

Sampling List Packings

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¹⁰ — Abstract —

¹¹ We initiate the study of approximately counting the number of list packings of a graph. The
¹² analogous problem for usual vertex coloring and list coloring has attracted substantial attention.
¹³ For list packing the setup is similar, but we seek a full decomposition of the lists of colors into
¹⁴ pairwise-disjoint proper list colorings. The existence of a list packing implies the existence of a
¹⁵ list coloring, but the converse is false. Recent works on list packing have focused on existence or
¹⁶ extremal results of on the number of list packings, but here we turn to the algorithmic aspects of
¹⁷ counting and sampling.

¹⁸ In graphs of maximum degree Δ and when the number of colors is at least $\Omega(\Delta^2)$, we give a fully
¹⁹ polynomial-time randomized approximation scheme (FPRAS) based on rapid mixing of a natural
²⁰ Markov chain (the Glauber dynamics) which we analyze with the path coupling technique. Some
²¹ motivation for our work is the investigation of an atypical spin system, one where the number of
²² spins for each vertex is much larger than the graph degree.

²³ **2012 ACM Subject Classification** Theory of computation → Random walks and Markov chains;
²⁴ Mathematics of computing → Approximation algorithms

²⁵ **Keywords and phrases** List packing, Graph colouring, Markov chains, Path coupling

²⁶ **Related Version** *Full Version:* <https://arxiv.org/abs/2402.03520>

²⁷ **Funding** *Ewan Davies:* supported in part by NSF grant CCF-2309707

²⁸ **Acknowledgements** We thank Colin McSwiggen and Semon Rezchikov for organizing Random
²⁹ Theory 2023, where this collaboration began. We thank Charlie Carlson and Guillem Perarnau for
³⁰ comments that led to improvements to the exposition of the paper.

³¹ 1 Introduction

³² The classic graph coloring problem is to determine, for a graph $G = (V, E)$ and number
³³ of colors q , whether there is a coloring $f : V \rightarrow [q]$ such that f is proper in the sense that
³⁴ $f(u) \neq f(v)$ for every edge $uv \in E$. List coloring emerged in the late 20th century as an
³⁵ adversarial version of this problem [27, 13]. To define list coloring, we take a list size $q \in \mathbb{N}$
³⁶ and assign to each $u \in V$ a list of allowed colors $L(u)$ of size q . One can think of the list
³⁷ assignment L as supplied by an adversary who might try to prevent the existence of a coloring
³⁸ respecting the lists. If, under any choices of such an assignment of lists, the graph G admits
³⁹ a proper coloring f such that $f(u) \in L(u)$ for every vertex u , then we say that G is q -list
⁴⁰ colorable (also known as q -choosable in some works). In some ways list coloring and classical
⁴¹ graph coloring are similar, e.g., the complete graph on n vertices requires $q \geq n$ for both

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42 problems. But all bipartite graphs can be colored with two colors, while $K_{n,n}$ requires lists
 43 of length $\Omega(\log n)$ in the list coloring setting [13].

44 List coloring also arises naturally in various other ways. Notably, if we take a classic
 45 graph coloring problem and pre-color some vertices then we can express the problem of
 46 completing the coloring through list coloring. We set $L(u)$ to be the subset of the colors not
 47 used by any pre-colored vertex in the neighborhood of u (though these lists may not have
 48 equal sizes). The idea of list coloring has been extended in several interesting directions,
 49 and in this work we study a very recent and structured variant known as list packing [6].
 50 The setup is the same as for list coloring with a fixed list size q , but instead of seeking just
 51 one proper coloring where each vertex is colored from its list, we seek q pairwise-disjoint
 52 proper colorings from the lists. We consider two colorings f and f' of a graph disjoint if, for
 53 all vertices u we have $f(u) \neq f'(u)$. We call this collection of q pairwise-disjoint proper list
 54 colorings a *list packing*. Given a graph G and list size q , if a list packing can be found for
 55 any lists of size q chosen by an adversary then we say that G is q -list packable.

56 One motivation for list packing in [6] is to challenge the state-of-the-art in list coloring.
 57 For example, a folklore result states that a bipartite graph of maximum degree Δ is q -list
 58 colorable for $q \geq (1 + o(1))\Delta / \log \Delta$ (see also [1] and a recent improvement of the leading
 59 constant due to Bradshaw [4]). Amongst other foundational results on list packing, the
 60 folklore result was matched for the significantly more structured notion of list packing in [6].
 61 A notable difference between known results for list packing and list coloring is that a general
 62 graph of maximum degree Δ is q -list packable for $q \geq 2\Delta$ (improved to $q \geq 2\Delta - 2$ for $\Delta \geq 4$
 63 in [8]); whilst such G are q -list colorable for $q \geq \Delta + 1$ (and even $q = \Delta$ if $\Delta \geq 3$ and G does
 64 not contain a clique on $\Delta + 1$ vertices [13]). Despite the gap in known results, there is scant
 65 evidence that more than $\Delta + 1$ colors are ever required for list packing [8, 6].

66 Early work on list packing has focused on the problem of existence, though many
 67 arguments for existence also provide efficient constructions of list packings. An extremal
 68 perspective on counting list packings was recently investigated by Kaul and Mudrock [21],
 69 and related problems where one seeks many list colorings or “flexible” list colorings are
 70 studied in [20, 7]. In this work we turn to the study of approximately counting the number
 71 of L -packings of a graph G with a fixed q -list assignment L . This is a natural question by
 72 analogy with the same questions for graph coloring and list coloring: is there an efficient
 73 procedure that, given a graph G and a number of colors q , approximates the number of
 74 proper q -colorings of G (e.g., to within a factor 2)? This question is well-studied in the
 75 field of approximate counting and sampling, and is an important test-bed for algorithmic
 76 techniques. A longstanding open question is the existence of such an approximate counting
 77 algorithm that works for all $q \geq \Delta + 1$ on graphs of maximum degree Δ . An influential
 78 collection of results using various techniques requires conditions such as $q \geq e\Delta + 1$ [2],
 79 $q \geq 2\Delta$ [17, 22], $q \geq 11\Delta/6$ [26], and even $q \geq (11/6 - \epsilon)\Delta$ for some small $\epsilon > 0$ [10, 9].
 80 Another branch of research on the algorithmic aspects of counting graph colorings seeks to
 81 sample perfectly uniformly from the set of proper q -colorings of a graph. The pioneering
 82 result is due to Huber [15], with improvements and new techniques supplied in a number of
 83 later works [23, 22, 3, 16]. Interestingly, one application of such perfect samplers is to design
 84 approximate counting algorithms that can be faster than analogous approaches which use
 85 approximate samplers.

