies that Lafforgue's $\mathcal{L}^{ss}(\pi')$, composed with the adjoint representation, is irreducible.

For essentially discrete series but not supercuspidal representations, we have a weaker globalization result which is nonetheless sufficient for our applications.

Proposition 3.11. *Assume the notation and hypotheses of Proposition 3.9 but let π be an essentially discrete series representation. Then there exists a cuspidal automorphic representation Π of G_K such that*

- $\mathcal{L}^{ss}(\Pi_z) = \mathcal{L}^{ss}(\pi)$;
- *the global parameter $\mathcal{L}(\Pi)$ is irreducible.*

In particular, the semisimple parameter $\mathcal{L}^{ss}(\pi)$ admits a completion to a pure Weil-Deligne parameter $\mathcal{L}(\pi)$ that is realized as the specialization at z of the global parameter $\mathcal{L}(\Pi)$, for any such Π.

Proof. Let f_π be a pseudo-coefficient of π, whose properties are recalled in [GLo, §8.1]. In [GLo, §8.2], a version of the simple trace formula was formulated as a working hypothesis. Applying Lemma A.1 in the appendix by R. Beuzart-Plessis, it follows as in the proof of [GLo, Proposition 8.2] that there is a cuspidal Π with $\Pi_x \xrightarrow{\sim} \pi'$ (with π' as in Proposition 3.9) for some place x and $Tr(\Pi_z)(f_\pi) \neq 0$. This implies (by [GLo, Lemma 8.1]) that Π_z has the same cuspidal support as π, and therefore that

$$\mathcal{L}^{ss}(\Pi_z) = \mathcal{L}^{ss}(\pi)$$

since the semisimple Genestier-Lafforgue parameter is compatible with parabolic induction. Now we do not know, nor care, whether Π_z is isomorphic to the discrete series π or not. Since $\mathcal{L}^{ss}(\Pi_x) = \mathcal{L}^{ss}(\pi')$ is irreducible, so is the global parameter $\mathcal{L}(\Pi)$, and we have produced the desired Π. □

Using Corollary 3.8, Proposition 3.9 and Proposition 3.11, the proof of Theorem 1.2 is now complete.

We note that the idea behind Corollary 3.8 and Proposition 3.9 was already applied in the case of classical groups in the proof of [GLo, Proposition 7.3]. Instead of using the Kloosterman representations of [HNY] to establish irreducibility, [GLo] used depth zero supercuspidal representations, but introduced a hypothesis because the irreducibility of the corresponding L-parameter has not (yet) been established for local fields of positive characteristic.

Corollary 3.12. *Under the hypotheses of Theorem 1.2, suppose G is semisimple. Then the composition of the adjoint representation of ${}^LG$ with the Weil-Deligne parameter $\mathcal{L}(\pi)$ of π is pure of weight 0.*

Proof. It follows from Proposition 3.11 that $Ad \circ \mathcal{L}(\pi)$ is a pure Weil-Deligne representation of some weight w. But since G, and therefore $\hat{G}$, is semisimple, the determinant of $Ad \circ \mathcal{L}(\pi)$ is pure of weight 0. It follows that w must equal 0. □

Remark 3.13. The proof actually shows that $\mathcal{L}(\pi)$ is ι-pure, in the sense of Remark 3.4, for every ι.

4. Local systems on open subsets of $\mathbb{P}^1$

The rest of the paper is devoted to establishing more refined properties of the semisimple parameter $\mathcal{L}^{ss}(\pi)$ of a supercuspidal representation π and, in particular, proving Theorem 1.4. For this, we shall

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need to appeal to a global argument which necessitates the construction of appropriate globalizations of supercuspidal representations by Poincaré series. Unlike the previous section, these globalizations will have to be constructed over the function field of $\mathbb{P}^1$, so that we may appeal to special properties of local systems on open subsets of $\mathbb{P}^1$. In this section, we shall discuss the requisite results about local systems on open subsets of $\mathbb{P}^1$.

4.1. Purity

Let $k = \mathbb{F}_q$ be a finite field of characteristic p. Let $Z, X \subset \mathbb{P}^1(k)$ be disjoint finite subsets, let $|Z|$ and $|X|$ be the corresponding reduced subschemes of $\mathbb{P}^1(k)$ and let $Y = \mathbb{P}^1_k \setminus |X| \cup |Z|$, viewed as a scheme over $Spec(k)$. Let L be a finite extension of $\mathbb{Q}_\ell$ and consider a local system $\mathcal{V}$ over Y with coefficients in L (a lisse ℓ-adic sheaf). Picking a point y of Y (the generic point, for example), and letting V denote the finite-dimensional vector space $\mathcal{V}_y$ over L, we can identify $\mathcal{V}$ with a representation

$$\rho : \pi_1(Y, y) \ \rightarrow \ Aut(V).$$

Let $F_z = k(Y)_z$ be the completion of $k(Y)$ at the point $z \in Z$. The monodromy of $\mathcal{V}$ at z is the restriction of ρ (up to conjugation) to a homomorphism

$$\rho_z : Gal(F_z^{sep}/F_z) \ \rightarrow \ Aut(V). \tag{4.1}$$

Let $I_z \subset Gal(F_z^{sep}/F_z)$ denote the inertia group and let $Frob_z \in Gal(F_z^{sep}/F_z)$ denote any geometric Frobenius element (i.e., any lift to an automorphism of F_z^{sep} of the automorphism $x \mapsto x^{\frac{1}{q}}$ of $\bar{k}$).

If Γ is a topological group and $\rho : \Gamma \ \rightarrow \ Aut(V)$ is a continuous homomorphism, we let ρ^{ss} denote its semisimplification – the sum of its Jordan-Hölder constituents.

Theorem 4.2 ([De80], Corollary 1.7.6). *Suppose that Z is not empty and, for all $z \in Z$, the following conditions are satisfied:*

- *ρ_z^{ss} is unramified;*
- *the Frobenius eigenvalues $\rho_z(Frob_z)$ are all Weil q-numbers of the same weight.*

Then $\mathcal{V}$ extends to a local system over $\mathbb{P}^1_k \setminus |X|$.

Proof. In fact, Deligne's result only requires the Frobenius eigenvalues to have the same weight for a single complex embedding; under this condition, $\rho_z(I_z)$ is a finite group. But we have assumed that ρ_z^{ss} is unramified; thus, $\rho_z(I_z)$ is unipotent. It follows that $\rho_z(I_z)$ must be trivial. □

Corollary 4.3. *Under the hypotheses of Theorem 4.2, suppose $X = \{0, \infty\}$ and $\mathcal{V}$ is a semisimple local system on Y with associated monodromy representation*

$$\rho : \pi_1(\mathbb{P}^1 \setminus |X|, y) \ \rightarrow \ Aut(V).$$

Suppose moreover that

(a) *the local monodromy representation of $\mathcal{V}$ at 0 and ∞ is tame,*
(b) *the semisimplification of the local monodromy representation at 0 **or** ∞ is abelian and Frobenius-semisimple.*

Then, possibly after replacing L by a finite extension L', $\mathcal{V}$ breaks up as the sum of 1-dimensional local systems.

Proof. Theorem 4.2 implies that $\mathcal{V}$ extends to a tame local system, also called $\mathcal{V}$, over $\mathbb{G}_m$. By [Laf02], Corollary VII.8, $\mathcal{V}$ can be written as a (finite) direct sum

$$\mathcal{V} = \bigoplus_{\chi} \mathcal{V}_\chi \otimes \chi,$$

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where each $\mathcal{V}_\chi$ is a mixed local system and χ is the pullback to $\mathbb{P}^1 \setminus |X|$ of a rank 1 ℓ-adic local system over $Spec(k)$ – in other words, a continuous ℓ-adic character of $Gal(\bar{k}/k)$. We thus reduce to the case where $\mathcal{V}$ is a mixed ℓ-adic local system, and since $\mathcal{V}$ is assumed semisimple, we may assume $\mathcal{V}$ to be irreducible.

Then the purity hypothesis at the points z in the nonempty set Z implies that $\mathcal{V}$ is itself pure. Since the tame fundamental group of $\mathbb{G}_{m,\bar{k}}$ is just $\prod_{p' \neq p} \mathbb{Z}_{p'}(1)$, it follows from the purity that the image J under ρ of the geometric fundamental group of $\mathbb{G}_m$ is finite and necessarily abelian.

Now one knows that the image of the tame inertia group I_0 at 0 in the tame fundamental group of $\mathbb{G}_{m,k}$ is equal to the tame fundamental group of $\mathbb{G}_{m,\bar{k}}$ (i.e., the tame geometric fundamental group of $\mathbb{G}_{m,k}$). The same is true for the tame inertia group at ∞. Hence, the hypothesis (b) on the local monodromy at 0 or ∞ implies that the conjugation action of $Gal(\bar{k}/k)$ on J is trivial, and thus, the image under ρ of the arithmetic fundamental group $\pi_1(\mathbb{P}^1 \setminus |X|, y)$ is also abelian; moreover, the image of $Gal(\bar{k}/k)$ is semisimple by hypothesis. This completes the proof. □

4.2. Kummer theory

Let $K = k(\mathbb{P}^1) = k(T)$ be the global function field of $\mathbb{P}^1$ and let K' denote the union of all abelian extensions of K of degree dividing $q-1$, which is the order of the group of roots of unity in K. Homomorphisms

$$\alpha : Gal(K'/K) \to \overline{\mathbb{Q}}_\ell^{\times}$$

are classified by Kummer theory:

$$\psi : K^\times/(K^\times)^{q-1} \xrightarrow{\sim} X(K) := Hom(Gal(K'/K), \overline{\mathbb{Q}}_\ell^{\times}). \tag{4.4}$$

Let $C_T \subset K^\times/(K^\times)^{q-1}$ be the subgroup generated by $k^\times$ and the parameter T and let $X_T = \psi(C_T)$. We also consider Kummer theory for the constant field k:

$$\psi_k : k^\times \xrightarrow{\sim} X(k) := Hom(Gal(k'/k), \overline{\mathbb{Q}}_\ell^{\times}), \tag{4.5}$$

where k' is the cyclic extension of k of order q^{q-1}. For $b \in k^\times$, $\beta \in Gal(k'/k)$, let us write

$$\beta^b := \psi_k(b)(\beta).$$

Any $z \in k^\times$ defines a closed point $z \in \mathbb{G}_{m,k}$; let $\Gamma_z \subset Gal(K'/K)$ denote its decomposition group and I_z its inertia group. An element $\alpha \in X_T$ is unramified at such a point z and thus defines a homomorphism

$$\alpha^{(z)} : \Gamma_z/I_z \xrightarrow{\sim} Gal(k'/k) \to \overline{\mathbb{Q}}_\ell^{\times}.$$

For $a = 1, \ldots, q-1$, we let $\alpha_a = \psi(T^a)$. Then we have the following reciprocity law:

Lemma 4.6. *For $z \in \mathbb{G}_m(k)$ and $a = 1, \ldots, q-1$, we have*

$$(\alpha_a)^{(z)} = \psi_k(az).$$

Proof. Let K_z denote the completion of K at z, $\mathcal{O}_z$ its integer ring, k_z its residue field, which the inclusion of k in K canonically identifies with k. The image of the element $T \in \mathcal{O}_z$ in k_z is identified with the element $z \in k^\times$, so the residue field of the unramified extension $\mathcal{O}_z[T^{1/(q-1)}]$ is just $k(z^{1/(q-1)})$. Both sides of the identity are defined by taking an element of the Galois group, observing that it acts on $T^{1/(q-1)} = z^{1/(q-1)}$ by multiplication by a $(q-1)$st root of unity and then composing with a fixed embedding of the $(q-1)$st roots of unity into $\overline{\mathbb{Q}}_\ell^{\times}$. The only difference is that $(\alpha_a)^{(z)}$ is calculated by first considering the action of the Galois group of K, restricting to the decomposition group, and observing

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that it factors through the Galois group of the residue field, while $\psi_k(az)$ is calculated by considering the action of the Galois group of the residue field directly, but these agree since the homomorphism from the decomposition group to the Galois group of the residue field is defined to be compatible with the action on elements of the residue field. □

Lemma 4.7. *Let $\mathcal{L}$ be a rank one local system on $\mathbb{G}_{m,k}$ and $z \in k^\times$. Then*

$$\mathrm{tr}(\mathrm{Frob}_z, \mathcal{L}) = \mathrm{tr}(\mathrm{Frob}_1, \mathcal{L})\,\mathrm{tr}(m_z^{-1}, \mathcal{L}),$$

where m_z is the element of the inertia group I_0 which acts by multiplication by z on the $(q-1)$st root of T.

Proof. We give two proofs of the lemma.

(i) (Class field theory) It suffices to show that $\mathrm{Frob}_z = \mathrm{Frob}_1 \cdot m_z^{-1}$ in the abelianization of the tame fundamental group of $\mathbb{G}_m$. Class field theory gives an isomorphism between this abelianization and the profinite completion of

$$K^\times \backslash \mathbb{A}_K^\times \Big/ \left(\prod_{\substack{y \in |\mathbb{P}^1_k| \\ y \notin \{0,\infty|}} \mathcal{O}_{K_y}^\times \times \prod_{y \in \{0,\infty\}} U_{K_y,1} \right),$$

where $U_{K_y,1}$ denotes the units of $\mathcal{O}_{K_y}$ that are congruent to 1 modulo the maximal ideal. Moreover,

- Frob_z corresponds to the idèle which is a uniformizer at z and the identity at all other places,
- Frob_1 corresponds to the idèle which is a uniformizer at 1 and the identity at all other places, and
- m_z corresponds to the idele which is z at 0 and the identity at all other places.

So it suffices to check that the first idèle class is the product of the other two idèle classes. This can be checked by multiplying by $\frac{T-z}{T-1} \in K^\times$, which is a uniformizer at z, the inverse of a uniformizer at 1, and z at 0.

(ii) (Kummer theory) We can construct a one-dimensional representation of $Gal(\overline{k}/k)$ on which Frob_q acts by multiplication by $\mathrm{tr}(\mathrm{Frob}_1, \mathcal{L})$. On tensoring $\mathcal{L}$ with the inverse representation, we reduce to the case where Frob_1 acts trivially on $\mathcal{L}$. Since the tame fundamental group of $\mathbb{G}_m$ is generated by the geometric tame fundamental group and Frobenius, the image of the Galois group acting on $\mathcal{L}$ must equal the image of the geometric tame fundamental group. The action of the geometric tame fundamental group on $\mathcal{L}$ must be by a Frob_q-invariant character, which means it factors through the Frob_q-coinvariant of the abelianization of the tame geometric fundamental group, which is $\mathbb{Z}/(q-1)$. So the action of the geometric fundamental group is by a character of order $q-1$. The result now follows from Lemma 4.6. □

5. Unramified Reductive Groups

We continue to assume that $K = k(\mathbb{P}^1)$, with $k = \mathbb{F}_q$. In this section, we establish some structural results for an unramified connected reductive group G over K. In particular, we establish a useful result (Lemma 5.1) which gives congruence conditions on the matrix entries of a parahoric subgroup of G (at a place of K) under a faithful algebraic representation. This parahoric entry inequality will be crucially used in the next few sections.

5.1. Unramified groups

By definition, a connected reductive group over K is unramified if it is quasi-split and split by an unramified extension of K. Such a G can, in fact, be defined over the finite field k. Hence, we assume that

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G is defined over k, with Z the identity component of its center and $B = T \cdot N$ a fixed Borel k-subgroup. Let $S \subset T$ be the maximal k-split subtorus in T, so that S is a maximal split torus of G over k.

