

Impossibility of latent inner product recovery via rate distortion

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Abstract—In a random geometric graph model or latent space model, the observed graph A on n vertices with average edge density p is assumed to be generated from latent locations z_1, \dots, z_n in \mathbb{R}^d associated with the n vertices. Given the graph A , it is of interest to estimate the inner products $\langle z_i, z_j \rangle$ which represents the geometry of the latent locations. In this note, assuming that the latent locations are Gaussian or spherical points, we show an impossibility result for inner product recovery when $d \gtrsim nh(p)$ where $h(p)$ is the binary entropy function. This matches the condition required for positive results on inner product recovery in the literature. The main technical ingredient of this work is a lower bound on the rate-distortion function of the Wishart distribution which is interesting in its own right.

Index Terms—Random geometric graphs, rate-distortion function, Wishart distribution

I. INTRODUCTION

Random graphs with latent geometric structures comprise an important class of network models used across a broad range of fields [6], [19], [26]. In a typical formulation of such a model, each vertex of a graph on n vertices is assumed to be associated with a latent location $z_i \in \mathbb{R}^d$ where $i = 1, \dots, n$. With $A \in \{0, 1\}^{n \times n}$ denoting the adjacency matrix of the graph, each edge A_{ij} follows the Bernoulli distribution with probability parameter $\kappa(z_i, z_j)$, where $\kappa : \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, 1]$ is a kernel function. In other words, the edges of the graph are formed according to the geometric locations of the vertices in a latent space. Given the graph A , the central question is then to recover the latent geometry, formulated as estimating the inner products $\langle z_i, z_j \rangle$ ¹.

In the study of this class of random graphs, a Gaussian or spherical prior is often imposed on the latent locations z_1, \dots, z_n , including in the early work on latent space models [17]–[20] and in the more recent work on random geometric graphs [3], [16], [23]. In particular, the isotropic spherical or Gaussian prior allows the latter line of work to use the theory of spherical harmonics to analyze spectral methods for estimating the latent inner products. For a class of kernels

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¹One can also formulate the problem as estimating the pairwise distances $\{\|z_i - z_j\|_2\}_{i,j=1}^n$ which is essentially equivalent to inner product estimation. The problem is not formulated as estimating the latent locations $\{z_i\}_{i=1}^n$ themselves, because the kernel function κ is typically invariant under an orthogonal transformation of z_1, \dots, z_n , making them non-identifiable.

including the step function $\kappa(z_i, z_j) = \mathbb{1}\{\langle z_i, z_j \rangle \geq \tau\}$ for a threshold τ , it is known (see Theorem 1.4 of [23]) that the inner products can be estimated consistently if $d \ll nh(p)$ where p is the average edge density of the graph and $h(p)$ is the binary entropy function. However, a matching negative result was not known (as remarked in Section 1.3 of [23]).

In this note, we close this gap by proving in Corollary 3 that it is information-theoretically impossible to recover the inner products in a random geometric graph model if $d \gtrsim nh(p)$, thereby showing that $d \asymp nh(p)$ is indeed the recovery threshold². In fact, it is not difficult to guess this negative result from entropy counting: It is impossible to recover n vectors in dimension d from $\binom{n}{2}$ binary observations with average bias p if $nd \gtrsim \binom{n}{2}h(p)$ since there is not sufficient entropy. And this argument does not rely on the specific model for generating the random graph A .

To formalize the entropy counting argument, we use a general approach from the rate-distortion theory [29]. See also [13], [27] for a modern introduction. The key step in this approach is a lower bound on the rate-distortion function of the estimand, i.e., $X \in \mathbb{R}^{n \times n}$ with $X_{ij} := \langle z_i, z_j \rangle$ in our case. If z_1, \dots, z_n are isotropic Gaussian vectors, then X follows the Wishart distribution. Therefore, our main technical work lies in estimating the rate-distortion function for the Wishart distribution (and its variant when z_1, \dots, z_n are on a sphere), which has not been done in the literature to the best of our knowledge. See Theorem 2.

The technical problem we solve is closely related to a work [22] on low-rank matrix estimation. To be more precise, Theorem VIII.17 of [22] proves a lower bound on the rate-distortion function of a rank- d matrix $X = ZZ^\top$ where $Z \in \mathbb{R}^{n \times d}$. Our proof partly follows the proof of this result but differs from it in two ways: First, the result of [22] assumes that Z is uniformly distributed on the Stiefel manifold, i.e., the columns of Z are orthonormal, while we assume that Z has i.i.d. Gaussian or spherical rows. Without the simplification from the orthonormality assumption, our

²Another related statistical problem is testing a random geometric graph model against an Erdős-Rényi graph model with the same average edge density [10]. This testing threshold, or detection threshold, is conjectured to be $d \asymp (nh(p))^3$, and the lower bound is still largely open. See [9], [10], [24].

proof requires different linear algebraic technicalities. Second, the result of [22] focuses on $d \leq n$, while we also consider the case $d > n$ which requires a completely different proof.

Finally, as a byproduct of our bound on the rate-distortion function of X , we show in Corollary 4 an impossibility result for one-bit matrix completion. While one-bit matrix completion has been studied extensively in the literature [7], [11], [14], less is known for the Bayesian model where a prior is assumed on the matrix X to be estimated [12], [25]. Similar to inner product estimation from a random geometric graph, the goal of one-bit matrix completion is to estimate a (typically low-rank) matrix X from a set of binary observations. It is therefore plausible that many techniques for random graphs can be used for one-bit matrix completion, and vice versa. Our result provides such an example.

