18

On F-Pure Inversion of Adjunction

Thomas Polstra, Austyn Simpson, and Kevin Tucker

Abstract

We analyze adjunction and inversion of adjunction for the F-purity of divisor pairs in characteristic $p > 0$. In this vein, we give a complete answer for principal divisors under $\mathbb{Q}$-Gorenstein assumptions but without divisibility restrictions on the index. We also give a detailed analysis relating the F-purity of the pairs $(R, \Delta + D)$ and that of $(R_D, \mathrm{Diff}_D(\Delta))$ motivated by Kawakita's log canonical inversion of adjunction via reduction to prime characteristic.

18.1 Introduction

Let $(R, \mathfrak{m}, k)$ be a local ring of prime characteristic $p > 0$ and $R \to F_*R$ be the Frobenius map, where F_*R denotes the Frobenius push-forward of R. For simplicity, assume that R is complete or F-finite (i.e. F_*R is a finite R-module). The ring R is said to be *F-pure* if $R \to F_*R$ splits. Let $x \in \mathfrak{m}$ be a nonzerodivisor of R. Under suitable hypotheses, Kawakita's breakthrough result on the inversion of adjunction of log canonical singularities [11] (when viewed through the lens of reduction to prime characteristic via [6]) predicts that the following are equivalent:

(1) The map $R \xrightarrow{\cdot F_* x^{p-1}} F_*R$ is split, *that is*, *the pair* $(R, \mathrm{div}_R(x))$ *is F-pure*;
(2) R/xR is F-pure.

Polstra was supported by NSF Grant DMS #2101890 and by a grant from the Simons Foundation, Grant Number 814268, MSRI.
Simpson was supported by NSF postdoctoral fellowship DMS #2202890.
Tucker was supported in part by NSF Grant DMS #2200716.
Acknowledgments We thank Mircea Mustaţă for suggesting the use of Manivel's trick in Section 18.5. We are also grateful to the anonymous referee for helpful comments on a previous draft.

https://doi.org/10.1017/9781009396233.019 Published online by Cambridge University Press

The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) are special cases of adjunction and inversion of adjunction of F-purity, respectively. While the forward direction requires no additional hypotheses, note that the converse (2) $\Rightarrow$ (1) is known to fail if R is not $\mathbb{Q}$-Gorenstein in light of counterexamples by Fedder [5] and Singh [22]. More generally, we may ask for the equivalence of (1) and (2) after incorporating an effective $\mathbb{Q}$-divisor Δ. Even after imposing $\mathbb{Q}$-Cartier assumptions on $K_R + \Delta$, however, the presence of p-torsion introduces additional subtleties that often require new methods to overcome.

The first contribution of this chapter is the following positive solution to F-pure inversion of adjunction along principal ideals for log $\mathbb{Q}$-Gorenstein pairs (R, Δ), provided the denominators of the coefficients of Δ are prime to p (but without any divisibility restrictions on the index of $K_R + \Delta$).

Theorem A (Theorem 18.2.7) *Let $(R, \mathfrak{m}, k)$ be an excellent local ring of prime characteristic $p > 0$ and $x \in \mathfrak{m}$ a nonzerodivisor such that R/xR is (G_1) and satisfies Serre's condition (S_2). Let $\Delta \geq 0$ be an effective $\mathbb{Q}$-divisor of R with components disjoint from $\operatorname{div}_R(x)$ such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier and $(p^e - 1)\Delta$ is integral for all $e \gg 0$ and divisible. Then the pair $(R, \Delta + \operatorname{div}_R(x))$ is F-pure if and only if $(R/xR, \Delta|_{\operatorname{div}(x)})$ is F-pure.*

Note that Theorem A applied to the case that R is a regular ring and $\Delta = 0$ is Fedder's criterion for a hypersurface to be F-pure [5, Theorem 1.12]. Moreover, Theorem A is a significant improvement over previous work by Polstra and Simpson [18, Theorem A], which affirmatively settled the weaker question of whether F-purity deforms in $\mathbb{Q}$-Gorenstein rings. When $\Delta = 0$, Theorem A may be derived from [18] using a trick commonly attributed to Manivel [14] together with an understanding of the behavior of F-purity under separable finite covers; for completeness, we give a detailed proof of this in Section 18.5. However, the cyclic cover techniques employed in [18] do not seem to lend themselves to the incorporation of a boundary in a straightforward manner. Our proof of Theorem A both circumvents these difficulties and recovers the main theorem of [18] while showing a more general result.

More generally, in Sections 18.3 and 18.4 we turn our attention to F-pure adjunction and inversion of adjunction beyond the case of principal ideals, that is, along a divisor D with components disjoint from those of the boundary divisor Δ. Specifically, we aim to relate the F-purity of $(R, \Delta + D)$ with that of $(R_D, \operatorname{Diff}_D(\Delta))$ where $\operatorname{Diff}_D(\Delta)$ is Shokurov's *different*. Our analysis yields the following results.

Theorem B (Adjunction of F-purity – Theorem 18.3.5) *Let* $(R, \mathfrak{m}, k)$ *be an excellent* (S_2) *and* (G_1) *local ring of prime characteristic* $p > 0$ *and let* K_R *be a choice of canonical divisor of* $\operatorname{Spec}(R)$*. Suppose that* $D \geq 0$ *is an effective integral* (S_2) *and* (G_1) *divisor, and let* $\Delta \geq 0$ *be an effective* $\mathbb{Q}$*-divisor on* $\operatorname{Spec} R$ *whose components are disjoint from* D *and such that* $(p^e - 1)\Delta$ *is integral for all sufficiently divisible* $e \gg 0$*. Suppose that* $K_R + D + \Delta$ *is* $\mathbb{Q}$*-Cartier. If* $(R, \Delta + D)$ *is F-pure, then* $(R_D, \operatorname{Diff}_D(\Delta))$ *is F-pure.*

Theorem C (Inversion of Adjunction of F-purity along a $\mathbb{Q}$-Cartier divisor – Corollary 18.4.2) *With the notation and assumptions of Theorem B, suppose further that*

(I) D *is* $\mathbb{Q}$*-Cartier;*
(II) for each $\mathbb{Q}$*-Cartier divisor* E *and* $\mathfrak{p} \in D \subseteq \operatorname{Spec}(R)$*, we have the inequality*

$$\operatorname{depth}(R(E)_{\mathfrak{p}}) \geq \min\{\operatorname{ht}(\mathfrak{p}), 3\}.$$

If the pair $(R_D, \operatorname{Diff}_D(\Delta))$ *is F-pure, then the pair* $(R, \Delta + D)$ *is F-pure.*

We remark that assumption (C) is necessary in Theorem C whenever D is $\mathbb{Q}$-Cartier by Lemma 18.3.4. More generally, we provide a characterization for when inversion of adjunction of an F-pure pair is satisfied in the absence of properties (C) and (C). By Corollary 18.3.3 and Lemma 18.4.1, if $(R_D, \operatorname{Diff}_D(\Delta))$ is F-pure, then $(R, \Delta + D)$ is F-pure if and only if for each p-power torsion divisor E of R, the R_D-module $R(E)/R(E - D)$ is (S_2).

A particularly notable feature of our analysis is the lack of index restrictions. Prior work by Schwede has shown Theorems B and C under the assumptions that $K_R + D + \Delta$ is $\mathbb{Q}$-Cartier of index not divisible by p and D is Cartier in codimension 2 [20]. In particular, Theorem A was solved by Schwede under the hypothesis that R is $\mathbb{Q}$-Gorenstein of index not divisible by p. Note also that the strongly F-regular version of inversion of adjunction was settled by Das in [3]. We have also strived throughout to avoid unnecessary F-finite restrictions.

Those familiar with Kawakita's theorem on log canonical inversion of adjunction may be surprised by the assumption that D is (S_2) and (G_1), which is slightly weaker than assuming D is normal and which is not a necessity in characteristic 0. Indeed, the content of [11] is that in equal characteristic 0, a pair $(R, \Delta + D)$ is log canonical if and only if $(R_{D^N}, \operatorname{Diff}_{D^N}(\Delta))$ is log canonical where D^N denotes the normalization of D. However, as with assumption (C) in the case that D is $\mathbb{Q}$-Cartier, our assumption is a necessity in prime characteristic. There are simple counterexamples to Theorem C if we consider the F-singularities of D^N instead. Indeed, let $R = \mathbb{F}_2[x, y, z]$ and $D = \operatorname{div}_R(x^2 - y^2 z)$. Then the normalization of $R_D = \mathbb{F}_2[x, y, z]/(x^2 - y^2 z)$ is regular with $\mathrm{Diff}_{D^N}(0)$ being a smooth divisor so that $(R_{D^N}, \mathrm{Diff}_{D^N}(0))$ is F-pure, but the pair $(\mathbb{F}_2[x,y,z], \mathrm{div}_R(x^2 - y^2 z))$ is not F-pure by Fedder's criterion (see also Theorem A). See [20, Example 8.4] for additional details, as well as the work of Miller and Schwede [16] for an analysis of the behavior of F-purity via the normalization map.

18.2 The $\mathbb{Q}$-Gorenstein Case

Consider a local ring $(R, \mathfrak{m}, k)$. Let $E_R(k)$ be an injective hull of k, and suppose that $R \to M$ is a map of R-modules. According to [10, Lemma 2.1(e)], $R \to M$ is pure if and only if the induced map $E_R(k) \to E_R(k) \otimes_R M$ is injective, a fact that we will use repeatedly. The main goal of this section is to give a proof of Theorem A. However, we first present a simple proof of F-pure (principal) inversion of adjunction when R is a Gorenstein ring since it will guide our subsequent investigations.

Example 18.2.1 (Inversion of Adjunction of F-purity, the Gorenstein case) Let $(R, \mathfrak{m}, k)$ be a local d-dimensional Gorenstein ring of prime characteristic $p > 0$, and $x \in R$ a nonzerodivisor. Let $\Delta \geq 0$ be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor with components disjoint from $\mathrm{div}_R(x)$ such that $(p^e - 1)\Delta$ is integral for all sufficiently divisible $e \gg 0$. We will show that the following are equivalent:

- $(R, \Delta + \mathrm{div}_R(x))$ is F-pure;
- $(R/xR, \Delta|_{\mathrm{div}_R(x)})$ is F-pure.

