One Dimensional Hyperbolic Conservation Laws: Past and Future

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1 Introduction

Aim of these notes is provide a brief review of the current well-posedness theory for hyperbolic systems of conservation laws in one space dimension, also pointing out open problems and possible research directions. They supplement the slides of the short course given by the author in Erice, May 2023, available at: sites.google.com/view/erice23/speakers-and-slides.

Section 2 introduces basic definitions, including the concept of weak solution, and various admissibility conditions. Section 3 describes several approximation methods. The main results on global existence of weak solutions and their continuous dependence on initial data are recalled in Sections 4 and 5. The recent advances, on the uniqueness of weak solutions that satisfy the Liu admissibility condition, are covered in greater detail in Section 6. The relevance of these results toward error bounds for all kinds of approximate solutions is discussed in Section 7, together with two specific open problems. Finally, Section 8 is devoted to solutions with possibly unbounded total variation, recalling the main known results and pointing out some research directions.

2 Basic concepts

2.1 Hyperbolic systems.

A system of conservation laws in one space dimension has the form

$$u_t + f(u)_x \;=\; 0 \tag{2.1}$$

In components, this can be written as

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} f_1(u) \\ \vdots \\ f_n(u) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.2}$$

Here $u = (u_1, \ldots, u_n)^T$ is the vector of **conserved quantities**, while $f = (f_1, \ldots, f_n)^T$ is the vector of **fluxes**. Conservation laws provide the fundamental mathematical models in continuum physics [44]. A primary example is provided by the Euler equations of gas dynamics, accounting for the conservation of mass, momentum and energy [46].

Smooth solutions to the system of PDEs (2.1) can be obtained by solving the equivalent quasilinear system

$$u_t + A(u)u_x = 0, \qquad \text{where} \quad A(u) \doteq Df(u). \tag{2.3}$$

We say that the system (2.1) is **strictly hyperbolic** if at every point u the Jacobian matrix $A(u) \doteq Df(u)$ has n real distinct eigenvalues, say

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u). \tag{2.4}$$

When this holds, one can find bases of right and left eigenvectors, say $\{r_1, \ldots, r_n\}$, $\{l_1, \ldots, l_n\}$, with

$$A(u)r_i(u) = \lambda_i(u)r_i(u), \qquad l_i(u)A(u) = \lambda_i(u)l_i(u), \qquad i = 1, \ldots, n. \tag{2.5}$$

Usually one chooses to normalize these vectors so that

$$\big|r_i(u)\big| = 1, \qquad l_i(u)r_j(u) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The behavior of eigenvalues of $Df(u)$ strongly affect the nature of solutions to (2.1). Following classical literature [54], we say that the i-th characteristic field is **genuinely nonlinear** if the directional derivative of the eigenvalue λ_i in the direction of the corresponding eigenvector $r_i(u)$ satisfies

$$\nabla\lambda_i(u) \cdot r_i(u) > 0 \qquad \text{for all } u. \tag{2.6}$$

On the other hand, we say that the i-th characteristic field is **linearly degenerate** if

$$\nabla\lambda_i(u) \cdot r_i(u) = 0 \qquad \text{for all } u. \tag{2.7}$$

Throughout the following we assume that the flux function f is at least twice continuously differentiable, so that the above derivatives are well defined.

Example 2.1. In Lagrangian coordinates, the Euler equations of **isentropic gas dynamics** take the form

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = 0. \end{cases} \tag{2.8}$$

Here ρ is the density of the gas, $v = \rho^{-1}$ is specific volume, u is the velocity and $p = p(v)$ is the pressure. A natural choice for the pressure is $p(v) = kv^{-\gamma}$, with $1 \leq \gamma \leq 3$. The eigenvalues of the Jacobian matrix

$$A \doteq Df = \begin{pmatrix} 0 & -1 \\ p'(v) & 0 \end{pmatrix}$$

are

$$\lambda_1 \;=\; -\sqrt{-p'(v)}\,, \qquad\qquad \lambda_2 \;=\; \sqrt{-p'(v)}.$$

Since $p'(v) < 0$, the system is strictly hyperbolic. A further computation reveals that both characteristic fields are genuinely nonlinear.

2.2 Weak solutions.

A key feature of hyperbolic conservation laws is that, even for initial data $\bar u \in \mathcal{C}^1$ (i.e., continuously differentiable), the gradient u_x of the solution may blow up at a finite time T (see Fig. 1). In order to prolong the solution also for $t > T$, one must work within a space of discontinuous functions, interpreting the equation (2.1) in distributional sense. In the following, $\mathbf{L}^1_{loc}(\Omega)$ denotes the space of locally integrable functions defined on an open subset $\Omega \subset \mathbb{R} \times \mathbb{R}$, with values in $\mathbb{R}^n$. Moreover, $\mathcal{C}^1_c(\Omega)$ denotes the space of continuously differentiable functions with compact support.

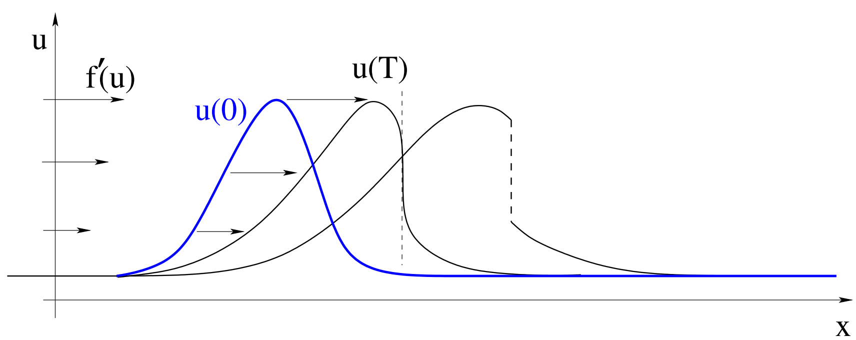

Figure 1: An example where the gradient u_x of the solution becomes unbounded at a finite time T. For $t > T$, the solution contains a shock and must be interpreted in distributional sense.

Definition 2.1. *Let $u = u(t,x)$ be a function defined on an open set $\Omega \subseteq \mathbb{R} \times \mathbb{R}$. We say that u is a* **weak solution** *to the system of conservation laws (2.1) if $u, f(u) \in \mathbf{L}^1_{loc}(\Omega)$ and*

$$\iint \big\{ u\phi_t + f(u)\phi_x \big\}\, dxdt \;=\; 0 \qquad \textit{for all} \quad \phi \in \mathcal{C}^1_c(\Omega).$$

Definition 2.2. *A function $u = u(t,x)$ is a* **weak solution** *to the* **Cauchy problem**

$$u_t + f(u)_x \;=\; 0, \qquad\qquad u(0,x) = \bar u(x), \qquad\qquad t \in [0,T], \tag{2.9}$$

if the map $t \mapsto u(t,\cdot)$ is continuous with values in $\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)$, satisfies the initial condition in (2.9), and moreover

$$\int_0^T \int_{-\infty}^{+\infty} \big\{ u\phi_t + f(u)\phi_x \big\}\, dxdt \;=\; 0 \qquad \textit{for all} \quad \phi \in \mathcal{C}^1_c\big(]0,T[\times\mathbb{R}\big).$$

The simplest example of a discontinuous solution to (2.1) is a single shock, shown in Fig. 2.

$$u(t,x) \;=\; \begin{cases} u^- & \text{if} \quad x < \lambda t\,, \\ u^+ & \text{if} \quad x > \lambda t\,. \end{cases} \tag{2.10}$$

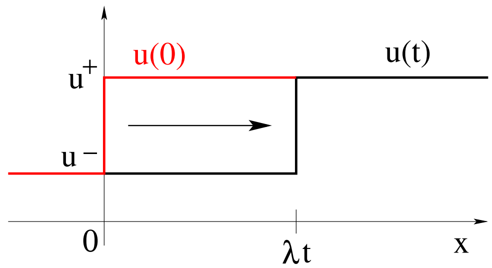

Figure 2: A shock with left and right states u^-, u^+, moving with speed λ.

In this case, an integration by parts shows that the above function is a weak solution if and only if the shock speed λ and the left and right states u^-, u^+ satisfy the **Rankine-Hugoniot equations**

$$\lambda \cdot (u^+ - u^-) \;=\; f(u^+) - f(u^-). \tag{2.11}$$

In other words, the vector equation (2.11) states that

$$[\text{speed}] \times [\text{jump in the state}] \;=\; [\text{jump in the flux}].$$

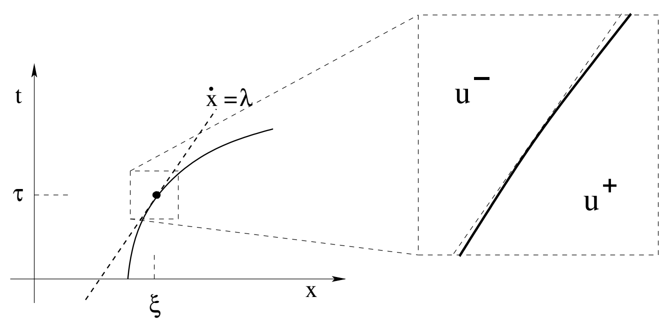

Figure 3: A point of approximate jump.

To state a version of the Rankine-Hugoniot conditions which applies to more general solutions, we introduce

Definition 2.3. *The function $u = u(t,x)$ has an* **approximate jump** *at the point $(\tau,\xi) \in \mathbb{R}^2$ if there exists vectors $u^+ \neq u^-$ and a speed λ such that, setting*

$$U(t,x) \;\doteq\; \begin{cases} u^- & \textit{if} \quad x < \lambda\, t, \\ u^+ & \textit{if} \quad x > \lambda\, t, \end{cases} \tag{2.12}$$

one has

$$\lim_{r\to 0+} \frac{1}{r^2} \int_{-r}^{r}\int_{-r}^{r} \Big| u(\tau+t,\, \xi+x) - U(t,x) \Big|\, dxdt \;=\; 0. \tag{2.13}$$

We say that u is **approximately continuous** *at the point (τ,ξ) if (2.13) holds with $u^+ = u^-$ (and λ arbitrary).*

Theorem 2.1. (Rankine-Hugoniot equations) *Let u be a bounded weak solution of (2.1), having an approximate jump at a point (τ,ξ). Then the left and right states u^-, u^+ and the speed λ satisfy the Rankine-Hugoniot equations (2.11).*

For a proof, see for example [17]. Writing the Rankine-Hugoniot equations in the form

$$\begin{aligned}\lambda\,(u^+ - u^-) &= f(u^+) - f(u^-) = \int_0^1 Df\big(\theta u^+ + (1-\theta)u^-\big)\cdot(u^+ - u^-)\,d\theta \\ &= A(u^+, u^-)\cdot(u^+ - u^-),\end{aligned}$$

we see that

(i) The jump $u^+ - u^-$ is an eigenvector of the averaged matrix $A(u^+, u^-)$.

(ii) The speed λ coincides with the corresponding eigenvalue.

2.3 Admissibility conditions.

In general, solutions to the Cauchy problem (2.9) may not be unique, as soon as discontinuities are present. To single out a unique weak solution one needs to impose further admissibility conditions on the shocks. These can be derived by three different approaches:

- Stability w.r.t. small perturbations.
- Vanishing viscosity approximations.
- Entropy dissipation.

1. A stability condition. Consider first a scalar conservation law. In this case, the Rankine-Hugoniot condition (2.11) simply states that the speed of a shock with left and right states u^-, u^+ must be

$$\lambda = \frac{f(u^+) - f(u^-)}{u^+ - u^-}\,. \tag{2.14}$$

Looking at the graph of the function $f(u)$ (see Fig. 4), this means

speed of the shock = slope of the secant line through u^-, u^+.

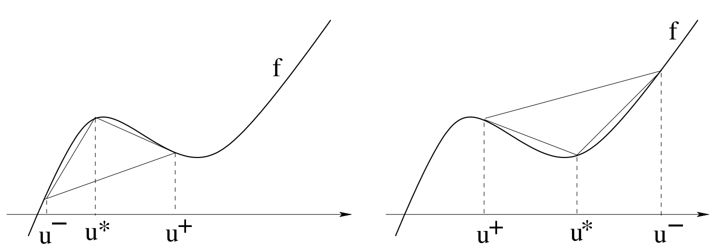

Figure 4: For a scalar conservation law, according to (2.14) the speed of a shock is the slope of a secant line to the graph of f.

To check the stability of the solution (2.10), we can perturb the shock by inserting an intermediate state $u^* \in [u^-, u^+]$

The original shock will be stable w.r.t. this perturbation iff

[speed of jump behind] $\geq$ [speed of jump ahead].

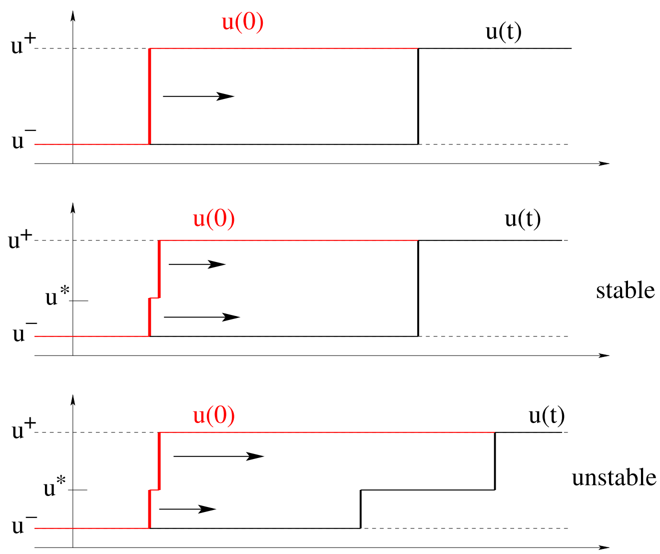

Figure 5: A solution containing a single shock (top figure) can be perturbed into a solution which initially contains two nearby shocks. If the shock behind travels faster than the shock ahead (center figure), then the original shock is stable. If the shock behind is slower than the shock ahead (lower figure), the original shock is unstable.

