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Γ-CONVERGENCE OF NONLOCAL DIRICHLET ENERGIES WITH PENALTY FORMULATIONS OF DIRICHLET BOUNDARY DATA*

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Abstract. We study nonlocal Dirichlet energies associated with a class of nonlocal diffusion models on a bounded domain subject to the conventional local Dirichlet boundary condition. The goal of this paper is to give a general framework to correctly impose the Dirichlet boundary condition in a nonlocal diffusion model. To achieve this, we formulate the Dirichlet boundary condition as a penalty term and use the theory of Γ-convergence to study the correct form of the penalty term. Based on the analysis of Γ-convergence, we prove that the Dirichlet boundary condition can be correctly imposed in a nonlocal diffusion model in the sense of Γ-convergence as long as the penalty term satisfies a few mild conditions. This work provides a theoretical foundation for the approximate Dirichlet boundary condition in a nonlocal diffusion model.

Key words. nonlocal model, Dirichlet energy, Γ-convergence, Dirichlet boundary

MSC codes. 49J45, 49J55, 62G20, 65D05, 65N12

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1. Introduction.

Nonlocal models have been extensively studied in various scientific disciplines [2, 3, 4, 9, 18, 23, 36, 40]. Among them, nonlocal models with operators that only consider nonlocal interactions of a limited range are of particular interest [11, 10], as they are closely related to peridynamic [36] and some meshless numerical-like smoothed particle hydrodynamics (SPH) [21, 26, 29]. In this paper, we concentrate on nonlocal energies associated with the nonlocal Laplacian that are the nonlocal counterpart of the local Laplacian. While various theoretical and numerical studies have been devoted to problems associated with nonlocal Laplacians, also called nonlocal diffusion models [14, 12, 28, 39, 45], establishing nonlocal analogues of the boundary conditions remains a topic of ongoing discussion. One approach is to extend the boundary to a small volume adjacent to the boundary, known as volume constraints [11]. Designing nonlocal models or volume constraints properly can achieve better convergence rates to their local limit, such as for Neumann boundary condition in one [38] and two dimensions [44]. The point integral method [25, 34] is also an effective approach to construct nonlocal approximations of the Poisson equation with the Neumann boundary condition. As for the Dirichlet boundary condition, the constant extension method [27] may be the most straightforward, but it only provides first-order convergence at best. Enforcing the no-slip condition for higher-order convergence may be costly due to the need to calculate particle distances from domain

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boundaries [22, 43]. Recently, there have been many studies devoted to constructing nonlocal models with respect to the Dirichlet condition in different implementations [15, 42, 24, 34, 35, 46, 32].

In this paper, we mainly analyze a nonlocal counterpart of the following well-known Dirichlet energy defined on a bounded domain $\Omega \subset \mathbb{R}^d$:

$$\int_\Omega |\nabla u(x)|^p dx, \quad p \in (1, \infty), \tag{1.1}$$

for functions $u \in W^{1,p}(\Omega)$, subject to Dirichlet boundary conditions on $\partial\Omega$.

Given a nonlocal horizon parameter $\delta > 0$ that controls the range of nonlocal interaction, a commonly studied nonlocal Dirichlet energy is given by

$$\frac{1}{\delta^p} \int_\Omega \int_\Omega R_\delta(|x-y|)|u(x) - u(y)|^p dy dx, \tag{1.2}$$

where $R_\delta(s) = \frac{1}{\delta^d} R\left(\frac{s^2}{\delta^2}\right)$ is a scaled nonlocal kernel.

It is well known that variational problems associated with the minimization of the functional (1.1) for a Dirichlet data a being a proper function prescribed on boundary $\partial\Omega$ lead to the following homogeneous Poisson equation, also called the p-Laplace equation or diffusion equation, with the Dirichlet boundary condition:

$$\begin{cases} \nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0 & \text{in } \Omega, \\ u = a & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

For example, for $p = 2$, we get the linear Dirichlet boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = a & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

For widely used kernels, particularly those smoothly defined R_δ, functions with a finite nonlocal energy are not expected to have sufficient regularity to have well-defined traces on $\partial\Omega$, thus making it hard to directly impose the Dirichlet boundary condition like that in (1.4). A possible remedy is to adopt the technique of heterogeneous localization that leads to improved regularity at the boundary; see [32] and the references cited therein.

There are other attempts to impose the Dirichlet boundary condition for the nonlocal diffusion model. For $p = 2$, by taking a constant $0 < \beta \ll 1$ and using the Robin boundary condition,

$$u + \beta \frac{\partial u}{\partial \mathbf{n}} = a,$$

to approximate the Dirichlet boundary condition $u = a$, a nonlocal model was proposed in [35] as follows:

$$\frac{4}{\delta^2} \int_\Omega R_\delta(|x-y|)(u(x) - u(y)) dy - \frac{2}{\beta} \int_{\partial\Omega} \bar{R}_\delta(|x-y|)(a(y) - u(y)) dS_y = 0,$$

where $\bar{R}_\delta(s) = \frac{1}{\delta^d} \bar{R}\left(\frac{s^2}{\delta^2}\right)$ and $\bar{R}(s) = \int_s^{+\infty} R(r) dr$. This nonlocal model is proved to converge to the local Laplace equation as $\delta, \beta \to 0$ [35], and an error estimate in terms of δ and β is also given. However, the symmetry is destroyed in the above nonlocal model such that it does not have variational form with nonlocal Dirichlet energy. By introducing $\frac{\partial u}{\partial \mathbf{n}}$ as an auxiliary variable, a nonlocal energy with penalty term can be derived [41].

$$
\begin{aligned}
F(u,a) = & \frac{1}{\delta^2}\int_\Omega\int_\Omega R_\delta(|x-y|)(u(x)-u(y))^2 dxdy \\
& + \int_{\partial\Omega} \frac{2}{\delta^2 \bar{\bar{\omega}}_\delta(x)} \left(\int_\Omega \bar{R}_\delta(|x-y|)(a(x)-u(y))dy\right)^2 dx,
\end{aligned}
\tag{1.5}
$$

where

$$
\bar{\bar{\omega}}_\delta(x) = \int_\Omega \bar{\bar{R}}_\delta(|x-y|)dy,
$$

$\bar{\bar{R}}_\delta(s) = \frac{1}{\delta^d}\bar{\bar{R}}\left(\frac{s^2}{\delta^2}\right)$, and $\bar{\bar{R}}(s) = \int_s^{+\infty}\bar{R}(r)dr$. It can be proved that minimal solution of the above energy function converges to the solution of (1.4) as $\delta \to 0$ and the convergence rate is $O(\delta)$ in the H^1 norm [41]. To get H^1 convergence, we need a specifically designed penalty term in (1.5). Previous research primarily focused on carefully designing penalty terms to ensure the best possible convergence of nonlocal models. In contrast, in this paper, we are more concerned with identifying the conditions under which the convergence of the nonlocal model can be guaranteed. Since these conditions should be as weak as possible, we are committed to studying a more general form of the penalty term.

Motivated by the penalty formulation of the Dirichlet boundary value problems for elliptic PDEs and nonlocal energy (1.5), we first consider the following nonlocal energy with a general penalty term:

$$
\begin{aligned}
& \frac{1}{\delta^p}\int_\Omega\int_\Omega R_\delta(|x-y|)|u(x)-u(y)|^p dxdy + B_p(u,a) \\
& \quad \text{where } B_p(u,a) = \int_{\partial\Omega}\left|\frac{1}{\delta}\int_\Omega K_\delta(|x-y|)(a(x)-u(y))dy\right|^p dx,
\end{aligned}
\tag{1.6}
$$

and K_δ is a nonlocal kernel that depends on the horizon parameter δ. Our interest is to give the correct form of K_δ such that the above nonlocal model converges to the local model with Dirichlet boundary condition as $\delta \to 0$. Since we want to get a general form of K_δ, Γ-convergence of the above nonlocal model is analyzed rather than other strong convergence studied in previous works.

In fact, Γ-convergence among functionals is significant in describing the relationship between nonlocal operators and local ones in semisupervised learning and other fields [5, 37, 31, 20, 30]. With a property of Γ-convergence, we can also demonstrate the convergence of the minimizers (or solutions of the stationary equations). Nevertheless, it does not provide any information about the convergence rate of the minimizers. Hence, Γ-convergence is a weaker convergence and allow us to consider more relaxed conditions for the penalty term.

For $p=2$, our Γ-convergence result covers the case of linear variational problems associated with the nonlocal energy (1.2) with $p=2$ and the boundary penalty term of the form

$$
E(u,a) = \frac{1}{\delta^2}\int_{\partial\Omega}\left(\int_\Omega K_\delta(|x-y|)(u(y)-a(x))dy\right)^2 dx
\tag{1.7}
$$

that has been previously considered in [41] for some special choices of K_δ connected to R_δ (see more details in Remark 2.5). In this paper, we work with more general choices of K_δ that enable the Γ-convergence analysis, which remains valid for more general p. The kernel K_δ only needs to fulfill some regularization conditions and does not need to have any connection with the kernel R_δ. Moreover, as another contribution, we also discuss the convergence of associated nonlocal eigenvalue problems. In addition, specializing to linear problems corresponding to $p=2$, we consider a penalty form more general than that in (1.7) to illustrate the broad applicability of the method developed here.

Based on the analysis of Γ-convergence, we establish a class of penalty models which are guaranteed to be the correct approximation of the local Dirichlet boundary condition. This is also the main contribution of this paper.

The rest of the paper is organized as follows. In section 2, we state the main results and all assumptions that we need. Some related work considering special cases of our results is discussed in section 3. In section 4, Γ-convergence and some estimations employed in our proof procedure are introduced. Γ-convergence of the nonlocal models and compactness results are demonstrated in section 5 and section 6, respectively. In section 7, we conclude this paper and present several aspects for future research.

2. Assumptions and main results.

Let $p>1$ be a finite constant. Suppose that Ω is a Lipschitz bounded domain in $\mathbb{R}^d$. K, R are two kernel functions satisfying the following regularity conditions:

(K1) $K, R: [0,\infty) \longrightarrow [0,\infty)$ belong to C^1;

(K2) K, R are monotonically decreasing;

(K3) $\mathrm{supp}(K) \subset [0, r_K^2]$ and $\mathrm{supp}(R) \subset [0, r_R^2]$ for some $r_K, r_R > 0$.