Recall that the pair $(R, \Delta + \mathrm{div}_R(x))$ is F-pure if and only if

$$R \to F^e_* R((p^e - 1)(\Delta + \mathrm{div}_R(x)))$$

is pure. There is a commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & F^e_* R((p^e - 1)(\Delta + \mathrm{div}_R(x))) \\ \Big\downarrow {=} & & \cong \Big\downarrow \cdot F^e_* x^{p^e-1} \\ R & \xrightarrow{\cdot F^e_* x^{p^e-1}} & F^e_* R((p^e - 1)\Delta). \end{array}$$

Similarly, $(R/xR, \Delta|_{\mathrm{div}_R(x)})$ is F-pure if and only if

$$R/xR \to (F^e_*(R/xR))((p^e - 1)\Delta|_{\mathrm{div}_R(x)})$$

is pure. To summarize, we aim to show that the following are equivalent:

- $R \xrightarrow{\cdot F^e_* x^{p^e-1}} F^e_* R((p^e - 1)\Delta)$ is pure;
- $R/xR \to F^e_* R/xR((p^e - 1)\Delta|_{\mathrm{div}_R(x)})$ is pure.

For each $e > 0$, let $\Theta_e = (p^e - 1)\Delta$. There exists e_0 so that for sufficiently divisible $e \gg 0$, Θ_e is a p^{e_0}-torsion integral divisor. Tensoring the map $R \to F^{e_0}_* R$ with $R(\Theta_e)$ and reflexifying over R give the map

$$R(\Theta_e) \to F^{e_0}_* R(p^{e_0}\Theta_e) \cong F^{e_0}_* R.$$

By [5], R is necessarily F-pure, and the F-purity of R then gives that the map $R(\Theta_e) \to F^{e_0}_* R$ is pure. In particular, any regular sequence on $F^{e_0}_* R$ is a regular sequence on $R(\Theta_e)$ and therefore $R(\Theta_e)$ is a Cohen–Macaulay R-module.[1] Even further, $R(\Theta_e)/xR(\Theta_e)$ is a Cohen–Macaulay R/xR-module; as R/xR is (S_2) and (G_1), we may therefore conclude that the R/xR-reflexification map is an isomorphism:

$$R(\Theta_e)/xR(\Theta_e) \xrightarrow{\cong} R/xR(\Theta_e|_{\mathrm{div}_R(x)}) = R/xR((p^e - 1)\Delta|_{\mathrm{div}_R(x)}).$$

Consider now the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^e_* R(\Theta_e) & \xrightarrow{\cdot F^e_* x} & F^e_* R(\Theta_e) & \longrightarrow & F^e_* R/xR(\Theta_e|_{\mathrm{div}_R(x)}) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \cdot F^e_* x^{p^e-1}} & & \uparrow{\scriptstyle F^e} & & \uparrow{\scriptstyle F^e} & & \\
0 & \longrightarrow & R & \xrightarrow{\cdot x} & R & \longrightarrow & R/xR & \longrightarrow & 0.
\end{array}$$

Recall that we wish to show the leftmost vertical map is pure if and only if the rightmost vertical map is pure. The modules R and $F^e_* R(\Theta_e)$ are Cohen–Macaulay. Therefore, all of the lower local cohomology modules of R and $F^e_* R(\Theta_e)$ vanish and we have the commutative diagram of local cohomology modules whose horizontal arrows are injective:

$$\begin{array}{ccc}
F^e_* H^{d-1}_{\mathfrak{m}}(R/xR(\Theta_e|_{\mathrm{div}_R(x)})) & \xrightarrow{\subseteq} & F^e_* H^d_{\mathfrak{m}}(R(\Theta_e)) \\
\uparrow{\scriptstyle F^e} & & \uparrow{\scriptstyle \cdot F^e_* x^{p^e-1}} \\
H^{d-1}_{\mathfrak{m}}(R/xR) & \xrightarrow{\subseteq} & H^d_{\mathfrak{m}}(R).
\end{array} \tag{18.1}$$

The local cohomology modules $H^{d-1}_{\mathfrak{m}}(R/xR) \cong (0 :_{H^d_{\mathfrak{m}}(R)} x) \cong E_{R/xR}(k)$ and $H^d_{\mathfrak{m}}(R) \cong E_R(k)$ are essential extensions of $k \cong (0 :_{H^d_{\mathfrak{m}}(R)} \mathfrak{m})$. Therefore, the left vertical map is injective if and only if the right vertical map is injective. We observe the following isomorphisms of the vertical maps in (18.1):

$$\begin{aligned}
&\Big(H^{d-1}_{\mathfrak{m}}(R/xR) \xrightarrow{F^e} F^e_* H^{d-1}(R/xR(\Theta_e|_{\mathrm{div}_R(x)}))\Big) \\
&\cong \big(R/xR \to F^e_*(R/xR(\Theta_e|_{\mathrm{div}_R(x)}))\big) \otimes_{R/xR} H^{d-1}_{\mathfrak{m}}(R/xR)
\end{aligned}$$

[1] Alternatively, a pure map splits after completion, and therefore one can verify the Cohen–Macaulay condition by examining induced maps of local cohomology modules.

https://doi.org/10.1017/9781009396233.019 Published online by Cambridge University Press

and

$$\left(H^d_{\mathfrak{m}}(R) \xrightarrow{\cdot F^e_* x^{p^e-1}} F^e_* H^d_{\mathfrak{m}}(R(\Theta_e))\right) \cong \left(R \xrightarrow{\cdot F^e_* x^{p^e-1}} F^e_* R(\Theta_e)\right) \otimes_R H^d_{\mathfrak{m}}(R).$$

Therefore, $R \xrightarrow{\cdot F^e_* x^{p^e-1}} F^e_* R(\Theta_e)$ is pure if and only if

$$R/xR \to F^e_*(R/xR(\Theta_e|_{\mathrm{div}_R(x)}))$$

is pure, as desired.

We briefly review the definition that we will use in this chapter of a $\mathbb{Q}$-Gorenstein ring and of a generalized divisor, as outside of the normal setting these notions may be somewhat unfamiliar. Let $(R, \mathfrak{m}, k)$ be an excellent equidimensional local ring satisfying conditions (S_2) and (G_1). We utilize the language of (generalized) divisors as introduced by Hartshorne in [7]. In particular, in this chapter, when we speak of a divisor Δ on R, or equivalently a Weil divisor Δ on R, we shall mean an *almost Cartier divisor* in the language of [7]. Explicitly, this means that Δ is represented by a finitely generated R-submodule $I \subseteq \mathrm{Tot}(R)$ of the total quotient ring of R (i.e. a fractional ideal) such that:

(1) $I_{\mathfrak{p}} = \mathrm{Tot}(R)_{\mathfrak{p}}$ for all minimal primes $\mathfrak{p} \subseteq R$;
(2) I is reflexive, that is, $I \to \mathrm{Hom}_R(\mathrm{Hom}_R(I, R), R)$ is an isomorphism.

A generalized divisor corresponding to a fractional ideal I is *effective* if $I \subseteq R$. Note that if R is normal, this notion of divisors corresponds to the usual one. We refer the reader to [7, section 2] or [13, Appendix A] for more details.

We assume that R has a canonical divisor K_R, that is, K_R is the class of divisor so that the corresponding fractional ideal $R(K_R)$ is a canonical module of R. We say that R is $\mathbb{Q}$-*Gorenstein* if there exists an integer $n > 0$ so that $nK_R \sim 0$. The smallest positive integer n with this property is the *index* of R. If R is $\mathbb{Q}$-Gorenstein and $x \in R$ is a nonzerodivisor so that R/xR is (S_2) and (G_1), then R/xR is also $\mathbb{Q}$-Gorenstein by [18, Proposition 2.6]. If $x \in R$ is such an element and if M is an R-module, then we write $M_{V(x)}$ to denote the R-module M/xM. If E is a divisor of R, then $R(E)$ is the corresponding fractional ideal. If E has components disjoint from $V(x)$, then $E|_{V(x)}$ denotes the class of the restricted divisor along $V(x)$. Let K_R be a choice of canonical divisor of $\mathrm{Spec}(R)$. By prime avoidance, we may assume that K_R has components disjoint from $V(x)$. Then the restricted divisor $K_{R/xR} := (K_R)|_{V(x)}$ is a choice of canonical divisor of R/xR. If E is a divisor of R with components disjoint from $V(x)$, then $\mathrm{Hom}_{R_{V(x)}}(\mathrm{Hom}_{R_{V(x)}}(R(E)_{V(x)}, R_{V(x)}), R_{V(x)}) \cong R_{V(x)}(E|_{V(x)})$. Note that if

https://doi.org/10.1017/9781009396233.019 Published online by Cambridge University Press

M is an R-module and x is a nonzerodivisor on M, then there is a short exact sequence

$$0 \to M \xrightarrow{\cdot x} M \to M_{V(x)} \to 0.$$

In particular, there is a right exact sequence of local cohomology modules

$$H^{d-1}_{\mathfrak{m}}(M_{V(x)}) \to H^d_{\mathfrak{m}}(M) \xrightarrow{\cdot x} H^d_{\mathfrak{m}}(M) \to 0.$$

Lemma 18.2.6 is an instance where this right exact sequence enjoys the property of being short exact. We first record an elementary observation.

Lemma 18.2.2 *Let $(R, \mathfrak{m}, k)$ be an excellent local (S_2) and (G_1) F-pure ring of prime characteristic $p > 0$. Suppose that E is a divisor on R. Then $R(E) \to F^e_* R(p^e E)$ is a pure map. Moreover, if E is torsion of index p^e and if $x_1, \ldots, x_t$ is a regular sequence on R, then $x_1, \ldots, x_t$ is a regular sequence on $R(E)$.*

Proof Consider the pure map $R \to F^e_* R$. We claim that the composition of maps $R(E) \to F^e_* R \otimes_R R(E) \to F^e_* R(p^e E)$ is a pure map. The assumptions that R is F-pure, (G_1) and (S_2) are unchanged by completion (by [13, Corollary 2.3], [15, Theorem 18.3], and [23, Tag 0339], respectively, the latter using the excellence assumption on R), as are the assumptions on the divisor E. Hence, we may assume that R is complete, so there is a splitting of $R \to F^e_* R$. Applying $- \otimes_R R(E)$ to this splitting and reflexifying, we find that there is a splitting of $R(E) \to F^e_* R(p^e E)$. In particular, $R(E) \to F^e_* R(p^e E)$ is pure.

