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# LAYER AND STABLE SOLUTIONS TO A NONLOCAL MODEL

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In celebration of Professor Vladimír Šverák's 65th birthday

Abstract. We study the layer and stable solutions of nonlocal problem

$$-\Delta u + F'(u)\,(-\Delta)^s\,F(u) + G'(u) = 0 \text{ in } \mathbb{R}^n$$

where $F \in C^2_{\text{loc}}(\mathbb{R})$ satisfies $F(0) = 0$ and $G$ is a double well potential. For $n = 2, s > 0$ and $n = 3$, $s \geq 1/2$, we establish the 1-d symmetry of layer solutions for this equation. When $n = 2$ and $F'$ is bounded away from zero, we prove the 1-d symmetry of stable solutions for this equation. Using a different approach, we also prove the 1-d symmetry of stable solutions for

$$F'(u)\,(-\Delta)^s\,F(u) + G'(u) = 0 \text{ in } \mathbb{R}^2.$$

1. **Introduction.** In this paper, we study the layer and stable solutions for the following model problem

$$-\Delta u + F'(u)\,(-\Delta)^s\,F(u) + G'(u) = 0 \text{ in } \mathbb{R}^n, \tag{1.1}$$

where $0 < s < 1$, $F$ is a $C^2_{\text{loc}}$ function satisfying $F(0) = 0$ and $G$ is a double well potential satisfying

$$G \in C^3(\mathbb{R}), \quad G(\pm 1) = 0 \text{ and } G(t) > 0 \text{ for } t \neq \pm 1. \tag{1.2}$$

Here the fractional Laplacian is defined as

$$(-\Delta)^s v(x) = c_n(s) P.V. \int_{\mathbb{R}^n} \frac{v(x) - v(y)}{|x-y|^{n+2s}} dy \tag{1.3}$$

where $c_n(s) = \pi^{-\frac{n}{2}} 2^{2s} \frac{\Gamma(\frac{n+2s}{2})}{\Gamma(2-s)} s(1-s)$ so that the Fourier symbol for $(-\Delta)^s$ is $|\xi|^{2s}$. It is known that the pointwise formula (1.3) gives a continuous function for $v \in C^{2s+\epsilon}(\Omega)$ ( or $C^{1,2s+\epsilon-1}(\Omega)$ if $s \geq 1/2$) for some $\epsilon > 0$ [42, 43] which satisfies

$$\int_{\mathbb{R}^n} \frac{|v(x)|}{1+|x|^{n+2s}} dx < \infty.$$

The energy functional associated with (1.1) is

$$J(u,\Omega) = \int_\Omega \left[\frac{1}{2}|\nabla u|^2 + G(u)\right] dx + \frac{c_n(s)}{4} \iint_{\mathbb{R}^n\times\mathbb{R}^n \setminus (\Omega^c\times\Omega^c)} \frac{|F(u)(x) - F(u)(y)|^2}{|x-y|^{n+2s}} dydx$$

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$$= J^L(u,\Omega) + \frac{c_n(s)}{4} J^s(u,\Omega). \tag{1.4}$$

A solution $u$ of (1.1) is called *a layer solution* if $u$ satisfies $\frac{\partial u}{\partial x_1} \geq 0$ and the asymptotic behavior

$$\lim_{x_1 \to -\infty} u(x_1, x') = -1, \qquad \lim_{x_1 \to \infty} u(x_1, x') = 1$$

for all $x' \in \mathbb{R}^{n-1}$. A solution $u$ is called a *stable* solution of (1.1) if the second local variation of $J(\cdot, \mathbb{R}^n)$ at $u$ is nonnegative, i.e.

$$\int_{\mathbb{R}^n} |\nabla \phi|^2 + G''(u)\phi^2 + 2\phi^2 F''(u)(-\Delta)^s F(u) + 2F'(u)\phi(-\Delta)^s (F'(u)\phi) \geq 0$$

for any $\phi \in C_0^2(\mathbb{R}^n)$. Our main interest in this paper is the 1-d symmetry of such solutions, namely if $u = u_0(a_0 \cdot x)$ for some function $u_0$ and a constant vector $a_0 \in \mathbb{R}^n$.

Our work is partly motivated by recent work by Cabré and Serra [12] where they studied the 1-d symmetry of layer solutions of nonlocal Allen-Cahn type equation

$$Lu + G'(u) = 0 \text{ in } \mathbb{R}^n, \tag{1.5}$$

here $G$ is a double well potential satisfying (1.2) and, for some $s^* \in (0,1)$,

$$Lu = \int_{[s^*,1]} (-\Delta)^s u \, d\mu(s)$$

with $\mu$ being a probablity measure supported in $[s^*, 1]$. Assuming either $n = 2$, $s^* > 0$ or $n = 3$, $s^* \geq \frac{1}{2}$, they proved the 1-d symmetry for layer solutions of (1.5). A special case covered by their theorem is the following equation

$$\sum_{i=1}^{K} \mu_i (-\Delta)^{s^i} u + G'(u) = 0 \text{ in } \mathbb{R}^n,$$

where $\mu_i > 0$, $\sum_{i=1}^K \mu_i = 1$, $0 < s_1 < \cdots < s_K \leq 1$.

