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Algebraic groups. The theory of group schemes of finite type over a field, by J. S. Milne, Cambridge University Press, 2022, ISBN 9781009018586

The theory of linear algebraic groups is concerned, broadly speaking, with the study of matrix groups using the techniques of group theory and algebraic geometry. Such groups arise naturally across many disciplines of mathematics, from algebra and number theory to geometry, topology, and mathematical physics. Some familiar examples include the general and special linear groups $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{SL}_n(\mathbb{C})$, the special orthogonal groups $\mathrm{SO}_n(\mathbb{C})$, and the symplectic groups $\mathrm{Sp}_{2n}(\mathbb{C})$. While these groups are often first encountered in a somewhat *ad hoc* fashion in courses on linear algebra, one of the outstanding features of the theory of linear algebraic groups is that it provides a unified perspective on the structure of all so-called reductive groups.

As we will see in Section 1, the theory developed in stages. Stemming in the late nineteenth century from essentially the same considerations that would later lead to the development of Lie group theory, the theory of reductive groups over algebraically closed fields came to fruition in the 1950s in the work of Chevalley, Kolchin, Borel, and others. Subsequently, in the 1960s, Grothendieck and his school substantially generalized these results to produce a powerful but technically very demanding theory of reductive group schemes, which was laid out in the three volumes of SGA 3 ([17]). Prior to the publication of Milne's book, most existing textbooks on algebraic groups operated within the classical framework of varieties over algebraically closed fields. While this perspective is adequate for analyzing structural issues, it becomes rather inconvenient when dealing with problems of a more arithmetic nature. By contrast, Milne uses the more flexible functorial point of view of scheme theory. However, by choosing to focus on affine group schemes of finite type over a field, he is able to avoid many of the technical complications that one encounters in SGA 3. Thus, Milne's book fulfills the dual purpose of providing an updated account of the theory of reductive groups while at the same time serving as an accessible entry point into the general theory of reductive group schemes.

1. Genesis of the theory of algebraic groups

To put Milne's exposition of the theory of algebraic groups in the appropriate context, we begin with a panoramic look at the development of the subject (we refer the reader to Armand Borel's collection of essays [4] for a highly informative historical account). Its origins can be traced back to the late nineteenth century to ideas of Sophus Lie and Émile Picard on developing an analogue of Galois theory for solutions of differential equations, work of Felix Klein on the "Erlangen Program" that aimed to connect geometry and group theory, results of Wilhelm Killing and Élie Cartan that provided a complete classification of simple Lie algebras, as well as four (largely overlooked) papers of Ludwig Maurer, in which he, in effect,

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defined and studied the Lie algebra of a complex algebraic group, established the Jordan decomposition, and analyzed various important classes of groups like tori and unipotent groups.

After a period of relative dormancy, the theory began to take on its modern form in the 1940s and 1950s in the works of Claude Chevalley and Ellis Kolchin. The starting point for Chevalley was the theory of Lie groups as developed by É. Cartan—Chevalley reworked and clarified many points of the theory, ultimately producing the first volume [10] of his treatise on Lie groups. Subsequently, Chevalley's attention shifted to more algebraic aspects. Using a formal version of the exponential map, Chevalley was able to generalize results of Maurer and others from groups over $\mathbb{C}$ to groups defined over an arbitrary algebraically closed field of characteristic 0, thus initiating a theory of linear algebraic groups in a much broader context (this theory is described in Chevalley's second volume [11]). At around the same time, Kolchin was concerned with putting the ideas of Lie and Picard in differential Galois theory on rigorous footing, which led him to consider algebraic groups. In contrast to Chevalley's approach, which was philosophically inspired by the analytic techniques of Lie theory and thus restricted to characteristic 0, Kolchin completely avoided Lie algebras and instead worked directly using group-theoretic and algebro-geometric considerations. This, in particular, allowed him to prove a number of substantial results over algebraically closed fields of *arbitrary* characteristic. The change of perspective introduced in Kolchin's work was so significant that Borel refers to the papers [24] and [25] as constituting the "birth certificate" of the theory of linear algebraic groups over general algebraically closed fields.