For the kernel R, we define a normalization constant

$$\sigma_R := \int_{\mathbb{R}^d} R(|z|^2)|z\cdot e_1|^p dz \tag{2.1}$$

that only depends on a kernel R with $e_1 := (1,0,\ldots,0)$. For an arbitrary positive constant δ, we also employ the scaled kernels $K_\delta(s) := \frac{1}{\delta^d}K(\frac{s^2}{\delta^2})$ and $R_\delta(s) := \frac{1}{\delta^d}R(\frac{s^2}{\delta^2})$. We consider the p-Laplace equation with Dirichlet boundary condition (1.3), where a is the trace of some function in $W^{1,p}(\Omega)$. Then the weak solution of (1.3) is also the minimizer of the following functional:

$$F(u) = \begin{cases} \int_\Omega |\nabla u(x)|^p dx & \text{if } u \in W^{1,p}, Tu = a \text{ on } \partial\Omega, \\ \infty & \text{otherwise,} \end{cases} \tag{2.2}$$

where T is the trace operator for $W^{1,p}$ function. We first establish a specific nonlocal model for (2.2):

$$\begin{aligned} F_n(u) = {} & \frac{1}{\delta_n^p}\int_\Omega\int_\Omega R_{\delta_n}(|x-y|)|u(x)-u(y)|^p dxdy \\ & + \int_{\partial\Omega}\left|\frac{1}{\delta_n}\int_\Omega K_{\delta_n}(|x-y|)(a(x)-u(y))dy\right|^p dx \end{aligned} \tag{2.3}$$

and prove the Γ-convergence from $F_n(u)$ to $\sigma_R F(u)$ as the following theorem.

Theorem 2.1. *Suppose that Ω is a Lipschitz bounded domain in $\mathbb{R}^d$. $1<p<\infty$ is a constant. K, R are two kernel functions satisfying* (K1)–(K3)*, with σ_R given by* (2.1)*. $\{\delta_n\}$ is a sequence of positive constants tending to* 0 *as $n\to\infty$. Then we have*

$$F_n \xrightarrow{\Gamma} \sigma_R F \quad \text{in } L^p(\Omega),$$

where F_n, F are defined as in (2.3), (2.2).

Subsequently, We consider the eigenfunctions of the Laplace operator with Dirichlet condition (2.4). In this part we set $p=2$.

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \|u\|_{L^2(\Omega)} = 1. \end{cases} \tag{2.4}$$

From the standard theory of second-order elliptic equations (for example, section 6.5 in [17]), if we denote by Σ the set of all eigenvalues of Laplace operator Δ, then $\Sigma = \{\lambda_k\}_{k=1}^{\infty}$, where

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \dots .$$

Moreover,

$$\lambda_1 = \min\{\|\nabla u\|_{L^2(\Omega)} \mid u \in H_0^1(\Omega), \|u\|_{L^2(\Omega)} = 1\},$$

and the corresponding eigenfunction μ_1 (which means (μ_1, λ_1) is the solution of (2.4)) is the minimizer of the following functional:

$$F_e^1(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u(x)|^2 dx & \text{if } u \in H_0^1, \|u\|_{L^2(\Omega)} = 1, \\ \infty & \text{otherwise.} \end{cases}$$

For $k \geq 1$, let $V_k = span\{\mu_1, \mu_2, \dots, \mu_k\}$. Then,

$$\lambda_{k+1} = \min\{\|\nabla u\|_{L^2(\Omega)} \mid u \in H_0^1(\Omega), \|u\|_{L^2(\Omega)} = 1, u \perp V_k\},$$

and μ_k is the minimizer of

$$F_e^{k+1}(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u(x)|^2 dx & \text{if } u \in H_0^1, \|u\|_{L^2(\Omega)} = 1, u \perp V_k, \\ \infty & \text{otherwise.} \end{cases} \tag{2.5}$$

For this problem, we select $a \equiv 0$, $p=2$; then the local functional (2.2) becomes

$$F(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u(x)|^2 dx & \text{if } u \in H_0^1, \\ \infty & \text{otherwise.} \end{cases} \tag{2.6}$$

Note that (2.5) is actually (2.6) with additional constraints. We can construct the nonlocal approximation of F_e^k by defining the unnormalized functionals:

$$F_{e,n}^k(u) = \begin{cases} F_n(u) & \text{if } \|u\|_{L^2(\Omega)} = 1, u \perp V_{k-1}^n, \\ \infty & \text{otherwise,} \end{cases} \tag{2.7}$$

where $V_0 = \emptyset$, $V_k^n = span\{\mu_1^n, \dots, \mu_k^n\}$ and μ_k^n is the minimizer of $F_{e,n}^k$. Moreover, with a kernel function W satisfying conditions (K1)–(K3) and

$$\int_{\mathbb{R}^d} W(|z|^2) dz = 1, \tag{2.8}$$

we can consider the inner product defined as

$$\langle u, v\rangle_n = \int_\Omega \int_\Omega W_{\delta_n}(|x-y|) u(x) v(y) dx dy$$

and the normalized functionals defined as

$$\tilde{F}_{e,n}^k(u) = \begin{cases} F_n(u) & \text{if } \langle u, u\rangle_n = 1,\ u \perp \tilde{V}_{k-1}^n, \\ \infty & \text{otherwise,} \end{cases} \tag{2.9}$$

where $W_\delta(s) := \frac{1}{\delta^d} W(\frac{s^2}{\delta^2})$, $\tilde{V}_0 = \emptyset$, $\tilde{V}_k^n = span\{\tilde{\mu}_1^n, \dots \tilde{\mu}_k^n\}$, and $\tilde{\mu}_k^n$ is the minimizer of $\tilde{F}_{e,n}^k$. We also have the Γ-convergence from $F_{e,n}^k$ or $\tilde{F}_{e,n}^k$ to F_e^k.

THEOREM 2.2. *Suppose that* Ω *is a Lipschitz bounded domain in* $\mathbb{R}^d$*.* K, R, W *are three kernel functions satisfying* (K1)–(K3)*, with* W *satisfying additionally* (2.8)*, and* σ_R *being given by* (2.1)*.* $\{\delta_n\}$ *is a sequence of positive constants tending to* 0 *as* $n \to \infty$*. Then we have*

$$F_{e,n}^k \xrightarrow{\Gamma} \sigma_R F_e^k \quad in\ L^2(\Omega),$$

and

$$\tilde{F}_{e,n}^k \xrightarrow{\Gamma} \sigma_R F_e^k \quad in\ L^2(\Omega),$$

for all $k \in \mathbb{N}$*, where* $F_{e,n}^k, \tilde{F}_{e,n}^k$*, and* F_e^k *are defined as in* (2.7)*,* (2.9)*, and* (2.5)*.*

Proceeding to focus on the case where $a \equiv 0$ and $p = 2$, we can consider a more general nonlocal functional F_n:

$$F_n(u) = \frac{1}{\delta_n^2} \int_\Omega \int_\Omega R_{\delta_n}(|x-y|)|u(x)-u(y)|^2 dxdy + E_n(u,0), \tag{2.10}$$

where the more general formulation of the boundary term is given by

$$E_n(u,a) = \int_{\partial\Omega} \int_\Omega \int_\Omega \rho_{\delta_n,x}(y,z)(u(y)-a(x))(u(z)-a(x))dydzdx \tag{2.11}$$

with a kernel $\rho_{\delta_n,x}(y,z)$ symmetric with respect to y and z. Theorem 2.1 offers the Γ-convergence for the special choice

$$\rho_{\delta_n,x}(y,z) = \frac{1}{\delta_n^2} K_{\delta_n}(|x-y|) K_{\delta_n}(|x-z|). \tag{2.12}$$

To maintain this property in general situations, some restrictions on $\rho_{\delta_n,x}(y,z)$ should be imposed. First, in order for $E_n(u,0)$ to capture information about the boundary, the kernel should rapidly decay or directly vanish when $|(y,z)-(x,x)|$ is large. Hence, we require $\rho_{\delta_n,x}$ to be compactly supported and the support set to be shrinking when n tends to infinity, ensuring increasingly precise delineation for the boundary. Specifically, there should exist a sequence of positive constants $\{c_n\}$, $\lim_{n\to\infty} c_n = 0$, such that $\rho_{\delta_n,x}$ is only nonzero when $|x-y|$, $|x-z| \le c_n$.

Moreover, as $n \to \infty$, we expect the convergence of the minimizers of $\{F_n\}$, which are confined only in $L^2(\Omega)$, to a minimizer of F in the L^2 norm. As the minimizer of F, we have $u \in H^1$ and $Tu \equiv 0$. Thus, intuitively speaking, the sequence of minimizers of $\{F_n\}$ should take small absolute value near the boundary. To quantify this and to overcome the possible lack of regularity, we pick a kernel $\hat{K}$ satisfying (K1)–(K3), $\hat{K}_\delta(s) = \frac{1}{\delta^d}\hat{K}(\frac{s^2}{\delta^2})$, and a regularized form of $\{u_n\}$:

$$\tilde{u}_n(x) := \frac{1}{\omega_{\delta_n}(x)} \int_\Omega \hat{K}_{\delta_n}(|x-y|)u_n(y)dy, \ \text{ and } \ \omega_{\delta_n}(x) := \int_\Omega \hat{K}_{\delta_n}(|x-y|)dy. \tag{2.13}$$

Note that the normalization coefficient $\omega_{\delta_n}(x)$ has uniformly positive lower and upper bounds with respect to δ_n as $\delta_n \to 0$. Should $\hat{K}_\delta(s)$ be $\frac{1}{\delta^d}\hat{K}(\frac{s^2}{\delta^2})$? Mollifications like (2.13) have been utilized in other works, such as, for example, [32]. We then require a coercivity condition,

$$C_n E_n(u_n, 0) \ge \|\tilde{u}_n\|_{L^2(\partial\Omega)}^2 \quad \forall u_n \in L^2(\Omega),$$

for some $\hat{K}$ and positive constants $\{C_n\}$ satisfying $\lim_{n\to\infty} C_n = 0$. Effectively, this means that the penalty functional is placed on matching with the boundary data, not directly by u_n but by a more regular $\tilde{u}_n$. As a summary, we have the following theorem.