Now suppose that E is torsion of index p^e and $x_1, \ldots, x_t$ is a regular sequence on R. Then $R(E) \to F^e_* R(p^e E) \cong F^e_* R$. The sequence $x_1^{p^e}, \ldots, x_t^{p^e}$ is a regular sequence on R and therefore $x_1, \ldots, x_t$ is a regular sequence on $F^e_* R$. It follows that $x_1, \ldots, x_t$ is a regular sequence on $R(E)$ as well since we have an injection $R(E) \otimes_R R/(x_1, \ldots, x_i) \hookrightarrow F^e_* R \otimes_R R/(x_1, \ldots, x_i)$ for every i. □

Lemma 18.2.3 *Let $(R, \mathfrak{m}, k)$ be a d-dimensional excellent local ring and $\varphi : M \to N$ a map of R-modules for which φ is an isomorphism in codimension 1. Then $H^d_{\mathfrak{m}}(M) \to H^d_{\mathfrak{m}}(N)$ is an isomorphism.*

Proof There exists a four-term exact sequence

$$0 \to K \to M \xrightarrow{\varphi} N \to C \to 0$$

so that K and C are not supported in codimension 1. Therefore, $H^i_{\mathfrak{m}}(K) = H^i_{\mathfrak{m}}(C) = 0$ for $i = d, d-1$. Split this exact sequence into two short exact sequences

$$0 \to K \to M \to M/K \to 0$$

and

$$0 \to M/K \xrightarrow{\varphi} N \to C \to 0.$$

Examine the long exact sequences of local cohomology modules to conclude that $H^d_{\mathfrak{m}}(M) \cong H^d_{\mathfrak{m}}(M/K) \cong H^d_{\mathfrak{m}}(N)$. □

We will frequently use finite generation of certain local cohomology modules when proving our main theorems. To that end, we the following.

Lemma 18.2.4 *Let $(R, \mathfrak{m}, k)$ be a complete equidimensional local ring of dimension d, and let M be a finitely generated R-module satisfying Serre's condition (S_t) for some $t < d$. Then $\ell_R(H^t_{\mathfrak{m}}(M)) < \infty$.*

Proof This is [13, Exercise 29] and follows from the proof of [13, Lemma 4.5]. □

Remark 18.2.5 *We remark that the equidimensionality assumption in Lemma 18.2.4 is harmless, as this result will only be applied to local rings that are (S_2) and are homomorphic images of Gorenstein local rings – such rings are always equidimensional (see for instance* [1, Remark 4.5]*,* [9, Remark 2.2(h)]*, or* [2, section 2.2]*).*

Lemma 18.2.6 *Let $(R, \mathfrak{m}, k)$ be a d-dimensional excellent local (S_2) and (G_1) ring of prime characteristic $p > 0$. Suppose that $x \in R$ is a nonzerodivisor such that R/xR is (S_2) and (G_1). Let E be a Weil divisor of R with the property that $R(E)_{V(x)}$ is reflexive. Then the natural map of local cohomology modules*

$$H^{d-1}_{\mathfrak{m}}\left(R(-E+K_R)_{V(x)}\right) \to H^d_{\mathfrak{m}}(R(-E+K_R))$$

is injective.

Proof Without loss of generality, we may assume that E has components disjoint from $V(x)$. In particular, the reflexification $R(E)_{V(x)} \to R_{V(x)}(E|_{V(x)})$ is an isomorphism. There is a short exact sequence

$$0 \to R(E) \xrightarrow{\cdot x} R(E) \to R_{V(x)}(E|_{V(x)}) \to 0. \tag{18.2}$$

The top local cohomology module $H^d_{\mathfrak{m}}(R(K_R))$ serves as the injective hull of the residue field. Therefore, we consider a choice of the Matlis duality functor $\operatorname{Hom}_R(-, H^d_{\mathfrak{m}}(R(K_R)))$. By Tensor–Hom adjunction

$$\operatorname{Hom}_R(R(E), H^d_{\mathfrak{m}}(R(K_R))) \cong R(-E) \otimes_R H^d_{\mathfrak{m}}(R(K_R)).$$

The local cohomology module $H^d_{\mathfrak{m}}(R(K_R))$ is a cokernel of a Čech complex, and tensor products preserve cokernels. Therefore,

$$R(-E) \otimes_R H^d_{\mathfrak{m}}(R(K_R)) \cong H^d_{\mathfrak{m}}(R(-E) \otimes_R R(K_R)).$$

https://doi.org/10.1017/9781009396233.019 Published online by Cambridge University Press

The reflexification map $R(-E) \otimes_R R(K_R) \to R(-E+K_R)$ is an isomorphism in codimension 1. Therefore, by Lemma 18.2.3,

$$H^d_{\mathfrak{m}}(R(-E) \otimes_R R(K_R)) \cong H^d_{\mathfrak{m}}(R(-E+K_R)).$$

Similarly, the Matlis dual of $R_{V(x)}(E|_{V(x)})$ is

$$\begin{aligned}
&\operatorname{Hom}_R(R_{V(x)}(E|_{V(x)}), H^d_{\mathfrak{m}}(R(K_R))) \\
&= \operatorname{Hom}_R(R_{V(x)}(E|_{V(x)}), 0 :_{H^d_{\mathfrak{m}}(R(K_R))} x) \\
&\cong \operatorname{Hom}_R(R_{V(x)}(E|_{V(x)}), H^{d-1}_{\mathfrak{m}}(R(K_{R/xR}))) \\
&\cong H^{d-1}_{\mathfrak{m}}(R_{V(x)}(-E|_{V(x)} + K_{R/xR})).
\end{aligned}$$

Therefore, the Matlis dual of (18.2) is the short exact sequence

$$\begin{aligned}
0 \to H^{d-1}_{\mathfrak{m}}(R_{V(x)}(-E|_{V(x)} + K_{R/xR})) &\to H^d_{\mathfrak{m}}(R(-E+K_R)) \\
&\xrightarrow{\cdot x} H^d_{\mathfrak{m}}(R(-E+K_R)) \to 0.
\end{aligned} \tag{18.3}$$

The reflexification map $R(-E+K_R)_{V(x)} \to R_{V(x)}(-E|_{V(x)} + K_{R/xR})$ is an isomorphism at the codimension 1 points of R/xR. Therefore,

$$H^{d-1}_{\mathfrak{m}}(R(-E+K_R)_{V(x)}) \cong H^{d-1}_{\mathfrak{m}}(R_{V(x)}(-E|_{V(x)} + K_{R/xR})). \tag{18.4}$$

We conclude the proof by plugging (18.4) into (18.3). □

We are now prepared to prove Theorem A.

Theorem 18.2.7 *Let $(R, \mathfrak{m}, k)$ be an excellent local ring of prime characteristic $p > 0$ and $x \in \mathfrak{m}$ a nonzerodivisor such that R/xR is (G_1) and (S_2). Let $\Delta \geq 0$ be an effective $\mathbb{Q}$-divisor of R with components disjoint from $\operatorname{div}_R(x)$ such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier and $(p^e - 1)\Delta$ is integral for all $e \gg 0$ sufficiently large and divisible. Then the pair $(R, \Delta + \operatorname{div}_R(x))$ is F-pure if and only if $(R/xR, \Delta|_{\operatorname{div}(x)})$ is F-pure.*

Proof As in the proof of Lemma 18.2.2, we may assume that R is complete. If $\dim(R) \leq 2$, then the assumption that R/xR is (G_1) implies that the ring R is Gorenstein since the property of being Gorenstein deforms [23, Tag 0BJJ]. The theorem follows for rings of dimension at most 2 by Example 18.2.1.

We assume that $\dim(R) \geq 3$. By induction on the dimension, we may assume that both R and R/xR are F-pure when localized at any non-maximal prime ideal in $V(x)$. Let $\Theta_e = (p^e - 1)\Delta$ and consider the commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^e_*R(p^eK_R + \Theta_e) & \xrightarrow{\cdot F^e_* x} & F^e_*R(p^eK_R + \Theta_e) & \longrightarrow & F^e_*R(p^eK_R + \Theta_e)_{V(x)} & \longrightarrow & 0 \\
 & & \uparrow {\scriptstyle \cdot F^e_* x^{p^e-1}} & & \uparrow {\scriptstyle F^e} & & \uparrow {\scriptstyle F^e} & & \\
0 & \longrightarrow & R(K_R) & \xrightarrow{\cdot x} & R(K_R) & \longrightarrow & R(K_R)_{V(x)} & \longrightarrow & 0.
\end{array}$$

https://doi.org/10.1017/9781009396233.019 Published online by Cambridge University Press

If $e \gg 0$ and sufficiently divisible, then $(p^e - 1)K_R + \Theta_e$ is an integral p^{e_0}-torsion divisor. By Lemma 18.2.6 applied to the divisor $E = 0$, we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & F^e_* H^{d-1}_{\mathfrak{m}}(R(p^e K_R + \Theta_e)_{V(x)}) & \xrightarrow{\xi} & F^e_* H^d_{\mathfrak{m}}(R(p^e K_R + \Theta_e)) & \xrightarrow{\cdot F^e_* x} & F^e_* H^d_{\mathfrak{m}}(R(p^e K_R + \Theta_e)) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle F^e} & & \uparrow{\scriptstyle \cdot F^e_* x^{p^e-1}} & & \uparrow{\scriptstyle F^e} & & \\ 0 & \longrightarrow & H^{d-1}_{\mathfrak{m}}(R(K_R)_{V(x)}) & \longrightarrow & H^d_{\mathfrak{m}}(R(K_R)) & \longrightarrow & H^d_{\mathfrak{m}}(R(K_R)) & \longrightarrow & 0. \end{array} \tag{18.5}$$

The leftmost vertical map of (18.5) is isomorphic to

$$(R_{V(x)} \xrightarrow{F^e} F^e_*(R_{V(x)}(\Theta_e|_{\mathrm{div}_R(x)}))) \otimes_{R/xR} H^{d-1}_{\mathfrak{m}}(R_{V(x)}(K_{R/xR}))$$

and the middle vertical map is isomorphic to

$$(R \xrightarrow{\cdot F^e_* x^{p^e-1}} F^e_* R(\Theta_e)) \otimes_R H^d_{\mathfrak{m}}(R(K_R)).$$

The local cohomology modules $H^{d-1}_{\mathfrak{m}}(R(K_R)_{V(x)})$ and $H^d_{\mathfrak{m}}(R(K_R))$ are essential extensions of the residue field k. If $(R, \Delta + \mathrm{div}_R(x))$ is F-pure, then the middle vertical map in (18.5) is injective, from which it follows that the leftmost vertical map in (18.5) is injective (i.e. that $(R/xR, \Delta|_{\mathrm{div}(x)})$ is F-pure). This concludes the proof of the forward direction.

We now prove the converse (i.e. inversion of adjunction), so suppose for the remainder of the proof that $(R/xR, \Delta|_{\mathrm{div}(x)})$ is F-pure. Once we know that the map ξ in (18.5) is injective, we will be able to conclude that $(R, \Delta + \mathrm{div}_R(x))$ is F-pure by once again equating the injectivity of the middle and leftmost verticle maps in (18.5). This will follow from the next claim.