II. MAIN RESULTS

In this section, we study the rate-distortion function for the Wishart distribution and its spherical variant. Let $I(X; Y)$ denote the mutual information between random variables X and Y . The rate-distortion function is defined as follows (see Part V of [27]).

Definition 1 (Rate-distortion function). *Let X be a random variable taking values in \mathbb{R}^ℓ , and let $P_{Y|X}$ be a conditional distribution on \mathbb{R}^ℓ given X . Let L be a distortion measure (or a loss function), i.e., a bivariate function $L : \mathbb{R}^\ell \times \mathbb{R}^\ell \rightarrow \mathbb{R}_{\geq 0}$. For $D > 0$, the rate-distortion function of X with respect to L is defined as*

$$R_X^L(D) := \inf_{P_{Y|X} : \mathbb{E}L(X, Y) \leq D} I(X; Y).$$

Our main technical result is the following lower bound on the rate-distortion function of a Wishart matrix.

Theorem 2 (Rate-distortion function of a Wishart matrix). *For positive integers n and d , let $Z := [z_1 \dots z_n]^\top \in \mathbb{R}^{n \times d}$ where the i.i.d. rows z_1, \dots, z_n follow either the Gaussian distribution $\mathcal{N}(0, \frac{1}{d} I_d)$ or the uniform distribution on the unit sphere $\mathcal{S}^{d-1} \subset \mathbb{R}^d$. Let $X := ZZ^\top$. Define a loss function³*

$$L(X, \hat{X}) := \frac{d}{n(n+1)} \|X - \hat{X}\|_F^2. \quad (1)$$

Let $n \wedge d := \min\{n, d\}$. There is an absolute constant $c > 0$ such that for any $D \in (0, c)$, we have

$$R_X^L(D) \geq cn(n \wedge d) \log \frac{1}{D}.$$

For $d < n$, the $n \times n$ matrix X is rank-deficient and is a function of $Z \in \mathbb{R}^{n \times d}$, so we expect the order nd for the rate-distortion function; for $d \geq n$, we expect the order n^2 considering the size of X . The matching upper bound on the rate-distortion function can be obtained using a similar argument as that in Section III-A for small d and through

³The normalization in the definition of L is chosen so that the trivial estimator $\mathbb{E}X = I_n$ of X has risk $\mathbb{E}L(X, \hat{X}) = 1$ in the case of Gaussian z_i , since $\mathbb{E}[X_{ij}^2] = \mathbb{E}[\langle z_i, z_j \rangle^2] = 1/d$ for $i \neq j$ and $\mathbb{E}[(X_{ii} - 1)^2] = \mathbb{E}[(\langle z_i, z_i \rangle - 1)^2] = 2/d$.

a comparison with the Gaussian distribution for large d (see Theorem 26.3 of [27]). Since it is in principle easier to obtain the upper bound and only the lower bound will be used in the downstream statistical applications, we do not state it here. Moreover, at the end of this section, we discuss the best possible constant c in the above lower bound. The bulk of the paper, Section III, will be devoted to proving Theorem 2. With this theorem in hand, we first establish corollaries for two statistical models via entropy counting.

Corollary 3 (Random geometric graph or latent space model). *Fix positive integers n, d and a parameter $p \in (0, 1)$. Suppose that we observe a random graph on n vertices with adjacency matrix A with average edge density p , i.e., $\sum_{(i,j) \in \binom{[n]}{2}} \mathbb{E}[A_{ij}] = \binom{n}{2}p$. Suppose that A is generated according to an arbitrary model from the latent vectors z_1, \dots, z_n given in Theorem 2, and the goal is to estimate the inner products $X_{ij} := \langle z_i, z_j \rangle$ in the norm L defined in (1). If $d \geq cnh(p)$ where $c > 0$ is any absolute constant and $h(p) := -p \log p - (1-p) \log(1-p)$ is the binary entropy function, then for any estimator \hat{X} measurable with respect to A , we have $\mathbb{E}L(X, \hat{X}) \geq D$ for a constant $D = D(c) > 0$.*

Proof. The estimand X , the observation A , and the estimator \hat{X} form a Markov chain $X \rightarrow A \rightarrow \hat{X}$. By the data processing inequality, we have

$$I(X; \hat{X}) \leq I(A; \hat{X}) \leq H(A),$$

where $H(A)$ denotes the entropy of A . Since $\sum_{(i,j) \in \binom{[n]}{2}} \mathbb{E}[A_{ij}] = \binom{n}{2}p$, by the maximum entropy under the Hamming weight constraint (see Exercise I.7 of [27]), we get

$$H(A) \leq \binom{n}{2}h(p).$$

If $\mathbb{E}L(X, \hat{X}) \leq D$, then combining the above inequalities with Theorem 2 gives

$$cn(n \wedge d) \log \frac{1}{D} \leq R_X^L(D) \leq I(X, \hat{X}) \leq \binom{n}{2}h(p).$$

Taking $D > 0$ to be a sufficiently small constant, we then get $n \wedge d < cnh(p)$, i.e., $d < cnh(p)$. \square

As a second application of Theorem 2, we consider one-bit matrix completion with a Wishart prior.