86 Motivated by simple and powerful ideas such as the Markov chain Monte Carlo approach
 87 to counting colorings due to Jerrum [17] (see also [25]), we seek similar results for list packing.
 88 The list packing problem, however, presents novel difficulties. Observe that finding a list
 89 coloring gets strictly easier as the list size q grows. Supposing that one has a technique that

90 works with lists of size q_0 , then given larger lists one can take arbitrary subsets of the lists
 91 of size q_0 and apply the technique in a black-box fashion. In contrast, for list packing with
 92 larger q we are required to find ever more list colorings which must also be pairwise-disjoint.
 93 Given lists of size $q > q_0$, it is not at all clear how to extend an arbitrary collection of q_0
 94 pairwise-disjoint list colorings to a full list packing. The methods of [8, 6] provide evidence
 95 that showing the *existence* of a list packing does get easier with larger q , albeit for less
 96 straightforward and general reasons. Whether *counting* list packings should get easier as q
 97 grows is another matter, but we confirm this principle in the setting of approximate counting
 98 by giving an algorithm that approximately counts list packings which works for all q large
 99 enough in terms of the maximum degree.

100 Approximate counting of combinatorial objects such as list colorings, independent sets,
 101 and matchings corresponds to approximating the so-called partition function of a spin system
 102 from statistical physics. We are interested in the study of counting and sampling list packings
 103 as it presents an unusual type of spin system. Typically, the number of spins available for
 104 each vertex in a spin system is small. In the Ising model of magnetism vertices take a spin
 105 from $\{+, -\}$, and in the (antiferromagnetic) Potts model associated with proper q -colorings,
 106 the spins are the q colors. Parameter ranges frequently studied for this model on graphs
 107 of maximum degree Δ include the case when q is close to $\Delta + 1$. The natural spin system
 108 associated with q -list packings, however, has $q!$ spins for each vertex. Since a vertex u has
 109 a list of q colors which we must decompose into q choices, one for each list coloring in the
 110 packing, the spins for u naturally correspond to permutations of the list $L(u)$. One of our
 111 contributions is to study a somewhat natural combinatorial problem which involves a spin
 112 system on bounded degree graphs with many more spins than commonly-studied examples.
 113 We hope that further insights into algorithmic techniques for approximate counting can be
 114 gained by studying an unusual spin system.

115 2 Results

116 Our main result is the existence of an approximate counting algorithm for list packings. We
 117 define an ϵ -approximation of a real number x as a real number y satisfying $e^{-\epsilon} \leq x/y \leq e^\epsilon$.
 118 We use the standard notion of a *fully polynomial-time randomized approximation scheme*
 119 (FPRAS) for a counting problem. This is a randomized algorithm that, given $\epsilon > 0$, yields
 120 an ϵ -approximation of the true answer in time polynomial in the input size and $1/\epsilon$.

121 Before we state the result, we need some more notation for list packings focusing on a
 122 specific list assignment L . Given a graph G we call an assignment $L : V(G) \rightarrow 2^{\mathbb{N}}$ of lists
 123 of colors to the vertices of G such that $|L(u)| = q$ for all vertices u a q -list assignment of
 124 G . A proper coloring f of G such that $f(u) \in L(u)$ for all vertices u is called an L -coloring,
 125 or a list coloring if the lists L are understood from context. Given a graph G and a q -list
 126 assignment L , we call a collection of q pairwise-disjoint L -colorings of G an L -packing. Note
 127 that an L -packing corresponds to permutation of the lists $L(v)$ for each vertex v .

128 **Theorem 1.** *There is an absolute constant C such that the following holds. For any $\Delta \geq 1$
 129 and $q \geq C\Delta^2$, let G be a graph of maximum degree Δ . Then for any q -list assignment L of
 130 G , there is an FPRAS for the number of L -packings of G .*

131 We prove this result by analyzing the so-called “heat-bath Glauber dynamics” for list
 132 packing. Briefly, the state space is the set of list packings of the graph and transitions involve
 133 resampling a uniform random valid permutation of the list of a uniform random choice of
 134 vertex. We define “valid” carefully and analyze this Markov chain later, noting here that

135 we prove Theorem 1 by establishing rapid mixing of this chain and hence an approximate
 136 sampling algorithm for a uniform L -packing under the same conditions.

137 A natural combinatorial problem on matchings in balanced bipartite graphs of large
 138 minimum degree emerges during the proof of Theorem 1, leading to a probabilistic result
 139 stated below that may be of independent interest. This is because the valid permutations of
 140 the list $L(v)$ correspond precisely to the perfect matchings in an auxiliary bipartite graph
 141 that we construct from the list packing on the neighbors of v . Throughout, we assume
 142 that Δ and q are fixed constants and do not analyze the case where they are allowed to
 143 depend on the number n of vertices of G as then algorithmic issues related to (perfectly)
 144 sampling perfect matchings in $2q$ -vertex bipartite graphs become trivial². We do not attempt
 145 to optimize the dependence of the running time on q or Δ .

146 Given a set A , we write \mathcal{U}_A for the uniform distribution on A . When a fixed q is clear
 147 from context, let d_C be the Cayley metric on the symmetric group S_q . That is, $d_C(\rho, \rho')$ is
 148 the minimum number of transpositions which one must compose with ρ to turn it into ρ' .
 149 This is merely graph distance on the Cayley graph of S_q generated by the transpositions. We
 150 associate perfect matchings in a balanced bipartite graph $H = ([q] \sqcup [q], E)$ with permutations
 151 $\rho \in S_q$ where $(i, \rho(i)) \in E$ for each i .

152 Given two random variables X, Y defined on the discrete probability spaces (Ω_X, p_X)
 153 and (Ω_Y, p_Y) , a coupling of X, Y is a random variable $\gamma = (X', Y')$ defined on a probability
 154 space $(\Omega_X \times \Omega_Y, p_\gamma)$ such that X and X' have identical distributions and Y and Y' have
 155 identical distributions.

156 Given a Markov chain Z_t with state space Ω and transition matrix P , a coupling of Z_t is
 157 a Markov chain (X_t, Y_t) with state space $\Omega \times \Omega$ and transition matrix \widehat{P} satisfying

$$158 \quad \sum_{y' \in \Omega} \widehat{P}((x, y), (x', y')) = P(x, x'),$$

$$159 \quad \sum_{x' \in \Omega} \widehat{P}((x, y), (x', y')) = P(y, y').$$

160 That is, each coordinate of the coupling is a faithful copy of Z_t , though the transitions of the
 161 coordinates are not necessarily independent.