Claim 18.2.8 *Suppose that E is an integral torsion divisor of index p^{e_0}. Then $R(E)_{V(x)}$ is an (S_2) R/xR-module.*

Proof of Claim 18.2.8 By assumption, R is F-pure whenever we localize at a non-maximal prime ideal $\mathfrak{p} \in V(x)$. By Lemma 18.2.2, the localized module $(R(E)_{V(x)})_{\mathfrak{p}}$ is an (S_2) $(R_{V(x)})_{\mathfrak{p}}$-module. Therefore, to show $R(E)_{V(x)}$ is (S_2), it suffices to show that $R(E)_{V(x)}$ has depth at least 2. Consider the following commutative diagram:

$$\begin{array}{ccccccccc} & & & & F^e_*R & \longrightarrow & F^e_*R_{V(x)} & \longrightarrow & 0 \\ & & & & \cong\uparrow & & \cong\uparrow & & \\ & & & & F^e_*R(p^eE) & \longrightarrow & F^e_*R(p^eE)_{V(x)} & \longrightarrow & 0 \\ & & & & F^e\uparrow & & F^e\uparrow & & \\ 0 & \longrightarrow & R(E) & \xrightarrow{\cdot x} & R(E) & \longrightarrow & R(E)_{V(x)} & \longrightarrow & 0. \end{array}$$

We cannot yet assert that the rightmost vertical map of this diagram is split. However, we claim that the rightmost map is split after applying $H^2_{\mathfrak{m}}(-)$. Indeed, the reflexification map $R(E)_{V(x)} \to R_{V(x)}(E|_{V(x)})$ is an isomorphism on the punctured spectrum of R/xR. Therefore, the induced map of local cohomology modules

$$H^2_{\mathfrak{m}}(R(E)_{V(x)}) \to H^2_{\mathfrak{m}}(R_{V(x)}(E|_{V(x)}))$$

is an isomorphism. Our assumptions imply that R/xR is F-pure. By Lemma 18.2.2, $R_{V(x)}(E|_{V(x)}) \to F^e_*R_{V(x)}(p^eE|_{V(x)}) \cong F^e_*R_{V(x)}$ is split. Therefore,

$$H^2_{\mathfrak{m}}(R(E)_{V(x)}) \cong H^2_{\mathfrak{m}}(R_{V(x)}(E|_{V(x)})) \to F^e_*H^2_{\mathfrak{m}}(R_{V(x)}(p^eE|_{V(x)}))$$

is split. Since R/xR is assumed to be (S_2), we know that $\operatorname{depth}(R) \geq 3$, hence $H^2_{\mathfrak{m}}(R) = 0$. Now consider the resulting diagram of local cohomology modules

$$\begin{array}{ccccccc} & & & & 0 = F^e_*H^2_{\mathfrak{m}}(R) & \longrightarrow & F^e_*H^2_{\mathfrak{m}}(R_{V(x)}) \\ & & & & \cong\uparrow & & \cong\uparrow \\ & & & & F^e_*H^2_{\mathfrak{m}}(R(p^eE)) & \longrightarrow & F^e_*H^2_{\mathfrak{m}}(R_{V(x)}(p^eE|_{V(x)})) \\ & & & & F^e\uparrow & & F^e\uparrow \\ H^1_{\mathfrak{m}}(R(E)_{V(x)}) & \xrightarrow{\subseteq} & H^2_{\mathfrak{m}}(R(E)) & \xrightarrow{\cdot x} & H^2_{\mathfrak{m}}(R(E)) & \longrightarrow & H^2_{\mathfrak{m}}(R_{V(x)}(E|_{V(x)})). \end{array} \tag{18.6}$$

The composition of the right vertical maps in (18.6) is split and therefore injective. One verifies by chasing the diagram (18.6) that $H^2_{\mathfrak{m}}(R(E)) \xrightarrow{\cdot x} H^2_{\mathfrak{m}}(R(E))$ is an onto map. Since $R(E)$ is an (S_2) R-module, $H^2_{\mathfrak{m}}(R(E))$ is finitely generated by Lemma 18.2.4. By Nakayama's lemma, we have $H^2_{\mathfrak{m}}(R(E)) = 0$, so $H^1_{\mathfrak{m}}(R(E)_{V(x)}) = 0$ as needed. □

Combining Claim 18.2.8 with Lemma 18.2.6 applied to the divisor $E = -(p^e-1)K_R - \Delta_e$ tells us that ξ is injective. Again, using that $H^{d-1}_{\mathfrak{m}}(R(K_R)_{V(x)})$ and $H^d_{\mathfrak{m}}(R(K_R))$ are essential extensions of k, the left vertical map of (18.5) is injective if and only if the middle vertical map of (18.5) is injective. Therefore, $(R_{V(x)}, \Delta|_{\operatorname{div}_R(x)})$ is F-pure if and only if $(R, \Delta + \operatorname{div}_R(x))$ is F-pure. □

https://doi.org/10.1017/9781009396233.019 Published online by Cambridge University Press

18.3 Adjunction of F-Purity along a Divisor

The primary objective of this section is to prove Theorem B (adjunction of F-purity), but we first give a brief description of the different $\mathrm{Diff}_D(\Delta)$.

18.3.1 The Different

Let R be a reduced excellent (S_2) and (G_1) local ring, D an effective integral (S_2) and (G_1) Weil divisor, and Δ an effective $\mathbb{Q}$-divisor with components disjoint from D such that the divisor class $K_R + D + \Delta$ is $\mathbb{Q}$-Cartier of index n. We choose a canonical divisor K_R so that $K_R = -D + G$ for some divisor G with components disjoint from D.

In particular, if $U \subseteq \mathrm{Spec}(R)$ is a dense open subscheme that is regular, then $(K_R + D)|_U$ is Cartier with components disjoint from $D \cap U$. It follows that $(K_R + D)|_{D\cap U}$ is a canonical divisor on $D \cap U$. The section $1 \in R(G)$ gives rise to a rational section of ω_D via the restriction mappings $R(G) \to R(G)|_U \to \omega_D|_{D\cap U}$, and we denote by K_D the corresponding canonical divisor. Note that if D is Cartier in codimension 2, then so too is $K_R + D$ as D is (G_1), and it follows that $K_D = (K_R + D)|_D$.

Suppose $n(K_R + D + \Delta) = \mathrm{div}_R(f)$ for some $f \in R$, and let $\overline{f}$ be its image in R_D. The *different of* Δ *along* D is the $\mathbb{Q}$-divisor $\mathrm{Diff}_D(\Delta) := \frac{1}{n}\mathrm{div}_D(\overline{f}) - K_D$ of R_D. This is independent of the choices of K_R and f; see [12, section 4.1] for further discussion. We recall some properties enjoyed by $\mathrm{Diff}_D(\Delta)$.

(1) $\mathrm{Diff}_D(\Delta)$ is effective and $0 \sim mn(K_R + D + \Delta)|_D = mn(\mathrm{Diff}_D(\Delta) + K_D)$ for every $m \in \mathbb{Z}$.
(2) Let V be an irreducible codimension 1 subvariety of D and view V simultaneously as an irreducible codimension 2 subvariety of $\mathrm{Spec}(R)$. Then $\mathrm{Diff}_D(\Delta)$ is not supported at V if and only if R and R_D are both regular at V and $V \notin \mathrm{Supp}(\Delta)$.
(3) If D is Cartier in codimension 2, then $\mathrm{Diff}_D(\Delta) = \Delta|_D$.

18.3.2 F-Purity

Let $(R, \mathfrak{m}, k)$ be a reduced local ring of prime characteristic $p > 0$ with total ring of fractions K. Suppose that R is (S_2) and (G_1). If Δ is a $\mathbb{Q}$-divisor, then

$$R(\Delta) = \{f \in K \mid \mathrm{div}_R(f) + \Delta \geq 0\}.$$

If $\Delta = \sum r_i[R/P_i]$ and $\lfloor \Delta \rfloor = \sum \lfloor r_i \rfloor [R/P_i]$ is the round down divisor, then $R(\Delta) = R(\lfloor \Delta \rfloor)$. If Δ is effective, then the pair (R, Δ) is *F-pure* if for all sufficiently divisible $e \gg 0$ the Frobenius maps $R \to F^e_* R((p^e - 1)\Delta)$ are pure.

https://doi.org/10.1017/9781009396233.019 Published online by Cambridge University Press

There are several competing notions of F-purity as it pertains to pairs — we direct the reader to Section 18.4.1 for more details. In particular, we provide a counterexample to sharp F-pure inversion of adjunction, at least at the level of generality of Theorem C (see Example 18.4.4).

Lemma 18.3.1 *Let $(R, \mathfrak{m}, k)$ be an excellent d-dimensional (S_2) and (G_1) local ring of prime characteristic $p > 0$, Δ an effective $\mathbb{Q}$-divisor, and K_R a choice of canonical divisor. Then the following are equivalent:*

(1) For all sufficiently divisible $e \gg 0$, the maps

$$R \to F^e_* R((p^e - 1)\Delta)$$

are pure, that is, the pair (R, Δ) is F-pure.

(2) There exists an e_0 so that for all sufficiently divisible $e \gg 0$, the maps

$$R \to F^{e+e_0}_* R(p^{e_0}(p^e - 1)\Delta)$$

are pure.

(3) For all sufficiently divisible $e \gg 0$, the maps of local cohomology

$$\begin{aligned} &H^d_{\mathfrak{m}}(R(K_R)) \to F^e_* H^d_{\mathfrak{m}}((p^e - 1)\Delta + p^e K_R) \\ &\cong (R \to F^e_* R((p^e - 1)\Delta)) \otimes_R H^d_{\mathfrak{m}}(R(K_R)) \end{aligned}$$

are injective.

(4) There exists an e_0 so that for all sufficiently divisible $e \gg 0$, the maps

$$\begin{aligned} H^d_{\mathfrak{m}}(R(K_R)) \quad &\to F^{e+e_0}_* H^d_{\mathfrak{m}}(p^{e_0}(p^e - 1)\Delta + p^{e+e_0} K_R) \\ &\cong (R \to F^{e+e_0}_* R(p^{e_0}(p^e - 1)\Delta)) \otimes_R H^d_{\mathfrak{m}}(R(K_R)) \end{aligned}$$

are injective.

Proof For all integers e, e_0, we can factor the map

$$R \to F^{e+e_0}_* R((p^{e+e_0} - 1)\Delta)$$

as

$$R \to F^e_* R((p^e - 1)\Delta) \to F^{e+e_0}_* (p^{e_0}(p^e - 1)\Delta) \subseteq F^{e+e_0}_* R((p^{e+e_0} - 1)\Delta).$$

This proves the equivalence of (1) and (2).