THEOREM 2.3. *Suppose that* Ω *is a Lipschitz bounded domain in* $\mathbb{R}^d$. R *is a kernel satisfying* (K1)–(K3), *and* σ_R *is given by* (2.1). $\{\delta_n\}$, $\{c_n\}$, $\{C_n\}$ *are sequences of positive constants tending to* 0 *as* $n \to \infty$. $\rho_{\delta_n,x}$ *is a kernel satisfying the following two conditions:*

(compact support) $\rho_{\delta_n,x}$ *is only nonzero when* $|x-y|, |x-z| \le c_n$,

(coercivity) $C_n E_n(u_n, 0) \ge \|\tilde{u}_n\|^2_{L^2(\partial\Omega)}$ *for any* $u_n \in L^2(\Omega)$,

where E_n, $\{\tilde{u}_n\}$ *are defined as in* (2.11) *and* (2.13) *for some* $\hat{K}$ *satisfying* (K1)–(K3). *Then, we have*

$$F_n \xrightarrow{\Gamma} \sigma_R F \quad \text{in } L^2(\Omega),$$

where F_n, F *are defined as in* (2.10), (2.6).

Remark 2.4. For general $p > 1$, a similar conclusion can be derived in the same way as in subsection 5.3. However, we do not yet have a unified formulation, like (2.11) for $p = 2$, that also works for more general $p \neq 2$, i.e., a formulation that covers both

$$E_n^1(u,a) = \int_{\partial\Omega} \left| \frac{1}{\delta_n} \int_\Omega K_{\delta_n}(|x-y|)(u(y) - a(x))dy \right|^p dx$$

and

$$E_n^2(u,a) = \frac{1}{\delta_n^p} \int_{\partial\Omega} \int_\Omega K_{\delta_n}(|x-y|)|u(y) - a(x)|^p dydx.$$

Hence, for the purpose of conciseness, we only discuss the $p = 2$ case in Theorem 2.3.

Remark 2.5. The main contribution of Theorem 2.3, in comparison with Theorem 2.1, is to formulate a more general boundary term. This general form contains a large class of kernel $\rho_{\delta_n,x}$ such as $\rho_{\delta_n,x}(y,z) = \frac{1}{\delta_n^2} K_{\delta_n}(|x-y|)\delta(|y-z|)$, where $\delta(\cdot)$ means the Dirac-delta measure. See more details in subsection 5.3. The subscript x in the kernel $\rho_{\delta_n,x}(y,z)$ means not only that it is centered at x but also that other parts of kernel expression can depend on x. For example, [38, 41] actually considered a special case covered by Theorem 2.3,

$$\rho_{\delta_n,x}(y,z) = \frac{2}{\delta^2 \bar{\bar{\omega}}_{\delta_n}(x)} \bar{R}_{\delta_n}(|x-y|)\bar{R}_{\delta_n}(|x-z|),$$

where $\bar{\bar{\omega}}_{\delta_n}$ is a bounded function depending on x. And in [33], another special case is considered:

$$\rho_{\delta_n,x}(y,z) = \frac{4}{\delta_n^2 \mu_{\delta_n}(x)} \bar{R}_{\delta_n}(|x-y|)\delta_n(|y-z|),$$

where $\mu_{\delta_n}(x)$ is also a bounded function.

With the Γ-convergence, the convergence of their minimizers follows.

THEOREM 2.6. *Suppose that* Ω *is a Lipschitz bounded domain in* $\mathbb{R}^d$. $1 < p < \infty$ *is a constant.* K, R *are two kernel functions satisfying* (K1)–(K3), *with* W *satisfying additionally* (2.8) *and* σ_R *being given by* (2.1). $\{\delta_n\}$ *is a sequence of positive constants tending to* 0 *as* $n \to \infty$. F_n, F *are defined as in* (2.3), (2.2). *Then any sequence* $\{u_n\} \subset L^p(\Omega)$ *satisfying*

$$\lim_{n\to\infty} (F_n(u_n) - \inf_{u \in L^p(\Omega)} F_n(u)) = 0$$

is relatively compact in $L^p(\Omega)$ *and*

$$\lim_{n\to\infty} F_n(u_n) = \min_{u\in L^p(\Omega)} \sigma_R F(u).$$

Furthermore, every cluster point of $\{u_n\}$ *is a minimizer of* F.

THEOREM 2.7. *Suppose that* Ω *is a Lipschitz bounded domain in* $\mathbb{R}^d$. K, R, W *are three kernel functions satisfying* (K1)–(K3) *with* W *satisfying additionally* (2.8) *and* σ_R *being given by* (2.1). $\{\delta_n\}$ *is a sequence of positive constants tending to* 0 *as* $n \to \infty$. $F^k_{e,n}, \tilde{F}^k_{e,n}$, *and* F^k_e *are defined as in* (2.7), (2.9), *and* (2.5). *Then any sequence* $\{u_n\} \subset L^2(\Omega)$ *satisfying*

$$\lim_{n\to\infty} (F^k_{e,n}(u_n) - \inf_{u\in L^2(\Omega)} F^k_{e,n}(u)) = 0$$

is relatively compact in $L^2(\Omega)$ *and*

$$\lim_{n\to\infty} F^k_{e,n}(u_n) = \min_{u\in L^2(\Omega)} \sigma_R F^k_e(u).$$

Furthermore, every cluster point of $\{u_n\}$ *is a minimizer of* F^k_e. *The conclusions still hold if* $\{F^k_{e,n}\}$ *is replaced by* $\{\tilde{F}^k_{e,n}\}$.

THEOREM 2.8. *Suppose that* Ω *is a Lipschitz bounded domain in* $\mathbb{R}^d$. R *is a kernel satisfying* (K1)–(K3), *and* σ_R *is given by* (2.1). $\{\delta_n\}$, $\{c_n\}$, $\{C_n\}$ *are sequences of positive constants tending to* 0 *as* $n \to \infty$. $\rho_{\delta_n,x}$ *is a kernel satisfying the following two conditions:*

(compact support) $\rho_{\delta_n,x}$ *is only nonzero when* $|x-y|, |x-z| \le c_n$,

(coercivity) $C_n E_n(u_n, 0) \ge \|\tilde{u}_n\|^2_{L^2(\partial\Omega)}$ *for any* $u_n \in L^2(\Omega)$,

where E_n *and* $\{\tilde{u}_n\}$ *are defined as in* (2.11) *and* (2.13) *for some* $\hat{K}$ *satisfying* (K1)–(K3), *and* F_n *and* F *are defined as in* (2.10), (2.6). *Then any sequence* $\{u_n\} \subset L^2(\Omega)$ *such that*

$$\lim_{n\to\infty} (F_n(u_n) - \inf_{u\in L^2(\Omega)} F_n(u)) = 0$$

is relatively compact in $L^2(\Omega)$ *and*

$$\lim_{n\to\infty} F_n(u_n) = \min_{u\in L^2(\Omega)} \sigma_R F(u).$$

Furthermore, every cluster point of $\{u_n\}$ *is a minimizer of* F.

3. Quantitative discussion.

All results in this paper are based on Γ-convergence. As shown by Lemma 4.2, this concept can lead to the convergence of minimizers but does not provide information about the convergence rate. Such weak conclusions allow us to consider a general class of nonlocal model. It is absolutely better if one can derive some quantitative results without strengthening our assumptions. But it may be difficult since the kernels of the boundary and interior term do not need any connection. Nevertheless, in some special cases, if the kernels are carefully designed and have good linkage, one can obtain a specific nonlocal model with stronger quantitative properties. For example, as mentioned in Remark 2.5, [41] studies the nonlocal model (2.10) with

$$\rho_{\delta_n,x}(y,z) = \frac{2}{\delta^2 \bar{\bar{\omega}}_{\delta_n}(x)} \bar{R}_{\delta_n}(|x-y|)\bar{R}_{\delta_n}(|x-z|),$$

where

$$\bar{\bar{\omega}}_{\delta_n}(x) := \int_\Omega \bar{\bar{R}}_{\delta_n}(|x-y|)dy,$$

and demonstrates the first order H^1 convergence of the minimizers (Theorem 3.2 in [41]). Select

$$\rho_{\delta_n,x}(y,z) = \frac{4}{\mu_{\delta_n}(x)}\bar{R}_{\delta_n}(|x-y|)\delta_n(|y-z|),$$

where $\mu_{\delta_n}(x) := \min\{2\delta_n, \max\{\delta_n^2, d(x)\}\}$ and $d(x) = \min_{y\in\partial\Omega}|x-y|$. It is shown in [33] that (2.10) with such $\rho_{\delta_n,x}$ fulfills the maximum principle and second order convergence for the minimizers (Theorem 5.1 in [33]).

Moreover, except for the convergence rate of minimizers, one may also directly explore the convergence rate of the nonlocal functional. In [7], the author proves that $\frac{\mathcal{F}_h-\mathcal{F}_0}{h^2}$ Γ-converges to a nonzero functional. $\mathcal{F}_0$, $\mathcal{F}_h$ are defined as follows:

$$\mathcal{F}_0(u) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(z) f\left(\left|\nabla u(x)\cdot\frac{z}{|z|}\right|\right)dzdx,$$
$$\mathcal{F}_h(u) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K_h(y-x) f\left(\frac{|u(y)-u(x)|}{|y-x|}\right)dydx,$$

where $f(\cdot)$ is a convex function that fulfills some conditions. This conclusion can be understood as nonlocal energy $\mathcal{F}_h$ second order converges to $\mathcal{F}_0$. If we choose $f(x) = x^p$ with $p>1$, $\mathcal{F}_h$ is quite similar to the interior term in our nonlocal model. The differences are that we change $\frac{1}{|y-x|}$ to $\frac{1}{h}$ and $\mathbb{R}^d$ to a bounded domain Ω. We believe that these differences are not essential and that a similar conclusion can be demonstrated in the cases we consider. However, much of the proof of this conclusion may be devoted to the treatment of interior terms, and the specific form of kernel in the boundary term has little influence. This goes against our topic of boundary terms and may be more suitable as future work. Specifically, consider the nonlocal model (2.10) for Theorem 2.3:

$$F_n(u) = \frac{1}{\delta_n^2}\int_\Omega\int_\Omega R_{\delta_n}(|x-y|)|u(x)-u(y)|^2dxdy + E_n(u,0)$$
$$=: I_{\delta_n}(u) + E_n(u,0).$$

Define

$$I_0(u) = \int_{\mathbb{R}^d}\int_\Omega R(|z|)|\nabla u(x)\cdot z|^p dxdz.$$

Suppose that with a method similar to that in [7], one can prove that $\frac{I_{\delta_n}-I_0}{\delta_n^2}$ Γ-converges to a nonzero functional $\mathcal{I}(u)$. With the coercivity condition in Theorem 2.3, there exists a sequence of positive constants $\{C_n\}$ tending to zero such that

$$C_nE_n(u,0) \geq \|\tilde{u}\|_{L^2(\partial\Omega)}$$

for all $u\in L^2(\Omega)$. Combining this with Lemma 4.3, we can prove that $\frac{F_n-I_0}{\delta_n^2} = \frac{I_{\delta_n}-I_0}{\delta_n^2} + \frac{1}{\delta_n^2}E_n(u,0)$ Γ-converges to

$$\tilde{\mathcal{I}}(u) = \begin{cases} \mathcal{I}(u) & \text{if } Tu\equiv 0 \text{ on } \partial\Omega,\\ \infty & \text{otherwise}\end{cases}$$

when $\frac{C_n}{\delta_n^2} \to 0$. So the specific form of boundary term E_n does not matter, whereas proving the conclusion that $\frac{I_{\delta_n}-I_0}{\delta_n^2}$ Γ-converges to a nonzero functional requires much more effort, especially for analyzing near the boundary of Ω.