In subsequent years, methods of algebraic geometry were imported in a systematic way into the subject. One of the notable developments was Borel's paper [1], which provided a detailed account of the theory of algebraic groups as it stood at the time, and, perhaps most significantly, highlighted the importance of connected solvable groups in the analysis of the structure of linear algebraic groups. Using these ideas, Chevalley and other participants of a seminar that he organized in Paris were then able to give in [12] a complete classification of semisimple algebraic groups over algebraically closed fields of arbitrary characteristic in terms of root systems. At this point, the theory of linear algebraic groups over an algebraically closed field had reached essentially full maturity. The first textbook account of this theory was given by Borel in [2]. Subsequently, more expanded treatments, along with various corrections and refinements, appeared in the books of Borel [3], Humphreys [21], and Springer [29], which, until the publication of Milne's book, were the standard references for the subject.

The next, and, for the present discussion, the final stage in the development of the theory of reductive groups came in the early 1960s. Namely, using the results of [12] as a starting point, Alexander Grothendieck and Michel Demazure initiated, in 1962, a seminar in which they developed a very general theory of reductive group schemes not just over arbitrary base fields (or even arbitrary base rings), but over arbitrary schemes. The resulting theory was assembled in the three volumes of SGA 3 ([17]; see also [18] and [19] for new editions of volumes 1 and 3, respectively). This theory has proven to be extremely useful over the years in various contexts, but what is perhaps most significant is that SGA 3 introduced a fundamentally new functorial perspective into the study of algebraic groups. Let us explain this in more detail.

The work of Chevalley, Kolchin, and Borel, described above, all takes place within the classical framework of algebraic varieties over algebraically closed fields, where, essentially by definition, a variety is identified with its set of points over the given algebraically closed field. More precisely, for an algebraically closed field K, Kolchin originally defined a subgroup G of $\mathrm{GL}_n(K)$ to be *algebraic* if it is the set of all invertible matrices whose coefficients annihilate a given set of polynomials in n^2 variables with coefficients in K—in other words, G is the intersection of $\mathrm{GL}_n(K)$ with a closed algebraic subvariety of $\mathrm{M}_n(K)$ (which one views as the affine space $\mathbb{A}_K^{n^2} = K^{n^2}$). Subsequently, the following *a priori* more general definition was adopted: an *affine algebraic group* over K is a closed subvariety G of some affine space $\mathbb{A}_K^{\ell}$ (i.e., the vanishing locus of a collection of polynomials in $K[x_1, \ldots, x_\ell]$) that is equipped with polynomial maps

$$m\colon G \times G \to G \text{ (multiplication)} \quad \text{and} \quad \iota\colon G \to G \quad \text{(inversion)} \tag{1}$$

and a distinguished element $e \in G$ (the identity) such that (G, e, m, ι) is a group. By working with the coordinate ring $K[G]$ of G (which is, by definition, the K-algebra of polynomial functions $G \to K$), one can show that G is in fact a Zariski-closed subgroup of some general linear group $\mathrm{GL}_n(K)$. (We note that $\mathrm{GL}_n(K)$ is itself an affine algebraic group since

$$\mathrm{GL}_n(K) = \left\{(X, y) \in \mathrm{M}_n(K) \times \mathbb{A}_K^1 \simeq \mathbb{A}_K^{n^2+1} \mid \det(X)y - 1 = 0\right\},$$

where $\det(X)$ is the determinant of the matrix X.)