Cabré and Serra's result is an extension of the famous De Giorgi conjecture for semilinear equation

$$-\Delta u + G'(u) = 0 \text{ in } \mathbb{R}^n, \tag{1.6}$$

which states that a solution of (1.6) which is montone in one direction is one dimensional for $n \leq 8$. The De Giorgi conjecture was completely solved for $n = 2$ and 3 [2, 4, 27] and proved by Savin [39] for $4 \leq n \leq 8$ under the additional limiting conditions $\lim_{x_1 \to \pm\infty} u(x_1, x') = \pm 1$. Partial results on De Giorgi conjecture can also be found in [6, 28]. A counterexample was constructed in [19] for $n \geq 9$.

De Giorgi's conjecture has been generalized to fractional Allen-Cahn equation

$$-\Delta^s u + G'(u) = 0 \text{ in } \mathbb{R}^n. \tag{1.7}$$

The fractional De Giorgi's conjecture holds for the case $n = 2, s \in (0,1)$ [13, 14, 15, 44], and for $n = 3$ and $s \geq \frac{1}{2}$ [9, 10]. Under the additional limit conditions, fractional De Giorgi's conjecture was proved for $n = 3$ and $s \in \left(0, \frac{1}{2}\right)$ by Dipierro, Serra and Valdinoci in [23] and by Savin in [40, 41] for $4 \leq n \leq 8$ and $s \in [\frac{1}{2}, 1)$. Under the additional limit condition, the best known results for fractional De Giorgi's conjecture in higher dimensions says there exists $\varepsilon_0 \in (0, \frac{1}{2}]$ such that for any $s \in (\frac{1}{2} - \varepsilon_0, 1]$, a layer solution is 1-d if $n \leq 8$ [23]. Similar conclusion holds when $n \leq 7$ if the solution is a minimizer. The limit condition is removed in [22] for $n = 3$ and $s \in \left(0, \frac{1}{2}\right)$. Figalli and Serra [24] solved the De Giorgi's conjecture for half-Laplacian when $n = 4$ (such result is *not* known for the classical case $s = 1$). A counter example for $n = 9$, $\frac{1}{2} < s < 1$ was announced in [18].

Stable solutions for Allen Cahn equation in the classical and fractional cases have also received a lot of attention over the years. The corresponding De Giorgi conjecture for stable solutions states that a stable solution to (1.6) or (1.7) is 1-d when $n \leq 7$. This conjecture holds true for $n = 2$ [4, 27] and remains a long standing open problem when $n \geq 3$ for the classical Allen Cahn equation (1.6). For fractional Allen Cahn equation, this conjecture was proved for $n = 2, 0 < s < 1$ (see [15] for the case $s = 1/2$ and [14, 44] for $0 < s < 1$ ) and $n = 3, s = 1/2$ [24]. Recently, Cabré, Cinti and Serra [11] established the stable De Gorigi conjecture for $n = 3$ and $s \in (s_*, 1/2)$ for some $s_* < 1/2$. For Allen-Cahn equations (1.6) and (1.7), it is known that layer solutions are stable solutions. Moreover, if any entire stable solution to in $\mathbb{R}^{n-1}$ is 1-d, then any layer solution to in $\mathbb{R}^n$ is 1-d for $s \in (0, 1)$ when $n \leq 3$, and for $s \in (1/2 - \varepsilon_0, 1)$ for some constant $\varepsilon_0 \in (0, 1/2)$ when $4 \leq n \leq 7$ (a proof can be found in the appendix in [29]).

A second motivation of our work comes from the ferromagnetic thin films. Letting $F(u) = \sin u$ and $G(u) = \frac{1}{2}(\sin u - h)^2$ in (1.1), where $h \in [0, 1)$ is a constant, we arrive at the following equation:

$$-\Delta u + \cos u \, (-\Delta)^s \sin u + \cos u(\sin u - h) = 0 \text{ in } \mathbb{R}^n. \tag{1.8}$$

When $n = 1$ and $s = \frac{1}{2}$, (1.8) reduces to

$$-\Delta\theta + (\sin\theta - h)\cos\theta + \cos\theta \, (-\Delta)^{\frac{1}{2}} \sin\theta = 0 \quad \text{in } \mathbb{R}. \tag{1.9}$$

Equation (1.9) is the associated Euler-Lagrange equation for energy functional

$$\mathcal{E}(\theta) = \frac{1}{2} \int_{\mathbb{R}} \left\{ |\theta_x|^2 + \frac{\nu}{2} \sin\theta \left( -\frac{d^2}{dx^2} \right)^{\frac{1}{2}} \sin\theta + (\sin\theta(x) - h)^2 \right\} dx, \tag{1.10}$$

which is the reduced magnetic energy per unit length for magnetization varying along one direction. Indeed, we consider a two-dimensional horizontal section of a magnetic sample that is infinite in the $x_1$ direction and $L-$periodic in $x_2$ direction, with external field applied in $\mathbf{e}_1$ direction, the reduced thin film energy (after suitable scaling) can be written as

