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The theory of F-rational signature

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Abstract. F-signature is an important numeric invariant of singularities in positive characteristic that can be used to detect strong F-regularity. One would like to have a variant that rather detects F-rationality, and there are two theories that aim to fill this gap: F-rational signature of Hochster and Yao and dual F-signature of Sannai. Unfortunately, several important properties of the original F-signature are unknown for these invariants. We find a modification of the Hochster–Yao definition that agrees with Sannai's dual F-signature and push further the united theory to achieve a *complete* generalization of F-signature.

1. Introduction

Let $(R, \mathfrak{m})$ be a commutative Noetherian local domain of positive characteristic p. The world of positive characteristic is driven by the Frobenius endomorphism $F\colon R \to R$ defined by $r \mapsto r^p$. A particular way to study this endomorphism is via the family of modules $F^e_* R$ obtained from R by iterated restriction of scalars so that $rF^e_* x = F^e_*(r^{p^e} x)$. Under mild assumptions, satisfied in most arithmetic or geometric settings, these modules are finitely generated; we shall assume this holds throughout the introduction. Kunz proved that these modules detect regularity [39]: $F^e_* R$ is free for all $e \in \mathbb{N}$ (or equivalently any $e \in \mathbb{N}$) if and only if R is regular. This result motivates a number of numerical measures of singularities in positive characteristic, including F-signature and Hilbert–Kunz multiplicity.

The first of such invariants, the Hilbert–Kunz multiplicity, was defined by Monsky in 1983 [46] as an extension of earlier work of Kunz [40]. If $\ell(\_)$ denotes the length over R and the dimension of R is d, the Hilbert–Kunz multiplicity of an ideal I with $\ell(R/I) < \infty$ is defined as $\mathrm{e}_{\mathrm{HK}}(I) = \lim_{e\to\infty} \frac{1}{p^{ed}} \ell(R/I^{[p^e]})$, where $I^{[p^e]} = \langle x^{p^e} \mid x \in I \rangle$ is the expansion of I over the e-iterated Frobenius. Similarly, the F-signature was formally defined by Huneke and Leuschke [35] building upon the earlier work of Smith and Van den Bergh [61] on R-module

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direct sum decomposition of $F^e_* R$. In our setting, it is given by

$$\mathrm{s}(R) = \lim_{e\to\infty} \frac{\max\{N \mid \text{there is a surjection } F^e_* R \twoheadrightarrow R^N\}}{\operatorname{rank} F^e_* R}.$$

Both $\mathrm{s}(R)$ and $\mathrm{e}_{\mathrm{HK}}(\mathfrak{m})$ are natural measures of singularity, as they encode asymptotically how far the modules $F^e_* R$ are from being free. An alternate perspective on the F-signature, pioneered in [67, 69] and borne out in [51], links the two invariants together and characterizes the F-signature as the infimum of all relative Hilbert–Kunz differences

$$\mathrm{s}(R) = \inf\{\mathrm{e}_{\mathrm{HK}}(I) - \mathrm{e}_{\mathrm{HK}}(\langle I, u\rangle) \mid u \notin I,\ \ell(R/I) < \infty\}.$$

A crucial property of the F-signature is that it detects strong F-regularity, a class of singularities central to the celebrated theory of tight closure pioneered by Hochster and Huneke [29]. (Strong) F-regularity can be viewed as the positive characteristic analogue of Kawamata log terminal singularities important to the minimal model program in higher-dimensional complex algebraic geometry [23, 24, 60]. Closely related to F-regularity, F-rationality has long been an important class of singularities in positive characteristic commutative algebra. Classically defined by the property that all ideals $\langle \underline{x}\rangle$ generated by a system of parameters $\underline{x} = x_1, \ldots, x_d$ are tightly closed [19], F-rationality can be interpreted geometrically as a positive characteristic analogue of rational singularities over the complex numbers [23, 45, 59].

Recent years have led to rapid advances in our understanding of the F-signature; focusing on those most relevant to our current purpose, we highlight the following five core properties of F-signature.

(a) *Existence*: the limit defining $\mathrm{s}(R)$ exists [63].

(b) *Detects F-regularity*: $\mathrm{s}(R) \geq 0$, and $\mathrm{s}(R) > 0$ if and only if R is strongly F-regular [4, Theorem 0.2].

(c) *Detects regularity*: $\mathrm{s}(R) \leq 1$, and $\mathrm{s}(R) = 1$ if and only if R is regular [35].

(d) *Compatible with localization*: $\mathrm{s}(R) \leq \mathrm{s}(R_{\mathfrak{p}})$ for every prime ideal $\mathfrak{p}$ (see [4, Proposition 1.3]).

(e) *Semicontinuity*: $\mathfrak{p} \mapsto \mathrm{s}(R_{\mathfrak{p}})$ is lower semicontinuous on Spec R (see [50, 51]).

Attempts have been made to find an invariant akin to F-signature which detects F-rationality rather than F-regularity. The first, due to Hochster and Yao [32], builds on the notion that relative Hilbert–Kunz multiplicity can be used to test for tight closure. The F-rational signature of R, denoted here by $\mathrm{s}_{\mathrm{rat}}(R)$, is defined as

$$\mathrm{s}_{\mathrm{rat}}(R) = \inf\{\mathrm{e}_{\mathrm{HK}}(\langle \underline{x}\rangle) - \mathrm{e}_{\mathrm{HK}}(\langle \underline{x}, u\rangle) \mid u \notin \langle \underline{x}\rangle,\ \underline{x} \text{ a system of parameters}\},$$

where the infimum is taken over all systems of parameters $\underline{x}$ and elements $u \notin \langle \underline{x}\rangle$. When R is Gorenstein, it is straightforward to check that $\mathrm{s}_{\mathrm{rat}}(R)$ and $\mathrm{s}(R)$ coincide (see [35]). Hochster and Yao show that $\mathrm{s}_{\mathrm{rat}}(R) > 0$ if and only if R is F-rational so that the F-rational signature detects F-rationality and satisfies the analogue of property (b) above. Moreover, interpreted appropriately, one can show the F-rational signature satisfies an analogue of existence (a) as well; this property is particularly important in practice as it allows for estimation and computation. However, a computation of Hochster and Yao [32, Example 7.4] (see also Remark 6.3) shows that $s_{\mathrm{rat}}(R) = 1$ does not determine regularity as in (c). Moreover, to our knowledge, it is unclear (and perhaps unlikely) that the F-rational signature satisfies analogues of properties (d) and (e) above.

Following the introduction of the F-rational signature, an alternate construction was introduced by Sannai [53] mimicking the original definition of F-signature directly. Called the dual F-signature of R and denoted here $s_{\mathrm{dual}}(R)$, the invariant is defined as

$$s_{\mathrm{dual}}(R) = \limsup_{e\to\infty} \frac{\max\{N \mid \text{there is a surjection } F^e_* \omega_R \twoheadrightarrow \omega_R^N\}}{\operatorname{rank} F^e_* \omega_R},$$

where R is assumed Cohen–Macaulay with a canonical module ω_R. Once again, when R is Gorenstein, it is clear that $s_{\mathrm{dual}}(R)$ and $s(R)$ coincide. Sannai shows further (relying heavily upon [32]) that $s_{\mathrm{dual}}(R) > 0$ if and only if R is F-rational. Moreover, $s_{\mathrm{dual}}(R)$ is known to detect regularity and to be compatible with localization as well, satisfying in total the analogues of properties (b), (c), and (d) above. However, outside of a small number of examples (cf. [27, 47]), it has remained open whether the limit defining the dual F-signature exists. Not only is this problematic when attempting to compute or estimate $s_{\mathrm{dual}}(R)$, it is also at the heart of the difficulty in attempting to show that the dual F-signature defines a lower semicontinuous function on $\operatorname{Spec}(R)$. Thus, in short, we are left to wonder if the dual F-signature indeed satisfies the analogue of properties (a) and (e). Note also that we will see in Example 3.2 that $s_{\mathrm{rat}}(R)$ can be strictly larger than $s_{\mathrm{rel}}(R)$.

In this paper, we bring together the two approaches used above, showing that a modified version of the Hochster–Yao invariant defined via relative Hilbert–Kunz multiplicity agrees with Sannai's dual F-signature defined via the maximal numbers of surjections. To that end, we introduce herein *the relative F-rational signature* $s_{\mathrm{rel}}(R)$ of R,

$$s_{\mathrm{rel}}(R) = \inf_{\langle \underline{x}\rangle \subset I} \frac{e_{\mathrm{HK}}(\langle \underline{x}\rangle) - e_{\mathrm{HK}}(I)}{\ell(R/\langle \underline{x}\rangle) - \ell(R/I)},$$

where the infimum is taken over all systems of parameters $\underline{x}$ and all ideals I properly containing $\langle \underline{x}\rangle$. Our first main results can be summarized as follows.

Theorem (Corollary 5.6, Theorem 5.10). *The limit defining the dual F-signature exists and equals the relative F-rational signature. Furthermore, the dual F-signature is lower semicontinuous and therefore satisfies* all *five core properties* (a)–(e) *of F-signature listed above.*

In the course of showing these results, the equivalence of different perspectives on the F-signature plays a prominent role. The equivalence $s_{\mathrm{dual}}(R) = s_{\mathrm{rel}}(R)$ stated in the theorem itself requires developing an intricate linear algebra result, based on a number of nontrivial matrix computations and inductions, which we separated out from the main body of the article in the appendix. The third perspective arises as a certain dual interpretation of the splitting ideals I_e, which were originally defined by Aberbach–Enescu [2] and Yao [68], developed in Section 4 using the Cartier formalism in the sense of Blickle [9].

To the extent possible, our goal is to present unified theory of F-rational and dual F-signature that is fully parallel to the established theory of F-signature. The newly observed properties of the dual F-signature immediately lead to a number of novel perspectives on previously established results. For example, the core properties imply that the set $\{\mathfrak{p} \mid s_{\mathrm{dual}}(R_{\mathfrak{p}}) > 0\}$ is equal to the F-rational locus of R and is open, so we recover a result of Vélez [64, Theorem 1.11]. However, we are also able to leverage these properties further – the equality

$$s_{\mathrm{dual}}(R) = s_{\mathrm{rel}}(R)$$

in particular opens the door to the use of sophisticated uniform convergence techniques from Hilbert–Kunz theory to establish a number of new and important results.

Theorem. *The dual F-signature satisfies the following properties.*

- *(Corollary 3.9) The dual F-signature deforms, i.e.,* $s_{\mathrm{dual}}(R) \geq s_{\mathrm{dual}}(R/xR)$ *for a regular element* $x \in R$.
- *(Corollaries 3.13 and 5.14)* $s_{\mathrm{dual}}(R) \geq s_{\mathrm{dual}}(S)$ *for a faithfully flat local map* $R \to S$, *with equality if the closed fiber is geometrically regular.*
- *(Theorem 5.11) The global dual F-signature in the sense of* [17] *exists and is equal to the minimum of localizations* $s_{\mathrm{dual}}(R_{\mathfrak{p}})$.
- *(Theorem 5.20)* $b_e(R)$ *admits a second coefficient, i.e., there exists a constant* β *such that*

$$b_e(R) = s_{\mathrm{dual}}(R)p^{ed} + \beta p^{e(d-1)} + O(p^{e(d-2)}).$$

In particular, while many of these results again parallel the theory of F-signature, in some cases, we see that the behavior of the dual F-signature is even better. Indeed, F-signature (as well as strong F-regularity) fails to deform without additional assumptions, and moreover, the last result on the second coefficient remains an important open question for the F-splitting numbers.

Finally, note that, while the dual F-signature is undefined when R is not assumed to be F-finite, the definitions of both the F-rational and relative F-rational signature are well-posed. While not the primary aim of this article, we additionally explore and verify a number of desirable properties of the relative F-rational signature in this setting.

Theorem. *The following statements hold.*

(a) $s_{\mathrm{rel}}(R) \geq 0$, *and* $s_{\mathrm{rel}}(R) > 0$ *if and only if* R *is F-rational (Corollary 3.7.*

(b) $s_{\mathrm{rel}}(R) \leq 1$, *and* $s_{\mathrm{rel}}(R) = 1$ *if and only if* R *is regular (Proposition 3.3).*

(c) $s_{\mathrm{rel}}(R) \leq s_{\mathrm{rel}}(R_{\mathfrak{p}})$ *for every prime* $\mathfrak{p}$ *(Proposition 3.12).*

(d) $s_{\mathrm{rel}}(R) \leq s_{\mathrm{rel}}(S)$ *for a faithfully flat local map* $R \to S$ *(Corollary 3.13).*

(e) s_{rel} *deforms (Corollary 3.9).*

1.1. Structure of the paper. After setting up definitions in Section 2, we define and study the relative F-rational signature in Section 3. The results of this section are developed without the F-finite assumption. In particular, the analogues of properties (c), (b), (d) of F-signature are shown to hold for the relative F-rational signature (Corollary 3.11, Propositions 3.3 and 3.12). We also present an appropriate inequality for flat extensions in Proposition 3.13 and for deformation in Corollary 3.9. The main technical result of this section is Proposition 3.5, which allows one to restrict the computation of the relative F-rational signature to socle ideals for a given system of parameters. In turn, this also allows us to utilize the results of Hochster–Yao and show independence of a given system of parameters as well (Corollary 3.7). Combined with the semicontinuity result of [57], we also deduce here that the relative F-rational signature is a minimum (rather than infimum) of relative Hilbert–Kunz multiplicities. This gives a new proof independent of [32, Section 3] showing that the positivity of either $s_{rat}(R)$ or $s_{rel}(R)$ detects F-rationality.

In Section 4, we give an equivalent definition of the relative F-rational signature based on the Cartier operator on the canonical module (Definition 4.3) in the F-finite setting. The importance of this perspective comes in part by allowing one to view the relative F-rational signature as the minimum of numerical function on the k-rational points on a Grassmann variety. This leads to further study of a technical extension of the relative rational signature that takes into account the relative Hilbert–Kunz multiplicities determined by the non-k-rational points as well, with a view towards properties such as semicontinuity where it is natural to consider behavior at all geometric points. We will show existence, uniform convergence, and semicontinuity of this generalized invariant. These proofs are somewhat novel and are likely of independent interest. Inspired by [55], our method is roughly based on two steps: we first apply uniform convergence methods introduced in [63] to translate the problem to a more tractable invariant, and second, we use the semicontinuity of the rank of a continuous matrix-valued function on a vector bundle. Restricting back to k-rational points gives a number of alternative arguments in the non-generalized setting as well. For instance, an alternative proof of the fact that the relative F-signature achieves its minimum via these methods is given (Corollary 4.22).

In Section 5, we apply the intricate linear algebra machinery of Appendix A to show that all of the different points of view give equal invariants, and in particular, $s_{rel}(R) = s_{dual}(R)$. This allows one to exploit all of our techniques together at once and establish the fundamental properties of the dual F-signature highlighted above: its existence as a limit (Corollary 5.9), its semicontinuity (Theorem 5.10), existence of the second coefficient (Theorem 5.20), invariance under regular morphisms (Corollary 5.14), and also the local-to-global property (Theorem 5.11). We note that our proof of the existence of the global dual F-signature utilizes semicontinuity more efficiently than [17] and can be used to significantly shorten it.

Finally, in Section 6, we present an approach for computing the F-rational signature of a toric singularity, and we finish the paper with some of the remaining open questions. The appendix contains the crucial linear algebra machinery.

2. Preliminaries

2.1. F-rational singularities. In [19], Fedder and Watanabe defined a local ring $(R, \mathfrak{m})$ to be *F-rational* if every parameter ideal is tightly closed, in parallel to the notion of *weakly F-regular*, due to Hochster and Huneke in [29], that asks every ideal to be tightly closed. F-rational rings are normal, and an F-rational ring which is an image of Cohen–Macaulay ring is Cohen–Macaulay itself. The theory of F-rational rings was further developed in [30].

It was shown in [29, Theorem 8.17] that tight closure is determined by Hilbert–Kunz multiplicity under mild assumptions. Thus we can restate F-rationality using so-called *relative Hilbert–Kunz multiplicity*. We provide a proof to illustrate where the assumptions are used.

Proposition 2.1. *Let* $(R, \mathfrak{m})$ *be a local ring. If* $\mathrm{e}_{\mathrm{HK}}(\langle \underline{x} \rangle) > \mathrm{e}_{\mathrm{HK}}(I)$ *for every system of parameters* $\underline{x}$ *and every ideal* $\underline{x} \subsetneq I$*, then* R *is F-rational. Moreover, if* R *is excellent (more generally, it suffices that* R *and* $\widehat{R}$ *have a common parameter test element; see also* [16, Proposition 3.2.2]*), then the converse holds.*

Proof. If R is not F-rational, then there exists a system of parameters which is not tightly closed, say $a \in \langle \underline{x} \rangle^* \setminus \langle \underline{x} \rangle$. Hence $\mathrm{e}_{\mathrm{HK}}(\langle \underline{x} \rangle) = \mathrm{e}_{\mathrm{HK}}(I)$ by [29, Theorem 8.17], which requires no assumptions.

If R is complete and F-rational, then, since an F-rational ring is a domain, the assumptions of the converse in [29, Theorem 8.17] are satisfied; thus $\mathrm{e}_{\mathrm{HK}}(\langle \underline{x} \rangle) > \mathrm{e}_{\mathrm{HK}}(I)$ for all $\underline{x} \subsetneq I$. Since Hilbert–Kunz multiplicity is not affected by completion, the relative Hilbert–Kunz condition determines whether $\widehat{R}$ is F-rational. The excellence (or the weaker assumption) is needed to descend F-rationality from $\widehat{R}$ to R; see the discussion in [32, proof of Theorem 4.1]. □

The definitions of F-rational signature [32] and normalized F-rational signature (Section 3) are motivated by this equivalence. In [58], Karen Smith restated F-rationality using tight closure in local cohomology. As observed by Hochster–Yao, the relative Hilbert–Kunz multiplicity can be also restated using local cohomology; see Proposition 3.6.

Motivated by the notion of *strong F-regularity*, Veléz gave definition of *strong F-rationality* [64]. However, while the equivalence of strong and weak F-regularity is a long-standing conjecture, Velez [64, Lemma 1.3, Proposition 1.6] showed that the two versions of F-rationality agree for F-finite domains, the assumption that we will impose from Section 4. Note that an F-finite ring is excellent [40] and is a quotient of a regular ring by [21]; thus it has a dualizing module: a maximal Cohen–Macaulay module of finite injective dimension and Cohen–Macaulay type 1. We refer to the Bruns–Herzog book [11, Section 3.3] for properties of dualizing (canonical) modules.

Theorem 2.2 (Veléz). *Let* $(R, \mathfrak{m})$ *be a Cohen–Macaulay F-finite reduced local ring and* ω_R *its dualizing module. Then* R *is F-rational if and only if, for every* $0 \neq c \in R$*, there are* $e \geq 1$ *and a homomorphism* $\phi \colon F^e_* \omega_R \to \omega_R$ *such that* $\phi(F^e_*(c\omega_R)) = \omega_R$*.*

2.2. Semicontinuity. We recall that function $f \colon X \to \mathbb{R}$ on a topological space X is lower semicontinuous if, for any $a \in \mathbb{R}$, the set $\{x \in X \mid f(x) > a\}$ is open. Semicontinuity is an essential property of a singularity invariant for multiple reasons, and we will present multiple consequences of semicontinuity in this paper. Let us start with several fundamental properties.

Theorem 2.3. *Let* f *be a lower semicontinuous function on a topological space* X*. The following properties hold.*

(a) *If* X *is quasi-compact, then* f *has a minimum.*

(b) *If* X *is Noetherian, then a lower semicontinuous function satisfies a descending chain condition, i.e., the set of its values does not contain infinite strictly decreasing sequences.*

(c) *In particular, if* X *is Noetherian, then the minimum of* f *is separated, i.e., there exists* $\varepsilon > 0$ *such that, for any* $y \in X$*, if* $f(y) - \min_{x \in X} f(x) < \varepsilon$*, then* $f(y) = \min_{x \in X} f(x)$*.*

(d) *If X is a Noetherian k-scheme, then f attains a minimum on ℓ-rational points for any $k \subseteq \ell$. This minimum is also separated.*

Proof. (a) The ordered family of sets $\{x \in X \mid f(x) > a\}$ forms an open cover of X; thus the quasi-compactness assumption implies that there exists a minimal a_0 such that

$$\{x \in X \mid f(x) > a\} \neq X.$$

This a_0 is the minimum of f.

(b) Suppose that $a_1 = f(x_1) > a_2 = f(x_2) > \cdots > a_i = f(x_i) > \cdots$ is an infinite decreasing sequence. Then $X_i := \{x \in X \mid f(x) > a_i\}$ form an increasing chain of open sets, but any such chain must stabilize because X is Noetherian. Hence $x_i \in X_i = X_{i-1}$ for $i \gg 0$, a contradiction.

(c) Since there is no infinite decreasing sequences, there is the second smallest value.

(d) The set of values on ℓ-rational points has a minimum because it cannot contain an infinite decreasing sequence. □

See Corollary 3.7 for an important application of (d).

2.2.1. An important example. A standard example of an upper semicontinuous function (e.g., [51, Lemma 2.2]) is the minimal number of generators of a finitely generated R-module M: $\mathfrak{p} \mapsto \dim_{k(\mathfrak{p})}(M \otimes_R k(\mathfrak{p}))$ defines an upper semicontinuous function on Spec R. From this example, we can build more. For example, if A is an Artinian k-algebra of finite length and J is an ideal of $A[\underline{X}]$, where $\underline{X}$ is a *set* of variables, then $\mathfrak{p} \mapsto \ell(A[\underline{X}]/J \otimes_{A[\underline{X}]} k(\mathfrak{p}))$ defines an upper semicontinuous function on Spec $k[\underline{X}]$.

Later, we will need a semicontinuity result on the Grassmannian. This was observed in [57, Remark 4.17], but its uniform convergence machinery requires R to be a finitely generated k-algebra. Instead, we may use that the Grassmannian parametrizes ideals in R, so we get uniform convergence in much easier way.

Theorem 2.4. *Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$, and let I be an $\mathfrak{m}$-primary ideal. Let $V = (I :_R \mathfrak{m})/I$ be the socle of I, and for any subspace $U \subseteq V$, denote $J_U = (I, U)$, the corresponding socle ideal. Then the function*

$$U \mapsto \frac{1}{\dim_k U}(\mathrm{e}_{\mathrm{HK}}(I) - \mathrm{e}_{\mathrm{HK}}(J_U))$$

is lower semicontinuous on the k-rational points of the Grassmannian of X.

Proof. First, let us fix e and show that the function

$$U \mapsto \ell(R/J_U^{[p^e]})$$

is upper semicontinuous on the k-rational points of the Grassmannian of rank r subspaces of V. This is a local question, so we may cover the Grassmannian by affine patches. After choosing a basis $e_1, \ldots, e_N$ of V, a patch is given by setting a fixed maximal minor of the generic $r \times N$-dimensional matrix to be the identity; the remaining $r \times (N - r)$ entries are coordinates. Let us organize these entries in a generic matrix X. Without loss of generality, the non-vanishing minor is the top one,

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ X_{1,1} & X_{1,2} & \cdots & X_{1,r} \\ \vdots & \vdots & \vdots & \vdots \\ X_{N-r,1} & X_{N-r,2} & \cdots & X_{N-r,r} \end{bmatrix}.$$

Let A denote the Artinian ring $R/(I^{[p^e]})$. We may choose a coefficient field k for A. The ideal $J_U^{[p^e]}$ is obtained by specializing $X_{i,j}$ in the ideal

$$A[X] \supset J^{[p^e]} = \left(e_1 + \sum_{i=1}^{N-r} X_{1,i}^{p^e} e_{r+i}, \dots, e_r + \sum_{i=1}^{N-r} X_{r,i}^{p^e} e_{r+i}\right).$$

Thus the function $U \mapsto \ell(R/J_U^{[p^e]})$ is upper semicontinuous; note that the restriction to k-rational closed points is still semicontinuous since this set is equipped with the induced topology.

Second, we recall that [63, Theorem 3.6] gives uniform convergence: there exist a constant $C > 0$ and a positive integer e_0 such that, for any ideal $J \supseteq I$ and any positive integer e,

$$\left| \mathrm{e_{HK}}(J) - \frac{1}{p^{(e+e_0)d}} \ell(R/J^{[p^{e+e_0}]}) \right| \leq C/p^e.$$

This implies uniform convergence of

$$\ell\left(\frac{J_U^{[p^e]}}{I^{[p^e]}}\right) = \ell(R/I^{[p^e]}) - \ell(R/J_U^{[p^e]})$$

to the relative Hilbert–Kunz multiplicity. It follows that

$$U \mapsto \frac{1}{\dim_k U}(\mathrm{e_{HK}}(I) - \mathrm{e_{HK}}(J_U))$$

is lower semicontinuous as the uniform convergent limit of lower semicontinuous functions. □

3. Relative F-rational signature

Definition 3.1. Let $(R, \mathfrak{m})$ be a local ring.

(a) The *F-rational signature* of R is defined as

$$\mathrm{s_{rat}}(R) = \inf_{\underline{x} \subset I} \{\mathrm{e_{HK}}(\langle \underline{x} \rangle) - \mathrm{e_{HK}}(I)\},$$

where the infimum is taken over all systems of parameters $\underline{x}$ and ideals $\langle \underline{x} \rangle \subset I$.

(b) The *relative F-rational signature* of R is

$$s_{rel}(R) = \inf_{\underline{x}\subset I} \frac{e_{HK}(\langle \underline{x}\rangle) - e_{HK}(I)}{\ell(R/\langle \underline{x}\rangle) - \ell(R/I)},$$

where the infimum is taken over all systems of parameters $\underline{x}$ and ideals $\langle \underline{x}\rangle \subset I$.

F-rational signature was defined by Hochster and Yao in [32]. Clearly, if $s_{rat}(R) > 0$ or $s_{rel}(R) > 0$, then R is F-rational by Proposition 2.1. However, as its name indicates, the converse also holds under the assumptions of Proposition 2.1 [32, Theorem 4.1]. In (a) above, it is enough to consider any fixed system of parameters [32, Theorem 2.4], and it is easy to see that one can restrict to socle ideals $I = \langle \underline{x}, u\rangle$, where $\langle \underline{x}\rangle : u = \mathfrak{m}$ (cf. Corollary 3.7 for a similar result for relative F-rational signature). Though the difference in the two definitions would seem small, the additional normalizing factor in the definition of relative F-rational signature is quite useful and leads to a number of desirable properties that are unknown (if not false) for F-rational signature.

Example 3.2. It is easy to find examples where $s_{rat}(R) \neq s_{rel}(R)$ in the toric case; see Section 6. Explicitly, let V_n be the nth Veronese subring of $k[[x, y]]$. Hochster and Yao [32, Example 7.4] computed that $s_{rat}(V) = 1 - \frac{1}{n}$. On the other hand, one can see that the relative Hilbert–Kunz multiplicity for the entire socle is $\frac{1}{2}$, so $s_{rat}(V_n) > s_{rel}(V_n)$ for $n \geq 3$.

Namely, if we take a system of parameters x^n, y^n, then the whole maximal ideal

$$x^n, x^{n-1}y, \dots, y^n$$

is the socle. We compute the Hilbert–Kunz multiplicity by passing to $S = k[[x, y]]$ as follows:

$$e_{HK}(\mathfrak{m}, V_n) = \frac{\ell(S/\langle x, y\rangle^n)}{[S : V_n]} = \frac{\binom{n+1}{2}}{n} = \frac{n+1}{2}.$$

Since $e_{HK}(\langle x^n, y^n\rangle, V_n) = \ell(V/\langle x^n, y^n\rangle) = n$, we get that

$$\frac{e_{HK}(\langle x^n, y^n\rangle, V_n) - e_{HK}(\mathfrak{m}, V_n)}{\ell(\mathfrak{m}/\langle x^n, y^n\rangle)} = \frac{1}{2}.$$

In [53, Example 3.17], Sannai has computed that the dual F-signature of any Veronese subring of $k[[x, y]]$ is $\frac{1}{2}$. Using Theorem 5.4, this will show that, in fact, $s_{rel}(V_n) = \frac{1}{2}$.

3.1. Measuring singularities. As a first step, we record that the relative F-rational signature is normalized so as to detect singularity. The original F-rational signature does not detect singularities (Remark 6.3). It is also not known whether it is bounded above by 1 (which we suspect may be false).