By (2.14), this means

$$\frac{f(u^*)-f(u^-)}{u^*-u^-} \;\geq\; \frac{f(u^+)-f(u^*)}{u^+-u^*}\,. \tag{2.15}$$

Interpreting the two sides of (2.15) as slopes of secant lines to the graph of f, as shown in Fig. 4 one obtains the following **stability conditions.**

(i) when $u^- < u^+$ the graph of f should remain above the secant line through u^-, u^+.

(ii) when $u^- > u^+$, the graph of f should remain below the secant line through u^-, u^+.

The two above cases are equivalent to one single inequality, namely

$$\text{speed of the shock } [u^-, u^+] \quad\leq\quad \text{speed of any intermediate shock } [u^-, u^*]. \tag{2.16}$$

For every intermediate state u^* between u^- and u^+, the stability of the shock thus requires

$$\frac{f(u^+)-f(u^-)}{u^+-u^-} \;\leq\; \frac{f(u^*)-f(u^-)}{u^*-u^-}\,. \tag{2.17}$$

The formulation (2.17) is particularly important, because it can be extended to any $n\times n$ strictly hyperbolic system of conservation laws. To state this general admissibility condition, we recall that, for any given left state $u^-\in\mathbb{R}^n$ and $i\in\{1,\ldots,n\}$, one can find a curve $s\mapsto S_i(s)$ of right states which can be connected to u^- by an i-shock. More precisely (see Fig. 6),

$$S_i(0) \;=\; u^-, \qquad\qquad \frac{d}{ds}S_i(s)\bigg|_{s=0} \;=\; r_i(u^-),$$

and for every s the Rankine-Hugoniot equations hold:

$$f\big(S_i(s)\big)-f(u^-) \;=\; \lambda_i(s)\big(S_i(s)-u^-\big), \tag{2.18}$$

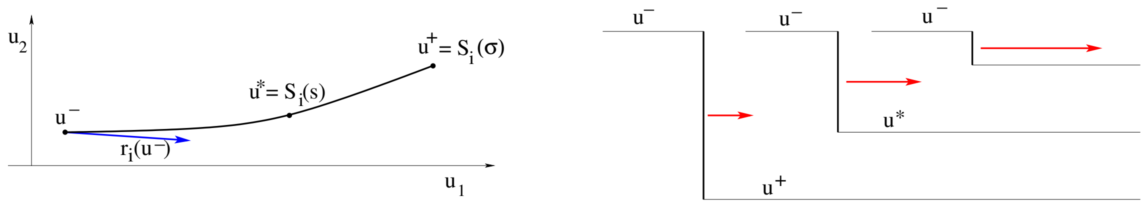

Figure 6: The i-shock curve $s \mapsto S_i(s)$ through the left state u^-.

for some speed $\lambda_i(s)$.

The Liu admissibility condition for general shocks can now be stated as follows.

Definition 2.4. *A shock of the i-th family, connecting the states u^- and $u^+ = S_i(\sigma)$ is* **Liu-admissible** *if the speeds of all intermediate shocks satisfy*

$$\lambda_i(s) \;\geq\; \lambda_i(\sigma) \qquad\qquad \textit{for all} \quad s \in [0, \sigma]. \tag{2.19}$$

2. Vanishing viscosity limits. From physical considerations, it is often natural to assume that the "good" solutions to the system of conservation laws (2.1) are those obtained as limit of solutions to vanishing viscosity approximations

$$u_t^\varepsilon + f(u^\varepsilon)_x \;=\; \varepsilon\, u_{xx}^\varepsilon\,. \tag{2.20}$$

In particular, we say that the shock solution (2.10) is **vanishing viscosity admissible** if it can be obtained as a limit of solutions to (2.20).

In the case of a single shock, the equivalence between the vanishing viscosity and the Liu admissibility condition for a single shock has been proved in [59] and in [8]. More generally, the analysis in [11] shows that every limit of vanishing viscosity approximations satisfies the Liu condition at every point of approximate jump.

3. Entropy admissibility condition. Given the hyperbolic system of conservation laws (2.1), a scalar function $\eta(u)$ is called an **entropy** with **entropy flux** $q(u)$ if

$$D\eta(u) \cdot Df(u) \;=\; Dq(u). \tag{2.21}$$

We observe that (2.21) is a system of n first order PDEs for the 2 functions η, q of the variables $(u_1, \ldots, u_n)$. In general, this is overdetermined and has no solution if $n > 2$. However, there are relevant physical systems where a nontrivial entropy can still be found.

By (2.21), every smooth solution to (2.1) satisfies the additional conservation law

$$\eta(u)_t + q(u)_x \;=\; D\eta(u)\, u_t + D\eta(u) Df(u) u_x \;=\; 0.$$

On the other hand, the entropy may not be conserved in the presence of shocks.

Definition 2.5. *Assume that the hyperbolic system of conservation laws (2.1) admits a* **convex entropy** *$\eta(u)$ with* **entropy flux** *$q(u)$. We say that a weak solution $u = u(t, x)$ is* **entropy admissible** *if it satisfies the inequality*

$$\eta(u)_t + q(u)_x \;\leq\; 0 \tag{2.22}$$

in distributional sense. That means

$$\iint \big\{\eta(u)\varphi_t + q(u)\varphi_x\big\}\, dxdt \;\geq\; 0 \qquad \textit{for all } \varphi \in \mathcal{C}^1_c\,, \quad \varphi \geq 0. \tag{2.23}$$

As a special case, the shock solution at (2.10) is entropy-admissible iff

$$\lambda\big[\eta(u^+) - \eta(u^-)\big] \;\geq\; q(u^+) - q(u^-). \tag{2.24}$$

It is well known that, if a convex entropy exists, every limit of vanishing viscosity approximations satisfies the entropy admissibility conditions [17, 44].

Remark 2.1. In the classical theory of gas dynamics, the second law of thermodynamics implies that the physical entropy should increase in time. To reconcile this fact with the decrease in the entropy stated at (2.22), it suffices to observe that the physical entropy is concave down, while the entropies considered by the mathematical theory are always convex functions. Hence the change in the sign.

3 Approximation methods

Several techniques for constructing approximate solutions to the Cauchy problem

$$u_t + f(u)_x \;=\; 0, \qquad\qquad u(0,\cdot) = \bar u, \tag{3.1}$$

have been considered in the literature. Generally speaking, these methods are known to converge to the exact solution in two main cases:

(i) For a scalar conservation law, based on Kuznetsov's estimates. See for example [51].

(ii) For general $n \times n$ systems, as long as the exact solution remains $\mathcal{C}^1$, i.e., continuously differentiable.

On the other hand, for weak solutions to $n\times n$ hyperbolic systems, the convergence of approximations requires a careful analysis. In various cases, the convergence still remains an open problem.

In many algorithms, a basic building block is provided by the **Riemann problem**, where the initial datum is piecewise constant with one single jump at the origin:

$$u(0,x) \;=\; \begin{cases} u^- & \text{if } \ x<0, \\ u^+ & \text{if } \ x>0. \end{cases} \tag{3.2}$$

For an $n \times n$ hyperbolic system, assuming that every characteristic field is either genuinely nonlinear or linearly degenerate, the general solution to the Riemann problem was first constructed by Lax [54]. As shown in Fig. 7, it consists of $n+1$ constant states

$$u^- = \omega_0\,, \;\; \omega_1\,, \qquad \cdots \qquad , \; \omega_n = u^+,$$

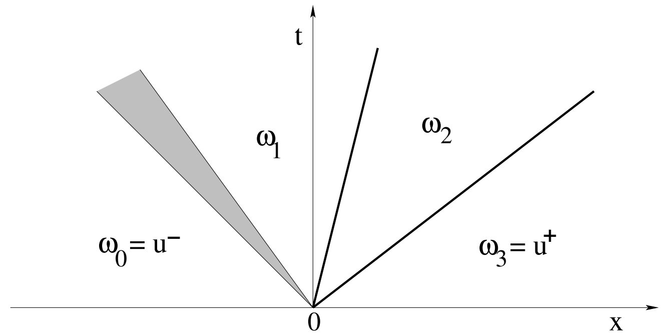

Figure 7: A typical solution to a Riemann problem for a 3×3 hyperbolic system.

where each couple of states (ω_{k-1}, ω_k) are separated by an admissible shock or by a centered rarefaction wave of the k-th family. Solutions $u = u(t,x)$ to a Riemann problem are self-similar, in the sense that they are invariant w.r.t. a rescaling symmetry:

$$u(t,x) \;=\; u(\theta t,\, \theta x) \qquad \text{for all } \theta > 0.$$

A brief survey of different approximation methods for the Cauchy problem (3.1) is given below.

1. The upwind Godunov scheme. To simplify the presentation, we shall assume that all characteristic speeds satisfy

$$\lambda_i(u) \in [0,1]. \tag{3.3}$$

This is not restrictive, because if $\lambda_i(u) \in [-M, M]$ one can achieve (3.3) by the simple coordinate change

$$\tilde{x} \;=\; x + Mt, \qquad\qquad \tilde{t} \;=\; 2Mt. \tag{3.4}$$

The Godunov (upwind) scheme starts by constructing a grid in the t-x plane with step size $\Delta t = \Delta x = \varepsilon$, see Fig. 8.

- The grid points are $(t_j, x_k) = (j \cdot \Delta t\,,\; k \cdot \Delta x)$.
- At each time t_j, $j \geq 0$, the approximate solution $u(t_j, \cdot)$ is piecewise constant with jumps at the points x_k:

$$u(t_j, x) \;=\; u_{j,k} \qquad\qquad \text{for} \quad x_k \leq x < x_{j,k+1}\,.$$

- For $t_j \leq t < t_{j+1}$ the solution is computed by solving the corresponding Riemann problems at each point of jump (t_j, x_k), for every integer k.
- At time t_{j+1} the solution is again approximated by a piecewise constant function, and the procedure can repeat.

In the Godunov scheme, at the time t_{j+1} the function

$$u(t_{j+1}-, \cdot) \;\doteq\; \lim_{t \to t_{j+1}-} u(t, \cdot)$$

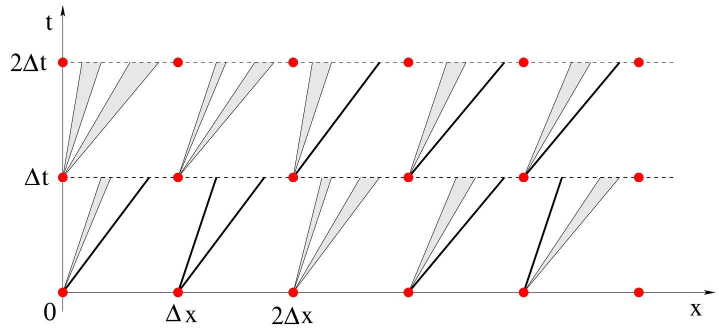

Figure 8: An approximate solution obtained by solving a Riemann problem at each node of the grid.

is replaced by a piecewise constant function, equal to its average on each interval $[x_k, x_{k+1}]$. Namely

$$u_{j+1,k} \doteq \frac{1}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} u(t_{j+1}-,\, x)\, dx\,. \tag{3.5}$$

A remarkable property of this scheme is that, in order to compute $u_{j+1,k}$ there is no need to actually construct the solution to a Riemann problem. Indeed, applying the divergence theorem on each square of the grid (see Fig. 9), by the conservation law (2.1) one immediately obtains

$$u_{j+1,k} = u_{j,k} + f(u_{j,k-1}) - f(u_{j,k}). \tag{3.6}$$

The finite difference scheme (3.6) is called the (upwind) Godunov scheme. While this is easy to implement numerically, a rigorous convergence analysis of this scheme is still lacking. Indeed, in view of the analysis in [7, 67], no a priori bounds on the total variation of approximate solutions can be established.

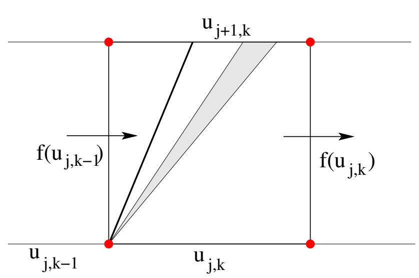

Figure 9: By the conservation law (2.1), the average value of the solution u on the top side of the square is equal to the value $u_{j+1,k}$ computed by the formula (3.6).

2. The Glimm scheme. This scheme is similar to the Godunov scheme, with one major difference. At each time t_j, we have to replace the function $u(t_j-,\cdot)$ with a piecewise constant function having jumps at the points x_k. In the Godunov scheme this is achieved by computing the average value on each interval $[x_k, x_{k+1}]$. In the Glimm scheme [47], the restarting is achieved by random sampling. Namely, inside each interval $[x_k, x_{k+1}]$ we choose a random point $x^*_{j,k}$. The value $u(t_j-, x^*_{j,k})$ of the solution to the Riemann problem at this particular point is taken to be the value of the function $u(t_j,\cdot)$ on the entire interval.