162 **► Lemma 2.** *There are constants C_1, C_2 such that the following hold. Suppose that $q \geq C_1\Delta$,
 163 and let $H = (V, E)$ be a bipartite graph in which $|V| = 2q$ and the bipartition is balanced.
 164 Suppose that H has minimum degree at least $q - \Delta$. Let $e \in E$ be an edge of H , let \mathcal{L} be the
 165 set of perfect matchings of H containing e , and let \mathcal{R} the set of perfect matchings of H not
 166 containing e . Then there is a coupling γ of $\mathcal{U}_\mathcal{R}$ and $\mathcal{U}_{\mathcal{L} \cup \mathcal{R}}$ such that*

$$167 \quad \mathbb{E}_{(\rho, \rho') \sim \gamma} [d_C(\rho, \rho')] \leq \frac{C_2 \Delta}{q}.$$

168 This lemma is the main bottleneck for improving the dependence of q on Δ in our main
 169 theorem. In particular, removing the factor of Δ would also improve Theorem 1 by a factor
 170 of Δ .

² Given a bipartite graph of constant order which contains a perfect matching, sampling a perfect matching uniformly at random can be done in constant time, e.g. by exhaustively enumerating the perfect matchings and selecting one at random.

171 3 Technical overview

172 We prove Theorem 1 by the standard Markov chain Monte Carlo approach and the path
 173 coupling technique of Bubley and Dyer [5]. We study an ergodic Markov chain—the heat-bath
 174 Glauber dynamics—whose stationary distribution is uniform on the list packings of a graph,
 175 and use path coupling to show rapid mixing. Then a well-known and generic reduction from
 176 counting to sampling yields Theorem 1. Path coupling reduces the potentially challenging
 177 task of proving rapid mixing of a Markov chain to designing a coupling on adjacent states
 178 according to some graph Γ on the state space Ω . We largely follow the notation of [12] and
 179 consider

180
$$\mathcal{M} := \{P^t p_0\}_{t=0}^{\infty},$$

181 an ergodic Markov chain on state space Ω with initial distribution p_0 and transition operator
 182 P . We denote the (unique) stationary distribution by π , and denote by $p_t := P^t p_0$ the
 183 distribution of the state of the chain \mathcal{M} after t steps. We use the standard notion of mixing
 184 time of Markov chains given by

185
$$t_{\text{mix}}(\epsilon) := \max_{p_0} \min \{t \geq 0 : d_{\text{TV}}(p_t, \pi) \leq \epsilon\},$$

186 where d_{TV} is total variation distance.

187 Typically, we write ω and ω' for states of the chain before a transition and $\sigma := P\omega$,
 188 $\sigma' := P\omega'$ for the (random) states after one step of the chain from ω and ω' respectively.

189 ▶ **Theorem 3** (Bubley and Dyer [5], see also [12]). *Let Ω be the state space of a Markov chain
 190 \mathcal{M} and let Γ be a weighted, directed graph on vertex set Ω with edge weights in \mathbb{N} . Let δ be the
 191 quasi-metric³ on Ω given by taking shortest paths in Γ , and suppose that $\delta(\omega, \omega') \leq D < \infty$
 192 for all $\omega, \omega' \in \Omega$.*

193 For each $(\omega, \omega') \in E(\Gamma)$, suppose that we have a coupling γ , of the random variables σ
 194 (with distribution $P\omega$) and σ' (with distribution $P\omega'$) for which $\mathbb{E}_{(\sigma, \sigma') \sim \gamma}[\delta(\sigma, \sigma')] \leq \beta \delta(\omega, \omega')$.
 195 Then if $\beta < 1$, we have $t_{\text{mix}}(\epsilon) \leq \log(D/\epsilon)/(1 - \beta)$.

196 The theorem may seem rather abstract, so we briefly discuss a well-known application
 197 to list coloring as a warm-up to the main argument for list packing. Let G be a graph of
 198 maximum degree Δ and let L be a q -list assignment of G . Let Ω be the set of L -colorings
 199 of G , and let \mathcal{M} be the heat-bath Glauber dynamics on Ω , defined as follows. Note that
 200 states $\omega \in \Omega$ are colorings and hence functions $V(G) \rightarrow \mathbb{N}$. A transition of \mathcal{M} from a state
 201 ω is performed by choosing a vertex $u \in V(G)$ uniformly at random, sampling a color c in
 202 $L(u) \setminus \omega(N(u))$, and moving to the state σ with $\sigma(u) = c$ and $\sigma(v) = \omega(v)$ for all $v \neq u$.
 203 That is, the transition operator P is defined via

- 204 1. sampling a vertex $u \sim \mathcal{U}_{V(G)}$,
 205 2. sampling a color $c \sim \mathcal{U}_{L(u) \setminus \omega(N(u))}$,
 206 3. defining σ by $\sigma(v) := \begin{cases} c & v = u \\ \omega(v) & v \neq u \end{cases}$, and moving to the state σ .

207 Note that $L(u) \setminus \omega(N(u))$ is the set of available colors for u . These colors are in the list $L(u)$
 208 but are not used by the coloring ω on the neighbors of u so setting $\omega(u)$ to one of them
 209 yields a valid list coloring. This chain is reversible, and when $q \geq \Delta + 2$ it is ergodic [18,
 210 Exercises 4.1] with stationary distribution uniform on Ω .

³ that is, a function which satisfies the conditions of a metric except symmetry

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211 For the purpose of constructing the couplings required by Theorem 3, let Γ be the
 212 weighted, directed graph on Ω where (ω, ω') is an edge if and only if the colorings ω, ω'
 213 differ at exactly one vertex and let all edge weights be 1. From the proof of ergodicity [18]
 214 it follows that we can take $D = (\Delta + 1)n$ in Theorem 3 and define couplings as follows.
 215 Let $(\omega, \omega') \in E(\Gamma)$ and suppose that the colorings ω and ω' differ at the vertex v^* . Sample
 216 $u \in V(G)$ uniformly at random and update the color of u in both chains (intuitively, this
 217 decision helps the chains coalesce). That is, the distribution γ of (σ, σ') is defined by the
 218 Markov transition $(\omega, \omega') \xrightarrow{P_\gamma} (\sigma, \sigma')$ itself defined by
 219 1. sampling a vertex $u \sim \mathcal{U}_{V(G)}$,
 220 2. sampling a pair of available colors (a, b) from a distribution γ_c such that

$$221 \quad \gamma_c := \arg \max_{\substack{\gamma' \text{ is a coupling of} \\ \mathcal{U}_{L(u) \setminus \omega(N(u))} \text{ and } \mathcal{U}_{L(u) \setminus \omega'(N(u))}}} \Pr_{(a,b) \sim \gamma'} (a = b) \quad (1)$$

222 3. defining σ and σ' by updating the color of u in each to a and b respectively:

$$223 \quad \sigma(v) := \begin{cases} a & v = u \\ \omega(v) & v \neq u \end{cases}, \quad \sigma'(v) := \begin{cases} b & v = u \\ \omega'(v) & v \neq u \end{cases},$$

224 and moving to the state (σ, σ') .