Proposition 3.3. *If $(R, \mathfrak{m})$ is a formally unmixed local ring, then $s_{rel}(R) \leq 1$ with equality if and only if R is regular.*

Proof. Suppose that $s_{rel}(R) \geq 1$. It follows from the definition that

$$\frac{e_{HK}(\langle \underline{x}\rangle) - e_{HK}(\mathfrak{m})}{\ell(R/\langle \underline{x}\rangle) - \ell(R/\mathfrak{m})} \geq 1$$

for any system of parameters $\underline{x}$. Since $\ell(R/\mathfrak{m}) = 1$, we obtain that

$$1 \geq \ell(R/\langle \underline{x} \rangle) - \mathrm{e}_{\mathrm{HK}}(\langle \underline{x} \rangle) + \mathrm{e}_{\mathrm{HK}}(\mathfrak{m}) \geq \mathrm{e}_{\mathrm{HK}}(\mathfrak{m}).$$

Because $\mathrm{e}_{\mathrm{HK}}(\mathfrak{m}) \geq 1$ and $\mathrm{e}_{\mathrm{HK}}(\langle \underline{x} \rangle) = \mathrm{e}(\langle \underline{x} \rangle) \leq \ell(R/\langle \underline{x} \rangle)$, where the former holds by Lech's lemma [42, Theorem 2] and the latter by [37, Proposition 11.1.10], the above inequality implies that $\mathrm{e}_{\mathrm{HK}}(\mathfrak{m}) = 1$ and R is regular by a result of Watanabe and Yoshida [66, Theorem 1.5].

The converse follows by noting that $\mathrm{e}_{\mathrm{HK}}(I) = \ell(R/I)$ for any $\mathfrak{m}$-primary ideal I of a regular local ring R, so $\mathrm{s}_{\mathrm{rel}}(R) = 1$. □

The same idea can be also used to show that R has mild singularities assuming $\mathrm{s}_{\mathrm{rel}}(R)$ is sufficiently close to one.

Corollary 3.4. *Let $(R, \mathfrak{m})$ be a formally unmixed local ring with an infinite residue field.*

(1) *If $\mathrm{s}_{\mathrm{rel}}(R) \geq 1 - \frac{\max\{\frac{1}{d!}, \frac{1}{\mathrm{e}(R)}\}}{\mathrm{e}(R)-1}$, then R is weakly F-regular.*
(2) *If $\mathrm{s}_{\mathrm{rel}}(R) \geq 1 - \frac{1}{(\mathrm{e}(R)-1)^2}$, then R is Gorenstein and F-regular.*

Proof. Take a system of parameters $\underline{x}$ that forms a minimal reduction of $\mathfrak{m}$ so that $\mathrm{e}(\langle \underline{x} \rangle) = \mathrm{e}(R)$. Suppose that $\mathrm{s}_{\mathrm{rel}}(R) \geq 1 - \varepsilon$ for some $\varepsilon > 0$. It follows from the definition that

$$\frac{\mathrm{e}_{\mathrm{HK}}(\langle \underline{x} \rangle) - \mathrm{e}_{\mathrm{HK}}(\mathfrak{m})}{\ell(R/\langle \underline{x} \rangle) - \ell(R/\mathfrak{m})} \geq 1 - \varepsilon.$$

Following the method of proof in Proposition 3.3, we obtain $\mathrm{e}_{\mathrm{HK}}(R) \leq 1 + \varepsilon(\mathrm{e}(R) - 1)$. The desired result now follows from that of Aberbach and Enescu [3, Corollaries 3.5, 3.6], which makes use of the expressions for ε appearing in statements (1) and (2). □

The residue field assumption can be removed once we establish Corollary 3.14.

3.2. Reduction to socle ideals. We want to prove that it is enough to take only socle ideals in the definition of $\mathrm{s}_{\mathrm{rel}}(R)$. This reduction is at the core of the theory and will allow to fix a system of parameters in the definition of $\mathrm{s}_{\mathrm{rel}}(R)$.

Proposition 3.5. *Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$. Then, for any $\mathfrak{m}$-primary ideal I, any ideal $J \supsetneq I$, and any element $x \in \mathfrak{m}$, there exists an ideal J' such that $I \subsetneq J' \subseteq J$, $xJ' \subseteq I$, and*

$$\frac{\mathrm{e}_{\mathrm{HK}}(I) - \mathrm{e}_{\mathrm{HK}}(J)}{\ell(R/I) - \ell(R/J)} \geq \frac{\mathrm{e}_{\mathrm{HK}}(I) - \mathrm{e}_{\mathrm{HK}}(J')}{\ell(R/I) - \ell(R/J')}.$$

Proof. Since both ideals are $\mathfrak{m}$-primary, there is an integer m such that $x^{m+1}J \subseteq I$. We will prove the claim by induction on m, with the trivial base case of $m = 0$.

By our assumption, the multiplication by x induces the exact sequence

$$0 \to \frac{I :_J x}{I} \to \frac{J}{I} \to \frac{I + xJ}{I} \to 0.$$

By applying the containment $(I :_J x)^{[p^e]} \subseteq I^{[p^e]} :_{J^{[p^e]}} x^{[p^e]}$ to the exact sequence

$$0 \to \frac{I^{[p^e]} :_{J^{[p^e]}} x^{[p^e]}}{I^{[p^e]}} \to \frac{J^{[p^e]}}{I^{[p^e]}} \to \frac{I^{[p^e]} + x^{p^e} J^{[p^e]}}{I^{[p^e]}} \to 0,$$

we get after taking limits that

$$\mathrm{e_{HK}}(I) - \mathrm{e_{HK}}(J) \geq \mathrm{e_{HK}}(I) - \mathrm{e_{HK}}(I :_J x) + \mathrm{e_{HK}}(I) - \mathrm{e_{HK}}(I + xJ).$$

Thus it follows from the inequality $\frac{a+c}{b+d} \geq \min\left(\frac{a}{b}, \frac{c}{d}\right)$ that

$$\begin{aligned}\frac{\mathrm{e_{HK}}(I) - \mathrm{e_{HK}}(J)}{\ell(R/I) - \ell(R/J)} &\geq \frac{\mathrm{e_{HK}}(I) - \mathrm{e_{HK}}(I + xJ) + \mathrm{e_{HK}}(I) - \mathrm{e_{HK}}(I :_J x)}{\ell(R/I) - \ell(R/J)} \\ &\geq \min\left\{\frac{\mathrm{e_{HK}}(I) - \mathrm{e_{HK}}(I + xJ)}{\ell(R/I) - \ell(R/(I + xJ))}, \frac{\mathrm{e_{HK}}(I) - \mathrm{e_{HK}}(I :_J x)}{\ell(R/I) - \ell(R/I :_J x)}\right\}.\end{aligned}$$

Depending on the minimizer, either $J' = I + xJ$ satisfies the assertion or we apply the induction hypothesis to $I :_J x$ and find $J' \subset I :_J x \subset J$ such that $xJ' \subseteq I$. □

We will now show that one can fix a system of parameters in the definition of $\mathrm{s_{rel}}(R)$ using the machinery built by Hochster and Yao. First, recall that the Peskine–Szpiro functor of a module M is defined as a module such that $F^e_* F^e(M) \cong M \otimes_R F^e_* R$ as $F^e_* R$-modules. If $L \subseteq H$, we will use $L_H^{[p^e]}$ to denote the image of $F^e(L)$ in $F^e(H)$. The following proposition combines [32, Proposition 2.3] and the proof of [32, Theorem 2.4].

Proposition 3.6. *Let $(R, \mathfrak{m})$ be a local Cohen–Macaulay ring of prime characteristic $p > 0$. If we denote $H = \mathrm{H}^d_{\mathfrak{m}}(R)$, then*

(a) *for every system of parameters $\underline{x}$ of R and ideal I such that $\langle \underline{x} \rangle \subset I$, there exists a submodule L of H isomorphic to $I/\langle \underline{x} \rangle$;*

(b) *given a finite length submodule L of H, there always exists a system of parameters $\underline{x}$ and ideal I, $\langle \underline{x} \rangle \subseteq I$ such that $I/\langle \underline{x} \rangle \cong L$;*

(c) *if $L \subseteq 0 :_H \mathfrak{m}$, the socle of the top local cohomology, then such I exists for any system of parameters $\underline{x}$.*

Moreover, via this identification, we have $\ell(L_H^{[p^e]}) = \ell(I^{[p^e]}/\langle \underline{x} \rangle^{[p^e]})$. In particular,

$$\lim_{e\to\infty} \frac{\ell(L_H^{[p^e]})}{p^{e \dim R}} = \mathrm{e_{HK}}(\langle \underline{x} \rangle) - \mathrm{e_{HK}}(I).$$

This proposition allows to consider F-rational signature as an invariant of the top local cohomology module

$$\mathrm{s_{rat}}(R) = \inf_{L \subseteq H} \left(\lim_{e\to\infty} \frac{\ell(L_H^{[p^e]})}{p^{e \dim R}} \right),$$

where the infimum is taken over all nonzero finite length submodules of H and immediately shows that $\mathrm{s_{rat}}(R)$ is independent of a system of parameters. Using Proposition 3.5, we will apply the same argument to $\mathrm{s_{rel}}(R)$.

Corollary 3.7. *Let $(R, \mathfrak{m})$ be a formally equidimensional local ring of characteristic $p > 0$. Then, for any system of parameters $\underline{x}$, we have*

$$\mathrm{s}_{\mathrm{rel}}(R) = \inf\left\{ \frac{\mathrm{e}_{\mathrm{HK}}(\langle \underline{x} \rangle) - \mathrm{e}_{\mathrm{HK}}(I)}{\ell(R/\langle \underline{x} \rangle) - \ell(R/I)} \;\middle|\; \langle \underline{x} \rangle \subset I \subseteq \langle \underline{x} \rangle : \mathfrak{m} \right\}.$$

Moreover, the infimum in the definition is, in fact, a minimum. Hence, if R is excellent, then $\mathrm{s}_{\mathrm{rel}}(R) > 0$ if and only if R is F-rational.

Proof. We note that the right-hand side does not change under completion, while the left-hand side can only decrease. Hence we assume that R is complete. If R is not Cohen–Macaulay, then it is enough to show that the right-hand side is 0. But by the colon-capturing [29, Theorem 7.15 (a)], $\langle \underline{x} \rangle$ is not tightly closed, so we can use $I = \langle \underline{x} \rangle^*$ to get 0 by [29, Theorem 8.17].

Let J be an arbitrary ideal containing a system of parameters $\underline{x}$. If $\langle m_1, \ldots, m_k \rangle = \mathfrak{m}$, then, after applying Proposition 3.5 k times, we obtain an ideal I such that

$$\mathfrak{m} I = \langle m_1, \ldots, m_k \rangle I \subseteq \langle \underline{x} \rangle$$

and

$$\frac{\mathrm{e}_{\mathrm{HK}}(\langle \underline{x} \rangle) - \mathrm{e}_{\mathrm{HK}}(J)}{\ell(R/\langle \underline{x} \rangle) - \ell(R/J)} \geq \frac{\mathrm{e}_{\mathrm{HK}}(\langle \underline{x} \rangle) - \mathrm{e}_{\mathrm{HK}}(I)}{\ell(R/\langle \underline{x} \rangle) - \ell(R/I)}.$$

This reduces $\mathrm{s}_{\mathrm{rel}}(R)$ to socle ideals. However, after fixing $\underline{x}$, Proposition 3.6 gives that, for $H = \mathrm{H}^d_{\mathfrak{m}}(R)$, we have

$$\begin{aligned} &\inf\left\{ \frac{\mathrm{e}_{\mathrm{HK}}(\langle \underline{x} \rangle) - \mathrm{e}_{\mathrm{HK}}(I)}{\ell(R/\langle \underline{x} \rangle) - \ell(R/I)} \;\middle|\; \langle \underline{x} \rangle \subset I \subseteq \langle \underline{x} \rangle : \mathfrak{m} \right\} \\ &\qquad = \inf\left\{ \frac{1}{\ell(L)} \lim_{e \to \infty} \frac{\ell(L_H^{[p^e]})}{p^{e \dim R}} \;\middle|\; 0 \neq L \subseteq 0 :_H \mathfrak{m} \right\}, \end{aligned}$$

so the left side is independent of $\underline{x}$ and must be equal to $\mathrm{s}_{\mathrm{rel}}(R)$.

Since $\underline{x}$ can be fixed, the existence of the minimum follows from semicontinuity in Theorem 2.4 and Theorem 2.3. We may now use Proposition 2.1 to characterize F-rationality. □

Remark 3.8. It is natural to expect that the minimum is achieved at the entire socle. However, Example 6.5 shows that the minimum might be given by a proper subspace.

Corollary 3.9. *Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$ which is a homomorphic image of a Cohen–Macaulay ring. If x is a regular element, then $\mathrm{s}_{\mathrm{rel}}(R) \geq \mathrm{s}_{\mathrm{rel}}(R/xR)$.*

Proof. We may assume that $\mathrm{s}_{\mathrm{rel}}(R/xR) > 0$, so R/xR is F-rational and R must be Cohen–Macaulay. We complete x to a system of parameters $x, \underline{y}$. Since R is Cohen–Macaulay,

$$\mathrm{e}_{\mathrm{HK}}(\langle x, \underline{y} \rangle) = \ell(R/\langle x, \underline{y} \rangle) = \mathrm{e}_{\mathrm{HK}}(\langle \underline{y} \rangle R/xR).$$

However, for an arbitrary ideal I containing x, we only have an inequality

$$\mathrm{e}_{\mathrm{HK}}(I) \leq \mathrm{e}_{\mathrm{HK}}(IR/xR)$$

(see [66, Proposition 2.13]). Since $\mathrm{s}_{\mathrm{rel}}(R)$ (resp. $\mathrm{s}_{\mathrm{rel}}(R/xR)$) can be computed on a fixed system of parameters $x, \underline{y}$ (resp. $\underline{y}$), the inequality now follows. □

As a corollary, we obtain that F-rationality deforms (proven in [30, Theorem 4.2 (h)] without excellence).

Corollary 3.10. *Let $(R, \mathfrak{m})$ be an excellent local ring of characteristic $p > 0$ which is a homomorphic image of a Cohen–Macaulay ring, and let $x \in \mathfrak{m}$ be a regular element. If R/xR is F-rational, then so is R.*

Proof. By Corollary 3.9 and Corollary 3.7, we have $\mathrm{s}_{\mathrm{rel}}(R) \geq \mathrm{s}_{\mathrm{rel}}(R/xR) > 0$, so R is F-rational. □

We can also derive an easy inequality that connects $\mathrm{s}_{\mathrm{rel}}(R)$ with $\mathrm{s}_{\mathrm{rat}}(R)$.

Corollary 3.11. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of characteristic $p > 0$. Then*

$$\mathrm{s}_{\mathrm{rat}}(R) \geq \mathrm{s}_{\mathrm{rel}}(R) \geq \frac{\mathrm{s}_{\mathrm{rat}}(R)}{\operatorname{type}(R)},$$

where $\operatorname{type}(R)$ *is the dimension of the socle of any system of parameters.*

Proof. The first inequality is clear from the definition. For the second, we note that, for any J such that $\mathfrak{m} J \subseteq \underline{x}$,

$$\frac{\mathrm{e}_{\mathrm{HK}}(\langle \underline{x} \rangle) - \mathrm{e}_{\mathrm{HK}}(J)}{\ell(R/\langle \underline{x} \rangle) - \ell(R/J)} \geq \frac{\mathrm{e}_{\mathrm{HK}}(\langle \underline{x} \rangle) - \mathrm{e}_{\mathrm{HK}}(J)}{\ell(R/\langle \underline{x} \rangle) - \ell(R/\langle \underline{x} \rangle : \mathfrak{m})}.$$

A classical result of Northcott [49] asserts that the denominator is independent of $\underline{x}$. The statement now follows after taking the infimums. □

3.3. Localization and flat extension. Another benefit of the normalized F-rational signature is that it satisfies the expected localization inequality $\mathrm{s}_{\mathrm{rel}}(R) \leq \mathrm{s}_{\mathrm{rel}}(R_{\mathfrak{p}})$ which is not known to hold for the original definition of Hochster and Yao. Namely, it is only known [32, Proposition 5.8] that $\mathrm{s}_{\mathrm{rat}}(R) \leq \mathrm{s}_{\mathrm{rat}}(R_{\mathfrak{p}})\alpha(\mathfrak{p})$, where $\alpha(\mathfrak{p}) \geq 1$ with equality if and only if $R/\mathfrak{p}$ is regular.

The following version of the proof was suggested to us by Pham Hung Quy. In the F-finite case, the proof is much easier; see Theorem 5.10.

Proposition 3.12. *Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$, and let $\mathfrak{p}$ be a prime ideal. Then $\mathrm{s}_{\mathrm{rel}}(R) \leq \mathrm{s}_{\mathrm{rel}}(R_{\mathfrak{p}})$.*

Proof. By induction on $\dim R/\mathfrak{p}$, we may also assume $\dim R/\mathfrak{p} = 1$. Let $\underline{x}$ be elements in R such that the images of $\underline{x}$ in $R_{\mathfrak{p}}$ form a system of parameters. Let $\mathfrak{p}, \mathfrak{p}_1, \ldots, \mathfrak{p}_k$ be minimal primes of $\underline{x}$. By prime avoidance, we may choose $u \in (\bigcap_{i=1}^k \mathfrak{p}_i) \setminus \mathfrak{p}$ and $v \in \mathfrak{p} \setminus \bigcup_{i=1}^k \mathfrak{p}_i$. For every $n \geq 1$, let $y_n = u^n + v$; then $\underline{x}, y_n$ is a system of parameters. By the associative formula for multiplicity, we have, for every integer $m \geq 1$,

$$\begin{aligned} \ell(R/\langle \underline{x}, y_n^m \rangle) &\geq \mathrm{e}(\langle y_n^m \rangle, R/\underline{x}) \\ &= nm\, \mathrm{e}(\langle u \rangle, R/\mathfrak{p}) \ell_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\langle \underline{x} \rangle) + \sum_{i=1}^k m\, \mathrm{e}(\langle v \rangle, R/\mathfrak{p}_i) \ell_{R_{\mathfrak{p}_i}}(R_{\mathfrak{p}_i}/\langle \underline{x} \rangle). \end{aligned}$$

By [56, Proposition 4.4], we also have

$$\lim_{m\to\infty}\frac{1}{m}\,\mathrm{e}_{\mathrm{HK}}(\langle \underline{x}, y_n^m\rangle) = n\,\mathrm{e}(\langle u\rangle, R/\mathfrak{p})\,\mathrm{e}_{\mathrm{HK}}(\langle \underline{x}\rangle R_{\mathfrak{p}}) + \sum_{i=1}^{k}\mathrm{e}(\langle v\rangle, R/\mathfrak{p}_i)\,\mathrm{e}_{\mathrm{HK}}(\langle \underline{x}\rangle R_{\mathfrak{p}_i}).$$

Let J be an arbitrary ideal in $R_{\mathfrak{p}}$ such that $\underline{x} \subset J$. Since $J \cap R$ is $\mathfrak{p}$-primary, we similarly obtain that for

$$\ell(R/\langle J\cap R, y_n^m\rangle) = \mathrm{e}(\langle y_n\rangle, R/J\cap R) = mn\,\mathrm{e}(\langle u\rangle, R/\mathfrak{p})\ell_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/J)$$

and

$$\lim_{m\to\infty}\frac{1}{m}\,\mathrm{e}_{\mathrm{HK}}(\langle J\cap R, y_n^m\rangle) = n\,\mathrm{e}(\langle u\rangle, R/\mathfrak{p})\ell_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/J).$$

For brevity, let us denote $C = \sum_{i=1}^{k}\mathrm{e}(\langle v\rangle, R/\mathfrak{p}_i)\ell_{R_{\mathfrak{p}_i}}(R_{\mathfrak{p}_i}/\langle \underline{x}\rangle)$. It follows from the above discussion that

$$\begin{aligned}\mathrm{s}_{\mathrm{rel}}(R) &\le \frac{\mathrm{e}_{\mathrm{HK}}(\langle \underline{x}, y_n^m\rangle) - \mathrm{e}_{\mathrm{HK}}(\langle J\cap R, y_n^m\rangle)}{\ell(R/\langle \underline{x}, y_n^m\rangle) - \ell(R/\langle J\cap R, y_n^m\rangle)}\\ &\le \frac{\mathrm{e}_{\mathrm{HK}}(\langle \underline{x}, y_n^m\rangle) - \mathrm{e}_{\mathrm{HK}}(\langle J\cap R, y_n^m\rangle)}{nm\,\mathrm{e}(u, R/\mathfrak{p})(\ell(R_{\mathfrak{p}}/\langle \underline{x}\rangle) - \ell(R_{\mathfrak{p}}/J)) + mC}.\end{aligned}$$

Therefore, after taking the limit as $m \to \infty$, we obtain that, for any n,

$$\mathrm{s}_{\mathrm{rel}}(R) \le \frac{n\,\mathrm{e}(u, R/\mathfrak{p})(\mathrm{e}_{\mathrm{HK}}(\underline{x}R_{\mathfrak{p}}) - \mathrm{e}_{\mathrm{HK}}(JR_{\mathfrak{p}})) + C}{n\,\mathrm{e}(u, R/\mathfrak{p})(\ell(R_{\mathfrak{p}}/\langle \underline{x}\rangle) - \ell(R_{\mathfrak{p}}/J)) + C}.$$

Therefore, after taking the limit as $n \to \infty$, we must have

$$\begin{aligned}\mathrm{s}_{\mathrm{rel}}(R) &\le \inf_{\underline{x}\subsetneq J}\frac{\mathrm{e}(u, R/\mathfrak{p})(\mathrm{e}_{\mathrm{HK}}(\underline{x}R_{\mathfrak{p}}) - \mathrm{e}_{\mathrm{HK}}(JR_{\mathfrak{p}}))}{\mathrm{e}(u, R/\mathfrak{p})(\ell(R_{\mathfrak{p}}/\langle \underline{x}\rangle) - \ell(R_{\mathfrak{p}}/J))}\\ &= \inf_{\underline{x}\subsetneq J}\frac{\mathrm{e}_{\mathrm{HK}}(\underline{x}R_{\mathfrak{p}}) - \mathrm{e}_{\mathrm{HK}}(JR_{\mathfrak{p}})}{\ell_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\underline{x}R_{\mathfrak{p}}) - \ell_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/JR_{\mathfrak{p}})} = \mathrm{s}_{\mathrm{rel}}(R_{\mathfrak{p}}).\end{aligned}$$ □

Next we study the behavior under flat extensions.

Proposition 3.13. *Let $(R, \mathfrak{m})$ and $(S, \mathfrak{m}_S)$ be local rings of characteristic $p > 0$ with a flat local map $R \to S$. Then $\mathrm{s}_{\mathrm{rel}}(R) \ge \mathrm{s}_{\mathrm{rel}}(S)$.*

Proof. First, we can take a minimal prime ideal Q of $\mathfrak{m}S$ and observe by Proposition 3.12 that $\mathrm{s}_{\mathrm{rel}}(S) \le \mathrm{s}_{\mathrm{rel}}(S_Q)$. Thus we assume that $\mathfrak{m}S$ is primary to the maximal ideal of S, i.e., the two rings have same dimension.

By flatness, we can tensor a composition series and get that, for an $\mathfrak{m}$-primary ideal I, $\ell(S/IS) = \ell(R/IR)\ell(S/\mathfrak{m}S)$. Thus

$$\frac{\mathrm{e}_{\mathrm{HK}}(\langle \underline{x}\rangle) - \mathrm{e}_{\mathrm{HK}}(I)}{\ell(R/\langle \underline{x}\rangle) - \ell(R/I)} = \frac{\mathrm{e}_{\mathrm{HK}}(\langle \underline{x}\rangle S) - \mathrm{e}_{\mathrm{HK}}(IS)}{\ell(S/\langle \underline{x}\rangle S) - \ell(S/IS)}.$$

Thus, because there are more ideals in S, we obtain that

$$\mathrm{s}_{\mathrm{rel}}(R) = \inf_{\underline{x}\subset I}\frac{\mathrm{e}_{\mathrm{HK}}(\langle \underline{x}\rangle) - \mathrm{e}_{\mathrm{HK}}(I)}{\ell(R/\langle \underline{x}\rangle) - \ell(R/I)} \ge \inf_{\underline{x}S\subset J}\frac{\mathrm{e}_{\mathrm{HK}}(\langle \underline{x}\rangle S) - \mathrm{e}_{\mathrm{HK}}(J)}{\ell(S/\langle \underline{x}\rangle S) - \ell(S/J)} = \mathrm{s}_{\mathrm{rel}}(S).$$ □

By using that a system of parameters can be fixed, we have the following result.

Corollary 3.14. *Let* $(R, \mathfrak{m})$ *be a local ring of characteristic* $p > 0$. *Then*

$$\mathrm{s}_{\mathrm{rel}}(R) = \mathrm{s}_{\mathrm{rel}}(R[[t]]) = \mathrm{s}_{\mathrm{rel}}(R(t)).$$

Proof. By Proposition 3.13, we have $\mathrm{s}_{\mathrm{rel}}(R) \geq \mathrm{s}_{\mathrm{rel}}(R(t))$, so we need to show the opposite inequality.

Let $S = (R[t])_{(\mathfrak{m},t)}$. As a first step, we observe that $\mathrm{s}_{\mathrm{rel}}(R) = \mathrm{s}_{\mathrm{rel}}(S)$. Namely, we take an arbitrary system of parameters $\underline{x}$ of R and observe that the map $I \mapsto (I, t)$ forms a bijection between socle ideals of $(\underline{x})$ and $(\underline{x}, t)$. Since $\ell(R/I) = \ell(S/(I, t)S)$, we can derive that this extension preserves the relative Hilbert–Kunz multiplicities, so the claim follows from Corollary 3.7. The same argument holds for $R[[t]]$. Now, $\mathrm{s}_{\mathrm{rel}}(S) \leq \mathrm{s}_{\mathrm{rel}}(S_{\mathfrak{m}S})$ by Proposition 3.12, and the assertion follows since $R(t) = S_{\mathfrak{m}S}$. □

Last, we give the following comparison result between F-signature and relative F-rational signature.

Proposition 3.15. *Let* $(R, \mathfrak{m})$ *be a Cohen–Macaulay local ring; then* $\mathrm{s}_{\mathrm{rel}}(R) \geq \mathrm{s}(R)$, *where the latter is defined for non-F-finite rings as in* [69, Definition 2.2].

Moreover, if $\mathrm{s}_{\mathrm{rel}}(R) > 0$, *then* $\mathrm{s}_{\mathrm{rel}}(R) = \mathrm{s}(R)$ *if and only if R is Gorenstein.*

Proof. We see that $\mathrm{s}_{\mathrm{rel}}(R) \geq \mathrm{s}(R)$ using [69, Theorem 1.3 (3)]. If R is Gorenstein, then, by the proof of [35, Theorem 11], $\mathrm{s}(R) = \mathrm{e}_{\mathrm{HK}}(\langle \underline{x} \rangle) - \mathrm{e}_{\mathrm{HK}}(\langle \underline{x}, u \rangle)$, where u generates the socle $(\langle \underline{x} \rangle : \mathfrak{m})/\langle \underline{x} \rangle$. However, Corollary 3.7 shows that $\mathrm{s}_{\mathrm{rel}}(R) = \mathrm{e}_{\mathrm{HK}}(\langle \underline{x} \rangle) - \mathrm{e}_{\mathrm{HK}}(\langle \underline{x}, u \rangle)$.

For the converse, [69, Remark 2.3] allows to extend the residue field without changing the F-signature, so we let S be such an extension, an F-finite faithfully flat R-algebra such that $S/\mathfrak{m}S$ is its residue field. Thus, by Proposition 3.13, $\mathrm{s}(R) = \mathrm{s}_{\mathrm{rel}}(R) \geq \mathrm{s}_{\mathrm{rel}}(S) \geq \mathrm{s}(S)$, so $\mathrm{s}_{\mathrm{rel}}(S) = \mathrm{s}(S)$. Note that it is now enough to show the statement for S, which is F-finite. The F-finite case will follow from [53, Proposition 3.10] after we will prove Corollary 5.9. □

3.4. Graded rings. We also want to remark that the F-rational signature of a graded ring can be computed using only homogeneous ideals.