Corollary 4 (One-bit matrix completion). *Fix positive integers n, d and a parameter $p \in (0, 1)$. Suppose that $X \in \mathbb{R}^{n \times n}$ is a rank- d matrix to be estimated. Assume the prior distribution of X as given in Theorem 2. For each entry $(i, j) \in [n]^2$, suppose that with probability p_{ij} , we have a one-bit observation $A_{ij} \in \{0, 1\}$ according to an arbitrary model, and with probability $1 - p_{ij}$, we do not have an observation, denoted as $A_{ij} = *$. Let p be the average probability of observations, i.e., $\sum_{i,j=1}^n p_{ij} = n^2p$. Let L be the loss function defined in (1). If $d \geq cn(h(p) + p)$ where $c > 0$ is any absolute constant and $h(p) := -p \log p - (1-p) \log(1-p)$, then for any estimator*

\hat{X} measurable with respect to A , we have $\mathbb{E}L(X, \hat{X}) \geq D$ for a constant $D = D(c) > 0$.

Proof. The argument is the same as the proof of Corollary 3, except the bound on the entropy of A . Let $Z \in \{0, 1\}^{n \times n}$ have Bernoulli(p_{ij}) entries such that $Z_{ij} = \mathbb{1}\{A_{ij} \neq *\}$. Then we have the conditional entropy $H(Z | A) = 0$. Conditional on any value of Z , the entropy of A is at most $\log 2^{\|Z\|_1}$. As a result,

$$H(A | Z) \leq \mathbb{E}_Z \log 2^{\|Z\|_1} = n^2 p \log 2.$$

We therefore obtain

$$\begin{aligned} H(A) &= H(A | Z) + I(Z; A) = H(A | Z) + H(Z) \\ &\leq n^2(h(p) + p \log 2). \end{aligned}$$

The rest of the proof is the same as that for the random geometric graph model. \square

Open problems. Several interesting problems are left open.

- *Sharp constant:* Recall that the lower bound on the rate-distortion function of the Wishart distribution in Theorem 2. While the order $n(n \wedge d) \log \frac{1}{D}$ is believed to be optimal, we did not attempt to obtain the sharp constant factor. In the case $d \geq n$, the rate-distortion function can be bounded from above by that of a Gaussian Wigner matrix, and the best leading constant is $1/4$ (see Theorems 26.2 and 26.3 of [27]). Indeed, the end result of Section III-B indeed shows a lower bound with the constant $1/4$ in the leading term if $D \rightarrow 0$. In the case $d/n \rightarrow 0$, Lemma 7 suggests that the best constant may be $1/2$, but we did not make the effort to obtain it as the end result. The most difficult situation appears to be when $d < n = O(d)$, in which case our techniques fail to obtain any meaningful constant factor.
- *Optimal rate:* Combined with the work [23], our result gives the recovery threshold $d \asymp nh(p)$ for random geometric graphs with Gaussian or spherical latent locations. However, it remains open to obtain an optimal lower bound on $\mathbb{E}L(X, \hat{X})$ as a function of d, n, p in the regime $d \ll nh(p)$. We believe the simple approach of entropy counting is not sufficient for obtaining the optimal rate and new tools need to be developed.
- *General latent distribution:* Existing positive and negative results for estimation in random geometric graph models are mostly limited to isotropic distributions of latent locations, such as Gaussian or spherical in [3], [16], [23] and this work. It is interesting to extend these results to more general distributions and metric spaces; see [4], [5] for recent work. Even for random geometric graphs with anisotropic Gaussian latent points, while there has been progress on the detection problem [8], [15], extending the recovery results to the anisotropic case remains largely open.

III. PROOF OF THEOREM 2

Let $c^* \in (0, 1)$ be some absolute constant to be determined later. We first consider the Gaussian model where

$z_i \sim \mathcal{N}(0, \frac{1}{d} I_d)$. The proof is split into three cases $d \leq c^*n$, $d \geq n$, and $c^*n < d < n$, proved in Sections III-A, III-B, and III-C respectively. We then consider the spherical model in Section III-D.

A. Case $d \leq c^*n$

To study the rate-distortion function of $X = ZZ^\top$, we connect it to the rate-distortion function of Z in the distortion measure to be defined in (2). The strategy is inspired by [22], but the key lemma connecting the distortion of X to that of Z is different. For $Z, \hat{Z} \in \mathbb{R}^{d \times d}$, define a loss function for recovering Z up to an orthogonal transformation

$$\ell(Z, \hat{Z}) := \frac{1}{n} \inf_{O \in \mathcal{O}(d)} \|Z - \hat{Z}O\|_F^2, \quad (2)$$

where $\mathcal{O}(d)$ denotes the orthogonal group in dimension d . The normalization is chosen so that $\mathbb{E}\ell(Z, EZ) = \mathbb{E}\ell(Z, 0) = 1$. We start with a basic linear algebra result.