Cokernels are preserved under tensor product and the top local cohomology of a module is the cokernel of the last non-trivial map of a Čech complex. Hence, the isomorphisms described in (3) and (4) are valid. Furthermore, as $H^d_{\mathfrak{m}}(R(K_R))$ serves as the injective hull of the residue field of R, purity of the map in (1) is equivalent to injectivity of the maps in (2). (2) $\Leftrightarrow$ (4) is similar. □

https://doi.org/10.1017/9781009396233.019 Published online by Cambridge University Press

Let E be a $\mathbb{Q}$-divisor and D an effective integral divisor of an (S_2) and (G_1) ring R. We use the following notation:

$$R(E)_D := \frac{R(E)}{R(E-D)}.$$

Lemma 18.3.2 *Let $(R, \mathfrak{m}, k)$ be a complete d-dimensional (S_2) and (G_1) local ring of prime characteristic $p > 0$. Suppose that D is an effective integral divisor such that the pair (R, D) is F-pure. If E is an integral Weil divisor of index p^{e_0}, then $R(E)_D$ is a direct summand of $F^e_*R_D$ for all sufficiently divisible $e \gg 0$.*

Proof For all sufficiently divisible $e \gg 0$, the composition of the maps

$$R \to F^e_*R \to F^e_*R((p^e-1)D)$$

is pure since we are assuming the pair (R, D) is F-pure. Let $\varphi_e : F^e_*R((p^e - 1)D) \to R$ be a splitting of $R \to F^e_*(R(p^e-1)D)$. Note that since D is effective, we have $F^e_*R \subseteq F^e_*R((p^e - 1)D)$ and so $\varphi_e(F^e_*R) \subseteq R$. Moreover, we have $\varphi_e(F^e_*R(-D)) = R(-D)$. To see this, simply tensor $\varphi_e : F^e_*R((p^e - 1)D) \to F^e_*R$ with $R(-D)$ and reflexify. Therefore, there are commutative diagrams

$$\begin{array}{ccccc}
0 & \longrightarrow & R(-D) & \longrightarrow & R \\
 & & \uparrow & & \uparrow{\scriptstyle \varphi_e} \\
0 & \longrightarrow & F^e_*R(-D) & \longrightarrow & F^e_*R \\
 & & \uparrow & & \uparrow \\
0 & \longrightarrow & R(-D) & \longrightarrow & R
\end{array}$$

and the composition of the vertical maps are the identity maps on their respective modules. Therefore, if we tensor by $R(E)$ and reflexify, we find that there is a commutative diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & R(E-D) & \longrightarrow & R(E) \\
 & & \uparrow & & \uparrow{\scriptstyle \tilde{\varphi}_e} \\
0 & \longrightarrow & F^e_*R(p^eE-D) & \longrightarrow & F^e_*R(p^eE) \\
 & & \uparrow{\scriptstyle \cong} & & \uparrow{\scriptstyle \cong} \\
0 & \longrightarrow & F^e_*R(-D) & \longrightarrow & F^e_*R.
\end{array}$$

The maps $\tilde{\varphi}_e$ are splittings of $R(E) \to F^e_*R(p^eE)$ and commutativity of this diagram provides us an inclusion

$$\tilde{\varphi}_e(F^e_*R(p^eE - D)) \subseteq R(E - D).$$

It follows that we can restrict φ_e to $F^e_*R(p^eE)_D \cong F^e_*R_D$ and produce maps $\tilde{\varphi}_{e,D} : F^e_*R_D \to R(E)_D$ that are splittings of

$$R(E)_D \to F^e_*R_D.$$

In particular, $R(E)_D$ can be realized as a direct summand of $F^e_*R_D$ as claimed. □

Corollary 18.3.3 *Let* $(R, \mathfrak{m}, k)$ *be an excellent d-dimensional* (S_2) *and* (G_1) *local ring of prime characteristic* $p > 0$. *Suppose that D is an effective integral divisor such that the pair* (R, D) *is F-pure and* R_D *is* (S_2) *and* (G_1)*. If E is an integral Weil divisor of index* p^{e_0}*, then the reflexification map* $R(E)_D \to R_D(E|_D)$ *is an isomorphism.*

Proof By Lemma 18.3.2, the module $R(E)_D$ is a direct summand of $F^e_*R_D$. We are assuming that R_D is (S_2) and (G_1). Therefore, $F^e_*R_D$ is (S_2) and so is any direct summand, hence the reflexification map $R(E)_D \to R_D(E|_D)$ is an isomorphism. □

Lemma 18.3.4 *Let* $(R, \mathfrak{m}, k)$ *be an excellent* (S_2) *and* (G_1) *d-dimensional local ring of prime characteristic* $p > 0$. *Suppose that there exists a non-zero reduced* $\mathbb{Q}$*-Cartier divisor D such that* (R, D) *is F-pure. If E is a* $\mathbb{Q}$*-Cartier divisor, then*

$$\operatorname{depth}(R(E)) \geq \min\{3, d\}.$$

Proof We may assume R is complete as all hypotheses and the desired conclusions are unaffected by completion. We are assuming that R is (S_2). Therefore, we may assume that R is of dimension at least 3 and show that $H^2_{\mathfrak{m}}(R(E)) = 0$. The composition of maps

$$R \to F^e_*R \subseteq F^e_*R((p^e - 1)D)$$

is pure and hence also split. Tensor with $R(E)$ and reflexify to conclude the composition of maps

$$R(E) \to F^e_*R(p^eE) \subseteq F^e_*R(p^eE + (p^e - 1)D)$$

is split. Consequently, the composition of maps of local cohomology modules

$$H^2_{\mathfrak{m}}(R(E)) \to H^2_{\mathfrak{m}}(F^e_*R(p^eE)) \subseteq H^2_{\mathfrak{m}}(F^e_*R(p^eE + (p^e - 1)D))$$

is injective. To conclude that $H^2_{\mathfrak{m}}(R(E)) = 0$, it suffices to show that the inclusion

$$R(p^eE) \subseteq R(p^eE + (p^e - 1)D)$$

https://doi.org/10.1017/9781009396233.019 Published online by Cambridge University Press

induces the 0-map of local cohomology modules

$$H^2_{\mathfrak{m}}(R(p^eE)) \to H^2_{\mathfrak{m}}(R(p^eE+(p^e-1)D))$$

for all sufficiently divisible $e \gg 0$.

The divisor E is $\mathbb{Q}$-Cartier. Therefore, there exists an e_0 so that for all sufficiently divisible $e \gg 0$, we have $R(p^eE) \cong R(p^{e_0}E)$. Let g_e be an element of the total ring of fractions of R so that $R(p^eE) \xrightarrow{\cdot g_e} R(p^{e_0}E)$ is an isomorphism. Suppose that D has $\mathbb{Q}$-Cartier index n and $nD = \operatorname{div}(f)$. For all $e \gg 0$, there exists $q_e \geq 1$ and $0 \leq r_e < n$ such that $p^e - 1 = q_e n + r_e$. Observe that $q_e \to \infty$ as $e \to \infty$. There are commutative diagrams:

$$\begin{array}{ccccc} R(p^eE) & & \xrightarrow{\quad\subseteq\quad} & & R(p^eE+(p^e-1)D) \\ {\scriptstyle \cdot g_e}\downarrow\cong & & & & \cong\downarrow{\scriptstyle \cdot g_e f^{q_e}} \\ R(p^{e_0}E) & \xrightarrow{\cdot f^{q_e}} & R(p^{e_0}E) & \xrightarrow{\subseteq} & R(p^{e_0}E+r_eD). \end{array}$$

Consider the induced maps of local cohomology modules:

$$\begin{array}{ccccc} H^2_{\mathfrak{m}}(R(p^eE)) & & \longrightarrow & & H^2_{\mathfrak{m}}(R(p^eE+(p^e-1)D)) \\ {\scriptstyle \cdot g_e}\downarrow\cong & & & & \cong\downarrow{\scriptstyle \cdot g_e f^{q_e}} \\ H^2_{\mathfrak{m}}(R(p^{e_0}E)) & \xrightarrow{\cdot f^{q_e}} & H^2_{\mathfrak{m}}(R(p^{e_0}E)) & \longrightarrow & H^2_{\mathfrak{m}}(R(p^{e_0}E+r_eD)). \end{array}$$

The module $R(p^{e_0}E)$ is (S_2), $f \in \mathfrak{m}$, and therefore $H^2_{\mathfrak{m}}(R(p^{e_0}E)) \xrightarrow{\cdot f^{q_e}} H^2_{\mathfrak{m}}(R(p^{e_0}E))$ is the 0-map for all $e \gg 0$. Therefore, the map

$$H^2_{\mathfrak{m}}(R(p^eE)) \to H^2_{\mathfrak{m}}(R(p^eE+(p^e-1)D))$$

is the 0-map as it can be factored through the 0-map. □

18.3.3 Adjunction of F-Purity

Theorem 18.3.5 (Adjunction of F-Purity) *Let $(R,\mathfrak{m},k)$ be an excellent d-dimensional (S_2) and (G_1) local ring of prime characteristic $p>0$. Suppose that D is an integral (S_2) and (G_1) divisor. Let Δ be an effective $\mathbb{Q}$-divisor of R with components disjoint from D such that $(p^e-1)\Delta$ is integral for all sufficiently divisible $e \gg 0$. Suppose that $K_R + D + \Delta$ is $\mathbb{Q}$-Cartier. If $(R, \Delta + D)$ is F-pure, then $(R_D, \operatorname{Diff}_D(\Delta))$ is F-pure.*

Proof There exists an integer e_0 so that for all sufficiently divisible $e \gg 0$,

$$\Delta_e := (p^e-1)(K_R + D + \Delta)$$

https://doi.org/10.1017/9781009396233.019 Published online by Cambridge University Press

is an integral divisor of index p^{e_0}. By Lemma 18.3.2, $R(-\Delta_e)_D$ is a direct summand of $F^e_* R_D$ for all sufficiently divisible $e \gg 0$. Therefore, $R(-\Delta_e)_D$ is an (S_2) R_D-module, that is,[2]

$$R(-\Delta_e)_D \cong R_D(-\Delta_e|_D) = R_D(-(p^e-1)(\mathrm{Diff}_D(\Delta)+K_D)).$$

We have shown the existence of the following short exact sequences for all sufficiently divisible $e \gg 0$:

$$0 \to R(-\Delta_e - D) \to R(-\Delta_e) \to R_D(-\Delta_e|_D) \to 0. \tag{18.7}$$

The Matlis dual of (18.7) produces the short exact sequence

$$0 \to H^{d-1}_{\mathfrak{m}}(R_D(\Delta_e|_D + K_D)) \to H^d_{\mathfrak{m}}(R(\Delta_e + K_R)) \to H^d_{\mathfrak{m}}(R(\Delta_e + D + K_R)) \to 0, \tag{18.8}$$

that is, the kernel of the natural map

$$H^d_{\mathfrak{m}}(R(\Delta_e + K_R)) \to H^d_{\mathfrak{m}}(R(\Delta_e + D + K_R))$$

induced by the inclusion $R(\Delta_e + K_R) \subseteq R(\Delta_e + D + K_R)$ is

$$H^{d-1}_{\mathfrak{m}}(R_D(\Delta_e|_D + K_D)).$$

Similarly, the Matlis dual of the short exact sequence

$$0 \to R(-D) \to R \to R_D \to 0$$

is

$$0 \to H^{d-1}_{\mathfrak{m}}(R_D(K_D)) \to H^d_{\mathfrak{m}}(R(K_R)) \to H^d_{\mathfrak{m}}(R(D + K_R)) \to 0. \tag{18.9}$$

Therefore, the kernel of $H^d_{\mathfrak{m}}(R(K_R)) \to H^d_{\mathfrak{m}}(R(K_R + D))$ induced from the inclusion $R(K_R) \subseteq R(K_R + D)$ is $H^{d-1}_{\mathfrak{m}}(R_D(K_D))$.