Proposition 3.16. *Let R be an* $\mathbb{N}$*-graded ring over a local ring* $(R_0, \mathfrak{m})$ *and*

$$M = \mathfrak{m} \oplus \bigoplus_{n>0} R_n.$$

Then $\mathrm{s}_{\mathrm{rel}}(R_M) = \mathrm{s}_{\mathrm{rel}}(R)$, *where the latter is computed in the graded category.*

Proof. We can choose a homogeneous system of parameters $\underline{x}$ of R to compute $\mathrm{s}_{\mathrm{rel}}(R_M)$. Now, for any finite colength ideal $I \subset R_M$, it is known that I and its initial form ideal

$$\operatorname{in} I = (I + R_{>0}) \cap R_0 \oplus (I + R_{>1}) \cap R_1 \oplus \cdots$$

have equal colengths. Moreover, one can easily see that

$$(\operatorname{in} I)^{[p^e]} \subseteq \operatorname{in} I^{[p^e]}, \quad \text{so} \quad \mathrm{e}_{\mathrm{HK}}(\operatorname{in}(I)) \geq \mathrm{e}_{\mathrm{HK}}(I).$$

Thus, by comparing I with its initial ideal, we see that

$$\mathrm{s_{rel}}(R_M) = \inf\left\{ \frac{\mathrm{e_{HK}}(\langle \underline{x} \rangle) - \mathrm{e_{HK}}(I)}{\ell(R/\langle \underline{x} \rangle) - \ell(R/I)} \,\middle|\, \underline{x} \subset I \right\}$$

can be computed using only homogeneous ideals, so it is then equal to $\mathrm{s_{rel}}(R)$. □

By the same technique, we derive an expected inequality with the associated graded ring.

Proposition 3.17. *Let* $(R, \mathfrak{m})$ *be a local ring of characteristic* $p > 0$*, and let* I *be an* $\mathfrak{m}$*-primary ideal. Then* $\mathrm{s_{rel}}(R) \geq \mathrm{s_{rel}}(\mathrm{gr}_I(R))$*, where the latter is computed in the graded category.*

Proof. We may assume that $\mathrm{gr}_I(R)$ is F-rational; otherwise, the statement is trivial. Note that $\mathrm{gr}_I(R)$ is a quotient of a polynomial ring over R/I which is Cohen–Macaulay, so $\mathrm{gr}_I(R)$ is Cohen–Macaulay and so is R by [31, Theorem 4.11].

We may use Corollary 3.14 to assume that the residue field is infinite. Then we may choose a regular sequence on $\mathrm{gr}_I(R)$ such that its lift to R is a minimal reduction of I (e.g., by the proof of [37, Corollary 8.6.2, Theorem 8.6.3]). The multiplicity of an ideal generated by a regular sequence is equal to its colength; thus the multiplicities of this common system of parameters in R and in the associated graded ring are equal. Hence, by comparing any socle ideal J with its initial ideal as in the previous proof, we see that

$$\begin{aligned}\mathrm{s_{rel}}(R) &= \inf\left\{ \frac{\mathrm{e_{HK}}(\langle \underline{x} \rangle) - \mathrm{e_{HK}}(J)}{\ell(R/\langle \underline{x} \rangle) - \ell(R/J)} \,\middle|\, \underline{x} \subset J \right\} \\ &\geq \inf\left\{ \frac{\mathrm{e_{HK}}(\langle \underline{x} \rangle) - \mathrm{e_{HK}}(\mathrm{in}\, I)}{\ell(\mathrm{gr}_I(R)/\langle \underline{x} \rangle) - \ell(\mathrm{gr}_I(R)/\,\mathrm{in}\, J)} \,\middle|\, \underline{x} \subset J \right\},\end{aligned}$$

and the latter is clearly greater than or equal to $\mathrm{s_{rel}}(\mathrm{gr}_I(R))$. □

Corollary 3.18. *Let* $(R, \mathfrak{m})$ *be a local ring of characteristic* $p > 0$*, let* I *be an* $\mathfrak{m}$*-primary ideal, and let* $S = R[It, t^{-1}]$*. Then, for any prime ideal* $\mathfrak{p}$ *of* S*,* $\mathrm{s_{rel}}(S_{\mathfrak{p}}) \geq \mathrm{s_{rel}}(\mathrm{gr}_I(R))$*, where the latter is computed in the graded category.*

Proof. The statement is trivial if $\mathrm{s_{rel}}(\mathrm{gr}_I(R)) = 0$, so as in the proof of Proposition 3.17, we may assume that $\mathrm{gr}_I(R)$ is Cohen–Macaulay. Since $\mathrm{gr}_I(R) = S/t^{-1}S$, for any $\mathfrak{p} \in \operatorname{Spec} S$ that contains t^{-1}, the localization $S_{\mathfrak{p}}$ is Cohen–Macaulay. Thus we use the inequalities from Corollary 3.9 and Proposition 3.12 to get that

$$\mathrm{s_{rel}}(S_{\mathfrak{p}}) \geq \mathrm{s_{rel}}((S/t^{-1}S)_{\mathfrak{p}}) \geq \mathrm{s_{rel}}(\mathrm{gr}_I(R)).$$

For primes that do not contain t^{-1}, we note that $S_{t^{-1}} \cong R[t, t^{-1}]$ and then use Proposition 3.12, Corollary 3.14, and Proposition 3.17. □

3.5. Finite extensions. The following result recovers F-rationality of direct summands of regular rings.

Proposition 3.19. *Let* $(R, \mathfrak{m}, k)$ *be a local domain of characteristic* $p > 0$*, and let* $(S, \mathfrak{n}, \ell)$ *be a module-finite domain extension of* R*. Then* $[S : R]\, \mathrm{s_{rel}}(R) \geq [\ell : k]\, \mathrm{s_{rel}}(S)$*.*

Proof. Let $\underline{x}$ be a system of parameters in R; then $\underline{x}$ is also a system of parameters in S because S is module-finite. By the formula for Hilbert–Kunz multiplicity in finite extensions [66, Theorem 2.7], $[S : R]\,\mathrm{e}_{\mathrm{HK}}(I) = [\ell : k]\,\mathrm{e}_{\mathrm{HK}}(IS)$ for any $\mathfrak{m}$-primary ideal. Because there can be ideals in S which are not extended from an ideal in R, it follows that

$$\begin{aligned} \mathrm{s}_{\mathrm{rel}}(R) &= \inf_{\underline{x} \subset I} \frac{\mathrm{e}_{\mathrm{HK}}(\langle \underline{x} \rangle) - \mathrm{e}_{\mathrm{HK}}(I)}{\ell(R/\langle \underline{x} \rangle) - \ell(R/I)} \\ &= \inf_{\underline{x} \subset I} \frac{[\ell : k]}{[S : R]} \frac{\mathrm{e}_{\mathrm{HK}}(\langle \underline{x} \rangle S) - \mathrm{e}_{\mathrm{HK}}(IS)}{\ell(S/\langle \underline{x} \rangle S) - \ell(S/IS)} \geq \frac{[\ell : k]}{[S : R]} \mathrm{s}_{\mathrm{rel}}(S). \end{aligned}$$

□

4. F-signature theories in the dualizing module

The goal of this section is to take our considerations to the dualizing module, where we can see generalizations of all three perspectives on F-signature. In this section, we will focus on a generalization of the definition of F-signature via so-called F-splitting or degeneracy ideals.

In order to relate this definition to the relative F-rational signature, we first need to transfer $\mathrm{s}_{\mathrm{rel}}(R)$ to the dualizing module. Since the results of Hochster–Yao already transferred the invariant to the top local cohomology module, it only remains to dualize Proposition 3.6 to obtain a theory of F-signature of the dualizing module based on the Cartier trace operator. This definition can be further generalized in the framework of Cartier modules introduced by Blickle [9], but we will not pursue such generalization in this work.

Definition 4.1. Let R be a ring of positive characteristic $p > 0$. A Cartier module (M, ϕ) is a finitely generated module M equipped with a p^{-1}-linear map $\phi\colon M \to M$. Equivalently, ϕ can be thought of as an R-module homomorphism $F_* M \to M$.

Remark 4.2. The canonical module ω_R of a Cohen–Macaulay ring is naturally a Cartier module via the trace map which is constructed as follows. By applying $\mathrm{Hom}_R(\bullet, \omega_R)$ to the Frobenius map, we obtain the trace map $\mathrm{Tr}^e\colon F^e_* \omega_R \cong \mathrm{Hom}(F^e_* R, \omega_R) \to \omega_R$ by the evaluation $\mathrm{Tr}^e(\alpha) = \alpha(1)$. Since $F^e_* R$ is a maximal Cohen–Macaulay module, we can dualize again and obtain that any map from $F^e_* \omega_R \cong \mathrm{Hom}(F^e_* R, \omega_R)$ to ω_R is a precomposition of trace, $\mathrm{Tr}^e(F^e_* r \times \bullet)$.

The next definition naturally extends the definition of F-signature used by Tucker in [63] and based on prior work of Yao [68] and Aberbach–Enescu [2].

Definition 4.3. Let $(R, \mathfrak{m}, k)$ be a local F-finite ring of characteristic $p > 0$ with a dualizing module ω_R, and let $W \neq 0$ be a quotient of $\omega_R/\mathfrak{m}\omega_R$. For $e \geq 1$, we define the submodule of $F^e_* \omega_R$,

$$Z_e(W) = \bigcap_{r \in R} \ker[F^e_* \omega_R \xrightarrow{\mathrm{Tr}^e(F^e_* r \times \bullet)} \omega_R \longrightarrow W].$$

It is clear that $\mathfrak{m}^{[p^e]}\omega_R \subseteq Z_e(W)$ for all W, so $\dim_k((F^e_* \omega_R)/Z_e(W))$ is finite and can be used to define the Cartier signature of W as

$$\mathrm{s}_{\mathrm{Tr}}(W) = \lim_{e \to \infty} \frac{\dim_k((F^e_* \omega_R)/Z_e(W))}{[k : k^{p^e}]p^{e \dim R}}.$$

Remark 4.4. If R is a domain, then, in the denominator,

$$p^{e \dim R}[k : k^{p^e}] = \operatorname{rank} F^e_* R = \operatorname{rank} F^e_* \omega_R,$$

where the first equality holds due to [40, Proposition 2.3] and the second because $\operatorname{rank} \omega_R = 1$.

Remark 4.5. An alternative interpretation of $Z_e(W)$ is closer to the standard definition of the splitting ideals I_e: for $N \subset \omega_R$, consider a submodule

$$Z_e(N) = \{x \in F^e_* \omega_R \mid \operatorname{Tr}^e_R(F^e_* Rx) \subseteq N\}.$$

It is easy to see that if $\ell(\omega_R/N) < \infty$, then $Z_e(N) = Z_e(\omega_R/N)$.

Remark 4.6. Because Tr^e generates $\operatorname{Hom}(F^e_* \omega_R, \omega_R)$, the elements of $Z_e(W)$ belong to the kernel of any map $F^e_* \omega_R \to \omega_R$.

We now define a notion of F-signature in ω_R; it will be later revisited in Definition 4.11.

Definition 4.7. Let $(R, \mathfrak{m}, k)$ be a local F-finite ring of characteristic $p > 0$. Then the (small) Cartier signature of R is

$$\widetilde{\mathrm{s}_{\mathrm{Tr}}}(R) := \inf_W \frac{\mathrm{s}_{\mathrm{Tr}}(W)}{\dim_k W},$$

where the infimum is taken over all nonzero quotients W of $\omega_R/\mathfrak{m}\omega_R$.

This definition is chosen so that it coincides with the dual of Proposition 3.6. In order to prove this, we first recall that an F-finite Cohen–Macaulay ring always has a dualizing module because Gabber [21, Remark 13.6] showed that an F-finite ring is an image of a regular ring.

Lemma 4.8. *Let $(R, \mathfrak{m}, k)$ be an F-finite Cohen–Macaulay local ring, and let ω_R be a dualizing module. Let $\underline{x}$ be a system of parameters and $\langle \underline{x} \rangle \subsetneq I \subseteq \langle \underline{x} \rangle : \mathfrak{m}$. Let $L = I/\langle \underline{x} \rangle$ and $W = \operatorname{Hom}_R(L, E)$, where E is the injective hull of the residue field. Then*

$$\dim_k((F^e_* \omega_R)/Z_e(W)) = [k : k^{p^e}]\ell(I^{[p^e]}/\langle \underline{x} \rangle^{[p^e]}).$$

Therefore, $\mathrm{e}_{\mathrm{HK}}(\langle \underline{x} \rangle) - \mathrm{e}_{\mathrm{HK}}(I) = \mathrm{s}_{\mathrm{Tr}}(W)$ *and* $\mathrm{s}_{\mathrm{rel}}(R) = \widetilde{\mathrm{s}_{\mathrm{Tr}}}(R)$.

Proof. Denote $M^\vee := \operatorname{Hom}_R(M, E)$. First, by Proposition 3.6, we view L as a vector space in $H = \mathrm{H}^d_\mathfrak{m}(R)$, so W is naturally a quotient of $H^\vee \otimes_R k \cong \omega_R \otimes_R k$. Hence $\mathrm{s}_{\mathrm{Tr}}(W)$ is defined.

By Proposition 3.6, we are interested in $\ell(L^{[p^e]}_H)$. Since

$$\ell(L^{[p^e]}_H)[k : k^{p^e}] = \ell(F^e_* L^{[p^e]}_H),$$

it will be more convenient to work with $F^e_* L^{[p^e]}_H$, i.e., the image of $F^e_* R \otimes_R L \to F^e_* R \otimes_R H$. Because tensor product is right-exact, we have an exact sequence

$$0 \to \operatorname{Hom}_R(F^e_* R \otimes_R H/L, E) \to \operatorname{Hom}_R(F^e_* R \otimes_R H, E) \to \operatorname{Hom}_R(F^e_* L^{[p^e]}_H, E) \to 0.$$

By the Hom-tensor adjunction, we obtain that

$$\operatorname{Hom}_R(F^e_* R \otimes_R H, E) \cong \operatorname{Hom}_R(F^e_* R, \operatorname{Hom}_R(H, E)) \cong \operatorname{Hom}_R(F^e_* R, \omega_R) \\ \cong \omega_{F^e_* R} \cong F^e_* \omega_R.$$

Similarly, $\operatorname{Hom}_R(F^e_* R \otimes_R H/L, E) \cong \operatorname{Hom}_R(F^e_* R, (H/L)^\vee)$ is a submodule of $F^e_* \omega_R$, so it is enough to show that $Z_e(W) = \operatorname{Hom}_R(F^e_* R, (H/L)^\vee)$.

By Remark 4.2, we identify $\operatorname{Tr}^e(F^e_* r \times \bullet)$ with the evaluation of $\operatorname{Hom}(F^e_* R, \omega_R)$ at $F^e_* r$. Then we have $Z_e(W) = \{\phi \in \operatorname{Hom}(F^e_* R, \omega_R) \mid \operatorname{Im}\phi \subseteq (H/L)^\vee\}$, which clearly coincides with $\operatorname{Hom}_R(F^e_* R, (H/L)^\vee)$. □

4.1. New perspective on existing results. Our new interpretation gives a more transparent proof of the localization property (Proposition 3.12) in the F-finite case.

Proposition 4.9. *Let $(R, \mathfrak{m})$ be an F-finite local ring of characteristic $p > 0$, and let $\mathfrak{p}$ be a prime ideal. Then $\widetilde{s_{\mathrm{Tr}}}(R) \leq \widetilde{s_{\mathrm{Tr}}}(R_\mathfrak{p})$.*

Proof. If R is not F-rational, then $\widetilde{s_{\mathrm{Tr}}}(R) = s_{\mathrm{rel}}(R) = 0$ and the claim is trivial. Thus we assume that R is Cohen–Macaulay and let ω_R be its dualizing module. By induction on $\dim R/\mathfrak{p}$, we may assume that $\dim R/\mathfrak{p} = 1$. Let x be a parameter modulo $\mathfrak{p}$. As $(\omega_R)_\mathfrak{p} = \omega_{R_\mathfrak{p}}$, for any submodule $\mathfrak{p}\omega_{R_\mathfrak{p}} \subseteq N \subset \omega_{R_\mathfrak{p}}$, we may associate a submodule $N' = N \cap \omega_R + y\omega_R$ of finite colength. Because $N \cap \omega_R$ is $\mathfrak{p}$-primary, x is a regular element on $\omega_R / N \cap \omega_R$ and, using that multiplicity is additive, we may compute that

$$\ell(\omega_R/N') = \mathrm{e}(x, \omega_R/N \cap \omega_R) = \mathrm{e}(x, R/\mathfrak{p})\ell_{R_\mathfrak{p}}(\omega_{R_\mathfrak{p}}/N).$$

If $L \subset \omega_R$ and $Z_e(L)$ are defined as in Remark 4.5, then one can check that

$$Z_e(L) :_{F^e_* \omega_R} x = Z_e(L :_{\omega_R} x).$$

In particular, $Z_e(N \cap \omega_R)$ is still $\mathfrak{p}$-primary. We also note that Tr localizes, so

$$(Z_e(N \cap \omega_R))_\mathfrak{p} = Z_e(N) \subseteq F^e_* \omega_{R_\mathfrak{p}}.$$

It is straightforward to check that $xF^e_* \omega_R + Z_e(N \cap \omega_R) \subseteq Z_e(N')$; thus

$$\begin{aligned} \ell(F^e_* \omega_R / Z_e(N')) &\leq \ell\big(F^e_* \omega_R / (xF^e_* \omega_R + Z_e(N \cap \omega_R))\big) \\ &= \mathrm{e}(x, F^e_* \omega_R / Z_e(N \cap \omega_R)) \\ &= \mathrm{e}(x, R/\mathfrak{p})\ell_{R_\mathfrak{p}}(F^e_* \omega_{R_\mathfrak{p}} / Z_e(N)). \end{aligned}$$

It follows that $s_{\mathrm{Tr}}(\omega_R/N') \leq \mathrm{e}(x, R/\mathfrak{p})\, s_{\mathrm{Tr}}(\omega_{R_\mathfrak{p}}/N)$; hence, by Lemma 4.8,

$$\begin{aligned} \widetilde{s_{\mathrm{Tr}}}(R_\mathfrak{p}) = \inf_{\mathfrak{p}\omega_{R_\mathfrak{p}} \subseteq N} \frac{s_{\mathrm{Tr}}(\omega_{R_\mathfrak{p}}/N)}{\ell_{R_\mathfrak{p}}(\omega_{R_\mathfrak{p}}/N)} &\geq \inf_{\mathfrak{p}\omega_{R_\mathfrak{p}} \subseteq N} \frac{s_{\mathrm{Tr}}(\omega_R/N')}{\ell(\omega_R/N')} \\ &\geq \inf_{\mathfrak{p}\omega_{R_\mathfrak{p}} \subseteq N} \frac{s_{\mathrm{Tr}}(\omega_R/N')}{\ell(\omega_R/N')} \geq \widetilde{s_{\mathrm{Tr}}}(R), \end{aligned}$$

where the last inequality holds by Proposition 3.6 and the proof of Lemma 4.8 because every such N' gives a system of parameters $\underline{x}$ and an ideal $I \supset \langle \underline{x} \rangle$. □

It is also more convenient to derive a deformation statement in the new language.

Proposition 4.10. *Let $(R, \mathfrak{m}, k)$ be an F-finite Cohen–Macaulay local ring, $x \in \mathfrak{m}$ a parameter, and ω_R the dualizing module of R. Then $\widetilde{s_{\mathrm{Tr}}}(R) \geq \widetilde{s_{\mathrm{Tr}}}(R/xR)$.*

Proof. Since $\omega_{R/xR} \cong \omega_R / x\omega_R$, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^e_*\omega_R & \xrightarrow{\times F^e_* x} & F^e_*\omega_R & \longrightarrow & F^e_*\omega_{R/xR} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Tr}_{\omega_R}} & & \downarrow{\scriptstyle \mathrm{Tr}_{\omega_R}(F^e_* x^{p^e-1}\times\bullet)} & & \downarrow{\scriptstyle \mathrm{Tr}_{\omega_{R/xR}}} & & \\ 0 & \longrightarrow & \omega_R & \xrightarrow{\times x} & \omega_R & \longrightarrow & \omega_{R/xR} & \longrightarrow & 0 \end{array}$$

that allows us to think about the trace map on $\omega_{R/xR}$ as a precomposition of the trace on ω_R. For any quotient W of $k \otimes_{R/xR} \omega_{R/xR} \cong k \otimes_R \omega_R$ and an element $r \in R$, we obtain the induced diagram

$$\begin{array}{ccccc} F^e_*\omega_R & \xrightarrow{\alpha} & F^e_*\omega_{R/xR} & \longrightarrow & 0 \\ \downarrow{\scriptstyle \mathrm{Tr}_{\omega_R}(F^e_* x^{p^e-1} r\times\bullet)} & & \downarrow{\scriptstyle \mathrm{Tr}_{\omega_{R/xR}}(F^e_* r\times\bullet)} & & \\ \omega_R & \longrightarrow & \omega_{R/xR} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ W & = & W & & \end{array}$$

which easily shows that $\alpha(Z_e(\omega_R, W) :_{F^e_*\omega_R} F^e_* x^{p^e-1}) \subseteq Z_e(\omega_{R/xR}, W)$. Hence

$$\dim_k \frac{F^e_*\omega_R}{Z_e(\omega_R, W) :_{F^e_*\omega_R} (F^e_* x^{p^e-1}) + F^e_* x\omega_R} \geq \dim_k \frac{F^e_*\omega_{R/xR}}{Z_e(\omega_{R/xR}, W)}.$$

Since $xF^e_*\omega_R \subseteq Z_e(\omega_R, W)$, we may filter

$$\begin{aligned} \dim_k F^e_*\omega_R / Z_e(W) &= \sum_{n=1}^{p^e} \dim_k \frac{Z_e(W) + F^e_* x^{n-1}\omega_R}{Z_e(W) + F^e_* x^n \omega_R} \\ &= \sum_{n=1}^{p^e} \dim_k \frac{F^e_* x^{n-1}\omega_R}{Z_e(W) \cap F^e_* x^{n-1}\omega_R + F^e_* x^n\omega_R} \\ &= \sum_{n=1}^{p^e} \dim_k \frac{F^e_*\omega_R}{Z_e(W) :_{F^e_*\omega_R} F^e_* x^{n-1}\omega_R + F^e_* x\omega_R} \\ &\geq p^e \dim_k \frac{F^e_*\omega_R}{Z_e(W) :_{F^e_*\omega_R} F^e_* x^{p^e-1}\omega_R + F^e_* x\omega_R} \\ &\geq p^e \dim_k \frac{F^e_*\omega_{R/xR}}{Z_e(W, \omega_{R/xR})}. \end{aligned}$$

Hence $s_{\mathrm{Tr}}(\omega_R, W) \geq s_{\mathrm{Tr}}(\omega_{R/xR}, W)$, and the assertion follows. □

4.2. Relative Hilbert–Kunz multiplicity on the Grassmannian. The Grassmann functor of a coherent sheaf E on a scheme X of rank n associates to any X-scheme Y the set of all equivalence classes $E \times_X Y \to F$, where F is locally free on Y of rank n. The Grassmannian scheme $\pi_n\colon \mathrm{Grass}(E,n) \to X$ represents the functor as $\mathrm{Hom}_X(Y, \mathrm{Grass}(E,n))$. The representing scheme is projective over S. We refer to [22, 48] for further background.

If we consider ω_R as a coherent sheaf on Spec R, then a point $x \in \mathrm{Grass}(\omega_R, n)$ can be thought of as a pair $\{k(x), W_x\}$ consisting of a field extension $k(x)$ of $\pi_n(x)$ and a rank n quotient W_x of $\omega_R \otimes k(x)$. This forces us to extend $\mathrm{s_{Tr}}$ to such quotients.

Definition 4.11. Let $(R, \mathfrak{m}, k)$ be a local F-finite ring of characteristic $p > 0$ with a dualizing module ω_R. If ℓ is a field extension of k and $W \neq 0$ is a quotient of $\omega_R \otimes_R \ell$, for $e \geq 1$, we define the submodule of $F_*^e \omega_R \otimes_R \ell$,

$$Z_e(W) = \bigcap_{r \in R} \ker[\ell \otimes_R F_*^e \omega_R \xrightarrow{1 \otimes \mathrm{Tr}^e(F_*^e r \times \bullet)} \ell \otimes_R \omega_R \longrightarrow W].$$

The Cartier signature of W is then defined as

$$\mathrm{s_{Tr}}(W) = \lim_{e \to \infty} \frac{\dim_\ell((\ell \otimes_R F_*^e \omega_R)/Z_e(W))}{[k : k^{p^e}] p^{e \dim R}}. \tag{4.1}$$

Definition 4.12. Let $(R, \mathfrak{m}, k)$ be a local F-finite ring of characteristic $p > 0$ with a dualizing module ω_R. Then the Cartier signature of R is

$$\mathrm{s_{Tr}}(R) := \inf\left\{ \frac{\mathrm{s_{Tr}}(W)}{\dim_\ell W} \;\middle|\; k \subseteq \ell \text{ is finite and } W \text{ is a nonzero quotient of } \ell \otimes_R \omega_R \right\}.$$

Before showing that this definition makes sense, i.e., that the limit in the definition exists, we want to make several useful observations.

The next observation is the key to the semicontinuity.

Lemma 4.13. *Let $(R, \mathfrak{m}, k)$ be a local F-finite ring of characteristic $p > 0$ with a dualizing module ω_R, let ℓ be a field extension of k, and let $\pi\colon \ell \otimes_R \omega_R \to W$ be a nonzero surjection of vector spaces over ℓ. If $r_1, \ldots, r_m \in R$ are such that $\{F_*^e r_i\}$ generate $F_*^e R$ as an R-module for some $e \geq 1$, then*

$$\begin{aligned} Z_e(W) &= \bigcap_{i=1}^{m} \ker[\ell \otimes_R F_*^e \omega_R \xrightarrow{1 \otimes \mathrm{Tr}^e(F_*^e r_i \times \bullet)} \ell \otimes_R \omega_R \xrightarrow{\quad \pi \quad} W] \\ &= \ker\Big[\ell \otimes_R F_*^e \omega_R \xrightarrow{\sum 1 \otimes \mathrm{Tr}(F_*^e r_i \times \bullet)} \bigoplus^{m} \ell \otimes_R \omega_R \xrightarrow{\quad \oplus\pi \quad} \bigoplus^{m} W\Big]. \end{aligned}$$

While we defined $\mathrm{s_{Tr}}(R)$ as the infimum over all finite extensions, this was done merely for convenience.

Lemma 4.14. *Let $(R, \mathfrak{m}, k)$ be a local ring. Then*

$$\begin{aligned} \mathrm{s_{Tr}}(R) &= \inf\left\{ \frac{\mathrm{s_{Tr}}(W)}{\dim_\ell W} \;\middle|\; \ell \otimes_R \omega_R \to W \to 0,\ \ell \textit{ is algebraic} \right\} \\ &= \inf\left\{ \frac{\mathrm{s_{Tr}}(W)}{\dim_{\bar{k}} W} \;\middle|\; \bar{k} \otimes_R \omega_R \to W \to 0 \right\}. \end{aligned}$$

Proof. If ℓ is algebraic (in particular, finite) over k, from a given surjection

$$\pi\colon \ell \otimes_R \omega_R \to W,$$

we may obtain a surjection $1 \otimes \pi\colon \bar{k} \otimes_R \omega_R \to \bar{k} \otimes_\ell W$ by tensoring. Let $W' = \bar{k} \otimes_\ell W$. We claim that $s_{\mathrm{Tr}}(W) = s_{\mathrm{Tr}}(W')$.

Let $F^e_* r_1, \dots, F^e_* r_m$ generate $F^e_* R$ as an R-module. Then, by Lemma 4.13,

$$\begin{aligned}\dim_\ell(\ell \otimes_R F^e_* \omega_R)/Z_e(W) \\ = \operatorname{rank}_\ell\Big[\ell \otimes_R F^e_* \omega_R \xrightarrow{\sum 1\otimes \mathrm{Tr}(F^e_* r_i \times \bullet)} \bigoplus^m \ell \otimes_R \omega_R \xrightarrow{\quad\oplus\pi\quad} \bigoplus^m W\Big].\end{aligned}$$

Because $\otimes_\ell \bar{k}$ is exact, we obtain that

$$\dim_\ell(\ell \otimes_R F^e_* \omega_R)/Z_e(W) = \dim_{\bar{k}}(\bar{k} \otimes_R F^e_* \omega_R)/Z_e(W'),$$

and the claim follows.

The claim easily implies the assertion. First,

$$\begin{aligned} s_{\mathrm{Tr}}(R) &\ge \inf\{s_{\mathrm{Tr}}(W)/(\dim_\ell W) \mid \ell \otimes_R \omega_R \to W \to 0,\ \ell \text{ is algebraic}\} \\ &\ge \inf\{s_{\mathrm{Tr}}(W)/(\dim_{\bar{k}} W) \mid \bar{k} \otimes_R \omega_R \to W \to 0\},\end{aligned}$$

where the first inequality holds since we have more extensions and the second because of the claim. It remains to observe that, for any surjection $\pi\colon \bar{k} \otimes_R \omega_R \to W$, we can find, by taking a basis of W, a finite extension ℓ of k and a surjection $\sigma\colon \ell \otimes_R \omega_R \to V$ such that $W = \bar{k} \otimes_\ell V$ and $\pi = 1 \times \sigma$. Thus $s_{\mathrm{Tr}}(R) \ge \inf\{s_{\mathrm{Tr}}(W) \mid \bar{k} \otimes_R \omega_R \to W \to 0\}$. □

In fact, it will follow from Corollary 4.22 that even including arbitrary extensions will not change the invariant.