Even if we prove the Γ-convergence above, there is still a gap for the convergence of minimizers. However, such convergence of functionals may derive the convergence of minimizers with some stronger assumptions. For example, a simple situation is that $\frac{I_{\delta_n}-I_0}{\delta_n^2}$ uniformly converges to a bounded functional $\mathcal{I}(u)$ and I_0 is strongly convex. In this case, we can obtain the second order convergence of the minimizers. This gap is also one of the obstacles we hope to overcome in future work.

4. Preliminaries.

For easy reference, we first recall the concept of Γ-convergence or functionals. We then present a technical lemma concerning the nonlocal energy with the kernel rescaled by different constants.

4.1. Γ-convergence.

The Γ-convergence proposed by De Giorgi is often used to study the convergence of minimizers of functionals under compactness assumptions. More detailed overviews about the Γ-convergence can be found in [6, 8].

Definition 4.1 (Γ-convergence). *Let X be a metric space and $F_n : X \to \mathbb{R} \cup \{-\infty, \infty\}$ be a sequence of functionals on X. We say that F_n Γ-converges to $F : X \to \mathbb{R} \cup \{-\infty, \infty\}$, which is also denoted by $F_n \xrightarrow{\Gamma} F(n \to \infty)$, if*

(1) *(liminf inequality) for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ converging to $x \in X$, we have $\liminf_{n\to\infty} F_n(x_n) \geq F(x)$.*
(2) *(limsup inequality) for any $x \in X$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ converging to x such that $\limsup_{n\to\infty} F_n(x_n) \leq F(x)$.*

The following lemma reveals the connection between Γ-convergence and the convergence of minimizers, which is also applied in [20, 37, 31, 19]. We include it here for completeness and easy reference.

Lemma 4.2 (convergence of minimizers). *Let X be a metric space and $F_n : X \to [0, \infty]$ Γ-converges to $F : X \to [0, \infty]$ which is not identically ∞. If there exists a relatively compact sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ such that*

$$\lim_{n\to\infty} (F_n(x_n) - \inf_{x\in X} F_n(x)) = 0$$

then we have

$$\lim_{n\to\infty} \inf_{x\in X} F_n(x) = \min_{x\in X} F(x)$$

and any cluster point of $\{x_n\}_{n\in\mathbb{N}}$ is a minimizer of F.

Proof. For any $y \in X$, we know that there exists a sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ satisfying the limsup inequality. So we have

$$F(y) \geq \limsup_{n\to\infty} F_n(y_n) \geq \limsup_{n\to\infty} \inf_{x\in X} F_n(x),$$

which yields

$$\min_{x\in X} F(x) \geq \limsup_{n\to\infty} \inf_{x\in X} F_n(x).$$

On the other hand, consider the sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ mentioned in the assumption. Let $\tilde{x}$ be one of the cluster points of $\{x_n\}_{n\in\mathbb{N}}$. Using the liminf inequality, we get

$$\liminf_{n\to\infty} \inf_{x\in X} F_n(x) = \liminf_{n\to\infty} F_n(x_n) \geq F(\tilde{x}) \geq \min_{x\in X} F(x).$$

Therefore,

$$\limsup_{n\to\infty} \inf_{x\in X} F_n(x) \le \min_{x\in X} F(x) \le F(\tilde{x}) \le \liminf_{n\to\infty} \inf_{x\in X} F_n(x),$$

and we can get the conclusion. ∎

4.2. Relation between kernels of different scales. The following technical lemma clarifies that when the kernel is rescaled by a constant factor, the nonlocal energy remains uniformly controlled by the original one. This conclusion is also indispensable in [34, 19, 13].

LEMMA 4.3. *Let R be a kernel satisfying* (K1)–(K3). *$p > 1$, $m > 0$ are finite constants and Ω is a Lipschitz bounded domain in $\mathbb{R}^d$. Then there exists a constant C depending on m, such that for all $\delta > 0$ and $u \in L^p(\Omega)$,*

$$\begin{aligned}\int_\Omega \int_\Omega & R_\delta(|x-y|)|u(x)-u(y)|^p dxdy \\ &\le C \int_\Omega \int_\Omega R_{m\delta}(|x-y|)|u(x)-u(y)|^p dxdy.\end{aligned}$$

Proof. When $m \ge 1$, by the monotone decreasing property of R, we have

$$R\left(\frac{|x-y|^2}{(m^{-1}\delta)^2}\right) \le R\left(\frac{|x-y|^2}{\delta^2}\right) \le R\left(\frac{|x-y|^2}{(m\delta)^2}\right).$$

Thus,

$$m^{-d} R_{m^{-1}\delta}(|x-y|) \le R_\delta(|x-y|) \le m^d R_{m\delta}(|x-y|).$$

The conclusion for $m \ge 1$ then follows easily and the case for $m < 1$ can be obtained from a telescoping argument, similar to those presented in [34, 13, 32]. ∎

5. Γ-convergence of nonlocal functionals.

5.1. Nonlocal model for Dirichlet problem. To prepare for the proof of Theorem 2.1, we first present some lemmas. The first one is about the property of the convolution between a kernel and a sequence of L^p functions which has a limit. It is well known that for a sequence of positive constants $\delta_n \to 0$, a L^1 kernel function R and an L^p function u, the equality

$$\lim_{n\to\infty} \|R_{\delta_n} * u - C_R u\|_{L^p} = 0 \tag{5.1}$$

holds for any $1 \le p < \infty$, where R_{δ_n} is the scaled kernel of R and $C_R := \int_{\mathbb{R}^d} R(|y|^2)dy$ is a constant only dependent on R. Replacing u in the above by a sequence $\{u_n\}$ that converges to u, we have a similar conclusion.

LEMMA 5.1. *Suppose that Ω is a domain in $\mathbb{R}^d$. R is a kernel function satisfying* (K1)–(K3). *$1 < p < \infty$. For a positive constant δ, $R_\delta(x) := \frac{1}{\delta^d} R(\frac{x^2}{\delta^2})$. $\delta_n \to 0$ and $u_n \to u$ in $L^p(\Omega)$ as $n \to \infty$. Then,*

$$\left\| \int_\Omega R_{\delta_n}(|x-y|)(u_n(x)-u_n(y))dy \right\|_{L^p(\Omega)} \longrightarrow 0, \ \text{as } n \to \infty.$$

Proof. Let $C_{R,n}(x) := \int_\Omega R_{\delta_n}(|x-y|)dy$ for $x \in \Omega$. Note that

$$\|R_{\delta_n} * u_n - C_{R,n}u_n\|_{L^p(\Omega)} \le \|R_{\delta_n} * (u_n - u)\|_{L^p(\Omega)} + \|C_{R,n}(u-u_n)\|_{L^p(\Omega)} + \|(C_{R,n} - C_R)u\|_{L^p(\Omega)} + \|R_{\delta_n} * u - C_R u\|_{L^p(\Omega)}.$$

The first two terms go to 0 by the convergence of u_n to u and boundedness of convolution with R_{δ_n} and the uniform bound of the function $\|C_{R,n}\|_{L^\infty(\Omega)} \le C_R$, while the third term goes to zero since $C_{R,n}(x) = C_R$ except for x in the layer $\{x \in \Omega, \mathrm{dist}(x,\Omega^c) < r_R\delta_n\}$ whose measure goes to 0. The last term follows from (5.1). ∎

In the functionals (2.3) under consideration, the Γ-convergence of the first term has been studied [37, Lemma 4.6]. We will directly use the liminf part, which is listed as follows.

LEMMA 5.2. *Let Ω, R, σ_R satisfy the same conditions as Theorem* 2.1. $1 < p < \infty$. *$u_n \to u$ in $L^p(\Omega)$, $\delta_n \to 0$, then*

$$\liminf_{n\to\infty} \frac{1}{\delta_n^p} \int_\Omega \int_\Omega R_{\delta_n}(|x-y|)|u_n(x) - u_n(y)|^p dxdy \ge \sigma_R E(u),$$

where

$$E(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u(x)|^p dx & \text{if } u \in W^{1,p}, \\ \infty & \text{otherwise.} \end{cases}$$

The main difference between Theorem 2.1 and the Lemma 4.6 in [37] is the additional term about the boundary $\partial\Omega$. To resolve it, the trace theorem that the $L^p(\partial\Omega)$ norm of the trace of a $W^{1,p}(\Omega)$ function can be controlled by the $W^{1,p}$ norm is useful. While the strong $W^{1,p}$ convergence from u_n to u does not follow directly from the derivation of the liminf inequality, it turns out that, with the following lemma, the weak $W^{1,p}$ convergence is enough.