$$\begin{aligned} \mathcal{E}(\mathbf{m}) = \frac{1}{2L} \int_{\mathbb{R}\times[0,L)} |\nabla \mathbf{m}|^2 \, dx + \frac{1}{2L} \int_{\mathbb{R}\times[0,L)} (\mathbf{m}\cdot\mathbf{e}_1 - h)^2 \, dx \\ + \frac{\nu}{8L^2} \int_{\mathbb{R}\times[0,L)} \int_{\mathbb{R}\times[0,L)} \frac{\nabla\cdot\mathbf{m}(\mathbf{x}) \, \nabla\cdot\mathbf{m}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} dx dx', \end{aligned} \tag{1.11}$$

where $\mathbf{m} : \mathbb{R}^2 \to \mathbb{S}^1$ is the unit magnetization vector in the film plane, $\mathbf{e}_i$ is the unit vector in the $i$-th coordinate. $h \in [0, 1)$ represents the rescaled strength of the applied field. Assuming $\mathbf{e}_2$ is the easy axis, let $\theta(x)$ be the phase angle between $\mathbf{m}$ and $\mathbf{e}_2$ in the counter-clockwise direction. Thus $\mathbf{m}(x) = (\sin\theta(x), \cos\theta(x))$, the reduced thin film energy (1.11) becomes

$$\mathcal{E}(\theta) = \frac{1}{2L} \int_{\mathbb{R}\times[0,L)} \left( |\nabla\theta(x)|^2 + \frac{\nu}{2} \left| (-\Delta)^{-\frac{1}{4}} (\nabla\cdot\mathbf{m}(x)) \right|^2 + (\sin\theta(x) - h)^2 \right) dx. \tag{1.12}$$

Under the assumption that $\theta$ varies only along $\mathbf{e}_1$ direction, (1.12) reduces to (1.10).

A minimizer of (1.10) subject to limiting conditions $\theta(-\infty) = \theta_h = \arcsin h$ and $\theta(\infty) = \pi - \arcsin h$ is called Néel wall. Néel wall has been studied extensively over the last few decades and its structure is well understood at current stage[1, 3, 7, 17, 21, 20, 25, 26, 30, 31, 33, 34, 35, 36, 37, 38]. One interesting question is the stability of Néel walls under arbitrary two dimensional perturbations. Our model (1.8) can be viewed as toy models to study the stability problem for Néel walls.

Following the arguments in [4, 27], the symmetry results for layer solutions of (1.6) are deduced from a Liouville type theorem. If a layer solution $u$ satisfies certain energy estimates, this Liouville type theorem would imply that $\frac{\partial_i u}{\partial_1 u}$ is a constant. This is equivalent to the 1-d symmetry of $u$. The symmetry results for solutions of (1.7) [9, 10, 13, 14, 15] are derived using the extension problem introduced by Caffarelli and Silverstre [16] and a Liouville theorem for the extended problem. A main contribution in [12] is the introduction of an extension problem and the related Liouville theorem for the extended problem for operators in the form of a sum of fractional Laplacians. We shall adapt their ideas to introduce a suitable extension problem for (1.1) and prove a generalized Liouvile type theorem (c.f. Theorem 2.11). The 1-d symmetry for the layer solutions of (1.1) can be deduced from the energy estimates and the generalized Liouville theorem.

To prove the 1-d symmetry for stable solutions of (1.1), we adapt ideas from [13, 27]. Under the additional assumptions that when $F$ is strictly monotone with $F'$ bounded away from zero, we prove the equivalence between the stability of $u$ and the existence of a positive solution to the linearized equation of (1.1) at $u$ (c.f. proposition 3.1). From this, energy estimates and the generalized Liouville theorem, we obtain the 1-d symmetry of stable solutions.

A different proof for the 1-d symmetry of stable solutions to Allen-Cahn equation relies on the suitable BV estimates for stable solutions [11, 24, 29]. As a generalization of this method, we study the stable solutions of the following equation.

$$F'(u)(-\Delta)^s F(u) + G'(u) = 0. \tag{1.13}$$

When $n = 2$ and $F$ is strictly monotone with derivative bounded away from zero, we can adapt the ideas in [11, 24, 29] to obtain suitable BV estimates for the stable solutions of (1.13). From those estimates, we obtain the 1-d symmetry of stable solutions of (1.13) when $n = 2$.

This paper is organized as follows. Section 2 discusses the 1-d symmetry of layer solutions for (1.1) and section 3 is devoted to the 1-d symmetry of stable solutions to (1.1) under the additional assumption that $F$ is strictly monotone. In the last section, we prove the 1-d symmetry for stable solutions of (1.13).

## 2. **1-d symmetry of layer solutions.**

In this section, we consider layer solutions of (1.1) . Our main theorem is the following.

**Theorem 2.1.** *Assume that $F \in C^2_{loc}(\mathbb{R})$ satisfies $F(0) = 0$ and $G \in C^3(\mathbb{R}^n)$ is a double well potential satisfying (1.2). Let $u \in L^\infty(\mathbb{R}^n)$ be a layer solution of (1.1) satisfying $(-\Delta)^s F(u) \in L^\infty$. If $n = 2$ and $s > 0$ or $n = 3$ and $s \geq \frac{1}{2}$, then $u(x) = u_0(a \cdot x)$ where $a \in \mathbb{R}^n$ is a unit vector and $u_0$ is a layer solution of*

$$-\Delta u + F'(u)(-\Delta)^s F(u) + G'(u) = 0 \text{ in } \mathbb{R}.$$

A crucial step in proving the 1-d symmetry for layer solutions is to establish sharp energy estimates in a ball of radius $R \geq 2$. Let $B_R = \{x \in \mathbb{R}^n : |x| < R\}$. We shall first prove an energy bound on $J(u, B_R)$ using the regularity results on solutions to (1.1) . Next we introduce an extension problem following the idea in [12]. Lastly we derive a Liouville theorem for the extension problem, which, together with the energy bound, would imply the 1-d symmetry for layer solutions of (1.1) .