4.3. Existence and uniform convergence. We will now show that our definitions make sense, i.e., the dimensions are finite and the limits exist. Furthermore, we will show that the convergence in Definition 4.11 is uniform.

Lemma 4.15. *Let $(R, \mathfrak{m}, k)$ be a local F-finite ring of characteristic $p > 0$ with a dualizing module ω_R. If ℓ is a field extension of k, then for any nonzero quotient W of $\omega_R \otimes_R \ell$,*

$$\dim_\ell(\ell \otimes_R F^e_* \omega)/Z_e(W) \le [k : k^{p^e}]\ell_R(\omega_R/\mathfrak{m}^{[p^e]}\omega_R) < \infty.$$

Proof. Observe that $\ell \otimes_R F^e_* \omega_R \cong \ell \otimes_k k \otimes_R F^e_* \omega \cong \ell \otimes_k F^e_* \omega_R/\mathfrak{m}^{[p^e]}\omega_R$. Thus

$$\dim_\ell(\ell \otimes_R F^e_* \omega/Z_e(W)) \le \dim_\ell \ell \otimes_R (F^e_* \omega_R/\mathfrak{m}^{[p^e]}\omega_R) = \dim_k F^e_* \omega_R/\mathfrak{m}^{[p^e]}\omega_R,$$

and the latter length is finite since ω_R is finitely generated and R is F-finite. □

Theorem 4.16. *Let $(R, \mathfrak{m}, k)$ be an F-finite reduced Cohen–Macaulay local ring of dimension d with a dualizing module ω_R. The limit in* (4.1) *exists and the convergence is uniform, i.e., there exists a constant C such that, for any $e \ge 1$, any extension ℓ of k, and any nonzero quotient W of $\ell \otimes_R \omega_R$,*

$$\left|\frac{\dim_\ell(\ell \otimes_R F^e_* \omega_R)/Z_e(W)}{[k : k^{p^e}]p^{ed}} - s_{\mathrm{Tr}}(W)\right| < \frac{C}{p^e}.$$

Proof. Let $\alpha_p = [k : k^p]p^d$. Since a Cohen–Macaulay local ring is equidimensional, by [40, Proposition 2.3], the ranks of $F_*\omega_{R_\mathfrak{p}}$ and $\bigoplus^{\alpha_p} \omega_{R_\mathfrak{p}}$ agree at any minimal prime ideal $\mathfrak{p}$. As in the proof of [63, Lemma 3.3], this gives us the exact sequence

$$\bigoplus^{\alpha_p} \omega_R \to F_*\omega_R \to T \to 0,$$

where $\dim T < d$ (i.e., $T = 0$ if $d = 0$). Thus the sequence

$$\bigoplus^{\alpha_p} F_*^e\omega_R \xrightarrow{\gamma_e} F_*^{e+1}\omega_R \longrightarrow F_*^e T \longrightarrow 0$$

is exact and, by tensoring it with ℓ, we obtain an exact sequence

$$\bigoplus^{\alpha_p} \ell \otimes_R F_*^e\omega_R \xrightarrow{1_\ell\otimes\gamma_e} \ell \otimes_R F_*^{e+1}\omega_R \xrightarrow{\pi_e} \ell \otimes_R F_*^e T \longrightarrow 0.$$

Claim 4.17. $(1_\ell \otimes \gamma_e)(\oplus^{\alpha_p} Z_e(W)) \subseteq Z_{e+1}(W)$.

Proof. By restricting γ_e to a summand and composing with an arbitrary multiple of Tr^{e+1}, we obtain the diagram

$$F_*^e\omega_R \longrightarrow F_*^{e+1}\omega_R \xrightarrow{\mathrm{Tr}^{e+1}(F_*^{e+1}r\times\bullet)} \omega_R.$$

Since the resulting map $F_*^e\omega_R \to \omega_R$ is necessarily a premultiple of Tr^e by the main property of the trace, the kernel of the induced map

$$\ell \otimes_R F_*^e\omega_R \longrightarrow \ell \otimes_R F_*^{e+1}\omega_R \xrightarrow{1\otimes\mathrm{Tr}^{e+1}(F_*^{e+1}r\times\bullet)} \ell \otimes_R \omega_R \longrightarrow W$$

contains $Z_e(W)$ by the definition. Since r was arbitrary, we see that

$$(1_\ell \otimes \gamma_e)(Z_e(W)) \subseteq Z_{e+1}(W). \qquad \square$$

The claim gives us the exact sequence

$$\bigoplus^{\alpha_p} \frac{\ell \otimes_R F_*^e\omega_R}{Z_e(W)} \xrightarrow{1_\ell\otimes\gamma_e} \frac{\ell \otimes_R F_*^{e+1}\omega_R}{Z_{e+1}(W)} \xrightarrow{\pi_e} \frac{\ell \otimes_R F_*^e T}{\pi_e(Z_{e+1}(W))} \longrightarrow 0,$$

which, as in the proof of Lemma 4.15, gives us the bound

$$\begin{aligned}\dim_\ell\left(\frac{\ell \otimes_R F_*^{e+1}\omega_R}{Z_{e+1}(W)}\right) - \alpha_p \dim_\ell\left(\frac{\ell \otimes_R F_*^e\omega_R}{Z_e(W)}\right) &\le \dim_\ell(\ell \otimes_R F_*^e T)\\ &\le [k : k^{p^e}]\ell_R(T/\mathfrak{m}^{[p^e]}T).\end{aligned}$$

For the second step, we consider the analogously obtained exact sequence

$$\ell \otimes_R F_*^{e+1}\omega_R \xrightarrow{1_\ell\otimes\delta_e} \bigoplus^{\alpha_p} \ell \otimes_R F_*^e\omega_R \xrightarrow{\rho_e} \ell \otimes_R F_*^e U \longrightarrow 0.$$

Claim 4.18. $(1_\ell \otimes \delta^e)(F_*^{e+1} Z_{e+1}(W)) \subseteq \oplus^{\alpha_p} F_*^e Z_e(W)$.

Proof. It is enough to show that if we compose $(1_\ell \otimes \delta_e)$ with the projection on one of the summands, then the image of $Z_{e+1}(W)$ is in $Z_e(W)$. Following the proof of the first claim, this reduces to the fact that

$$F_*^{e+1}\omega_R \xrightarrow{\delta_e} \bigoplus^{\alpha_p} F_*^e \omega_R \longrightarrow F_*^e \omega_R$$

is necessarily a premultiple of the trace again. □

Thus we have the exact sequence

$$\frac{\ell \otimes_R F_*^{e+1}\omega_R}{Z_{e+1}(W)} \to \bigoplus^{\alpha_p} \frac{\ell \otimes_R F_*^e \omega_R}{Z_e(W)} \to \frac{\ell \otimes_R F_*^e U}{\rho(Z_e(W))} \to 0,$$

which by Lemma 4.15 gives us the bound

$$\begin{aligned} \alpha_p \dim_\ell\left(\frac{\ell \otimes_R F_*^e \omega_R}{Z_e(W)}\right) - \dim_\ell\left(\frac{\ell \otimes_R F_*^{e+1}\omega_R}{Z_{e+1}(W)}\right) &\le \dim_\ell(\ell \otimes_R F_*^e U) \\ &= [k : k^{p^e}]\ell_R(U/\mathfrak{m}^{[p^e]}U). \end{aligned}$$

After combining and dividing the inequalities by $[k : k^{p^{e+1}}]p^{(e+1)d}$, we get that

$$\begin{aligned} &\left|\frac{\dim_\ell(\ell \otimes_R F_*^e \omega_R)/Z_e(W)}{[k : k^{p^e}]p^{ed}} - \frac{\dim_\ell(\ell \otimes_R F_*^{e+1}\omega_R)/Z_{e+1}(W)}{[k : k^{p^{e+1}}]p^{(e+1)d}}\right| \\ &\qquad \le \frac{\max\{\ell_R(T/\mathfrak{m}^{[p^e]}T), \ell_R(U/\mathfrak{m}^{[p^e]}U)\}}{[k : k^p]p^{(e+1)d}}. \end{aligned} \tag{4.2}$$

By [46, Lemma 1.1], the right-hand side is bounded above by D/p^e for some constant $D \ge 0$. The theorem then follows from [51, Lemma 3.5]. □

The proof also shows uniform, independent of a prime ideal, convergence on Spec R.

Corollary 4.19. *Let R be an F-finite reduced Cohen–Macaulay ring of dimension d. If* Spec R *is connected, then there exists a constant C such that, for all $\mathfrak{p} \in$* Spec(R)*, all field extensions $k(\mathfrak{p}) \subseteq \ell$, all nonzero quotients W of $\omega_R \otimes_R \ell$, and all $e \ge 1$, we have*

$$\left|\frac{\dim_\ell(\ell \otimes_R F_*^e \omega_R)/Z_e(W)}{[k(\mathfrak{p}) : k(\mathfrak{p})^{p^e}]p^{e \operatorname{ht}\mathfrak{p}}} - s_{\mathrm{Tr}}(W)\right| < \frac{C}{p^e}.$$

Proof. By [40, Corollary 2.7], $\alpha_p = [k(\mathfrak{p}) : k(\mathfrak{p})^p]p^{\operatorname{ht}\mathfrak{p}}$ is independent of $\mathfrak{p}$. Hence, as in the proof of Theorem 4.16, we may choose the exact sequences

$$\bigoplus^{\alpha_p} \omega_R \to F_*\omega_R \to T \to 0 \quad \text{and} \quad F_*\omega_R \to \bigoplus^{\alpha_p} \omega_R \to U \to 0,$$

where $T_\mathfrak{q} = U_\mathfrak{q} = 0$ for every minimal prime $\mathfrak{q}$. By [50, Proposition 3.3], we can find a constant $D > 0$ such that, for all $\mathfrak{p} \in$ Spec R,

$$\frac{\max\{\ell_{R_\mathfrak{p}}(T_\mathfrak{p}/\mathfrak{p}^{[p^e]}T_\mathfrak{p}), \ell_{R_\mathfrak{p}}(U_\mathfrak{p}/\mathfrak{p}^{[p^e]}U_\mathfrak{p})\}}{[k(\mathfrak{p}) : k(\mathfrak{p})^p]p^{(e+1)\operatorname{ht} p}} < \frac{D}{p^e}$$

and then use this bound in (4.2) of Theorem 4.16. □

For our main result, it will be important to interchange the infimum and the limit.

Corollary 4.20. *Let R be an F-finite reduced Cohen–Macaulay ring of dimension d. If* $\operatorname{Spec} R$ *is connected, then there exists a constant C such that, for all $\mathfrak{p} \in \operatorname{Spec}(R)$ and all $e \geq 1$, we have, for $\alpha_e(\mathfrak{p}) = [k(\mathfrak{p}) : k(\mathfrak{p})^{p^e}]p^{e \operatorname{ht} \mathfrak{p}}$,*

$$\left| \frac{1}{\alpha_e(\mathfrak{p})} \inf \left\{ \frac{\dim_\ell(\ell \otimes_R F^e_* \omega_{R_\mathfrak{p}})/Z_e(W)}{\dim_\ell W} \;\middle|\; [\ell : k(\mathfrak{p})] < \infty,\ W \neq 0,\ \ell \otimes_R \omega_R \to W \to 0 \right\} - s_{\mathrm{Tr}}(R) \right| < \frac{C}{p^e}.$$

Proof. A standard application of uniform convergence shows that, in a local ring $R_\mathfrak{p}$, we can interchange the infimum and the limit as follows:

$$\begin{aligned} s_{\mathrm{Tr}}(R) &= \inf_{\ell \otimes_R \omega_R \to W} \frac{s_{\mathrm{Tr}}(W)}{\dim_\ell W} = \inf_{\ell \otimes_R \omega_R \to W} \frac{1}{\dim_\ell W} \lim_{e \to \infty} \frac{\dim_\ell((\ell \otimes_R F^e_* \omega_\mathfrak{p})/Z_e(W))}{[k(\mathfrak{p}) : k(\mathfrak{p})^{p^e}]p^{e \operatorname{ht} \mathfrak{p}}} \\ &= \lim_{e \to \infty} \frac{1}{[k(\mathfrak{p}) : k(\mathfrak{p})^{p^e}]p^{e \operatorname{ht} \mathfrak{p}}} \inf_{\ell \otimes_R \omega_R \to W} \frac{1}{\dim_\ell W} \dim_\ell((\ell \otimes_R F^e_* \omega_\mathfrak{p})/Z_e(W)). \end{aligned}$$

But since the appearing constants are independent of $\mathfrak{p}$ by Corollary 4.19, we get that this convergence is also uniform in $\mathfrak{p}$. □

4.4. Semicontinuity. We extended s_{Tr} to a function on $\operatorname{Grass}(\omega_R, n)$ and will now show its semicontinuity, which will imply several other good properties.

Theorem 4.21. *Let R be an F-finite reduced Cohen–Macaulay ring of characteristic $p > 0$ with a connected spectrum, and let ω_R be a dualizing module. Let $\pi_n \colon \mathcal{B}_n \to \operatorname{Spec} R$ be the rank n Grassmannian of the coherent sheaf ω_R. Then $s_{\mathrm{Tr}} \colon \mathcal{B}_n \to \mathbb{R}$ is a lower semicontinuous function.*

Proof. Let $\mathcal{Q}$ be the universal quotient bundle of $\mathcal{B}_n$. Let $r_1, \dots, r_\mu \in R$ be such that they generate $F^e_* R$ as an R module. Then we may define $\phi_i \colon F^e_* \omega_R \to \omega_R$ by $x \mapsto \operatorname{Tr}^e(F^e_* r_i x)$ and consider

$$g_n \colon \pi_n^* F^e_* \omega_R \xrightarrow{\sum \pi_n^* \phi_i} \bigoplus_{i=1}^{\mu} \pi_n^* \omega_R \longrightarrow \bigoplus_{i=1}^{\mu} \mathcal{Q},$$

where the last map is given by the construction of $\mathcal{Q}$. The rank of the image of the composition is a lower semicontinuous function (e.g., because non-vanishing of a minor is an open condition). If $x \in \mathcal{B}_n$ is a point such that $\pi_n(x) = \mathfrak{p}$, then $k(x)$ is a field extension of $k(\mathfrak{p})$ and x represents a rank n quotient W_x of $\omega_R \otimes_R k(x)$. Thus, at x, we have the map

$$g_{n,e}(x) \colon F^e_* \omega_R \otimes_R k(x) \xrightarrow{\sum \phi_i \otimes 1} \bigoplus_{i=1}^{\mu} \omega_R \otimes_R k(x) \longrightarrow \bigoplus_{i=1}^{\mu} W_x,$$

which coincides with Lemma 4.13. Note that $s_{\mathrm{Tr}}(x)$ is defined as $s_{\mathrm{Tr}}(W_x)$.

Furthermore, let $\alpha_e(x) = [\pi_n(x) : \pi_n(x)^{p^e}]p^{e \operatorname{ht} \pi_n(x)}$ and note that rank $g_{n,e}(x)/\alpha_e(x)$ then coincides with the sequence used in the definition of $s_{\mathrm{Tr}}(W_x)$. Hence Theorem 4.19 establishes its uniform convergence independent of x. Because $\operatorname{Spec} R$ is connected, by [40, Corollary 2.7], $\alpha_e(x)$ is a constant; it does not depend on x. Thus $\operatorname{rank} g_{n,e}(x)/\alpha_e(x)$ is a lower semicontinuous function. Therefore,

$$\mathrm{s}_{\mathrm{Tr}}(x) = \lim_{e\to\infty} \frac{\operatorname{rank} g_{n,e}(x)}{\alpha_e(x)}$$

is lower semicontinuous because it is the uniform limit of lower semicontinuous functions. □

Corollary 4.22. *Let $(R, \mathfrak{m}, k)$ be an F-finite Cohen–Macaulay local ring, and let ω_R be a dualizing module. Then the infimum in the definition of $\mathrm{s}_{\mathrm{Tr}}(R)$ is achieved.*

Proof. If R is not F-rational, then $\mathrm{s}_{\mathrm{Tr}}(W) = 0$ by the tight closure characterization and Lemma 4.8. Hence we may assume that R is a domain. By Theorem 4.21, s_{Tr} is lower semicontinuous on $\mathcal{B}_n$ for each n. Thus s_{Tr} has a minimum on $\mathcal{B}_n$, and this minimum is achieved at a closed point x. Because π_n is projective, $\pi_n(x) = \mathfrak{m}$. Furthermore, it follows from Nullstellensatz that $k(x)$ is a finite extension of k. Therefore,

$$\mathrm{s}_{\mathrm{Tr}}(R) = \min\Big\{\frac{1}{n} \min_{x\in\mathcal{B}_n} \mathrm{s}_{\mathrm{Tr}}(x) \;\Big|\; 1 \le n \le \dim_k \omega_R/\mathfrak{m}\omega_R\Big\}. \qquad \square$$

Remark 4.23. Lemma 4.8 allows to view $\mathrm{s}_{\mathrm{rel}}(R)$ as the infimum of the generalized F-signature function s_{Tr} on k-rational points of the Grassmannian. Hence, by (d) of Theorem 2.3, we obtain a different proof of Corollary 3.7 in the F-finite case. It should be noted that, for non-k-rational points, the two functions are different: we will show in the next section (Theorem 5.4 and Corollary 5.9) that $\mathrm{s}_{\mathrm{Tr}}(R) = \widetilde{\mathrm{s}_{\mathrm{Tr}}}(R)$, i.e., the minimum on k-rational points is equal to the global minimum, while Corollary 3.7 considers relative Hilbert–Kunz multiplicity after the field extension.

Semicontinuity also implies that the minimum is separated, i.e., there is the second smallest value. This result also holds for non-F-finite rings by using [57] as in Corollary 3.7.

Corollary 4.24. *Let R be an F-finite reduced Cohen–Macaulay ring such that* Spec R *is connected. Then*

$$\mathfrak{p} \mapsto \mathrm{s}_{\mathrm{Tr}}(R_\mathfrak{p}) := \inf\Big\{\frac{\mathrm{s}_{\mathrm{Tr}}(W)}{\dim_\ell W} \;\Big|\; k(\mathfrak{p}) \subseteq \ell \text{ is finite and } W \neq 0 \text{ is a quotient of } \omega_{R_\mathfrak{p}} \otimes_{R_\mathfrak{p}} \ell\Big\}$$

is a lower semicontinuous function.

Proof. We need to show that $\{\mathfrak{p} \mid \mathrm{s}_{\mathrm{Tr}}(R_\mathfrak{p}) \le a\}$ is closed for all $a \in \mathbb{R}$. Let

$$\pi_n \colon \mathcal{B}_n \to \operatorname{Spec} R$$

be the rank n Grassmannian. Because π_n is projective and s_{Tr} is lower semicontinuous, the set $Z_n(\le a) := \pi_n(\{x \in \mathcal{B}_n \mid \mathrm{s}_{\mathrm{Tr}}(x) \le a\})$ is closed for all $a \in \mathbb{R}$. Clearly, we have

$$Z_n(\le a) = \big\{\mathfrak{p} \in \operatorname{Spec} R \;\big|\; \inf\{\mathrm{s}_{\mathrm{Tr}}(x) \mid \mathfrak{p} = \pi_n(x)\} \le a\big\}.$$

By the definition of Grassmannian, the points $x \in \pi_n^{-1}(\mathfrak{p})$ parametrize all possible extensions of $k(\mathfrak{p})$ and all possible quotients of $\omega_R \otimes_R k(x)$ of rank n. Furthermore, from the proof of Corollary 4.22, we know that this infimum is achieved at x such that $k(x)$ is finite over $k(\mathfrak{p})$.

Thus, if we let N be such that ω_R can be generated by N elements as an R-module, then $\bigcup_{1 \leq n \leq N} Z_n(\leq na)$ is closed and coincides with $\{\mathfrak{p} \mid s_{\mathrm{Tr}}(R_{\mathfrak{p}}) \leq a\}$ due to the equality

$$\bigcup_{1 \leq n \leq N} Z_n(\leq na) = \left\{ \mathfrak{p} \in \operatorname{Spec} R \;\middle|\; \begin{array}{l} \dfrac{s_{\mathrm{Tr}}(W)}{\dim_\ell(W)} \leq a \text{ for some } [\ell : k(\mathfrak{p})] < \infty \\ \text{and } \omega_R \otimes_R \ell \to W \to 0 \end{array} \right\}. \qquad \square$$

5. New properties of the dual F-signature

In this section, we proceed to study the dual F-signature. The powerful linear algebra machinery of the appendix will show that all three perspectives on F-rational signature are equivalent. By combining the available techniques, this will allow to greatly advance the theory of dual F-signature, in particular, due to the powerful uniform convergence techniques of Hilbert–Kunz theory available for the relative F-rational signature.

Let us start by recalling the definition given by Sannai in [53].

Definition 5.1. Let $(R, \mathfrak{m}, k)$ be an F-finite Cohen–Macaulay local ring. Let ω_R be the dualizing module of R. For any e, let $b_e(R)$ be the largest integer N such that there exists a surjection

$$F^e_* \omega_R \to \bigoplus^N \omega_R \to 0.$$

Then the dual F-signature of R is defined as

$$s_{\mathrm{dual}}(R) = \limsup_{e \to \infty} \frac{b_e(R)}{p^{e \dim R}[k : k^{p^e}]}.$$

Remark 5.2. In [53], the dual F-signature of R was defined under the assumption that R is reduced. This restriction is not essential because R must be reduced if $s_{\mathrm{dual}}(R) > 0$.

Namely, suppose there is a surjection $F^e_* \omega_R \to \omega_R \to 0$. If a is a nilpotent element such that $a^{p^e} = 0$, then $a F^e_* \omega_R = 0$. It follows that $a \omega_R = 0$, which is a contradiction with faithfulness of ω_R (see [7, (1.8)]).

Remark 5.3. Sannai observed in [53, Lemma 3.6] that there is a useful one-to-one correspondence, arising from duality, between surjections

$$F^e_* \omega_R \to \bigoplus^{b_e} \omega_R \to 0$$

and injections

$$0 \to \bigoplus^{b_e} R \to R^{1/p^e} \to M \to 0,$$

where M is maximal Cohen–Macaulay. In particular, this shows that $s_{\mathrm{dual}}(R) \geq s(R)$.

We now easily get inequalities connecting the theories of F-rational signature.

Theorem 5.4. *Let $(R, \mathfrak{m})$ be an F-finite Cohen–Macaulay local ring. Then*

$$s_{\mathrm{rat}}(R) \geq s_{\mathrm{rel}}(R) = \widetilde{s_{\mathrm{Tr}}}(R) \geq s_{\mathrm{Tr}}(R) \geq s_{\mathrm{dual}}(R) \geq s(R).$$

Proof. The last inequality was established in [53, Proposition 3.8]. The inequalities $s_{\mathrm{rat}}(R) \geq s_{\mathrm{rel}}(R)$ and $\widetilde{s_{\mathrm{Tr}}}(R) \geq s_{\mathrm{Tr}}(R)$ follow from the definitions. It was proved in Lemma 4.8 that $s_{\mathrm{rel}}(R) = \widetilde{s_{\mathrm{Tr}}}(R)$, so it remains to show the second to last inequality.

Let ω_R be the dualizing module. By tensoring the definition of $b_e(R)$, for any field extension ℓ and any quotient W of $\ell \otimes_R \omega_R$, there is a surjection

$$\ell \otimes_R F^e_* \omega_R \to \bigoplus^{b_e(R)} \ell \otimes_R \omega_R \to \bigoplus^{b_e(R)} W.$$

Since the original $b_e(R)$ surjective maps were necessarily multiples of Tr^e by Remark 4.2, the map induces a surjection

$$(\ell \otimes_R F^e_* \omega_R)/Z_e(W) \to \bigoplus^{b_e(R)} W \to 0,$$

and the inequality $s_{\mathrm{Tr}}(R) \geq s_{\mathrm{dual}}(R)$ follows. □

We refine the theorem in the following uniform relation needed both for showing the existence and semicontinuity of the dual F-signature.

Theorem 5.5. *Let R be an F-finite ring and ω_R its dualizing module. There is a constant C such that, for all $\mathfrak{p} \in \operatorname{Spec} R$ and for all $e \geq 1$, we have*

$$\begin{aligned} b_e(R_\mathfrak{p}) + C &\geq \min\left\{ \frac{\dim_{k(\mathfrak{p})}(F^e_* \omega_{R_\mathfrak{p}}/Z_e(W))}{\dim_{k(\mathfrak{p})} W} \;\middle|\; W \neq 0,\ \omega_R \otimes_R k(\mathfrak{p}) \to W \to 0 \right\} \\ &\geq b_e(R_\mathfrak{p}). \end{aligned}$$

Proof. The second inequality was observed in the proof of Theorem 5.4, so it remains to show the first inequality. Let us denote

$$N_e(\mathfrak{p}) = \min\left\{ \frac{\dim_{k(\mathfrak{p})}(F^e_* \omega_{R_\mathfrak{p}}/Z_e(W))}{\dim_{k(\mathfrak{p})} W} \;\middle|\; W \neq 0,\ \omega_R \otimes_R k(\mathfrak{p}) \to W \to 0 \right\}.$$

As a first step, we assume that R is a local ring with the maximal ideal $\mathfrak{m}$ and the residue field k. Let $X = F^e_* \omega_R/\mathfrak{m}^{[p^e]}\omega_R$ and $Y = \omega_R/\mathfrak{m}\omega_R$. Note that any map $F^e_* \omega_R \to \omega_R/\mathfrak{m}\omega_R$ factors through $F^e_* \omega_R/\mathfrak{m}^{[p^e]}\omega_R$, so we let $H \subseteq \operatorname{Hom}(X, Y)$ consist of homomorphisms induced by $\operatorname{Hom}_R(F^e_* \omega_R, \omega_R)$. By Corollary A.12, by taking $C = P(\dim_k Y)$ for the polynomial $P(T) = T^2(T^2 - 1)/6$, we can build a surjection

$$F^e_* \omega_R/\mathfrak{m}^{[p^e]}\omega_R \to \bigoplus^{N_e(\mathfrak{m})-C} \omega_R/\mathfrak{m}\omega_R \to 0$$

which descended from R-module maps. By Nakayama's lemma, it can be lifted to a surjection $F^e_* \omega_R \to \bigoplus^{N_e(\mathfrak{m})-C} \omega_R$. Thus $b_e(R) \geq N_e(\mathfrak{m}) - C$.

Second, in all other cases, we let ν be any integer such that there are ν elements that generate ω_R. Since $\dim_{k(\mathfrak{p})} \omega_{R_\mathfrak{p}}/\mathfrak{p}\omega_{R_\mathfrak{p}} \leq \nu$ for all $\mathfrak{p}$, and $P(T)$ is monotone by Corollary A.12, the theorem follows from the first case with $C = P(\nu)$. □

Combining the theorem with Lemma 4.8, we obtain a connection with relative Hilbert–Kunz multiplicities.

Corollary 5.6. *Let R be an F-finite ring. There is a constant C such that, for any $\mathfrak{p} \in \operatorname{Spec} R$ and any system of parameters $\underline{x_{\mathfrak{p}}}$ of $R_{\mathfrak{p}}$, we have*

$$b_e(\mathfrak{p}) + C \geq [k(\mathfrak{p}) : k(\mathfrak{p})^{p^e}] \min\left\{ \frac{\ell_{R_\mathfrak{p}}(I^{[p^e]}/\langle \underline{x_\mathfrak{p}} \rangle^{[p^e]})}{\ell_{R_\mathfrak{p}}(I/\langle \underline{x_\mathfrak{p}} \rangle)} \;\middle|\; \langle \underline{x_\mathfrak{p}} \rangle \subset I \subseteq \langle \underline{x_\mathfrak{p}} \rangle :_{R_\mathfrak{p}} \mathfrak{p} \right\}$$
$$\geq b_e(\mathfrak{p}).$$

Remark 5.7. From the optimal criterion for two-dimensional vector spaces in Theorem A.4, by appropriately modifying Corollary A.12 and Theorem 5.5, we obtain the exact equality

$$b_e(\mathfrak{p}) = [k(\mathfrak{p}) : k(\mathfrak{p})^{p^e}] \min\left\{ \frac{\ell_{R_\mathfrak{p}}(I^{[p^e]}/\langle \underline{x_\mathfrak{p}} \rangle^{[p^e]})}{\ell_{R_\mathfrak{p}}(I/\langle \underline{x_\mathfrak{p}} \rangle)} \;\middle|\; \langle \underline{x_\mathfrak{p}} \rangle \subset I \subseteq \langle \underline{x_\mathfrak{p}} \rangle :_{R_\mathfrak{p}} \mathfrak{p} \right\}$$

whenever type $R(\mathfrak{p}) = 2$.