More precisely:

- We choose a random sequence of numbers $\theta_1, \theta_2, \theta_3, \ldots$ uniformly distributed on $[0, 1]$.

- At each time t_j, for every k we consider the random point $x^*_{j,k} = (k + \theta_j) \cdot \Delta x$ and define

$$u(t_j, x) \;=\; u(t_j-, x^*_{j,k}) \qquad \text{for all } x \in [x_k,\, x_{k+1}[\,. \tag{3.7}$$

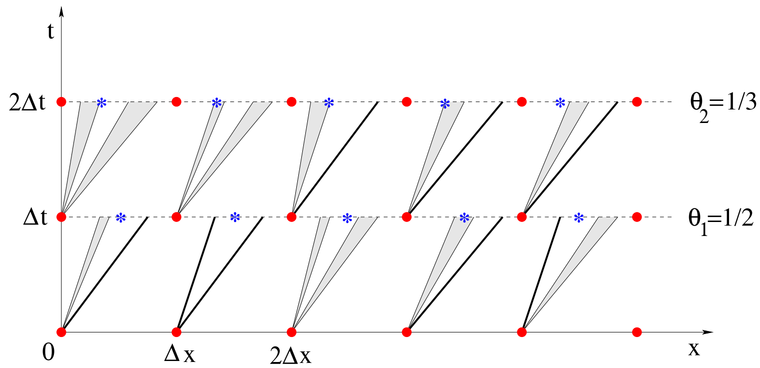

Figure 10: An approximate solution obtained by solving a Riemann problem at each node of the grid. In the Glimm scheme, at each time $t_j = j\,\Delta t$ the solution is sampled at the points marked by an asterisk, depending om the random sequence $\theta_1, \theta_2, \ldots$

As later proved by T.P.Liu [58], instead of a random sequence one can use a deterministic sequence of numbers $\theta_1, \theta_2, \theta_3, \ldots \in [0,1]$ which is **uniformly distributed**, so that

$$\lim_{N\to\infty} \frac{\#\{j\ ;\ \ 1 \le j \le N,\ \ \theta_j \in [0,\lambda]\ \}}{N} \;=\; \lambda \qquad \text{for each } \lambda \in [0,1]. \tag{3.8}$$

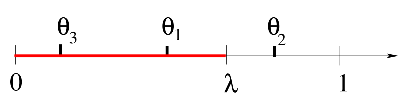

Figure 11: A sequence of numbers in the interval $[0,1]$. For every λ, the percentage of points θ_i, $1 \le i \le N$ which fall inside the subinterval $[0,\lambda]$ should approach λ as $N \to \infty$.

A simple way of generating such a sequence is to write decimal digits in inverse order:

$$\theta_1 = 0.1\,, \quad \ldots \quad , \quad \theta_{759} = 0.957\,, \quad \ldots \quad , \quad \theta_{39022} = 0.22093\,, \quad \ldots \tag{3.9}$$

The relevance of the assumption (3.8) is illustrated in Fig. 12. As in (2.10), consider a solution containing a single shock, traveling with speed $\lambda \in \,]0,1[\,$.

Fix a time interval $[0,T]$ and take $\Delta x = \Delta t = T/N$. Call $x(t_j)$ the location of the shock at time t_j, in the approximate solution. By construction (see Fig. 12, left), we have

- If $\theta_j \le \lambda$, then $x(t_j) = x(t_{j-1}) + \Delta x$.
- If $\theta_j > \lambda$, then $x(t_j) = x(t_{j-1})$.

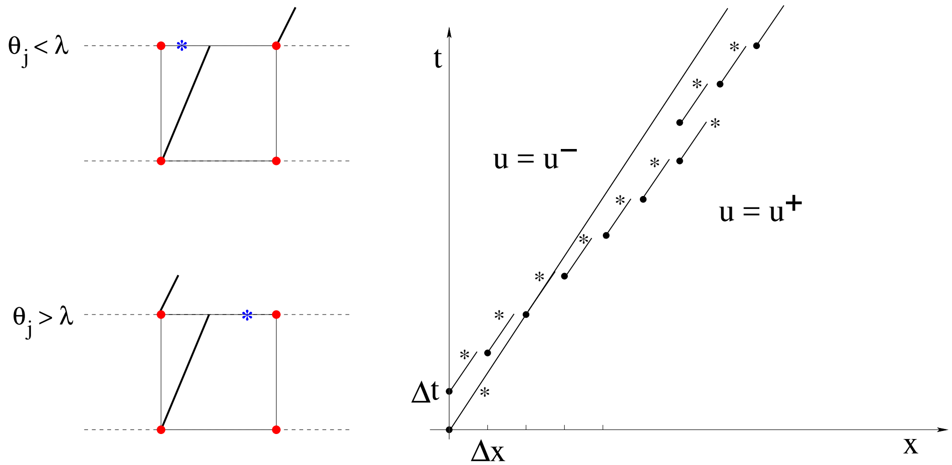

Figure 12: Left: at each time step, the position of the shock remains unchanged if $\theta_j > \lambda$, while it jumps forward if $\theta_j < \lambda$. Right: a single shock solution with speed $\lambda \in [0,1]$ and an approximate solution constructed by the Glimm scheme. As the grid size approaches zero, convergence to the exact solution is achieved if and only if the limit in (3.8) holds.

Hence, at time T, the position of the shock in the approximate solution is

$$\begin{aligned} x(T) &= \#\Big\{j\,;\quad 1\leq j\leq N,\quad \theta_j\in[0,\lambda]\ \Big\}\cdot\Delta t \\ &= \frac{\#\Big\{j\,;\quad 1\leq j\leq N,\quad \theta_j\in[0,\lambda]\ \Big\}}{N}\cdot T\ \to\ \lambda T \qquad\qquad \text{as}\quad N\to\infty, \end{aligned}$$

provided that the uniform distribution assumption (3.8) holds.

3. Front tracking approximations. In the Glimm scheme, the Riemann problems are solved on a fixed grid in the t-x plane. In a front tracking algorithm, the points where new Riemann problems are solved depend on the solution itself.

- The construction starts by approximating the initial data $\bar u \in \mathbf{L}^1$ with a piecewise constant function $\bar v$.
- At each point where $\bar v$ has a jump, an approximate solution to the Riemann problem is constructed, within the class of piecewise constant functions. As shown in Fig. 13, this solution can be prolonged up to the first time t_1 where two fronts interact.
- At time t_1, we construct a piecewise constant approximate solution to the new Riemann problem generated by the interaction, and prolong the solution until a further interaction occurs.
- By inductively solving the new Riemann problems at the times $t_2, t_3, \ldots$ where two fronts interact, a piecewise constant approximate solution is constructed for all times $t \geq 0$.

For a single conservation law, the front tracking method was first introduced by Dafermos [41]. To apply this method to $n\times n$ systems, one needs to make sure that the number of wave fronts does not become infinite in finite time. This requires some technical provision, such as the

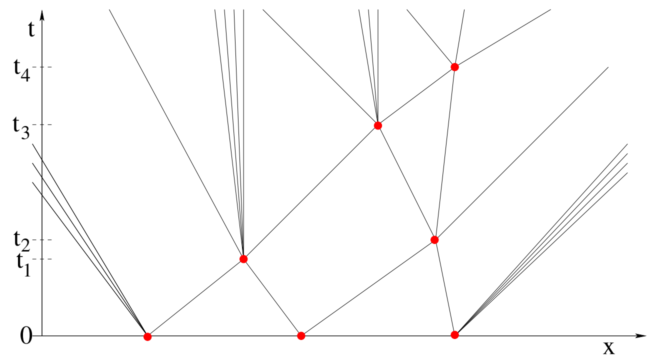

Figure 13: An approximate solution constructed by front tracking.

introduction of “non-physical fronts” [6, 14]. For a comprehensive presentation we refer to [17, 44, 51].

4. Vanishing viscosity approximations. Starting with the hyperbolic system (2.1) and adding a small diffusion term, one obtains the quasilinear parabolic system:

$$u_t^\varepsilon + A(u^\varepsilon)u_x^\varepsilon \;=\; \varepsilon u_{xx}^\varepsilon, \qquad\qquad u(0,x) = \bar u(x). \tag{3.10}$$

Here $A(u) = Df(u)$ is the $n\times n$ Jacobian matrix of the flux function. Letting $\varepsilon \to 0+$, it is expected that the solutions to (3.10) will converge to the unique solution to the hyperbolic Cauchy preblem (3.1). For initial data with small total variation, a rigorous proof of this convergence was given in [11].

5. Jin-Xin relaxation approximations. These are obtained by solving the second order wave equation

$$u_t + f(u)_x \;=\; \varepsilon(u_{xx} - u_{tt})\,, \qquad\qquad u(0,x) = \bar u(x).$$

As $\varepsilon \to 0$, uniform BV bounds and convergence to a unique limit have been proved by S. Bianchini [10]. The convergence rate has not been studied in detail.

6. The method of lines. In this case, approximate solutions are obtained by discretizing space while keeping time continuous.

Fix a mesh size $\Delta x = \varepsilon > 0$. Approximating the partial derivative $f(u)_x$ by a finite difference, the system of conservation laws (2.1) is replaced by a countable family of ODEs

$$\frac{d}{dt}U_k(t) \;=\; \frac{f(U_{k-1}(t)) - f(U_k(t))}{\varepsilon}\,, \qquad\qquad k \in \mathbb{Z},$$

for the variables $U_k(t) \approx u(t, k\varepsilon)$, see Fig. 14.

As $\varepsilon \to 0$, uniform BV bounds and convergence to a unique limit have been proved by S. Bianchini [9].

7. Backward Euler approximations. In this case we discretize time while keeping space continuous. Choosing $\Delta t = \varepsilon$ as time step, the Backward Euler approximation takes the form

$$u(t+\varepsilon) \;=\; u(t) - \varepsilon f(u(t+\varepsilon))_x\,. \tag{3.11}$$

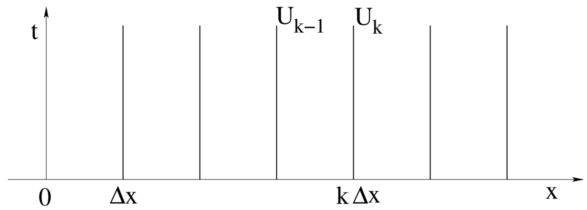

Figure 14: The method of lines.

Setting

$$v(x) \ = \ u(t,x), \qquad w(x) \ = \ u(t+\varepsilon,x), \qquad A(w) = Df(w),$$

at every time step one needs to solve

$$w(x) \ = \ v(x) - \varepsilon Df(w(x))\, w_x(x),$$

which leads to the ODE

$$w'(x) \ = \ A^{-1}(w(x)) \left(\frac{v(x)-w(x)}{\varepsilon} \right). \tag{3.12}$$

By performing a change of coordinates similar to (3.4), one can assume that all characteristic speeds (i.e., all eigenvalues of the Jacobian matrix $A(u) = Df(u)$) are contained inside the interval $[1,2]$. This guarantees that in (3.12) all matrices $A(w)$ have a uniformly bounded inverse. For a fixed $\varepsilon > 0$, existence and uniqueness of $\mathbf{L}^1$ solutions to (3.12) have been studied in [39], together with traveling wave profiles. However, as $\varepsilon \to 0$, uniform BV bounds and convergence to a unique limit remain an open question.

8. Periodic mollifications. These approximations are again constructed by discretizing time. Fix $\varepsilon > 0$ and set $t_k = \varepsilon k$, $k = 0,1,2,\ldots$ On each subinterval $[t_k, t_{k+1}[$, the function u is defined to be a classical solution:

$$u_t + Df(u) u_x \ = \ 0, \qquad\qquad t \in [t_k, t_{k+1}[\,.$$

At each time t_k, before any shock is formed, the solution is restarted by performing a convolution with a mollifying kernel:

$$u(t_k) \ = \ J_\varepsilon * u(t_k-).$$

Letting $\varepsilon \to 0$, we expect that the approximations should converge to an admissible weak solution to the original system (2.1). This is well known in the scalar case [51]. However, for general $n \times n$ systems, uniform BV bounds and convergence to a unique limit have not been proved.

9. Nonlinear diffusion approximations. These take the form

$$u_t + f(u)_x \ = \ \varepsilon \big(B(u) u_x \big)_x\,, \tag{3.13}$$

where $B(u)$ is a (possibly degenerate) $n \times n$ diffusion matrix. Since in many physical systems the viscosity depends on the macroscopic variables, it would be of great interest to prove rigorous convergence results for solutions of (3.13). However, apart from the case where B is a constant, invertible matrix [11], establishing uniform BV bounds and convergence to a unique limit as $\varepsilon \to 0$ remains a challenging open problem.

4 Global existence of weak solutions

If the initial datum $\bar{u} : \mathbb{R} \mapsto \mathbb{R}^n$ is smooth, a unique local in time $\mathcal{C}^1$ solution to the Cauchy problem (3.1) can be constructed by the method of characteristic, as the fixed point of a contractive transformation [17, 66]. However, for large times, as shown in Fig. 1 the gradient of the solution can become unbounded. Global in time solutions can only be obtained in a space of discontinuous functions.

For scalar conservation laws, also in several space dimensions, a general existence-uniqueness theorem was proved in the famous paper by Kruzhkov [53]. Shortly afterwards, an alternative proof based on the theory of nonlinear contractive semigroups was given by Crandall [40].