Lemma 5. *Let $A, B \in \mathbb{R}^{n \times d}$. For the loss functions L and ℓ defined by (1) and (2) respectively, we have*

$$\ell(A, B) \leq \sqrt{\frac{n+1}{n} L(AA^\top, BB^\top)}.$$

Proof. Consider the polar decompositions $A = (AA^\top)^{1/2}U$ and $B = (BB^\top)^{1/2}V$ where $U, V \in \mathcal{O}(d)$. Then we have

$$\begin{aligned} \ell(A, B) &= \frac{1}{n} \inf_{O \in \mathcal{O}(d)} \|A - BO\|_F^2 \\ &\leq \frac{1}{n} \|(AA^\top)^{1/2}U - (BB^\top)^{1/2}V(V^\top U)\|_F^2 \\ &= \frac{1}{n} \|(AA^\top)^{1/2} - (BB^\top)^{1/2}\|_F^2. \end{aligned}$$

The Powers–Størmer inequality [28] gives

$$\|(AA^\top)^{1/2} - (BB^\top)^{1/2}\|_F^2 \leq \|AA^\top - BB^\top\|_*,$$

where $\|\cdot\|_*$ denotes the nuclear norm. In addition, AA^\top and BB^\top are at most rank d , so

$$\begin{aligned} \ell(A, B) &\leq \frac{1}{n} \|AA^\top - BB^\top\|_* \\ &\leq \frac{\sqrt{d}}{n} \|AA^\top - BB^\top\|_F = \sqrt{\frac{n+1}{n} L(AA^\top, BB^\top)}. \end{aligned}$$

\square

Next, we relate the rate-distortion function of $X = ZZ^\top$ in the loss L to the rate-distortion function of Z in the loss ℓ .

Lemma 6. *Let Z and X be defined as in Theorem 2, and let L and ℓ be defined by (1) and (2) respectively. Recall the notation of the rate-distortion function in Definition 1. For $D > 0$, we have*

$$R_X^L(D) \geq R_Z^\ell(\sqrt{8D}).$$

Proof. Fix a conditional distribution $P_{Y|X}$ such that,

$$\mathbb{E}L(X, Y) \leq D.$$

Define

$$\tilde{Z} = \underset{W \in \mathbb{R}^{n \times d}}{\operatorname{argmin}} \|Y - WW^\top\|_F,$$

where the non-unique minimizer \tilde{Z} is chosen arbitrarily. Then we have

$$\|ZZ^\top - \tilde{Z}\tilde{Z}^\top\|_F \leq \|ZZ^\top - Y\|_F + \|Y - \tilde{Z}\tilde{Z}\|_F \leq 2\|ZZ^\top - Y\|_F.$$

In other words,

$$L(ZZ^\top, \tilde{Z}\tilde{Z}^\top) \leq 4L(X, Y).$$

By Lemma 5,

$$\ell(Z, \tilde{Z}) \leq \sqrt{2L(ZZ^\top, \tilde{Z}\tilde{Z}^\top)} \leq \sqrt{8L(X, Y)}.$$

Jensen's inequality then yields

$$\mathbb{E}\ell(Z, \tilde{Z}) \leq \mathbb{E}\sqrt{8L(X, Y)} \leq \sqrt{8\mathbb{E}L(X, Y)} \leq \sqrt{8D}.$$

Let O be a uniform random orthogonal matrix over $\mathcal{O}(d)$, independent from everything else. In view of the definition of ℓ , we have

$$\mathbb{E}\ell(ZO, \tilde{Z}) = \mathbb{E}\ell(Z, \tilde{Z}) \leq \sqrt{8D}.$$

Therefore, by the definition of the rate-distortion function R_Z^ℓ (see Definition 1),

$$I(ZO; \tilde{Z}) \geq R_{ZO}^\ell(\sqrt{8D}) = R_Z^\ell(\sqrt{8D}),$$

where the equality follows from the orthogonal invariance of the distribution of Z .

Next, we note that

$$I(ZO; \tilde{Z}) \leq I(ZZ^\top; \tilde{Z}).$$

(In fact, equality holds because the reverse inequality is trivial by data processing.) To see this, given ZZ^\top , take any $A \in \mathbb{R}^{n \times d}$ such that $ZZ^\top = AA^\top$, and let Q be a uniform random orthogonal matrix over $\mathcal{O}(d)$ independent from everything else. Since $A = ZP$ for some $P \in \mathcal{O}(d)$, we have $(AQ, \tilde{Z}) = (ZPQ, \tilde{Z}) \stackrel{d}{=} (ZO, \tilde{Z})$, where $\stackrel{d}{=}$ denotes equality in distribution. Hence, the data processing inequality gives $I(ZZ^\top; \tilde{Z}) \geq I(AQ; \tilde{Z}) = I(ZO; \tilde{Z})$.

Combining the above two displays and recalling that \tilde{Z} is defined from Y , we apply the data processing inequality again to obtain

$$I(X; Y) \geq I(ZZ^\top; \tilde{Z}\tilde{Z}^\top) \geq R_Z^\ell(\sqrt{8D}).$$

Minimizing $P_{Y|X}$ subject to the constraint $\mathbb{E}L(X, Y) \leq D$ yields the rate-distortion function $R_X^L(D)$ on the left-hand side, completing the proof. \square

Lemma 7. *Let Z be defined as in Theorem 2, let ℓ be defined by (2), and let R_Z^ℓ be given by Definition 1. There is an absolute constant $C > 0$ such that for any $D \in (0, 1/4)$, we have*

$$R_Z^\ell(D) \geq \frac{nd}{2} \log \frac{1}{4D} - \frac{d^2}{2} \log \frac{C}{D}.$$

Proof. Fix a conditional distribution $P_{\hat{Z}|Z}$ such that $\mathbb{E}\ell(Z, \hat{Z}) \leq D$. Let $O = O(Z, \hat{Z}) \in \mathcal{O}(d)$ be such that