Let

$$\rho\colon G \to \mathrm{Aut}(V)$$

be a faithful algebraic representation of G defined over k and let $e_1, \ldots, e_n$ be a basis of V on which S acts diagonally via the eigencharacter $\lambda_i \in X^*(S)$ on e_i.

By base change, we obtain from the above the analogous objects over any k-algebra. In particular, we have

- $S_K \subset B_K = T_K \cdot N_K \subset G_K$ over $K = k(\mathbb{P}^1) = k(t)$; to simplify notation, we shall suppress the subscript K. Note that S is still a maximal K-split torus in G.
- At each place y of K, with associated local completion K_y and ring of integers $\mathcal{O}_y \subset K_y$, we have the $\mathcal{O}_y$-group schemes

$$S_y \subset B_y = T_y \cdot N_y \subset G_y.$$

  Note, however, that S_y need no longer be a maximal K_y-spit torus in G_y; we let $A_y \supset S_y$ be a maximal split torus over $\mathcal{O}_y$. For $y \in \mathbb{P}^1(k)$, we have $A_y = S_y$.
- At each place y of K, $G(\mathcal{O}_y)$ is a hyperspecial maximal compact subgroup of $G(K_y)$. One has the reduction-mod-p map

$$G(\mathcal{O}_y) \longrightarrow G(k_y),$$

  where k_y is the residue field of K_y. The preimage of $B(k_y) \subset G(k_y)$ in $G(\mathcal{O}_y)$ is an Iwahori subgroup I_y of $G(K_y)$. Its pro-p radical $I(p)$ is the preimage of the unipotent radical $N(k_y)$ of $B(k_y)$.
- For each place y of K, we have a faithful algebraic representation $\rho_y\colon G_y \to \mathrm{Aut}(V_y)$. If $y \in \mathbb{P}^1(k)$, then the basis $\{e_i\}$ is a basis of eigenvectors for the maximal K_y-split torus $S_y = A_y$.

5.2. Buildings

For each place y of K, consider the (extended) Bruhat-Tits building $\mathcal{B}(G_y)$ which contains the apartment $\mathcal{A}(A_y)$ associated to the maximal K_y-split torus A_y. The hyperspecial maximal compact subgroup $G(\mathcal{O}_y)$ determines a basepoint in $\mathcal{A}(A_y)$, giving an identification

$$\mathcal{A}(A_y) \xrightarrow{\sim} X_*(A_y) \otimes \mathbb{R},$$

with $G(\mathcal{O}_y)$ the stabilizer of the origin $0 \in X_*(A_y)$. For places $y \notin \mathbb{P}^1(k)$, we shall only need to consider this basepoint 0 and its stabilizer $G(\mathcal{O}_y)$.

Now consider only those $y \in \mathbb{P}^1(k)$, so that $A_y = S_y$. In this case, one has a natural identification of the apartment $\mathcal{A}(A_y)$ with $X_*(S) \otimes \mathbb{R}$. In particular, the apartments $\mathcal{A}(A_y)$ are naturally identified with each other as y varies in $\mathbb{P}^1(k)$.

5.3. Parahoric entry inequality

In this section, we prove a technical lemma on the matrix entries of elements of parahoric subgroups of G over the local field $K_y = k((t))$ (for $y \in \mathbb{P}^1(k)$) that will be useful for us later on.

Lemma 5.1. *Fix a local field $F = k((t))$ (which we may think of as K_y for $y \in \mathbb{P}^1(k)$) and let*

- *G be an unramified semisimple group (as introduced above), with maximal F-split torus S contained in a Borel subgroup $B = TN$, all defined over k;*

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- *c be a point in the apartment of S in the Bruhat-Tits building $\mathcal{B}(G)$ with associated parahoric subgroup G_c;*
- *$\rho\colon G \to \mathrm{Aut}(V)$ be a faithful algebraic representation of G defined over k, with S-eigenbasis $e_1, \dots, e_n$ with associated eigencharacters $\lambda_i \in X^*(S)$ on e_i.*

Then for any $g \in G_c \subset G(F)$, we have

$$v(\rho_{ij}(g)) \geq c \cdot (\lambda_i - \lambda_j), \tag{5.2}$$

where $v : F \to \mathbb{Z}$ is the normalized valuation on F and $\rho_{ij}(g)$ denotes the (i, j)-th entry of the matrix of $\rho(g)$ relative to the basis $\{e_i\}$.

More generally, if g lies in the Moy-Prasad subgroup $G_{c,r+}$ for some $r \geq 0$, then we have

$$v(\rho_{ij}(g) - \delta_{ij}) > c \cdot (\lambda_i - \lambda_j) + r. \tag{5.3}$$

Proof. We give the proof in the G_c case and then describe the modifications necessary for $G_{c,r+}$.

If (5.2) holds for g_1 and g_2, then it holds for $g_1 g_2$, so that it suffices to verify (5.2) for a set of generators of the parahoric G_c. The parahoric G_c is generated by the subgroups U_ψ for those affine functions ψ on the apartment of S such that

$$\psi(c) \geq 0.$$

Here, ψ is an affine function of the form $\psi(w) = b \cdot w - s$ for a root $b \in X^*(S)$ and a real number s. If $U_b \subset G$ is the (restricted) root group associated to b, then one has an isomorphism over k:

$$i_b : U_b \xrightarrow{\sim} \mathrm{Res}_{k_b/k} \mathbb{G}_a,$$

where k_b is the field of definition of the root b and $\mathbb{G}_a$ is the additive group, and the subgroup U_ψ can be described as

$$U_\psi = \{u \in U_b(F) : v(i_b(u)) \geq s\},$$

where v is the unique extension of v to k_b.

To obtain (5.2), it suffices to check that the following two statements hold for $g \in U_b$:

(a) If $\rho_{ij}(g) \neq 0$, then $\lambda_i - \lambda_j = nb$ for some $n \geq 0$;
(b) $\rho_{ij}(g)$ is a homogeneous polynomial function on U_b defined over k of degree n.

Indeed, if (a) and (b) hold, then we have, for $g \in U_\psi$

$$v(\rho_{ij}(g)) \geq ns \geq ns + n\psi(c) = nb \cdot c = (\lambda_i - \lambda_j) \cdot c,$$

as desired.

To prove (a) and (b), we use lifting to characteristic 0. The group G (which is defined over k) has a lift from the field k to the Witt vectors $W(k)$, unique up to not-necessarily-unique isomorphism. Since irreducible algebraic representations of G are classified by Galois orbits of highest weights, and the Galois action for k and $W(k)$ are the same, we may lift the representation V from the field k to its ring of Witt vectors $W(k)$. Both properties in (a) and (b) are preserved by reduction mod p and thus may be checked over the fraction field of $W(k)$, where U_b is obtained by exponentiating the root subspace of b. In characteristic 0, both (a) and (b) follow readily from the representation theory of SL_2.

We now indicate the modifications needed for the statement for G_{c,r^+}. The inequality (5.3) is also stable under matrix multiplication since $r \geq 0$. Because the subgroups $G_{c,r+}$ are generated by U_ψ for those ψ with $\psi(c) > r$ and the subgroups $T^a_{n\delta}$ for the maximal torus T containing S, for $n > r$, it suffices to verify (5.3) for these subgroups. Here, a is an (absolute) simple root of T (over $\overline{k}$), with associated

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root subgroup SL_2 (over $\overline{k}$), T^a is the 1-dimensional torus (over $\overline{k}$) which is the image of the coroot

$$a^\vee : \mathbb{G}_m \to T^a \subset T$$

and

$$T^a_{n\delta} = \{a^\vee(x) \in T^a(F^{ur}) \cap G(F) : v(x-1) \geq n\}.$$

We begin with the elements of U_ψ as above. That the elements of such U_ψ satisfy (5.3) follow from (a) and (b) above and the additional fact that ρ_{ij} is the constant function 1 on U_b if $n = 0$. Indeed, these imply that

$$v(\rho_{ij}(g)) \geq ns \geq ns + n\psi(c) + nr = nb \cdot c + nr > (\lambda_i - \lambda_j) \cdot c + r$$

if $n > 0$ and $v(\rho_{ij}(g) - 1) = \infty$ if $n = 0$.

Hence, it remains to show that elements $g \in T^a_{n\delta}$ satisfy (5.3). Because T is the centralizer of S in G, we see that

$$g \in T^a_{n\delta} \text{ and } \lambda_i \neq \lambda_j \Longrightarrow \rho_{ij}(g) = 0.$$

Thus, it suffices to show

$$v(\rho_{ij}(g) - \delta_{ij}) \geq n > r \quad \text{if } \lambda_i = \lambda_j.$$

Now the action of the elements $g \in T^a$ and therefore the matrix entries ρ_{ij} (restricted to T^a) are given by polynomial functions on $\mathbb{G}_m$ over $\overline{k}$, which are equal to δ_{ij} at the identity element 1. Thus, the difference between $\rho_{ij}(g)$ and δ_{ij} has valuation at least n for $g \in T^a_{n\delta}$, as desired. □

6. Open compact subgroups of $G(\mathbb{A}_K)$

After the preparatory local lemma above, we return to the global setting of §5.1 where G is an unramified semisimple group over $K = k(t)$. We will thus use the notation of §5.1 and §5.2. Our goal is to construct some open compact subgroups of $G(\mathbb{A}_K)$ with desirable properties.

With $F = k((t))$, let $U^* \subset G(F)$ be a fixed maximal compact subgroup. We choose an open compact subgroup of $G(\mathbb{A}_{k(t)})$ of the form $U = \prod_x U_x$, where x runs over the places of K, with the following properties.

(a) At every $z \in \mathbb{G}_m(k) \subset \mathbb{P}^1(k)$, U_z equals the fixed U^*, with respect to a chosen isomorphism

$$G(K_z) \xrightarrow{\sim} G(F);$$

(b) For $x \notin |\mathbb{P}^1(k)|$, U_x is the hyperspecial maximal compact subgroup $G(\mathcal{O}_x)$;
(c) The subgroups U_0 and U_∞ are parahoric subgroups that remain to be chosen.

Proposition 6.1. *Let G be unramified and simply-connected. The subgroups U_0 and U_∞ can be chosen in such a way that*

$$G(K) \cap U = T(k).$$

Proof. Let $\Lambda = X_*(S)$ be the cocharacter lattice of S. We have remarked that for $z \in \mathbb{P}^1(k)$, we have a natural identification of $\Lambda \otimes \mathbb{R}$ as an apartment in the Bruhat-Tits building of G_z. Let $c \in \Lambda \otimes \mathbb{R}$ be the fixed point of $U^* \subset G(F) \cong G(K_z)$; because G is simply-connected, U^* is the parahoric subgroup G_c. For $y \notin \mathbb{P}^1(k)$, we also have a natural inclusion $\Lambda \otimes \mathbb{R} \hookrightarrow \mathcal{A}(A_y)$. Choose $a, b \in \Lambda \otimes \mathbb{R}$ such that

$$a + b + (q-1)c = 0$$

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and such that a is sufficiently generic in the sense that $a \cdot v$ is not an integer for all v in a finite set of nonzero elements of $X^*(S)$ to be chosen later.

For a place v of K, define $f(v)$ by

$$\begin{cases} f(0) = a, \\ f(\infty) = b \\ f(z) = c \text{ for } z \in \mathbb{G}_m(k), \\ f(x) = 0, \text{ for } x \notin |\mathbb{P}^1(k)|. \end{cases}$$

Under this definition, we take U_v to be the parahoric subgroup $G_{f(v)}$ associated to $f(v)$.

We are now ready to prove that $G(K) \cap U = T(k)$. To do this, we use the faithful algebraic representation (ρ, V) fixed in §5.1, equipped with a basis $e_1, \dots, e_n$, where e_i is an eigenvector of S with eigencharacter $\lambda_i \in X^*(S)$. Applying Lemma 5.1, we see that for $g \in U \cap G(K)$ with $\rho_{ij}(g) \neq 0$ (for a fixed pair (i, j)), we have

$$0 = \sum_v v(\rho_{ij}(g)) \geq \sum_v f(v) \cdot (\lambda_i - \lambda_j) = 0 \cdot (\lambda_i - \lambda_j) = 0.$$

Because both sides are 0, the inequality is sharp, so

$$v(\rho_{ij}(g)) = f(v) \cdot (\lambda_i - \lambda_j)$$

for all v. This implies that $f(v) \cdot (\lambda_i - \lambda_j)$ is an integer, and, in particular, that $a \cdot (\lambda_i - \lambda_j)$ is an integer. However, for our genericity assumption on a, we may assume that $a \cdot (\lambda_i - \lambda_j)$ is not an integer for any pair (i, j) for which $\lambda_i \neq \lambda_j$.

We conclude that, for $g \in U$, $\rho_{ij}(g) = 0$ unless $\lambda_i = \lambda_j$. In other words, $\rho(g)$ commutes with $\rho(S)$, and thus, $g \in T(K)$. Furthermore, for a pair (i, j) with $\rho_{ij}(g) \neq 0$, we have

$$v(\rho_{ij}(g)) = f(v) \cdot (\lambda_i - \lambda_j) = f(v) \cdot 0 = 0$$

for all v. This shows that $\rho_{ij}(g) \in k$, and so $g \in T(k)$. We have thus shown that $U \cap G(K) = T(k)$, as desired. □

The following is then obvious.

Corollary 6.2. *Under the hypotheses of Proposition 6.1, one can choose Iwahori subgroups $I_0 \subset U_0$ and $I_\infty \subset U_\infty$ such that, if $I_\infty(p) \subset I_\infty$ is the maximal pro-p subgroup, then*

$$G(K) \cap \prod_{x \neq 0, \infty} U_x \times I_\infty(p) \times I_0 = \{1\}.$$

Remark 6.3. Suppose $U_z = G(\mathcal{O}_z)$ for $z \in \mathbb{G}_m(k)$. Then in Proposition 6.1, we can take $U_0 = I^+$ and $U_\infty = I^-$ to be respectively the upper and lower Iwahori subgroups; by this, we mean that U_0 (resp. U_∞) is the Iwahori subgroup corresponding to the positive (resp. negative) root system corresponding to our fixed Borel subgroup B. Then

$$G(K) \cap U \subset G(K) \cap \prod_x G(\mathcal{O}_x).$$

The right-hand side is the group of global maps from $\mathbb{P}^1_k$ to the affine group G_k. Any such map must be constant because $\mathbb{P}^1$ is projective and thus must belong to $G(k)$ because it is defined over k. Thus,

$$G(K) \cap U = (U_0 \times U_\infty) \cap G(k) = T(k).$$

The open compact subgroups of $G(\mathbb{A}_K)$ built in this section will be used in the next section to constuct some Poincaré series.

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7. Proof of Theorem 1.4(i)

This section is devoted to the proof of Theorem 1.4(i). In fact, we prove the following slightly stronger statement.

Theorem 7.1. *Suppose F is the nonarchimedean local field $k((t))$, where $k = \mathbb{F}_q$ is a finite field of order $q = p^m$ for some prime p. Let G be an unramified connected reductive group over F which is not a torus. Assume $q > 5$ and π is an irreducible supercuspidal representation of $G(F)$ that can be obtained as the induction of a representation of a compact open (modulo center) subgroup $U \subset G(F)$:*

$$\pi \xrightarrow{\sim} \mathrm{c-Ind}_U^{G(F)} \tau$$

for some (irreducible) smooth representation τ of U with coefficients in $\overline{\mathbb{Q}}_\ell$.