Consider the following commutative diagrams with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^e_* R(\Delta_e + K_R) & \longrightarrow & F^e_* R(\Delta_e + D + K_R) & \longrightarrow & F^e_* R(\Delta_e + D + K_R)_D & \longrightarrow & 0 \\
 & & \uparrow F^e & & \uparrow F^e & & \uparrow F^e & & \\
0 & \longrightarrow & R(K_R) & \longrightarrow & R(D + K_R) & \longrightarrow & R(D + K_R)_D & \longrightarrow & 0.
\end{array}$$

The short exact sequences (18.8) and (18.9) give us commutative diagrams of local cohomology modules with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^e_* H^{d-1}_{\mathfrak{m}}(R_D(\Delta_e|_D + K_D)) & \longrightarrow & F^e_* H^d_{\mathfrak{m}}(R(\Delta_e + K_R)) & \longrightarrow & F^e_* H^d_{\mathfrak{m}}(R(\Delta_e + D + K_R)) & \longrightarrow & 0 \\
 & & \uparrow F^e & & \uparrow F^e & & \uparrow F^e & & \\
0 & \longrightarrow & H^{d-1}_{\mathfrak{m}}(R_D(K_D)) & \longrightarrow & H^d_{\mathfrak{m}}(R(K_R)) & \longrightarrow & H^d_{\mathfrak{m}}(R(D + K_R)) & \longrightarrow & 0.
\end{array} \tag{18.10}$$

[2] Note that this isomorphism requires that D is (G_1).

https://doi.org/10.1017/9781009396233.019 Published online by Cambridge University Press

In light of Lemma 18.2.3, the left vertical map in (18.10) is isomorphic to

$$(R_D \to F^e_* R((p^e - 1)\,\mathrm{Diff}_D(\Delta))) \otimes_R H^{d-1}_{\mathfrak{m}}(R_D(K_D))$$

and the right vertical map is isomorphic to $(R \to F^e_* R((p^e - 1)(\Delta + D))) \otimes_R H^d_{\mathfrak{m}}(R(K_R))$. We are assuming that $(R, \Delta + D)$ is F-pure; therefore, the middle map is injective by Lemma 18.3.1. Since the middle vertical map is injective, so too is the left vertical map. Therefore, the pair $(R_D, \mathrm{Diff}_D(\Delta))$ is F-pure by Lemma 18.3.1. □

Remark 18.3.6 *The proof of Theorem 18.3.5 contains an important observation that will be necessary to proving the theorem's converse. Namely, if* $(R, \Delta + D)$ *is* F*-pure and* $(p^e - 1)\Delta$ *is integral for all sufficiently divisible* $e \gg 0$*, then*

$$R(-\Delta_e)|_D \to R_D(-\Delta_e|_D)$$

is an isomorphism for all such e.

18.4 Inversion of Adjunction of F-Purity

Lemma 18.4.1 (Key Lemma) *Let* $(R, \mathfrak{m}, k)$ *be an excellent d-dimensional* (S_2) *and* (G_1) *local ring of prime characteristic* $p > 0$*. Suppose that D is a reduced* (S_2) *and* (G_1) *divisor. Let* Δ *be an effective* $\mathbb{Q}$*-divisor with components disjoint from D. Suppose that* $K_R + D + \Delta$ *is* $\mathbb{Q}$*-Cartier and* $(p^e - 1)\Delta$ *is integral for all sufficiently divisible* $e \gg 0$*. Suppose that* $(R_D, \mathrm{Diff}_D(\Delta))$ *is* F*-pure. Then the following are equivalent:*

(1) The pair $(R, \Delta + D)$ *is* F*-pure.*
(2) If E is a p-power torsion divisor, then $R(E)_D = R(E)/R(E - D)$ *is an* (S_2) R_D*-module.*

Proof If the pair $(R, \Delta + D)$ is F-pure, then the pair (R, D) is F-pure. Therefore, if E is a torsion divisor of index p^{e_0}, then $R(E)_D$ is a direct summand of $F^e_* R_D$ for all sufficiently divisible $e \gg 0$ by Lemma 18.3.2. We are assuming that R_D is (S_2). Therefore, $R(E)_D$ is (S_2) since it is a direct summand of an (S_2) R_D-module. This completes the forward direction $(1) \Rightarrow (2)$.

We introduce notation that will be used in the proof of the converse $(2) \Rightarrow (1)$. For each $e \in \mathbb{N}$, let $\Delta_e = (p^e - 1)(K_R + D + \Delta)$. There exists an integer e_0 so that for all sufficiently divisible $e \gg 0$, the divisor $-\Delta_e$ is a torsion of index p^{e_0}. By assumption, the reflexification map $R(-\Delta_e)_D \to R_D(-\Delta_e|_D)$ is an isomorphism for all sufficiently divisible $e \gg 0$. There are short exact sequences

$$0 \to R(-\Delta_e - D) \to R(-\Delta_e) \to R_D(-\Delta_e|_D) \to 0.$$

Matlis duality provides us short exact sequences (again using that R_D is (S_2))

$$0 \to H^{d-1}_{\mathfrak{m}}(R_D(\Delta_e|_D + K_D)) \to H^d_{\mathfrak{m}}(R(\Delta_e + K_R)) \to H^d_{\mathfrak{m}}(R(\Delta_e + D + K_R)) \to 0.$$

Consequently, the kernel of the natural map $H^d_{\mathfrak{m}}(R(\Delta_e + K_R)) \to H^d_{\mathfrak{m}}(R(\Delta_e + D + K_R))$ induced by the inclusion $R(\Delta_e + K_R) \to R(\Delta_e + D + K_R)$ is $H^{d-1}_{\mathfrak{m}}(R_D(\Delta_e|_D + K_D))$. Similarly, there are short exact sequences,

$$0 \to R(-D) \to R \to R_D \to 0$$

and Matlis duality provides us short exact sequences

$$0 \to H^{d-1}_{\mathfrak{m}}(R_D(K_D)) \to H^d_{\mathfrak{m}}(R(K_R)) \to H^d_{\mathfrak{m}}(R(K_R + D)) \to 0.$$

Therefore, the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & F^e_* R(\Delta_e + K_R) & \to & F^e_* R(\Delta_e + K_R + D) & \to & F^e_* R(\Delta_e + K_R + D)_D & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & R(K_R) & \to & R(K_R + D) & \to & R(K_R)_D & \to & 0
\end{array}$$

induces the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & F^e_* H^{d-1}_{\mathfrak{m}}(R_D(\Delta_e|_D + K_D)) & \to & F^e_* H^d_{\mathfrak{m}}(R(\Delta_e + K_R)) & \to & F^e_* H^d_{\mathfrak{m}}(R(\Delta_e + K_R + D)) & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & H^{d-1}_{\mathfrak{m}}(R_D(K_D)) & \to & H^d_{\mathfrak{m}}(R(K_R)) & \to & H^d_{\mathfrak{m}}(R(K_R + D)) & \to & 0.
\end{array} \tag{18.11}$$

The left vertical map in (18.11) is isomorphic to

$$(R_D \to F^e_* R_D((p^e - 1)\,\mathrm{Diff}_D(\Delta))) \otimes_{R_D} H^{d-1}_{\mathfrak{m}}(R_D(K_D)). \tag{18.12}$$

Since $(R_D, \mathrm{Diff}_D(\Delta))$ is F-pure, the map in (18.12) is injective by Lemma 18.3.1. Since $H^{d-1}_{\mathfrak{m}}(R_D(K_D))$ and $H^d_{\mathfrak{m}}(R(K_R))$ are essential extensions of the residue field, a simple chase of a socle element in the diagram (18.11) shows that the middle vertical map is similarly injective. Moreover, this map is isomorphic to

$$(R \to F^e_* R((p^e - 1)(\Delta + D))) \otimes_R H^d_{\mathfrak{m}}(R(K_R)).$$

In particular, the pair $(R, \Delta + D)$ is F-pure by Lemma 18.3.1. This completes the proof of (2) $\Rightarrow$ (1). $\square$

https://doi.org/10.1017/9781009396233.019 Published online by Cambridge University Press

Corollary 18.4.2 (Inversion of adjunction along a $\mathbb{Q}$-Cartier divisor) *Let $(R, \mathfrak{m}, k)$ be an excellent (S_2) and (G_1) ring of prime characteristic $p > 0$ and let K_R be a choice of canonical divisor of* $\operatorname{Spec}(R)$. *Suppose that $D \geq 0$ is an effective integral (S_2) and (G_1) divisor, and let $\Delta \geq 0$ be an effective $\mathbb{Q}$-divisor on* $\operatorname{Spec} R$ *whose components are disjoint from those of D and such that $(p^e - 1)\Delta$ is integral for all sufficiently divisible $e \gg 0$. Suppose that $K_R + D + \Delta$ is $\mathbb{Q}$-Cartier. Suppose that*

(I) D is $\mathbb{Q}$-Cartier;
(II) for each $\mathbb{Q}$-Cartier divisor E and $\mathfrak{p} \in D \subseteq$ $\operatorname{Spec}(R)$ *that*

$$\operatorname{depth}(R(E)_{\mathfrak{p}}) \geq \min\{\operatorname{ht}(\mathfrak{p}), 3\}.$$

If $(R_D, \operatorname{Diff}_D(\Delta))$ is F-pure, then $(R, \Delta + D)$ is F-pure.