LEMMA 5.3. *Suppose that Ω is a Lipschitz bounded domain and $1 < p < \infty$. Let $\{u_n\} \subset W^{1,p}(\Omega)$ satisfy that $\sup_n \|u_n\|_{W^{1,p}(\Omega)} < \infty$, and $u_n \to u$ in $L^p(\Omega)$ for some $u \in W^{1,p}(\Omega)$ with $\|Tu_n\|_{L^p(\partial\Omega)} \to 0$ as $n \to \infty$. Then we have $Tu = 0$ on $\partial\Omega$, in the sense of trace space.*

Proof. By the reflexivity of $W^{1,p}(\Omega)$, the trace theorem (see, for example, [16]) and compact embeddings of Sobolev spaces, we can see that u is the weak limit of u_n in $W^{1,p}(\Omega)$, and Tu is both the weak limit of Tu_n in $W^{1-1/p,p}(\partial\Omega)$ and the strong limit in $L^p(\partial\Omega)$. Thus $Tu = 0$. ∎

When processing the boundary term, as stated in section 2, we actually transform u_n into a more regular form $\{\tilde{u}_n\}$ defined as in (2.13). For the gradient of $\tilde{u}_n$, we have the following L^p-estimate; see similar results presented in [32].

LEMMA 5.4. *Let Ω, R satisfy the same conditions as Theorem* 2.1. $1 < p < \infty$. *$\tilde{u}_n$ is defined as in* (2.13)*. We have the following estimate about $\nabla\tilde{u}_n$:*

$$\|\nabla\tilde{u}_n\|_{L^p(\Omega)} \le \frac{C}{\delta_n}\left(\int_\Omega\int_\Omega R_{\delta_n}(|x-y|)|u_n(x) - u_n(y)|^p dxdy\right)^{\frac{1}{p}},$$

where C is a constant depending only on $\hat{K}, R$, and Ω.

Proof. By the definitions,

$$\begin{aligned}\nabla \tilde{u}_n(x) &= \frac{1}{\omega_{\delta_n}(x)^2}\bigg(\int_\Omega\int_\Omega \hat{K}_{\delta_n}(|x-y|)\nabla_x \hat{K}_{\delta_n}(|x-z|)u_n(z) \\ &\quad - \nabla_x \hat{K}_{\delta_n}(|x-y|)\hat{K}_{\delta_n}(|x-z|)u_n(z)dydz\bigg) \\ &= \frac{1}{\omega_{\delta_n}(x)^2}\left(\int_\Omega\int_\Omega \nabla_x \hat{K}_{\delta_n}(|x-y|)\hat{K}_{\delta_n}(|x-z|)(u_n(y)-u_n(z))dydz\right).\end{aligned}$$

Therefore, using the Hölder inequality, we have

$$\begin{aligned}\|\nabla \tilde{u}_n\|_{L^p} &\le C_1 \left\| \int_\Omega\int_\Omega \nabla_x \hat{K}_{\delta_n}(|x-y|)\hat{K}_{\delta_n}(|x-z|)(u_n(y)-u_n(z))dydz\right\|_{L^p} \\ &\le \frac{C_2}{\delta_n^{\frac{1}{p^*}}}\left\{\int_\Omega\int_\Omega \left(\int_\Omega |\nabla_x \hat{K}_{\delta_n}(|x-y|)|\hat{K}_{\delta_n}(|x-z|)dx\right)|u_n(y)-u_n(z)|^p dydz\right\}^{\frac{1}{p}},\end{aligned}$$

where $\frac{1}{p^*} = 1 - \frac{1}{p}$. Denote

$$\tilde{K}_{\delta_n}(y,z) := \int_\Omega \nabla_x \hat{K}_{\delta_n}(|x-y|)\hat{K}_{\delta_n}(|x-z|)dx.$$

Recalling the condition $\operatorname{supp}(\hat{K}) \subset [0, r_{\hat{K}}]$, it is obvious that $\tilde{K}_{\delta_n}(y,z) = 0$ when $|y-z| > 2\delta_n r_{\hat{K}}$. According to the regularity of kernel $\hat{K}$ and R, we may assume that there exist some constants $k_1, k_2, r_1 > 0$ such that

$$\hat{K}(x) \le k_1,\ |\hat{K}'(x)| \le k_2\ \forall x \ge 0, \quad \text{and} \quad R(x) \ge r_1\ \forall x \in \left[0, \frac{r_R}{2}\right].$$

For any y, z with $|y-z| \le 2\delta_n r_{\hat{K}}$,

$$\begin{aligned}\tilde{K}_{\delta_n}(y,z) &\le \frac{k_2}{\delta_n^{d+1}}\int_\Omega \hat{K}_{\delta_n}(|x-z|)dx \le \frac{k_2}{\delta_n^{d+1}}\int_{\mathbb{R}^d} \hat{K}_{\delta_n}(|x-z|)dx \\ &\le \frac{C_{\hat{K}}}{\delta_n^{d+1}} \le \frac{C_{\hat{K},R}}{\delta_n} R_{\frac{4r_{\hat{K}}}{r_R}\delta_n}(|y-z|).\end{aligned}$$

Hence,

$$\begin{aligned}\|\nabla \tilde{u}_n\|_{L^p} &\le \frac{C_2}{\delta_n^{\frac{1}{p^*}}}\left\{\int_\Omega\int_\Omega \tilde{K}_{\delta_n}(y,z)|u_n(y)-u_n(z)|^p dydz\right\}^{\frac{1}{p}} \\ &\le \frac{C_3}{\delta_n}\left\{\int_\Omega\int_\Omega R_{\frac{4r_{\hat{K}}}{r_R}\delta_n}(|y-z|)|u_n(y)-u_n(z)|^p dydz\right\}^{\frac{1}{p}} \\ &\le \frac{C}{\delta_n}\left\{\int_\Omega\int_\Omega R_{\delta_n}(|y-z|)|u_n(y)-u_n(z)|^p dydz\right\}^{\frac{1}{p}}.\end{aligned}$$

The last inequality holds owing to Lemma 4.3. ◻

With the preparation above, we can start to prove Theorem 2.1. First, we simplify the problem into the situation of $a \equiv 0$. To do this, we consider a $W^{1,p}$ function v whose trace is a. Then we can transform the functionals by translation:

$$F^v(u) := F(v+u),\ F_n^v := F_n(v+u).$$

Specifically,

$$F_n^v(u) = \frac{1}{\delta_n^p}\int_\Omega\int_\Omega R_{\delta_n}(|x-y|)|u(x)+v(x)-u(y)-v(y)|^p dxdy + \int_{\partial\Omega}\left|\frac{1}{\delta_n}\int_\Omega K_{\delta_n}(|x-y|)(a(x)-u(y)-v(y))dy\right|^p dx \tag{5.2}$$

and

$$F^v(u) = \begin{cases} \int_\Omega |\nabla u(x)+\nabla v(x)|^p dx & \text{if } u\in W^{1,p}, Tu\equiv 0 \text{ on } \partial\Omega, \\ \infty & \text{otherwise.}\end{cases} \tag{5.3}$$

It is obvious that the Γ-convergence from F_n to F is equivalent to the one from F_n^v to F^v. And we have the following lemma to show that the latter is similar to Theorem 2.1 when $a\equiv 0$.

LEMMA 5.5. *Let F^v, F_n^v be defined as in* (5.2), (5.3). *Then it is sufficient for Theorem* 2.1 *to prove the Γ-convergence from $\tilde{F}_n^v$ to $\sigma_R F^v$, where*

$$\tilde{F}_n^v(u) = \frac{1}{\delta_n^p}\int_\Omega\int_\Omega R_{\delta_n}(|x-y|)|u(x)+v(x)-u(y)-v(y)|^p dxdy + \int_{\partial\Omega}\left|\frac{1}{\delta_n}\int_\Omega K_{\delta_n}(|x-y|)u(y)dy\right|^p dx. \tag{5.4}$$

Proof. According to the definitions, we only need to demonstrate that

$$\left\|\frac{1}{\delta_n}\int_\Omega K_{\delta_n}(|x-y|)(v(y)-a(x))dy\right\|_{L^p(\partial\Omega)} \to 0 \text{ as } n\to\infty.$$

Consider the extension $\bar{v}$ of v on the convex hull $\bar{\Omega}$ of Ω and a sequence of C^1 approximations $\{v_n\}$ of $\bar{v}$, such that $\|v_n-\bar{v}\|_{W^{1,p}(\bar{\Omega})} = o(\delta_n^{1+\frac{d}{p}})$. By the trace theorem, this gives $\|v_n - a\|_{L^p(\partial\Omega)} = o(\delta_n^{1+\frac{d}{p}}) = o(\delta_n)$. Note that

$$\begin{aligned}
&\left\|\frac{1}{\delta_n}\int_\Omega K_{\delta_n}(|x-y|)(v(y)-a(x))dy\right\|_{L^p(\partial\Omega)} \\
&\leq \left\|\frac{1}{\delta_n}\int_\Omega K_{\delta_n}(|x-y|)(v_n(x)-a(x))dy\right\|_{L^p(\partial\Omega)} \\
&\quad + \left\|\frac{1}{\delta_n}\int_\Omega K_{\delta_n}(|x-y|)(v_n(y)-v(y))dy\right\|_{L^p(\partial\Omega)} \\
&\quad + \left\|\frac{1}{\delta_n}\int_\Omega K_{\delta_n}(|x-y|)(v_n(y)-v_n(x))dy\right\|_{L^p(\partial\Omega)} \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

For the first and second terms, they tend to zero since $\{v_n\}$ converges to v at a sufficiently rapid rate.

$$\begin{aligned}
I_1 &= \left\|\frac{1}{\delta_n}\int_\Omega K_{\delta_n}(|x-y|)dy(v_n(x)-a(x))\right\|_{L^p(\partial\Omega)} \\
&= O\left(\frac{1}{\delta_n}\|v_n(x)-a(x)\|_{L^p(\partial\Omega)}\right) \\
&= o(1),
\end{aligned}$$

and

$$\begin{aligned}
I_2^p &= \int_{\partial\Omega} \left| \frac{1}{\delta_n} \int_\Omega K_{\delta_n}(|x-y|)(v_n(y)-v(y))dy \right|^p dx \\
&= O\left(\int_{\partial\Omega} \int_\Omega \frac{1}{\delta_n^p} K_{\delta_n}(|x-y|)|v_n(y)-v(y)|^p dydx \right) \\
&= O\left(\frac{1}{\delta_n^{p+d}} \|v_n - v\|^p_{L^p(\Omega)} \right) \\
&= o(1).
\end{aligned}$$