Then we have

$$\mathcal{L}^{ss}(\pi) \text{ is pure} \Longrightarrow f \circ \mathcal{L}^{ss}(\pi) = f \circ \mathcal{L}(\pi) \text{ is ramified},$$

where $f : {}^L G \longrightarrow {}^L G/Z(\hat{G})$ is the natural projection map.

Because ${}^L G/Z(\hat{G})$ is the L-group of the simply-connected cover G_{sc} of the derived group G_{der} of G, Theorem 2.3(vi) implies (by applying it to the natural maps $G_{sc} \to G_{der} \to G$) that it suffices to prove the above theorem under the additional hypothesis that G is k-simple and simply-connected.

7.1. Purity and ramification

We shall prove Theorem 1.4(i) by a global argument. Hence, we assume the setup and notation from §5.1. In particular, $K = k(\mathbb{P}^1) = k(t)$ and G is an unramified simply connected simple group over k.

Before stating the theorem below, we choose a special character of T. The adjoint representation of $\hat{G}$ splits as a sum of irreducible representations, each corresponding to a simple factor, up to isogeny, of $G_{\overline{k}}$. The group W_k acts on these irreducible representations. Let $\mathfrak{v}_1, \ldots, \mathfrak{v}_n$ be an orbit of this action. Let $\hat{\alpha}_i$ be the highest weight of $\mathfrak{v}_i$, a coroot of G, and let α_i be the corresponding root of G. Let $\alpha \in X^*(T)$ be given by $\alpha = \sum_{i=1}^n \alpha_i$. Then since the action of W_i permutes the $\mathfrak{v}_i$s, it permutes the $\hat{\alpha}_i$s, and thus the α_is, so α is W_k-invariant.

The following theorem is the key step in proving Theorem 1.4(i).

Theorem 7.2. *Let $F = k((t))$ with $k = \mathbb{F}_q$ and $q > 5$, and let σ be a supercuspidal representation of $G(F)$ (with G unramified over k). Let $K = k(\mathbb{P}^1)$ and suppose there exists a cuspidal automorphic representation Π of $G(\mathbb{A}_K)$, with the following properties:*

(a) *At every $z \in \mathbb{G}_m(k) \subset \mathbb{P}^1(k)$, $\Pi_z \xrightarrow{\sim} \sigma$.*
(b) *The representation Π_∞ has a vector invariant under the pro-p Iwahori subgroup $I_\infty(p)$ in Corollary 6.2.*
(c) *The representation Π_0 is a constituent of a principal series $\mathrm{Ind}_{B(F)}^{G(F)} \chi$, where χ is a tame (i.e., depth 0) character of $T(F)$, whose restriction χ_k to the subgroup $T(k) \subset T(F)$ arises in the following way:*

$$\chi_k : T(k) \xrightarrow{\alpha} k^\times \xrightarrow{\mu} \overline{\mathbb{Q}}_\ell^\times$$

where μ is a faithful character of $k^\times$.
(d) *The local components Π_y of Π for all $y \notin \mathbb{P}^1(k)$ are $G(\mathcal{O}_y)$-unramified.*

Then

$$\mathcal{L}^{ss}(\sigma) \text{ is pure} \Longrightarrow \mathcal{L}^{ss}(\sigma) \text{ is ramified}.$$

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Proof. Assume $\mathcal{L}^{ss}(\sigma)$ is unramified; we shall derive a contradiction.

Applying Lafforgue, let

$$\mathcal{L}(\Pi) : \mathrm{Gal}(K^{sep}/K) \longrightarrow {}^L G = \hat{G} \rtimes W_k$$

be the global Galois representation associated to Π as in Theorem 2.1. By property (d), $\mathcal{L}(\Pi)_x$ is unramified for all places x of $\mathbb{P}^1$ outside of $\mathbb{P}^1(k)$. However, by property (b) and (c), it follows by Theorem 2.3 (iv) that $\mathcal{L}(\Pi)_0^{ss}$ and $\mathcal{L}(\Pi)_\infty^{ss}$ are tamely ramified. For $z \in \mathbb{G}_m(k)$, with decomposition group $\Gamma_z \subset \pi_1(\mathbb{P}^1 \setminus |\mathbb{P}^1(k)|)$, it follows by property (a) and Theorem 2.1(iii) that

$$\mathcal{L}(\Pi)_z^{ss} \xrightarrow{\sim} \mathcal{L}^{ss}(\sigma). \tag{7.3}$$

By our hypothesis, $\mathcal{L}(\Pi)_z^{ss}$ is thus unramified for all $z \in \mathbb{G}_m(k) = k^\times$.

We shall now examine in greater detail the restriction of $\mathcal{L}(\Pi)$ to the local decomposition group Γ_0 at $0 \in k$. By property (c) and Theorem 2.3 (iv), one has

$$\mathcal{L}(\Pi)_0^{ss} : \Gamma_0 = \mathrm{Gal}(F^{sep}/F) \longrightarrow {}^L T \longrightarrow {}^L G,$$

where the first map is the L-parameter of the character χ of $T(F)$. By the discussion in [DR, §4.3], one has

$$\mathrm{Hom}(T(k), \overline{\mathbb{Q}}_l^\times) \cong \mathrm{Hom}_{\mathrm{Frob}}(I_t, \hat{T}) \cong \mathrm{Hom}_{\mathrm{Frob}}(k_n^\times, \hat{T}),$$

where $k_n = \mathbb{F}_{q^n}$ is the splitting field of T over k, I_t is the tame inertia group of Γ_0 and $\mathrm{Hom}_{\mathrm{Frob}}$ stands for Frobenius-equivariant homomorphisms. Hence, the restriction of $\mathcal{L}(\Pi)_0^{ss}$ to the inertia group (which factors to the tame inertia group) determines and is determined by the restriction χ_k of χ to $T(k)$. The special form of χ_k given in property (c) and the fact that α is fixed by Frob thus imply that the restriction of $\mathcal{L}(\Pi)_0^{ss}$ to I_t factors as

$$\mathcal{L}(\Pi)_0^{ss}|_{I_t} : I_t \longrightarrow k^\times \xrightarrow{\mu} \overline{\mathbb{Q}}_l^\times \xrightarrow{\alpha} \hat{T} \longrightarrow {}^L G. \tag{7.4}$$

Let $V = \otimes_{i=1}^n \mathfrak{v}_i$ be the representation of ${}^L G = \hat{G} \rtimes W_k$ where $\hat{G}$ acts on each $\mathfrak{v}_i$ and W_i permutes the factors. The composite $V \circ \mathcal{L}(\Pi)^{ss}$ gives rise to a (semisimple) local system $\mathcal{V}(\Pi, V)$ on $\mathbb{P}^1 \setminus |\mathbb{P}^1(k)|$ over k. However, since $\mathcal{L}(\Pi)_z^{ss}$ is unramified, the purity of $\mathcal{L}^{ss}(\sigma)$ and Theorem 4.2 then imply that $\mathcal{V}(\Pi, V)$ extends to a local system on $\mathbb{G}_m$ over k.

The representation V is obtained as a tensor product of the representations $\mathfrak{v}_i$ of $\hat{G}$, which do not extend to representations of $\hat{G} \rtimes W_k$, but do extend to representations of $\hat{G} \rtimes W_{k_n}$ for k_n the degree n extension, since the stabilizer of $\mathfrak{v}_i$ in the action of W_k on irreducible factors of the adjoint representation of $\hat{G}$ is W_{k_n}. Thus, we similarly have that the composite $\mathfrak{v}_i \circ \mathcal{L}(\Pi)^{ss}|_{\mathrm{Gal}(K^{sep}/k_n(t))}$ gives rise to a (semisimple) local system $\mathcal{V}(\Pi, \mathfrak{v}_i)$ on $\mathbb{P}^1 \setminus |\mathbb{P}^1(k)|$ over k_n, which extends to a local system on $\mathbb{G}_m$ over k_n.

The above discussion shows that the local system $\mathcal{V}(\Pi, \mathfrak{v}_i)$ on $\mathbb{G}_m$ over k_n satisfies the hypotheses of Corollary 4.3. In particular,

(a) the local monodromy representations at 0 and ∞ are tamely ramified;
(b) the local monodromy representation at 0 has abelian image.

Corollary 4.3 thus implies that $\mathcal{V}(\Pi, \mathfrak{v}_i)$ breaks up as the sum of rank 1 local systems $\mathcal{L}_1, \ldots, \mathcal{L}_N$ over k_n.

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Applying Lemma 4.7 to the $\mathcal{L}_i$'s, and a standard formula for the trace on a tensor product representation of an operator that permutes the tensor factors, one sees that for each $z \in \mathbb{G}_m(k)$,

$$\mathrm{tr}(\mathrm{Frob}_z, \mathcal{V}(\Pi, V)) = \mathrm{tr}(\mathrm{Frob}_z^n, \mathcal{V}(\Pi, \mathfrak{v}_1)) = \sum_{j=1}^{N} \mathrm{tr}(\mathrm{Frob}_z^n, \mathcal{L}_j) = \sum_{j=1}^{N} \mathrm{tr}(\mathrm{Frob}_1^n, \mathcal{L}_j) \cdot \mathrm{tr}(m_z^{-n}, \mathcal{L}_j). \tag{7.5}$$

Here, it is crucial that we consider the tensor induction instead of the usual induction because the trace of Frob_z on the usual induction would be identically zero as soon as $n > 1$, and so studying this trace would not be helpful. We take $i = 1$ without loss of generality because $\mathrm{tr}(\mathrm{Frob}_z^n, \mathcal{V}(\Pi, \mathfrak{v}_i))$ is independent of i.

Let us now evaluate $\mathrm{tr}(m_z, \mathcal{L}_j)$. By (7.4), and observing that the eigenvalues of $\hat{T}$ on $\mathfrak{v}_1$ are the roots of $(\hat{G}, \hat{T})$ contained in the simple factor corresponding to $\mathfrak{v}_1$, we see that

$$\mathrm{tr}(m_z, \mathcal{L}_j) = \mu(z)^{\langle \alpha, \beta_j \rangle}, \tag{7.6}$$

where β_j is either a root of $\hat{G}$ (or equivalently a coroot of G) contained in $\mathfrak{v}_1$ or the zero element in $X^*(\hat{T})$.

Now note that

(a) μ is a character of order $q-1$ of $k^\times$;
(b) the function

$$z \mapsto \mathrm{tr}(\mathrm{Frob}_z, \mathcal{V}(\Pi, V))$$

is a constant function of z, since the local representation at each $z \in \mathbb{G}_m(k)$ is just $\mathcal{L}^{ss}(\sigma)$.

By orthogonality of characters (of $k^\times$) and (7.5), we thus have

$$\begin{aligned} 0 &= \sum_{z \in k^\times} \mu(z)^{-2} \cdot \mathrm{tr}(\mathrm{Frob}_z, \mathcal{V}(\Pi, V)) \\ &= \sum_{z \in k^\times} \mu(z)^{-2} \cdot \sum_{j=1}^{N} \mathrm{tr}(\mathrm{Frob}_1^n, \mathcal{L}_j) \cdot \mu(z)^{\langle \alpha, \beta_j \rangle} \\ &= \sum_{j=1}^{N} \mathrm{tr}(\mathrm{Frob}_1^n, \mathcal{L}_j) \cdot \sum_{z \in k^\times} \mu(z)^{-2} \cdot \mu(z)^{\langle \alpha, \beta_j \rangle} \\ &= (q-1) \cdot \sum_{\substack{j: \\ \langle \alpha, \beta_j \rangle \equiv 2 \bmod (q-1)}} \mathrm{tr}(\mathrm{Frob}_1^n, \mathcal{L}_j). \end{aligned} \tag{7.7}$$

Furthermore, we have

$$\langle \alpha, \beta_j \rangle = \langle \sum_{i=1}^{n} \alpha_i, \beta_j \rangle = \sum_{i=1}^{n} \langle \alpha_i, \beta_j \rangle = \langle \alpha_1, \beta_j \rangle$$

since for $i \neq 1$, α_i is a root of a different simple factor from β_j and thus is orthogonal to β_j.

Now one of the β_j's is the highest weight $\alpha_1^\vee$ of $\mathfrak{v}_1$, for which one has $\langle \alpha, \alpha_1^\vee \rangle = \langle \alpha_1, \alpha_1^\vee \rangle = 2$. For all other β_j, one has

$$-2 \leq \langle \alpha, \beta_j \rangle < 2.$$

Hence, since $\langle \alpha, \beta_j \rangle \in \mathbb{Z}$, we have

$$\langle \alpha, \beta_j \rangle \in \{-2, -1, 0, 1\}.$$

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Since we have assumed $q > 5$, this implies that there is only one term appearing in the sum in (7.7) – namely, the term corresponding to the highest root $\alpha_1^\vee$ of $\hat{G}$. Hence, we deduce the desired contradiction

$$(q-1) \cdot \mathrm{tr}(\mathrm{Frob}_1^n, \mathcal{L}_j) = 0,$$

which is impossible as $\mathcal{L}_j$ is a one-dimensional representation.

We have thus completed the proof of Theorem 7.2. □

7.2. *Poincare series I*

We continue with the proof of Theorem 1.4(i), under the hypothesis that G is semisimple and simply-connected. Let $F = k((t))$ and let σ be a supercuspidal representation of $G(F)$. We say σ contains an $\mathfrak{s}$-type if there is an open compact subgroup $U^* \subset G(F)$ and an irreducible representation ρ of U^* such that

$$\sigma \xrightarrow{\sim} c\text{-}Ind_{U^*}^{G(F)} \rho. \tag{7.8}$$

Theorem 7.9. *Let $K = k(\mathbb{P}^1)$. Let $F = k((t))$ and let σ be a supercuspidal representation of $G(F)$. Suppose*

$$\sigma \text{ contains an } \mathfrak{s}\text{-type } (U^*, \rho). \tag{7.10}$$

Then there exists a cuspidal automorphic representation Π of $G(K)$ satisfying the conditions (a)–(d) of Theorem 7.2.