2.1. **Regularity.** Let $Ku = -\Delta u + F'(u)\left(-\Delta\right)^s F(u)$. For $\Omega \subset \mathbb{R}^n$, we call $u$ a solution to (1.1) in $\Omega$ *in the sense of distribution* if

$$\begin{aligned}&\int_\Omega \left[\nabla u \cdot \nabla\varphi + G'(u)\varphi\right] dx\\ &\quad+\frac{c_n(s)}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n\setminus(\Omega^c\times\Omega^c)} \frac{\left(F(u)(x)-F(u)(y)\right)\left(F'(u)(x)\varphi(x)-F'(u)(y)\varphi(y)\right)}{|x-y|^{n+2s}}dydx\\ &=0 \end{aligned} \tag{2.1}$$

for all $\varphi \in C_c^\infty(\Omega)$.

Our regularity result states the following.

**Proposition 2.2.** *Let $u \in L^\infty(\mathbb{R}^n)$ satisfies (1.1) in the sense of distribution and $(-\Delta)^s F(u) \in L^\infty(\mathbb{R}^n)$, then $u \in C^{2,\gamma}(\mathbb{R}^n)$ and $\|u\|_{C^{2,\gamma}(\mathbb{R}^n)} \le C$ for some $\gamma > 0$ and $C = C(n,\ s, F, G, \gamma)$ is a constant depending only on $n, s, F, G, \gamma$.*

The proof of propositon 2.2 follows from the following propositions in [43].

**Proposition 2.3.** *(Proposition 2.8 in [43]) Let $w=(-\Delta)^\sigma u$. Assume $w\in C^{0,\alpha}(\mathbb{R}^n)$ and $u \in L^\infty(\mathbb{R}^n)$ for $\alpha \in (0,1]$ and $\sigma > 0$.*

- *If $\alpha + 2\sigma \le 1$, then $u \in C^{0,\alpha+2\sigma}(\mathbb{R}^n)$. Moreover*
$$\|u\|_{C^{0,\alpha+2\sigma}(\mathbb{R}^n)} \le C\left(\|u\|_{L^\infty} + \|w\|_{C^{0,\alpha}}\right)$$
*for a constant $C$ depending only on $n, \alpha, \sigma$.*
- *If $\alpha + 2\sigma > 1$, then $u \in C^{1,\alpha+2\sigma-1}(\mathbb{R}^n)$. Moreover*
$$\|u\|_{C^{1,\alpha+2\sigma-1}(\mathbb{R}^n)} \le C\left(\|u\|_{L^\infty} + \|w\|_{C^{0,\alpha}}\right)$$
*for a constant $C$ depending only on $n, \alpha, \sigma$.*

**Proposition 2.4.** *(Proposition 2.9 in [43]) Let $w=(-\Delta)^\sigma u$. Assume $w\in L^\infty(\mathbb{R}^n)$ and $u \in L^\infty(\mathbb{R}^n)$ for $\sigma > 0$.*

- *If $2\sigma \le 1$, then $u \in C^{0,\alpha}(\mathbb{R}^n)$ for any $\alpha < 2\sigma$. Moreover*
$$\|u\|_{C^{0,\alpha}(\mathbb{R}^n)} \le C\left(\|u\|_{L^\infty} + \|w\|_{L^\infty}\right)$$
*for a constant $C$ depending only on $n, \alpha, \sigma$.*
- *If $2\sigma > 1$, then $u \in C^{1,\alpha}(\mathbb{R}^n)$ for any $\alpha < 2\sigma - 1$. Moreover*
$$\|u\|_{C^{1,\alpha}(\mathbb{R}^n)} \le C\left(\|u\|_{L^\infty} + \|w\|_{L^\infty}\right)$$
*for a constant $C$ depending only on $n, \alpha, \sigma$.*

*Proof of proposition 2.2.* Let $v = (-\Delta)^s u$. (2.1) can be written as

$$\left(-\Delta\right)^{1-s} v = w\ ,\ w = -F'(u)\left(-\Delta\right)^s F(u) - G'(u). \tag{2.2}$$