225 Observe that γ_c is defined as the coupling on the uniform distributions of available colors,
 226 $\mathcal{U}_{L(u) \setminus \omega(N(u))}$ and $\mathcal{U}_{L(u) \setminus \omega'(N(u))}$, which maximizes the probability the colors are the same.
 227 The important property of this definition in terms of applying Theorem 3 is that this coupling
 228 minimizes the expectation of the discrete metric on the sample.

229 If $u \in N(v^*)$ the sets of available colors for u can be different making γ_c nontrivial.
 230 In this case, let $C = \omega(N(u) \setminus \{v^*\})$ and note that the two sets of available colors are
 231 $A = (L(u) \setminus C) \setminus \{\omega(v^*)\}$ and $B = (L(u) \setminus C) \setminus \{\omega'(v^*)\}$. We wish to couple \mathcal{U}_A and \mathcal{U}_B such
 232 that the probability of choosing the same color is maximized, and the best general coupling
 233 is not too hard to find.

234 ▶ **Lemma 4** (See e.g., [18, Lemma 4.10]). *Let U be a finite set and $A, B \subset U$. Then there is
 235 a coupling γ_c of \mathcal{U}_A and \mathcal{U}_B such that*

$$236 \quad \Pr_{(a,b) \sim \gamma_c} (a = b) = \frac{|A \cap B|}{\max\{|A|, |B|\}},$$

237 where (a, b) is a random element of $A \times B$ chosen according to the coupling γ_c .

238 Let $m = \max\{|A|, |B|\}$. We have $A \cap B = (L(u) \setminus C) \setminus \{\omega(v^*), \omega'(v^*)\}$. Checking the
 239 four cases according to whether each of $\omega(v^*)$ and $\omega'(v^*)$ are in $L(u) \setminus C$, we observe that
 240 $|A \cap B| \geq m - 1$. By Lemma 4, we can ensure that the two chains choose the same color
 241 with probability at least $1 - 1/m$. Considering the definitions of A and B , we also have
 242 $m \geq q - \Delta$. Returning to the analysis of the coupling γ described above, we have

$$243 \quad \mathbb{E}_{(\sigma, \sigma') \sim \gamma} [\delta(\sigma, \sigma')] \leq 1 + \frac{1}{n} \left(-1 + \frac{\Delta}{q - \Delta} \right).$$

244 This comes from the facts that $\delta(\omega, \omega') = 1$ by assumption, the probability $1/n$ that we
 245 successfully reduce the distance by 1 in an update to v^* , and the probability of at most
 246 Δ/n that we choose to update a neighbor of v^* , and in this case fail to choose the same
 247 color in the coupling of the color choice in each chain given by Lemma 4 (which occurs

248 with probability at most $1/(q - \Delta)$ given that we update a neighbor of v^*). Set β to the
 249 right-hand side above and solve for $\beta < 1$ to obtain $q > 2\Delta$.

250 Well-known works that first studied this technique [17, 5, 18] give a similar proof, though
 251 Jerrum [17] manually constructed a coupling of the Markov chain and did not appeal to
 252 path coupling. With path coupling, it is slightly easier to study a variant of the Markov
 253 chain known as the “Metropolis Glauber dynamics” where the transition is defined slightly
 254 differently, though the same lower bound on q is required in the argument for this chain.

255 Our argument for list packing follows the same outline as above using the heat-bath
 256 dynamics for list packings, but it is much harder to construct the coupling. In particular,
 257 we need an analogue of Lemma 4 for the much more intricate combinatorial setting of list
 258 packings. We describe this in the next subsection.

259 3.1 Coupling perfect matchings

260 In the list packing setting, each spin is a permutation of a list. The central problem one
 261 faces when adapting the above sketch to list packing is the issue of coupling the choice of
 262 available permutations in the case that we are updating the spin of u in two copies of the
 263 Glauber dynamics which differ at a neighbor $v \in N(u)$. It turns out (see e.g. [6] and earlier
 264 works such as [24]) that there is an auxiliary bipartite graph in which available permutations
 265 for u correspond to perfect matchings.

266 Given a graph G with q -list assignment L , let Ω be the set of L -packings of G , and
 267 let $\omega \in \Omega$. Let f_1, \dots, f_q be the L -colorings represented by ω . To be explicit, suppose
 268 that for each vertex v of G we write $L(v)$ in ascending order as $c_{1,v}, \dots, c_{q,v}$. Then we
 269 identify $V(G)$ with $[n]$ and consider Ω as a subset of S_q^n such that with $v \in [n]$ we define
 270 $f_1, \dots, f_q : V(G) \rightarrow \mathbb{N}$ by $f_i(v) = c_{\omega_v(i),v}$, where we interpret ω_v as a permutation of $L(v)$ so
 271 that $\omega_v(i)$ is the color assigned to v in the i th coloring in the packing. This notation has
 272 the advantage that many of the dependencies are explicit, but to avoid a proliferation of
 273 subscripts we omit them where it is possible to fix some context in advance.

274 For a vertex $u \in V(G)$, we construct the *availability graph* $H_u = H_u(G, L, \omega)$ as follows.
 275 We consider the vertex set of H_u as $[q] \sqcup [q]$, the disjoint union of two copies of the set
 276 $[q]$. The left copy consists of “packing indices” and the right copy consists of “list indices”.
 277 For clarity, the edges of H_u are considered oriented from left to right so that (i, j) joins
 278 packing index i to list index j . Suppose that $L(u) = \{c_1, \dots, c_q\}$ is supplied in some fixed
 279 order. Then in H_u we include each edge $(i, j) \in [q]^2$ such that c_j is an available color for
 280 u in the coloring f_i . That is, (i, j) is an edge of H_u if and only if $c_j \notin f_i(N(u))$. One can
 281 check the definitions and observe that perfect matchings in H_u correspond to the available
 282 permutations $\rho \in S_q$ for u in the sense that setting ω_u to an available ρ yields a valid list
 283 packing. The heat-bath Glauber dynamics for L -packings thus works in much the same way
 284 as for L -colorings. The transition from a state ω is defined by choosing a vertex $u \in V(G)$
 285 uniformly at random, and then a perfect matching in $H_u(G, L, \omega)$ uniformly at random. It is
 286 straightforward to check that the chain is reversible and has uniform stationary distribution;
 287 we prove that it is ergodic (for large enough q) in Lemma 7.