$\frac{1}{n}\|\hat{Z}O - Z\|_F^2 = \ell(Z, \hat{Z})$. Then we have $\mathbb{E}\|\hat{Z}O - Z\|_F^2 \leq nD$. Let $N(\mathcal{O}(d), \epsilon)$ be an ϵ -net of $\mathcal{O}(d)$ with respect to the Frobenius norm, where $\epsilon^2 = \frac{nD}{\mathbb{E}\|Z\|_F^2} \wedge d$. For $O = O(Z, \hat{Z})$, choose $\hat{O} = \hat{O}(Z, \hat{Z}) \in N(\mathcal{O}(d), \epsilon)$ such that $\|\hat{O} - O\|_F^2 \leq \epsilon^2$. Define $W := \hat{Z}\hat{O}$. We have

$$\begin{aligned} \mathbb{E}\|W - Z\|_F^2 &= \mathbb{E}\|\hat{Z}\hat{O} - Z\|_F^2 = \mathbb{E}\|\hat{Z} - Z\hat{O}^{-1}\|_F^2 \\ &\leq 2\mathbb{E}\|\hat{Z} - ZO^{-1}\|_F^2 + 2\mathbb{E}\|ZO^{-1} - Z\hat{O}^{-1}\|_F^2 \\ &\leq 2\mathbb{E}\|\hat{Z}O - Z\|_F^2 + 2\mathbb{E}\|Z\|_F^2\|O^{-1} - \hat{O}^{-1}\|_F^2 \\ &\leq 2nD + 2\epsilon^2\mathbb{E}\|Z\|_F^2 = 4nD, \end{aligned}$$

where $\|\cdot\|$ denotes the spectral norm.

By Theorem 26.2 of [27] (with d replaced by nd and σ^2 replaced by $1/d$), the rate-distortion function of Z with respect to the Frobenius norm $L_0(Z, W) := \|Z - W\|_F^2$ is

$$R_Z^{L_0}(D) = \frac{nd}{2} \log \frac{n}{D}. \quad (3)$$

Since $\mathbb{E}\|W - Z\|_F^2 \leq 4nD$, we obtain

$$I(Z; W) \geq R_Z^{L_0}(4nD) = \frac{nd}{2} \log \frac{1}{4D}.$$

Moreover, we have

$$\begin{aligned} I(Z; W) &\leq I(Z; \hat{Z}, \hat{O}) = I(Z; \hat{Z}) + I(Z; \hat{O} | \hat{Z}) \\ &\leq I(Z; \hat{Z}) + H(\hat{O}), \end{aligned}$$

where the three steps follow respectively from the data processing inequality, the definition of conditional mutual information $I(Z; \hat{O} | \hat{Z})$, and a simple bound on the mutual information by the entropy. The above two inequalities combined imply

$$I(Z; \hat{Z}) \geq I(Z; W) - H(\hat{O}) \geq \frac{nd}{2} \log \frac{1}{4D} - H(\hat{O}).$$

Since $\hat{O} \in N(\mathcal{O}(d), \epsilon)$, the entropy $H(\hat{O})$ can be bounded by the metric entropy of $\mathcal{O}(\epsilon)$. By Theorem 8 of [30], there is an absolute constant $C_0 > 1$ such that the covering number of $\mathcal{O}(d)$ with respect to the Frobenius norm is at most $\left(\frac{\sqrt{C_0}d}{\epsilon}\right)^{d^2}$ for any $\epsilon \in (0, \sqrt{d})$. We have

$$\epsilon = \sqrt{\frac{nD}{\mathbb{E}\|Z\|_F^2}} \wedge \sqrt{d} \geq c_1 \sqrt{dD}$$

for an absolute constant $c_1 > 0$, where the bound follows from the concentration of $\|Z\|$ at order $O(\frac{\sqrt{n}+\sqrt{d}}{\sqrt{d}})$ (see, e.g., Corollary 5.35 of [31]) and that $d \leq n$. Therefore,

$$H(\hat{O}) \leq \log|N(\mathcal{O}(d))| \leq \frac{d^2}{2} \log \frac{C_0 d}{\epsilon^2} \leq \frac{d^2}{2} \log \frac{C_0}{c_1^2 D}.$$

Putting it together, we obtain

$$I(Z; \hat{Z}) \geq \frac{nd}{2} \log \frac{1}{4D} - \frac{d^2}{2} \log \frac{C_0}{c_1^2 D},$$

finishing the proof in view of the definition of $R_Z^L(D)$. \square

Combining Lemmas 6 and 7, we conclude that

$$R_X^L(D) \geq \frac{nd}{2} \log \frac{1}{4\sqrt{8D}} - \frac{d^2}{2} \log \frac{C}{\sqrt{8D}} \geq \frac{nd}{8} \log \frac{1}{D}$$

provided that $D \in (0, c^*)$ and $d \leq c^*n$ for a sufficiently small constant $c^* > 0$.

B. Case $d \geq n$

In the case $d \geq n$, the Wishart distribution of $X = ZZ^\top$ has a density on the set of symmetric matrices $\mathbb{R}^{n(n+1)/2}$, and we can apply the Shannon lower bound [29] on the rate-distortion function. See Equation (26.5) and Exercise V.22 of the book [27] (with the norm taken to be the Euclidean norm and $r = 2$) for the following result.