Since $F \in C^2_{\text{loc}}(\mathbb{R})$ and $G \in C^3(\mathbb{R}^n)$, by assumption on $u$, we have $w \in L^\infty(\mathbb{R}^n)$. If $s < \frac{1}{2}$, apply Proposition 2.4 to (2.2), we have

$$\begin{aligned}\|v\|_{C^{1,\delta}(\mathbb{R}^n)} &\le C\left(\|v\|_{L^\infty} + \|w\|_{L^\infty}\right)\\ &\le C\left(\|v\|_{L^\infty} + M_0\left\|(-\Delta)^s F(u)\right\|_{L^\infty} + L_0\right),\end{aligned} \tag{2.3}$$

for any $\delta < 1-2s$, where $M_0$, $L_0$ are upper bounds for $||F'(u)||_{L^\infty}$ and $||G'(u)||_{L^\infty}$ respectively and $C$ is a constant depending only on $n$, $\delta$ and $s$. Fix $\delta$, apply proposition 2.3 (see also classial Riesz potential estimates [32]) to $v = (-\Delta)^s u$, we have

$$\|u\|_{C^{0,\delta+2s}(\mathbb{R}^n)} \le C\left(\|u\|_{L^\infty} + \|v\|_{C^{0,\delta}}\right), \tag{2.4}$$

and (since $\delta/2+2s>2s$),

$$\begin{aligned} \|v\|_{L^\infty} &\le C\,\|u\|_{C^{0,\delta/2+2s}}\,, \\ \|(-\Delta)^s F(u)\|_{L^\infty} &\le C\,\|F(u)\|_{C^{0,\delta/2+2s}} \le C\left(\|u\|_{C^{0,\delta/2+2s}}+M_0\right). \end{aligned} \tag{2.5}$$

Combining (2.3) , (2.4) and (2.5) , we have

$$\|u\|_{C^{0,\ \delta+2s}(\mathbb{R}^n)} \le C\left(\|u\|_{L^\infty}+\|u\|_{C^{0,\delta/2+2s}}+1\right).$$

By interpolation inequality, we conclude

$$\|u\|_{C^{0,\ \delta+2s}(\mathbb{R}^n)} \le C\left(\|u\|_{L^\infty}+1\right),$$

where $C=C(n,s,M_0,L_0,\delta)$ is a constant independent of $u$. This gives

$$\|(-\Delta)^s F(u)\|_{C^{0,\delta}} \le \|F(u)\|_{C^{0,\ \delta+2s}(\mathbb{R}^n)} \le C\left(\|u\|_{C^{0,\ \delta+2s}(\mathbb{R}^n)}+M_0\right).$$

Therefore

$$-\Delta u = -F'(u)\,(-\Delta)^s F(u) - G'(u)$$

with $w=-F'(u)\,(-\Delta)^s F(u)-G'(u)\in C^{0,\delta}$. The $C^{2,\delta}$ regularity and estimates for $u$ follows from standard Hölder estimates for Laplace equation. $s\ge\frac{1}{2}$ case can be proved similarly. ☐

2.2. **Energy bound.** Let

$$\Phi_{n,s}(R)=\begin{cases} R^{n-1}\left(R^{1-2s}-1\right)(1-2s)^{-1} & \text{if } s\ne\frac{1}{2}, \\ R^{n-1}\ln R & \text{if } s=\frac{1}{2}. \end{cases}$$

we establish the following energy bound for layer solutions.

**Proposition 2.5.** *Let $u\in L^\infty(\mathbb{R}^n)$ be a layer solution of (1.1) with $(-\Delta)^s F(u)\in L^\infty$, then $J(u,B_R)\le C\Phi_{n,s}(R)$ for some constant $C=C(n,s,F,G)$.*

A main step in the proof of Proposition 2.5 is the following estimates on energy difference.

**Proposition 2.6.** *Let $u$ be a layer solution of (1.8) which is monotone in the $x_1$ direction. Define $\overline{u}:\mathbb{R}^n\to\mathbb{R}$ by $\overline{u}(x_1,x')=\overline{u}(x')=\lim_{x_1\to\infty}u(x_1,x')$. Then there exists a constant $C=C(n,s)$ such that*

$$J(u,B_R)-J(\overline{u},B_R)\le C\Phi_{n,s}(R)$$

*for all $R\ge 2$.*

*Proof.* We consider the slided function $u^t(x_1,x')=u(t+x_1,x')$. Direct calculation via integration by parts shows

$$\begin{aligned} \frac{d}{dt}J\left(u^t,B_R\right) &= \left\langle u^t,\partial_t u^t\right\rangle_{B_R}+\int_{B_R}G'(u^t)\partial_t u^t dx \\ &= \int_{B_R}\left[Ku^t+G'(u^t)\right]\partial_t u^t dx \\ &\quad + c_n(s)\int_{B_R^c}dx\int_{B_R}dy\frac{F(u^t)(x)-F(u^t)(y)}{|x-y|^{n+2s}}\partial_t u^t \\ &= c_n(s)\int_{B_R^c}dx\int_{B_R}dy\frac{F(u^t)(x)-F(u^t)(y)}{|x-y|^{n+2s}}\partial_t u^t, \end{aligned}$$

here we used $Ku^t+G'(u^t)=0$ everywhere in $\Omega$ in the last step.