While this perspective is adequate for establishing structural results, it becomes rather inconvenient in more arithmetic matters, where one not only needs to consider algebraic groups in the context of nonalgebraically closed fields (usually global fields), but also bring into play the completions of the base field, the associated ring of adeles, and reductions modulo the maximal ideals of the ring of integers. Such problems can be alleviated by instead adopting a more *functorial* approach provided by Grothendieck's scheme theory. To illustrate the general idea, first suppose that K is an algebraically closed field. Then by Hilbert's Nullstellensatz, for any affine algebraic variety V over K, we have a bijection between the set $V(K)$ of K-points of V and the set $\mathrm{Specm}(K[V])$ of maximal ideals of the coordinate ring of V. The latter set is, in turn, in bijection with the set of K-algebra homomorphisms $\mathrm{Hom}_{K\text{-alg.}}(K[V], K)$. One can then consider, for any commutative K-algebra B, the set of B-points $V(B) := \mathrm{Hom}_{K\text{-alg.}}(K[V], B)$, and one observes that a K-algebra homomorphism $f\colon B \to B'$ induces a map of sets

$$V(B) \to V(B'), \quad \varphi \mapsto f \circ \varphi.$$

Thus, the variety V defines a *functor*, called the *functor of points* of V,

$$\underline{\mathbf{V}}\colon \underline{\mathsf{K\text{-}alg.}} \to \underline{\mathsf{Set}}$$

from the category $\underline{\mathsf{K\text{-}alg.}}$ of commutative K-algebras to the category $\underline{\mathsf{Set}}$ of sets that is *represented* by the K-algebra $K[V]$.

Next, if G is an affine algebraic group over K, then the existence of the morphisms in (1) together with the identity element $e \in G(K)$ implies that we have K-algebra homomorphisms

$$\Delta\colon K[G] \to K[G] \otimes_K K[G], \quad S\colon K[G] \to K[G], \quad \epsilon\colon K[G] \to K \tag{2}$$

(called the *co-multiplication*, *co-inverse* or *antipode*, and *co-identity*, respectively) that satisfy some compatibility relations forced by the group axioms in G. In other words, $K[G]$ is a finitely generated, commutative *Hopf algebra* over K. For example, if $G = \mathrm{SL}_n(K)$, then

$$K[G] = K[T_{11}, T_{12}, \ldots, T_{nn}]/(\det(T_{ij}) - 1).$$

Since the group operation is matrix multiplication, we see that

$$\Delta(T_{ij}) = \sum_{k=1}^{n} T_{ik} \otimes T_{kj}. \tag{3}$$

Next, using the formula for the inverse of a matrix in terms of the classical adjoint, we find that

$$S(T_{ij}) = (-1)^{i+j} \det(T_{rs})_{r\neq j, s\neq i}. \tag{4}$$

Finally, since the identity element is the $(n \times n)$ identity matrix I_n, we have

$$\epsilon(T_{ij}) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases} \tag{5}$$

The Hopf algebra structure on $K[G]$ implies that for any commutative K-algebra B, the set

$$G(B) := \mathrm{Hom}_{K\text{-alg}}(K[G], B),$$

is, in fact, a group, and, moreover, any K-algebra homomorphism $B \to B'$ induces a group homomorphism $G(B) \to G(B')$. So, in this more specialized case, the functor

$$\underline{\mathbf{G}}\colon \underline{\mathsf{K\text{-}alg.}} \to \underline{\mathsf{Group}}$$

represented by the Hopf algebra $K[G]$ takes values in the category $\underline{\mathsf{Group}}$ of *groups*. The essential difference between the functorial point of view and the classical approach is that one now shifts the focus from the group of points $G(K)$ over a fixed algebraically closed field K to *the functor of points* $\underline{\mathbf{G}}$.