Lemma 8 (Shannon lower bound [29]). *Let Y be a continuous random vector with a density on \mathbb{R}^N . For $D > 0$, let $R_Y^{L_0}(D)$ be the rate-distortion function of Y with respect to the Euclidean norm $L_0(Y, \hat{Y}) := \|Y - \hat{Y}\|_2^2$. Let $\mathbf{h}(Y)$ denote the differential entropy of Y . Then we have*

$$R_Y^{L_0}(D) \geq \mathbf{h}(Y) - \frac{N}{2} \log \frac{2\pi e D}{N}.$$

As a result, for the loss L defined by (1) and the random matrix X distributed over $\mathbb{R}^{n(n+1)/2}$, we have

$$R_X^L(D) \geq \mathbf{h}(X) - \frac{n(n+1)}{4} \log \frac{4\pi e D}{d}.$$

The differential entropy $\mathbf{h}(X)$ of the Wishart matrix X is known.

Lemma 9 (Differential entropy of a Wishart matrix [21]). *For X defined in Theorem 2, we have*

$$\begin{aligned} \mathbf{h}(X) &= \frac{n(n+1)}{2} \log \frac{2}{d} + \log \Gamma_n\left(\frac{d}{2}\right) \\ &\quad - \frac{d-n-1}{2} \psi_n\left(\frac{d}{2}\right) + \frac{nd}{2}, \end{aligned}$$

where Γ_n is the multivariate gamma function and ψ_n is the multivariate digamma function.

The above two results combined give the lower bound

$$\begin{aligned} R_X^L(D) &\geq \frac{n(n+1)}{2} \log \frac{2}{d} + \log \Gamma_n\left(\frac{d}{2}\right) \\ &\quad - \frac{d-n-1}{2} \psi_n\left(\frac{d}{2}\right) + \frac{nd}{2} - \frac{n(n+1)}{4} \log \frac{4\pi e D}{d} \\ &= \frac{nd}{2} + \frac{n(n+1)}{4} \log \frac{1}{\pi e D d} + \log \Gamma_n\left(\frac{d}{2}\right) \\ &\quad - \frac{d-n-1}{2} \psi_n\left(\frac{d}{2}\right). \end{aligned} \tag{4}$$

We now analyze the functions Γ_n and γ_n . By Stirling's approximation for the gamma function (see Equation 6.1.40 of [1]), we have $\log \Gamma(x + 1/2) \geq x \log(x + 1/2) - x - 1/2 + \frac{\log(2\pi)}{2}$

for $x \geq 0$. Together with the definition of the multivariate gamma function Γ_n , this gives

$$\begin{aligned} \log \Gamma_n\left(\frac{d}{2}\right) &= \frac{n(n-1)}{4} \log \pi + \sum_{i=1}^n \log \Gamma\left(\frac{d+1-i}{2}\right) \\ &\geq \frac{n(n-1)}{4} \log \pi \\ &\quad + \sum_{i=1}^n \left(\frac{d-i}{2} \log \frac{d+1-i}{2} - \frac{d+1-i}{2} + \frac{\log(2\pi/e)}{2} \right) \\ &\geq \frac{n^2}{4} \log(\pi e) - \frac{nd}{2} + \sum_{i=1}^n \left(\frac{d-i}{2} \log \frac{d+1-i}{2} \right) - O(n). \end{aligned}$$

Moreover, by Equation (2.2) of [2], the digamma function satisfies $\log x - \frac{1}{x} < \psi(x) < \log x$ for $x > 0$. Combining this with the definition of the multivariate digamma function ψ_n , we obtain

$$\begin{aligned} \frac{d-n-1}{2} \psi_n\left(\frac{d}{2}\right) &= \frac{d-n-1}{2} \sum_{i=1}^n \psi\left(\frac{d+1-i}{2}\right) \\ &\leq \frac{d-n-1}{2} \sum_{i=1}^n \log \frac{d+1-i}{2} + O(n), \end{aligned}$$

where we note that the $O(n)$ term is only necessary in the case that $d = n$ and $\frac{d-n-1}{2}$ is negative.

Plugging the above two estimates into (4), we see that

$$\begin{aligned} R_X^L(D) &\geq \frac{n(n+1)}{4} \log \frac{1}{Dd} \\ &\quad + \sum_{i=1}^n \left(\frac{n+1-i}{2} \log \frac{d+1-i}{2} \right) - O(n). \end{aligned} \tag{5}$$

If $d \geq 2n$, then

$$\begin{aligned} R_X^L(D) &\geq \frac{n(n+1)}{4} \log \frac{1}{Dd} \\ &\quad + \left(\log \frac{d+1-n}{2} \right) \sum_{i=1}^n \frac{n+1-i}{2} - O(n) \\ &= \frac{n(n+1)}{4} \log \frac{1}{D} + \frac{n(n+1)}{4} \log \frac{d+1-n}{2d} \\ &\quad - O(n) \\ &\geq \frac{n(n+1)}{4} \log \frac{1}{D} - O(n^2). \end{aligned}$$