288 An important consideration when G has maximum degree Δ is that H_u has minimum
 289 degree $q - \Delta$. This follows from the properties of a list packing: at packing index i any color
 290 that is not available must be used by f_i on a neighbor of u and there are at most Δ such
 291 neighbors. Similarly, for a color c_j with color index j , any packing index i in which c_j is
 292 not available is explained by c_j being used by f_i on $N(u)$. Since the colorings in a packing
 293 are pairwise-disjoint, each such index must be due to distinct neighbors of u , of which there
 294 are at most Δ . It is useful to observe that for any $\omega \in S_q^n$, even one that may not be a

proper list packing in the sense that the q list colorings it represents may not be proper, the definition of availability graph still makes sense and the observation on the minimum degree still applies. That is, the key property of q -list packings on graphs of maximum degree Δ that yields the minimum degree bound $q - \Delta$ is that the list colorings represented are pairwise-disjoint. This fact is convenient in the proof of Lemma 5.

To construct the coupling we consider the weighted, directed graph Γ on Ω where (ω, ω') is an edge if and only if the list packings ω and ω' differ at exactly one vertex. For the edge $(\omega, \omega') \in E(\Gamma)$, let v be the unique vertex at which the two packings differ, and assign weight $d_C(\omega_v, \omega'_v)$ to the edge. As with list coloring, the key computation is how much the expected distance changes for one step of the coupling in the case that we start at (ω, ω') and update a neighbor u of the unique vertex v at which ω and ω' differ. We prove the following result which plays the role of Lemma 4 in our proof.

► **Lemma 5.** *There is a universal constant C such that if $q > C\Delta^2$ the following holds.*

Let $(\omega, \omega') \in E(\Gamma)$ be an edge of weight ψ in Γ and let v be the unique vertex at which ω and ω' differ. Let $u \in N(v)$ and consider the availability graphs H and H' for u in the packings ω and ω' respectively.

Then there exists a coupling γ of the uniform distributions on perfect matchings on H and H' which satisfies

$$\mathbb{E}_{(\rho, \rho') \sim \gamma} [d_C(\rho, \rho')] \leq \frac{\psi}{2\Delta}.$$

Note that in the conclusion any bound strictly better than ψ/Δ suffices for an application of Theorem 3, and one might expect to obtain results in the case of exactly ψ/Δ as well [17, 5, 12]. We do not attempt to optimize the constant C in our argument and hence $\psi/(2\Delta)$ is sufficient. Lemma 5 is a simple corollary of Lemma 2, and with these results in hand the rest of the argument for Theorem 1 is standard.

3.2 Organization

In Section 4 we define a Markov chain on list packings, prove ergodicity and establish rapid mixing given our results on coupling perfect matchings. In Section 5 we prove Lemmas 2 and 5. We conclude with some remarks in Section 6.

4 Glauber dynamics for list packing

In this section we fix an n -vertex graph G of maximum degree Δ , a q -list assignment L of G , and let Ω be the set of L -packings of G .

We consider heat-bath Glauber dynamics $\mathcal{M} = \mathcal{M}(G, L)$ for list packing. Given a state $\omega \in \Omega$ and a vertex $u \in V(G)$, we say that a permutation ρ in S_q is *available* for u in ω if setting ω_u to ρ yields a valid list packing. Transitions of \mathcal{M} are defined as follows. From state $\omega \in \Omega$, choose vertex $u \in V(G)$ uniformly at random, choose an available permutation $\rho \in S_q$ uniformly at random, and let the new state be σ given by $\sigma_u = \rho$ and $\sigma_v = \omega_v$ for $v \neq u$. It is straightforward to check that this is a reversible Markov chain on Ω with uniform stationary distribution. The question of ergodicity is less straightforward, though the standard argument for list coloring adapts easily. We require a simple corollary of Hall's classic result on perfect matchings in bipartite graphs.

► **Lemma 6** (Corollary of Hall's theorem [14]). *Let H be a bipartite graph with q vertices on each side and minimum degree $d \geq q/2$. Then H contains a perfect matching.*

337 **Sketch proof.** Let the bipartition be $A \sqcup B$ and consider a subset $X \subset A$. If $1 \leq |X| \leq d$
338 then $|N(X)| \geq d \geq |X|$. If instead $d+1 \leq |X| \leq q$ then $N(X)$ must be all of B since for each
339 neighborhood of a vertex in B must intersect X . The result follows from Hall's theorem. \blacktriangleleft

340 **► Lemma 7.** *For $q \geq 2\Delta + 2$ the Markov chain \mathcal{M} is ergodic.*

341 **Proof.** Because there are self-transitions at every state, it suffices to show that the (finite)
342 state space is connected.

343 Every state $\omega \in \Omega$ can be connected to an arbitrary $\omega' \in \Omega$ as follows. Label the vertices
344 of G with the integers $1, \dots, n$ arbitrarily, and for $i = 1, \dots, n$, sequentially turn ω_i into ω'_i
345 as follows. If setting ω_i equal to ω'_i is not possible (i.e. ω'_i is not available for i in ω) then
346 it's because some neighbor j of i with $j > i$ uses a color at a particular packing index which
347 conflicts with ω_i . To solution is to repack each such neighbor j in ω , i.e. change ω_j to a new
348 permutation of $L(j)$, in turn such that the repacking is proper *and* avoids any such conflicts.
349 These repacking steps are steps of the chain.

350 To perform the repacking, we find a perfect matching in a suitable modification of the
351 availability graph $H'_j = H_j(G, L, \omega')$. Recall that perfect matchings in H'_j correspond to the
352 available permutations ρ for u in ω' . We have an additional condition on the permutation ρ
353 that we seek, namely that ρ does not correspond to a color choice for j that is incompatible
354 with turning ω_i into ω'_i . We can encode this in the availability graph by deleting an edge
355 (a, b) such that the b -th color in $L(j)$ is used at packing index a in color choices represented
356 by ω'_i . After this modification, the availability graph has minimum degree $q - \Delta - 1$, so the
357 condition $q \geq 2\Delta + 2$ allows for an application of Lemma 6. This shows that the necessary
358 repackings exist, and thus that Ω is connected by transitions of the chain that occur with
359 positive probability. \blacktriangleleft

360 **► Theorem 8.** *The mixing time $t_{\text{mix}}(\epsilon)$ of \mathcal{M} is at most $O(n \log(n/\epsilon))$.*

361 **Proof.** We apply Theorem 3 with the following coupling defined on edges of the weighted
362 graph Γ on Ω such that (ω, ω') is an edge of weight $d_C(\omega_v, \omega'_v)$ whenever ω and ω' differ at a
363 single vertex v . Lemma 7 shows that Γ is connected, and we note that the diameter D is at
364 most $(\Delta + 1)(q - 1)n = O(n)$. This is because the sequence of steps constructed in Lemma 7
365 consists of edges of Γ , the total number of steps is at most $(\Delta + 1)n$ because we repack each
366 vertex v at most once for each neighbor of v to avoid conflict and at most once more to agree
367 with ω' . As the Cayley distance on S_q takes values in $\{0, 1, \dots, q - 1\}$, the diameter bound
368 follows.