Proof. The construction follows the method of [GLo]. Hypothesis (7.10) implies that σ has a matrix coefficient φ^* with support in U^*, with the property that $\varphi^*(1) = 1$. Using this U^*, let $U = \prod_x U_x \subset G(\mathbb{A}_K)$ be the open compact subgroup constructed in Proposition 6.1. Define a function $\varphi = \prod_x \varphi_x : G(\mathbb{A}_K) \to \mathbb{C}$ as follows:

(a) At every $z \in \mathbb{G}_m(k) \subset \mathbb{P}^1(k)$, $\varphi_z = \varphi^*$;
(b) φ_∞ is the characteristic function of the group $I_\infty(p)$ of Corollary 6.2;
(c) φ_0 is the character $\chi_k : I_0/I_0(p) \cong T(k) \to \mathbb{C}^\times$, where χ_k is as in the proof of Theorem 7.2;
(d) For $x \notin |\mathbb{P}^1(k)|$, φ_x is the characteristic function of the hyperspecial maximal compact subgroup $G(\mathcal{O}_x)$;

Now define the Poincaré series $P_\varphi : G(K)\backslash G(\mathbf{A}_K) \to \mathbb{C}$:

$$P_\varphi(g) = \sum_{\gamma \in G(K)} \varphi(\gamma \cdot g).$$

The sum converges absolutely, as in [GLo]. It follows from Corollary 6.2 that

$$P_\varphi(1) = \prod_{z \in \mathbb{G}_m(k)} \varphi^*(1) = 1.$$

In particular, $P_\varphi \neq 0$. The spectral decomposition of P_φ yields the desired cuspidal automorphic representation Π. □

Corollary 7.11. *Let G be an unramified reductive group over a nonarchimedean local field F of characteristic p, with constant field k of order $q > 5$. Suppose that G is not a torus and p does not divide the order of the Weyl group of G. Then if σ is pure, $f \circ \mathcal{L}^{ss}(\sigma)$ is ramified (where $f : {}^L G \to {}^L G/Z(\hat{G})$). In particular, Theorem 1.4(i) holds.*

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Proof. As mentioned at the beginning of the section, we may assume G is simply-connected. Thanks to the results of [Fi], the supercuspidal representation σ of $G(F)$ contains an $\mathfrak{s}$-type by our hypothesis on p. The claim now follows immediately from Theorems 7.2 and 7.9. □

Theorem 7.12. *Let $F = k((t))$ and let σ be an irreducible representation of $G(F)$, with G unramified reductive over F. Suppose the Genestier-Lafforgue parameter $\mathcal{L}^{ss}(\sigma)$ is pure and unramified. Suppose every supercuspidal representation of every Levi factor of $G(F)$ contains an $\mathfrak{s}$-type. This is true in particular in the following cases:*

- *p does not divide the order of the Weyl group of G [Fi];*
- *p is odd and G is an unramified classical group (unitary, symplectic or special orthogonal) [St].*

Then σ is an irreducible constituent of an unramified principal series representation of $G(F)$.

Proof. In any case, σ is an irreducible constituent of a representation of the form $Ind_{P(F)}^{G(F)}\tau$ for some parabolic subgroup $P(F) = L(F) \cdot N(F) \subset G(F)$, where $L(F)$ is a Levi subgroup of $P(F)$, $N(F)$ its unipotent radical, and τ is a supercuspidal representation of $L(F)$. Since the Genestier-Lafforgue parametrization is compatible with parabolic induction by Theorem 2.3, it follows that $\mathcal{L}^{ss}(\tau)$ is unramified. Since τ is supercuspidal, it follows from Theorems 7.9 and 7.2 (or more simply Corollary 7.11) that L must be a torus and τ an unramified character of $L(F)$. □

8. Proof of Theorem 1.4(ii): wild ramification

The goal of this section is to prove Theorem 1.4(ii). For the convenience of the reader, we restate the result here:

Theorem 8.1. *Let G be an unramified reductive group over a local field $F = k((t))$ of equal characteristic p, and J an open subgroup of a parahoric subgroup G_a. Let σ be a representation of J such that $\pi = \text{c-Ind}_J^{G(F)}\sigma$ is irreducible.*

If J is sufficiently small and $|k| = q > 5$, then $f \circ \mathcal{L}^{ss}(\pi)$ (and hence, $f \circ \mathcal{L}(\pi)$) is wildly ramified (where $f : {}^LG \to {}^LG/Z(\hat{G})$).

8.1. Sufficiently small subgroups

Let us first give the precise definition of being 'sufficiently small', recalling that G is defined over k.

Definition 8.2. (i) A finite subgroup H of $G(k)$ is *sufficiently small* if there is a Borel k-subgroup B of G such that $B(k) \cap H$ is trivial.

(ii) For a parahoric subgroup G_a of $G(F)$, we say $J \subseteq G_a$ is *sufficiently small* if its projection to the reductive group $G_a/G_{a,0+}$ is sufficiently small.

For example, the maximal unipotent subgroup of a Borel subgroup is a sufficiently small subgroup of $G(k)$ (by the Bruhat decomposition), and $G_{a,0+}$ is a sufficiently small subgroup of $G(F)$.

8.2. An open compact subgroup of $G(\mathbb{A}_K)$

As for Theorem 1.4(i), we may and shall assume that G is simply-connected when proving Theorem 8.1. The proof will be via a global argument. Hence, we will be working over $K = k(\mathbb{P}^1)$ and make use of the notation of §5.1. We will need to construct appropriate globalizations of π by Poincaré series, and the purpose of this section is to construct appropriate open compact subgroups of $G(\mathbb{A}_K)$.

Recall from §5.2 that for $z \in \mathbb{P}^1(k)$, the apartments $\mathcal{A}(S_z)$ in the buildings $\mathcal{B}(G_z)$ can be naturally identified with $X_*(S) \otimes \mathbb{R}$ and hence with each other. Without loss of generality, we may assume that the parahoric subgroup G_a in Theorem 8.1 is associated to a point $a \in \mathcal{A}(S_0) = X_*(S) \otimes \mathbb{R}$. Let b be a small perturbation of the point $-a$ in $\mathcal{A}(S_\infty)$. Specifically, we choose b so that $a + b$ is a small rational

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multiple of a cocharacter which lies in the interior of the positive Weyl chamber of $X_*(S) \otimes \mathbb{R}$. Because $a+b$ does not lie on the walls of the Weyl chamber, b does not lie on any wall of the apartment that also contains $-a$, and because $a+b$ is small, b does not lie on any wall of the apartment that does not contain $-a$. So b lies in the interior of a Weyl alcove and thus gives rise to an Iwahori subgroup $G_b \subset G(K_\infty)$.

For each place v of K, set

$$f(v) = \begin{cases} a, & \text{if } v = 0; \\ b, & \text{if } v = \infty; \\ 0, & \text{for all other } v. \end{cases}$$

Let

$$C = \prod_v G_{f(v)} \subset G(\mathbb{A}_K)$$

be the associated compact open subgroup of $G(\mathbb{A}_K)$ and write $C_v = G_{f(v)}$, so that for $v \notin \{0, \infty\}$, C_v is the standard hyperspecial maximal compact subgroup $G(\mathcal{O}_v)$.

We now note the following:

Lemma 8.3. *The natural map*

$$C \cap G(K) \to C_0 \to G_a \to G_a/G_{a,0+}$$

is injective with image contained in a Borel subgroup of $G_a/G_{a,0+}$.

Proof. Let $\lambda\colon \mathbb{G}_m \to S$ be a cocharacter which is a negative multiple of $a+b$. (This exists because $a+b$ is a small rational multiple of a cocharacter, and we may multiply by any negative integer that cancels the denominator of that rational to obtain a cocharacter.) The image of λ in $G_a/G_{a,0+}$ continues to define a cocharacter of the maximal split torus of $G_a/G_{a,0+}$. Since λ lies in the interior of a Weyl chamber, it defines a Borel subgroup B_λ over k (under the 'dynamical' definition of parabolic subgroups). We shall show that the image of $C \cap G(K)$ in $G_a/G_{a,0+}$ belongs to B_λ. For this, we need to show that for any $g \in C \cap G(K)$,

$$\lambda(t) \cdot g \cdot \lambda(t)^{-1} \text{ remains bounded as } t \to 0.$$

To verify this, we shall apply Lemma 5.1 to the faithful algebraic representation $\rho\colon G \to \mathrm{Aut}(V)$ of G over k that was fixed in §5.1. Recall that V is equipped with a k-basis $e_1, \ldots, e_n$ of V, where e_i is an eigenvector of S with eigencharacter λ_i. For any $g \in C \cap G(K)$ and (i,j) such that $\rho_{ij}(g) \neq 0$, we deduce by the product formula, Lemma 5.1, and the definition of $f(v)$, that

$$0 = \sum_v v(\rho_{ij}(g)) \geq \sum_v \lceil f(v) \cdot (\lambda_i - \lambda_j) \rceil = \lceil a \cdot (\lambda_i - \lambda_j) \rceil + \lceil b \cdot (\lambda_i - \lambda_j) \rceil. \tag{8.4}$$

Now since b is a small perturbation of $-a$, (8.4) can only be satisfied if $a \cdot (\lambda_i - \lambda_j)$ is an integer and $b \cdot (\lambda_i - \lambda_j) \leq -a \cdot (\lambda_i - \lambda_j)$. Indeed, b is a small perturbation of $-a$, so we may assume that $b \cdot (\lambda_i - \lambda_j)$ is arbitrarily close to $-a \cdot (\lambda_i - \lambda_j)$. In particular, if $a \cdot (\lambda_i - \lambda_j)$ lies in the interval $(n, n+1)$ for some integer n, then $b \cdot (\lambda_i - \lambda_j)$ lies in the interval $(-n-1, -n)$, and so the sum of their ceilings is $n + 1 + (-n) = 1 > 0$.

With λ chosen as in the beginning of the proof, we thus deduce that for $g \in C \cap G(K)$ such that $\rho_{ij}(g) \neq 0$, we have $(a+b) \cdot (\lambda_i - \lambda_j) \leq 0$, and thus,

$$\lambda \cdot (\lambda_i - \lambda_j) \geq 0.$$

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It follows that $\lambda(t)g\lambda(t)^{-1}$ is bounded as $t \to 0$, since all the nonvanishing coordinates ρ_{ij} of g are eigenvectors under the conjugation action by $\lambda(t)$ with eigenvalue a positive power of t. This implies that the image of g in $G_a/G_{a,0+}$ lies in the Borel subgroup B_λ, as desired.

It remains to show that if $g \in C \cap G(K)$ has trivial image in $G_a/G_{a,0+}$, then g is the identity element. If $\rho_{ij}(g) \neq 0$, then because $a \cdot (\lambda_i - \lambda_j)$ is an integer and $b \cdot (\lambda_i - \lambda_j) \leq -a \cdot (\lambda_i - \lambda_j)$, the right side of (8.4) is 0. So every inequality is sharp; that is,

$$v_0(\rho_{ij}(g)) = a \cdot (\lambda_i - \lambda_j).$$

However, by Lemma 5.1(ii), if $g \in G_{a,0+}$, then

$$v_0(\rho_{ij}(g)) - \delta_{ij}) > a \cdot (\lambda_i - \lambda_j).$$

Thus, if $i \neq j$, we obtain a contradiction from the assumption $\rho_{ij}(g) \neq 0$. Hence, $\rho_{ij}(g) = 0$ for $i \neq j$. However, if $i = j$, then $\rho_{ij}(g) \in \mathcal{O}_v$ for every place of K, and thus lies in k, but since $v_0(\rho_{ii}(g) - 1) > 0$, we see that $\rho_{ii}(g) = 1$. Thus, if $g \in G_{0,a+}$, then $\rho(g)$ is the identity, so that g is the identity since ρ is faithful. □

8.3. Poincaré series II

Using the above lemma, we will construct two different Poincaré series for G which are unramified outside $\{0, \infty\}$ and whose local representation at 0 is π, but with different local representations at ∞.

Let B_λ be a Borel subgroup of $G_a/G_{a,0+}$ containing the image of $C \cap G(K)$. Let H be the image of J in $G_a/G_{a,0+}$. Since we are assuming that J is sufficiently small, there exists g in $G_a/G_{a,0+}$ with $J \cap gB_\lambda g^{-1}$ trivial. By conjugating J by a lift of g, we may assume $g = 1$. Set

$$C' = J \times \prod_{v \neq 0} C_v \subset C \subset G(\mathbb{A}_K).$$

Then the previous lemma implies that natural map $C' \cap G(K) \to G_a/G_{a,0+}$ is injective with image contained in $H \cap B = \{1\}$. Hence, $C' \cap G(K)$ is trivial.

Recall that $G_b \subset G(K_\infty)$ is an Iwahori subgroup, so that the quotient $G_b/G_{b,0+}$ is a torus of rank $r \geq \dim S$ and hence has at least $(q-1)^r$ points. It thus has a nontrivial character χ because $q > 2$. We can view σ and $\sigma \otimes \chi$ as representations of C' by projection onto the components at 0 and ∞. Now apply the Poincaré series construction to obtain

- an automorphic function f_1 created from a matrix coefficient of σ,
- an automorphic function f_2 created from a matrix coefficient of $\sigma \otimes \chi$.

These Poincaré series are nonzero because $C' \cap G(K)$ is trivial, so that the sum defining the series has only one nonzero term. From the spectral expansion of f_1 and f_2, we obtain two cuspidal automorphic representations Π_1 and Π_2 which are unramified outside $\{0, \infty\}$ and such that

- at 0, the local components of Π_1 and Π_2 are both isomorphic to π;
- at ∞, $\Pi_{1,\infty}$ has nonzero Iwahori-fixed vectors, whereas $\Pi_{2,\infty}$ does not. Indeed, both these representations are subquotients of principal series representations induced from characters of $T(K)$ whose restrictions to $T(k)$ are 1 and χ, respectively.

8.4. Proof of Theorem 8.1

We can now conclude the proof of Theorem 8.1. Consider the global parameters $\mathcal{L}(\Pi_1)$ and $\mathcal{L}(\Pi_2)$ associated to Π_1 and Π_2 above. These are both unramified outside $\{0, \infty\}$. Moreover,

$$\mathcal{L}(\Pi_1)_0^{ss} \cong \mathcal{L}(\Pi_2)_0^{ss}, \quad \text{but} \quad \mathcal{L}(\Pi_1)_\infty^{ss} \ncong \mathcal{L}(\Pi_2)_\infty^{ss}.$$

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Indeed, at ∞, the associated local Genestier-Lafforgue parameters are tame, and their restrictions to the tame inertia group at ∞ correspond under class field theory to 1 and $\chi \neq 1$ (up to semisimplification).

Now suppose, for the sake of contradiction, that the local Genestier-Lafforgue parameter of π is tame. Thus, $\mathcal{L}(\Pi_1)$ and $\mathcal{L}(\Pi_2)$ factor through the tame fundamental group of $\mathbb{G}_{m,k}$. In the tame fundamental group of $\mathbb{G}_{m,k}$, the inertia subgroups at 0 and ∞ are the same subgroup, both being equal to the tame geometric fundamental group. Since $\mathcal{L}(\Pi_1)$ and $\mathcal{L}(\Pi_2)$ are isomorphic when restricted to the tame inertia group at 0 and semisimplified, they must also be isomorphic when restricted to the tame inertia group at ∞ and semisimplified. This contradicts what we showed above: that $\mathcal{L}(\Pi_1)$ is trivial while $\mathcal{L}(\Pi_2)$ is nontrivial when restricted to the tame inertia group at ∞ and semisimplified. This gives the desired contradiction and completes the proof of Theorem 8.1.

8.5. *A special case*

When the group J is contained in the principal congruence subgroup of a hyperspecial maximal compact, there is a shorter argument, though it requires the purity of $\mathcal{L}^{ss}(\pi)$ as a hypothesis and appeals to Theorem 7.2.

Proposition 8.5. *Under the hypotheses of Proposition 8.1, suppose G_a is a hyperspecial maximal compact and J is contained in its principal congruence subgroup $G_{a,0+}$. Let π be a pure supercuspidal representation of $G(K)$ compactly induced from an irreducible representation σ of J. Then $\mathcal{L}^{ss}(\pi)$ is wildly ramified.*

Proof. We construct a Poincaré series as in Theorem 7.9 using a function φ which at ∞ is a matrix coefficient of σ and is a characteristic function of the standard hyperspecial maximal compact $U_v = G(\mathcal{O}_v)$ at all other places v. Now the support of φ_∞ is contained in J, and $J \times \otimes_{v \neq \infty} U_v \cap G(K) = \{1\}$. Thus, the Poincaré series P_φ does not vanish at 1, and the space of cusp forms it generates contains a cuspidal automorphic representation Π that is unramified except at ∞. So $\mathcal{L}(\Pi)$ is a parameter with values in ${}^L G$ that is unramified away from ∞. However, by Theorem 7.2, $\mathcal{L}^{ss}(\pi)$ is ramified. Since a tamely ramified local system on $\mathbb{A}^1$ is unramified, it follows that the ramification at ∞ must be wild. □

9. Questions

In this section, we raise a number of natural questions which are suggested by our results.