For $n \leq d < 2n$, we first note that the term $\frac{n+1-i}{2} \log \frac{d+1-i}{2}$ with $i = n$ can be dropped from the sum in (5), because $\frac{n+1-n}{2} \log \frac{d+1-n}{2} < 0$ only if $d = n$, in which case the negative quantity $\frac{1}{2} \log \frac{1}{2}$ is subsumed by the $-O(n)$ term. Furthermore, since the function $x \mapsto \frac{n+1-x}{2} \log \frac{d+1-x}{2}$ is

decreasing on $[1, n]$, we have

$$\begin{aligned} & \sum_{i=1}^{n-1} \left(\frac{n+1-i}{2} \log \frac{d+1-i}{2} \right) \\ & \geq \int_1^n \frac{n+1-x}{2} \log \frac{d+1-x}{2} dx \\ & = \frac{2dn-d^2}{4} \log \frac{d}{d+1-n} + \frac{n^2-1}{4} \log(d+1-n) \\ & \quad + O(n^2), \end{aligned}$$

where the integral can be evaluated explicitly, but we suppress $O(n^2)$ terms for brevity. Plugging this back into (5), we obtain

$$\begin{aligned} R_X^L(D) & \geq \frac{n(n+1)}{4} \log \frac{1}{D} \\ & \quad + \frac{2dn-d^2-n^2+1}{4} \log \frac{d}{d+1-n} - O(n^2). \end{aligned}$$

Since $2dn-d^2-n^2 \leq 0$ and $\log \frac{d}{d+1-n} \leq \frac{n-1}{d+1-n} \leq \frac{n-1}{d-n}$, it holds that

$$\begin{aligned} \frac{2dn-d^2-n^2+1}{4} \log \frac{d}{d+1-n} & \geq \frac{2dn-d^2-n^2}{4} \cdot \frac{n-1}{d-n} \\ & = -\frac{1}{4}(d-n)(n-1). \end{aligned}$$

(While the above argument relied on $d > n$ due to the presence of $d-n$ in the denominator, the conclusion clearly holds for $d=n$.) Consequently, we again have

$$R_X^L(D) \geq \frac{n(n+1)}{4} \log \frac{1}{D} - O(n^2).$$

This readily implies the desired lower bound.

C. Case $c^*n < d < n$

This case can be easily reduced to the case $d \geq n$. Fix a conditional distribution $P_{Y|X}$ such that $\mathbb{E}L(X, Y) \leq D$. Let X_d be the top left $d \times d$ principal minor of X and define Y_d similarly. Then X_d clearly has the Wishart distribution as X in Theorem 2 with n replaced by d . Let L_d be the loss L in (1) with n replaced by d . Then we have

$$\begin{aligned} L_d(X_d, Y_d) & = \frac{d}{d(d+1)} \|X_d - Y_d\|_F^2 \\ & \leq \frac{d}{(c^*)^2 n(n+1)} \|X - Y\|_F^2 = \frac{1}{(c^*)^2} L(X, Y), \end{aligned}$$

so $\mathbb{E}L_d(X_d, Y_d) \leq D/(c^*)^2$. Applying the result for the case $d=n$, we get

$$\begin{aligned} I(X_d; Y_d) & \geq \frac{d(d+1)}{4} \log \frac{(c^*)^2}{D} - O(d^2) \\ & \geq \frac{c^*nd}{4} \log \frac{(c^*)^2}{D} - O(nd). \end{aligned}$$

Since $I(X; Y) \geq I(X_d; Y_d)$, to complete the proof, it remains to take $D \leq c$ for a sufficiently small constant $c > 0$ depending only on c^* and the hidden constant in $O(nd)$.

D. Spherical case

We now consider the case $Z = [z_1 \dots z_n]^\top$ and $X = ZZ^\top$ where z_1, \dots, z_n are i.i.d. uniform random vectors over the unit sphere $\mathcal{S}^{d-1} \subset \mathbb{R}^d$. The proof is via a reduction from the Gaussian case. Let w_1, \dots, w_n be i.i.d. $\mathcal{N}(0, \frac{1}{d} I_d)$ vectors and let $\beta_i := \|w_i\|_2$, so that $z_i = w_i/\beta_i$ and $w_i = \beta_i z_i$. Let $B \in \mathbb{R}^{n \times n}$ be the diagonal matrix with β_1, \dots, β_n on its diagonal. Let $Y = BXB$. Then Y has the distribution of X in the case where z_1, \dots, z_n are Gaussian vectors, so the result of the Gaussian case gives

$$R_Y^L(D) \geq cn(n \wedge d) \log \frac{1}{D}. \quad (6)$$

Fix a conditional distribution $P_{\hat{X}|X}$ such that $\mathbb{E}L(X, \hat{X}) \leq D$. Let g_1, \dots, g_n be i.i.d. $\mathcal{N}(0, \delta^2)$ random variables independent from everything else, where $\delta > 0$ is to be chosen. Define $\hat{\beta}_i := \beta_i + g_i$, and let $\hat{B} \in \mathbb{R}^{n \times n}$ be the diagonal matrix with $\hat{\beta}_1, \dots, \hat{\beta}_n$ on its diagonal. Define $\hat{Y} := \hat{B}\hat{X}\hat{B}$. Since z_i is independent from β_i , we see that (X, \hat{X}) is independent from (B, \hat{B}) . Hence,

$$I(Y; \hat{Y}) \leq I(X, B; \hat{X}, \hat{B}) = I(X; \hat{X}) + I(B; \hat{B}).$$

For the term $I(B; \hat{B})$, the independence across the pairs $(\beta_i, \hat{\beta}_i)$ for $i = 1, \dots, n$ implies

$$I(B; \hat{B}) = \sum_{i=1}^n I(\beta_i; \hat{\beta}_i) = nI(\beta_1; \hat{\beta}_1).$$