By Propositon 2.2, $\|u^t\|_{C^{2,\gamma}} \leq C$ for some constant $C$ indepedent of $t$. Recall that $u$ is monotone in $x_1$ direction, we have $u^t \to \overline{u}$ in $\mathrm{C}^{2,\gamma}_{\mathrm{loc}}(\mathbb{R}^n)$ and $|u^t(x) - u^t(y)| \leq C\min\{1, |x-y|\}$. Therefore

$$\begin{aligned}
&J(u, B_R) - J(\overline{u}, B_R) \\
&= -\int_0^\infty \frac{d}{dt} J\left(u^t, B_R\right) dt \\
&= -c_n(s) \int_0^\infty dt \int_{B_R^c} dx \int_{B_R} dy \frac{F(u^t)(x) - F(u^t)(y)}{|x-y|^{n+2s}} \partial_t u^t \\
&\leq Cc_n(s) \int_0^\infty dt \int_{B_R^c} dx \int_{B_R} dy \frac{\min\{1, |x-y|\}}{|x-y|^{n+2s}} \partial_t u^t \\
&= Cc_n(s) \int_{B_R^c} dx \int_{B_R} dy \frac{\min\{1, |x-y|\}}{|x-y|^{n+2s}} \int_0^\infty \partial_t u^t dt \\
&\leq Cc_n(s) \|u\|_{L^\infty(\mathbb{R}^n)} \Phi_{n,s}(R).
\end{aligned}$$

Here we used $\partial_t u^t = \partial_{x_1} u^t \geq 0$ and estimate (claim 4.1 in [12])

$$\int_{B_R^c} dx \int_{B_R} dy \frac{\min\{1, |x-y|\}}{|x-y|^{n+2s}} \leq C\Phi_{n,s}(R). \qquad \square$$

*Proof of Proposition 2.5.* Proposition **2.5** follows immediately from Proposition **2.6** since $\overline{u} = 1$ and $J(\overline{u}, B_R) = 0$. $\square$

2.3. **Extension problem.** Let $\Omega \subset \overline{\mathbb{R}^{n+1}_+}$ be a relatively open Lipschitz domain and $\Omega^+ = \Omega \cap \{\lambda > 0\}$, $\underline{\Omega} = \Omega \cap \{\lambda = 0\}$. Given any $\underline{w}$ defined on $\underline{\Omega}$, we consider the energy functional

$$\widetilde{J}(\underline{w}, w, \Omega) = \int_{\underline{\Omega}} \left[\frac{1}{2}|\nabla \underline{w}|^2 + G(\underline{w})(x)\right] dx + \widetilde{I}(w, \Omega),$$

where

$$\widetilde{I}(w, \Omega) = \frac{d(s)}{2} \int_{\Omega+} \lambda^{1-2s} |\nabla w|^2 \, dx d\lambda$$

with $d(s) = \frac{2^{2s}\Gamma(s)}{\Gamma(1-s)}$ and $w$ is the $s-$extension of $F(\underline{w})$ to the upper half space $\mathbb{R}^{n+1}_+$, i.e.

$$\begin{cases} \nabla \cdot \left(\lambda^{1-2s} \nabla w\right) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ w(x, 0) = F(\underline{w})(x) & \text{on } \{\lambda = 0\}. \end{cases}$$

We have the following estimates.

**Lemma 2.7.** *Let $u, v$ be such that $J^s(u, \mathbb{R}^n) < \infty$ and $J^s(v, \mathbb{R}^n) < \infty$ and $u \equiv v$ outside $B_R$, let $\widetilde{u}$, $\widetilde{v}$ be the $s-$extension of $F(u)$ and $F(v)$ to $\mathbb{R}^{n+1}_+$ respectively, then*

$$\widetilde{I}\left(\widetilde{u}, \mathbb{R}^{n+1}_+\right) = J^s(u, \mathbb{R}^n)$$

*and*

$$\widetilde{I}\left(\widetilde{u}, \mathbb{R}^{n+1}_+\right) - \widetilde{I}\left(\widetilde{v}, \mathbb{R}^{n+1}_+\right) = J^s(u, B_R) - J^s(v, B_R).$$

*Proof.* We first assume $u \in C_c^\infty(\mathbb{R}^n)$, integrating by parts, we have

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}_+} d(s) \lambda^{1-2s} |\nabla \widetilde{u}|^2 \, dx d\lambda \\
&= -d(s) \int_{\mathbb{R}^{n+1}_+} \widetilde{u} \nabla \cdot \left(\lambda^{1-2s} \nabla \widetilde{u}\right) - d(s) \lim_{\lambda \to 0^+} \int_{\mathbb{R}^n} \widetilde{u}(x, \lambda) \lambda^{1-2s} \partial_\lambda \widetilde{u}(x, \lambda) \, dx
\end{aligned}$$

$$= \int_{\mathbb{R}^n} F(u)\,(-\Delta)^s\, F(u)dx$$
$$= \frac{c_n\,(s)}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(F(u)\,(x) - F(u)\,(y))^2}{|x-y|^{n+2s}} dxdy.$$

First conclusion follows. General case follows via approximation argument. Second claim can be proved similarly. □

Next we derive energy estimates for $\widetilde{\mathcal{J}}$. We first quote the following estimates from [12].

**Lemma 2.8** (Lemma 5.7 [12], see also Proposition 4.6 in [13]). *Assume that $|w| \leq C_1$ and $|\nabla w| \leq C_2$ in $\mathbb{R}^n$. Then, for $s \in (0,1)$, the $s$-extension $\tilde{w}_s$ of $w$ satisfies*

$$|\tilde{w}_s| \leq C_1 \text{ and } |\nabla_x \tilde{w}_s| \leq C_2 \tag{2.6}$$

*in all $\mathbb{R}^{n+1}_+$. Moreover,*

$$|\nabla_x \tilde{w}_s| + |\partial_\lambda \tilde{w}_s| \leq \frac{CC_1}{\lambda} \text{ for } \lambda > 0, \tag{2.7}$$

*where $C$ depends only on $n$ (and not on $s$).*

Let

$$C_R = \{(x,\lambda), |x| < R, 0 \leq \lambda < R\}$$

be the open cylinder in $\mathbb{R}^{n+1}_+$ with height $R$ in the $\lambda$ direction and bottom $B_R \in \mathbb{R}^n$, we have the following relation between $\widetilde{I}\,(\tilde{u}, C_R)$ and $J^s\,(u, B_R)$.