9.1. *Positive depth representations*

In the context of Theorem 8.1, for a given reductive group G over a local field F, one may ask for a better understanding (or even a classification) of all supercuspidal representations satisfying the condition of Theorem 8.1 (i.e., that can be induced from a sufficiently small open compact (modulo center) subgroup J). In particular, one may ask if they can be understood in the framework of J. K. Yu's construction of supercuspidal representations. For example, one might hope that if the smallest twisted Levi subgroup in the twisted Levi sequence in a Yu datum is a torus, then the resulting supercuspidal representation should satisfy the condition of Theorem 8.1 or a slightly modified version of it.

9.2. *Examples of pure supercuspidals*

Theorem 1.4(i) has the serious condition that π is a pure supercuspidal representation. It will be good to have some examples of pure supercuspidal representations in addition to the ones used in §3.3. For example, the desiderata of the LLC suggests that if π is a generic supercuspidal representation of a quasi-split G, then π is pure. In the rest of this section, we show this for certain generic supercuspidal representations of depth 0, conditionally on the independence of ℓ result assumed in [DL].

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9.3. Poincaré series III

We begin by giving yet another construction of Poincaré series. Let us work in the context of §5.1 again, so that G is an unramified semisimple group over k.

Proposition 9.1. *Let π be a depth 0 generic supercuspidal representation of $G(F)$ (with $F = k((t))$) of the form*

$$\pi \xrightarrow{\sim} c - Ind_{G(\mathcal{O}_F)}^{G(F)} \sigma,$$

where σ is a generic cuspidal representation of the finite reductive group $G(k)$. There is a globally generic representation Π of G such that

- *$\Pi_0 \cong \pi$ and $\Pi_\infty \cong \pi^\vee$;*
- *for all other places v of $K = k(t)$, Π_v is $G(\mathcal{O}_v)$-unramified.*

Proof. Since σ is a generic representation of $G(k)$, one can find a generic character

$$\psi : N(k) \longrightarrow \mathbb{C}^\times$$

such that the $(N(k), \psi)$-eigenspace of σ is nonzero, in which case it is 1-dimensional. Let us fix a nonzero vector w such that

$$\sigma(n) \cdot w = \psi(n) \cdot w \quad \text{for all } n \in N(k). \tag{9.2}$$

Fixing a $G(k)$-invariant inner product $\langle -, - \rangle$ on σ, we may assume that $\langle w, w \rangle = 1$.

Now we may consider the function on $G(K_0)$ defined by

$$f_0(g) = \begin{cases} \langle \sigma(g) \cdot w, w \rangle, & \text{if } g \in G(\mathcal{O}_0); \\ 0, & \text{if } g \notin G(\mathcal{O}_0). \end{cases}$$

This is a matrix coefficient of π, which is built out of a matrix coefficient of σ supported on $G(\mathcal{O}_0)$. Likewise, we define a function on $G(K_\infty)$ by

$$f_\infty(g) = \begin{cases} \langle w, \sigma(g) \cdot w \rangle, & \text{if } g \in G(\mathcal{O}_\infty); \\ 0, & \text{if } g \notin G(\mathcal{O}_\infty), \end{cases}$$

which is a matrix coefficient of $\overline{\pi} \cong \pi^\vee$. Define a locally constant compactly supported function $f = \prod_v f_v$ on $G(\mathbb{A}_K)$ by requiring that f_0 and f_∞ are as defined above and f_v is the characteristic function of $G(\mathcal{O}_v)$ for all other v. Then consider the Poincaré series

$$P_f(g) = \sum_{\gamma \in G(K)} f(\gamma g).$$

Because of the support conditions on f, one has

$$P_f(1) = \sum_{\gamma \in G(k)} f_0(\gamma) \cdot f_\infty(\gamma) = \sum_{\gamma \in G(k)} |\langle \sigma(\gamma) \cdot w, w \rangle|^2.$$

This sum is certainly nonzero, so that P_f is nonzero.

To see if P_f is globally generic, let us first construct an appropriate automorphic generic character

$$\Psi = \prod_v \Psi_v : N(K) \backslash N(\mathbb{A}_K) \longrightarrow \mathbb{C}^\times.$$

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Note that one has

$$N(k)\backslash N(\widehat{O}) \xrightarrow{\sim} N(K)\backslash N(\mathbb{A}_K),$$

where $\widehat{O} = \prod_v \mathcal{O}_v$. We define Ψ by requiring that

- $\Psi_v = 1$ on $N(\mathcal{O}_v)$ for all $v \neq 0$ or ∞,
- the restriction of Ψ_0 to $N(\mathcal{O}_0)$ is obtained as

$$\Psi_0 : N(\mathcal{O}_0) \longrightarrow N(k) \xrightarrow{\psi} \mathcal{C}^\times,$$

- likewise, the restriction of Ψ_∞ to $N(\mathcal{O}_\infty)$ is obtained as

$$\Psi_\infty : N(\mathcal{O}_\infty) \longrightarrow N(k) \xrightarrow{\psi^{-1}} \mathcal{C}^\times.$$

Hence, for $n \in N(\widehat{O})$, one has

$$\Psi(n) = \psi(n_0) \cdot \overline{\psi(n_\infty)},$$

so that Ψ is indeed trivial on the diagonally embedded $N(k) \hookrightarrow N(\widehat{O})$.

We can now compute the (N, Ψ)-Whittaker-Fourier coefficient of P_f:

$$\begin{aligned}&\int_{N(K)\backslash N(\mathbb{A}_K)} \overline{\Psi(n)} \cdot P_f(n)\, dn \\ =&\int_{N(k)\backslash N(\widehat{\mathcal{O}})} \overline{\Psi(n)} \cdot \sum_{\gamma \in G(K)} f(\gamma n)\, dn.\end{aligned}$$

For $n \in N(\widehat{O})$,

$$\gamma n \in \mathrm{supp}(f) \Longrightarrow \gamma \in G(\widehat{O}) \cap G(K) = G(k).$$

Moreover, with $n \in N(\widehat{O})$, one deduces by (9.2) that

$$f(\gamma n) = \langle \sigma(\gamma \cdot n_0) \cdot w, w\rangle \cdot \langle w, \sigma(\gamma \cdot n_\infty) \cdot w\rangle = \psi(n_0) \cdot \overline{\psi(n_\infty)} \cdot |\langle \sigma(\gamma) w, w\rangle|^2.$$

Hence, one has

$$\begin{aligned}&\int_{N(K)\backslash N(\mathbb{A}_K)} \overline{\Psi(n)} \cdot P_f(n)\, dn \\ =&\int_{N(k)\backslash N(\widehat{\mathcal{O}})} \overline{\psi(n_0)} \cdot \psi(n_\infty) \cdot \sum_{\gamma \in G(k)} \psi(n_0) \cdot \overline{\psi(n_\infty)} \cdot |\langle \sigma(\gamma) w, w\rangle|^2\, dn. \\ =&\mathrm{Vol}(N(k)\backslash N(\widehat{O})) \cdot \sum_{\gamma \in G(k)} |\langle \sigma(\gamma) w, w\rangle|^2,\end{aligned}$$

which is nonzero. Thus, P_f is (N, Ψ)-generic. □

9.4. Purity of generic depth zero representations

Now we can sketch our strategy for showing that a depth 0 generic supercuspidal representation is pure. We use the following:

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Expected Theorem 9.3 (Dat-Lanard). Let $\pi \in \mathcal{A}(G,F)$ be a depth 0 supercuspidal representation of $G(F)$ (in the sense of [DR]). Then the semisimple parameter $\mathcal{L}^{ss}(\pi)$ is (at most) tamely ramified.

Dat and Lanard prove this in [DL], conditionally on the independence of ℓ of the ℓ-adic parameter $\mathcal{L}^{ss}(\pi)$.[3]

Using this, we can now show the following:

Theorem 9.4. *Assume that G is semisimple over k and $p = \mathrm{char}(k)$ is a good prime for G. Then for any depth 0 generic supercuspidal representation of $G(F)$, the associated semisimple parameter $\mathcal{L}^{ss}(\pi)$ is pure.*

Proof. Consider the global Lafforgue parameter $\mathcal{L}(\Pi)$ associated to the globally generic cuspidal representation Π of Proposition 9.1. Since Π is unramified outside 0 and ∞, it gives rise to a semisimple local system $\mathcal{L}(\Pi, Ad)$ on $\mathbb{G}_m/k$, where Ad is the adjoint representation of ${}^LG$. Expected Theorem 9.3 implies that $\mathcal{L}(\Pi, Ad)$ is tamely ramified. It follows that, for a finite extension k'/k of the constant field, the restriction $\mathcal{L}(\Pi, Ad)_{k'}$ of $\mathcal{L}(\Pi, Ad)^{ss}$ to $\mathbb{G}_m/k'$ is a sum of 1-dimensional tamely ramified ℓ-adic local systems:

$$\mathcal{L}(\Pi, Ad)_{k'} \xrightarrow{\sim} \oplus L_i,$$

where each L_i is pure of weight w_i. Let Ω be the set of weights w_i that occur.

Now suppose $\mathcal{L}^{ss}(\pi)$ is not pure. Since π belongs to the discrete series, G is semisimple and p is good for G, it follows from Theorem 1.2 and Corollary 3.12 that the weights that occur in $\mathcal{L}(\pi)^{ss}$ are integral and two of them differ by at least 2.

Thus, there are two weights w_i and w_j in Ω with $|w_i - w_j| \geq 2$. This would imply that for $z \in \mathbb{G}_m(k)$, the Satake parameter of the unramified representation Π_z also has at least two weights that differ by at least 2. But Π_z is unitary and generic because Π is cuspidal and globally generic. The existence of two distinct integral weights that differ by at least 2 is then ruled out by the main theorem of [CH]. □

10. Base change and incorrigible representations

As we mentioned in the introduction, though our ramification result in Theorem 1.4 may seem rather weak, it could, in fact, serve as a starting point, in conjunction with the global input of automorphic base change, in the (long) journey towards establishing the local Langlands correspondence for a general reductive group G over local function fields. In this section, we would like to elaborate on this.

10.1. Tempered base change

Let $E = k((t))$ for a finite field k and let G be a connected reductive group over E. For any finite separable field extension F of E, let $\mathcal{T}(G,F) \subset \mathcal{A}(G,F)$ denote the set of irreducible tempered (admissible) representations of $G(F)$ with coefficients in C and let $P(\mathcal{T}(G,F))$ be the set of nonempty subsets of $\mathcal{T}(G,F)$. We write $\mathcal{L}^{ss}_F$ for the semisimple Langlands parametrization of (2.2) for $\mathcal{A}(G,F)$, with a subscript to indicate the base field.

Definition 10.1. Say G admits tempered base change if, for every pair $F \subset F'$ of finite extensions of E, with F'/F a cyclic extension of prime order, there is a map

$$BC_{F'/F} : \mathcal{T}(G,F) \to P(\mathcal{T}(G,F'))$$

so that for any $\pi \in \mathcal{T}(G,F)$ and any $\pi' \in BC_{F'/F}(\pi)$,

$$\mathcal{L}^{ss}_{F'}(\pi') = \mathcal{L}^{ss}_F(\pi)\,|_{W_{F'}}. \tag{10.2}$$

[3] In the most recent version of their paper (April 2024), the authors refer to an announcement that Scholze has proved this independence.

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We may naturally regard $BC_{F'/F}$ as a map from $P(\mathcal{T}(G,F))$ to $P(\mathcal{T}(G,F'))$. Of course, it is expected that every G admits tempered base change and that the set $BC(\pi)$ is characterized intrinsically inside $\mathcal{A}(G,F)$ in terms of π by relations of distribution characters. It is also assumed that $BC(\pi)$ is always a finite set, but we do not make that assumption. There is a brief discussion of the existence of tempered base change at the end of this section.

Definition 10.3. Suppose G admits tempered base change. The supercuspidal representation π of $G(F)$ is **incorrigible** if for any sequence

$$F = F_0 \subset F_1 \subset \cdots \subset F_r \tag{10.4}$$

of extensions, where F_i/F_{i-1} is cyclic of prime order for all $i \geq 1$, the set

$$BC_{F_r/F}(\pi) := BC_{F_r/F_{r-1}}(BC_{F_{r-1}/F_{r-2}} \dots (BC_{F_1/F}(\pi)\dots))$$

contains a supercuspidal member.

Proposition 10.5. *Assume G admits tempered base change and $p \nmid |W|$. Then for any extension F/E, no pure supercuspidal representation of $G(F)$ is incorrigible. More generally, suppose G has the property that, for every Levi subgroup $L(F)$ of $G(F)$, every supercuspidal representation of $L(F)$ contains an $\mathfrak{s}$-type. Then no pure supercuspidal representation of $G(F)$ is incorrigible.*

Proof. Let π be a pure supercuspidal representation of $G(F)$. The image of the inertia group under $\mathcal{L}_F^{ss}(\pi)$ is finite, so there is a sequence of cyclic extensions of prime order, as in (10.4), such that, if π_r is any member of $BC_{F_r/F}(\pi)$, we have

$$\mathcal{L}_{F_r}^{ss}(\pi_r) = \mathcal{L}_F^{ss}(\pi)\mid_{W_{F_r}}$$

is unramified. It then follows from Corollary 7.12 that π_r is a constituent of an unramified principal series representation. In particular, no member of $BC_{F_r/F}(\pi)$ can be supercuspidal. □

10.2. Existence of tempered base change

Suppose for the moment that F is a p-adic field. A procedure for defining cyclic base change can be constructed using the methods of [Lab99]. Suppose first that π is supercuspidal with central character of finite order. Then there is

(i) A totally real number field K with a local place w such that $K_w \xrightarrow{\sim} F$;
(ii) A totally real cyclic extension K'/K with $K'_w = K' \otimes_K F \xrightarrow{\sim} F'$,
(iii) A connected reductive group $\mathcal{G}$ over K, with $\mathcal{G}(K_w) \xrightarrow{\sim} G(F)$ and
(iv) $\mathcal{G}(K_\sigma)$ compact modulo center for all archimedean places σ of K,
(v) And an automorphic representation Π such that $\Pi_w \xrightarrow{\sim} \pi$ and
(vi) Π_{v_i} isomorphic to a Steinberg representation at any chosen finite set of places v_i.

Then it is proved in [Lab99], using the stable twisted trace formula, that there is a nonempty collection $\{\Pi'\}$ of automorphic representations of $\mathcal{G}_{K'}$ such that, at every place u at which Π and K'/K are unramified, Π'_u is the unramified base change of Π_u. We can then define $BC(\pi)$ to be the collection of Π'_w for all such $K, K', \mathcal{G}, \Pi$. The stable twisted characters of such collections of Π'_w are related to the stable character of π, but in the absence of a canonical notion of L-packet, we omit the precise statement.