To bring arbitrary fields into the picture, suppose now that A is a finitely generated commutative algebra over a field F, and assume that A has the structure of a Hopf algebra defined by F-algebra homomorphisms (Δ, S, ϵ) as in (2). Then, as above, for any F-algebra B, the set $\mathrm{Hom}_{F\text{-alg.}}(A, B)$ is a group, and we obtain a functor

$$\underline{\mathbf{G}}\colon \underline{\mathsf{F\text{-}alg.}} \to \underline{\mathsf{Group}}, \quad B \mapsto \underline{\mathbf{G}}(B) := \mathrm{Hom}_{F\text{-alg.}}(A, B).$$

In more geometric terms, recall that to any commutative ring R, one associates an affine scheme $(X, \mathcal{O}_X)$, where $X = \mathrm{Spec}(R)$ is the topological space of all prime ideals of R endowed with the Zariski topology, and $\mathcal{O}_X$ is the corresponding structure sheaf—for simplicity, this scheme is usually denoted by $\mathrm{Spec}(R)$. Moreover, if R has the structure of a commutative F-algebra, then we have a morphism $\mathrm{Spec}(R) \to \mathrm{Spec}(F)$, so $\mathrm{Spec}(R)$ is in fact an affine scheme over F. It is known that the correspondence

$$R \mapsto \mathrm{Spec}(R)$$

sets up a contravariant equivalence between the category $\underline{\mathsf{F\text{-}alg.}}$ of commutative F-algebras and the category $\underline{\mathsf{AffSch/F}}$ of affine schemes over F (see, e.g., [20, Chapter 2] for the contruction and some basic properties of affine schemes). Now, it turns out that under this equivalence, the tensor product of F-algebras $R_1 \otimes_F R_2$ corresponds to the (fiber) product of affine schemes $\mathrm{Spec}(R_1) \times_F \mathrm{Spec}(R_2)$. So, the fact that A is a Hopf algebra implies that the associated affine scheme $G = \mathrm{Spec}(A)$ comes equipped with morphisms

$$m\colon G \times_F G \to G, \quad \iota\colon G \to G, \quad e\colon \mathrm{Spec}(F) \to G$$

satisfying the usual group axioms. In other words, G is an *affine group scheme* of finite type over the field F. Furthermore, one shows that for any F-algebra B, we have

$$G(B) := \mathrm{Hom}_{\mathrm{Sch}/F}(\mathrm{Spec}(B), G) = \mathrm{Hom}_{F\text{-alg.}}(A, B) = \underline{\mathbf{G}}(B).$$

Thus, $\underline{\mathbf{G}}$ is the functor represented by G in the category $\underline{\mathsf{AffSch/F}}$.

Returning to our previous example, suppose $F = \mathbb{Q}$, and consider the $\mathbb{Q}$-algebra

$$A = \mathbb{Q}[T_{11}, T_{12}, \ldots, T_{nn}]/(\det(T_{ij}) - 1)$$

equipped with the Hopf algebra structure given by the analogues over $\mathbb{Q}$ of the maps Δ, S, and ϵ described in (3), (4), and (5), respectively. Then it is straightforward to see that for any commutative $\mathbb{Q}$-algebra B, we have

$$G(B) = \mathrm{SL}_n(B) = \{X \in \mathrm{M}_n(B) \mid \det(X) = 1\}.$$

The focus of Milne's book is on affine group schemes G of finite type over an arbitrary field F (which, by definition, come equipped with a structure morphism $G \to \mathrm{Spec}(F)$). While this is, of course, only a special case of the theory developed in SGA 3 (where group schemes with a structure morphism $G \to S$ to an arbitrary scheme S are considered), the main advantage is that in this case, certain subtle algebro-geometric considerations can be carried out fairly explicitly with Hopf algebras. Now, while the theory of affine group schemes has been around for quite some time, prior to the publication of Milne's book, the most significant resources where the reader could get acquainted with the subject were the books of Demazure and Gabriel [16], and Waterhouse [32]; however, neither discusses reductive group schemes in detail. To the best of the reviewer's knowledge, Milne's book is thus the first complete textbook account of the structure theory of reductive groups over fields that is written in scheme-theoretic language, and, as such, it is a very welcome addition to the literature. Moreover, once readers are familiar with the overall picture presented in Milne's book, they will be in a better position to tackle the general theory of SGA 3.