**Lemma 2.9.** *Let $u \in C^{2,\gamma}\,(\mathbb{R}^n)$ with $\|u\|_{C^{2,\gamma}(\mathbb{R}^n)} \leq C_0$. Let $\widetilde{u}$ be the $s-$extension of $F(u)$ to $\mathbb{R}^{n+1}_+$, then for $R \geq 2$, we have*

$$\left|\widetilde{I}\,(\tilde{u}, C_R) - J^s\,(u, B_R)\right| \leq CM_0^2 C_0 \Phi_{n,s}\,(R)\,.$$

*Here $C$ is a constant depending only on $n, s$ and $M_0 = ||F(u)||_{C^1[-C_0,C_0]}$.*

*Proof.* Since $F \in C^1(\mathbb{R})$ and $\|u\|_{C^{2,\gamma}(\mathbb{R}^n)} \leq C_0$, there exists $M_0 > 0$ such that

$$|F(u)\,(x) - F(u)\,(y)| \leq M_0\,|u\,(x) - u\,(y)|\,, \;\; |\nabla F(u)| \leq M_0\,|\nabla u|\,,$$

and

$$\begin{aligned}
2\widetilde{I}\,(\tilde{u}, C_R) &= \int_{C_R} d\,(s)\,\lambda^{1-2s}\,|\nabla \tilde{u}|^2\, dxd\lambda \\
&= \int_{\partial B_R} \int_0^R d(s)\lambda^{1-2s} \tilde{u}_s \frac{\partial \tilde{u}_s}{\partial \nu} dSd\lambda \\
&\quad + \int_{B_R} d(s) R^{1-2s} (\tilde{u}_s \partial_\lambda \tilde{u}_s)_{\lambda=R} dx - \int_{B_R} \left(\lim_{\lambda \searrow 0} d(s)\lambda^{1-2s} \partial_\lambda \tilde{u}_s\right) \tilde{u}_s dx
\end{aligned} \tag{2.8}$$

Apply bounds (2.6) and (2.7) from Lemma 2.8 to $w = F(u)$ with $C_1 = M_0$, $C_2 = M_0 C_0$, we obtain

$$\begin{aligned}
&\left|\int_{\partial B_R} \int_0^R d(s)\lambda^{1-2s} \tilde{u}_s \frac{\partial \tilde{u}_s}{\partial \nu} dSd\lambda\right| \\
&\leq d(s) M_0^2 C_0 |\partial B_R| \int_0^R \min\{\lambda^{1-2s}, \lambda^{-2s} d\lambda\} \leq CM_0^2 C_0 \Phi_{n,s}(R).
\end{aligned} \tag{2.9}$$

Here we used $d(s)/(1-s) \leq C$ for $s \in (0,1)$ in the last inequality.

Similarly, we can bound the second term of (2.8) by

$$\left|\int_{B_R} d(s)R^{1-2s}(\tilde{u}_s\partial_\lambda\tilde{u}_s)_{\lambda=R}dx\right| \tag{2.10}$$
$$\leq C|B_R|d(s)R^{1-2s}M_0^2C_0/R \leq CM_0^2C_0\Phi_{n,s}(R).$$

To estimate the third term of (2.8), by the well known identity

$$\frac{c_n(s)}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n\setminus(\Omega^c\times\Omega^c)}\frac{(u(x)-u(y))\,(v(x)-v(y))}{|x-y|^{n+2s}}$$
$$=\int_\Omega v(x)(-\Delta)^s u(x)dx + c_n(s)\int_{\Omega^c}dx\int_\Omega dy\frac{u(x)-u(y)}{|x-y|^{n+2s}}v(x)$$

which holds for any $u, v \in C^2(\mathbb{R}^n)$, we have

$$\frac{c_n(s)}{2}J^s(u,B_R) = \int_{B_R}F(u(x))(-\Delta)^s F(u(x))dx$$
$$+ c_n(s)\int_{B_R^c}dx\int_{B_R}dy\frac{F(u(x))-F(u(y))}{|x-y|^{n+2s}}F(u(x)).$$