We will combine these results with the following uniform convergence result that easily follows from [50, Theorem 3.6].

Theorem 5.8. *Let R be an F-finite ring. There exists a constant D such that, for any $\mathfrak{p} \in \operatorname{Spec} R$ and any $\mathfrak{p}$-primary ideal I, we have*

$$\left| \frac{1}{p^{e \operatorname{ht} \mathfrak{p}}} \min\left\{ \frac{\ell_{R_\mathfrak{p}}(J^{[p^e]}R_\mathfrak{p}/I^{[p^e]}R_\mathfrak{p})}{\ell_{R_\mathfrak{p}}(JR_\mathfrak{p}/IR_\mathfrak{p})} \;\middle|\; I \subset J \subseteq \mathfrak{p} \right\} - \inf\left\{ \frac{\mathrm{e}_{\mathrm{HK}}(IR_\mathfrak{p}) - \mathrm{e}_{\mathrm{HK}}(JR_\mathfrak{p})}{\ell_{R_\mathfrak{p}}(JR_\mathfrak{p}/IR_\mathfrak{p})} \;\middle|\; I \subset J \subseteq \mathfrak{p} \right\} \right| \leq \frac{D}{p^e}.$$

Proof. By setting $q_2 \to \infty$ in [50, Theorem 3.6], we obtain a constant $D > 0$ such that, for any $\mathfrak{p} \in \operatorname{Spec} R$ and any pair of $\mathfrak{p}$-primary ideals $I \subseteq J$,

$$\left| \frac{1}{p^{e \operatorname{ht} \mathfrak{p}}} \ell\left(\frac{J^{[p^e]}R_\mathfrak{p}}{I^{[p^e]}R_\mathfrak{p}} \right) - \left(\mathrm{e}_{\mathrm{HK}}(IR_\mathfrak{p}) - \mathrm{e}_{\mathrm{HK}}(JR_\mathfrak{p}) \right) \right| \leq \frac{D}{p^e} \ell\left(\frac{JR_\mathfrak{p}}{IR_\mathfrak{p}} \right).$$

In order to finish the proof, it remains to remove the absolute value and take the infimums,

$$\inf_{I \subset J} \frac{\mathrm{e}_{\mathrm{HK}}(IR_\mathfrak{p}) - \mathrm{e}_{\mathrm{HK}}(JR_\mathfrak{p})}{\ell_{R_\mathfrak{p}}(JR_\mathfrak{p}/IR_\mathfrak{p})} - \frac{D}{p^e} \leq \inf_{I \subset J} \frac{\ell_{R_\mathfrak{p}}(J^{[p^e]}R_\mathfrak{p}/I^{[p^e]}R_\mathfrak{p})}{p^{e \operatorname{ht} \mathfrak{p}} \ell_{R_\mathfrak{p}}(JR_\mathfrak{p}/IR_\mathfrak{p})}$$
$$\leq \inf_{I \subset J} \frac{\mathrm{e}_{\mathrm{HK}}(IR_\mathfrak{p}) - \mathrm{e}_{\mathrm{HK}}(JR_\mathfrak{p})}{\ell_{R_\mathfrak{p}}(JR_\mathfrak{p}/IR_\mathfrak{p})} + \frac{D}{p^e}. \qquad \square$$

5.1. Dual F-signature exists and is semicontinuous. Now, we can easily show that the dual F-signature exists.

Corollary 5.9. *Let $(R, \mathfrak{m})$ be an F-finite Cohen–Macaulay local ring and ω_R its dualizing module. Then*

$$\mathrm{s}_{\mathrm{dual}}(R) = \lim_{e \to \infty} \frac{b_e(R)}{\operatorname{rank} F_*^e \omega_R}$$

exists and is equal to $\mathrm{s}_{\mathrm{rel}}(R)$.

Proof. By Corollary 5.6, it is enough to show that, for a system of parameters $\underline{x}$,

$$\lim_{e\to\infty} \frac{1}{p^{e \dim R}} \min\left\{ \frac{\ell(I^{[p^e]}/\langle \underline{x}\rangle^{[p^e]})}{\ell(I/\langle \underline{x_{\mathfrak{p}}}\rangle)} \;\middle|\; \langle \underline{x}\rangle \subset I \subseteq \langle \underline{x}\rangle : \mathfrak{m} \right\}$$

exists. This follows from Theorem 5.8. □

Combining the corollary with Theorem 5.4 and Corollary 4.24 shows that the dual F-signature defines a lower semicontinuous function on the spectrum $\mathfrak{p} \mapsto \mathrm{s}_{\mathrm{dual}}(R_{\mathfrak{p}})$. However, it is easy to give a direct proof avoiding Corollary 4.24.

Theorem 5.10. *Let R be an F-finite Cohen–Macaulay locally equidimensional ring. If* Spec R *is connected, then the convergence of*

$$\frac{b_e(\mathfrak{p})}{p^{e \dim R_{\mathfrak{p}}}[k(\mathfrak{p})\colon k(\mathfrak{p})^{p^e}]} \to \mathrm{s}_{\mathrm{dual}}(R_{\mathfrak{p}})$$

is uniform on Spec R *and* $\mathfrak{p} \mapsto \mathrm{s}_{\mathrm{dual}}(R_{\mathfrak{p}})$ *is lower semicontinuous.*

Proof. We may assume that Spec R is connected because semicontinuity can be checked on components. Furthermore, since F-rationality coincides with regularity for Artinian rings, we may assume that $\dim R > 0$.

We start by proving lower semicontinuity of the function

$$\mathfrak{p} \mapsto b_e(\mathfrak{p}) := \max\left\{ N \;\middle|\; F^e_* \omega_{R_{\mathfrak{p}}} \to \bigoplus^N \omega_{R_{\mathfrak{p}}} \to 0 \text{ is exact} \right\}.$$

We can lift a surjection by collecting denominators: for any $\mathfrak{p}$, there is an element $s \notin \mathfrak{p}$ such that $F^e_* \omega_{R_s} \to \bigoplus^{b_e(\mathfrak{p})} \omega_{R_s} \to 0$ is exact. Thus $b_e(\mathfrak{q}) \ge b_e(\mathfrak{p})$ for any $\mathfrak{q} \in D(s)$. Therefore, for any $\mathfrak{p}$ such that $b_e(\mathfrak{p}) > a$, there is an open set $\mathfrak{p} \in D(s)$ satisfying the same inequality; hence the set $\{\mathfrak{q} \mid b_e(\mathfrak{q}) > a\}$ is open.

Because Spec R is connected, by [40, Corollary 2.7], for any $e \ge 1$, the function

$$\mathfrak{p} \mapsto \alpha_e(\mathfrak{p}) := p^{e \dim R_{\mathfrak{p}}}[k(\mathfrak{p}) : k(\mathfrak{p})^{p^e}]$$

is constant on Spec R. Clearly, $\alpha_e(\mathfrak{p}) \ge p^e$. Thus, by Corollary 5.6, there is a constant C such that, for all $\mathfrak{p}$,

$$\begin{aligned}\frac{b_e(\mathfrak{p})}{\alpha_e(\mathfrak{p})} + \frac{C}{p^e} &\ge \frac{b_e(\mathfrak{p}) + C}{\alpha_e(\mathfrak{p})}\\ &\ge \frac{1}{p^{e \dim R_{\mathfrak{p}}}} \min\left\{ \frac{\ell(I^{[p^e]}/\langle \underline{x_{\mathfrak{p}}}\rangle^{[p^e]})}{\ell(I/\langle \underline{x_{\mathfrak{p}}}\rangle)} \;\middle|\; \langle \underline{x_{\mathfrak{p}}}\rangle \subset I \subseteq \langle \underline{x_{\mathfrak{p}}}\rangle : \mathfrak{p} \right\} \ge \frac{b_e(\mathfrak{p})}{\alpha_e(\mathfrak{p})}.\end{aligned}$$

Hence Theorem 5.8 and Corollary 5.9 imply that

$$\left| \frac{b_e(\mathfrak{p})}{p^{e \dim R_{\mathfrak{p}}}[k(\mathfrak{p}) : k(\mathfrak{p})^{p^e}]} - \mathrm{s}_{\mathrm{dual}}(R_{\mathfrak{p}}) \right| \le \frac{C + D}{p^e}.$$

This finishes the proof because the uniform limit of semicontinuous functions is semicontinuous. □

Note that semicontinuity is a vast generalization of [64, Theorem 1.11], where it was shown that the F-rational locus, i.e., the set $\{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{s}_{\mathrm{dual}}(R_{\mathfrak{p}}) > 0\}$, is open.

5.2. Global dual F-signature. De Stefani, Polstra, and Yao defined global versions of F-signature and Hilbert–Kunz multiplicity in [17]. A similar definition can be made for the dual F-signature: if R is a Cohen–Macaulay F-finite ring with a dualizing module ω_R, then we define $b_e(\omega_R)$ by the formula in Definition 5.1. It seems that $b_e(\omega_R)$ may depend on the choice of ω_R, but this does not affect the dual F-signature: it follows from a result of Baidya, see (5.1) in the next proof, that $|b_e(\omega_R) - b_e(\omega'_R)| \leq \dim R$ for any two dualizing modules ω_R, ω'_R.

We will now give an analogue of the main result of [17]. Our treatment is based on a deeper use of semicontinuity and greatly shortens [17] since we do not need to show the existence of the global dual F-signature separately.

Theorem 5.11. *Let R be a Cohen–Macaulay F-finite domain with a dualizing module ω_R. Then*

$$s_{\mathrm{dual}}(R) := \lim_{e\to\infty} \frac{b_e(\omega_R)}{\operatorname{rank} F^e_* \omega_R} = \min\{s_{\mathrm{dual}}(R_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec} R\}.$$

In particular, the limit defining $s_{\mathrm{dual}}(R)$ exists and does not depend on the choice of ω_R.

Proof. By Theorem 2.3, $s = \min\{s_{\mathrm{dual}}(R_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec} R\}$ exists due to semicontinuity, so the right-hand side is defined. Similarly, the function $\mathfrak{p} \mapsto b_e(R_\mathfrak{p})$ also has a minimum by semicontinuity. Note that $\dim R < \infty$ by [40, Proposition 1.1]. Then

$$(5.1) \quad \min\{b_e(R_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec} R\} - \dim R \leq b_e(\omega_R) \leq \min\{b_e(R_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec} R\},$$

where the first inequality holds by [8, Theorem 1.1] and the second holds by localizing the definition. Using the inequalities, it is enough to show that

$$\frac{\min\{b_e(R_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec} R\}}{\operatorname{rank} F^e_* R}$$

converges to s. We will derive this using semicontinuity and uniform convergence of the dual F-signature obtained in Theorem 5.10.

By Theorem 2.3, there exists $\varepsilon > 0$ such that, for every $\mathfrak{p}$, we have either $s_{\mathrm{dual}}(R_\mathfrak{p}) = s$ or $s_{\mathrm{dual}}(R_\mathfrak{p}) > s + \varepsilon$. By uniform convergence established in Theorem 5.10, for all $e \gg 0$, we have

$$\left| s_{\mathrm{dual}}(R_\mathfrak{p}) - \frac{b_e(R_\mathfrak{p})}{\operatorname{rank} F^e_* R} \right| < \frac{\varepsilon}{2}.$$

Note that

$$\operatorname{rank} F^e_* R = p^{e \dim R_\mathfrak{p}} [k(\mathfrak{p}) : k(\mathfrak{p})^{p^e}]$$

by [40, Proposition 2.3]. Therefore, if $s_{\mathrm{dual}}(R_\mathfrak{p}) \neq s$, then for any $\mathfrak{m}$ such that $s_{\mathrm{dual}}(R_\mathfrak{m}) = s$, we have

$$\frac{b_e(R_\mathfrak{p})}{\operatorname{rank} F^e_* R} > s_{\mathrm{dual}}(R_\mathfrak{p}) - \frac{\varepsilon}{2} > s + \frac{\varepsilon}{2} > \frac{b_e(R_\mathfrak{m})}{\operatorname{rank} F^e_* R}.$$

Thus $\min\{b_e(R_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec} R\} = \min\{b_e(R_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec} R, s_{\mathrm{dual}}(R_\mathfrak{p}) = s\}$ for all $e \gg 0$. Then, by uniform convergence,

$$\lim_{e\to\infty} \frac{b_e(\omega_R)}{\operatorname{rank} F^e_* R} = \lim_{e\to\infty} \frac{\min\{b_e(R_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec} R\}}{\operatorname{rank} F^e_* R}$$
$$= \lim_{e\to\infty} \frac{\min\{b_e(R_\mathfrak{p}) \mid \mathfrak{p}, s_{\mathrm{dual}}(R_\mathfrak{p}) = s\}}{\operatorname{rank} F^e_* R} = s. \qquad \square$$

5.3. Geometrically regular fibers. As observed in Proposition 3.13, F-rational signature does not increase in flat extensions, and it is natural to search for conditions that ensure equality. This seems to be a difficult question, perhaps due to the lack of complete understanding of the conditions that guarantee that F-rationality passes from R to S. We will present a generalization of a result of Vélez [64, Theorem 3.1], asserting that F-rationality is preserved when $R \to S$ is smooth, by proving that $s_{\mathrm{dual}}(R) = s_{\mathrm{dual}}(S)$ if the closed fiber is *geometrically regular*. Note that $R \to S$ is flat with a geometrically regular fiber if and only if $R \to S$ is formally smooth.

There are further results in the literature that concern the transfer of rationality from R to S (see [18, 26, 64]). In particular, Aberbach and Enescu [1] relaxed the assumption to requiring *geometric F-rationality* of the closed fiber.

Lemma 5.12. *Let $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, \ell)$ be a flat local homomorphism of F-finite rings such that the closed fiber $k \to S/\mathfrak{m}S$ is geometrically regular. Then $[k : k^p] = [\ell : \ell^p]$.*

Proof. Let L be the fraction field of the regular domain $S/\mathfrak{m}S$. Because L is geometrically regular over k, it is separable. Thus $[L : L^p] = [k : k^p]p^{\operatorname{tr.deg}_k L}$. On the other hand, $\operatorname{tr.deg}_k L = \dim S/\mathfrak{m}S$, so $[L : L^p] = [\ell : \ell^p]p^{\operatorname{tr.deg}_k L}$ by [40, formula 2.2]. □

Theorem 5.13. *Let $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, \ell)$ be a flat local homomorphism of F-finite rings such that the closed fiber $k \to S/\mathfrak{m}S$ is geometrically regular. Then*

$$b_e(R)p^{e \dim S/\mathfrak{m}S} \leq b_e(S).$$

Proof. By a theorem of André [5, page 297], the homomorphism $R \to S$ is regular, i.e., *all* fibers are geometrically regular. Thus we may apply the Radu–André theorem [6, 52] to learn that the relative Frobenius map $S \otimes_R F^e_* R \to F^e_* S$ is faithfully flat. Note that $S \otimes_R F^e_* R$ is still a local ring due to $F^e_* R$ being purely inseparable and that $F^e_* S$ is a finite module over $S \otimes_R F^e_* R$ because S is F-finite. It follows that $F^e_* S$ is a free module over $S \otimes_R F^e_* R$. Its rank can be found after first tensoring with $\otimes_{F^e_* R} F^e_* k$, which yields the map $S/\mathfrak{m}S \otimes_k F^e_* k \to F^e_* S/\mathfrak{m}S$, and further tensoring with the quotient field L of $S/\mathfrak{m}S$ to see that it is enough to compute the rank of L^{1/p^e} over $L \otimes_k k^{1/p^e}$. Since L is separable over L, this rank is equal to $p^{e \operatorname{tr.deg}_k L}$.

By the first paragraph, there is an isomorphism $\bigoplus^{p^{e \dim S/\mathfrak{m}S}} (S \otimes_R F^e_* R) \cong F^e_* S$, which implies that

$$F^e_* \omega_S \cong \operatorname{Hom}_S(F^e_* S, \omega_S) \cong \bigoplus^{p^{e \dim S/\mathfrak{m}S}} \operatorname{Hom}_S(S \otimes_R F^e_* R, \omega_S).$$

Since S is flat and $S/\mathfrak{m}S$ is Gorenstein, we have $\omega_S = \omega_R \otimes_R S$. This leads to a further isomorphism $\operatorname{Hom}_S(S \otimes_R F^e_* R, \omega_S) \cong S \otimes_R F^e_* \omega_R$. Thus we can build a surjection

$$F^e_* \omega_S \to \bigoplus^{b_e(R) p^{e \dim S/\mathfrak{m}S}} \omega_S$$

by tensoring $F^e_* \omega_R \to \bigoplus^{b_e(R)} \omega_R \to 0$ with S and taking an appropriate direct sum. This finishes the proof. □

Corollary 5.14. *Let $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, \ell)$ be a flat local homomorphism of F-finite rings such that the closed fiber $k \to S/\mathfrak{m}S$ is geometrically regular. Then $s_{\text{dual}}(R) = s_{\text{dual}}(S)$.*

Proof. We combine Theorem 5.13, Proposition 3.13, and Corollary 5.9. □

Remark 5.15. A special case of the corollary is when the residue field extension $k \to \ell$ is separable. Recall, that a field extension $k \subset L$ is *separable* if $L \otimes_k k^{1/p}$ is still a field (hence, equivalently, L is geometrically reduced over L) and is *separably generated* if it can be presented as a separable algebraic extension of a purely transcendental extension of k. The second notion is due to Mac Lane [44] who showed that every finitely generated separable extension is separably generated. However, Mac Lane also observed that $L = k(t^{1/p^\infty})$ is separable, but is not separably generated.

5.4. A transformation rule. In [12], it was established that, for an extension $R \to S$ which is étale in codimension one, there is a transformation rule connecting F-signatures. Such rule is impossible for F-rational signature because an example of Singh (see [54, Examples 6.5, 6.6] and [54, Theorem 4.2] for the background) shows that F-rationality may not transfer from R to S.

5.5. Second coefficient. It was shown in [36] that Hilbert–Kunz functions have a second coefficient in an excellent normal local ring with a perfect residue field. Subsequent works have shown this result holds with somewhat weaker assumptions: an unpublished manuscript of Hochster and Yao demonstrates that, essentially, only Serre's (R1) condition is needed. For rings over a perfect field, this was also independently shown in [13].

We will now prove a similar result for the dual F-signature. In order to do so, we will work with the relative Hilbert–Kunz multiplicity and follow Huneke's alternative proof from [34] of the main result in [36]. While the proof in [34] was stated for normal rings with perfect residue field, a close inspection shows that it works in merely F-finite rings. In our proof, we track the dependency of a number of constants so as to give uniform control over the correction terms. Our careful handling is further motivated by the proof of [34, Lemma 7.5]: it appears that it has a small inaccuracy, which we explain and fix, and its last part is left as an exercise, which we believe requires a mild generalization of [34, Lemma 7.2].

Theorem 5.16 (Huneke). *Let $(R, \mathfrak{m})$ be an F-finite local normal domain of dimension $d \geq 2$ and characteristic $p > 0$, and let $\underline{x}$ be a fixed system of parameters. For any torsion module N, there exists a positive constant $C(N)$ with the following property: for all ideals $\langle \underline{x} \rangle \subseteq I$, there exists $\gamma(I, N) \in \mathbb{R}$ such that, for all $e \geq 1$,*

$$|\ell(\operatorname{Tor}_1(N, R/I^{[p^e]})) - \gamma(I, N)p^{e(d-1)}| < C(N)p^{e(d-2)}.$$

Proof. The result essentially follows from the proof of [34, Theorem 7.8], but the constant $C(N)$ needs to be chosen to work uniformly for all $\langle \underline{x} \rangle \subseteq I$. In order to verify this claim, we will carefully trace through the proof and the preceding results of [34, Section 7].

Step 1 (Uniform and extended [34, Lemma 7.2]). We will show that, for any finitely generated R-module torsion module T and a finite generated R-module M, there exists a constant $C_2(T,M)$ such that, for all ideals I containing $\underline{x}$,

$$\ell(\mathrm{Tor}_1(T, M/I^{[p^e]}M)) \le C_2(T,M)p^{e\dim T}.$$

We start with the case of $M = R$ covered by Huneke. In the proof, he shows that, for any finitely generated R-module T and any ideal I containing $\underline{x}$, there is a bound

$$\begin{aligned}\ell(\mathrm{Tor}_1(T, R/I^{[p^e]})) &\le C(T,\underline{x})p^{e\dim T} + \ell(I/\langle\underline{x}\rangle)\ell(T/\mathfrak{m}^{[p^e]}T)\\ &\le C(T,\underline{x})p^{e\dim T} + \ell(R/\langle\underline{x}\rangle)\ell(T/\mathfrak{m}^{[p^e]}T),\end{aligned}$$

where $C(T,\underline{x})$ is given by applying [34, Theorem 7.3] to the Koszul complex of $\underline{x}$ and does not depend on I. Since the Hilbert–Kunz function converges [46], it follows from the above equation that there is a constant $C_2(T)$ such that, for all ideals I containing $\underline{x}$, we have

$$\ell(\mathrm{Tor}_1(T, R/I^{[p^e]})) \le C_2(T)p^{e\dim T}.$$

For an arbitrary M, we tensor an exact sequence

$$0 \to \Omega \to \bigoplus^N R \to M \to 0$$

with $R/I^{[p^e]}$ to get an exact sequence

$$0 \to \Omega/(I^{[p^e]}\Omega + A_e) \to \bigoplus^N R/I^{[p^e]} \to M/I^{[p^e]}M \to 0.$$

After tensoring with T, it is possible to estimate the Tor-module of interest as

$$\begin{aligned}\ell(\mathrm{Tor}_1(T, M/I^{[p^e]}M)) &\le N\ell(\mathrm{Tor}_1(T, R/I^{[p^e]})) + \ell(T\otimes_R \Omega/I^{[p^e]}\Omega)\\ &\le NC_2(T)p^{e\dim T} + \ell(T\otimes_R \Omega/\langle\underline{x}\rangle^{[p^e]}\Omega)\\ &\le NC_2(T)p^{e\dim T} + C_3p^{e\dim T},\end{aligned}$$

where the last bound is given by Hilbert–Kunz theory because $\dim \Omega\otimes_R T \le \dim T$. The proof is now finished after setting $C_2(T,M) = NC_2 + C_3$.

Step 2 (Uniform and extended version of [34, Lemma 7.4]). We will show that, for any finitely generated R-module T of dimension at most $d-2$ and a finitely generated R-module M, there exists a constant $C_4(T,M)$ such that, for all ideals I containing $\underline{x}$,

$$\ell(\mathrm{Tor}_2(T, M/I^{[p^e]}M)) \le C_4(T,M)p^{e(d-2)}.$$

Following Huneke's proof, we deduce for any R-module T annihilated by a regular sequence x, y and generated by N elements a bound

$$\ell(\mathrm{Tor}_2(T, M/I^{[p^e]}M)) \le N\ell(\mathrm{Tor}_2(R/\langle x,y\rangle, M/I^{[p^e]}M)) + \ell(\mathrm{Tor}_1(T', M/I^{[p^e]}M)),$$

where T' is an $R/\langle x,y\rangle$-syzygy of T. Using the Koszul resolution of $R/(x,y)$, we bound

$$\ell(\mathrm{Tor}_2(T, M/I^{[p^e]}M)) \le N\ell(\mathrm{Tor}_1(R/\langle x,y\rangle, M/I^{[p^e]}M)) + \ell(\mathrm{Tor}_1(T', M/I^{[p^e]}M)).$$

The result of the first step bounds the length of Tor-modules and shows that we may take $C_4(T,M) = NC_2(R/\langle x,y\rangle, M) + C_2(T',M)$.

Step 3 (Uniform version of [34, Lemma 7.5]). We need to show that, for any torsion-free finitely generated R-module, there exists a constant $C_5(M)$ such that, for all ideals I containing $\underline{x}$,

$$\ell(\mathrm{Tor}_1(M, R/I^{[p^e]})) \leq C_5(M) p^{e(d-2)}.$$

Huneke's proof first passes to the double dual M^{**} by observing that

$$\begin{aligned}\ell(\mathrm{Tor}_1(M, R/I^{[p^e]})) &\leq \ell(\mathrm{Tor}_1(M^{**}, R/I^{[p^e]})) + \ell(\mathrm{Tor}_1(M^{**}/M, R/I^{[p^e]})) \\ &\qquad + \ell(\mathrm{Tor}_2(M^{**}/M, R/I^{[p^e]})) \\ &\leq \ell(\mathrm{Tor}_1(M^{**}, R/I^{[p^e]})) \\ &\qquad + (C_2(M^{**}/M) + C_4(M^{**}/M)) p^{e(d-2)}.\end{aligned}$$

Thus we may assume that M is reflexive.

Let $\to F_1 \to F_0 \to M \to 0$ be a part of a free resolution of M. Let Z_e be the kernel of the induced map

$$(F_1 \to F_0) \otimes R/I^{[p^e]},$$

and let B_e be the image of the induced map

$$(F_2 \to F_1) \otimes R/I^{[p^e]}.$$

Because $\mathrm{Tor}_1(M, R/I^{[p^e]}) = Z_e/B_e$, we derive an exact sequence

$$0 \to \mathrm{Tor}_1(M, R/I^{[p^e]}) \to F_1/B_e \to F_1/Z_e \to 0. \tag{5.2}$$

By tensoring the exact sequence defining the first syzygy Ω_1 of M, $F_2 \to F_1 \to \Omega_1 \to 0$, with $R/I^{[p^e]}$, we identify[1)] $F_1/B_e \cong \Omega_1/I^{[p^e]}\Omega_1$.

As explained in Huneke's proof, one can choose a regular sequence x, y so that $\langle x, y\rangle$ annihilates all $\mathrm{Tor}_1(M, \bullet)$. It follows that tensoring (5.2) with $R/\langle x, y\rangle$ yields the bound

$$\begin{aligned}\ell(\mathrm{Tor}_1(M, R/I^{[p^e]})) &\leq \ell(\mathrm{Tor}_1(R/\langle x, y\rangle, F_1/Z_e)) + \ell(\Omega_1/\langle x, y, I^{[p^e]}\rangle\Omega_1) \\ &\leq \ell(\mathrm{Tor}_1(R/\langle x, y\rangle, F_1/Z_e)) + \ell(\Omega_1/\langle x, y, \langle \underline{x}\rangle^{[p^e]}\rangle\Omega_1) \\ &\leq \ell(\mathrm{Tor}_1(R/\langle x, y\rangle, F_1/Z_e)) + C p^{e(d-2)}\end{aligned}$$

from Hilbert–Kunz theory. We estimate the remaining Tor-module by tensoring

$$0 \to F_1/Z_e \to F_0/I^{[p^e]}F_0 \to M/I^{[p^e]}M \to 0$$

with $R/\langle x, y\rangle$ and obtain that

$$\begin{aligned}\ell(\mathrm{Tor}_1(R/\langle x, y\rangle, F_1/Z_e)) &\leq \ell(\mathrm{Tor}_2(R/\langle x, y\rangle, M/I^{[p^e]}M)) \\ &\qquad + \ell(\mathrm{Tor}_1(R/\langle x, y\rangle, F_0/I^{[p^e]}F_0)) \\ &\leq C_4(R/\langle x, y\rangle, M) p^{e(d-2)} + C_2(R/\langle x, y\rangle, F_0) p^{e(d-2)}.\end{aligned}$$

The assertion follows.

1) The proof in [34] seems to claim that $F_1/Z_e \cong \Omega_1/I^{[p^e]}\Omega_1$.

Step 4 (Uniform [34, Corollary 7.6]). The assertion follows by replacing [34, Lemma 7.5] by its uniform version. Hence, for any R-module M and any $i \geq 2$, there exists a constant $C_{6,i}$ such that $\ell(\mathrm{Tor}_i(M, R/I^{[p^e]})) \leq C_{6,i}(M) p^{e(d-2)}$ for all $\langle \underline{x} \rangle \subseteq I$.