2. An overview of Milne's book

Due to limitations of space, we will not attempt to give a comprehensive account of the contents of Milne's book, and will only briefly comment on a few key points. Roughly speaking, the book can be divided into four parts.

In the first part (consisting of Chapters 1–8), Milne discusses the basic definitions and foundational material, some of it in the context of general (not necessarily affine) group schemes of finite type over a field. Significant results of this part include the following. In Chapter 3, Milne proves Cartier's theorem that all affine group schemes of finite type over a field of characteristic 0 are smooth (or, equivalently, geometrically reduced). This result highlights one of the major differences between the classical and scheme-theoretic perspectives. Classically, one deals only with varieties, which are (geometrically) reduced by definition, and thus algebraic groups are always smooth in *any* characteristic. By contrast, nonreduced group schemes appear quite naturally over fields of positive characteristic, and, in fact, are an essential part of the theory. For instance, let F be a field and consider the functor

$$\mu_3 \colon \underline{\mathsf{F\text{-}alg.}} \to \underline{\mathsf{Group}}, \quad B \mapsto \{b \in B \mid b^3 = 1\}.$$

This functor is clearly represented by the F-algebra $F[X]/(X^3-1)$, and thus $G = \mathrm{Spec}(F[X]/(X^3-1))$ is an affine group scheme of finite type over F. If $\mathrm{char} F \neq 3$, then this is just the familiar group scheme of third roots of unity. On the other hand, if $\mathrm{char} F = 3$, then since $X^3 - 1 = (X-1)^3$, we see that G is not reduced. Although $\mu_3(K) = \{1\}$ for all fields K containing F (so this group scheme carries no information in the classical setting), μ_3 is not the trivial group scheme since $\mu_3(B)$ may be nontrivial if B has nilpotent elements.

Perhaps the most substantial result of the first part is the Barsotti–Chevalley theorem (proved in Chapter 8), which states that if G is a connected algebraic group over a perfect field, then G contains a unique connected *affine* normal subgroup N such that G/N is an abelian variety. In effect, this result allows one to reduce the study of algebraic groups into two essentially disjoint cases: one focusing on abelian varieties (i.e., algebraic groups whose underlying varieties are *projective*), and the other dealing with linear (or affine) algebraic groups. Despite its conceptual significance, this theorem is rarely seen in textbooks, so readers will appreciate Milne's decision to include it. For a different exposition of this result, one can also consult [13]. Following the proof of the Barsotti–Chevalley theorem, Milne deals exclusively with affine group schemes.

The second part, comprising Chapters 9–11, lays some of the groundwork that will be needed later for a systematic study of reductive groups. Chapter 9 establishes the existence and various properties of Jordan decompositions (which is essentially an extension of the Jordan canonical form, covered in linear algebra, to the setting of arbitrary affine algebraic groups). Although this material is standard, the exposition is notable for emphasizing the Tannakian philosophy that, in a precise sense, places an algebraic group and its category of representations on equal footing. Chapter 10 is devoted to Lie algebras of algebraic groups—again, while this material is very familiar, Milne's account differs from the standard sources [3], [21], and [29] by taking a more functorial point of view. Chapter 11 covers the theory of finite group schemes, i.e., group schemes represented by *finite-dimensional* Hopf algebras. Despite its great importance for number theory and arithmetic geometry, this topic is typically omitted from textbooks on algebraic groups, so readers will likely find this chapter very useful.

In the third part (Chapters 12–16), Milne carries out a systematic study of solvable algebraic groups, starting with diagonalizable groups and unipotent groups, and proceeding to general trigonalizable groups. Although the exposition focuses mainly on various standard core items (such as the classification of tori by their character lattices, and the Lie–Kolchin theorem that all smooth connected solvable groups are trigonalizable over a finite extension of the base field), some innovative and less frequently encountered points include the following. First, in Chapter 13, Milne discusses actions of tori on schemes, culminating with the Białynicki–Birula decomposition. In Chapter 14, Milne presents a side-by-side comparison of unipotent groups over fields of characteristic 0 (where one can use the exponential map, going back to Chevalley's second volume [11]) and over fields of positive characteristic (where the picture is considerably more complicated but where one still has a reasonable classification of so-called elementary unipotent groups using certain rings of twisted polynomials). Finally, in Chapter 16, the reader can find, among other things, a proof of the classification of connected one-dimensional algebraic groups that requires fewer algebro-geometric prerequisites than the argument in Borel's book [3].