For the last term, using the Taylor expansion, we have

$$\begin{aligned}
I_3^p &= O\left(\int_{\partial\Omega} \int_\Omega K_{\delta_n}(|x-y|) \frac{1}{\delta_n^p} |v_n(x)-v_n(y)|^p dydx \right) \\
&= O\left(\int_{\partial\Omega} \int_{\bar\Omega} K_{\delta_n}(|x-y|) \frac{1}{\delta_n^p} |v_n(x)-v_n(y)|^p dydx \right) \\
&= O\left(\int_{\partial\Omega} \int_{\bar\Omega} \int_0^1 K_{\delta_n}(|x-y|) \frac{1}{\delta_n^p} |(y-x)\cdot\nabla v_n(x+t(y-x))|^p dtdydx \right) \\
&= O\left(\int_{\partial\Omega} \int_{|z|\le r_K, x+\delta_n z\in\bar\Omega} \int_0^1 K(|z|^2)|z\cdot\nabla v_n(x+t\delta_n z)|^p dtdzdx \right) \\
&= O\left(\int_{\partial\Omega} \int_0^1 \int_{|z|\le r_K, x+\delta_n z\in\bar\Omega} |\nabla v_n(x+t\delta_n z)|^p dzdtdx \right) \\
&= O(\|\nabla v_n\|_{L^p(\tilde\Omega_{\delta_n})}),
\end{aligned}$$

where $\tilde\Omega_{\delta_n} := \{x\in\bar\Omega \big| dist(x,\partial\Omega) \le \delta_n r_K\}$ and the last equation holds since

$$\int_{|z|\le r_K, x+\delta_n z\in\bar\Omega} |\nabla v_n(x+t\delta_n z)|^p dz \le \|\nabla v_n\|_{L^p(\tilde\Omega_{\delta_n})}$$

for all fixed $x\in\partial\Omega,\ t\in(0,1]$. Note that the measure of $\tilde\Omega_{\delta_n}$ tends to zero. Hence,

$$\|\nabla v_n\|_{L^p(\tilde\Omega_{\delta_n})} \le \|\nabla\bar v\|_{L^p(\tilde\Omega_{\delta_n})} + \|\bar v - v_n\|_{W^{1,p}(\bar\Omega)} = o(1). \qquad \square$$

To complete the proof of Theorem 2.1, we show the Γ-convergence of $\tilde F_n^v$ to F^v. The proof is divided into two parts according to Definition 4.1.

Lemma 5.6 (the liminf inequality). *Suppose that Ω is a Lipschitz bounded domain in $\mathbb{R}^d$. K,R are two kernel functions satisfying* (K1)–(K3). $1<p<\infty$. $\delta_n\to 0$ *as $n\to\infty$. Then for any $u_n\to u$ in $L^p(\Omega)$,*

$$\liminf_{n\to\infty} \tilde F_n^v(u_n) \ge \sigma_R F^v(u),$$

where $\tilde F_n^v, F^v$ are defined as in (5.4), (5.3).

Proof. Without loss of generality, we can suppose that $\liminf_{n\to\infty}\tilde F_n^v(u_n)<\infty$. From Lemma 5.2, we have

$$\liminf_{n\to\infty} \frac{1}{\delta_n^p} \int_\Omega\int_\Omega R_{\delta_n}(|x-y|)|u_n(x)+v(x)-u_n(y)-v(y)|^p dxdy \ge \sigma_R E(u+v),$$

which also yields that $u \in W^{1,p}$. We then claim that $Tu \equiv 0$ on $\partial\Omega$. Once the claim is verified, we get

$$\begin{aligned}&\liminf_{n\to\infty} \tilde{F}_n^v(u_n)\\ &\quad \geq \liminf_{n\to\infty} \frac{1}{\delta_n^p} \int_\Omega \int_\Omega R_{\delta_n}(|x-y|)|u_n(x)+v(x)-u_n(y)-v(y)|^p dxdy\\ &\quad \geq \sigma_R E(u+v)\\ &\quad = \sigma_R F^v(u)\end{aligned}$$

and the proof is complete. To show the claim, note that we also have

$$\liminf_{n\to\infty} \int_{\partial\Omega} \left| \frac{1}{\delta_n} \int_\Omega K_{\delta_n}(|x-y|)u_n(y)dy \right|^p dx < \infty.$$

Define

$$\tilde{u}_n(x) := \frac{1}{\omega_{\delta_n}(x)} \int_\Omega K_{\delta_n}(|x-y|)u_n(y)dy, \text{ and } \omega_{\delta_n}(x) := \int_\Omega K_{\delta_n}(|x-y|)dy.$$

The above property implies the fact that there exists a subsequence (still denoted by $\{\tilde{u}_n\}$ for simplification of notation) of $\{\tilde{u}_n\}$ satisfying

$$\lim_{n\to\infty} \|T\tilde{u}_n\|_{L^p(\partial\Omega)} = 0$$

owing to the positive lower bound of $\omega_{\delta_n}(x)$. For the $W^{1,p}$ estimation, first,

$$\|\tilde{u}_n\|_{L^p(\Omega)} \leq C_1 \left(\int_\Omega \int_\Omega K_{\delta_n}(|x-y|)u_n^p(y)dydx \right)^{\frac{1}{p}} \leq C_2 \|u_n\|_{L^p(\Omega)},$$

which yields that the $L^p(\Omega)$ norm of $\{\tilde{u}_n\}$ is uniformly bounded. Moreover,

$$\begin{aligned}&\limsup_{n\to\infty} \left\| \frac{1}{\delta_n} R_{\delta_n}^{\frac{1}{p}}(|x-y|)(v(x)-v(y)) \right\|_{L^p(\Omega\times\Omega)}\\ &\quad = \limsup_{n\to\infty} \left\| \frac{1}{\delta_n} R_{\delta_n}^{\frac{1}{p}}(|x-y|)\nabla v(x)\cdot(x-y) \right\|_{L^p(\Omega\times\Omega)}\\ &\quad = \limsup_{n\to\infty} \left(\int_\Omega \int_\Omega R_{\delta_n}(|x-y|)|\nabla v(x)\cdot \frac{x-y}{\delta_n}|^p dxdy \right)^{\frac{1}{p}} < \infty.\end{aligned}$$

So the assumption $\liminf_{n\to\infty} \tilde{F}_n^v(u_n) < \infty$ also yields that

$$\begin{aligned}&\liminf_{n\to\infty} \frac{1}{\delta_n} \left(\int_\Omega \int_\Omega R_{\delta_n}(|x-y|)|u_n(x)-u_n(y)|^p dxdy \right)^{\frac{1}{p}}\\ &\leq \left\| \frac{1}{\delta_n} R_{\delta_n}^{\frac{1}{p}}(|x-y|)(u(x)-u(y)+v(x)-v(y)) \right\|_{L^p(\Omega\times\Omega)}\\ &\quad + \left\| \frac{1}{\delta_n} R_{\delta_n}^{\frac{1}{p}}(|x-y|)(v(x)-v(y)) \right\|_{L^p(\Omega\times\Omega)} < \infty.\end{aligned}$$

Therefore, with Lemma 5.4, we get the conclusion that the $W^{1,p}$ norm of $\{\tilde{u}_n\}$ is uniformly bounded:

$$\sup_n \|\tilde{u}_n\|_{W^{1,p}(\Omega)} < \infty.$$

Finally, the transform from u_n to $\tilde{u}_n$ preserves the L^p convergence due to Lemma 5.1:

$$\begin{aligned}\|\tilde{u}_n - u\|_{L^p(\Omega)} &= \left\| \frac{1}{\omega_{\delta_n}(x)} \int_\Omega K_{\delta_n}(|x-y|)(u_n(y)-u(x))dy \right\|_{L^p(\Omega)} \\ &\le C_3 \left\| \int_\Omega K_{\delta_n}(|x-y|)(u_n(y)-u(x))dy \right\|_{L^p(\Omega)} \\ &\le C_3 \Bigg(\left\| \int_\Omega K_{\delta_n}(|x-y|)(u_n(y)-u_n(x))dy \right\|_{L^p(\Omega)} \\ &\quad + \left\| \int_\Omega K_{\delta_n}(|x-y|)(u_n(x)-u(x))dy \right\|_{L^p(\Omega)} \Bigg) \\ &\le C_3 \left\| \int_\Omega K_{\delta_n}(|x-y|)(u_n(y)-u_n(x))dy \right\|_{L^p(\Omega)} \\ &\quad + C_4 \|u_n - u\|_{L^p(\Omega)} \longrightarrow 0, \quad (\text{as } n \to \infty).\end{aligned}$$

Hence, with Lemma 5.3, $\|Tu\|_{L^p(\partial\Omega)} = 0$. ∎

For the limsup inequality, the following lemma is useful. It provides the connection between the nonlocal Dirichlet energy and the local version defined on $W^{1,p}(\Omega)$. The proof can be done using the Taylor expansion; see proofs in, for example, [5].

LEMMA 5.7. *Suppose that Ω is a Lipschitz bounded domain in $\mathbb{R}^d$. R is a kernel satisfying* (K1)–(K3). *Then, for all $u \in W^{1,p}(\Omega)$, $\delta > 0$,*

$$\frac{1}{\delta^p} \int_\Omega \int_\Omega R_\delta(|x-y|)|u(x)-u(y)|^p dxdy \le \sigma_R \|\nabla u\|^p_{L^p(\Omega)}.$$

LEMMA 5.8 (the limsup inequality). *Suppose that Ω is a Lipschitz bounded domain in $\mathbb{R}^d$. K, R are two kernel functions satisfying* (K1)–(K3). *$\delta_n \to 0$ as $n \to \infty$. Then for any u in $L^p(\Omega)$, there exists a sequence $\{u_n\}$ converging to u in $L^p(\Omega)$ and*

$$\limsup_{n\to\infty} \tilde{F}^v_n(u_n) \le \sigma_R F^v(u),$$

where $\tilde{F}^v_n, F^v$ are defined as in (5.4), (5.3).