For $n \times n$ hyperbolic systems, the first global existence theorem for weak solutions to the Cauchy problem (3.1) was proved in a celebrated paper by Glimm [47].

Theorem 4.1. (Global existence of weak solutions). *Consider the one-dimensional Cauchy problem (3.1) for a strictly hyperbolic system of conservation laws, where each characteristic field is either linearly degenerate or genuinely nonlinear.*

Then there exists a constant $\delta > 0$ such that, if the initial data satisfies

$$\bar{u} \in \mathbf{L}^1(\mathbb{R};\ \mathbb{R}^n), \qquad\qquad \textit{Tot.Var.}\{\bar{u}\} \leq \delta,$$

then (3.1) has a weak solution $u = u(t, x)$ defined for all $t \geq 0$.

If the system admits a convex entropy, a global solution exists which is entropy admissible.

The proof is achieved by constructing a sequence of approximate solutions $(u_m)_{m\geq 1}$ according to the Glimm scheme (see Fig. 10). Here we let the grid size $\Delta t, \Delta x \to 0$ as $m \to \infty$.

- By carefully estimating the strength of new waves produced by nonlinear wave interactions, one obtains a uniform bound on the total variation of all approximate solutions. Namely if Tot.Var.$\{\bar{u}\}$ is sufficiently small, then Tot.Var.$\{u_m(t, \cdot)\}$ remains small for all times $t \geq 0$ and every $m \geq 1$.
- By Helly's compactness theorem, one obtains a convergent subsequence $u_m \to u$ in $\mathbf{L}^1_{loc}(\mathbb{R}_+ \times \mathbb{R}\,;\ \mathbb{R}^n)$.
- Using the assumption that the sequence $(\theta_k)_{k\geq 1}$ used for random sampling (3.7) is uniformly distributed on $[0, 1]$, one proves that with probability one the limit function $u = u(t, x)$ is a weak solution to the Cauchy problem.

For many years, the Glimm scheme provided the only tool for a rigorous analysis of weak solutions to hyperbolic conservation laws. Among the first such studies, the asymptotic behavior of solutions as $t \to +\infty$ was analyzed by T.P.Liu [60]. The assumption of genuine nonlinearity or linear degeneracy of each characteristic field was removed in [61]. Alternative proofs, relying on front tracking approximations, were later given in [4, 6, 14].

We remark that, in all these results, the smallness of the initial data is a key assumption which has never been removed. This leads to the following question:

If the total variation of the initial data $u(0,\cdot) = \bar{u}(\cdot)$ is bounded but possibly large, does the total variation of the solution $u(t,\cdot)$ remain bounded for all times $t > 0$, or can it blow up in finite time?

A counterexample constructed by Jenssen [52] shows that, in some cases, the total variation of a weak solution can indeed blow up in finite time. However, the hyperbolic system considered in this example does not admit any strictly convex entropy. In particular, the construction does not apply to any of the systems of conservation laws which are relevant for continuum physics.

Open Problem #1. *Consider a strictly hyperbolic system of conservation laws (2.1), admitting a strictly convex entropy, and where each characteristic field is either linearly degenerate or genuinely nonlinear.*

- *Construct an example of an entropy admissible weak solution $u = u(t,x)$ whose total variation blows up in finite time.*
- *Or else, prove that every solution, whose total variation is initially bounded, remains with bounded variation for all times $t > 0$.*

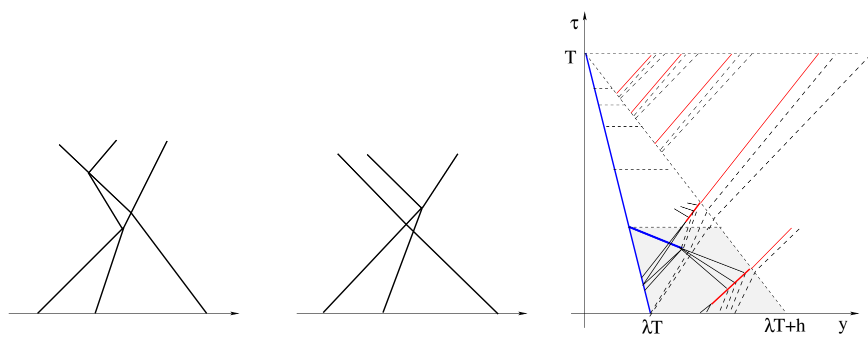

Figure 15: Left: if their speeds are slightly changed, the three wave fronts will interact in different order, producing new waves of different strengths. Right: a sketch of the interaction pattern considered in [18], leading to blow up of the total variation in finite time.

The question of global BV bounds versus finite time blow up is not resolved even for the 2×2 system of isentropic gas dynamics (2.8). The recent analysis in [18] only points out how difficult the problem really is. By slightly changing the wave speeds, one can arrange so that wave fronts cross each other in a different order (see Fig. 15). For the same initial data, one can construct approximate solutions whose total variation remains bounded, and other approximate solutions whose total variation blows up in finite time, depending on the interaction pattern. It is hard to say what happens for the exact solution.

5 Continuous dependence on initial data

For a wide class of evolution equations, continuous dependence on initial data is achieved by showing that the distance between any two solutions satisfies the differential inequality

$$\frac{d}{dt}\big\|u(t) - v(t)\big\| \;\le\; C\big\|u(t) - v(t)\big\|, \tag{5.1}$$

for some constant C. In turn, by Gronwall's lemma this implies

$$\|u(t)-v(t)\| \;\leq\; e^{Ct}\,\|u(0)-v(0)\|\,. \tag{5.2}$$

This approach can be applied to scalar conservation laws, even in a multidimensional space $\mathbb{R}^d$. As proved in [40, 53], a scalar conservation law generates a contractive semigroup on $\mathbf{L}^1(\mathbb{R}^d)$. Namely, for every couple of entropy admissible solutions u, v one has

$$\|u(t)-v(t)\|_{\mathbf{L}^1} \;\leq\; \|u(s)-v(s)\|_{\mathbf{L}^1} \qquad\qquad \text{for all } 0\leq s<t. \tag{5.3}$$

On the other hand, the inequality (5.1) does not hold for weak solutions to hyperbolic systems. As shown in Fig. 16, the $\mathbf{L}^1$ distance between two nearby solutions can increase rapidly during short time intervals.

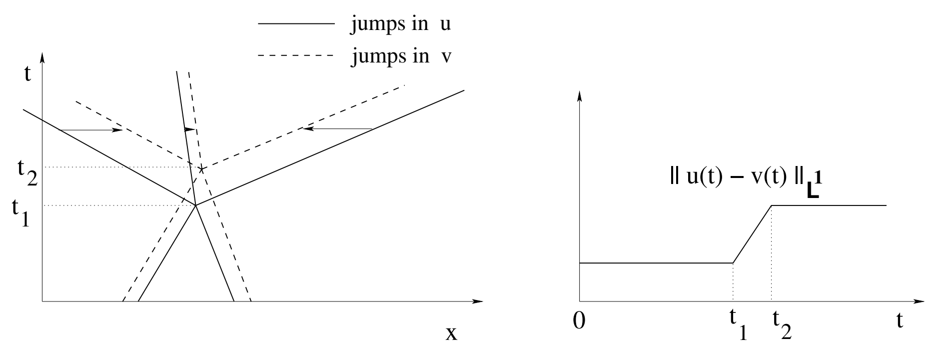

Figure 16: A solution u which initially contains two approaching shocks, and a second solution v which differs from u only in the location of the shocks. During the short time interval $[t_1, t_2]$ the solution u contains three shocks, while v still has two. The $\mathbf{L}^1$ distance between the two solutions remains constant for $t \in [0, t_1]$ and for $t \geq t_2$, but increases rapidly during the interval $[t_1, t_2]$.

We observe that the Glimm scheme does not provide insight on the continuous dependence of solutions. Indeed, the approximate solutions constructed by the Glimm scheme do not depend continuously on initial data. With reference to Fig. 17, left, an arbitrarily small change in the speed of a shock may place it to the right or to the left of the sampling point $x^*_{j,k}$. In this case, the piecewise constant approximation changes value on the entire interval $[x_k, x_{k+1}]$.

Understanding how weak solutions depend on initial data was the primary motivation for developing an alternative approximation scheme based on wave front tracking [14]. As shown in Fig. 17, let u be a piecewise constant approximate solution, and let v be a perturbed solution obtained by slightly shifting the location of the jumps in u. At any time $t \geq 0$, the $\mathbf{L}^1$ distance between the two solutions is measured by

$$\|u(t)-v(t)\|_{\mathbf{L}^1} \;\approx\; \sum_\alpha |\sigma_\alpha|\cdot|\xi_\alpha| \;=\; \sum_\alpha [\text{jump strength}]\times[\text{shift}]. \tag{5.4}$$

By carefully estimating how the right hand side of (5.4) changes at interaction times, one obtains a bound on the distance between the two solutions, at every time $t \geq 0$.

Theorem 5.1. *Let (2.1) be a strictly hyperbolic $n\times n$ system of conservation laws, and assume that each characteristic field is either genuinely nonlinear or linearly degenerate. Then there exists a domain $\mathcal{D} \subset \mathbf{L}^1(\mathbb{R};\mathbb{R}^n)$ and a semigroup $S : \mathcal{D}\times\mathbb{R}_+ \mapsto \mathcal{D}$ with the following properties.*

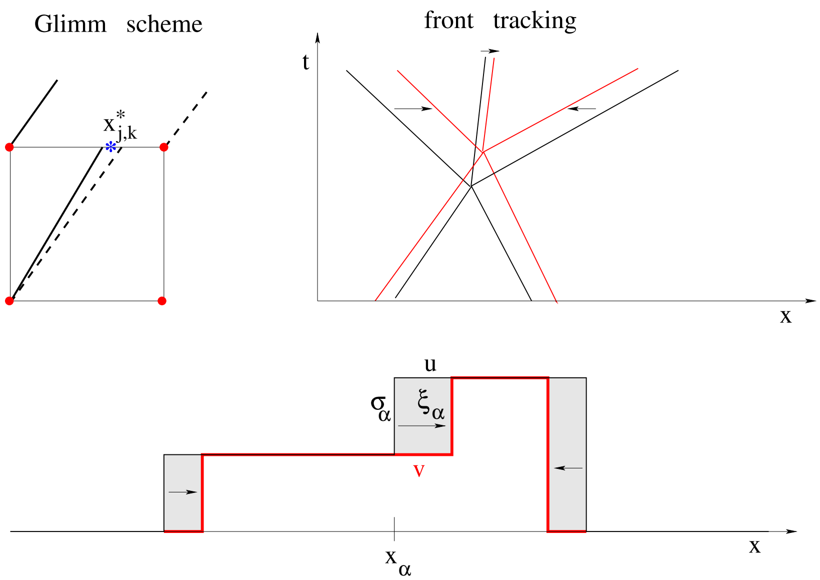

Figure 17: For the Glimm scheme, approximate solutions do not depend continuously on the initial data. On the other hand, by shifting the location of the wave fronts, one can understand how front tracking approximations are affected by changes in the initial data.

(i) The domain $\mathcal{D}$ contains all functions $\bar{u} \in \mathbf{L}^1$ with sufficiently small total variation.

(ii) The map $(\bar{u}, t) \mapsto u(t, \cdot) \doteq S_t\bar{u}$ is a uniformly Lipschitz continuous semigroup:

$$S_0\bar{u} = \bar{u}, \qquad S_s(S_t\bar{u}) = S_{s+t}\bar{u},$$

$$\big\|S_t\bar{u} - S_s\bar{v}\big\|_{\mathbf{L}^1} \leq L \cdot \big(\|\bar{u} - \bar{v}\|_{\mathbf{L}^1} + |t - s|\big) \qquad \textit{for all } \ \bar{u}, \bar{v} \in \mathcal{D}, \quad s, t \geq 0.$$

(iii) For every initial data $\bar{u} \in \mathcal{D}$, the trajectory $t \mapsto u(t, \cdot) = S_t\bar{u}$ provides an admissible solution to the Cauchy problem (3.1).

Trajectories of the semigroup can be obtained as limits of front tracking approximations. This theorem was first proved in [20] in the case of 2×2 systems. A more elaborate proof, valid for $n \times n$ systems, was later worked out in [22]. These earlier proofs relied on a homotopy method: the distance between two solutions u, v was estimated by constructing a 1-parameter family of solutions u^θ, $\theta \in [0, 1]$, connecting u with v. At every time $t \geq 0$, the length of the path $\theta \mapsto u^\theta(t, \cdot)$ provides a bound on the distance $\big\|u(t) - v(t)\big\|_{\mathbf{L}^1}$.

Using ideas introduced by T.P.Liu and T.Yang [63], an alternative proof was given in [31]. This approach relies on the construction of a Lyapunov functional $\Phi : \mathcal{D} \times \mathcal{D} \mapsto \mathbb{R}_+$ with the following properties.

- Φ is equivalent to the $\mathbf{L}^1$ distance:

$$\frac{1}{C} \cdot \big\|v - u\big\|_{\mathbf{L}^1} \leq \Phi(u, v) \leq C \cdot \big\|v - u\big\|_{\mathbf{L}^1},$$

  for every couple of piecewise constant functions $u, v \in \mathbf{L}^1(\mathbb{R}; \mathbb{R}^n)$ with small total variation.