369 Let (ω, ω') be an edge of Γ of weight ψ . We define the coupling as follows. We choose
370 $u \in V(G)$ uniformly at random and update u in both packings. If $u \notin N(v)$ then the sets of
371 available permutations of $L(u)$ in both packings are identical and we choose one uniformly at
372 random to use in both chains. If $u = v$ this results in a distance of zero, else the distance is
373 unchanged. If $u \in N(v)$ then we use the coupling of Lemma 5 to choose the permutations of
374 $L(u)$ in the packings. Let (σ, σ') be the random state after one step of the coupling started
375 from (ω, ω') . Then

$$\mathbb{E}[\delta(\sigma, \sigma')] \leq \psi + \frac{1}{n} \left(-\psi + \Delta \frac{\psi}{2\Delta} \right) = \psi \left(1 - \frac{1}{2n} \right).$$

377 Theorem 3 now gives mixing time $t_{\text{mix}}(\epsilon) = O(n \log(n/\epsilon))$. \blacktriangleleft

378 The proof of Theorem 1 using Theorem 8 is now entirely standard. We sketch the
379 argument here. Counting list packings is a *self-reducible* problem in the sense of [19] and

380 so having an almost-uniform sampler, which follows from running the Markov chain for
 381 polynomially many steps, is equivalent to having an FPRAS. Concretely, construct a sequence
 382 $G = G_m \supset \dots \supset G_1 \supset G_0 = (V(G), \emptyset)$ of graphs by starting from G and removing an
 383 arbitrary edge to form the next member of the sequence. For a fixed q -list assignment L of
 384 G we can let Ω_i be the set of L -packings of G_i and write

$$385 \quad |\Omega_m| = |\Omega_0| \prod_{i=1}^m \frac{|\Omega_i|}{|\Omega_{i-1}|}.$$

386 We have $|\Omega_0| = (q!)^n$ and can estimate each ratio in the product as it's the probability
 387 that when we choose a uniformly random $\omega \in \Omega_{i-1}$ we have $\omega \in \Omega_i$. We also note that
 388 $\frac{1}{1+q!} \leq \frac{|\Omega_i|}{|\Omega_{i-1}|} \leq 1$ because $\Omega_i \subset \Omega_{i-1}$ and we can construct the following bipartite graph B
 389 on $(\Omega_{i-1} \setminus \Omega_i) \sqcup \Omega_i$. Let uv be the edge removed from G_i to form G_{i-1} , and include the edge
 390 (ω, ω') in B if ω' can be obtained from Ω by permuting the list of u . Then any $\omega \in \Omega_{i-1} \setminus \Omega_i$
 391 is connected to at least one element in Ω_i by Lemma 6, and there are at most $q!$ ways to
 392 permute $L(u)$ so from the other side the degrees are at most $q!$. We do not attempt to
 393 optimize this argument; more intricate arguments yield stronger lower bounds, but we are
 394 merely interested in a bound independent of n . This observation lets us repeat the analysis
 395 of Jerrum [17] for the case of colorings in the setting of list packings. Briefly, we use multiple
 396 copies of the almost uniform sampler offered by the Markov chain to estimate each ratio. To
 397 make this work, one has to bound the variance of the estimator for each ratio and manage
 398 the overall error with Chebyshev's inequality, but this is standard. See e.g., [17].

399 5 Coupling matchings

400 In this section we prove Lemmas 2 and 5. We first collect some results on auxiliary bipartite
 401 graphs.

402 ▶ **Lemma 9.** *Let $G = (V, E)$ be a bipartite graph with bipartition $V = L \cup R$. Let μ_L and
 403 μ_R be probability distributions over L and R . Let*

$$404 \quad p = \min_{A \subseteq L} [1 - \mu_L(A) + \mu_R(N(A))].$$

405 *There exists a coupling γ of μ_L and μ_R such that when $(v, w) \sim \gamma$, with probability p , vw is
 406 an edge of G .*

407 **Proof.** This follows from the max-flow min-cut theorem after introducing a source vertex s ,
 408 a sink vertex t , and introducing the following edges:

- 409 1. for $v \in L$, the edge sv with capacity $\mu_L(v)$,
- 410 2. for $v \in R$, the edge vt with capacity $\mu_R(v)$,
- 411 3. for each edge $e \in G$, an edge with capacity ∞ .

412 A max flow corresponds to a coupling γ of the type we require with maximum probability
 413 that vw is an edge. The value of the max flow is the same as the value of the min cut. If S, T
 414 is a finite cut with $s \in S, t \in T$, letting $A = S \cap L, B = S \cap R$, we must have $N(A) \subseteq B$,
 415 and the value is $\mu_L(A^c) + \mu_R(B) \geq \mu_L(A^c) + \mu_R(N(A))$. Equality is achieved for $B = N(A)$.
 416 Taking the minimum over A then gives the lemma. ◀

417 ▶ **Corollary 10.** *Let $G = (V, E)$ be a bipartite graph with bipartition $V = L \cup R$. Suppose that
 418 each vertex in L has degree contained in $[m_L, M_L]$ and each vertex in R has degree contained
 419 in $[m_R, M_R]$. Then there exists a coupling γ of the uniform distributions \mathcal{U}_L and \mathcal{U}_R on L
 420 and R respectively such that when $(v, w) \sim \gamma$, with probability at least $\frac{m_L m_R}{M_L M_R}$, vw is an edge
 421 of G .*

422 **Proof.** For each subset $A \subseteq L$, we can bound the number of edges between A and $N(A)$:

423
$$m_L|A| \leq |E(A, N(A))| \leq M_R|N(A)|.$$

424 We also have

425
$$m_R|R| \leq |E| \leq M_L|L|.$$

426 Then for any set $A \subseteq L$,

427
$$\frac{|N(A)|/|R|}{|A|/|L|} = \frac{|N(A)|}{|A|} \cdot \frac{|L|}{|R|} \geq \frac{m_L m_R}{M_L M_R}.$$

428 Then for any such A ,

429
$$1 - \mathcal{U}_L(A) + \mathcal{U}_R(N(A)) \geq 1 - \mathcal{U}_L(A) + \frac{m_L m_R}{M_L M_R} \cdot \mathcal{U}_L(A) \geq \frac{m_L m_R}{M_L M_R}.$$