When Labesse defined his construction in [Lab99], the stabilization of the trace formula had not yet been established in either the twisted or the untwisted setting. Arthur reduced the stabilization to the fundamental lemma in a series of papers shortly thereafter, and a few years later, the main step in the proof of the fundamental lemma was completed by Ngô. The stabilization of the twisted trace formula is contained in [MW]; see also [CHLN] for references for the untwisted case. Labesse needed hypothesis (vi) in the above list in order to work with a simplified version of the trace formula. In principle, (vi) is no

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longer necessary, but the necessary consequences of the full stabilized trace formula have not yet been established (see [CHLN, §CHL.IV.B] for an example of the kind of work required). Thus for practical purposes, the construction in [Lab99] still provides the most complete definition of local supercuspidal base change in general.

Since we are only looking for semisimple Langlands parameters, we can reduce a general tempered π to the supercuspidal case by means of parabolic induction. Unfortunately, although Fargues and Scholze have defined a semisimple parametrization of all irreducible admissible representations of $G(F)$, when F is a p-adic field, there is no way to relate the parameter attached to π to that attached to the elements of the set $BC(\pi)$ defined by Labesse's construction, unless we know how to attach Galois parameters to the globalizations Π as in (v). If we could do that, then the relation between the parameters of Π_u and Π'_u at unramified places would suffice, by Chebotarev density, to establish the relation (10.2) at the place of interest. This reasoning has, in fact, been applied for most classical groups, and for G_2, but it is not available in general.

Now suppose $F = k((t))$ as above. We can certainly globalize π as in the number field case, and then [Laf18] supplies the necessary global parametrization that is missing in the setting of number fields. However, the stable twisted trace formula is lacking for function fields, so for most G, tempered base change is not (yet) available. It should nevertheless be (relatively) straightforward, although time-consuming, for specialists to prove the necessary statements, starting with the construction of the non-invariant trace formula by Labesse and Lemaire [LL], when p is large relative to the group G.

10.3. Base change of large prime degree and fields of small order

Let π be a supercuspidal representation as in Theorems 7.1 and 7.2. Those theorems are stated under the hypothesis $q = |k| > 5$, which concerns the size of the residue field k rather than the characteristic. Theorem 7.2 of [BFHKT] asserts that there is a constant $c(\pi)$ such that, if $m > c(\pi)$, and F'/F is cyclic of prime degree m, then there exists a base change π' of π to F', in the sense that the $\mathcal{L}^{ss}(\pi')$ is the restriction of $\mathcal{L}^{ss}(\pi)$ to the Weil group of F'. In particular, we may assume $F' = k'((t))$ is an unramified extension, where $[k' : k] = m$ and therefore $|k'| = q^m$. In particular, the hypotheses on q are satisfied for π'. Write π_m instead of π'. Then $\mathcal{L}^{ss}(\pi)$ is unramified if and only if $\mathcal{L}^{ss}(\pi_m)$ is unramified. We thus have the following alternative:

(i) For some $m \geq \sup(3, c(\pi))$, π_m is supercuspidal, and then the results of Theorems 7.1 and 7.2 remain valid for π, without any hypothesis on q;
(ii) Or else, for all $m \geq \sup(3, c(\pi))$, π_m is not supercuspidal.

Although the second alternative is clearly absurd, we do not see how to exclude it by available means. At least for the toral supercuspidals considered by Chan and Oi, the results of [BFHKT, §8] show that π_m remains supercuspidal for almost all prime m. However, we have not proved that their parameters are pure.

The above alternatives do seem to provide a route to proving that a given pure supercuspidal π is not incorrigible, even when $|k| \leq 5$. If we are in the first alternative, then the residue field satisfies the hypotheses of Theorems 7.1 and 7.2. We can then define what it means for π to be incorrigible if we have access to cyclic base change of all prime degrees – not just those guaranteed by [BFHKT] – and argue as in the previous sections. Under the second (absurd) alternative, we have already reduced the supercuspidality by base change, and we can then argue by induction on the size of the cuspidal support.

Too much attention should not be given to these remarks, however. The restriction on $|k|$ is only relevant when the residue characteristic is 2, 3 or 5. These are the primes that tend to divide the order of Weyl groups. So it would be unnatural to try to formulate unconditional results based on these observations.

10.4. The case of $GL(n)$

We again assume F is a local field of positive characteristic. We shall illustrate the strategy discussed above for $G = GL(n)$. It has been proved by Henniart and Lemaire in [HeLe] that $G = GL(n)$ admits

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tempered base change; they even prove that $BC_{F'/F}(\pi)$ is a singleton for any π. Moreover, every supercuspidal representation of every Levi subgroup of $G(F)$ contains an $\mathfrak{s}$-type [BK]. Thus, the above discussion applies. Before drawing the relevant conclusion, we make the following observation.

The following lemma may be derived from the work of Bushnell and Kutzko.

Lemma 10.6. *Let π be a supercuspidal representation of $GL(n,F)$ for F a local field. One of the following is true:*

(a) *There exists a principal hereditary order $\mathfrak{A} \subset M_n(F)$, a field $E \subset M_n(F)$ which is an extension of F of degree > 1, such that $E^\times$ normalizes $\mathfrak{A}$, and, writing $\mathfrak{B}$ for the centralizer of E in $\mathfrak{A}$ and $\mathfrak{P}$ for the maximal ideal of $\mathfrak{A}$, a representation Λ of $E^\times \cdot (\mathfrak{A}^\times \cap (\mathfrak{B}+\mathfrak{P}))$ such that π is the induced representation of Λ.*

(b) *There exists a representation Λ of $F^\times GL_n(\mathcal{O}_F)$ such that π is the induced representation of Λ.*

Proof. By [BK, Theorem 8.4.1], there exists a simple type (J,λ) in G such that $\pi \mid J$ contains λ. The definition [BK, Definition 5.5.10] of simple type splits into two cases, (a) and (b).

The case (b) will be easier to handle – so much so that it is not even necessary to give the definition of this case. Instead, we note that in this case, by [BK, Theorem 8.4.1(iii)], there exists a representation Λ of $F^\times \mathbf{U}(\mathfrak{A})$, where $\mathbf{U}(\mathfrak{A})$ is the unit group of a hereditary order $\mathfrak{A}$, such that $\pi = \mathrm{Ind}(\Lambda)$. Because $\mathfrak{A}$ is a hereditary order, $\mathbf{U}(\mathfrak{A})$ is a compact open subgroup of $GL(n,F)$ and thus, up to conjugation, is contained in the maximal compact $GL_n(\mathcal{O}_F)$, By first inducing Λ from $F^\times \mathbf{U}(\mathfrak{A})$ to $F^\times GL_n(\mathcal{O}_F)$, we obtain case (b) above.

Now we turn to case (a). Let A be the algebra of $n \times n$ matrices over F. In this case, $\mathfrak{A}$ is a principal $\mathcal{O}_F$-order in A and $[\mathfrak{A}, n, 0, \beta]$ is a simple stratum. Let us unpack this definition. The order $\mathfrak{A}$ is a compact open $\mathcal{O}_F$-subalgebra of A. We write $\mathfrak{P}$ for the Jacobson radical of $\mathfrak{A}$. We let β be an element of A and take n to be a positive integer. We define $E = F[\beta]$, $\mathfrak{B}$ the centralizer of E in $\mathfrak{A}$, and $\mathfrak{P}$ the maximal ideal of $\mathfrak{A}$. We set [BK, Definitions 1.4.3 and 1.4.5]

$$k_0(\beta,\mathfrak{A}) = \max\{k \in \mathbb{Z} \mid \{x \in \mathfrak{A} \mid \beta x - x\beta \in \mathfrak{P}^k\} \not\subset \mathfrak{B}+\mathfrak{P}\}.$$

Then the assumption that $[\mathfrak{A}, n, 0, \beta]$ is a simple stratum means that $\mathfrak{A}$ is hereditary, $E^\times$ normalizes $\mathfrak{A}$, $v_{\mathfrak{A}}(\beta) = -n$, and $0 < -k_0(\beta,\mathfrak{A})$ [BK, Definition 1.5.5].

In this context, [BK, Definition 3.1.8] defines an algebra $\mathfrak{J}(\beta)$ by an inductive procedure. For us, the only relevant feature of this definition is that it is contained in $\mathfrak{B}+\beta^{\lfloor \frac{r+1}{2} \rfloor}$ where $r = -k_0(\beta,\mathfrak{A}) > 0$ ([BK, 3.1.6]), and thus, $\mathfrak{J} \subseteq \mathfrak{B}+\mathfrak{P}$. We can next define [BK, (3.1.14)] a group $J(\beta,\mathfrak{A})$ as the intersection of $\mathfrak{J}$ with the group of units $\mathbf{U}^0(\mathfrak{A}) = \mathfrak{A}^\times$.

We can finally state the conclusion of [BK, Theorem 8.4.1(ii)] in this case, which is that there is a representation Λ of $E^\times J(\beta,\mathfrak{A})$ such that $\pi = \mathrm{Ind}(\Lambda)$. We will use only this structure and will not concern ourselves with the exact construction of Λ. If $E = F$, then, because $J(\beta,\mathfrak{A})$ is contained in a maximal compact subgroup, we are again in case (b), so we may assume $E \neq F$.

Since $J(\beta,\mathfrak{A})$ is contained in $\mathfrak{A}^\times \cap (\mathfrak{B}+\mathfrak{P})$, we can induce Λ from $E^\times J(\beta,\mathfrak{A})$ to $E^\times \cdot (\mathfrak{A}^\times \cap (\mathfrak{B}+\mathfrak{P}))$, and we are in case (a) above. □

Lemma 10.7. *If π satisfies case (b) of Lemma 10.6, then π is pure and ramified.*

Proof. The determinants of elements of $GL_n(\mathcal{O}_F)$ must have zero valuation, and because $F^\times$ consists of scalars, the determinants of its elements must have valuation divisible by n. It follows that, if α is the character of $GL(n,F)$ defined by $\alpha(g) = e^{\frac{2\pi i}{n} v(\det(g))}$, then

$$\pi \otimes \alpha = \mathrm{Ind}(\Lambda) \otimes \alpha = \mathrm{Ind}(\Lambda \otimes \alpha) = \mathrm{Ind}(\Lambda) = \pi.$$

By [GLa, Remarque 0.2], it follows that the Genestier-Lafforgue parameter σ_π is stable under tensor-product with the one-dimensional unramified representation of W_F sending Frobenius to $e^{2\pi i/n}$. Let a be a matrix representing that isomorphism. Then $\sigma_\pi(\mathrm{Frob}_q) \circ a = e^{2\pi i/n} a \circ \sigma_\pi(\mathrm{Frob}_q)$, so a is conjugate

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to $e^{2\pi i/n}a$. Thus, the set of eigenvalues of a is stable under multiplication by $e^{2\pi i/n}$, so a has n distinct eigenvalues, and thus, a has a one-dimensional eigenspace. This one-dimensional eigenspace is a one-dimensional representation χ of the subgroup of W_F corresponding to the degree n unramified extension F_n of F, which, because Frobenius permutes the eigenspaces, gives an isomorphism $\mathrm{Ind}_{W_{F_n}}^{W_F}\chi \cong \sigma_\pi$. Since χ is a one-dimensional representation, it is automatically pure of some weight (possibly non-integral), so σ_π is pure. By Theorem 1.4(i), σ_π is ramified. □

We now assume given π in case (a) of Lemma 10.6, and prepare for the construction of a Poincairé series by choosing compact subgroups at each place.

It follows from an elementary calculation or [BK, p. 76 and p. 183] that a principal hereditary order $\mathfrak{A}$, after choosing a basis, can be put into the following form: We choose a divisor e of n, and then we take all $n \times n$ matrices decomposed into $n/e \times n/e$ blocks of size $e \times e$, where all the blocks on the diagonal and above have entries in the ring of integers $\mathcal{O}_F$, and all the the blocks below the diagonal have entries in the maximal ideal of $\mathcal{O}_F$. We fix for the remainder of this argument a basis where $\mathfrak{A}$ has this form.

We can choose a finite field $\mathbb{F}_q$ such that $F \cong \mathbb{F}_q((t))$ and fix such an isomorphism. This identifies F with the local field $\mathbb{F}_q(t)_0$ of $\mathbb{P}^1_{\mathbb{F}_q}$ at 0. We now define subgroups U_x of $GL_n(\mathbb{F}_q(t)_x)$ for each place x of $\mathbb{F}_q(t)$. For $x \neq 0, \infty$, we take U_x to be the standard maximal compact subgroup. We take U_∞ to consist of block-diagonal matrices in $GL_n(\mathbb{F}_q((t^{-1})))$ where all the blocks on the diagonal and below are in $M_e(\mathbb{F}_q[[t^{-1}]])$, all the blocks above the diagonal are in $M_e(t^{-1}\mathbb{F}_q[[t^{-1}]])$, and all the diagonal blocks, modulo t^{-1} are in n/e fixed Borel subgroups of GL_n, to be chosen in Lemma 10.8. We take U_0 to be $E^\times(\beta, \mathfrak{A})$.

Let $S = GL_n(\mathbb{F}_q(t)) \cap \prod_x U_x$.

Lemma 10.8. *For a suitable choice of Borel subgroups, the group S is contained in a maximal torus T of $GL_n(\mathbb{F}_q)$, and furthermore, the n-dimensional vector space, when viewed as a representation of S, is a sum of n/k characters each repeated k times for some $k > 1$ a divisor of n.*

Proof. First observe that the valuation of the determinant of any element of U_x for any $x \neq 0$ is 0. It follows that for $g \in S$, $\det g$ has valuation 0 at every place away from 0 and thus by the product formula has valuation 0 at 0 as well. The valuation of the determinant of every element of $\mathfrak{A}^\times$ is 0, so g must be the product of an element of $\mathfrak{A}^\times \cap (\mathfrak{B} + \mathfrak{P})$ with a unit $u \in E^\times$. Now every unit of $E^\times$ is contained in $\mathfrak{A}^\times$ by [BK, Proposition 1.2.1(i)]. Because $u \in E^\times$ and thus u commutes with β, we have $u \in \mathfrak{B}$. Thus,

$$g \in u \cdot (\mathfrak{A}^\times \cap (\mathfrak{B} + \mathfrak{P})) = \mathfrak{A}^\times \cap (\mathfrak{B} + \mathfrak{P}).$$

In particular, g, at 0, is contained in $\mathfrak{A}$. Together with our assumptions at other places, we conclude that every matrix entry of g is integral at each place of $\mathbb{F}_q(t)$ and hence, in fact, lies in $\mathbb{F}_q$, so $g \in GL_n(\mathbb{F}_q)$. Furthermore, by the integrality conditions at 0, we see that every $e \times e$ block of g below the diagonal vanishes, and by the integrality conditions at ∞, every $e \times e$ block of g above the diagonal vanishes. So g is a block-diagonal matrix (i.e., $g \in (GL_e(\mathbb{F}_q))^{n/e}$). Furthermore, each of the diagonal matrices must be contained in some fixed Borel, to be chosen later.

We now use the fact that E/F is a nontrivial extension. Thus, it either ramifies or contains an extension of the residue field.