- Φ is non-increasing in time, along couples of (front tracking approximate) solutions:

$$\Phi(u(t), v(t)) \leq \Phi(u(s), v(s)) \qquad \text{for all } t \geq s \geq 0. \tag{5.5}$$

Given two piecewise constant functions u, v, the functional $\Phi(u, v)$ is defined as follows. For each $x \in \mathbb{R}$, we uniquely determine intermediate states

$$u(x) = \omega_0(x), \quad \omega_1(x), \quad \ldots, \quad \omega_n(x) = v(x),$$

such that every pair $\big(\omega_{i-1}(x), \omega_i(x)\big)$ is joined by a (possibly non-admissible) shock of the i-th family (see Fig. 18). Calling $q_i(x)$ the strength of this shock, the $\mathbf{L}^1$ distance between u and v can now be estimated as

$$\|u - v\|_{\mathbf{L}^1} \approx \int_{-\infty}^{+\infty} \sum_{i=1}^{n} |q_i(x)| \, dx \,.$$

To achieve the decreasing property (5.5), suitable weights W_i must be inserted. Roughly speaking, $W_i(x)$ measures the total strength of waves in u and in v that approach an i-shock located at x. The functional Φ thus takes the form

$$\Phi(u, v) = \sum_{i=1}^{n} \int_{-\infty}^{\infty} |q_i(x)| \, W_i(x) \, dx \,.$$

For all details we refer to [31]. This approach greatly simplified the earlier proofs, and is now adopted in most textbooks on the subject [17, 44, 51].

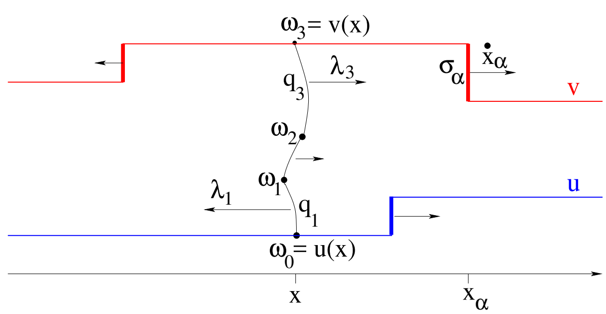

Figure 18: Constructing the functional $\Phi(u, v)$. At each point x, the strengths $q_1(x), \ldots, q_n(x)$ of the shocks connecting $u(x)$ with $v(x)$ can be regarded as the scalar components of the jump $\big(u(x), v(x)\big)$.

All previous results were proved in the same setting as in Glimm's theorem, where each characteristic field is either linearly degenerate or genuinely nonlinear. Eventually, this assumption was entirely removed by the approach based on vanishing viscosity approximations [11]. In this case, one does not even need that the hyperbolic system be in conservation form.

Theorem 5.2. *Consider the Cauchy problem for a strictly hyperbolic system with small viscosity*

$$u_t + A(u)u_x = \varepsilon \, u_{xx} \,, \qquad u(0, x) = \bar{u}(x) \,. \tag{5.6}$$

If Tot.Var.$\{\bar{u}\}$ is sufficiently small, then (5.6) admits a unique solution $u^\varepsilon(t, \cdot) = S_t^\varepsilon \bar{u}$, defined for all $t \geq 0$. Moreover, for some constants C, L independent of ε, one has

$$\text{Tot.Var.}\big\{S_t^\varepsilon \bar{u}\big\} \leq C \, \text{Tot.Var.}\{\bar{u}\} \,, \qquad \textbf{(BV bounds)}$$

$$\left\|S_t^\varepsilon \bar u - S_t^\varepsilon \bar v\right\|_{\mathbf{L}^1} ~\leq~ L\,\|\bar u - \bar v\|_{\mathbf{L}^1}\,. \qquad\qquad (\mathbf{L^1}\ \textbf{stability})$$

As $\varepsilon\to 0$, the solutions u^ε converge to the trajectories of a semigroup S such that

$$\left\|S_t \bar u - S_t \bar v\right\|_{\mathbf{L}^1} ~\leq~ L\,\|\bar u - \bar v\|_{\mathbf{L}^1} \qquad\qquad \textit{for all } t\geq 0\,.$$

If the system is in conservation form: $A(u) = Df(u)$ for some flux function f, then every trajectory $t\mapsto u(t,\cdot) = S_t\bar u$ provides a weak solution to the Cauchy problem

$$u_t + f(u)_x ~=~ 0, \qquad\qquad u(0,x) = \bar u(x).$$

Moreover, the Liu admissibility conditions (2.19) are satisfied at every point of approximate jump.

For a proof, see [11] or the lecture notes [15]. These vanishing viscosity limits can be regarded as the unique **viscosity solutions** of the hyperbolic Cauchy problem

$$u_t + A(u)u_x = 0 \qquad\qquad u(0,x) = \bar u(x)\,.$$

We remark that all results stated in Theorem 5.2 hold for the system (5.6) with "artificial viscosity", where the diffusion acts uniformly on all components $(u_1, u_2,\ldots,u_n)$ of the solution. In several physical models, the diffusion acts differently on different components, and depends on the state u as well. This leads to

Open Problem #2. *Extend the results of Theorem 5.2 to hyperbolic systems with nonlinear diffusion:*

$$u_t + A(u)u_x ~=~ \varepsilon\big(B(u)u_x\big)_x\,, \qquad\qquad u(0,\cdot) = \bar u\,, \tag{5.7}$$

where $B(u)$ is a positive semidefinite viscosity matrix. Assuming that the initial data $\bar u$ has small total variation, prove uniform bounds on Tot.Var.$\{u(t,\cdot)\}$ and study the limit of these solutions as $\varepsilon\to 0+$.

In the conservative case where $A(u) = Df(u)$, we expect that the vanishing viscosity limit should be unique, and provide a weak solution to the hyperbolic system (2.1). On the other hand, in the non-conservative case, as $\varepsilon\to 0+$ different limits may well be obtained, depending on the choice of the viscosity matrices $B(u)$ in (5.7).

A major advance was recently achieved by Chen, Kang and Vasseur [36], for the 2×2 system of barotropic gas dynamics in Eulerian coordinates with physical viscosity:

$$\left\{\begin{array}{rl} \rho_t + (\rho u)_x &=~ 0,\\ (\rho u)_t + (\rho u^2)_x + p(\rho)_x &=~ \varepsilon\big(b(\rho)u_x\big)_x\,. \end{array}\right. \tag{5.8}$$

Here ρ and u are respectively the density and the velocity of the gas, while $p(\rho) = \rho^\gamma$ is the pressure, with $\gamma > 1$. The coefficient $b(\rho)$ accounts for a density-dependent viscosity. For initial data $(\bar\rho,\bar u)$ with small total variation, when $\varepsilon = 0$ it is well known that the hyperbolic system (5.8) has a unique entropy weak solution, corresponding to a trajectory of the semigroup described in Theorem 5.1. The analysis in [36] shows that this solution can be obtained as the unique limit of solutions to the viscous equations (5.8), as $\varepsilon\to 0$. Remarkably, the proof does not require any a priori bound on the total variation of the approximating solutions. Instead, it is based on careful estimates of the relative entropy, using the so-called "method of a-contraction".

6 Uniqueness of weak solutions

Having constructed a Lipschitz semigroup of admissible weak solutions, which are limits of vanishing viscosity approximations (and of front tracking approximations as well), it becomes entirely clear which is the unique "good" solution to the Cauchy problem (3.1). Namely, the semigroup trajectory $t \mapsto u(t,\cdot) = S_t\bar{u}$. From this point of view, uniqueness becomes a marginal issue in the overall theory. Some authors barely mention the problem [62], focusing instead all the attention on the continuous dependence on initial data.

On the other hand, a general uniqueness theorem can be quite useful if we want to study the convergence of different approximation methods. Without a uniqueness result, one may even suspect that these algorithms converge to different limit solutions.

As soon as a semigroup of solutions has been constructed, to establish a uniqueness theorem it suffices to come up with a set of conditions that uniquely characterize the semigroup trajectories. Various ways to do this have been worked out in [24, 29, 30]. The proofs rely on the following elementary error estimate (see Fig. 19).

Lemma 6.1. *Let $S : \mathcal{D} \times [0,\infty[\mapsto \mathcal{D}$ be a Lipschitz semigroup satisfying*

$$\|S_t u - S_s v\| \;\leq\; L \cdot \|u - v\| + L' \cdot |t - s|\,.$$

Then, for every Lipschitz continuous map $w : [0,T] \mapsto \mathcal{D}$ one has

$$\big\|w(T) - S_T w(0)\big\| \;\leq\; L \cdot \int_0^T \left\{ \liminf_{h\to 0+} \frac{\big\|w(t+h) - S_h w(t)\big\|}{h} \right\} dt\,. \tag{6.1}$$

For a proof, see [16, 17]. The integrand on the right hand side of (6.1) can be interpreted as the **instantaneous error rate** of the approximate solution $w(\cdot)$ at time t.

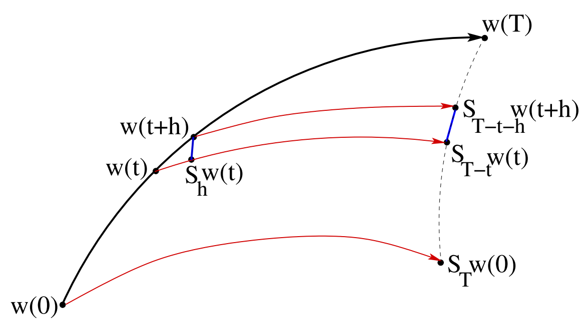

Figure 19: The distance between $w(T)$ and $S_T w(0)$ is bounded by the length of the path $t \mapsto S_{T-t}w(t)$, $t \in [0,T]$. Since the semigroup amplifies distances at most by a factor L, this leads to the formula (6.1).

To prove that a solution $u = u(t,x)$ of the Cauchy problem (3.1) coincides with the semigroup trajectory, it now suffices to show that

$$\liminf_{h\to 0+} \frac{\big\|u(\tau+h) - S_h u(\tau)\big\|_{\mathbf{L}^1}}{h} \;=\; 0 \tag{6.2}$$

for a.e. time $\tau \in [0,T]$.

To fix ideas, w.l.o.g. we assume that all wave speeds (i.e., all eigenvalues of the matrices $A(u) = Df(u)$) are contained in the interval $[-1, 1]$. Given the function $u(\tau, \cdot)$, following the approach introduced in [16] we split the real line inserting points

$$-\infty \doteq x_0 < x_1 < x_2 < \cdots < x_N \doteq +\infty,$$

so that (see Fig. 20)

$$\text{Tot.Var.}\Big\{u(\tau,\cdot)\,;\ ]x_{i-1}, x_i[\,\Big\} < \varepsilon, \qquad\qquad i = 1, 2, \ldots, N. \tag{6.3}$$

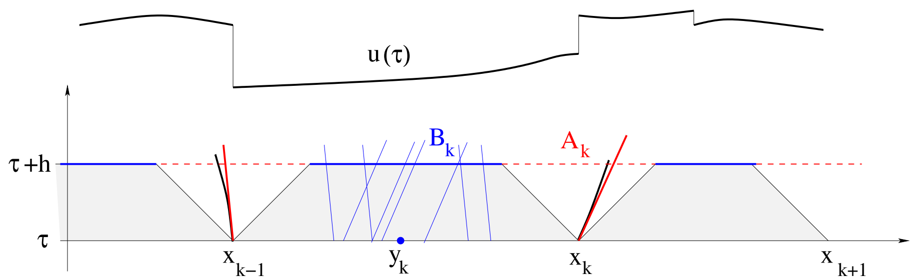

Figure 20: Given the function $u(\tau, \cdot)$, we insert points x_k so that the total variation on each open interval $]x_{k-1}, x_k[$ is $< \varepsilon$.

We now estimate

$$\begin{aligned}
&\frac{1}{h}\int_{-\infty}^{\infty}\Big|u(\tau+h,x) - S_h u(\tau)(x)\Big|\,dx \\
&= \sum_k \frac{1}{h}\int_{x_k-h}^{x_k+h}\Big|u(\tau+h,x) - S_h u(\tau)(x)\Big|\,dx + \sum_k \frac{1}{h}\int_{x_{k-1}+h}^{x_k-h}\Big|u(\tau+h,x) - S_h u(\tau)(x)\Big|\,dx \\
&= \sum_k A_k(h) + \sum_k B_k(h)\,.
\end{aligned} \tag{6.4}$$

We claim that, by choosing $\varepsilon > 0$ small, the limit as $h \to 0$ of the right hand side of (6.4) can be made arbitrarily small. Indeed, assume that at (τ, x_k) the solution u is either approximately continuous, or has an approximates jump, as in (2.13). Then

- On each interval $[x_k - h, x_k + h]$, the function $u(\tau + h, \cdot)$ and the semigroup solution $S_h u(\tau)$ are both compared with the piecewise constant function

$$U_k(t,x) \doteq \begin{cases} u^+ \doteq u(\tau, x_k+) & \text{if} \quad x > x_k + \lambda(t-\tau), \\ u^- \doteq u(\tau, x_k-) & \text{if} \quad x < x_k + \lambda(t-\tau). \end{cases}$$

- On each of the remaining intervals $[x_{k-1} + h, x_k - h]$ the function $u(\tau + h, \cdot)$ and the semigroup solution $S_h u(\tau)$ are both compared with the solution W_k of the linear Cauchy problem with constant coefficients

$$w_t + A_k w_x = 0 \qquad\qquad w(\tau, x) = u(\tau, x), \tag{6.5}$$

where $A_k \doteq Df\big(u(\tau, y_k)\big)$ for some $y_k \in\ ]x_{k-1}, x_k[\,$.