The fourth part, consisting of Chapters 17–25, is the technical heart of the book. In Chapters 17–23, Milne develops in detail the structure theory of split reductive groups and their representations in terms of their root data. Much of this material goes back to the work of Chevalley's seminar [12]; however, Milne's account is significant in that it provides the first complete and essentially self-contained textbook treatment of the root datum of a reductive group, as well as the Isogeny and Existence theorems, that is written in scheme-theoretic language. Another notable point is Milne's exposition of Chevalley's theorem on the unipotent radical: while the classical argument is deeply embedded in the general structure theory and cannot be easily extracted from there, Milne presents an alternative geometric approach due to Luna, which is based largely on the Białynicki–Birula decomposition and is thus, in many respects, more transparent. Finally, in Chapter 24, Milne describes the construction of all almost-simple algebraic groups in terms of algebras with involution, and in Chapter 25, he briefly touches upon the structure theory of nonsplit reductive groups.

3. Afterword

In summary, Milne's book provides a comprehensive yet accessible exposition of the theory of linear algebraic groups over fields that is written in scheme-theoretic language. One can expect that in the coming years, it will become a dominant source for those seeking the first entry point into the subject with an eye towards arithmetic applications of (affine) algebraic groups.

We would like to conclude this review by mentioning a few references that the interested reader may wish to explore for a variety of more technical and specialized topics. First, as previously mentioned, Milne focuses primarily on affine group schemes of finite type over a field. If one is instead looking to learn about the general theory of reductive group schemes, then, besides the original volumes of SGA 3, one can consult Conrad's notes [14] for a lucid account of the theory that, in addition, incorporates various developments in algebraic geometry (particularly in the theory of stacks and algebraic spaces) that came into being after the initial publication of SGA 3.

Next, while Milne presents the complete structure theory for split reductive groups, he stops short of discussing in detail the extension of this theory to nonsplit groups, which is particularly important for those working with algebraic groups over nonalgebraically closed fields. A structure theory for reductive groups over arbitrary fields, based on the relative root system, was developed by Borel and Tits in the papers [5] and [6], and a classification of semisimple algebraic groups in terms of so-called Tits indices was given by Tits in [30]. An overview of this material appears in [3, Chapter V] and a fairly detailed account can be found in

[29, Chapters 12–17]. This theory has been generalized to the case of pseudo-reductive groups by Conrad, Gabber, and Prasad [15]. We should also mention that the study of semisimple algebraic groups over nonalgebraically closed fields relies heavily on techniques from Galois cohomology and the theory of algebras with involution—for details on these, the reader can consult [23] and [26] (see also the new edition [27]).

The last topic that we would like to mention, which does not appear in Milne's book, is the arithmetic theory of algebraic groups, which is concerned with the study of various properties of the groups of points of algebraic groups defined over fields of arithmetic interest. For groups over number fields, an extensive account that culminates in the proof of the local-global principle for simply connected semisimple groups is given in [26]. In a different direction, a detailed structure theory for reductive groups over a field complete with respect to a discrete valuation (such as the field $\mathbb{Q}_p$ of p-adic numbers) was developed by Bruhat and Tits in [7], [8], and [9], and an overview of this theory was given by Tits in [31]. A new approach to Bruhat–Tits theory, which avoids some of the original combinatorial arguments and instead systematically utilizes certain techniques of SGA 3, recently appeared in the book of Kaletha and Prasad [22]. Finally, some aspects of the developing arithmetic theory of algebraic groups over fields other than global are discussed in [28].

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