We handle the case when E is ramified first. In this case, E contains an element x whose determinant does not have valuation divisible by n, and thus the action of x on $\mathfrak{A}$ must permute the n/e diagonal blocks. The action on the blocks divides them into $n/(ek)$ orbits of size k for some $k > 1$ dividing n/e. Because elements of $\mathfrak{B}$ commute with E, and thus in particular with x, their value mod $\mathfrak{p}_F$ on one block in the orbit determines their value on every other block in the orbit. So, mod $\mathfrak{P}$, the elements of $\mathfrak{B}$ lie in $M_e(\mathbb{F}_q)^{n/(ek)}$, and the units of $\mathfrak{B}$ lie in $GL_e(\mathbb{F}_q)^{n/(ek)}$. To define U_∞, we choose n/e Borel subgroups of $GL_e(\mathbb{F}_q)$ such that, in each orbit, at least one of the Borel subgroups is sent by x to a Borel in general position with respect to the next Borel in the orbit. It follows that each element of $S \subset GL_e(\mathbb{F}_q)^{n/(ek)}$ lies in the intersection of two Borels in general position and thus lies in the maximal torus, which

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is contained in a maximal torus of GL_n. Furthermore, the n-dimensional representation of S is the restriction of the n-dimensional representation of $GL_e(\mathbb{F}_q)^{n/(ek)}$, which is the sum of $n/(ek)$ standard representations, each repeated k times, so as a representation of S, it is the sum of n/k characters, each repeated k times, as desired.

We handle the case when E contains a residue field extension $\mathbb{F}_{q^k}$ next. In this case, E contains an element x which generates the residue field extension. Because x is a unit, x lies in $\mathfrak{A}$. Restricting to each of the n/e diagonal blocks and modding out by $\mathfrak{p}_F$, the element x satisfies the characteristic polynomial of a generator of $\mathbb{F}_{q^k}$ and thus generates a subfield isomorphic to $\mathbb{F}_{q^k}$. Thus, the centralizer of x in each block is isomorphic to $GL_{e/k}(\mathbb{F}_{q^k})$, and every unit in $\mathfrak{B} + \mathfrak{P}$, restricted to each diagonal block and reduced modulo $\mathfrak{p}_F$, must lie in this $GL_{e/k}(\mathbb{F}_{q^k})$.

We choose a Borel as follows: Fix a $\mathbb{F}_{q^k}$-basis of the e-dimensional vector space $v_1, \ldots, v_{e/k}$, and then extend the sequence

$$v_1, \ldots, v_{e/k}, xv_{e/k}, xv_{e/k-1}, \ldots, xv_1$$

arbitrarily to an $\mathbb{F}_q$-basis of this space. For each block, choose a Borel consisting of linear transformations which are upper-triangular with respect to this basis, and use these to define U_∞. Then every $g \in S$, restricted to this block, will lie in the intersection of $GL_{e/k}(\mathbb{F}_{q^k})$ with this Borel. Hence, g will be upper-triangular with respect to the $\mathbb{F}_{q^k}$-basis $v_1, \ldots, v_k$ and upper-triangular with respect to the $\mathbb{F}_{q^k}$-basis $xv_{e/k}, xv_{e/k-1}, \ldots, xv_1$, and thus $\mathbb{F}_{q^k}$-diagonal with respect to the basis $v_1, \ldots, v_k$. Furthermore, g must act on $v_1, \ldots, v_k$ by multiplication by elements of $\mathbb{F}_q$, since otherwise it would not preserve the $\mathbb{F}_q$-subspace generated by $v_1, \ldots, v_k$. So, in fact, g is contained in the diagonal of $GL_{e/k}(\mathbb{F}_q)$ inside $GL_{e/k}(\mathbb{F}_{q^k})$ inside $GL_e(\mathbb{F}_q)$. Passing from $GL_{e/k}(\mathbb{F}_q)$ to $GL_e(\mathbb{F}_q)$ in this way sends the maximal torus to a subset of the maximal torus, as desired, and expresses the e-dimensional standard representation of GL_e as a sum of k copies of the e/k-dimensional standard representation of $GL_{e/k}$. Hence, the sum of k copies of e/k characters of S, as desired. □

Note that U_∞ is an Iwahori subgroup of $GL_n(\mathbb{F}_q((t^{-1})))$ and thus surjects onto $((\mathbb{F}_q)^\times)^n$.

Lemma 10.9. *If $q > 2$, any (one-dimensional) character of S may be extended to a nontrivial character of U_∞.*

Furthermore, if $q > 3$, any character of S may be extended to two characters, χ_a, χ_b, both pulled back from $((\mathbb{F}_q)^\times)^n$, where χ_a is trivial on at least one copy of $\mathbb{F}_q^\times$ and χ_b is trivial on no copies of $\mathbb{F}_q^\times$.

Proof. For each case, note that the subgroup U_∞ is an Iwahori subgroup of $GL_n(\mathbb{F}_q((t^{-1})))$, so its quotient by its maximal pro-p subgroup is $((\mathbb{F}_q)^\times)^n$. Since S is a subgroup of the maximal torus of $GL_n(\mathbb{F}_q)$ by Lemma 10.8, its order is prime to p, and so the quotient of U_∞ by its maximal pro-p subgroup is faithful. Thus, χ_S is a character of the image of S inside $((\mathbb{F}_q)^\times)^n$.

For the first case, it suffices to extend χ_S^{-1} to a nontrivial character of $(\mathbb{F}_q^\times)^n$ and inflate to U_∞. We can do this unless χ_S is trivial and $S = ((\mathbb{F}_q)^\times)^n$, which contradicts our claim that the characters of S are repeated unless $\mathbb{F}_q^\times$ is trivial, which implies $q = 2$.

For the second case, if $q > 3$, note that since the standard representation of S is the sum of n/k characters repeated k times, the image of S inside $(\mathbb{F}_q^\times)^n$ is contained in $(\mathbb{F}_q^\times)^{n/k}$, repeated k times. It follows that when we extend χ_S^{-1} to a character χ of $(\mathbb{F}_q^\times)^n$, which can be viewed as a tuple of n characters $\chi_1, \ldots, \chi_n$ of $\mathbb{F}_q$, we can choose any tuple of characters as long as the product of k characters in each of n/k orbits takes some fixed value depending on the orbit. So we can certainly choose $k - 1$ of the characters in each orbit to be trivial, and the last one to take the fixed value, to produce χ_a. To produce χ_b, we choose $k - 2$ of the characters in each orbit to be an arbitrary nontrivial character. The product of the two remaining characters is then determined. Since $q > 3$, there are more than two characters, and so we can choose the next character to be a character which is neither trivial nor the determined product. It follows that the last character is nontrivial, so χ_b indeed contains no trivial characters. □

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We are now ready to construct our Poincaré series. Since S is a finite abelian group, λ restricted to $S \subset E^\times J(\beta, \mathfrak{A})$ is a sum of one-dimensional characters. Let χ_S be one of these.

Proposition 10.10. *Assume the residue field of F has more than 2 elements. Then every supercuspidal representation of $GL(n, F)$ is ramified. Moreover, if the residue field of F has more than 3 elements, then every supercuspidal representation of $GL(n, F)$ is either ramified and pure or wildly ramified.*

Proof. Let $F = \mathbb{F}_q((t))$. Fix a supercuspidal rerpesentation π. We apply Lemma 10.6. In case (b), π is pure and ramified by Lemma 10.7, so we may reduce to case (a), where we will prove that π is ramified if $q > 2$ and wildly ramified if $q > 3$.

To do this, we construct a Poincaré series, using all the notation above. Since S is a finite abelian group, λ restricted to $S \subset E^\times J(\beta, \mathfrak{A})$ is a sum of one-dimensional characters. Let χ_S be one of these. We extend χ_S^{-1} to a nontrivial character χ of U_∞ using Lemma 10.9. We take the representation $\lambda \otimes \chi$ of $\prod_x U_x$, which is trivial by construction on $S = GL_n(F) \cap \prod_x U_x$. Using a matrix coefficient of this representation, we can construct a Poincaré series. After decomposing into Hecke eigenforms, this series generates an automorphic representation Π which is unramified outside of 0 and ∞, of type $\mathrm{Ind}(\lambda) = \pi$ at 0, and whose local factor at ∞ contains a vector that transforms according to χ under U_∞. By Lafforgue, associated to this representation is a global parameter σ_Π which matches σ_π at 0, and is unramified away from 0 and ∞.

Furthermore, at ∞, the local factor of Π is contained in a parabolic induction from the Borel of a tamely ramified character of the maximal torus, which, restricted to the $\mathbb{F}_q$-points of the maximal torus, is χ. Thus, by the compatibility of Genestier-Lafforgue with parabolic induction and the local Langlands correspondence for GL_1, the Genestier-Lafforgue parameter at ∞ is a sum of n tamely ramified characters whose restriction to the inertia group is trivial if and only if the restriction of χ to the corresponding copy of $\mathbb{F}_q^\times$ is trivial. Thus, the restriction of the global parameter σ_Π to ∞ is tamely ramified and is unipotent if and only if χ is trivial.

If π is unramified, then σ_Π is unramified away from 0 and ∞ and has at worst unipotent ramification at 0. Because σ_Π is unramified away from 0 and ∞ and tamely ramified at 0 and ∞, the restrictions of σ_Π to the inertia groups at 0 and ∞ are isomorphic, which contradicts the fact that σ_Π is unipotent at 0 and σ_Π is non-unipotent at ∞. So, in fact, σ_π is ramified.

If $q > 3$ and π is tamely ramified, then we apply the second case of Lemma 10.9 to produce two characters, χ_a and χ_b, and follow the same procedure above to produce two globalizations Π_a, Π_b. Again, $\sigma_{\Pi_a}, \sigma_{\Pi_b}$ are tame at 0 and ∞, so the restrictions of σ_{Π_a} to the inertia subgroup at ∞ is isomorphic to its restriction to the inertia subgroup at 0 and thus isomorphic, up to semisimplification, to σ_π. The same is true for σ_{Π_b}, so σ_{Π_a} and σ_{Π_b} are isomorphic up to semisimplification. However, the semisimplification of σ_{Π_a} is a sum of one-dimensional characters, at least one trivial, whereas the semisimplification of σ_{Π_b} is a sum of one-dimesional characters, all nontrivial, so they cannot be isomorphic. This is a contradiction; hence, π is wildly ramified in this case. □

Corollary 10.11. *Let π be a supercuspidal representation of $GL(n, F)$. Then there is a sequence of cyclic extensions as in (10.4) such that $BC_{F_r/F}(\pi)$ is an irreducible constituent of an unramified principal series representation.*

Proof. By 7.12 and induction, we know that we can find a sequence of cyclic extensions as in (10.4) such that $BC_{F_r/F}(\pi)$ is an irreducible constituent of a principal series representation whose parameter is unramified. By Proposition 10.10, such a principal series must be a parabolic induction from the Borel, as a supercuspidal on any Levi except the maximal torus would have a ramified Langlands parameter. By the local Langlands conjecture for GL_1, it must be an induction of an unramified representation of the maximal torus (i.e., an unramified principal series). □

Corollary 10.11 is the key point in (almost) every proof of the local Langlands correspondence for $GL(n)$. It was already mentioned in the introduction that in [LRS93, HT01, He00], this result is obtained as a consequence of Henniart's numerical correspondence, whereas in [Sch13], it is proved by

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a geometric argument involving nearby cycles in the local model. Starting with Corollary 10.11, one obtains the full local correspondence by an inductive study of the fibers of local base change.

Here, we obtain Corollary 10.11 by a completely different argument, based on the geometric considerations of [Laf18], the global base change of [HeLe] and the existence of types for supercuspidal representations [BK]. It is questionable whether this argument is genuinely new because the existence of the global parametrization in [Laf18], combined with the converse theorem as in [Laf02], already suffices to prove the local as well as the global correspondence. However, the reasoning used here can be applied to any G for which global base change and the existence of types is known.

11. Applications to groups over p-adic fields

In this final section, we speculate on how our results in positive characteristic could have impact on analogous questions in characteristic zero local fields, via the principle of close local fields.

Let $F = k((t))$ be a local field of characteristic p as before and let $F^\sharp$ be a p-adic field. We write $\mathcal{O}_F, \mathcal{O}_{F^\sharp}$ for their integer rings, and $m_F, m_{F^\sharp}$ for the corresponding maximal ideals.

Definition 11.1. Let n be a positive integer. We say that F and $F^\sharp$ are **n-close** if

$$\mathcal{O}_F/m_F^n \xrightarrow{\sim} \mathcal{O}_{F^\sharp}/m_{F^\sharp}^n$$

as rings.

We write $F^?$ for either F or $F^\sharp$. Let G be a connected reductive group over $F^?$. The *depth* of a parameter $\phi \in \mathcal{G}^{ss}(G,F)$ is defined to be the maximum n such that ϕ is trivial on the subgroup $I_{F^?}^n$ of the inertia group $I_{F^?}$, where we are using the upper numbering. Let $\mathcal{G}^{ss,n}(G,F) \subset \mathcal{G}^{ss}(G,F)$ denote the subset of parameters of depth at most n.

Theorem 11.2 (Deligne). *If F and $F^\sharp$ are n-close, then there is a natural bijection*

$$\mathcal{G}^{ss,n}(G,F) \simeq \mathcal{G}^{ss,n}(G,F^\sharp). \tag{11.3}$$

Assume now that G is split and let $I_n(F^?) \subset G(\mathcal{O}_{F^?})$ denote the n-th Iwahori filtration subgroup, as defined in [G, §3]; thus, $I = I_0$ is the usual Iwahori subgroup, and I_n is the kernel of the reduction map $\mathbf{I}(\mathcal{O}_{F^?}) \to \mathbf{I}(\mathcal{O}_{F^?}/m_{F^?}^n)$. Let $H(G(F^?), I_n)$ denote the Hecke algebra of $I_n(F^?)$-biinvariant functions on $G(F^?)$ with coefficients in the algebraically closed field C. Let $\mathcal{A}^n(G,F^?) \subset \mathcal{A}(G,F)$ denote the subset of equivalence classes of irreducible admissible representations generated by vectors fixed under $I_n(F^?)$. Any $\pi \in \mathcal{A}^n(G,F^?)$ is then determined up to isomorphism by the representation of $H(G(F^?), I_n)$ on its invariant subspace $\pi^{I_n(F^?)}$. The following is Ganapathy's refinement of a theorem of Kazhdan:

Theorem 11.4 (Ganapathy-Kazhdan, [G]). *If $F^\sharp$ and F are n-close, then there is a natural isomorphism*

$$H(G(F), I_n) \xrightarrow{\sim} H(G(F^\sharp), I_n) \tag{11.5}$$

of finitely-generated C-algebras. Moreover, there is a bijection

$$\mathcal{A}^n(G,F) \simeq \mathcal{A}^n(G,F^\sharp) \tag{11.6}$$

with the property that, if $\pi \in \mathcal{A}^n(G,F)$ corresponds to $\pi^\sharp \in \mathcal{A}^n(G,F^\sharp)$, then the invariant subspaces $\pi^{I_n(F)}$ and $\pi^{\sharp,I_n(F^\sharp)}$ are isomorphic as modules with respect to the isomorphism (11.5).

In view of these results, it would be unnatural not to make the following conjecture:

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Conjecture 11.7. *For any positive integer n, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{A}^n(G/F) & \xrightarrow{Genestier-Lafforgue} & \mathcal{G}^{ss,n}(G/F) \\ {\scriptstyle (11.6)}\downarrow & & \downarrow{\scriptstyle (11.3)} \\ \mathcal{A}^n(G/F^\sharp) & \xrightarrow[Fargues-Scholze]{} & \mathcal{G}^{ss,n}(G/F^\sharp). \end{array}$$

Here, the top line is the parametrization of Genestier and Lafforgue (Theorem 2.1 (iii)), while the bottom line is the parametrization defined by Fargues and Scholze in [FS] for groups over p-adic fields. Fargues and Scholze have also constructed a semisimple parametrization for groups over local fields of positive characteristic. Li Huerta has shown in [LH23] that this latter parametrization is compatible with Lafforgue's global parametrization and therefore coincides with the Genestier-Lafforgue parametrization. Thus, Conjecture 11.7 also asserts that the two Fargues-Scholze parametrizations are compatible with the Deligne-Kazhdan correspondence.