Proof By Lemma 18.4.1, we only require that $R(E)_D$ is an (S_2) R_D-module whenever E is a p-power torsion divisor. We first remark that if $r \in R$ and $s \in R(E)$, then $rs \in R(E - D)$ if and only if $r \in R(-D)$. Therefore, $R(E)_D = R(E)/R(E - D)$ is a torsion-free R_D-module. In particular, if $\dim R \leq 2$, then R_D is at most one-dimensional and $R(E)_D$ is a torsion-free (and therefore a maximal Cohen–Macaulay) R_D-module. By induction, we may assume that $\dim R \geq 3$ and aim to show that $H^1_{\mathfrak{m}}(R(E)_D) = 0$.

There are short exact sequences

$$0 \to R(E - D) \to R(E) \to R(E)_D \to 0$$

and therefore $H^1_{\mathfrak{m}}(R(E)_D) \subseteq H^2_{\mathfrak{m}}(R(E - D))$. The divisor $E - D$ is torsion and therefore the latter local cohomology module is 0 by assumption. □

Lemma 18.3.4 demonstrates a necessary condition for F-pure inversion of adjunction, namely

$$\operatorname{depth}(R(E)_{\mathfrak{p}}) \geq \min\{\operatorname{ht}(\mathfrak{p}), 3\}$$

for all $\mathbb{Q}$-Cartier divisors E and $\mathfrak{p} \in D \subseteq \operatorname{Spec}(R)$. This assumption is vacuous whenever the ambient ring is strongly F-regular by [17, Corollary 3.3]. Moreover, we can replace this hypothesis with the milder assumption that for all $\mathfrak{p} \in D \subseteq \operatorname{Spec}(R)$

$$\operatorname{depth}(R_{\mathfrak{p}}) \geq \min\{\operatorname{ht}(\mathfrak{p}), 3\}$$

and still obtain F-pure inversion of adjunction under the hypothesis that D is a p-power torsion divisor.

Corollary 18.4.3 (Inversion of adjunction along a p-power torsion divisor) *Let $(R, \mathfrak{m}, k)$ be an excellent (S_2) and (G_1) ring of prime characteristic $p > 0$ and*

https://doi.org/10.1017/9781009396233.019 Published online by Cambridge University Press

let K_R be a choice of canonical divisor of $\operatorname{Spec}(R)$. *Suppose that $D \geq 0$ is an effective integral* (S_2) *and* (G_1) *divisor, and let $\Delta \geq 0$ be an effective $\mathbb{Q}$-divisor on* $\operatorname{Spec} R$ *whose components are disjoint from those of D and such that $(p^e - 1)\Delta$ is integral for all sufficiently divisible $e \gg 0$. Suppose that $K_R + D + \Delta$ is $\mathbb{Q}$-Cartier. Suppose further that D is a p-power torsion divisor and that for all $\mathfrak{p} \in D \subseteq \operatorname{Spec}(R)$ that*

$$\operatorname{depth}(R_{\mathfrak{p}}) \geq \min\{\operatorname{ht}(\mathfrak{p}), 3\}.$$

If $(R_D, \operatorname{Diff}_D(\Delta))$ *is F-pure, then* $(R, \Delta + D)$ *is F-pure.*

Proof As in the proof of Corollary 18.4.2, we may assume that R is of dimension at least 3. By induction on the dimension of R, we may assume that $(R, \Delta + D)$ is F-pure when localized at a non-maximal point of D. In light of Lemma 18.4.1, the reflexification map $R(E)_D \to R_D(E|_D)$ is an isomorphism on the punctured spectrum. Therefore, it suffices to show $H^1_{\mathfrak{m}}(R(E)_D) = 0$ to conclude that $R(E)_D$ is an (S_2) R_D-module.

As the reflexification map $R(E)_D \to R_D(E|_D)$ is an isomorphism on the punctured spectrum, we observe

- $H^2_{\mathfrak{m}}(R(E)_D) \cong H^2_{\mathfrak{m}}(R_D(E|_D))$, and
- $H^2_{\mathfrak{m}}(R(E)_D) \to F^e_* H^2_{\mathfrak{m}}(R_D(p^e E|_D))$ is an injective map as R_D is F-pure.

Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^e_* R(p^e E - D) & \longrightarrow & F^e_* R(p^e E) & \longrightarrow & F^e_* R(p^e E)_D & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & R(E - D) & \longrightarrow & R(E) & \longrightarrow & R(E)_D & \longrightarrow & 0
\end{array}$$

and the resulting commutative diagram of local cohomology modules:

$$\begin{array}{ccccccccc}
0 & \to & F^e_* H^1_{\mathfrak{m}}(R(p^e E)_D) & \to & F^e_* H^2_{\mathfrak{m}}(R(p^e E - D)) & \to & F^e_* H^2_{\mathfrak{m}}(R(p^e E)) & \to & F^e_* H^2_{\mathfrak{m}}(R(p^e E)_D) \\
 & & \uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle \beta} & & \uparrow{\scriptstyle \gamma} & & \uparrow{\scriptstyle \delta} \\
0 & \longrightarrow & H^1_{\mathfrak{m}}(R(E)_D) & \longrightarrow & H^2_{\mathfrak{m}}(R(E - D)) & \longrightarrow & H^2_{\mathfrak{m}}(R(E)) & \xrightarrow{\zeta} & H^2_{\mathfrak{m}}(R(E)_D).
\end{array}$$

If $e \gg 0$, then $p^e E \sim 0$ in which case $H^1_{\mathfrak{m}}(R(p^e E)_D) = H^2_{\mathfrak{m}}(R(p^e E)) = H^2_{\mathfrak{m}}(R) = 0$ by assumption and the diagram simplifies as

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & F^e_* H^2_{\mathfrak{m}}(R(p^e E - D)) & \longrightarrow & 0 & \longrightarrow & F^e_* H^2_{\mathfrak{m}}(R(p^e E)_D) \\
 & & \uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle \beta} & & \uparrow{\scriptstyle \gamma} & & \uparrow{\scriptstyle \delta} \\
0 & \to & H^1_{\mathfrak{m}}(R(E)_D) & \longrightarrow & H^2_{\mathfrak{m}}(R(E - D)) & \longrightarrow & H^2_{\mathfrak{m}}(R(E)) & \xrightarrow{\zeta} & H^2_{\mathfrak{m}}(R(E)_D).
\end{array}$$

The map δ is injective. Therefore, ζ is the 0-map and we have a surjection $H^2_{\mathfrak{m}}(R(E-D)) \twoheadrightarrow H^2_{\mathfrak{m}}(R(E))$. We mention that up to this point, we have not utilized that D is a p-power torsion divisor.

We may repeat the argument from the previous paragraph after replacing E with the likewise p-power torsion divisor $E-D$ to obtain a surjection $H^2_{\mathfrak{m}}(R(E-2D)) \twoheadrightarrow H^2_{\mathfrak{m}}(R(E-D))$. More generally, for any $n \in \mathbb{N}$, we obtain surjections

$$H^2_{\mathfrak{m}}(R(E-nD)) \twoheadrightarrow H^2_{\mathfrak{m}}(R(E-(n-1)D)) \twoheadrightarrow \cdots \twoheadrightarrow H^2_{\mathfrak{m}}(R(E-D)) \twoheadrightarrow H^2_{\mathfrak{m}}(R(E)).$$

Choose n so that $nD = \operatorname{div}_R(f) \sim 0$. Then there is a commutative diagram

$$\begin{array}{ccc} H^2_{\mathfrak{m}}(R(E-nD)) & \xrightarrow{\cong} & H^2_{\mathfrak{m}}(R(E)) \\ {\scriptstyle \cdot f}\uparrow{\scriptstyle \cong} & & \uparrow{\scriptstyle =} \\ H^2_{\mathfrak{m}}(R(E)) & \xrightarrow{\cdot f} & H^2_{\mathfrak{m}}(R(E)). \end{array}$$

Consequently, $H^2_{\mathfrak{m}}(R(E)) = fH^2_{\mathfrak{m}}(R(E))$. The module $H^2_{\mathfrak{m}}(R(E))$ is finitely generated by Lemma 18.2.4 since $R(E)$ is (S_2) and $\dim R \geq 3$. By Nakayama's lemma, the module $H^2_{\mathfrak{m}}(R(E)) = 0$. Similarly, $H^2_{\mathfrak{m}}(R(E-D)) = 0$ because $E-D$ is a p-power torsion divisor. Therefore, $H^1_{\mathfrak{m}}(R(E)_D) = 0$ as needed since $H^1_{\mathfrak{m}}(R(E)_D)$ is a submodule of $H^2_{\mathfrak{m}}(R(E-D)) = 0$. □

18.4.1 Closing Remarks on Theorems B and C

Techniques used here might be useful in proving other cases of F-pure inversion of adjunction. For example, there are cases to consider whenever the boundary divisor Δ does not have the property that $(p^e-1)\Delta$ is integral for sufficiently divisible $e \gg 0$. Arguments using our techniques will require additional assumptions on the modules $R((p^e-1)\Delta) = R(\lfloor (p^e-1)\Delta \rfloor)$ for sufficiently divisible $e \gg 0$. For example, one could develop a version of Lemma 18.4.1 under the additional hypothesis that $\lfloor (p^e-1)(K_R+D+\Delta)\rfloor$ is $\mathbb{Q}$-Cartier for all sufficiently large and divisible $e \gg 0$.

Alternatively, one can consider novel roundings of the divisors $(p^e-1)\Delta$ when defining a competing notion of an F-pure pair (R,Δ). For example, Schwede defined a pair (R,Δ) to be *sharply F-pure* if for all sufficiently divisible $e \gg 0$ the maps

$$R \to F^e_* R(\lceil (p^e-1)\Delta \rceil)$$

are pure [19]. Sharp F-purity has a distinct advantage whenever $\Delta + D$ is $\mathbb{Q}$-Cartier:

$$\operatorname{Hom}_R(F^e_* R(\lceil (p^e-1)\Delta \rceil), R) \cong F^e_* R(-(p^e-1)(\Delta + K_R)).$$

Regardless, there are simple counterexamples to sharp F-pure inversion of adjunction, even if the ambient ring R is nonsingular.