430 The conclusion follows from Lemma 9. ◀

431 We are now ready to prove Lemma 2, which we restate for convenience.

432 **► Lemma 2.** *There are constants C_1, C_2 such that the following hold. Suppose that $q \geq C_1\Delta$,
433 and let $H = (V, E)$ be a bipartite graph in which $|V| = 2q$ and the bipartition is balanced.
434 Suppose that H has minimum degree at least $q - \Delta$. Let $e \in E$ be an edge of H , let \mathcal{L} be the
435 set of perfect matchings of H containing e , and let \mathcal{R} the set of perfect matchings of H not
436 containing e . Then there is a coupling γ of $\mathcal{U}_\mathcal{R}$ and $\mathcal{U}_{\mathcal{L} \cup \mathcal{R}}$ such that*

437
$$\mathbb{E}_{(\rho, \rho') \sim \gamma} [d_C(\rho, \rho')] \leq \frac{C_2 \Delta}{q}.$$

438 **Proof.** Label the bipartitions by $[q]$. Without loss of generality the edge e is $(1, 1)$. We
439 define an auxiliary bipartite graph on $\mathcal{L} \cup \mathcal{R}$ as follows. For $\rho \in \mathcal{L}$ and $\rho' \in \mathcal{R}$, connect ρ and
440 ρ' by an edge if $\rho(1 \ i \ j) = \rho'$ for some $i, j \in [q] \setminus \{1\}$ with $i \neq j$, i.e., they differ by a 3-cycle
441 containing 1. Note that here we use standard group-theoretic notation for permutations in
442 S_q so that e.g. $\rho(1 \ i \ j)$ is the product of the permutation ρ and the permutation $(1 \ i \ j)$ (in
443 standard cycle notation).

444 We bound the degrees of arbitrary matchings $\rho \in \mathcal{L}$ and $\rho' \in \mathcal{R}$. Given ρ , we count the
445 number of i, j for which $\rho(1 \ i \ j) \in \mathcal{R}$. Let $N(i) = \{j : (i, j) \in E\}$ and $N'(j) = \{i : (i, j) \in E\}$.
446 Given $\rho \in \mathcal{L}$, we know that $\rho(1) = 1$. We choose $j \in N'(1) \setminus \{1\}$; there are at least
447 $q - \Delta - 1$ choices. Any possible value of i must be in the set $S_- \setminus \{j\}$, where $S_- :=$
448 $\{\rho^{-1}(k) : k \in N(1) \setminus \{1\}\}$; $S_- \setminus \{j\}$ has size at least $q - \Delta - 2$. A valid pair (i, j) is exactly
449 one where $j \in N'(1) \setminus \{1\}$, $i \in S_- \setminus \{j\}$, and $(i, \rho(j)) \in E$. Since at most $\Delta + 1$ of edges
450 $(i, \rho(j))$, $j \in N'(1) \setminus \{1\}$ can land outside E , at least $q - 2\Delta - 3$ of these edges are valid,
451 i.e., there are at least $(q - \Delta - 1)(q - 2\Delta - 3)$ valid choices of (i, j) . There are at most
452 $(q - 1)(q - 2)$ choices.

453 Next, given $\rho' \in \mathcal{R}$, we count the number of $i, j \in [q] \setminus \{1\}$, $i \neq j$ for which $\rho' = \rho(1 \ i \ j)$
454 where $\rho \in \mathcal{L}$. First, note we must have $\rho'(j) = 1$. The requirement on i is that $i \in N'(\rho'(1))$
455 and $\rho'(i) \in N(j)$. There are at least $q - \Delta - 2$ indices besides 1 and j satisfying each
456 condition, so at least $q - 2\Delta - 4$ possible indices. There are at most $q - 2$ choices. By
457 Corollary 10, there is a coupling γ of $\mathcal{U}_\mathcal{L}$ and $\mathcal{U}_\mathcal{R}$ such that with probability at least

458
$$p := \frac{(q - \Delta - 1)(q - 2\Delta - 3)}{(q - 1)(q - 2)} \cdot \frac{q - 2\Delta - 4}{q - 2},$$

459 $(\rho, \rho') \in E$ are connected by an edge and hence have Cayley distance at most two. For
 460 appropriate choices of $C_1, K > 0$, $1 - p \leq K\Delta/q$ for $q \geq C_1\Delta$. For this coupling γ , we hence
 461 have

$$462 \quad \mathbb{E}_{(\rho, \rho') \sim \gamma}[d_C(\rho, \rho')] \leq \frac{K\Delta}{q} \cdot q + \left(1 - \frac{K\Delta}{q}\right) \cdot 2 \leq K'\Delta$$

463 for an appropriate constant $K' > 0$.

464 Next, note that we can define a coupling between $\mathcal{U}_{\mathcal{L} \cup \mathcal{R}}$ and $\mathcal{U}_{\mathcal{R}}$ as a mixture of the
 465 identity coupling between $\mathcal{U}_{\mathcal{R}}$ and $\mathcal{U}_{\mathcal{R}}$ (with probability $\frac{|\mathcal{R}|}{|\mathcal{L}|+|\mathcal{R}|}$) and the above coupling
 466 (with probability $\frac{|\mathcal{L}|}{|\mathcal{L}|+|\mathcal{R}|}$). The expected distance for this coupling is then

$$467 \quad \frac{|\mathcal{L}|}{|\mathcal{L}|+|\mathcal{R}|} \cdot K'\Delta \leq \frac{q-2}{(q-2)+(q-\Delta-1)(q-2\Delta-3)} \cdot K'\Delta \leq \frac{C_2\Delta}{q}$$

468 for appropriate $C_2 > 0$, as needed. \blacktriangleleft

469 \blacktriangleright **Lemma 5.** *There is a universal constant C such that if $q > C\Delta^2$ the following holds.*

470 Let $(\omega, \omega') \in E(\Gamma)$ be an edge of weight ψ in Γ and let v be the unique vertex at which
 471 ω and ω' differ. Let $u \in N(v)$ and consider the availability graphs H and H' for u in the
 472 packings ω and ω' respectively.

473 Then there exists a coupling γ of the uniform distributions on perfect matchings on H
 474 and H' which satisfies

$$475 \quad \mathbb{E}_{(\rho, \rho') \sim \gamma}[d_C(\rho, \rho')] \leq \frac{\psi}{2\Delta}.$$

476 **Proof.** Formally, we proceed by induction on ψ , though we need a more general statement
 477 for the induction hypothesis. It is convenient to relax the requirement that ω and ω' are valid
 478 list packings. For the application, it is important that the pair (ω, ω') are valid list packings
 479 which agree on every vertex except u , but we construct the coupling of perfect matchings
 480 in H and H' using a sequence of *near-valid* list packings $\omega'' \in S_q^n$ in the sense that ω''
 481 agrees with ω and ω' on all vertices except u , but we allow the (pairwise-disjoint) colorings
 482 it represents to have monochromatic edges incident to v . We can still construct availability
 483 graphs for u in these near-valid packings and consider their sets of perfect matchings for the
 484 purposes of constructing an eventual coupling of the perfect matchings in H and H' . With
 485 these definitions in place, the generalization that we prove by induction is the statement
 486 obtained by replacing the assumption that ω and ω' are valid packings with the assumption
 487 that they are near-valid.