The following corollary is an immediate consequence of the conjecture and the results of the previous sections.

Corollary 11.8. *Let $F^\sharp$ be a finite extension of $\mathbb{Q}_p$ with residue field k of characteristic p. Let $n \geq 0$ and assume there is a local field F of characteristic p such that $F^\sharp$ and F are n-close. Let G be a split semisimple group over F. Suppose $|k| > 5$ and p does not divide the order of the Weyl group of G. Finally, suppose Conjecture 11.7 holds for the p-adic field $F^\sharp$.*

Let σ be a supercuspidal representation of $G(F^\sharp)$. Suppose the semisimple Fargues-Scholze parameter $\mathcal{L}^{ss}(\sigma)$ attached to σ is pure. Then $\mathcal{L}^{ss}(\sigma)$ is ramified.

More generally, let σ be any pure irreducible representation of $G(F^\sharp)$ with unramified Fargues-Scholze parameter. Then σ is an irreducible constituent of an unramified principal series representation.

Since every p-adic field has a local field F of characteristic p that is 0-close to $F^\sharp$, we have the following special case of Corollary 11.8

Corollary 11.9. *Let $F^\sharp$ be a finite extension of $\mathbb{Q}_p$ with residue field k of characteristic p. Let G be a split semisimple group over F. Suppose $|k| > 5$ and p does not divide the order of the Weyl group of G. Finally, suppose Conjecture 11.7 holds for the p-adic field $F^\sharp$. Then any pure depth zero supercuspidal representation of $G(F^\sharp)$ has ramified Fargues-Scholze parameter.*

One can hope to derive consequences of Conjecture 11.7 for general supercuspidal representations of a general p-adic field $F^\sharp$, replacing $F^\sharp$ by a Galois extension F' with an n-close F for any n. This is the strategy followed by Henniart in his proof of the numerical local correspondence for p-adic fields in [He88]; the Galois extension F' in his construction was obtained by applying the results of [AC] on cyclic base change. For general groups over p-adic fields, one can define a version of local cyclic base change using the global methods of [Lab99], as explained in §10.2. In order to apply this to prove results about purity, we need to know the answers to the following questions:

Question 11.10. Does the Fargues-Scholze parametrization commute with cyclic base change, as described in §10.2?

Question 11.11. Let $F'/F^\sharp$ be a cyclic extension of prime order. Suppose $\pi \in \mathcal{A}^n(G/F^\sharp)$, and π' belongs to the base change of π to $G(F')$. Is $\pi' \in \mathcal{A}^n(G/F')$?

A. Globalization of discrete series, by Raphaël Beuzart-Plessis

Let X be a smooth proper algebraic curve X over $\mathbb{F}_q$, with function field $K = \mathbb{F}_q(X)$. Let G be a connected reductive group over K and let Z^0 be the neutral component of its center. Recall that for $v \in |X|$ and π_v a discrete series of $G(K_v)$ a *pseudo-coefficient* of π_v is a function $\varphi_{\pi_v} \in C_c^\infty(G(K_v)/Z^0(K_v), \xi_v^{-1})$,

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where ξ_v denotes the restriction of the central character of π_v to $Z^0(K_v)$, such that for every tempered irreducible representation π_v' of $G(K_v)$ on which $Z^0(K_v)$ acts through the character ξ_v, we have

$$\text{Trace } \pi_v'(\varphi_{\pi_v}) = \begin{cases} 1 & \text{if } \pi_v' \simeq \pi_v; \\ 0 & \text{if } \pi_v' \not\simeq \pi_v. \end{cases}$$

The existence of pseudo-coefficients can be deduced from the Trace Paley-Wiener theorem [BDK]. Moreover, it follows from Langlands classification that if π_v' is an arbitrary smooth irreducible representation of $G(K_v)$, on which $Z^0(K_v)$ acts through the character ξ_v, if Trace $\pi_v'(\varphi_{\pi_v}) \neq 0$, then π_v' has the same cuspidal support as π_v.

Let $\xi : Z^0(K)\backslash Z^0(\mathbf{A}_K) \to \mathbf{C}^\times$ be a continuous character.

Let T be the universal Cartan of G (i.e., the torus quotient of a Borel subgroup in any quasi-split inner form of G). Let $S_{ram} \subset |X|$ be the finite subset of places where G or ξ is ramified and choose for every $v \in |X| \setminus S_{ram}$ a hyperspecial maximal compact subgroup $J_v \subset G(K_v)$ such that $J_v = G(\mathcal{O}_v)$ for almost all v. By the Satake isomorphism, (J_v-)unramified irreducible representations of $G(K_v)$ are parametrized by $\widehat{T}_v//W_v$, where $\widehat{T}_v$ denotes the complex torus of unramified characters of $T(K_v)$ and $W_v = Norm_{G(K_v)}(T)/T(K_v)$ the Weyl group. We denote by $(\widehat{T}_v//W_v)_{\xi_v}$ the subvariety of unramified characters $\chi \in \widehat{T}_v$ such that $\chi\,|_{Z^0(K_v)} = \xi_v$.

The purpose of this appendix is to prove the following lemma.

Lemma A.1. *Let $S, S' \subset |X|$ be disjoint subsets of places of X with $S' \cap S_{ram} = \emptyset$. For each $v \in S$, let π_v be a discrete series representation whose central character restricted to $Z^0(K_v)$ equals ξ_v, with at least one $v_0 \in S$ such that π_{v_0} is supercuspidal. Also, for $v \in S'$, let $X_v \subset (\widehat{T}_v//W_v)_{\xi_v}$ be a proper closed subset for the Zariski topology. Then there exists a (necessarily cuspidal) automorphic representation $\Pi \simeq \bigotimes'_{x\in|X|} \Pi_x$ of $G(\mathbf{A}_K)$, on which $Z^0(\mathbf{A})$ acts through the character ξ, such that*

(i) *For all $v \in S$, Π_v has nonzero trace against a pseudo-coefficient for π_v (in particular, Π_v and π_v have the same cuspidal support),*
(ii) *For all $v \in S'$, Π_v is an unramified representation with Satake parameter in $(\widehat{T}_v//W_v)_{\xi_v} \setminus X_v$.*

Proof. This follows from an application of the Deligne-Kazhdan simple trace formula [DKV, §A.1 p.41]. More precisely, we consider test functions $f = \otimes_v f_v \in C_c^\infty(G(\mathbf{A}_K)/Z^0(\mathbf{A}_K), \xi^{-1})$ satisfying

(i) For each $v \in S$, $f_v = \varphi_{\pi_v}$ is a pseudo-coefficient of π_v. Moreover, for $v = v_0$ we assume, as we may, that f_{v_0} is a matrix coefficient of $\pi_{v_0}^\vee$;
(ii) For each $v \in S'$, $f_v(g) = \int_{Z^0(K_v)} f_v'(z^{-1}g)\xi_v(z)dz$, where $f_v' \in \mathcal{H}(G(K_v), J_v)$ is a J_v-spherical function whose Satake transform (a regular function on $\widehat{T}_v//W_v$) vanishes identically on X_v but not on $(\widehat{T}_v//W_v)_{\xi_v}$;
(iii) For almost all v, f_v equals the function f_v^0 which coincides with $zk \mapsto \xi_v(z)^{-1}$ on $Z^0(K_v)J_v$ and is zero outside;
(iv) For all $\delta \in G(K)$, if the $G(\mathbf{A}_K)$-conjugacy class of δ intersects the support of f, then δ is regular elliptic.

Then the Deligne-Kazhdan simple trace formula can be applied to f yielding the following identity:

$$\sum_{\Pi} \text{Trace } \Pi(f) = \sum_{\delta} \iota(\delta)^{-1} vol(G_\delta(K)Z^0(\mathbf{A}_K)\backslash G_\delta(\mathbf{A}_K))O_\delta(f), \tag{A.2}$$

where the left sum runs over cuspidal automorphic representations Π of $G(\mathbf{A}_K)$ with central character ξ, the right sum runs over elliptic regular conjugacy classes in $G(K)/Z^0(K)$, G_δ (resp. G_δ^+) denotes

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the centralizer of δ (resp. $Z^0\delta$) in G, $\iota(\delta) = [G^+_\delta(K) : G_\delta(K)]$ and

$$O_\delta(f) = \prod_v \underbrace{\int_{G_\delta(K_v)\backslash G(K_v)} f_v(g_v^{-1}\delta g_v)dg_v}_{=:O_\delta(f_v)}$$

stands for the orbital integral of f at δ. Moreover, both sums are finite.

The choice of the test functions f_v for $v \in S \cup S'$ implies that the only nonzero contributions to the left-hand side of (A.2) come from cuspidal representations satisfying the conclusion of the lemma. Thus, it suffices to see that f can be arranged so that the right-hand side of (A.2) is nonzero. For this, we need an auxiliary lemma.

Lemma A.3. *We can find local test functions f_v, for $v \in S \cup S'$, satisfying conditions (i) and (ii) above as well as a regular elliptic element $\delta \in G(K)$ such that $G^+_\delta = G_\delta$ and $O_\delta(f_v) \neq 0$ for every $v \in S \cup S'$.*

Proof. If weak approximation holds for the inclusion $G(K) \hookrightarrow \prod_{v\in S\cup S'} G(K_v)$, then the argument is pretty standard. More precisely, we can choose any set of test functions $(f_v)_{v\in S\cup S'}$ satisfying (i) and (ii). Indeed, by definition of the Satake transform, for every $v \in S'$, the hyperbolic regular orbital integrals of f_v are not identically zero (where by *hyperbolic regular orbital integral* we mean one that is associated to a regular element in the maximal torus of a Borel subgroup), whereas by [BP], for each $v \in S$, the elliptic regular orbital integrals of f_v are not identically zero. From this, weak approximation and the local constancy of regular semisimple orbital integrals, we deduce the existence of $\delta \in G(K)$ that is elliptic regular in $G(K_v)$ for every $v \in S$ (hence, in particular, δ is regular elliptic in $G(K)$ since $S \neq \emptyset$) and such that $O_\delta(f_v) \neq 0$ for every $v \in S\cup S'$. Moreover, we can also certainly arrange to have $G^+_\delta = G_\delta$ as the subset of regular semisimple elements $\delta_v \in G(K_v)$ satisfying $G^+_{\delta_v} = G_{\delta_v}$ is open and dense.

To deal with the general case (that is including the case where weak approximation fails), we introduce the closure $\overline{G(K)}$ of $G(K)$ in $G(K_{S\cup S'}) = \prod_{v\in S\cup S'} G(K_v)$. According to the argument at the beginning of [PR, Proof of Proposition 7.9 p.419] (which is valid regardless of the characteristic of the global field), $\overline{G(K)}$ is an open subgroup of finite index in $G(K_{S\cup S'})$. Let $v \in S \cup S'$ and let H_v be a normal open subgroup of finite index of $G(K_v)$ contained in $\overline{G(K)} \cap G(K_v)$. Then we just need to check the existence of a test function f_v satisfying condition (i) or (ii) above (according to whether $v \in S$ or $v \in S'$) and a regular semisimple element $\delta \in H_v$ such that $O_\delta(f_v) \neq 0$. First we consider the case $v \in S'$. Then the image of $H_v \cap T(K_v)$ in $T(K_v)/T(\mathcal{O}_v)$ is a subgroup of finite index corresponding to a finite quotient $\widehat{T}'_v$ of $\widehat{T}_v$. Then we look for a regular function on $\widehat{T}_v//W_v$ vanishing identically on X_v but not on $(\widehat{T}_v//W_v)_{\xi_v}$ and whose push forward to $\widehat{T}'_v$ via the projection $\widehat{T}_v \to \widehat{T}'_v$ is nonzero. Such a function is readily seen to exist. Consider now the case $v \in S$ and let f_v be a pseudo-coefficient of π_v. By [BP], it suffices to show that the restriction of the Harish-Chandra character Θ_{π_v} of π_v to the subset $H_{v,ell-rs}$ of elliptic regular semisimple relements in H_v is nonzero. For δ a finite dimensional representation of $G(K_v)/H_v$, $\pi_v \otimes \delta$ is a direct sum of discrete series with Harish-Chandra character $\Theta_{\pi_v\otimes\delta} = \Theta_{\pi_v}\Theta_\delta$ where Θ_δ denotes the usual character of δ. Then if $\Theta_{\pi_v}|_{H_{v,ell-rs}} = 0$, the restriction of $\sum_\delta \frac{1}{\dim\delta}\Theta_{\pi_v\otimes\delta}$ to the elliptic regular semisimple locus of $G(K_v)$ would be zero, and this would contradict the elliptic orthogonality relations of [BP] on Harish-Chandra characters of discrete series. □

Let $(f_v)_{v\in S\cup S'}$ and $\delta \in G(K)$ as in the above lemma. Then we choose the functions $f_v \in C^\infty_c(G(K_v)/Z^0(K_v), \xi_v^{-1})$ for $v \notin S \cup S'$ such that $O_\delta(f_v) \neq 0$ for all v and $f_v = f^0_v$ for almost all v (this is certainly possible thanks to the condition $G^+_\delta = G_\delta$). Then the set of semisimple conjugacy classes of $G(K)$ whose $G(\mathbf{A}_K)$-conjugacy classes meet the support of f is finite. Hence, up to shrinking the support of f_v at some auxiliary place $v \notin S \cup S'$, we may assume that this set only consists in the orbit of δ. The function f then satisfies condition (iii) above and the right-hand side of the trace formula (A.2) is reduced to the term corresponding to δ. As $O_\delta(f) \neq 0$ by construction, this implies that the left-hand side does not vanish either, and we are done. □

https://doi.org/10.1017/fmp.2024.10 Published online by Cambridge University Press

Acknowledgements. The authors thank Hélène Esnault, Tony Feng, Jessica Fintzen, Dennis Gaitsgory, Guy Henniart, Tasho Kaletha, Jean-Pierre Labesse, Bertrand Lemaire and Zhiwei Yun for helpful discussions of various aspects of the paper. We benefited from exchanges with Laurent Fargues and Peter Scholze regarding the material in the final section. We thank Xinwen Zhu for directing our attention to his article [XZ] with Daxin Xu, which allowed us to remove a restriction in the statement of Theorem 1.2. We are especially grateful to Luis Lomelí for his role, in collaboration with one of us, in developing the main techniques on which our results are based, and for his comments in the early stages of this project. Finally, we thank Raphael Beuzart-Plessis for providing us with the appendix, allowing us to prove Theorem 1.2 for discrete series representations.

Competing interests. The authors have no competing interest to declare.

Financial support. The work of Beuzart-Plessis was funded by the European Union ERC Consolidator Grant, RELANTRA, project number 101044930. Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or the European Research Council. Neither the European Union nor the granting authority can be held responsible for them.

W. T. Gan was partially supported by a Singapore government MOE Tier 1 grant R-146-000-320-114.

M. Harris was partially supported by NSF Grant DMS-2001369 and by a Simons Foundation Fellowship, Award number 663678.

W. Sawin was partially supported by NSF grant DMS-2101491 and served as a Clay Research Fellow.

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