These comparisons are based on two lemmas. Equivalent results were proved in Theorem 2.6 of [17] and in [26], respectively. Since they play a crucial role in the uniqueness results, we include here the complete proofs.

Lemma 6.2. *Assume that the map $t \mapsto u(t,\cdot)$ is Lipschitz continuous with values in $\mathbf{L}^1(\mathbb{R})$. Moreover, let (τ,ξ) be a point of approximate jump for u, so that (2.13) holds. Then*

$$\lim_{h\to 0+} \frac{1}{h}\int_{-h}^{h} \Big| u(\tau+h,\xi+x) - U(h,x)\Big|\,dx \;=\; 0. \tag{6.6}$$

Proof. Assume that, on the contrary, there exists a decreasing sequence $h_m \to 0$ such that

$$\frac{1}{h_m}\int_{-h_m}^{h_m} \Big| u(\tau+h_m,\xi+x) - U(h_m,x)\Big|\,dx \;\geq\; \delta \;>\; 0 \qquad\qquad \text{for all } m\geq 1.$$

By Lipschitz continuity, for some constant L this implies

$$\int_{-h_m}^{h_m} \Big| u(\tau+h_m+s,\xi+x) - U(h_m+s,x)\Big|\,dx \;\geq\; \delta h_m - Ls \qquad\qquad \text{for all } m\geq 1,\ s\geq 0.$$

Setting

$$\bar s = \frac{\delta}{L}\,, \qquad\qquad r_m \;=\; h_m(1+\bar s),$$

we now obtain

$$\begin{aligned}
&\frac{1}{r_m^2}\int_{-r_m}^{r_m}\int_{-r_m}^{r_m} \Big| u(\tau+t,\,\xi+x) - U(t,x)\Big|\,dxdt \\
&\quad\geq\; \frac{1}{(1+\bar s)^2 h_m^2}\int_{h_m}^{(1+\bar s)h_m}\int_{-h_m}^{h_m} \Big| u(\tau+t,\,\xi+x) - U(t,x)\Big|\,dxdt \\
&\quad\geq\; \frac{1}{(1+\bar s)^2 h_m^2}\int_0^{\bar s h_m} (\delta h_m - Ls)\,ds \;=\; \frac{1}{(1+\bar s)^2}\cdot\frac{\delta^2}{2L} \;>\; 0.
\end{aligned}$$

This contradicts the assumption (2.13). □

Estimating the difference $u(\tau+h,\cdot) - W_k(\tau+h,\cdot)$ requires more work. Indeed, here we are approximating u with the solution to a linearized problem. This is accurate as long as u remains close to a constant, not only at time τ, but on the entire trapezoidal domain

$$\Gamma_k \;=\; \Big\{(t,x)\,;\quad t\in[\tau,\tau+h],\quad x\in J_k(t)\Big\}, \tag{6.7}$$

$$J_k(t) \;\doteq\; \,]x_{k-1}+(t-\tau),\ x_k-(t-\tau)[\,, \tag{6.8}$$

as shown in Fig. 20. For a solution constructed by front tracking, or by the Glimm scheme, the assumption that $u(\tau,\cdot)$ has small total variation on $]x_{k-1},x_k[$ implies that u has few waves (and hence is nearly constant) also on the domain of dependence Γ_k in (6.7). However, in principle this may not be true for more general weak solutions. For example, the analysis in [7] shows that the Godunov scheme can ampify the total variation by an arbitrarily large factor.

For this reason, additional assumptions were required in earlier papers. Namely "Tame Variation" in [29], "Tame Oscillation" in [24], and "Bounded variation along space-like curves" in [30]. For a class of 2×2 systems, the recent analysis in [37] has shown that any entropy admissible weak solution taking values in the domain of the semigroup always satisfies the bounded variation condition in [30]. Therefore, one does not need this additional assumption to achieve uniqueness.

Following [23, 26], we show here that uniqueness holds also for fully general $n \times n$ systems, without any of the previous regularity assumptions.

To appreciate the underlying idea, consider the scalar function

$$V(t) \;=\; \text{Tot.Var.}\big\{u(t,\cdot)\,;\; J_k(t)\big\},$$

where $J_k(t)$ is the interval introduced in (6.8). Notice that V is lower semicontinuous, hence measurable. Assume that τ is a Lebesgue point for V. Since $V(\tau) \leq \varepsilon$, this implies that the set of times where $V(t) > 2\varepsilon$ is very small. Indeed,

$$\lim_{h\to 0+} \frac{\text{meas}\big\{t \in [\tau, \tau+h]\,;\;\; V(t) > 2\varepsilon\big\}}{h} \;=\; 0. \tag{6.9}$$

At time $\tau + h$ the difference between u and the solution W_k to the linearized equation (6.5) can be estimated as

$$\begin{aligned} E_k(h) \;&\doteq\; \int_{x_{k-1}+h}^{x_k-h} \big|u(\tau+h,x) - W_k(\tau+h,x)\big|\,dx \\ &=\; \mathcal{O}(1)\cdot \int_\tau^{\tau+h} \text{Tot.Var.}\big\{u(t,\cdot);\, J_k(t)\big\}\cdot \Big\|Df\big(u(t,\cdot)\big) - Df\big(u(\tau,y_k)\big)\Big\|_{\mathbf{L}^\infty}\,dt \qquad (6.10) \\ &=\; \mathcal{O}(1)\cdot \int_\tau^{\tau+h} V(t)\cdot \big\|u(t,\cdot) - u(\tau,y_k)\big\|_{\mathbf{L}^\infty}\,dt. \end{aligned}$$

If $V(t) \leq 2\varepsilon$ for all $t \in [\tau, \tau+h]$, we would be in the Tame Variation case. Both factors in the integrand on the right hand side of (6.10) have size $\mathcal{O}(1)\cdot\varepsilon$. Hence $E_k = \mathcal{O}(1)\cdot h\varepsilon^2$, as proved in [17, 16, 29]. In the general case, there is an additional error, measured by how much the solution u can change during the intervals of time where $V(t) > 2\varepsilon$. If τ is a Lebesgue point for V, since the map $t \mapsto u(t,\cdot)$ is Lipschitz continuous, by (6.9) we obtain the slightly weaker bound

$$E_k(h) \;=\; \mathcal{O}(1)\cdot h\varepsilon^2 + o(h),$$

where the Landau symbol $o(h)$ denotes a higher order infinitesimal as $h \to 0$. We now state a more precise result in this direction. As before, we assume that all characteristic speeds are contained in the interval $[-1, 1]$.

Lemma 6.3. *Let $t \mapsto u(t,\cdot)$ be a weak solution to the strictly hyperbolic system (2.1), Lipschitz continuous with values in $\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)$. Assume that*

(i) Tot.Var.$\big\{u(\tau,\cdot)\big\} \;\leq\; M$ for all $t \geq 0$.

(ii) Tot.Var.$\big\{u(\tau,\cdot)\,;\;]a,b[\big\} \;\leq \varepsilon$,

(iii) the time τ is a Lebesgue point of the function

$$V(t) \;=\; \textit{Tot.Var.}\Big\{u(t,\cdot)\,;\; \big]a + (t-\tau)\,,\; b - (t-\tau)\big[\Big\}. \tag{6.11}$$

Call W the solution to the linearized equation

$$w_t + \widetilde{A} w_x \;=\; 0, \qquad\qquad w(\tau,x) = u(\tau,x) \quad \text{for} \quad x \in [a,b], \tag{6.12}$$

with $\widetilde{A} = Df\big(u(\tau,\xi)\big)$ for some $\xi \in]a,b[$. Then there holds

$$\limsup_{h\to 0+} \frac{1}{h}\int_{a+h}^{b-h} \Big| u(\tau+h,x) - W(\tau+h,x)\Big| dx \;=\; \mathcal{O}(1)\cdot\varepsilon^2. \tag{6.13}$$

Proof. By Theorem 4.3.1 in [44], the bound (i) on the total variation implies that the map $t \mapsto u(t,\cdot)$ is Lipschitz continuous with values in $\mathbf{L}^1(\mathbb{R};\,\mathbb{R}^n)$.

Let $\lambda_i(u)$, $l_i(u)$, $r_i(u)$, $i = 1,\ldots,n$, be respectively the eigenvalues and the left and right eigenvectors of the matrix $A(u) = Df(u)$. For notational convenience, call $\tilde u = u(\tau,\xi)$, and let $\tilde\lambda_i, \tilde l_i, \tilde r_i$ the corresponding eigenvalues and left and right eigenvectors of the matrix $\widetilde{A} = Df(\tilde u)$.

Since W solves the linear problem (6.12), one has

$$\tilde l_i \cdot W(t,x) \;=\; \tilde l_i \cdot W\big(\tau, x-(t-\tau)\tilde\lambda_i\big) \;=\; \tilde l_i \cdot u\big(\tau, x-(t-\tau)\tilde\lambda_i\big).$$

Following the proof of Theorem 9.4 in [17], fix any two points

$$\zeta', \zeta'' \;\in\; J(t) \;\doteq\; \big]a+(t-\tau),\, b-(t-\tau)\big[\,. \tag{6.14}$$

Assuming $\zeta' < \zeta''$, consider the quantity

$$\begin{aligned} E_i(\zeta',\zeta'') \;&\doteq\; \tilde l_i \cdot \int_{\zeta'}^{\zeta''} \big(u(t,x) - W(t,x)\big)\, dx \\ &=\; \tilde l_i \cdot \int_{\zeta'}^{\zeta''} \Big(u(t,x) - u\big(\tau,\; x-(t-\tau)\tilde\lambda_i\big)\Big)\, dx. \end{aligned} \tag{6.15}$$

We apply the divergence theorem to the vector $\big(u,\, f(u)\big)$ on the domain

$$D_i \;\doteq\; \Big\{(s,x);\;\; s\in[\tau,\; t],\;\; \zeta' - (t-s)\tilde\lambda_i \le x \le \zeta'' - (t-s)\tilde\lambda_i\Big\}, \tag{6.16}$$

shown in Fig. 21.

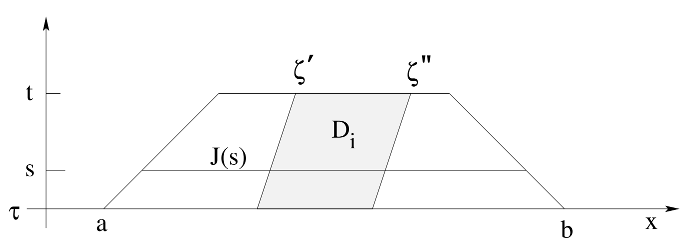

Figure 21: The domain D_i considered at (6.16).

Since u satisfies the conservation equation (2.1), the difference between the integral of u at the top and at the bottom of the domain D_i is measured by the inflow from the left side minus the outflow from the right side of D_i. From (6.15) it thus follows

$$\begin{aligned}
E_i(\zeta',\zeta'') = \quad & \int_\tau^t \tilde{l}_i \cdot \Big(\big(f(u) - \tilde{\lambda}_i u\big)(s,\ \zeta' - (t-s)\tilde{\lambda}_i) \Big)\, ds \\
& \quad - \int_\tau^t \tilde{l}_i \cdot \Big(\big(f(u) - \tilde{\lambda}_i u\big)(s,\ \zeta'' - (t-s)\tilde{\lambda}_i) \Big)\, ds \\
= \quad & \int_\tau^t l_i(\tilde{u}) \cdot \Big(\big(f(u'(s)) - \tilde{\lambda}_i\,(\tilde{u})\, u'(s)\big) - \big(f(u''(s)) - \lambda_i(\tilde{u}) u''(s)\big) \Big)\, ds \\
= \quad & \int_\tau^t F\left(\tilde{u}, u'(s), u''(s)\right)\, ds,
\end{aligned} \tag{6.17}$$

where we set

$$u'(s) \doteq u\big(s,\ \zeta' - (t-s)\tilde{\lambda}_i\big), \qquad u''(s) \doteq u\big(s,\ \zeta'' - (t-s)\tilde{\lambda}_i\big),$$

$$F\left(u, u_1, u_2\right) \ \doteq\ l_i(u) \cdot \Big(\big(f(u_1) - \lambda_i(u) u_1\big) - \big(f(u_2) - \lambda_i(u) u_2\big) \Big).$$

Observing that

- $F\left(u, u_2, u_2\right) = 0,$
- $D_{u_1} F\left(u, u_1, u_2\right) = l_i(u) \cdot (Df(u_1) - \lambda_i(u) I),$
- $D_{u_1} F\left(u, u, u_2\right) = l_i(u) \cdot (Df(u) - \lambda_i(u) I) = 0,$

one obtains

$$\begin{aligned}
F\left(u, u_1, u_2\right) \ &=\ F\left(u, u_1, u_2\right) - F\left(u, u_2, u_2\right) \\
&=\ \int_0^1 D_{u_1} F\left(u, u_2 + \sigma\left(u_1 - u_2\right), u_2\right)\ d\sigma \cdot \left(u_1 - u_2\right) \\
&=\ \int_0^1 \left[D_{u_1} F\left(u, u_2 + \sigma\left(u_1 - u_2\right), u_2\right) - D_{u_1} F\left(u, u, u_2\right)\right]\ d\sigma \cdot \left(u_1 - u_2\right) \\
&=\ \mathcal{O}(1) \cdot \big(\left|u_1 - u\right| + \left|u_2 - u\right|\big) \cdot \left|u_1 - u_2\right|.
\end{aligned}$$