Step 5 (Uniform [34, Corollary 7.7]). The proof shows that, for an exact sequence $0 \to T_1 \to T_2 \to T_3 \to 0$ of torsion modules, we can bound

$$\Big|\sum_{i=1}^{3}(-1)^{i+1}\ell(\mathrm{Tor}_1(T_i, R/I^{[p^e]}))\Big| \leq \ell(\mathrm{Tor}_2(T_3, R/I^{[p^e]})) + \sum_{i=1}^{3}(-1)^{i+1}\ell(T_i/I^{[p^e]}T_i).$$

Since R is a domain, we can find $c \neq 0$ that annihilates T_1, T_2, T_3. Because Hilbert–Kunz multiplicity is additive in short exact sequences and converges uniformly [63, Theorem 3.6], by working in $R/(c)$, we can find a constant D such that, for all I containing $\underline{x}$, there is a bound

$$|\ell(T_3/I^{[p^e]}T_3) + \ell(T_1/I^{[p^e]}T_1) - \ell(T_2/I^{[p^e]}T_2)| < Dp^{e(d-2)}.$$

Therefore, we take $C_7 = C_{6,2}(T_3) + D$ to bound

$$\Big|\sum_{i=1}^{3}(-1)^{i+1}\ell(\mathrm{Tor}_1(T_i, R/I^{[p^e]}))\Big| \leq C_7 p^{e(d-2)}.$$

Step 6 (The proof of the assertion). Last, we trace the proof [34, Theorem 7.8] to show that it works in the F-finite case and provides a uniform constant. First, it is explained that we may reduce to $N = R/Q$, where Q is a height one prime. Observe that

$$[k(Q) : k(Q)^p] = p^{d-1}[k : k^p]$$

by [40, Corollary 2.7], so [34, (10) in the proof of Theorem 7.8] can be replaced with

$$|p^{d-1}[k : k^p]\ell(\mathrm{Tor}_1(R/Q, R/I^{[p^e]})) - \ell(\mathrm{Tor}_1((R/Q)^{1/p}, R/I^{[p^e]}))| \leq (C_2(T) + C_4(T))p^{e(d-2)}. \tag{5.3}$$

From the long exact sequence for the tensor product, we derive that the quantity

$$\ell(\mathrm{Tor}_1((R/Q)^{1/p}, R/I^{[p^e]})) - \ell(Q^{1/p} \otimes_R R/I^{[p^e]}) + \ell(R^{1/p} \otimes_R R/I^{[p^e]}) - \ell((R/Q)^{1/p} \otimes_R R/I^{[p^e]})$$

is non-negative and is bounded above by $\ell(\mathrm{Tor}_1(R^{1/p}, R/I^{[p^e]}))$. Thus

$$\Big|\frac{\ell(\mathrm{Tor}_1((R/Q)^{1/p}, R/I^{[p^e]}))}{[k : k^p]} - \ell(Q/I^{[p^{e+1}]}Q) + \ell(R/I^{[p^{e+1}]}) - \ell(R/\langle Q, I^{[p^{e+1}]}\rangle)\Big| \leq \frac{C_5(R^{1/p})}{[k : k^p]} p^{e(d-2)}.$$

Huneke notes that the alternating sum of lengths can be computed by tensoring

$$0 \to Q \to R \to R/Q \to 0$$

with $R/I^{[p^{e+1}]}$, so it follows that

$$\left|\frac{\ell(\mathrm{Tor}_1((R/Q)^{1/p}, R/I^{[p^e]}))}{[k:k^p]} - \ell(\mathrm{Tor}_1(R/Q, R/I^{[p^{e+1}]}))\right| \le \frac{C_5(R^{1/p})}{[k:k^p]} p^{e(d-2)}.$$

By plugging this into (5.3), we get that

$$|p^{d-1}\ell(\mathrm{Tor}_1(R/Q, R/I^{[p^e]})) - \ell(\mathrm{Tor}_1(R/Q, R/I^{[p^{e+1}]}))| \le \frac{C_5(R^{1/p}) + C_2(T) + C_4(T)}{[k:k^p]} p^{e(d-2)},$$

which allows to invoke [51, Lemma 3.5 (iii)] and deduce the existence of

$$\gamma(I, R/Q) := \lim_{e\to\infty} \frac{1}{p^{e(d-1)}} \ell(\mathrm{Tor}_1(R/Q, R/I^{[p^e]}))$$

and estimate the convergence rate by

$$|\ell(\mathrm{Tor}_1(R/Q, R/I^{[p^e]})) - p^{e(d-1)}\gamma(I, R/Q)| \le 2\frac{C_5(R^{1/p}) + C_2(T) + C_4(T)}{[k:k^p]} p^{e(d-2)}.$$ □

Corollary 5.17 ([34, Proposition 7.9, Corollary 7.10]). *Let $(R, \mathfrak{m})$ be an F-finite local normal domain of dimension $d \ge 2$ and characteristic $p > 0$, and let $\underline{x}$ be a system of parameters. Let M be a finitely generated torsion-free R-module. There exists a constant $C_9(M) \in \mathbb{R}$ such that, for any ideal I that contains $\underline{x}$, there exists a constant $\gamma(I, M)$ such that*

$$|\ell(\mathrm{Tor}_0(M, R/I^{[p^e]})) - r\ell(\mathrm{Tor}_0(R, R/I^{[p^e]})) - \gamma(I, M)p^{e(d-1)}| < C_9(M)p^{e(d-2)}.$$

In particular, for any ideal I that contains $\underline{x}$, there exists $\gamma(I, R^{1/p}) \in \mathbb{R}$ such that

$$|[k:k^p]\ell(\mathrm{Tor}_0(R, R/I^{[p^{e+1}]})) - p^d[k:k^p]\ell(\mathrm{Tor}_0(R, R/I^{[p^e]})) - \gamma(I, R^{1/p})p^{e(d-1)}| < C_9(R^{1/p})p^{e(d-2)}.$$

Proof. Huneke's proof of [34, Proposition 7.9] applies verbatim to the first statement by replacing his references to [34, Lemma 7.5, Theorem 7.8] by the uniform versions obtained in Theorem 5.16 and his appeal to the convergence of the Hilbert–Kunz sequence by the uniform convergence estimate from [63, Theorem 3.6].

The second statement can be obtained by taking $M = R^{1/p}$ and noting that its rank is $p^d[k:k^p]$ and that

$$\ell(\mathrm{Tor}_0(R, R^{1/p} \otimes_R R/I^{[p^e]})) = \ell(R^{1/p} \otimes_R R/I^{[p^e]}) = [k:k^p]\ell(R/I^{[p^{e+1}]}).$$ □

Theorem 5.18. *Let $(R, \mathfrak{m})$ be an F-finite local normal domain of dimension $d \ge 2$, and let $\underline{x}$ be a system of parameters. Then there exists a constant $C \ge 0$ such that, for every ideal $\underline{x} \in I$, there exists $\beta(I)$ such that*

$$|\ell(R/I^{[p^e]}) - \mathrm{e}_{\mathrm{HK}}(I)p^{ed} - \beta(I)p^{e(d-1)}| < Cp^{e(d-2)}.$$

Proof. The assertion is a uniform version of [34, Theorem 7.11] and is obtained from its proof by replacing $\gamma(R^{1/p})$ with $\gamma(R^{1/p})/[k : k^p]$ in the definition of ϵ_q, replacing the reference to [34, Corollary 7.10] by the uniform estimate in Corollary 5.17, and quantifying the geometric series trick through [51, Lemma 3.5 (iii)]. □

Corollary 5.19. *Let $(R, \mathfrak{m})$ be an F-finite local normal domain of dimension $d \geq 2$, and let $\underline{x}$ be a system of parameters. Then there exists a constant $C \geq 0$ such that, for every ideal $\langle \underline{x} \rangle \subseteq I$, there exists $\beta(I)$ such that*

$$|\ell(I^{[p^e]}/\langle \underline{x} \rangle^{[p^e]}) - (\mathrm{e}_{\mathrm{HK}}(\langle \underline{x} \rangle) - \mathrm{e}_{\mathrm{HK}}(I))p^{ed} + \beta(I)p^{e(d-1)}| < 2Cp^{e(d-2)}.$$

Proof. This follows from Theorem 5.18 by using the estimates for $\langle \underline{x} \rangle$ and I. □

Theorem 5.20. *Let $(R, \mathfrak{m}, k)$ be an F-finite F-rational local domain of dimension $d \geq 2$. Then there exists a constant β such that*

$$\frac{b_e(R)}{[k : k^{p^e}]} = \mathrm{s}_{\mathrm{dual}}(R)p^{ed} + \beta p^{e(d-1)} + O(p^{e(d-2)}).$$

Proof. As discussed in Remark 4.23, semicontinuity of the relative Hilbert–Kunz multiplicity on the Grassmannian of the socle implies that there exists $\varepsilon > 0$ such that, whenever $\mathrm{e}_{\mathrm{HK}}(\langle \underline{x} \rangle) - \mathrm{e}_{\mathrm{HK}}(I) < (\mathrm{s}_{\mathrm{rel}}(R) + \varepsilon)\ell(I/\langle \underline{x} \rangle)$ for an ideal $\underline{x} \in I \subseteq \langle \underline{x} \rangle : \mathfrak{m}$, then

$$\mathrm{e}_{\mathrm{HK}}(\langle \underline{x} \rangle) - \mathrm{e}_{\mathrm{HK}}(I) = \mathrm{s}_{\mathrm{rel}}(R)\ell(I/\langle \underline{x} \rangle).$$

Due to the uniform convergence (Theorem 5.8), there exists $e_0 > 0$ such that, for all socle ideals I and all $e \geq e_0$,

$$\left| \frac{\ell(I^{[p^e]}/\langle \underline{x} \rangle^{[p^e]})}{p^{ed}} - (\mathrm{e}_{\mathrm{HK}}(\langle \underline{x} \rangle) - \mathrm{e}_{\mathrm{HK}}(I))p^{ed} + \beta(I)p^{e(d-1)} \right| < \frac{\varepsilon}{2}.$$

Hence, if I, J are arbitrary socle ideals such that $\mathrm{e}_{\mathrm{HK}}(\langle \underline{x} \rangle) - \mathrm{e}_{\mathrm{HK}}(I) = \mathrm{s}_{\mathrm{rel}}(R)\ell(I/\langle \underline{x} \rangle)$ and $\mathrm{e}_{\mathrm{HK}}(\langle \underline{x} \rangle) - \mathrm{e}_{\mathrm{HK}}(J) > \mathrm{s}_{\mathrm{rel}}(R)\ell(J/\langle \underline{x} \rangle)$, then for all $e \geq e_0$,

$$\frac{\ell(J^{[p^e]}/\langle \underline{x} \rangle^{[p^e]})}{p^{ed}\ell(J/\langle \underline{x} \rangle)} > \frac{\mathrm{e}_{\mathrm{HK}}(\langle \underline{x} \rangle) - \mathrm{e}_{\mathrm{HK}}(J)}{\ell(J/\langle \underline{x} \rangle)} - \frac{\varepsilon}{2} > \mathrm{s}_{\mathrm{rel}}(R) + \frac{\varepsilon}{2} > \frac{\ell(I^{[p^e]}/\langle \underline{x} \rangle^{[p^e]})}{p^{ed}\ell(I/\langle \underline{x} \rangle)}.$$

Therefore, by Corollary 5.6, for all $e \geq e_0$,

$$b_e(R) = [k : k^{p^e}] \min\left\{ \frac{\ell(I^{[p^e]}/\langle \underline{x} \rangle^{[p^e]})}{\ell(I/\langle \underline{x} \rangle)} \;\middle|\; \begin{array}{l} \langle \underline{x} \rangle \subset I \subseteq \langle \underline{x} \rangle : \mathfrak{m}, \\ \dfrac{\mathrm{e}_{\mathrm{HK}}(\langle \underline{x} \rangle) - \mathrm{e}_{\mathrm{HK}}(I)}{\ell(I/\langle \underline{x} \rangle)} = \mathrm{s}_{\mathrm{rel}}(R) \end{array} \right\} + O(1).$$

Now, for any socle ideal I such that $\mathrm{e}_{\mathrm{HK}}(\langle \underline{x} \rangle) - \mathrm{e}_{\mathrm{HK}}(I) = \mathrm{s}_{\mathrm{rel}}(R)\ell(I/\langle \underline{x} \rangle)$, consider the sequence

$$c_e(I) = \frac{1}{p^{e(d-1)}}\left(\frac{\ell(I^{[p^e]}/\langle \underline{x} \rangle^{[p^e]})}{\ell(I/\langle \underline{x} \rangle)} - p^{ed}\,\mathrm{s}_{\mathrm{rel}}(R) \right).$$

By Corollary 5.19, this sequence converges uniformly, at the rate $2C/p^e$ independent of I, to $\beta(I)/\ell(I/\langle \underline{x} \rangle)$. Thus, by taking the infimum in the inequality

$$c_e(I) - 2C/p^e \leq \beta(I)/\ell(I/\langle \underline{x} \rangle) \leq c_e(I) + 2C/p^e,$$

we obtain that $\inf_I c_e(I)$ converges, at the same rate, to $\inf_I \beta(I)/\ell(I/\langle \underline{x} \rangle)$. Therefore,

$$\frac{b_e(R)}{[k : k^{p^e}]} = \mathrm{s}_{\mathrm{rel}}(R) p^{ed} + \inf\left\{ \frac{\beta(I)}{\ell(I/\langle \underline{x} \rangle)} \;\middle|\; \frac{\mathrm{e}_{\mathrm{HK}}(\langle \underline{x} \rangle) - \mathrm{e}_{\mathrm{HK}}(I)}{\ell(I/\langle \underline{x} \rangle)} = \mathrm{s}_{\mathrm{rel}}(R) \right\} p^{e(d-1)} + O(p^{e(d-2)}). \qquad \square$$

Corollary 5.21. *Let $(R, \mathfrak{m}, k)$ be an F-finite $\mathbb{Q}$-Gorenstein F-rational local domain of dimension $d \geq 2$ with a perfect residue field. Then*

$$b_e(R) = \mathrm{s}_{\mathrm{dual}}(R) p^{ed} + O(p^{e(d-2)}).$$

Proof. By [41], the second coefficient $\beta(I)$ is zero for every $\mathfrak{m}$-primary ideal I. □

6. A formula for toric varieties

Besides Gorenstein examples where dual F-signature and F-signature coincide, we do not have many examples where the dual F-signature was computed. In [53], Sannai computed dual F-signature of the Veronese subrings of $k[x, y]$ (Example 3.2). More generally, the results of Nakajima in [47] can be used for computations in cyclic quotients of $k[x, y]$. In [27], Hashimoto studied the dual F-signature of invariant subrings and was able to characterize vanishing of $\mathrm{s}_{\mathrm{dual}}(\omega_R)$ representation-theoretically even in the non-Cohen–Macaulay case. In particular, he showed that $\mathrm{s}_{\mathrm{dual}}(\omega_{R^G}) > 1/|G|$ whenever it is positive.

We suspect that it might be easier to work with the Hilbert–Kunz definition and will now discuss F-rational signature in the toric case. Note that cyclic quotient surface singularities are exactly two-dimensional toric singularities [20, Section 2.2]. Hilbert–Kunz multiplicity of monomial ideals in toric rings was computed combinatorially by Watanabe [65]. We extend his idea and will start with a recipe for computing $\mathrm{s}_{\mathrm{Tr}}(\omega_R/N)$ where $\mathfrak{m}\omega_R \subseteq N$ is torus-invariant.

Specifically, consider a lattice L (i.e., a group isomorphic to $\mathbb{Z}^n$) and a convex rational polyhedral cone $\sigma \subset L \otimes_{\mathbb{Z}} \mathbb{R}$. We can always assume that $L = \mathbb{Z}^n$, but sometimes, it is more convenient to work with a proper sublattice of $\mathbb{Z}^n$. Let $M = \mathrm{Hom}(L, \mathbb{Z})$ be the dual lattice, the dual cone is defined as $\sigma^\vee = \{u \in M \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle u, v \rangle \geq 0,\ v \in \sigma\}$, and let $R = k[\sigma^\vee \cap M]$ be a monomial subring of a Laurent polynomial ring $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] = k[\mathbb{Z}^n]$. It is invariant under the torus action $T\colon x_i \mapsto t_i x_i$ with $t_i \in k^\circ$.

We say that $\sigma^\vee$ is pointed if $\sigma^\vee \cap -\sigma^\vee = 0$ or, equivalently, if σ spans $L \otimes_{\mathbb{Z}} \mathbb{R}$. In this case, $R = k[\sigma^\vee \cap M]$ has a distinguished maximal ideal generated by all nontrivial monomials. It corresponds to the unique fixed point of the torus action on the toric variety Spec R. We will denote this ideal by $\mathfrak{m}$.

By the work of Hochster [28], R is Cohen–Macaulay, and it is also known that one can choose a torus-invariant dualizing ideal [15] corresponding to the interior $(\sigma^\vee)^\circ$ of the cone. From now on, we will use ω_R to denote this ideal.

Suppose that $N \subset \omega_R$ is a monomial ideal where ω_R/N has finite length, and we identify ω_R/N with the k-vector space with basis given by the finitely many monomials in the complement $\omega_R \setminus N$. Following the interpretation of $s_{\mathrm{Tr}}(\omega_R/N)$ in Remark 4.6, we are searching for monomials

$$x^u \in \omega_R^{1/p^e} \quad \text{such that} \quad \mathrm{Tr}_R(r^{1/p^e} x^u) \notin N \text{ for some } r \in R.$$

It is known (see [33, page 1780]) that Tr^e is a projection on the lattice: for $u \in 1/p^e M$,

$$\mathrm{Tr}^e(x^u) = \begin{cases} x^u & \text{if } u \in M, \\ 0 & \text{otherwise.} \end{cases}$$

Hence we need $x^u \in \omega_R^{1/p^e}$ such that $x^{u+v} \in \omega_R \setminus N$ for some $x^v \in R^{1/p^e} = k[\frac{1}{p^e} M \cap \sigma^\vee]$, i.e.,

$$u \in P_e := \bigcup \Big\{ \frac{1}{p^e} M \cap (\sigma^\vee)^\circ \cap (a - \sigma^\vee) \;\Big|\; x^a \in \omega_R \setminus N \Big\}.$$

Thus we see that

$$s_{\mathrm{Tr}}(\omega_R/N) = \lim_{e\to\infty} \frac{\dim_{k^{1/p^e}} \omega_R^{1/p^e}/Z_e(I)}{p^{e \dim R}} = \lim_{e\to\infty} \frac{|P_e|}{p^{e \dim R}}.$$

Since $\dim R = \dim \sigma^\vee$, from the Ehrhart theory, we obtain that the limit is the normalized volume of a region,

$$s_{\mathrm{Tr}}(\omega_R/N) = \mathrm{vol}\big(\bigcup \{\sigma^\vee \cap (a - \sigma^\vee) \mid x^a \in \omega_R \setminus N\}\big)/\mathrm{vol}(M), \tag{6.1}$$

where $\mathrm{vol}(M)$ is the Euclidean volume of an elementary parallelepiped of the lattice.

We will now show that F-rational signature can be computed from the toric quotients only.

Proposition 6.1. *Let $R = k[\sigma^\vee \cap M]$ be an affine pointed toric ring and*

$$\mathcal{T} = \{a \in (\sigma^\vee)^\circ \cap M \mid a - m \notin (\sigma^\vee)^\circ \text{ for all } 0 \neq m \in \sigma^\vee \cap M\}$$

a finite set of lattice points corresponding to the monomials $x^a \in \omega_R \setminus \mathfrak{m}\omega_R$. Then

$$s_{\mathrm{rel}}(R) = \frac{1}{\mathrm{vol}(M)} \min_{\emptyset \neq S \subseteq \mathcal{T}} \frac{1}{|S|} \mathrm{vol}\big(\bigcup \{\sigma^\vee \cap (a - \sigma^\vee) \mid a \in S\}\big).$$

Proof. It follows from (6.1) and the discussion preceding it that the invariant on the right-hand side is the infimum computed over the subset of monomial quotients ω_R/N of ω_R with $\mathfrak{m}\omega_R \subseteq N$, so it is clearly no less than $\widetilde{s_{\mathrm{Tr}}}(R) = s_{\mathrm{rel}}(R)$. Furthermore, the right-hand side is independent of the ground field by (6.1). Since $s_{\mathrm{rel}}(R)$ cannot increase when the ground field is extended, it is enough to show the equality at some field extension. Thus we may assume that k is algebraically closed.

Since ω_R is T-invariant, T acts on the Grassmannian of rank n quotients, so we follow the construction in Theorem 4.21 to show that $s_{\mathrm{Tr}}(\omega_R/N) = s_{\mathrm{Tr}}(\omega_R/t \cdot N)$ for any $t \in T$. Namely, choose a T-invariant set $\{r_i\}_{i=1}^{m_e}$ such that $F_*^e r_i$ generate $F_*^e R$ as an R-module and let

$$g_e(N)\colon F_*^e \omega_R/\mathfrak{m}^{[p^e]}\omega_R \xrightarrow{\oplus \mathrm{Tr}^e(F_*^e r_i \cdot \bullet)} \bigoplus \omega_R/\mathfrak{m}\omega_R \longrightarrow \bigoplus \omega_R/N.$$

Since T acts invertibly and the first map is T-invariant, one can easily check that

$$\ker g_e(t \cdot N) = t \cdot \ker g_e(N).$$

Thus $\operatorname{rank} g_e(N) = \operatorname{rank} g_e(t \cdot N)$, so $s_{\mathrm{Tr}}(N) = s_{\mathrm{Tr}}(t \cdot N)$.

Because k is algebraically closed, the infimum in $\widetilde{s_{\mathrm{Tr}}}(R)$ is attained by Corollary 4.22. Then the locus of the Grassmannian that minimizes $s_{\mathrm{Tr}}(N)$ is T-invariant and, because the Grassmannian is projective, we may apply the Borel fixed point theorem [10, Theorem 10.4] to conclude that there is a T-invariant minimizer.

It remains to show that the minimizer is monomial.[2)] For any $f \in I$, we can write

$$f = \sum_{i=-a}^{b} x_1^i p_i(x_2, \dots, x_n)$$

as an element of the Laurent polynomial ring. Take g to be the generator of the multiplicative group of $F_{p^{a+b}} \subset k$; then $1, \dots, g^{a+b}$ are all distinct. Since I is T-invariant, multiplying by the powers of the element $t = (g, 1, \dots, 1) \in T$ provides that

$$g^{a\ell}(t^\ell \cdot f) = x_1^{-a} p_{-a}(x_2, \dots, x_n) + g^\ell x_1^{-a+1} p_{-a+1}(x_2, \dots, x_n) + \cdots \\ + g^{(a+b)\ell} x_1^b p_b(x_2, \dots, x_n) \in I$$

for $\ell = 0, \dots, a+b$. Since the Vandermonde matrix is invertible, each $x_1^i p_i(x_2, \dots, x_n) \in I$ for any $i = -a, \dots, b$ as it is a linear combination of the $g^{a\ell}(t^\ell \cdot f)$. After repeating the process for $x_2, \dots, x_n$, it is easy to see that I is monomial. □

Note that, in the above result, the number of monomials $x^a \in \omega_R \setminus \mathfrak{m}\omega_R$ is finite but may be large in number, whereby the number of subsets S involved in applying Proposition 6.1 is an issue when doing computations. Moreover, the volume of the union in (6.1) for a given S can be computed from the basic volumes $\operatorname{vol}(\sigma^\vee \cap (a - \sigma^\vee))$ by inclusion-exclusion, but this does not give an efficient algorithm.

Example 6.2. It is sometimes easier to work with a sublattice. We can consider the nth Veronese subring of the polynomial ring in d variables as a toric variety for the sublattice $L \subset \mathbb{Z}^d$ formed by vectors whose sum of components is divisible by n and the positive orthant as the cone. If e_i are a standard basis of $\mathbb{Z}^d$, then $ne_1, e_1 - e_i, i \geq 2$, form a basis of L, so we can easily compute that $\operatorname{vol}(L) = n$.

Integral points in ω_V are vectors $a = (a_1, \dots, a_d)$ such that all components are positive and their sum is divisible by n. One can further see that an integral point $a \in \omega_V \setminus \mathfrak{m}\omega_V$ is such that $\sum a_i = \lceil \frac{d}{n} \rceil n$ since any smaller sum will have some $a_i = 0$ (so it is not in ω_R) and any larger sum can be decreased without violating positivity of a_i (so it is in $\mathfrak{m}\omega_R$). For any such point, we can easily compute the volume $\operatorname{vol}(\sigma^\vee \cap (a - \sigma^\vee)) = \prod a_i$. This volume is minimized when all but one components are equal to 1, giving the minimum of $\lceil \frac{d}{n} \rceil n - d + 1$. See Hochster–Yao [32, Example 7.4] for another approach.

2) An alternative to the elementary proof given here can be found in [14, Lemma 1.1.16], noting that the referenced result [62, Theorem 3.2.3] is characteristic independent.

It is well known that there are $\binom{N-1}{d-1}$ integer points such that $a_i > 0$, $\sum a_i = N$. If we use all of them in (6.1), we obtain a shape which is best described as a "building block pyramid". The volume of it is equal to the number of integer points with positive coordinates such that $\sum a_i \leq N$, which can be seen by identifying each integer unit cube with its vertex with the largest coordinates (e.g., top-right for a square). Hence the volume of that region is

$$\sum_{k=1}^{N} \binom{N-1}{d-1} = \binom{N}{d}.$$

Thus, using $N = \lceil \frac{d}{n} \rceil n$, we obtain from (6.1) that the relative Hilbert–Kunz multiplicity of the entire socle is

$$\frac{\binom{N}{d}}{n\binom{N-1}{d-1}} = \frac{N}{dn} = \frac{1}{d}\left\lceil \frac{d}{n} \right\rceil.$$

Note that $\lceil \frac{d}{n} \rceil n \geq d$ with equality if and only if d is divisible by n, which is itself equivalent, by the socle formula, to V being Gorenstein. Hence $s_{\text{rat}}(V) = s_{\text{rel}}(V)$ if and only if V is Gorenstein because $s_{\text{rat}}(V) = (\lceil \frac{d}{n} \rceil n - d + 1)/n \geq \frac{1}{d} \lceil \frac{d}{n} \rceil = s_{\text{rel}}(V)$.

Remark 6.3. The second Veronese of $k[x, y, z]$ is an example of a singular ring such that $s_{\text{rat}}(V) = 1$.

Question 6.4. Is $s_{\text{rel}}(V_n) = \frac{1}{d} \lceil \frac{d}{n} \rceil$?

The equality is easy to verify when the Cohen–Macaulay type is close to d. For example, in the simplest non-Gorenstein case, such as $n = 2$ or $n = d + 1$, we have $\lceil \frac{d}{n} \rceil n = d + 1$. Then $\omega_R \setminus \mathfrak{m}\omega_R$ has d integer points and they have the form $(1, \ldots, 1, 2, \ldots, 1)$. By symmetry, any collection of k points will have the volumes $2 + (k - 1)$, so the relative Hilbert–Kunz multiplicity is $(k + 1)/(kn)$, which is minimized for $k = d$. With slightly more effort, one can also verify combinatorially the next case $\lceil \frac{d}{n} \rceil n = d + 2$.

Example 6.5. Suppose $C \subseteq \mathbb{R}^3$ is the strongly convex rational polyhedral cone with rays through the points $[0, 0, 1]$, $[0, 2, 1]$, $[3, 0, 1]$, and $[1, -1, 1]$. In other words, C is the cone over the polytope pictured in Figure 1 in the $z = 1$ plane. Let k be any F-finite field of characteristic $p > 0$ and consider $R = k[C \cap \mathbb{Z}^n]$ and $\mathfrak{m}$ the homogeneous maximal ideal. We view

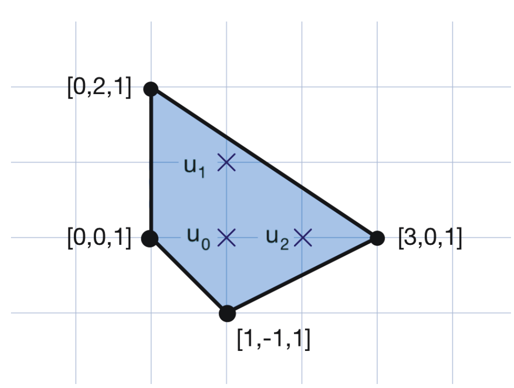

Figure 1

ω_R as $k[C^\circ \cap \mathbb{Z}^n]$, the ideal of R generated by the monomials corresponding to the interior lattice points of C. One checks that $u_0 = [1,0,1]$, $u_1 = [1,1,1]$, $u_2 = [2,0,1]$ correspond to a minimal set of generators for ω_R, i.e., the images of the corresponding monomials give a k-basis for the vector space $\omega_R/\mathfrak{m}\omega_R$. In particular, we see that the type of R is three.