488 The base case is $\psi = 0$ in which the trivial coupling suffices as H and H' , and hence their
 489 sets of perfect matchings, are identical.

490 The induction step follows from Lemma 2. The fact that $d_C(\omega_v, \omega'_v) = \psi$ means that
 491 there is a sequence of transpositions τ_1, \dots, τ_ψ such that $\omega_v = \tau_\psi \cdots \tau_1 \omega'_v$. Let H'' be the
 492 availability graph of the vertex u in the near-valid packing ω'' such that $\omega''_v = \tau_\psi \omega_v$ and
 493 ω'' agrees with ω on all other vertices. By induction, there is a coupling γ' of the uniform
 494 distributions on perfect matchings in H'' and H' such that

$$495 \quad \mathbb{E}_{(\rho'', \rho') \sim \gamma'}[d_C(\rho'', \rho')] \leq \frac{\psi-1}{2\Delta}.$$

496 Without loss of generality, suppose that ω_v is the identity. Let $\tau_\psi = (i\ j)$, and note that
 497 this gives

$$498 \quad E(H) \setminus E(H'') \subset \{(i, j), (j, i)\} \quad \text{and} \quad E(H'') \setminus E(H) \subset \{(i, i), (j, j)\}.$$

499 This is because any difference between the edges of H and H' is explained by applying τ to
500 ω_v . When τ is a transposition, we swap the packing index of the coloring at which two colors
501 in the list of v are used, which can swap two edges of the complement of the availability
502 graph. It can be the case that the swapped color indices refer to different colors in $L(u)$
503 which is why we do not have equality.

504 We start with the case that

$$505 \quad E(H) \setminus E(H'') = \{(i, j), (j, i)\} \quad \text{and} \quad E(H'') \setminus E(H) = \{(i, i), (j, j)\},$$

506 the other cases are similar. Let X be the set of perfect matchings in H and let X'' be the
507 set of perfect matchings in H'' . We seek a coupling of \mathcal{U}_X and $\mathcal{U}_{X''}$ which we will combine
508 with the coupling γ' to obtain the desired result.

509 We apply Lemma 2 to H and $H - (i, j)$, yielding a coupling γ_1 of \mathcal{U}_X and \mathcal{U}_Y , where
510 Y is the set of perfect matchings in $H - (i, j)$. We can apply Lemma 2 again to $H - (i, j)$
511 and $H - (i, j) - (j, i)$, yielding a coupling γ_2 of \mathcal{U}_Y and \mathcal{U}_Z , where Z is the set of perfect
512 matchings in $H - (i, j) - (j, i)$. There is a slight technicality here as the minimum degree of
513 $H - (i, j)$ is $q - \Delta - 1$, but this can be handled by setting Δ to $\Delta + 1$ and adjusting the
514 constants slightly. Analogously, starting from H'' , we construct a coupling γ''_1 of $\mathcal{U}_{X''}$ and
515 $\mathcal{U}_{Y''}$ and a coupling γ''_2 of $\mathcal{U}_{Y''}$ and \mathcal{U}_Z , where Y'' is the set of perfect matchings in $H'' - (i, i)$.
516 A careful composition of these couplings gives the result. The composition of these couplings
517 yields a distribution on $X \times Y \times Z \times Y'' \times X''$ which is uniform on each individual set in the
518 Cartesian product, and such that the expected distance between permutations from adjacent
519 sets in the Cartesian product is at most $C_2\Delta/q$. Taking the first and last coordinate yields a
520 coupling γ'' of \mathcal{U}_X and $\mathcal{U}_{X''}$ such that

$$521 \quad \mathbb{E}_{(\pi, \pi'') \sim \gamma''} [d_C(\pi, \pi'')] < \frac{4C_2\Delta}{q}.$$

522 This can be combined with γ' obtained by induction in the same way. Simple composition
523 yields a distribution on $X \times X'' \times X'$ which is uniform on each individual set in the Cartesian
524 product. Taking the first and last coordinates we have a coupling γ of \mathcal{U}_X and $\mathcal{U}_{X'}$ such that

$$525 \quad \mathbb{E}_{(\pi, \pi') \sim \gamma} [d_C(\pi, \pi')] < \frac{\psi - 1}{2\Delta} + \frac{4C_2\Delta}{q}.$$

526 Since we assume $q \geq C\Delta^2$, for a large enough C this is at most $\psi/(2\Delta)$ as required. The
527 other cases proceed similarly, but require fewer applications of Lemma 2 and yield a stronger
528 upper bound. \blacktriangleleft

529 6 Concluding remarks

530 Many natural questions remain unanswered. We have chosen to extend some of the most
531 fundamental techniques for counting list colorings to list packings, but there are many more
532 recent improvements to consider. Dyer and Greenhill [11] study a Markov chain on (list)
533 colorings whose transitions are defined by properly recoloring both endpoints of a uniform
534 random edge and show that its mixing time is less than that of Glauber dynamics studied
535 in [17, 25]. The flip dynamics employed by Vigoda [26] for counting colorings is an important
536 technique that gets significantly below the number of colors required by Jerrum's approach.
537 Extending Vigoda's approach to list *coloring* was first done by Chen et al. [10], roughly 20
538 years after Vigoda's breakthrough. Though they also surpassed a significant barrier at $11\Delta/6$
539 colors related to 1-step contractions in Hamming distance of two colorings that differ at a

540 vertex. The history of this problem suggests that further generalizations, e.g. to list packing
 541 may not be straightforward. While one can consider analogous dynamics for list packings and
 542 hope to reduce the bound on q in Theorem 1, we do not have results showing the existence
 543 of list packings in graphs of maximum degree Δ for fewer than $2\Delta - 2$ colors. In another
 544 direction, the use of more advanced Markov chain techniques to give perfect sampling of list
 545 packings could be interesting.

546 We finish with a natural conjecture on approximately counting list packings.

547 ▶ **Conjecture 11.** *For each Δ and $q \geq 2\Delta$ there is an FPRAS for counting the number of
 548 q -list packings of graphs of maximum degree Δ .*

549 At the time of writing, we know of no reason that the lower bound on q cannot be reduced
 550 to, say, $\Delta + 1$. The value 2Δ represents a significant barrier in the sense that the existence of
 551 a list packing when $q \geq 2\Delta$ is an elementary consequence of Hall's theorem (though arguably
 552 not entirely trivial).

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