Proof. We can suppose that $F^v(u) < \infty$, which yields $u \in W^{1,p}$ and $Tu \equiv 0$. With the density of C^∞ functions in $W^{1,p}$ space, we can choose $\{u_n\}$ to be a sequence of smooth functions that converges to u with respect to the $W^{1,p}(\Omega)$ norm. We additionally require $u_n \equiv 0$ in $\Omega^c_{\delta_n}$, where $\Omega_{\delta_n} := \{x \in \Omega \big| \text{dist}(x, \Omega^c) \ge r_K \delta_n\}$. This condition can be attained by multiplying a smooth approximation of the indicator function $\mathbf{1}_{\Omega_{\delta_n}}$.

For such $\{u_n\}$, it is obvious that

$$\left(\int_{\partial\Omega} \left| \frac{1}{\delta_n} \int_\Omega K_{\delta_n}(|x-y|) u_n(y) dy \right|^p dx \right)^{\frac{1}{p}} = 0.$$

To simplify the notation, we denote $u_n + v, u + v$ by u^v_n, u^v. With Lemma 5.7,

$$\frac{1}{\delta_n^p} \int_\Omega \int_\Omega R_{\delta_n}(|x-y|)|u^v_n(x) - u^v_n(y)|^p dxdy \le \sigma_R \|\nabla u^v_n(z)\|^p_{L^p(\Omega)} \; \forall n > o.$$

As a result, with $u^v_n \to u^v$ in $W^{1,p}(\Omega)$,

$$\begin{aligned}\limsup_{n\to\infty} \tilde{F}^v_n(u_n) &= \limsup_{n\to\infty} \frac{1}{\delta_n^p} \int_\Omega \int_\Omega R_{\delta_n}(|x-y|)(u^v_n(x) - u^v_n(y)|^p dxdy \\ &\le \sigma_R \limsup_{n\to\infty} \|\nabla u^v_n\|^p_{L^p(\Omega)} = \sigma_R \|\nabla u^v\|^p_{L^p(\Omega)} = \sigma_R F^v(u).\end{aligned}$$ ∎

Finally, with Lemmas 5.5, 5.6, and 5.8, the proof of Theorem 2.1 is complete.

5.2. Nonlocal model for eigenvalue problem. Recall that $F_{e,n}^k$, $\tilde{F}_{e,n}^k$, and F_e^k defined as in (2.7), (2.9), and (2.5) actually equal to F_n or F if u satisfies the constraints. With the Γ-convergence of F_n to F proved in Theorem 2.1, we only need to complete the remaining proof.

Proof of Theorem 2.2. We apply the induction with respect to k.

Unnormalized case, liminf inequality: Without loss of generality, we can assume that $\liminf_{n\to\infty} F_{e,n}^k(u_n) < \infty$. Then there exists a subsequence $\{n_l\}$ such that $\|u_{n_l}\|_{L^2(\Omega)} = 1$ and $u_{n_l} \perp V_{k-1}^n$ $\forall l$. Note that $u_{n_l} \to u$ in $L^2(\Omega)$, so we have $\|u\|_{L^2(\Omega)} = 1$. For $k = 1$, $u \perp V_0$ is trivial since $V_0 = \emptyset$. For $k > 1$, suppose that $F_{e,n}^l$ Γ-converges to F_e^l for all $l < k$. Then with the compactness result in section 6 and the property of Γ-convergence (Lemma 4.2), we have $\mu_l^n \to \mu_l$ in L^2 norm for all $l < k$. Hence,

$$\langle u, \mu_l \rangle = \lim_{n\to\infty} \langle u_n, \mu_l^n \rangle = 0$$

for all $l < k$, which means that $u \perp V_{k-1}$ and

$$\liminf_{n\to\infty} F_{e,n}^k(u_n) = \liminf_{n\to\infty} F_n(u_n) \le \sigma_R F(u) = \sigma_R F_e^k(u).$$

Unnormalized case, limsup inequality: Supposing that $F(u) < \infty$, from Lemma 5.8 with $v \equiv 0$, we get a sequence $\{u_n\}$ converging to u in L^2 norm. Meanwhile, u_n is smooth and supported in Ω_{δ_n}. For $k = 1$, replacing u_n with $\frac{u_n}{\|u_n\|_{L^2}}$ and using the same process in Lemma 5.8, we can prove the limsup inequality in the unnormalized case. For $k > 1$, we consider a smooth approximation ν_m^n for a basis μ_m^n of V_{k-1}^n. ν_m^n is also supported in Ω_{δ_n} and $\|\mu_m^n - \nu_m^n\|_{L^2} \to 0$. Let

$$\tilde{u}_n = u_n - \sum_{m=1}^{k-1} \alpha_m^n \nu_m^n, \quad \text{where} \quad \alpha^n := \begin{pmatrix} \alpha_1^n \\ \vdots \\ \alpha_{k-1}^n \end{pmatrix} = G_n^{-1} b_n,$$

$$G_n = \begin{pmatrix} \langle \mu_1^n, \nu_1^n \rangle & \dots & \langle \mu_1^n, \nu_{k-1}^n \rangle \\ \vdots & \ddots & \vdots \\ \langle \mu_{k-1}^n, \nu_1^n \rangle & \dots & \langle \mu_{k-1}^n, \nu_{k-1}^n \rangle \end{pmatrix}, \quad b_n = \begin{pmatrix} \langle \mu_1^n, u_n \rangle \\ \vdots \\ \langle \mu_{k-1}^n, u_n \rangle \end{pmatrix}$$

and $\langle u, v \rangle = \int_\Omega uv dx$ is the inner product of $L^2(\Omega)$. G_n is invertible for large enough n because $G_n \to I$ as $n \to \infty$. It can be verified that with such modification, $\tilde{u}_n$ is still smooth, is supported in Ω_{δ_n}, and becomes perpendicular to V_{k-1}^n. Furthermore, with the induction assumption, $b_n \to \mathbf{0}$ as $n \to \infty$ so the L^2 convergence is also retained. Finally, by replacing $\tilde{u}_n$ with $\frac{\tilde{u}_n}{\|\tilde{u}_n\|_{L^2}}$ the conclusion can be proved in the same way as in Lemma 5.8.

Normalized case: The proof in the normalized case is similar to the one in the unnormalized case with the fact that

$$\lim_{n\to\infty} |\langle u, v_n \rangle_n - \langle u, v \rangle| = 0$$

for all $u \in L^2(\Omega)$ and $v_n \to v$ in the L^2 norm. Actually, by extending the domain of v, v_n, and u to $\mathbb{R}^d$ with zero-value outside Ω, with Lemma 5.1,

$$
\begin{aligned}
&|\langle u, v_n\rangle_n - \langle u, v\rangle| \\
&\quad = \left| \int_\Omega \int_\Omega W_{\delta_n}(|x-y|)u(x)v_n(y)dxdy - \int_\Omega u(x)v(x)dx \right| \\
&\quad = \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W_{\delta_n}(|x-y|)u(x)v_n(y)dxdy - \int_{\mathbb{R}^d} u(x)v(x)dx \right| \\
&\quad \le \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W_{\delta_n}(|x-y|)|v(x)-v_n(y)|dy|u(x)|dx \\
&\quad \le \left\| \int_\Omega W_{\delta_n}(|x-y|)|v(x)-v_n(y)|dy \right\|_{L^2(\mathbb{R}^d)} \|u\|_{L^2(\mathbb{R}^d)} \to 0, \text{ as } n\to\infty.
\end{aligned}
$$

The proof is complete. ∎

5.3. Nonlocal model with general boudary term. The proof of Theorem 2.1 allows us to extract the requirements for the boundary terms in the nonlocal model. To fulfill the liminf inequality, we need $\{\tilde{u}_n\}$ to satisfy the conditions of Lemma 5.3. Among them, the part involved in the boundary term is that $\|\tilde{u}_n\|_{L^2(\partial\Omega)}$ should tend to 0 as $n\to\infty$. This is the reason why we require the coercivity presented in Theorem 2.3. Regarding the limsup inequality, the construction in Lemma 5.8 can be employed to maintain the boundary term at 0 as long as the kernel has a gradually shrinking compact support.

Proof of Theorem 2.3. The liminf inequality. From Lemma 5.2, we only need to demonstrate that $Tu\equiv 0$ on $\partial\Omega$ with $u_n\to u$ in L^2 norm and

$$\liminf_{n\to\infty} F_n(u_n) < \infty.$$

The coercivity of boundary term $E(u,0)$ derives that there exists a kernel K satisfying (K1)–(K3) such that

$$\lim_{n\to\infty} \|\tilde{u}_n\|_{L^2(\partial\Omega)} = 0$$

up to a subsequence. $\tilde{u}_n\to u$ in L^2 norm with Lemma 5.1. And Lemma 5.4 gives the uniform boundedness of $\|\tilde{u}_n\|_{H^1(\Omega)}$. Therefore, using Lemma 5.3, the proof of the liminf inequality is completed.

The limsup inequality. With a method similar to that in Lemma 5.8, we can suppose $F(u)<\infty$ and select $\{u_n\}$ as a sequence of C^∞ approximations converging to u with respect to H^1 norm. Moreover, we require $u_n\equiv 0$ in $\Omega_{c_n}^c$, where $\Omega_{c_n} := \{x\in\Omega \big| \operatorname{dist}(x,\Omega^c)\ge c_n\}$. Such a property of $\{u_n\}$ leads to the vanishing of boundary term $E_n(u,0)$ because the intersection of the support sets of the kernel $\rho_{\delta_n,x}$ and u_n is empty. As for the first term, using Lemma 5.7, we have

$$\frac{1}{\delta_n^2}\int_\Omega\int_\Omega R_{\delta_n}(|x-y|)|u_n(x)-u_n(y)|^2 dxdy \le \sigma_R \|\nabla u_n(z)\|^2_{L^2(\Omega)}$$

for all n. Hence,

$$
\begin{aligned}
&\limsup_{n\to\infty} F_n(u_n) \\
&\quad = \limsup_{n\to\infty} \frac{1}{\delta_n^2}\int_\Omega\int_\Omega R_{\delta_n}(|x-y|)|u_n(x)-u_n(y)|^2 dxdy \\
&\quad \le \limsup_{n\to\infty} \sigma_R\|\nabla u_n\|^2_{L^2(\Omega)} = \sigma_R\|\nabla u\|^2_{L^2(\Omega)} = F(u).
\end{aligned}
$$

∎

The construction in Theorem 2.1 is not the only one that fulfills our sufficient conditions. Another example is, as we have mentioned in Remark 2.5, selecting $\rho_{\delta_n,x}(y,z) = \frac{1}{\delta_n^2} K_{\delta_n}(|x-y|)\delta(|x-y|)$. In this case, our nonlocal functional is formulated as

$$
\begin{aligned}
F_n(u) = & \frac{1}{\delta_n^2}\int_\Omega\int_\Omega R_{\delta_n}(|x-y|)|u(x)-u(y)|^2 dxdy \\
& + \frac{1}{\delta_n^2}\int_{\partial\Omega}\int_\Omega K_{\delta_n}(|x-y|)u(y)^2 dydx,
\end{aligned} \tag{5.5}
$$

and we have the following corollary due to Theorem 2.3.