For each subset of indices $\emptyset \neq S \subseteq \{0,1,2\}$, let N_S be the R-submodule of ω_R generated by $\mathfrak{m}\omega_R$ and x^{u_i} for $i \in \{0,1,2\} \setminus S$. We have that $W_S = \omega_R/N_S$ is then a k-vector space with basis given by the images of x^{u_i} for $i \in S$. To make our use of (6.1) transparent, consider the rational polytopes $P_i = C \cap (u_i - C)$ for $i = 0,1,2$; we then have that $\mathrm{s}_{\mathrm{Tr}}(W_S)$ is the Euclidean volume of $\bigcup_{i\in S} P_i$ divided by the number of elements in S. One computes

$$\begin{aligned} &\mathrm{s}_{\mathrm{Tr}}(W_{\{0\}}) = 136/441, \quad \mathrm{s}_{\mathrm{Tr}}(W_{\{0,1\}}) = 187/882, \quad \mathrm{s}_{\mathrm{Tr}}(W_{\{0,1,2\}}) = 101/588,\\ &\mathrm{s}_{\mathrm{Tr}}(W_{\{1\}}) = 167/882, \quad \mathrm{s}_{\mathrm{Tr}}(W_{\{0,2\}}) = 89/441,\\ &\mathrm{s}_{\mathrm{Tr}}(W_{\{2\}}) = 80/441, \quad \mathrm{s}_{\mathrm{Tr}}(W_{\{1,2\}}) = 571/3528 \end{aligned}$$

and checks that the smallest value achieved is thus $\mathrm{s}_{\mathrm{dual}}(R) = \mathrm{s}_{\mathrm{rel}}(R) = 571/3528$. In contrast, taking the minimum of the $\mathrm{s}_{\mathrm{Tr}}(W_{\{i\}})$ for $i = 0,1,2$ gives $\mathrm{s}_{\mathrm{rat}}(R) = 80/441$, which is strictly larger. Moreover, unlike what was seen for Veronese subrings in the previous example, $\mathrm{s}_{\mathrm{rel}}(R)$ is also not achieved by taking the (normalized) relative Hilbert–Kunz multiplicity for the entire socle modulo a parameter ideal. Explicitly, note that we have

$$\mathrm{s}_{\mathrm{Tr}}(W_{\{0,1,2\}}) = \mathrm{s}_{\mathrm{Tr}}(\omega_R/\mathfrak{m}\omega_R) = \frac{\mathrm{e}_{\mathrm{HK}}(I) - \mathrm{e}_{\mathrm{HK}}(I:\mathfrak{m})}{\ell((I:\mathfrak{m})/(I))} = 101/588$$

for an ideal $I \subseteq \mathfrak{m}$ corresponding to a parameter ideal of $R_{\mathfrak{m}}$.

7. Some open questions

This work opens a number of natural questions inspired by the existing theory of Hilbert–Kunz multiplicity and F-signature. We want to highlight questions that were touched but not resolved in this work.

Beyond F-finite. One benefit of $\mathrm{s}_{\mathrm{rel}}(R)$ is that it is defined via Hilbert–Kunz theory, and the definition makes sense for any local ring of positive characteristic. We developed Section 3 without the F-finite hypothesis and showed that $\mathrm{s}_{\mathrm{rel}}(R)$ has many good properties. However, all further results are tied to the dual F-signature.

Question 7.1. Is $\mathrm{s}_{\mathrm{rel}}(R)$ semicontinuous without the F-finite assumption?

We suspect that, for this question, one needs an interpretation of dual F-signature for non-F-finite rings. This is related to the following question since we would like to get rid of the residue field extension appearing in the definition of the dual F-signature.

Question 7.2. Is b_e always divisible by $[k : k^{p^e}]$?

Note that this will follow if one could remove the constant C from Corollary 5.6.

Good fibers.

Question 7.3. If $R \to S$ is a flat local map, under which conditions is $\mathrm{s}_{\mathrm{rel}}(R) = \mathrm{s}_{\mathrm{rel}}(S)$?

In particular, it is desirable to show that $\mathrm{s}_{\mathrm{rel}}(R) = \mathrm{s}_{\mathrm{rel}}(S)$ for a regular map. One way to achieve this would be to reduce to the F-finite case by means of the so-called Γ-*construction* [30]. This motivates the following question.

Question 7.4. If Γ varies over the cofinite subsets of a p-base of a coefficient field k of R, then does the equality $\sup_\Gamma \mathrm{s}_{\mathrm{rel}}(R^\Gamma) = \mathrm{s}_{\mathrm{rel}}(R)$ hold? Is the supremum achieved, i.e., is $\sup_\Gamma \mathrm{s}_{\mathrm{rel}}(R^\Gamma) = \mathrm{s}_{\mathrm{rel}}(R^{\Gamma'})$ for all sufficiently small Γ'?

Here, note that we have always have $\mathrm{s}_{\mathrm{rel}}(R^\Gamma) \le \mathrm{s}_{\mathrm{rel}}(R)$, and moreover, R is F-rational if and only if $\mathrm{s}_{\mathrm{rel}}(R^{\Gamma'}) > 0$ for all sufficiently small Γ'.

Question 7.5. Let $(R, \mathfrak{m}, k)$ be a complete F-finite Cohen–Macaulay local ring, and let ℓ be a finite separable field extension of k. Do R and $S := R \mathbin{\widehat{\otimes}_k} \ell$ have equal Sannai's sequences b_e?

Rees algebras. We explored a connection with Rees algebras in Corollary 3.18, but it is likely that one can say more. For example, it was conjectured in [25] and proved in [38] that, for an $\mathfrak{m}$-primary ideal I, the extended Rees algebra $R[It, t^{-1}]$ is F-rational if and only if R and the Rees algebra $R[It]$ are F-rational. It is desirable to give a connection in terms of F-rational signature akin to Corollary 3.18.

Remark 7.6. A very recent work [43] by Shiji Lyu made progress on Questions 7.1, 7.4.

A. A criterion for simultaneous injection of vector spaces

Throughout this section, we will work with vector spaces over a field k. For finite-dimensional vector spaces V, W and subspaces $U \subseteq V$ and $H \subseteq \mathrm{Hom}_k(V, W)$, we denote by $H(U) := \sum_{h \in H} h(U)$ the total image of U under H.

Definition A.1. We shall say that there are n simultaneous injections from V to W in H provided there exist $\phi_1, \dots, \phi_n \in H$ such that the induced map

$$\Phi = (\phi_1, \dots, \phi_n) \colon \bigoplus_{i=1}^{n} V \to W$$

is an injection. We will use $\mathrm{MaxInj}(H)$ to denote the maximal non-negative integer n such that there are n simultaneous injections from V to W in H.

Given n simultaneous injections, for any k-vector subspace U of V, we must have

$$n \cdot \dim(U) = \dim\Big(\Phi\Big(\bigoplus_{i=1}^{n} U\Big)\Big) = \dim\Big(\sum_{i=1}^{n} \phi_i(U)\Big) \le \dim(H(U)), \tag{A.1}$$

and so $n \le \min_{0\neq U\subseteq V}\lfloor \dim(H(U))/\dim(U)\rfloor$, where U varies over all of the non-zero subspaces of V. The problem we seek to address here is the optimality of this upper bound, and the main technical result of this section is the following criterion.

Theorem A.2. *Let k be a field and suppose V is a finite-dimensional vector space over k. Then there exists a positive constant C with the following property: for any finite-dimensional vector space W and vector subspace $H \subseteq \operatorname{Hom}_k(V, W)$, we have*

$$0 \le \min_{0\neq U\subseteq V}\left\lfloor \frac{\dim(H(U))}{\dim(U)}\right\rfloor - \operatorname{MaxInj}(H) \le C, \tag{A.2}$$

where U varies over all non-zero subspaces of V.

Remark A.3. Our proof will show that the constant C appearing in Theorem A.2 can be taken to be

$$C = \sum_{i=1}^{\dim(V)-1} i(\dim(V)-i)\cdot\dim(V) = \frac{1}{6}(\dim(V))^2\big((\dim(V))^2-1\big)$$

independently of the ground field k. However, we believe this bound to be far from optimal. In particular, when working over an infinite field k and using general k linear combinations of maps in H appropriately, we believe it is possible to exhibit a quadratic bound in terms of $\dim(V)$. This should not be a surprise because new injections can appear in H after extending from a finite field. For example, over the field $F_2 = \mathbb{Z}/2\mathbb{Z}$, let $V = W = F_2^{\oplus 3}$ and consider the subspace

$$H = \left\{ A = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix},\ B = \begin{bmatrix} 1 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix},\ A+B = \begin{bmatrix} 0 & 0 & 0\\ 1 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right\}$$

of $\operatorname{Hom}_{F_2}(V, W)$. There is no injection in H, but $xA + B$ is an injection over

$$F_4 = F_2[x]/(x^2+x+1).$$

When $\dim(V) = 2$, we obtain a much sharper theorem.

Theorem A.4. *Let k be a field and suppose V is a two-dimensional vector space over k. For any finite-dimensional vector space W and vector subspace $H \subseteq \operatorname{Hom}_k(V, W)$, we have*

$$\operatorname{MaxInj}(H) = \min_{0\neq U\subseteq V}\left\lfloor \frac{\dim(H(U))}{\dim(U)}\right\rfloor,$$

where U varies over all non-zero subspaces of V.

The following example shows that the assumption $\dim(V) = 2$ is essential in Theorem A.4.

Example A.5. Let k be an arbitrary field and set $V = W = k^{\oplus 3}$ with standard basis vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$. Consider the linear transformations from V to W given by the matrices

$$f = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad g = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

and let H be the linear span of f, g, h in $\operatorname{Hom}_k(V, W)$. By verifying that

$$\det(\lambda f + \mu g + \nu h) = 0$$

for all values of λ, μ, ν, we see that there are no injections in H. Yet

$$\min_{0 \neq U \subseteq V} \lfloor \dim(H(U))/\dim(U) \rfloor = 1,$$

as it is easy to verify that $H(V) = W$ and that $\dim H(k\langle \vec{v} \rangle) = 2$ for any $0 \neq \vec{v} \in V$.

Turning first towards a proof of Theorem A.4, we start with an elementary lemma.

Lemma A.6. *Let V and W be two finite-dimensional vector spaces over a field k. If $\phi, \xi \in \operatorname{Hom}_k(V, W)$ are such that $\operatorname{Im}\phi \cap \operatorname{Im}\xi = 0$, then $\operatorname{rank}(\phi + \xi) \geq \operatorname{rank}\phi$ and the inequality is strict provided there exists $\vec{v} \in \ker\phi$ such that $\xi(\vec{v}) \neq 0$.*

Proof. If $\vec{v} \in \ker(\phi + \xi)$, then $\phi(\vec{v}) = -\xi(\vec{v}) \in \operatorname{Im}\phi \cap \operatorname{Im}\psi = 0$. Thus we see that $\ker(\phi + \xi) = (\ker\phi \cap \ker\xi) \subseteq \ker\phi$, and the desired inequality $\operatorname{rank}(\phi + \xi) \geq \operatorname{rank}\phi$ follows and is strict provided $\ker\phi \cap \ker\xi \subsetneq \ker\phi$. □

Proposition A.7. *Let k be a field, V a two-dimensional k-vector space, and $n \geq 1$ an integer. Let W be a finite-dimensional k-vector space and H a subspace of $\operatorname{Hom}(V, W)$ such that, for every $0 \neq U \subseteq V$, we have $\dim H(U) \geq n \dim U$. Then $\operatorname{MaxInj}(H) \geq n$, i.e., there is an injection $\bigoplus^n V \to W$, where each component is in H.*

Proof. We proceed by induction starting with $n = 1$. Suppose all maps in H have rank at most 1. Take any $0 \neq h \in H$ and let $U = \ker h$. By the assumption, there is $g \in H$ such that $g(U) \neq 0$. We must have $\operatorname{Im} g = \operatorname{Im} h$, or $g + h$ has rank 2 by Lemma A.6. Since $\dim H(V) \geq 2$, there is $f \in H$ such that $\operatorname{Im} f \not\subseteq \operatorname{Im} g$. This gives a contradiction since either $f + g$ or $f + h$ must have rank 2 by Lemma A.6.

Now, assume that the $n + 1$-level condition holds, i.e., $\dim H(U) \geq (n + 1)\dim U$ for all $0 \neq U \subseteq V$. By induction, we find independent injections $\phi_1, \ldots, \phi_n$. Set $W' = W/\operatorname{Im}\phi_1$. If the n-level condition holds for $H' \subseteq \operatorname{Hom}(V, W')$, then there is an injection $\bigoplus^n V \to W'$ by induction which then lifts to the required injection $\bigoplus^{n+1} V \to W$.

Thus we assume that H' does not satisfy the n-level condition, i.e., there is a one-dimensional subspace U such that $\dim H'(U) \leq n - 1$, forcing that $\dim H(U) = n + 1$ and $\phi_1(V) \subseteq H(U)$. This can only happen if $\phi_1(V) + \phi_2(U) + \ldots + \phi_n(U) = H(U)$ since the dimensions are equal. We now pass to $\overline{W} = W/\sum_{i=2}^n \phi_i(V)$. For any $\phi \in H$, we denote by $\overline{\phi}$ the induced map $V \to \overline{W}$ and define $\overline{H}$ analogously. Since the original $\phi_1, \ldots, \phi_n$ are independent injections, we still have that $\overline{\phi}_1(V) = \overline{H}(U)$. Hence $\dim \overline{H}(U) = 2$ and $\dim \overline{H}(V) \geq 4$.

It remains to build two independent injections $U \to \overline{W}$ because their lifts will be independent with $\phi_2, \dots, \phi_n$.

Let $\overline{\psi}$ be such that $\overline{\phi}_1(V) = \overline{H}(U) = \overline{\psi}(U) + \overline{\phi}_1(U)$. First, assume that $\overline{\psi}$ is an injection, and let $\vec{v} \notin U$ be such that $\overline{\phi}_1(\vec{v}) \in \overline{\psi}(U)$. Since $\dim \overline{H}(V) \geq 4$, there is $\overline{g}$ such that $\overline{g}(V) \not\subseteq \overline{\psi}(V) + \overline{\phi}_1(V)$. If $\overline{g}, \overline{\psi}$ are independent injections, then we are done. Otherwise, clearly, $\overline{g}(\vec{v}) \notin \overline{\psi}(V) + \overline{\phi}_1(V)$, so we may apply Lemma A.6 in $\overline{W}/\psi(V)$ to show that $\overline{g} + \overline{\phi}_1$ and $\overline{\psi}$ are independent injections.

Last, suppose that $\overline{\psi}$ is not an injection and fix $0 \neq \vec{e} \in \ker \overline{\psi}$. As $\dim \overline{H}(V) \geq 4$, there is $\overline{h}$ such that $\overline{h}(V) \not\subseteq \operatorname{Im} \phi_1 = \overline{H}(U)$. By the choice of $\overline{\psi}$, we have $\vec{e} \notin U$, so $\overline{h}(\vec{e}) \notin \overline{H}(U)$. If $\overline{h}(U) \not\subseteq \overline{\phi}_1(U)$, then $\overline{h}$ is injective and we reduce to the previous case by replacing ψ with h. Otherwise, if $\overline{h}(U) \subseteq \overline{\phi}_1(U)$, then $\overline{h}(V) \cap \overline{\psi}(V) = 0$, so $\overline{h} + \overline{\psi}$ is injective by Lemma A.6 and we reduce to the previous case since $\overline{\phi}_1(V) = (\overline{\psi} + \overline{h})(U) + \overline{\phi}_1(U)$. □

Proof of Theorem A.4. From (A.1), we have that $n \dim(U) \leq \dim H(U)$ for all subspaces $0 \neq U \subseteq V$, and hence

$$n \leq \min_{0 \neq U \subseteq V} \left\lfloor \frac{\dim H(U)}{\dim(U)} \right\rfloor.$$

Moreover, we must have $\dim H(U_0) < (n+1)\dim(U_0)$ for some $0 \neq U_0 \subseteq V$ by Proposition A.7. Altogether this gives

$$n \leq \min_{0 \neq U \subseteq V} \left\lfloor \frac{\dim(H(U))}{\dim(U)} \right\rfloor \leq \frac{\dim(H(U_0))}{\dim(U_0)} \leq n,$$

and so equality must hold throughout completing the proof. □

Our proof of Theorem A.2 runs similarly to the proof of Theorem A.4 above, though the requisite inductive constructions in Theorem A.9 and Corollary A.10 are quite a bit more involved than that of Proposition A.7. Additionally, the elementary result below is used to avoid using general linear combinations over finite fields.

Lemma A.8. *Let U, W be vector spaces over a field k. Suppose that*

$$\phi_1, \dots, \phi_N \in \operatorname{Hom}(U, W)$$

are such that $\Phi = (\phi_1, \dots, \phi_N) \colon \bigoplus^N U \to W$ is an injection. If Z is a subspace of W such that $\dim(Z \cap \operatorname{Im} \Phi) = d \leq N$, then omitting some d of the $\phi_1, \dots, \phi_N$ will yield an injection $\bigoplus^{N-d} U \to W$ with image disjoint from Z. In other words, after reordering $\phi_1, \dots, \phi_N$, one can ensure that $Z \cap (\sum_{i=d+1}^N \operatorname{Im} \phi_i) = 0$.

Proof. We proceed by induction on d, noting first that the lemma is a tautology when $d = 0$. Now, assume the statement holds for all $0 \leq n < d$ and we have an injection

$$\Phi = (\phi_1, \dots, \phi_N) \colon \bigoplus^N U \to W$$

and a subspace $Z \subseteq W$ with $\dim(Z \cap \operatorname{Im} \Phi) = d \leq N$. Let $0 \neq \vec{v} \in (Z \cap \operatorname{Im} \Phi)$ and denote

$$\Phi_j := (\phi_1, \dots, \widehat{\phi}_j, \dots, \phi_N).$$

Since $\bigcap_{i=1}^{N} \operatorname{Im} \Phi_j = 0$, it follows that $\vec{v} \notin \operatorname{Im} \Phi_j$ for some j which we may assume to be 1. In particular, $\Phi_1 \colon \bigoplus^{N-1} U \to W$ is an injection with $Z \cap \operatorname{Im} \Phi_1 \subsetneq Z \cap \operatorname{Im} \Phi$ so that

$$\dim(Z \cap \operatorname{Im} \Phi_1) = n \leq d - 1 \leq N - 1.$$

Using the induction assumption on Φ_1 and Z, it follows that we can reorder $\phi_2, \dots, \phi_N$ to achieve $0 = (Z \cap (\sum_{i=n+2}^{N} \operatorname{Im} \phi_i)) \supseteq (Z \cap (\sum_{i=d+1}^{N} \operatorname{Im} \phi_i))$ as desired. □

Theorem A.9. *Let V and W be finite-dimensional vector spaces over a field k, and H a subspace of* $\operatorname{Hom}_k(V, W)$. *Suppose $n \geq 0$ and $1 \leq d < \dim V$ are integers and assume the following conditions are satisfied.*

(a) *There exist $\phi_1, \dots, \phi_n \in H$ giving an injection $\Phi = (\phi_1, \dots, \phi_n) \colon \bigoplus_{i=1}^{n} V \to W$.*

(b) *We have $\dim(H(U)) > n \dim(U)$ for any non-zero subspace $0 \neq U \subseteq V$.*

(c) *Writing $m := (\dim(V) - d) \cdot \dim(V) + 1$, there exist $\psi_1, \dots, \psi_m \in H$ so that*

$$\dim\left(\operatorname{Im} \Phi + \sum_{j=1}^{\ell} \operatorname{Im} \psi_j\right) \geq d + \dim\left(\operatorname{Im} \Phi + \sum_{j=1}^{\ell-1} \operatorname{Im} \psi_j\right)$$

for $\ell = 1, \dots, m$. In other words, we have that each ψ_ℓ has rank at least d modulo $\operatorname{Im} \Phi + \sum_{j=1}^{\ell-1} \operatorname{Im} \psi_j$.

Then there are maps $\widetilde{\phi}_1, \dots, \widetilde{\phi}_n, \psi \in H$ so that $\widetilde{\Phi} = (\widetilde{\phi}_1, \dots, \widetilde{\phi}_n) \colon \bigoplus^n V \to W$ is an injection and ψ has rank at least $d + 1$ modulo $\operatorname{Im} \widetilde{\Phi}$, i.e.,

$$\dim\left(\operatorname{Im} \psi + \sum_{i=1}^{n} \operatorname{Im} \widetilde{\phi}_i\right) \geq d + 1 + \dim\left(\sum_{i=1}^{n} \operatorname{Im} \widetilde{\phi}_i\right).$$

Proof. In order to have the rank at least d modulo $\operatorname{Im} \Phi + \sum_{j=1}^{\ell-1} \operatorname{Im} \psi_j$, each ψ_ℓ must have rank at least d modulo $\operatorname{Im} \Phi$. The assertion follows trivially if any ψ_ℓ has rank at least $d + 1$ modulo $\operatorname{Im} \Phi$, so we may assume each ψ_ℓ has rank exactly d modulo either $\operatorname{Im} \Phi$ or $\operatorname{Im} \Phi + \sum_{j=1}^{\ell-1} \operatorname{Im} \psi_j$.

Let $W' = W / \operatorname{Im} \Phi$, and for any $\phi \in \operatorname{Hom}_k(V, W)$, we denote by $\phi' \in \operatorname{Hom}_k(V, W')$ the map $V \xrightarrow{\phi} W \to W'$ induced by the quotient. Since the rank of ψ'_ℓ does not change after going modulo $\sum_{j=1}^{\ell-1} \operatorname{Im} \psi'_j$, we have $\operatorname{Im} \psi'_\ell \cap (\sum_{j=1}^{\ell-1} \operatorname{Im} \psi'_j) = 0$ for any $1 \leq \ell \leq m$, and in particular, $\operatorname{Im} \psi'_i \cap \operatorname{Im} \psi'_j = 0$ for any $i \neq j$. If ever $\ker \psi'_i \neq \ker \psi'_j$ for some $i \neq j$, then $\operatorname{rank}(\psi'_i + \psi'_j) > d$ by Lemma A.6, and the assertion follows. Hence assume now that all of these kernels are equal, and set $K = \ker \psi'_\ell$ for all $1 \leq \ell \leq m$.

Let U be a vector space complement of K in V so that $\dim(U) = d = \dim(V) - \dim(K)$ and $V = U + K$ with $U \cap K = 0$. Observe that $\operatorname{Im} \psi'_\ell = \psi'_\ell(U)$, so the restriction of each ψ'_ℓ to U is injective as $\operatorname{rank} \psi'_\ell = \dim(U) = d$. Moreover, setting

$$\Psi' = (\psi'_1, \dots, \psi'_m) \colon \bigoplus^{m} V \to W',$$

we similarly have that $\Psi'|_{\oplus^m U}$ is an injection as $\dim(\sum_{j=1}^{m} \psi'_j(U)) = dm$. In particular, it follows that each ψ'_ℓ has rank d modulo $\sum_{j=1,\, j \neq \ell}^{m} \operatorname{Im} \psi'_j$, i.e., $\operatorname{Im} \psi_\ell \cap (\sum_{j=1,\, j \neq \ell}^{m} \operatorname{Im} \psi'_j) = 0$. Note that we may permute $\psi_1, \dots, \psi_m$ as needed below while preserving our setup.

If $H(K) \not\subseteq \operatorname{Im}\Phi$, then we will find $h \in H$ and $\vec{v} \in K$ such that $h'(\vec{v}) \neq 0$. Since $\operatorname{Im}\psi_\ell$ are disjoint from each other, at most $\operatorname{rank} h' \leq \dim(V) < m$ of the $\operatorname{Im}\psi'_\ell$ can intersect $\operatorname{Im} h'$ nontrivially, so after reordering, we may assume $\operatorname{Im}\psi'_1 \cap \operatorname{Im} h' = 0$ (as in the proof of Lemma A.8). By Lemma A.6, $\operatorname{rank}(\psi'_1 + h') > \operatorname{rank}\psi'_1 = d$ and the assertion follows. Thus we may assume going forward that $H(K) \subseteq \operatorname{Im}\Phi$. In particular, note that this implies $\operatorname{Im}\Phi \neq 0$ so we must have $n \geq 1$.

Since $\dim(H(K)) > n\dim(K)$, there is some $h \in H$ with $h(K) \not\subseteq \Phi(K)$. Because

$$h(K) \subseteq H(K) \subseteq \operatorname{Im}\Phi \cong \bigoplus^n V$$

and an element of $\bigoplus^n V$ is in $\bigoplus^n K$ if and only if each of the components is in K, we may reorder the ϕ_i so that

$$h(K) \not\subseteq \Big(\phi_1(K) + \sum_{i=2}^n \operatorname{Im}\phi_i\Big)$$

and choose $\vec{v} \in K$ with

$$h(\vec{v}) \notin \Big(\phi_1(K) + \sum_{i=2}^n \operatorname{Im}\phi_i\Big).$$

Let

$$\bar{W} = W \Big/ \Big(\sum_{i=2}^n \operatorname{Im}\phi_i\Big),$$

and for any $\phi \in \operatorname{Hom}_k(V, W)$, we shall denote by $\bar{\phi} \in \operatorname{Hom}_k(V, \bar{W})$ the map $V \xrightarrow{\phi} W \to \bar{W}$ induced by the quotient by $\sum_{i=2}^n \operatorname{Im}\phi_i$. It now suffices to find an injection $\bar{\phi}$ and a map $\bar{\psi}$ that has rank $d+1$ modulo $\bar{\phi}$. We break the rest of the proof up into two cases depending on whether we could assume that this h is one of ψ_i.

Case 1. *There is some $1 \leq \ell \leq m$ with $\psi_\ell(K) \not\subseteq \Phi(K)$, i.e., $h = \psi_\ell$.*

Since $\dim(\operatorname{Hom}_k(K, \operatorname{Im}\bar{\phi}_1)) = \dim(K) \cdot \dim(V) = (\dim(V) - d) \cdot \dim(V) < m$, there must be a nontrivial linear combination $\xi = \sum_{i=1}^m \alpha_i \psi_i$ with $\alpha_1, \dots, \alpha_m \in k$ not all zero and $K \subseteq \ker\bar{\xi}$. If $0 \neq \vec{u} \in U$, then $\xi'(\vec{u}) = \Psi'(\alpha_1\vec{u}, \dots, \alpha_m\vec{u}) \neq 0$ as $\Psi'|_{\oplus^m U}$ is an injection and $\alpha_j \neq 0$ for some j. Thus $\bar{\xi}(\vec{u}) \neq 0$ as well, so it follows that $\bar{\xi}|_U$ is an injection, $\operatorname{rank}\bar{\xi} = d$, and $K = \ker\bar{\xi}$. Moreover, since $\ker\bar{\psi}_\ell \neq K$, ξ cannot be a scalar multiple of ψ_ℓ, and we must have that $\alpha_j \neq 0$ for some $j \neq \ell$.

We will now show that $\phi := \phi_1 + \xi$ and $\psi := \psi_\ell$ are the required maps. Let us first check that $\operatorname{Im}\bar{\xi} \cap (\operatorname{Im}\bar{\phi}_1 + \operatorname{Im}\bar{\psi}_\ell) = 0$. We already know that

$$\operatorname{Im}\bar{\xi} = \bar{\xi}(U) \quad \text{and} \quad \operatorname{Im}\bar{\phi}_1 + \operatorname{Im}\bar{\psi}_\ell = \operatorname{Im}\bar{\phi}_1 + \bar{\psi}_\ell(U).$$

Thus if $\bar{\xi}(\vec{u}) = \bar{\phi}_1(\vec{a}) + \bar{\psi}_\ell(\vec{v})$, then $\vec{u}, \vec{v} \in U$ and $\bar{\xi}(\vec{u}) - \bar{\psi}_\ell(\vec{v}) \in \operatorname{Im}\bar{\Psi}(\oplus^m U) \cap \operatorname{Im}\bar{\phi}_1$. As $\Psi'|_{\oplus^m U}$ is injective, it follows that $\vec{u}, \vec{v} = 0$ as ξ' is an injection modulo ψ'_ℓ. Thus we conclude $\operatorname{Im}\bar{\xi} \cap (\operatorname{Im}\bar{\phi}_1 + \operatorname{Im}\bar{\psi}_\ell) = 0$. In particular, we have $0 = \operatorname{Im}\bar{\xi} \cap \operatorname{Im}\bar{\phi}_1$, giving that $\bar{\phi} = \bar{\phi}_1 + \bar{\xi}$ is injective by applying Lemma A.6 and using that $\bar{\phi}_1$ is injective.