We have $\text{Var}(\beta_1) = \text{Var}(\|w_1\|_2) = \frac{1}{d}(d-2\frac{\Gamma((d+1)/2)^2}{\Gamma(d/2)^2}) \leq 1/(2d)$ using the variance of the χ_d distribution and basic properties of the gamma function. Let $g' \sim \mathcal{N}(0, 1/(2d))$. Then the Gaussian saddle point theorem (see Theorem 5.11 of [27]) gives

$$I(\beta_1; \hat{\beta}_1) \leq I(g'; g' + g_1) = \frac{1}{2} \log \left(1 + \frac{1}{2d\delta^2} \right).$$

The above three displays combined yield

$$I(X; \hat{X}) \geq I(Y; \hat{Y}) - \frac{n}{2} \log \left(1 + \frac{1}{2d\delta^2} \right). \quad (7)$$

It remains to bound $I(Y; \hat{Y})$ from below. To this end, note that

$$\begin{aligned} \|\hat{Y} - Y\|_F^2 & = \|\hat{B}\hat{X}\hat{B} - BXB\|_F^2 \\ & \leq 2\|\hat{B}\hat{X}\hat{B} - \hat{B}X\hat{B}\|_F^2 + 2\|\hat{B}X\hat{B} - BXB\|_F^2 \\ & = 2 \sum_{i,j=1}^n \hat{\beta}_i^2 \hat{\beta}_j^2 (\hat{X}_{ij} - X_{ij})^2 \\ & \quad + 2 \sum_{i,j=1}^n X_{ij}^2 (\hat{\beta}_i \hat{\beta}_j - \beta_i \beta_j)^2. \end{aligned}$$

Since $\hat{\beta}_i = \beta_i + g_i$, we have $\mathbb{E}[\hat{\beta}_i^2] = \mathbb{E}[\beta_i^2] + \mathbb{E}[g_i^2] = 1 + \delta^2$. Moreover, we have $\mathbb{E}[X_{ii}^2] = \mathbb{E}[(z_i^\top z_i)^2] = 1$ and $\mathbb{E}[X_{ij}^2] = \mathbb{E}[(z_i^\top z_j)^2] = 1/d$ for $i \neq j$. Finally,

$$\begin{aligned} \mathbb{E}[(\hat{\beta}_i \hat{\beta}_j - \beta_i \beta_j)^2] & = \mathbb{E}[(\beta_i g_j + \beta_j g_i + g_i g_j)^2] \\ & = 2\delta^2 + \mathbb{E}[g_i^2 g_j^2] + 2\mathbb{E}[\beta_i \beta_j] \mathbb{E}[g_i g_j] \end{aligned}$$

so $\mathbb{E}[(\hat{\beta}_i^2 - \beta_i^2)^2] = 4\delta^2 + 3\delta^4$ and $\mathbb{E}[(\hat{\beta}_i \hat{\beta}_j - \beta_i \beta_j)^2] = 2\delta^2 + \delta^4$ for $i \neq j$. Since $\hat{\beta}_1, \dots, \hat{\beta}_n$ are independent and B, \hat{B}, X are mutually independent, we conclude that

$$\begin{aligned} \mathbb{E}\|\hat{Y} - Y\|_F^2 &\leq 2(1 + \delta^2)^2 \mathbb{E}\|\hat{X} - X\|_F^2 + 2n(4\delta^2 + 3\delta^4) \\ &\quad + 2\frac{n(n-1)}{d}(2\delta^2 + \delta^4) \\ &\leq 8\frac{n(n+1)}{d}D + 14\frac{n}{d}D + 6\frac{n(n-1)}{d^2}D, \end{aligned}$$

where we used that $\mathbb{E}L(X, \hat{X}) \leq D$ for the loss L defined in (1) and chose $\delta^2 = D/d < 1$. Hence, we have $\mathbb{E}L(Y, \hat{Y}) \leq 28D$. This together with (6) implies that

$$I(Y; \hat{Y}) \geq cn(n \wedge d) \log \frac{1}{28D}.$$

Plugging this bound into (7), we obtain

$$I(X; \hat{X}) \geq cn(n \wedge d) \log \frac{1}{28D} - \frac{n}{2} \log \left(1 + \frac{1}{2D}\right).$$

The above bound completes the proof if $d \geq C$ for some constant $C > 0$ depending only on c . For the case $d \leq C$ (in fact, for the entire case $d \leq c^*n$), it suffices to note that the proof in Section III-A also works for the spherical model. To be more precise, there are only three places where the Gaussianity assumption is used. First, the proof of Lemma 6 uses the orthogonal invariance of the distribution of the rows of Z , which is also true for the spherical model where z_i is uniform over S^{d-1} . Second, (3) uses the rate-distortion function of the entry-wise Gaussian matrix Z . In the case where Z have i.i.d. rows distributed uniformly over S^{d-1} , it suffices to replace this formula by a lower bound: by Theorems 27.17 and 24.8 of [27], we have

$$R_Z^{L_0}(D) \geq \frac{n(d-1)}{2} \log \frac{1}{D} - nC_2$$

for an absolute constant $C_2 > 0$, which is sufficient for the rest of the proof. Third, the proof of Lemma 7 also uses that $\mathbb{E}\|Z\|^2$ is of order $\frac{n+d}{d}$, which is obviously true if d is of constant size and the rows of Z are on the unit sphere.

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