COROLLARY 5.9. *Suppose that* Ω *is a Lipschitz bounded domain in* $\mathbb{R}^d$. R *is a kernel satisfying* (K1)–(K3). K *is a nonnegative compactly supported kernel with a uniform positive lower bound in a neighborhood of the origin. Then, we have*

$$F_n \xrightarrow{\Gamma} \sigma_R F \quad \text{in } L^2(\Omega),$$

where F_n, F *are defined as in* (5.5), (2.6).

Proof. $\rho_{\delta_n,x}$ has a shrinking compact support since

$$\rho_{\delta_n,x}(y,z) = \frac{1}{\delta_n^2}K_{\delta_n}(|x-y|)\delta(|y-z|)$$

and the kernel K is compactly supported.

Note that K is nonnegative and there exist constants $c_1,\ c_2 > 0$ such that $K(s) < c_1$ for all $s \in [0, c_2]$. Define a new kernel $\hat{K}: [0,\infty) \to [0,\infty)$,

$$\hat{K}(s) := \begin{cases} \dfrac{c_1}{c_2^2}(s-c_2)^2 & \text{if } s \le c_2, \\ 0 & \text{if } s > c_2. \end{cases}$$

It can be easily verified that $\hat{K}$ satisfies (K1)–(K3) and $K(s) \ge \hat{K}(s)$ for all $s \ge 0$. Hence, with $\hat{K}_\delta$ being the rescaled kernel defined by $\hat{K}$, we have

$$
\begin{aligned}
E_n(u,0) &= \frac{1}{\delta_n^2}\int_{\partial\Omega}\int_\Omega K_{\delta_n}(|x-y|)u(y)^2 dydx \\
&\ge \frac{1}{\delta_n^2}\int_{\partial\Omega}\int_\Omega \hat{K}_{\delta_n}(|x-y|)u(y)^2 dydx \\
&\ge \frac{C}{\delta_n^2}\int_{\partial\Omega}\int_\Omega \hat{K}_{\delta_n}(|x-y|)dy \int_\Omega \hat{K}_{\delta_n}(|x-y|)u(y)^2 dydx \\
&\ge \frac{C}{\delta_n^2}\int_{\partial\Omega}\left(\int_\Omega \hat{K}_{\delta_n}(|x-y|)u(y)dy\right)^2 dx,
\end{aligned}
$$

which means that the functional E_n associated with $\rho_{\delta_n,x}$ is coercive as defined in Theorem 2.3 for $C_n = \frac{\delta_n^2}{C}$ and

$$\tilde{u}_n = \frac{1}{w_{\delta_n}(x)}\int_\Omega \hat{K}_{\delta_n}(|x-y|)u(y)dy.$$

Therefore, with Theorem 2.3, $\{F_n\}$ Γ-converges to F. ∎

6. Compactness. In this section, we demonstrate that any minimizing sequence of the nonlocal functionals $\{F_n\}$ defined as in (2.3) or (2.10), where $\{F_{e,n}^k\}$ is defined as in (2.7) and $\{\tilde{F}_{e,n}^k\}$ is defined as in (2.9), is relatively compact in $L^p(\Omega)$. In the literature, such kinds of compactness results have been studied for nonlocal functions using the techniques developed in [5, 30]. For smooth kernels, one can also directly work the compactness of the mollified sequences; see, for example, [32]. In our case, this corresponds to the use of mollification given by (2.13). Note that if $\{u_n\}$ is a minimizing sequence of $\{F_n\}$, then $\sup_n\{F_n(u_n)\}<\infty$. Hence it is sufficient to show that $\{u_n\}$ is relatively compact if one of the following three conditions holds: $\sup_n\{F_n(u_n)\}<\infty$, $\sup_n\{F_{e,n}^k(u_n)\}<\infty$, and $\sup_n\{\tilde{F}_{e,n}^k(u_n)\}<\infty$. Recall that $F_{e,n}^k$ and $\tilde{F}_{e,n}^k$ are F_n with additional constraints added, which means that $F_{e,n}^k(u)$, $\tilde{F}_{e,n}^k(u)\geq F_n(u)$ for all $u\in L^2(\Omega)$. As for F_n, the relative compactness can be derived mainly by the interior term of the nonlocal functional defined as follows:

$$F_n^i=\frac{1}{\delta_n}\left(\int_\Omega\int_\Omega R_{\delta_n}(|x-y|)|u(x)-u(y)|^p dxdy\right)^{\frac{1}{p}}. \tag{6.1}$$

LEMMA 6.1. *Suppose that Ω is a Lipschitz bounded domain in $\mathbb{R}^d$. $1<p<\infty$. R is a kernel satisfying* (K1)–(K3). *$\{\delta_n\}$ is a sequence of positive constants tending to 0 as $n\to\infty$. $\{u_n\}$ is a bounded sequence in $L^p(\Omega)$ and satisfies*

$$\sup_n F_n^i(u_n)<\infty,$$

where F_n^i is defined as in (6.1)*. Then $\{u_n\}$ is a relatively compact sequence in $L^p(\Omega)$.*

Proof. Recalling $\tilde{u}_n$ defined as in (2.13), we have shown that $\|\tilde{u}_n-u_n\|_{L^p(\Omega)}\to 0$ as $n\to\infty$ in Lemma 5.1. Hence, $\{\tilde{u}_n\}$ is also bounded in $L^p(\Omega)$. With the condition $\sup_n F_n^i(u_n)<\infty$ and Lemma 5.4, $\{\tilde{u}_n\}$ is actually bounded in $W^{1,p}(\Omega)$. Hence, $\{\tilde{u}_n\}$ is relatively compact in $L^p(\Omega)$ due to the Rellich–Kondrachov theorem (see, for example, Theorem 6.3 in [1]). So is $\{u_n\}$, because $\{\tilde{u}_n\}$ and $\{u_n\}$ are asymptotically approximated in $L^p(\Omega)$. ∎

Note that $\{u_n\}$ is required to be bounded in Lemma 6.1. We claim that this requirement can be deduced by the boundedness of $\{F_n(u_n)\}$ and no additional assumptions are needed. Recall the Poincaré inequality (see, for example, section 5.8.1 of [17]). Let Ω be a Lipschitz bounded domain in $\mathbb{R}^d$, $1\leq p\leq\infty$. Then there exists a constant C, depending only on d,p, and Ω, such that

$$\|u-(u)_\Omega\|_{L^p(\Omega)}\leq C\|\nabla u\|_{L^p(\Omega)} \tag{6.2}$$

for each function $u\in W^{1,p}(\Omega)$ and $(u)_\Omega:=\frac{1}{|\Omega|}\int_\Omega udx$ is the average of u over Ω. In the nonlocal model F_n defined as in (2.3), the interior term and the boundary term can be considered as the approximation of $\|\nabla u\|_{L^p(\Omega)}$ and $\|u\|_{L^p(\partial\Omega)}^p$, respectively. Hence, with the aid of the Poincaré inequality, we can establish its nonlocal counterpart as the following lemma.

LEMMA 6.2. *Suppose that Ω is a Lipschitz bounded domain in $\mathbb{R}^d$. $1<p<\infty$. K,R are two kernel functions satisfying* (K1)–(K3). *$\{\delta_n\}$ is a sequence of positive constants tending to 0 as $n\to\infty$. $\{u_n\}\subset L^p(\Omega)$ is a sequence satisfying $\sup_n F_n(u_n)<\infty$, where F_n is defined as in* (2.3)*. Then*

$$\sup_n\|u_n\|_{L^p(\Omega)}<\infty.$$

Proof. Define

$$\tilde{u}_n(x) := \frac{1}{\omega_{\delta_n}(x)} \int_\Omega \hat{K}_{\delta_n}(|x-y|) u_n(y) dy, \text{ and } \omega_{\delta_n}(x) := \int_\Omega \hat{K}_{\delta_n}(|x-y|) dy.$$

With Lemma 5.4 and the condition that $\sup_n F_n(u_n) < \infty$, we have $\sup_n \|\nabla \tilde{u}_n\|_{L^p(\Omega)} < \infty$ and $\sup_n \|\tilde{u}_n\|_{L^p(\partial\Omega)} < \infty$. By the classical Poincaré inequality, there is a constant $C_1(p,d,\Omega) > 0$ such that

$$\|u\|_{L^p(\Omega)} \leq C_1(p,d,\Omega)(\|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^p(\partial\Omega)}), \; \forall u \in W^{1,p}(\Omega).$$

So $\sup_n \|\tilde{u}_n\|_{L^p(\Omega)} < \infty$. Meanwhile, as $n \to \infty$, we have $\|\tilde{u}_n - u_n\|_{L^p(\Omega)} \to 0$ by the proof of Lemma 5.1. Hence, we have the desired uniform bound of $\{\|u_n\|_{L^p(\Omega)}\}$. ☐

For F_n defined as in (2.10), we can obtain the boundedness of $\{u_n\}$ with the same method in Lemma 6.2 under the coercivity assumption. With the compactness result above, Theorems 2.6 to 2.8 can be demonstrated with the *Γ*-convergence Theorems 2.1 to 2.3 and its property Lemma 4.2.

7. Conclusion. In this paper, we propose a penalty formulation for some variational nonlocal Dirichlet problems. Sufficient conditions for the boundary terms of these models to ensure *Γ*-convergence are presented. Based on this work, there are several aspects for future research. First, the coercivity proposed in Theorem 2.3 may be difficult to verify in certain situations. Alternative conditions that are more intuitive, albeit stronger, may be explored. The conditions studied in this work are only sufficient, and it would be valuable to investigate necessary conditions as well. Furthermore, it is also worthy of consideration to extend the study here to models such as biharmonic equations, Stokes systems, and other linear and nonlinear equations of broad interest.

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