It remains to show that $\bar{\psi}_\ell$ has rank at least $d+1$ modulo $\operatorname{Im}(\bar{\phi}_1 + \bar{\xi})$. To that end, let us first check that $\bar{\psi}_\ell$ has rank at least $d+1$ modulo $\bar{\phi}_1(K)$. By our choice of $\vec{v} \in K$ above, we have that $\bar{\psi}_\ell(\vec{v}) \in \operatorname{Im}\bar{\phi}_1 \setminus \bar{\phi}_1(K)$. Put $T = U + k\vec{v}$, which has dimension $d+1$. Suppose we have $\vec{u} \in U$ and $\lambda \in k$ with $\overline{\psi}_\ell(\vec{u} + \lambda\vec{v}) \in \overline{\phi}_1(K)$. It follows that $\overline{\psi}_\ell(\vec{u}) \in \operatorname{Im}\overline{\phi}_1$, and so also $\psi'_\ell(\vec{u}) = 0$, which gives $\vec{u} = 0$ as $\ker\psi'_\ell = K$. Thus $\overline{\psi}_\ell(\lambda\vec{v}) = \lambda\overline{\psi}_\ell(\vec{v}) \in \overline{\phi}_1(K)$, which yields $\lambda = 0$ as $\overline{\psi}_\ell(\vec{v}) \notin \overline{\phi}_1(K)$. It follows that $\overline{\psi}_\ell|_T$ is injective modulo $\overline{\phi}_1(K)$, i.e., $\overline{\psi}_\ell$ is injective on T and $\overline{\psi}_\ell(T) \cap \overline{\phi}_1(K) = 0$. To conclude the stronger statement that $\overline{\psi}_\ell$ has rank at least $d+1$ modulo $\operatorname{Im}(\overline{\phi}_1 + \overline{\xi})$, it suffices verify $\operatorname{Im}(\overline{\phi}_1 + \overline{\xi}) \cap \overline{\psi}_\ell(T) = 0$. Suppose we have some $\vec{w} \in V$ with $\overline{\phi}_1(\vec{w}) + \overline{\xi}(\vec{w}) \in \overline{\psi}_\ell(T)$. It follows that $\overline{\xi}(\vec{w}) \in \operatorname{Im}\overline{\xi} \cap (\operatorname{Im}\overline{\phi}_1 + \operatorname{Im}\overline{\psi}_\ell) = 0$ and $\vec{w} \in \ker\overline{\xi} = K$. Thus we must have $\overline{\phi}_1(\vec{w}) \in \overline{\psi}_\ell(T) \cap \overline{\phi}_1(K) = 0$, so that $\overline{\phi}_1(\vec{w}) + \overline{\xi}(\vec{w}) = 0$ and hence $\operatorname{Im}(\overline{\phi}_1 + \overline{\xi}) \cap \overline{\psi}_\ell(T) = 0$.

Case 2. *For all* $1 \le \ell \le m$, $\psi_\ell(K) \subseteq \Phi(K) = \sum_{i=1}^n \phi_i(K)$.

Since

$$\dim\Big(\sum_{j=1}^m \overline{\psi}_j(U)\Big) = \dim\Big(\sum_{j=1}^m \psi'_j(U)\Big) = md,$$

we see that $\dim(\sum_{j=1}^m \overline{\psi}_j(U))$ remains unchanged modulo $\operatorname{Im}\overline{\phi}_1$, and thus

$$\Big(\sum_{j=1}^m \overline{\psi}_j(U)\Big) \cap \operatorname{Im}\overline{\phi}_1 = 0.$$

Set $Z = \overline{h}(U) + \operatorname{Im}\overline{\phi}_1$. Then

$$Z \cap \Big(\sum_{j=1}^m \overline{\psi}_j(U)\Big) \cong \frac{Z \cap \big(\sum_{j=1}^m \overline{\psi}_j(U)\big)}{Z \cap \big(\sum_{j=1}^m \overline{\psi}_j(U)\big) \cap \operatorname{Im}\overline{\phi}_1} \subseteq \frac{Z}{\operatorname{Im}\overline{\phi}_1} = \frac{\overline{h}(U) + \operatorname{Im}\overline{\phi}_1}{\operatorname{Im}\overline{\phi}_1} = h'(U),$$

and in particular,

$$\dim\Big(Z \cap \Big(\sum_{j=1}^m \overline{\psi}_j(U)\Big)\Big) \le \dim(h'(U)) \le \dim(U) = d.$$

Applying Lemma A.8 to $\overline{\psi}_1|_U, \dots, \overline{\psi}_m|_U \in \operatorname{Hom}(U, \overline{W})$ and Z, it follows that, after reordering $\psi_1, \dots, \psi_m$, we may assume

$$Z \cap \Big(\sum_{j=d+1}^m \overline{\psi}_j(U)\Big) = (\overline{h}(U) + \operatorname{Im}\overline{\phi}_1) \cap \Big(\sum_{j=d+1}^m \overline{\psi}_j(U)\Big) = 0. \tag{A.3}$$

Since $\overline{\psi}_{d+1}|_K, \dots, \overline{\psi}_m|_K \in \operatorname{Hom}_k(K, \overline{\phi}_1(K))$ and

$$\dim(\operatorname{Hom}_k(K, \overline{\phi}_1(K))) = (\dim(V) - d)^2 < m - d,$$

there must be a nontrivial linear combination $\xi = \sum_{i=d+1}^m \alpha_i\psi_i$ with $\alpha_{d+1}, \dots, \alpha_m \in k$ not all zero and $K \subseteq \ker\overline{\xi}$. If $0 \ne \vec{u} \in U$, then $\xi'(\vec{u}) = \Psi'(0, \dots, 0, \alpha_{d+1}\vec{u}, \dots, \alpha_m\vec{u}) \ne 0$ as $\Psi'|_{\oplus^m U}$ is an injection and $\alpha_j \ne 0$ for some $d+1 \le j \le m$. Thus $\overline{\xi}(\vec{u}) \ne 0$ as well, so it follows $\overline{\xi}|_U$ is an injection, $\operatorname{rank}\overline{\xi} = d$, and $K = \ker\overline{\xi}$. Since $m \ge \dim(V) + 1 \ge d + 2$, we may choose ψ_ℓ with $d+1 \le \ell \le m$ so that $\alpha_j \ne 0$ for some $d+1 \le j \le m$ and $j \ne \ell$. We know $\overline{\psi}_\ell|_U$ is injective as $\psi'_\ell|_U$ is injective, and it follows from (A.3) that $\overline{\psi}_\ell(U) \cap \overline{h}(U) = 0$ so that $\overline{\psi}_\ell + \overline{h}$ restricts to an injection on U by Lemma A.6. Furthermore, let us argue that

$$\overline{\xi}(U) \cap (\overline{\psi}_\ell(U) + Z) = \overline{\xi}(U) \cap (\overline{\psi}_\ell(U) + \overline{h}(U) + \operatorname{Im}\overline{\phi}_1) = 0. \tag{A.4}$$

Suppose $\vec{u}, \vec{u}', \vec{u}'' \in U$ and $\vec{w} \in V$ with $\overline{\xi}(\vec{u}) = \overline{\psi}_\ell(\vec{u}') + \overline{h}(\vec{u}'') + \overline{\phi}_1(\vec{w})$. Then

$$-\overline{\psi}_\ell(\vec{u}') + \sum_{i=d+1}^{m} \alpha_i \overline{\psi}_i(\vec{u}) \in Z \cap \left(\sum_{j=d+1}^{m} \overline{\psi}_j(U) \right) = 0$$

by (A.3), which implies

$$\Psi'(0, \ldots, 0, \alpha_{d+1}\vec{u}, \ldots, \alpha_{\ell-1}\vec{u}, \alpha_\ell\vec{u} - \vec{u}', \alpha_{\ell+1}\vec{u}, \ldots, \alpha_m\vec{u}) = 0,$$

giving that $\alpha_j \vec{u} = 0$ by the injectivity of $\Psi'|_{\oplus^m U}$ as $j \neq \ell$; thus we must have $\vec{u} = 0$, and (A.4) follows.

We will now show that $\phi := \phi_1 + \xi$ and $\psi := \psi_\ell + h$ are the two required maps. Notice first that, since $\operatorname{Im}\overline{\xi} \cap \operatorname{Im}\overline{\phi}_1 = \overline{\xi}(U) \cap \operatorname{Im}\overline{\phi}_1 = 0$ by (A.4) and $\overline{\phi}_1$ is injective, Lemma A.6 implies that $\overline{\phi}_1 + \overline{\xi}$ remains injective. To finish, we need to show that $\overline{\psi}_\ell + \overline{h}$ has rank at least $d + 1$ modulo $\operatorname{Im}(\overline{\phi}_1 + \overline{\xi})$. By our choice of $\vec{v} \in K$ above, we have that $\overline{h}(\vec{v}) \in \operatorname{Im}\overline{\phi}_1 \setminus \overline{\phi}_1(K)$. Put $T = U + k\vec{v}$, which has dimension $d + 1$. Suppose we have $\vec{u} \in U$ and $\lambda \in K$ with

$$(\overline{\psi}_\ell + \overline{h})(\vec{u} + \lambda\vec{v}) \in \operatorname{Im}(\overline{\phi}_1 + \overline{\xi}).$$

As $V = U + K$, suppose $\vec{u}' \in U$ and $\vec{w} \in K$ with

$$(\overline{\psi}_\ell + \overline{h})(\vec{u} + \lambda\vec{v}) = (\overline{\phi}_1 + \overline{\xi})(\vec{u}' + \vec{w}).$$

Then, as $\vec{v}, \vec{w} \in K = \ker\overline{\xi}$ and $H(K) \subseteq \operatorname{Im}\Phi$, we see

$$\overline{\xi}(\vec{u}') = \overline{\psi}_\ell(\vec{u}) + \overline{h}(\vec{u}) + \overline{\psi}_\ell(\lambda\vec{v}) + \overline{h}(\lambda\vec{v}) - \overline{\phi}_1(\vec{u}' + \vec{w}) \in (\overline{\psi}_\ell(U) + \overline{h}(U) + \operatorname{Im}\overline{\phi}_1),$$

and it follows that $\overline{\xi}(\vec{u}') = 0$ by (A.4), and also $\vec{u}' = 0$ because $\overline{\xi}|_U$ is an injection. Rearranging once again, we have

$$\overline{\psi}_\ell(\vec{u}) = -\overline{h}(\vec{u}) - \overline{\psi}_\ell(\lambda\vec{v}) - \overline{h}(\lambda\vec{v}) + \overline{\phi}_1(\vec{w}) \in \overline{h}(U) + \operatorname{Im}\overline{\phi}_1 = Z,$$

and it follows that $\overline{\psi}_\ell(\vec{u}) = 0$ by (A.3), and also $\vec{u} = 0$ because $\overline{\psi}_\ell|_U$ is an injection. Using that $\vec{w} \in K$ and $\psi_\ell(K) \subseteq (\sum_{i=1}^n \phi_i(K))$, this leaves

$$\overline{h}(\lambda\vec{v}) = \overline{\phi}_1(\vec{w}) - \overline{\psi}_\ell(\lambda\vec{v}) \in \overline{\phi}_1(K),$$

which is only possible if $\lambda = 0$ as $h(\vec{v}) \notin \overline{\phi}_1(K)$. Putting all of this together, we conclude that, given $\vec{u} \in U$ and $\lambda \in k$, we have $(\overline{\psi}_\ell + \overline{h})(\vec{u} + \lambda\vec{v}) \in \operatorname{Im}(\overline{\phi}_1 + \overline{\xi})$ only when $\vec{u} = 0$ and $\lambda = 0$. It follows that $\overline{\psi}_\ell + \overline{h}$ restricts to an injection on T which persists modulo $\operatorname{Im}(\overline{\phi}_1 + \overline{\xi})$, which concludes the proof. □

Corollary A.10. *Let V and W be finite-dimensional vector spaces over a field k, and H a subspace of* $\operatorname{Hom}_k(V, W)$. *Suppose $n \geq 0$ and $1 \leq d \leq \dim V$ are integers, and assume the following conditions are satisfied.*

(a) *There exist $\phi_1, \ldots, \phi_n \in H$ giving an injection $(\phi_1, \ldots, \phi_n)\colon \bigoplus_{i=1}^n V \to W$.*

(b) *We have*

$$\dim H(U) \geq n \dim U + 1 + \sum_{i=1}^{d-1} i(\dim V - i) \dim V$$

for any non-zero subspace $0 \neq U \subseteq V$.

Then there are maps $\widetilde{\phi}_1, \dots, \widetilde{\phi}_n, \psi \in H$ *so that* $\widetilde{\Phi} = (\widetilde{\phi}_1, \dots, \widetilde{\phi}_n) \colon \bigoplus_{i=1}^n V \to W$ *is an injection and* ψ *has rank at least* d *modulo* $\operatorname{Im} \widetilde{\Phi}$*, i.e.,*

$$\dim\left(\operatorname{Im} \psi + \sum_{i=1}^n \widetilde{\phi}_i\right) \geq d + \dim\left(\sum_{i=1}^n \widetilde{\phi}_i\right).$$

Proof. If $d = 1$, we have $\dim(H(V)) \geq n \dim V + 1 = n \dim(\sum_{i=1}^n \operatorname{Im} \phi_i)$. Taking $\psi \in H$ with $\psi(V) \not\subseteq \sum_{i=1}^n \operatorname{Im} \phi_i$, we see that $\widetilde{\phi}_1 = \phi_1, \dots, \widetilde{\phi}_n = \phi_n, \psi \in H$ give a suitable collection of maps. Proceeding inductively, assume now the conclusion holds for some $d \geq 1$. Suppose we have finite-dimensional vector spaces V, W with $\dim(V) \leq d$ admitting n simultaneous injections from V to W in $H \subseteq \operatorname{Hom}_k(V, W)$ and so that

$$\dim(H(U)) \geq n \dim(U) + 1 + \sum_{i=1}^d i(\dim V - i) \dim V \tag{A.5}$$

for any non-zero subspace $0 \neq U \subseteq V$. We need to find a map in H with rank at least $d + 1$ modulo the image of some n potentially different simultaneous injections from V to W in H.

By our induction assumption, there are $\phi_1^{(1)}, \dots, \phi_n^{(1)}, \psi_1 \in H$ with

$$\dim\left(\sum_{i=1}^n \operatorname{Im} \phi_i^{(1)} + \operatorname{Im} \psi_1\right) \geq \dim\left(\sum_{i=1}^n \operatorname{Im} \phi_i^{(1)}\right) + d = n \dim(V) + d.$$

We proceed to define $\psi_1, \dots, \psi_\ell \in H$ and $\phi_1^{(\ell)}, \dots, \phi_n^{(\ell)} \in H$ recursively until either the desired conclusion is satisfied or we reach $\ell = (\dim V - d) \dim V + 1$. Assume we have

$$\psi_1, \dots, \psi_{\ell-1} \in H \quad \text{and} \quad \phi_1^{(\ell-1)}, \dots, \phi_n^{(\ell-1)} \in H$$

so that $\dim(\sum_{i=1}^n \operatorname{Im} \phi_i^{(\ell-1)}) = n \dim(V)$ and

$$\dim\left(\sum_{i=1}^n \operatorname{Im} \phi_i^{(\ell-1)} + \sum_{j=1}^{\ell'} \operatorname{Im} \psi_j\right) \geq \dim\left(\sum_{i=1}^n \operatorname{Im} \phi_i^{(\ell-1)} + \sum_{j=1}^{\ell'-1} \operatorname{Im} \psi_j\right) + d$$

for all $1 \leq \ell' < \ell$. In particular, we also have

$$\dim\left(\sum_{i=1}^n \operatorname{Im} \phi_i^{(\ell-1)} + \operatorname{Im} \psi_{\ell'}\right) \geq \dim\left(\sum_{i=1}^n \operatorname{Im} \phi_i^{(\ell-1)}\right) + d$$

for any $1 \leq \ell' < \ell$. If this inequality is ever strict, we are done as

$$\widetilde{\phi}_1 = \phi_1^{(\ell-1)}, \dots, \widetilde{\phi}_n = \phi_n^{(\ell-1)}, \psi = \psi_{\ell'} \in H$$

give the desired maps, so we shall assume we have equality for all $1 \leq \ell' < \ell$. Moreover, if we have

$$(\psi_{\ell'}^{(\ell-1)})^{-1}\left(\sum_{i=1}^n \operatorname{Im} \phi_i^{(\ell-1)}\right) \neq (\psi_{\ell''}^{(\ell-1)})^{-1}\left(\sum_{i=1}^n \operatorname{Im} \phi_i^{(\ell-1)}\right)$$

for some $1 \leq \ell'' < \ell' < \ell$, Lemma A.6 gives that the rank of $\psi_{\ell'} + \psi_{\ell''}$ is at least $d + 1$ modulo $\sum_{i=1}^n \operatorname{Im} \phi_i^{(\ell-1)}$, and again, we are done with

$$\widetilde{\phi}_1 = \phi_1^{(\ell-1)}, \dots, \widetilde{\phi}_n = \phi_n^{(\ell-1)}, \psi = \psi_{\ell'} + \psi_{\ell''} \in H,$$

giving the desired maps. Thus we may assume

$$K_\ell = (\psi_{\ell'}^{(\ell-1)})^{-1}\Big(\sum_{i=1}^{n} \operatorname{Im}\phi_i^{(\ell-1)}\Big)$$

is independent of $1 \le \ell' < \ell$. Picking U_ℓ to be a complement of K_ℓ, we have that U_ℓ has dimension d. Let

$$\bar{W} = W/\Big(\sum_{j=1}^{\ell-1} \psi_j(U_\ell)\Big),$$

and for any $\phi \in \operatorname{Hom}_k(V, W)$, we shall denote by $\bar{\phi} \in \operatorname{Hom}_k(V, \bar{W})$ the map $V \xrightarrow{\phi} W \to \bar{W}$ induced by quotienting out by $\sum_{j=1}^{\ell-1} \psi_j(U_\ell)$. Note that $\psi_{\ell'}(U_\ell) = \operatorname{Im}\psi_{\ell'}$ modulo either

$$\sum_{i=1}^{n} \operatorname{Im}\phi_i^{(\ell-1)} \quad \text{or} \quad \sum_{i=1}^{n} \operatorname{Im}\phi_i^{(\ell-1)} + \sum_{j=1}^{\ell'-1} \operatorname{Im}\psi_j$$

for $1 \le \ell' < \ell$ so that

$$\dim\Big(\sum_{j=1}^{\ell-1} \psi_j(U_\ell)\Big) = (\ell-1)d.$$

Consider also $\bar{H} = \{\bar{\phi} \mid \phi \in H\} \subseteq \operatorname{Hom}_k(V, \bar{W})$. We have that $\bar{\phi}_1^{(\ell-1)}, \dots, \bar{\phi}_n^{(\ell-1)}$ give n simultaneous injections from V to $\bar{W}$ in $\bar{H}$, and for any subspace $0 \ne U \subseteq V$, we compute

$$\begin{aligned}
\dim \bar{H}(U) &\ge \dim H(U) - \dim\Big(\sum_{i=1}^{\ell-1} \psi_i(U_\ell)\Big) \\
&\ge n\dim(U) + 1 + \sum_{i=1}^{d} i(\dim V - i)\dim V - (\ell-1)d \\
&\ge n\dim(U) + 1 + \sum_{i=1}^{d-1} i(\dim V - i)\dim V + ((\dim V - d)\dim V + 1 - \ell)d \\
&\ge n\dim(U) + 1 + \sum_{i=1}^{d-1} i(\dim V - i)\dim V
\end{aligned}$$

since $\ell \le (\dim V - d)\dim V + 1$. Thus, by our induction assumption, there are maps

$$\phi_1^{(\ell)}, \dots, \phi_n^{(\ell)}, \psi_\ell \in H$$

so that

$$\dim\Big(\sum_{i=1}^{n} \operatorname{Im}\bar{\phi}_i^{(\ell)} + \operatorname{Im}\bar{\psi}_\ell\Big) \ge \dim\Big(\sum_{i=1}^{n} \operatorname{Im}\bar{\phi}_i^{(\ell)}\Big) + d,$$

$$\dim\Big(\sum_{i=1}^{n} \operatorname{Im}\bar{\phi}_i^{(\ell)}\Big) = n\dim(V).$$

In particular, it follows that

$$\dim\Big(\sum_{i=1}^{n} \operatorname{Im}\phi_i^{(\ell)}\Big) = n\dim(V).$$

If the rank of any $\psi_{\ell'}$ modulo $\sum_{i=1}^{n} \operatorname{Im}\phi_i^{(\ell)}$ is at least $d+1$ for some $1 \le \ell' < \ell$, we are again done, so we may assume this rank is at most d. As $\dim(\sum_{i=1}^{n} \operatorname{Im}\phi_i^{(\ell)})$ does not change modulo $\sum_{j=1}^{\ell-1} \psi_j(U_\ell)$, we similarly must have that

$$\dim\Bigl(\sum_{j=1}^{\ell-1} \psi_j(U_\ell)\Bigr) = (l-1)d$$

does not change modulo $\sum_{i=1}^{n} \operatorname{Im}\phi_i^{(\ell)}$. Thus

$$(\psi_1, \dots, \psi_{\ell-1})\colon \bigoplus_{j=1}^{\ell-1} U_\ell \to W$$

is injective and remains so after going modulo $\sum_{i=1}^{n} \operatorname{Im}\phi_i^{(\ell)}$. Moreover, we must have that $\psi_{\ell'}(U_\ell) = \operatorname{Im}\psi_{\ell'}$ modulo either

$$\sum_{i=1}^{n} \operatorname{Im}\phi_i^{(\ell)} \quad \text{or} \quad \sum_{i=1}^{n} \operatorname{Im}\phi_i^{(\ell)} + \sum_{j=1}^{\ell'-1} \operatorname{Im}\psi_j$$

for $1 \le \ell' < \ell$, and also,

$$\begin{aligned}
\dim\Bigl(\sum_{i=1}^{n} \operatorname{Im}\phi_i^{(\ell)} + \sum_{j=1}^{\ell} \operatorname{Im}\psi_j\Bigr) &= \dim\Bigl(\sum_{i=1}^{n} \operatorname{Im}\phi_i^{(\ell)} + \sum_{j=1}^{\ell-1} \psi_j(U_\ell) + \operatorname{Im}\psi_\ell\Bigr) \\
&= \dim\Bigl(\sum_{i=1}^{n} \operatorname{Im}\bar{\phi}_i^{(\ell)} + \operatorname{Im}\bar{\psi}_\ell\Bigr) \\
&\ge \dim\Bigl(\sum_{i=1}^{n} \operatorname{Im}\bar{\phi}_i^{(\ell)}\Bigr) + d \\
&= \dim\Bigl(\sum_{i=1}^{n} \operatorname{Im}\phi_i^{(\ell)} + \sum_{j=1}^{\ell-1} \psi_j(U_\ell)\Bigr) + d \\
&= \dim\Bigl(\sum_{i=1}^{n} \operatorname{Im}\phi_i^{(\ell)} + \sum_{j=1}^{\ell-1} \operatorname{Im}\psi_j\Bigr) + d.
\end{aligned}$$

This completes our recursive construction, as it now follows $\dim(\sum_{i=1}^{n} \operatorname{Im}\phi_i^{(\ell)}) = n\dim(V)$ and

$$\dim\Bigl(\sum_{i=1}^{n} \operatorname{Im}\phi_i^{(\ell)} + \sum_{j=1}^{\ell'} \operatorname{Im}\psi_j\Bigr) \ge \dim\Bigl(\sum_{i=1}^{n} \operatorname{Im}\phi_i^{(\ell)} + \sum_{j=1}^{\ell'} \operatorname{Im}\psi_j\Bigr) + d$$

for all $1 \le \ell' \le \ell$.

To finish the proof, we need only address the remaining case where the recursion above proceeded all the way to $\ell - (\dim(V) - d)\dim(V) + 1$. However, the desired conclusion now follows from Theorem A.9 because conditions (a) and (c) from Theorem A.9 are satisfied by $\phi_1^{(\ell)}, \dots, \phi_n^{(\ell)}$ and $\psi_1, \dots, \psi_\ell$, and condition (b) is immediate from (A.5). □

Corollary A.11. *Let k be an arbitrary field, let V, W be finite-dimensional vector spaces over k, and let H be a subspace of $\operatorname{Hom}_k(V, W)$. Suppose that, for some $n \geq 0$ and any $0 \neq U \subseteq V$, we have*

$$\dim H(U) \geq (n-1)\dim U + 1 + \sum_{i=1}^{\dim V - 1} i(\dim V - i)\dim V.$$

Then there is an injection $\bigoplus^n V \to W$ where all components are in H.

Proof. We use induction on n, noting first that the base case $n = 0$ is trivially satisfied. Assume now that the statement holds for some $n \geq 0$ and we have finite-dimensional k-vector spaces V, W and $H \subseteq \operatorname{Hom}_k(V, W)$ with

$$\dim H(U) \geq n \dim U + 1 + \sum_{i=1}^{\dim V - 1} i(\dim V - i)\dim V.$$

The induction hypothesis implies there exist $\phi_1, \ldots, \phi_n \in H$ so that

$$(\phi_1, \ldots, \phi_n) \colon \bigoplus^n V \to W$$

is an injection. Applying Corollary A.10, we have $\psi \in H$ with full rank modulo the image of the simultaneous injections $\widetilde{\phi}_1, \ldots, \widetilde{\phi}_n \in H$, giving an injection

$$(\widetilde{\phi}_1, \ldots, \widetilde{\phi}_n, \psi) \colon \bigoplus^{n+1} V \to W$$

with all components in H as desired. □

Proof of Theorem A.2. Combining (A.1) and Corollary A.11, we must have

$$n\dim(U) \leq \dim H(U) \leq n\dim(U) + \sum_{i=1}^{\dim V - 1} i(\dim V - i)\dim V$$

for all subspaces $0 \neq U \subseteq V$. Dividing through by $\dim(U)$, it follows the constant

$$C := \sum_{i=1}^{\dim V - 1} i(\dim V - i)\dim V = \frac{1}{6}(\dim(V))^2((\dim(V))^2 - 1)$$

satisfies (A.2). □

To conclude this section, we exhibit a dual formulation of the above results that is tailored towards our desired applications.

Corollary A.12. *The polynomial $P(T) = \frac{1}{6}T^2(T^2 - 1) \in \mathbb{Q}[T]$ is an increasing function on positive integers with the following property: for any integer $n \geq 1$, any field k, all finite-dimensional vector spaces X, Y over k, and all subspaces $H \subseteq \operatorname{Hom}_k(X, Y)$ so that*

$$\dim\Big(X/\Big(\bigcap_{h \in H} \ker(\pi_Z \circ h)\Big)\Big) \geq (n + P(\dim Y))\dim Z,$$

for all nontrivial quotients $\pi_Z \colon Y \to Z \neq 0$, there exists a surjection $X \to \bigoplus^n Y$ with all components in H.

Proof. Let $h_1, \dots, h_\ell$ be a basis of H. For any surjection $\pi_Z: Y \to Z \neq 0$ of vector spaces, observe first that

$$\bigcap_{h \in H} \ker(\pi_Z \circ h) = \bigcap_{i=1}^{\ell} \ker(\pi_Z \circ h_i).$$

Writing Φ_Z for the composition

$$X \xrightarrow{(h_1,\dots,h_\ell)} \bigoplus^{\ell} Y \xrightarrow{\oplus \pi_Z} \bigoplus^{\ell} Z,$$

our assumptions give rank $\Phi_Z \geq (n + P(\dim Y)) \dim Z$ for all nontrivial quotients Z of Y.

We let $(\_)^* = \operatorname{Hom}_k(\_, k)$ and use duality for finite-dimensional vector spaces over k. Put $H^* = \{\phi^* \mid \phi \in H\} \subseteq \operatorname{Hom}_k(Y^*, X^*)$. For every nontrivial subspace $0 \neq U \subseteq Y^*$, the rank of

$$\bigoplus^{\ell} U \longrightarrow \bigoplus^{\ell} Y^* \xrightarrow{(h_1^*,\dots,h_\ell^*)} X^*$$

equals the rank of Φ_{U^*} and so is at least $(n + P(\dim Y)) \dim U$. It follows that

$$\dim H^*(U) \geq (n + P(\dim Y)) \dim U \quad \text{for all } 0 \neq U \subseteq Y^*.$$

Applying Theorem A.2, we have that there exists an injection

$$\bigoplus^{n} Y^* \xrightarrow{(\phi_1^*,\dots,\phi_n^*)} X^* \quad \text{for some } \phi_1^*, \dots, \phi_n^* \in H^*.$$

Dualizing yields a surjection

$$X \xrightarrow{(\phi_1,\dots,\phi_n)} \bigoplus^{n} Y \quad \text{with } \phi_1, \dots, \phi_n \in H$$

as desired. □

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