In turn, this yields

$$E_i(\zeta',\zeta'') \ =\ \mathcal{O}(1) \cdot \int_\tau^t \big|u'(s) - u''(s)\big| \cdot \Big(\big|u'(s) - \tilde{u}\big| + \big|u''(s) - \tilde{u}\big| \Big)\, ds. \tag{6.18}$$

Recalling the definitions at (6.11) and (6.14), for any $x \in J(s)$ we now compute

$$\big|u'(s) - \tilde{u}\big| \ \leq\ V(s) + \big|u\big(s,x\big) - u\left(\tau,x\right)\big| + \left|u\left(\tau,x\right) - \tilde{u}\right| \ \leq\ V(s) + \big|u\big(s,x\big) - u\left(\tau,x\right)\big| + V(\tau).$$

Integrating w.r.t. x over the interval $J(s)$, dividing by its length and using the Lipschitz continuity of the map $t \mapsto u(t,\cdot)$, we obtain

$$\begin{aligned}
\left|u'(s) - \tilde{u}\right| \ &\leq\ V(s) + V(\tau) + \frac{1}{\mathrm{meas}\big(J_k(s)\big)} \int_{J_k(s)} \big|u(s,x) - u(\tau,x)\big|\, dx \\
&=\ V(s) + \varepsilon + \mathcal{O}(1) \cdot (s - \tau) \doteq g(s).
\end{aligned} \tag{6.19}$$

An entirely similar estimate holds for $\big|u''(s)-\tilde u\big|$. Therefore

$$\begin{aligned} E_i(\zeta',\zeta'') &= \mathcal{O}(1)\cdot\int_\tau^t \big|u'(s)-u''(s)\big|\cdot g(s)\,ds \\ &= \mathcal{O}(1)\cdot\int_\tau^t \text{Tot.Var.}\Big\{u(s)\,;\ \big]\zeta'-(t-s)\tilde\lambda_i,\ \zeta''-(t-s)\tilde\lambda_i\big]\Big\}\cdot g(s)\,ds \\ &= \mathcal{O}(1)\cdot\mu_i\big(]\zeta',\zeta'']\big). \end{aligned}$$

Here μ_i is the Borel measure defined by

$$\mu_i\big(]c,d[\big) = \int_\tau^t \text{Tot.Var.}\Big\{u(s)\,;\ \big]c-(t-s)\tilde\lambda_i,\ d-(t-s)\tilde\lambda_i\big[\Big\}\cdot g(s)\,ds,$$

for any open interval $]c,d[\subset J(t)$.

As proved in Lemma 9.3 of [17], one has

$$\begin{aligned} \int_{J(t)}\big|u(t,x)-W(t,x)\big|\,dx &= \mathcal{O}(1)\cdot\sum_{i=1}^n\int_{J(t)}\Big|\tilde l_i\cdot\big(u(t,x)-W(t,x)\big)\Big|\,dx \\ &= \mathcal{O}(1)\cdot\sum_{i=1}^n\mu_i\left(J(t)\right) = \mathcal{O}(1)\cdot\int_\tau^t V(s)\cdot g(s)\,ds. \end{aligned}$$

In turn, this implies

$$\frac{1}{t-\tau}\int_{J(t)}\big|u(t,x)-W(t,x)\big|\,dx = \mathcal{O}(1)\cdot\left(\frac{\|g\|_\infty}{t-\tau}\int_\tau^t\big|V(s)-V(\tau)\big|\,ds+\frac{V(\tau)}{t-\tau}\int_\tau^t g(s)\,ds\right). \tag{6.20}$$

We now observe that, for all $s>\tau$ sufficiently close to τ, the function g introduced at (6.19) satisfies

$$g(s) \le V(s)+\varepsilon+\mathcal{O}(1)\cdot(t-\tau)\,. \tag{6.21}$$

Since $t=\tau$ is a Lebesgue point for V, taking the limit of (6.20) as $t\to\tau+$ we thus obtain

$$\limsup_{t\to\tau+}\frac{1}{t-\tau}\int_{J(t)}\big|u(t,x)-W(t,x)\big|\,dx = \mathcal{O}(1)\cdot V(\tau)\left(V(\tau)+\varepsilon\right) = \mathcal{O}(1)\cdot\varepsilon^2, \tag{6.22}$$

proving (6.13). □

Going back to instantaneous error estimate (6.4), by Lemma 6.2 it follows

$$\lim_{h\to0+} A_k(h) = 0 \qquad\qquad \text{for all } k=1,\ldots,N-1. \tag{6.23}$$

Moreover, using Lemma 6.3, for some constant C and every $k=1,\ldots,N$ we obtain

$$\limsup_{h\to0+} B_k(h) \le C\,\varepsilon^2. \tag{6.24}$$

Observing that the number of intervals in the partition is $N=\mathcal{O}(1)\cdot\varepsilon^{-1}$, we conclude

$$\limsup_{h\to0+}\left(\sum_{k=1}^{N-1}A_k(h)+\sum_{k=1}^N B_k(h)\right) \le 0+N\cdot C\varepsilon^2 = \mathcal{O}(1)\cdot\varepsilon. \tag{6.25}$$

Since $\varepsilon>0$ can be chosen arbitrarily small, this implies (6.2).

We can now state the main uniqueness theorem in [23].

Theorem 6.1. *Let (2.1) be a strictly hyperbolic $n\times n$ system of conservation laws and consider the semigroup of vanishing viscosity solutions $S : \mathcal{D} \times \mathbb{R}_+ \mapsto \mathcal{D}$, constructed in Theorem 5.2. Then, any weak solution $t \mapsto u(t,\cdot) \in \mathcal{D}$, which takes values in the domain of the semigroup and satisfies the Liu admissibility conditions (2.19) at every point of approximate jump, coincides with a semigroup trajectory:*

$$u(t,\cdot) \;=\; S_t u(0) \qquad\qquad \textit{for all } t \geq 0. \tag{6.26}$$

Notice that, by Theorem 5.2, every limit of the vanishing viscosity approximations (5.6) is a weak solution to (2.9) and satisfies the Liu admissibility conditions. The above result provides a converse: every weak solution to (2.9) which is Liu-admissible (and has suitably small total variation, so it lies within the domain $\mathcal{D}$) actually coincides with a semigroup trajectory. Therefore it is obtained as the unique limit of the viscous approximations (5.6).

Sketch of the proof. 1. The assumption that $u = u(t,x)$ is a weak solution and its total variation Tot.Var.$\big\{u(t,\cdot)\big\}$ remains uniformly bounded implies that $u : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}^n$ is a BV function of the two variables t, x. By a general structure theorem, the set of its approximate jumps is countably rectifiable [1], i.e., it can be covered by countably many Lipschitz curves (see Fig. 22, left). However, since u is a solution to a hyperbolic system, at each point of jump the Rankine-Hugoniot equations hold. In particular, the speed of these jumps must be uniformly bounded. As proved in [23], the set of approximate jumps is contained in the graphs of countably many Lipschitz functions (see Fig. 22, left).

$$x \;=\; \phi_\ell(t), \qquad\qquad \ell \geq 1. \tag{6.27}$$

To simplify our notation, w.l.o.g. we shall assume that all characteristic speeds $\lambda_i(u)$ are contained in the interval $[-1,1]$ and all functions ϕ_ℓ have Lipschitz constant 1.

2. In addition to the functions ϕ_ℓ we consider the countably many functions

$$\phi^{\xi+}(t) \;=\; \xi + t, \qquad\qquad \phi^{\zeta-}(t) \;=\; \xi - t, \tag{6.28}$$

where $\xi \in \mathbb{Q}$ is rational. We relabel the set of all these functions as

$$\{\phi_\ell\,;\; \ell \geq 1\} \cup \{\phi^{\xi+}\,;\; \xi \in \mathbb{Q}\} \cup \{\phi^{\zeta-}\,;\; \zeta \in \mathbb{Q}\} \;=\; \{\psi_j\,;\; j \geq 1\}, \tag{6.29}$$

and consider the countably many functions

$$W_{ij}(t) \;\doteq\; \begin{cases} \text{Tot.Var.}\Big\{u(t,\cdot)\,;\;\; \big]\psi_i(t),\, \psi_j\big[\,\Big\} & \text{if } \;\psi_i(t) < \psi_j(t), \\ \qquad\qquad 0 & \text{otherwise.} \end{cases} \tag{6.30}$$

We observe that each function W_{ij} is measurable. Therefore there exists a null set $\mathcal{N}$ such that every $\tau \in \mathbb{R}_+ \setminus \mathcal{N}$ is a Lebesgue point for all the countably many functions W_{ij}.

3. As shown in Fig. 23, we now insert points $y_1 < y_2 < \cdots < y_N$ so that the total variation of $u(\tau,\cdot)$ on each open interval $]y_{k-1}, y_k[$ is $< \varepsilon$.

- If $u(\tau,\cdot)$ has a jump at y_k, then by construction (τ, y_k) lies on the graph of one of the functions ϕ_ℓ.

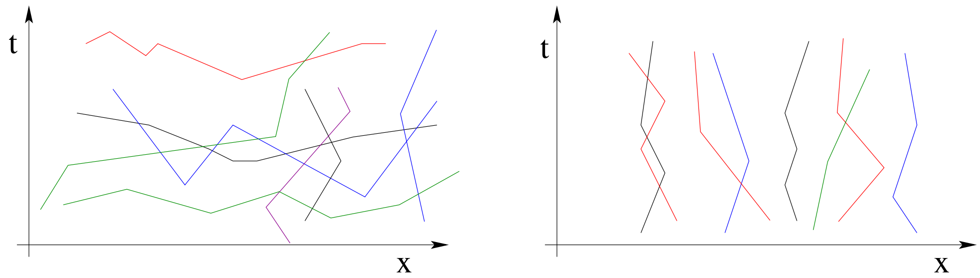

Figure 22: Left: for a general BV function $u : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}^n$ the set of approximate jumps can be covered by countably many Lipschitz curves. Right: if u is a solution to (2.1), all of its jumps travel with bounded speed. Hence the set of jumps can be covered by graphs of uniformly Lipschitz functions.

- If $u(\tau, \cdot)$ is continuous at y_k, then we can find two nearby points $y_k' < y_k < y_k''$, lying on one of the rational lines in (6.28), namely

$$y_k' \;=\; \phi^{\xi+}(\tau), \qquad\qquad y_k'' = \phi^{\zeta-}(\tau), \qquad \text{for some} \;\; \xi, \zeta \in \mathbb{Q},$$

  and furthermore

$$\text{Tot.Var.}\big\{u(\tau, \cdot)\,;\; [y_k', y_k'']\big\} \;<\; \varepsilon.$$

In the end, we can cover the real line with finitely many points y_k and open intervals $I_k = \,]a_k, b_k[\,$ with the following properties.

(i) At each point y_k, the function $u(\tau, \cdot)$ has an approximate jump, satisfying the Liu admissibility condition. By Lemma 6.2 this implies

$$\lim_{h\to 0+} \frac{1}{h} \int_{y_k - h}^{y_k + h} \Big| u(t, x) - U(t - \tau, x - y_k) \Big| \, dx \;=\; 0. \tag{6.31}$$

(ii) On each open interval I_k the total variation satisfies $\text{Tot.Var.}\big\{u(\tau, \cdot)\,;\; I_k\big\} < 2\varepsilon$. Moreover, both endpoints of I_k lie on the graph of one of the functions ψ_j at (6.29), say

$$a_k = \psi_i(\tau), \qquad\qquad b_k = \psi_j(\tau),$$

for some $i, j \geq 1$. Since all functions ϕ_ℓ have Lipschitz constant ≤ 1, for $t \geq \tau$ we have

$$\text{Tot.Var.}\Big\{u(t, \cdot)\,;\;\; \big]a_k + (t - \tau),\, b_k - (t - \tau)\big[\,\Big\} \;\leq\; \text{Tot.Var.}\Big\{u(t, \cdot)\,;\;\; \big]\psi_i(t),\, \psi_j(t)\big[\,\Big\}.$$

Therefore, calling W_k the solution to the linearized equation

$$w_t + \widetilde{A}_k w_x \;=\; 0, \qquad\qquad w(\tau, x) = u(\tau, x), \qquad \widetilde{A}_k \doteq Df\left(u\Big(\tau, \frac{a_k + b_k}{2}\Big)\right),$$

by Lemma 6.3 it follows

$$\limsup_{h\to 0+} \frac{1}{h} \int_{a_k + h}^{b_k - h} \Big| u(\tau + h, x) - W_k(\tau + h, x) \Big| dx \;=\; \mathcal{O}(1) \cdot \varepsilon^2. \tag{6.32}$$

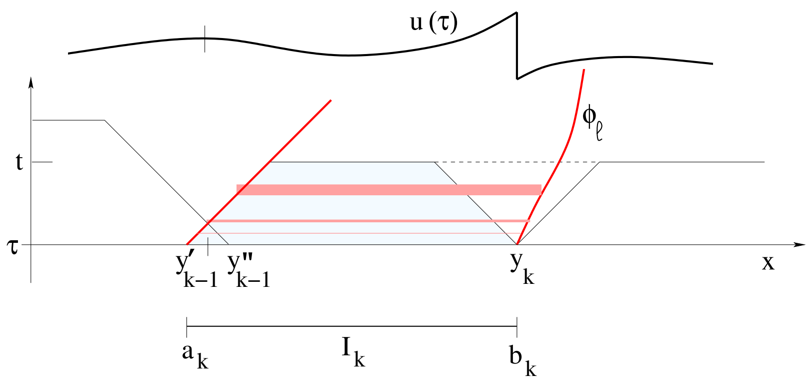

Figure 23: Covering the real line with points y_k where $u(\tau,\cdot)$ has a jump, and open intervals $I_k =]a_k, b_k[$ where the total variation is $< 2\varepsilon$.