Example 18.4.4 Let $R = \mathbb{F}_p[x,y,z]$, $\Delta = \frac{1}{p}[R/(x+y^p+z^p)]$, and $D = V(x)$. The pair $(R_D, \Delta|_D)$ is sharply F-pure, but $(R, \Delta+\mathrm{div}_R(x))$ is not sharply F-pure. To see this, first notice that $[R/(x+y^p+z^p)]|_D = p[R_D/(y+z)]$. Therefore, $\Delta|_D = [R_D/(y+z)]$. The pair $(R_D, \Delta|_D)$ is (sharply) F-pure. Indeed, R_D is isomorphic to the regular local ring $\mathbb{F}_p[y,z]$ and $(y+z)^{p^e-1} \notin \mathfrak{m}^{[p^e]}$ for all e.

Observe that

$$\lceil (p^e-1)\Delta \rceil = \left\lceil \frac{p^e-1}{p} \right\rceil \left[\frac{R}{(x+y^p+z^p)} \right] = p^{e-1} \left[\frac{R}{(x+y^p+z^p)} \right].$$

In particular, the pair $(R, \Delta+D)$ is not sharply F-pure as $(x+y^p+z^p)^{p^{e-1}} x^{p^e-1} \in \mathfrak{m}^{[p^e]}$ for all $e \in \mathbb{N}$.

We conclude this section by highlighting an application of our results (and those of [18]) to a question of Enescu.

Question 18.4.5 (see [4, Question 3.5] and surrounding discussion) *Does there exist an excellent local F-rational ring $(R, \mathfrak{m})$ such that for all nonzerodivisors $x \in R$, the ring R/xR is not F-pure?*

Example 18.4.6 By either Theorem A or [18, Theorem A], any $\mathbb{Q}$-Gorenstein isolated singularity $(R, \mathfrak{m})$ which is F-rational but not F-pure will provide an affirmative answer to Question 18.4.5. In particular, the ring

$$R = \left(\frac{\mathbb{F}_2[x,y,z,w]}{(x^3+y^3+z^3+w^3)} \right)^{(n)}_{\mathfrak{m}}, \qquad n \geq 2$$

(where $(-)^{(n)}$ denotes the nth Veronese subring and $\mathfrak{m}$ is its homogeneous maximal ideal) is such an example by [21, Example 6.3].

18.5 Manivel's Trick

Recall the following theorem on deformation of F-purity.

Theorem 18.5.1 ([18]) *Let $(R, \mathfrak{m}, k)$ be a local F-finite $\mathbb{Q}$-Gorenstein ring of prime characteristic $p > 0$. Suppose that $x \in \mathfrak{m}$ is a nonzerodivisor such that R/xR is (G_1), (S_2), and F-pure. Then R is F-pure.*

We will demonstrate in this section how a special case of Theorem A follows from Theorem 18.5.1 using a variant of a trick due to Manivel [14].

https://doi.org/10.1017/9781009396233.019 Published online by Cambridge University Press

Proof of Theorem A when R is F-finite and $\Delta = 0$ Assume now that $(R, \mathfrak{m}, k)$ is an F-finite $\mathbb{Q}$-Gorenstein local ring of prime characteristic p, and $x \in \mathfrak{m}$ is a nonzerodivisor such that R/xR is (G_1) and (S_2), and so that R/xR is F-pure. Suppose $n \in \mathbb{Z}_{>0}$ is relatively prime to p and consider the module finite extension ring $U_n := R[Y]/(Y^n - x)$. Note first that U_n is reduced, as U_n is generically étale over R. Also, U_n is a free R-module of rank n, from which it follows that U_n remains (S_2) and f is a nonzerodivisor on U_n. Consequently, if y denotes the image of Y in U_n, y is also a nonzerodivisor on U_n, and moreover $U_n/(y) = R/xR$ is (S_2), (G_1), and F-pure. Since $\mathrm{Hom}_R(U_n, R) \cong U_n$, for example, by taking the projection onto the R-factor corresponding to y^{n-1}, we also see that U_n is (G_1) as it is Gorenstein over the Gorenstein locus of R. Similarly, U_n remains $\mathbb{Q}$-Gorenstein. By Theorem 18.5.1 (applied to the localizations at the finitely many maximal ideals of U_n), it follows that U_n is F-pure.

Letting $n = p + 1$, consider the following diagram:

$$\begin{array}{ccc} R & \xrightarrow[\gamma]{1 \mapsto F_* x^{p-1}} & F_* R \\ {\scriptstyle 1 \mapsto y^p}\Big\downarrow{\scriptstyle \beta} & & {\scriptstyle \delta}\Big\downarrow{\scriptstyle F_* 1 \mapsto F_* y} \\ U_{p+1} & \xrightarrow[1 \mapsto F_* 1]{\alpha} & F_* U_{p+1}. \end{array}$$

To see that this commutes, note that we have

$$\delta(\gamma(1)) = \delta(F_* x^{p-1}) = F_* x^{p-1} y = F_* y^{(p+1)(p-1)} y = F_* y^{p^2}$$

and also

$$\alpha(\beta(1)) = \alpha(y^p) = y^p \alpha(1) = y^p F_* 1 = F_* y^{p^2}.$$

Since U_{p+1} is F-pure, the bottom rightward arrow α is a split inclusion of U_{p+1}-modules, hence also of R-modules. The left downward arrow β is also a split inclusion of R-modules. It follows that the composite mapping $\alpha \circ \beta = \delta \circ \gamma$ is a split inclusion of R-modules, and so in particular γ is as well. By definition, we see that the pair $(R, \mathrm{div}_R(x))$ is F-pure. □

Remark 18.5.2 *One can also apply Manivel's original trick in the the preceding alternate proof of Theorem A when R is F-finite and* $\Delta = 0$ *proof instead. With the same notation as in the first paragraph of the proof, instead consider the commutative diagram*

https://doi.org/10.1017/9781009396233.019 Published online by Cambridge University Press

$$\begin{array}{ccc} R & \xrightarrow[\gamma]{1\mapsto F^e_* x^{(1-1/n)(p^e-1)}} & F^e_* R \\ {\scriptstyle 1\mapsto y^{n-1}}\big\downarrow{\scriptstyle\beta} & & {\scriptstyle\delta}\big\downarrow{\scriptstyle F^e_*1\mapsto F^e_* y^{n-1}} \\ U_n & \xrightarrow[1\mapsto F^e_*1]{\alpha} & F^e_* U_n \end{array}$$

where $e \gg 0$ *is sufficiently large and divisible so that* $n|(p^e - 1)$*. Since* U_n *is F-pure, the bottom rightward arrow* α *is a split inclusion of* U_n*-modules, hence also of R-modules. The left downward arrow* β *is also a split inclusion of R-modules. It follows that the composite mapping* $\alpha \circ \beta = \delta \circ \gamma$ *is a split inclusion of R-modules, and so in particular* γ *is as well. By definition, we see that the pair* $(R, (1 - 1/n)\mathrm{div}_R(x))$ *is F-pure. Letting* $n \to \infty$ *and using* [8, Theorem 4.9]*, it follows that* $(R, \mathrm{div}_R(x))$ *is F-pure.*

References

[1] Y. Aoyama, *Some basic results on canonical modules*, Kyoto J. Math. 23 (1983), no. 1, 85–94.

[2] K. Česnavicius, *Macaulayfication of Noetherian schemes*, Duke Math. J. 170 (2021), no. 7, 1419–1455.

[3] O. Das, *On strongly F-regular inversion of adjunction*, J. Algebra 434 (2015), 207–226.

[4] F. Enescu, *On the behavior of F-rational rings under flat base change*, J. Algebra 233 (2000), no. 2, 543–566.

[5] R. Fedder, *F-purity and rational singularity*, Trans. Amer. Math. Soc. 278 (1983), no. 2, 461–480.

[6] N. Hara and K.-I. Watanabe, *F-regular and F-pure rings vs. log terminal and log canonical singularities*, J. Algebraic Geom. 11 (2002), no. 2, 363–392.

[7] R. Hartshorne, *Generalized divisors on Gorenstein schemes*, K-Theory 8 (1994), no. 3, 287–339.

[8] D. J. Hernandez, *F-purity of hypersurfaces*, Math. Res. Lett. 19 (2012), no. 2, 389–401.

[9] M. Hochster and C. Huneke, *Indecomposable canonical modules and connectedness*, Commutative algebra: Syzygies, multiplicities, and birational algebra (South Hadley, MA, 1992), pp. 197–208, Contemp. Math., 159, Amer. Math. Soc., Providence, RI, 1994.

[10] M. Hochster and C. Huneke, *Applications of the existence of big Cohen-Macaulay algebras*, Adv. Math. 113 (1995), no. 1, 45–117.

[11] M. Kawakita, *Inversion of adjunction on log canonicity*, Invent. Math. 167 (2007), no. 1, 129–133.

https://doi.org/10.1017/9781009396233.019 Published online by Cambridge University Press

[12] J. Kollar, *Singularities of the minimal model program*, Cambridge Tracts in Mathematics, 200, Cambridge University Press, Cambridge, 2013, With the collaboration of Sándor Kovács.
[13] L. Ma and T. Polstra, *F-singularities: A commutative algebra approach*, www.math.purdue.edu/~ma326/F-singularitiesBook.pdf, 2021.
[14] L. Manivel, *Un théoreme de prolongement L^2 de sections holomorphes d'un fibré hermitien.*, Mathematische Zeitschrift 212 (1993), no. 1, 107–122.
[15] H. Matsumura, *Commutative ring theory*, 2nd ed., Cambridge Studies in Advanced Mathematics, 8, Cambridge University Press, Cambridge, 1989, xiv+320 pp, Translated from the Japanese by M. Reid.
[16] L. E. Miller and K. Schwede, *Semi-log canonical vs F-pure singularities*, J. Algebra 349 (2012), 150–164.
[17] Z. Patakfalvi and K. Schwede, *Depth of F-singularities and base change of relative canonical sheaves*, J. Inst. Math. Jussieu 13 (2014), no. 1, 43–63.
[18] T. Polstra and A. Simpson, *F-purity deforms in $\mathbb{Q}$-Gorenstein rings*, International Mathematics Research Notices (2022), to appear.
[19] K. Schwede, *Generalized test ideals, sharp F-purity, and sharp test elements*, Math. Res. Lett. 15 (2008), no. 6, 1251–1261.
[20] K. Schwede, *F-adjunction*, Algebra Number Theory 3 (2009), no. 8, 907–950.
[21] A. Singh, *Veronese subrings and tight closure*, Pacific Journal of Mathematics 192 (2000), 399–413.
[22] A. K. Singh, *F-regularity does not deform*, Amer. J. Math. 121 (1999), no. 4, 919–929.
[23] T. Stacks Project Authors, *Stacks Project*, https://stacks.math.columbia.edu.