Since the semigroup trajectory $v(t,\cdot) = S_{t-\tau}u(\tau)$ satisfies the same estimates as $u(t,\cdot)$, combining (6.31)-(6.32) we conclude that, at any Lebesgue time $\tau \in \mathbb{R}_+ \setminus \mathcal{N}$,

$$\limsup_{h\to 0+} \frac{\big\|u(\tau+h) - S_h u(\tau)\big\|_{\mathbf{L}^1}}{h} \;\leq\; \sum_{k=1}^{N} C_1 \cdot \varepsilon^2 \;\leq\; C_2\varepsilon, \tag{6.33}$$

for some constants C_1, C_2. Indeed, the number of intervals I_k in the partition is $N = \mathcal{O}(1)\cdot\varepsilon^{-1}$. Since $\varepsilon > 0$ can be chosen arbitrarily small, this achieves the proof. □

7 Error estimates

Having constructed a Lipschitz semigroup of admissible solutions to (2.1), it is of interest to estimate the $\mathbf{L}^1$ distance between an approximate solution constructed by one of the algorithms described in Section 3 and the exact solution.

Results in this direction were proved in [16] for the front tracking method, in [5, 33, 13] for the Glimm scheme, and in [28, 35] for vanishing viscosity approximations.

More precisely, consider the $n \times n$ hyperbolic system (2.1), assuming that all characteristic fields are genuinely nonlinear. The estimate in [35] shows that the distance between the solution u^ε of the viscous approximation (3.10) and the exact solution $u(t) = S_t\bar u$ to the Cauchy problem (3.1) can be estimated as

$$\big\|u^\varepsilon(t,\cdot) - u(t,\cdot)\big\|_{\mathbf{L}^1} \;=\; \mathcal{O}(1)\cdot(1+t)\,\text{Tot.Var.}\{\bar u\}\,\sqrt{\varepsilon}\,|\ln\varepsilon|. \tag{7.1}$$

Next, consider an approximate solution u^{Glimm} constructed by the Glimm scheme, with a grid of step size $\Delta t = \Delta x = \varepsilon$. Choosing sampling points as in (3.9), the analysis in [33] has established a similar convergence rate:

$$\lim_{\varepsilon\to 0} \frac{\big\|u^{Glimm}(t,\cdot) - u(t,\cdot)\big\|_{\mathbf{L}^1}}{\sqrt{\varepsilon}\,|\ln\varepsilon|} \;=\; 0 \qquad\qquad \text{for all } t > 0. \tag{7.2}$$

For other approximation methods, such as periodic mollifications, the backward Euler scheme, or fully discrete numerical schemes, no a priori BV bounds are currently available. In particular, it is known that the Godunov scheme can amplify the total variation by an arbitrarily large factor [7]. For this reason, it seems more promising to look for *a posteriori* error bounds. Namely, assume that an approximate solution to the Cauchy problem (3.1) has been constructed, whose total variation remains small for all times $t \in [0, T]$. Using this additional information, we seek a bound on the error

$$\left\| u^{approx}(t,\cdot) - u^{exact}(t,\cdot) \right\|_{\mathbf{L}^1}. \tag{7.3}$$

We outline here an approach which is in a sense "universal", i.e., it does not make reference to any particular approximation method.

Given $\varepsilon > 0$, consider an approximate solution $u = u(t,x)$ with the following properties.

Definition 7.1. *Let (2.1) be an $n\times n$ strictly hyperbolic systems of conservation laws, endowed with a strictly convex entropy η, with entropy flux q. We say that $u = u(t,x)$ is an ε-* **approximate solution** *to the Cauchy problem (2.9) if $\left\| u(0,\cdot) - \bar u \right\|_{\mathbf{L}^1} \leq \varepsilon$ and moreover the following holds.*

(AL$_\varepsilon$) Approximate Lipschitz continuity:

$$\|u(\tau,\cdot) - u(\tau',\cdot)\|_{\mathbf{L}^1} \ \leq\ M\,|\tau - \tau'| + \varepsilon \qquad\qquad \textit{for all } \tau, \tau' \geq 0$$

(P$_\varepsilon$) Approximate conservation law and approximate entropy inequality:

For every strip $[\tau, \tau'] \times \mathbb{R}$ and every test function $\varphi \in \mathcal{C}^1_c(\mathbb{R}^2)$, one has

$$\begin{aligned} &\left| \int u(\tau,x)\varphi(\tau,x)\,dx - \int u(\tau',x)\varphi(\tau,x)\,dx + \int_\tau^{\tau'}\!\!\int \big\{ u\varphi_t + f(u)\varphi_x \big\}\,dx\,dt \right| \\ &\qquad \leq\ \varepsilon(\tau' - \tau + \varepsilon)\,\|\varphi\|_{W^{1,\infty}}\,. \end{aligned} \tag{7.4}$$

Moreover, assuming $\varphi \geq 0$, one has the entropy inequality

$$\begin{aligned} &\int \eta(u(\tau,x))\varphi(\tau,x)\,dx - \int \eta(u(\tau',x))\varphi(\tau',x)\,dx + \int_\tau^{\tau'}\!\!\int \big\{ \eta(u)\varphi_t + q(u)\varphi_x \big\}\,dxdt \\ &\qquad \geq\ -\,\varepsilon(\tau' - \tau + \varepsilon)\,\|\varphi\|_{W^{1,\infty}}. \end{aligned} \tag{7.5}$$

In the above setting, the paper [19] has established **a posteriori** error estimates, assuming that the total variation of the ε-approximate solution remains small, so that $u(t,\cdot)$ remains within the domain of the semigroup. However, the estimates in [19] also required a "post processing algorithm", tracing the location of the large shocks in the approximate solution. This is related to the assumptions of "tame variation", "tame oscillation" or "bounded variation along space-like curves" which were used respectively in [29], [25] and in [30] to prove uniqueness of solutions. In essence, these additional assumptions rule out configurations such as the one shown in Fig. 24.

The recent paper [26] has shown that, for a system endowed with a strictly convex entropy, these additional regularity conditions are not needed to achieve uniqueness:

Theorem 7.1. *Let (2.1) be a strictly hyperbolic $n \times n$ system, where each characteristic field is either genuinely nonlinear or linearly degenerate, and which admits a strictly convex entropy $\eta(\cdot)$. Then every entropy admissible weak solution $u : [0,T] \mapsto \mathcal{D}$, coincides with a semigroup trajectory.*

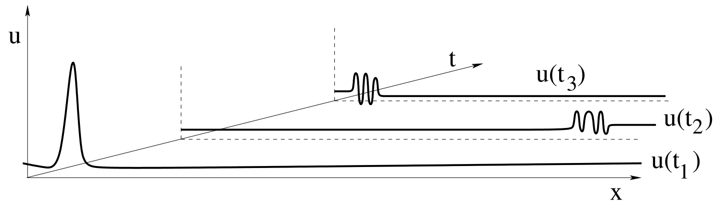

Figure 24: An approximate solution u where the total variation remains small at all times. However, oscillations appear and disappear at different regions on the x-t plane.

We observe that, in the setting of the above theorem, the dissipation of a single entropy suffices to single out the Liu-admissible shocks.

As proved in [26], the compactness of the family of approximate solutions, together with the uniqueness of the limit, yields a uniform convergence rate:

Corollary 7.1. *In the above setting, given $T, R > 0$, there exists a function $\varepsilon \mapsto \varrho(\varepsilon)$ with the following properties.*

(i) ϱ is continuous, nondecreasing, with $\varrho(0) = 0$.

(ii) Let $t \mapsto u(t,\cdot) \in \mathcal{D}$ be an ε-approximate solution to (3.1), with $u(t,\cdot)$ supported inside the interval $[-R, R]$ for all $t \in [0,T]$. Then one has

$$\left\|u(t) - S_t\bar{u}\right\|_{\mathbf{L}^1} \leq \varrho(\varepsilon) \qquad \textit{for all } t \in [0,T]. \tag{7.6}$$

This corollary shows that such a "universal rate of convergence" must exist. However, it does not offer clues on how the function $\rho(\cdot)$ looks like.

Open Problem #3. *Let (2.1) be a strictly hyperbolic $n\times n$ system, where each characteristic field is either genuinely nonlinear or linearly degenerate, and which admits a strictly convex entropy $\eta(\cdot)$. Provide an asymptotic estimate on the universal convergence rate $\varrho(\cdot)$ in (7.6), as $\varepsilon \to 0$.*

Based on the earlier estimates (7.1)-(7.2), in the genuinely nonlinear case one may conjecture that $\varrho(\varepsilon) \approx \varepsilon^{1/2}|\ln \varepsilon|$.

The key feature of the bound (7.6) is that it holds for any ε-approximate solution satisfying **($\mathbf{AL}_\varepsilon$)-($\mathbf{P}_\varepsilon$)**, regardless of the method used to construct the approximation. All the algorithms considered in Section 3 generate ε-approximate solutions, in the sense of Definition 7.1. See Section 6 in [19] for details.

One can speculate whether a similar universal convergence rate can be valid for general $n \times n$ systems, not necessarily endowed with a strictly convex entropy. For these systems, semigroup trajectories are characterized by the Liu admissibility condition, as in Definition 2.4. To reach our goal, we should replace the ε-approximate entropy condition (7.5) with some sort of ε-approximate Liu condition. This leads to

Open Problem #4. *Introduce a definition of "ε-approximate Liu admissible solution", valid for general $n \times n$ hyperbolic systems, possibly not endowed with a strictly convex entropy.*

A bit more precisely, what is needed here is a suitable definition such that the following properties will be satisfied.

- Approximate solutions with small total variation constructed by the various methods described in Section 3 should all satisfy the ε-approximate Liu condition, with $\varepsilon \to 0$ as the step size in the approximation (or the viscosity coefficient) approaches zero.
- Given a convergent sequence of approximations $u_n \to u$, if each u_n is an ε_n-approximate Liu admissible solution with $\varepsilon_n \to 0$, then the limit solution u should be Liu-admissible in the original sense.

8 Solutions with unbounded variation

As remarked in Section 4, for solutions with large initial data it is a hard open question to decide whether the total variation remains bounded for all times. It is thus natural to consider solutions in the larger space $\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)$, possibly with unbounded variation.

For 2×2 systems, existence of weak solutions with $\mathbf{L}^\infty$ data was proved in a fundamental paper by DiPerna [45], based on compensated compactness. See also [44, 64, 68] for a comprehensive account of this approach. Existence of $\mathbf{L}^\infty$ solutions remains a largely open problem for general $n \times n$ systems.

Unfortunately, compensated compactness works as a "black box". It provides an abstract result on the existence of solutions, but it does not yield information about uniqueness, continuous dependence, or the qualitative structure of these solutions. Some of the few results on the regularity of $\mathbf{L}^\infty$ solutions can be found in [38, 49].

In this direction, it would be of interest to construct a continuous semigroup of admissible solutions, defined on a domain larger than BV.

Open Problem #5. *Given an $n \times n$ hyperbolic system of conservation laws, extend the semigroup of vanishing viscosity solutions to a larger domain $\widetilde{\mathcal{D}} \subseteq \mathbf{L}^\infty(\mathbb{R};\,\mathbb{R}^n)$, also containing functions with unbounded variation.*

A continuous semigroup of solutions defined on the entire space $\mathbf{L}^\infty(\mathbb{R};\,\mathbb{R}^n)$ was constructed in [25] for some Temple class systems, and more recently in [27] for 2×2 systems in triangular form. But apart from a few special cases the problem is wide open.

As suggested in [2], in general it may not be possible to construct a continuous semigroup defined on the entire space $\mathbf{L}^\infty$. Instead, one could consider some intermediate domain $\widetilde{\mathcal{D}} \subset \mathbf{L}^\infty$, borrowing ideas from the theory of intermediate spaces used in the analysis of parabolic equations [50, 65]. Of course, we do not expect that the extended semigroup will be Lipschitz continuous. Its modulus of continuity will strongly depend on the regularity properties of functions $u \in \widetilde{D}$. For some very recent results in this direction, see [3, 32].

In addition to compensated compactness, another approach is worth mentioning here. In their classical memoir [48], Glimm and Lax consider the Cauchy problem for a genuinely nonlinear 2×2 system. Assuming that the initial data $\bar{u}$ has sufficiently small $\mathbf{L}^\infty$ norm, they prove that a global weak solution exists, globally in time. Indeed, the total variation (which initially may well be infinite) becomes locally finite at every time $t > 0$. See also [12] for a much shorter proof, based on front tracking approximations. The uniqueness and continuous dependence of these solutions still remains an open problem.

In the opposite direction, it would also be of interest to find examples of Cauchy problems admitting multiple solutions. In [34] a 3×3 strictly hyperbolic system has been constructed, together with bounded, measurable initial data, leading to an infinite number of solutions. However, this example does not have physical relevance because the system does not admit convex entropies. We thus conclude with

Open Problem #6. *Construct an example of an $n \times n$ strictly hyperbolic system, endowed with a strictly convex entropy, together with initial data $\bar{u} \in \mathbf{L}^\infty(\mathbb{R}; \mathbb{R}^n)$, such that the Cauchy problem admits two distinct entropy admissible solutions.*

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