Recall that

$$-\lim_{\lambda\searrow 0}d(s)\lambda^{1-2s}\tilde{u}_s(x,\lambda)\partial_\lambda\tilde{u}_s(x,\lambda) = F(u(x))(-\Delta)^s F(u(x)),$$

therefore

$$2\widetilde{I}\left(\widetilde{u},C_R\right) - 2J^s\left(u,B_R\right)$$
$$=\int_{C_R}d\left(s\right)\lambda^{1-2s}\left|\nabla\widetilde{u}\right|^2dxd\lambda - \int_{B_R}F(u)\left(-\Delta\right)^s F(u)dx$$
$$-\,c_n\left(s\right)\int_{B_R^c}dx\int_{B_R}dy\frac{F(u)\left(x\right)F(u)\left(y\right)}{|x-y|^{n+2s}}F(u)\left(x\right)$$
$$=\int_{\partial B_R}\int_0^R d\left(s\right)\lambda^{1-2s}\widetilde{u}\frac{\partial\widetilde{u}}{\partial\nu}dSd\lambda + \int_{B_R}d\left(s\right)R^{1-2s}\left.\widetilde{u}\partial_\lambda\widetilde{u}\right|_{\lambda=R}dx$$
$$-\quad\int_{B_R}\lim_{\lambda\to 0^+}d\left(s\right)\lambda^{1-2s}\widetilde{u}\partial_\lambda\widetilde{u}dx - \int_{B_R}F(u)\left(-\Delta\right)^s F(u)dx$$
$$-\,c_n\left(s\right)\int_{B_R^c}dx\int_{B_R}dy\frac{F(u)\left(x\right)-F(u)\left(y\right)}{|x-y|^{n+2s}}F(u)\left(x\right)$$
$$=\int_{\partial B_R}\int_0^R d\left(s\right)\lambda^{1-2s}\widetilde{u}\frac{\partial\widetilde{u}}{\partial\nu}dSd\lambda + \int_{B_R}d\left(s\right)R^{1-2s}\left.\widetilde{u}\partial_\lambda\widetilde{u}\right|_{\lambda=R}dx$$
$$-\,c_n\left(s\right)\int_{B_R^c}dx\int_{B_R}dy\frac{F(u)\left(x\right)-F(u)\left(y\right)}{|x-y|^{n+2s}}F(u)\left(x\right).$$

From (2.9), (2.10) and

$$\left|\int_{B_R^c}dx\int_{B_R}dy\frac{F(u)\left(x\right)-F(u)\left(y\right)}{|x-y|^{n+2s}}F(u)\left(x\right)\right|$$
$$\leq CM_0^2C_0\int_{B_R^c}dx\int_{B_R}dy\frac{\min\{1,|x-y|\}}{|x-y|^{n+2s}}$$
$$\leq CM_0^2C_0\Phi_{n,s}(R),$$

we conclude the lemma. □

Next we obtain the following energy estimate on $\widetilde{\mathcal{J}}\left(\widetilde{u}, C_R\right).$

**Lemma 2.10.** *Let $u$ be a layer solution in $\mathbb{R}^n$ of (1.1) and $\widetilde{u}$ is the $s-$extenson of $F(u)$ to $\mathbb{R}^{n+1}_+$. Then $\widetilde{\mathcal{J}}\left(u, \widetilde{u}, C_R\right) \leq C\Phi_{n,s}\left(R\right),$ where $C$ depends only on $n, s, F, G$.*

*Proof.* Since
$$\tilde{J}(u, \tilde{u}, C_R) - J(u, B_R) = \tilde{I}(\tilde{u}, C_R) - J^s(u, B_R),$$
the conclusion follows from Proposition 2.2, Proposition 2.5 and Lemma 2.9. □

2.4. **Liouville type theorem and the 1-d symmetry of layer solutions.** In this section, we prove the following Liouville type theorem.

**Theorem 2.11.** *Let $\sigma$ satisfy*
$$\begin{cases} -\sigma\nabla\cdot\left(\lambda^{1-2s}\varphi^2\nabla\sigma\right) \leq 0 & \text{in } \mathbb{R}^{n+1}_+ \\ \sigma\left(x,0\right) = \underline{\sigma}\left(x\right) & \text{on } \mathbb{R}^n \\ -\underline{\sigma}\underline{\varphi}^2 \lim_{\lambda\to 0^+} d\left(s\right)\lambda^{1-2s}\partial_\lambda\sigma - \underline{\sigma}\nabla\cdot\left(\rho^2\nabla\underline{\sigma}\right) \leq 0 & \text{on } \mathbb{R}^n, \end{cases} \tag{2.11}$$
*where $\varphi$ and $\rho$ are continuous functions defined on $\mathbb{R}^{n+1}_+$ and $\mathbb{R}^n$ respectively and $\underline{\varphi}(x) = \varphi(x,0)$. Assume $\lambda^{1-2s}\varphi^2|\nabla\sigma|^2 \in L^1_{loc}(\overline{\mathbb{R}^{n+1}_+})$ and for $R \geq 2$,*
$$d\left(s\right)\int_{C_R}\lambda^{1-2s}\left(\varphi\sigma\right)^2 dxd\lambda + \int_{B_R}\left(\rho^2\underline{\sigma}\right)^2 dx \leq CR^2G\left(R\right) \tag{2.12}$$
*for some constant $C$ independent of $R$ and some nondecreasing function $G:\mathbb{R}_+\to\mathbb{R}_+$ satisfying*
$$\sum_{j=1}^{\infty}\frac{1}{G\left(2^{j+1}\right)} = +\infty.$$
*Then $\sigma$ is constant.*

*Proof.* Since $\sigma$ satisfies (2.11), we have
$$\nabla\cdot\left(\sigma\lambda^{1-2s}\varphi^2\nabla\sigma\right) \geq \lambda^{1-2s}\varphi^2\left|\nabla\sigma\right|^2.$$
Set
$$H\left(R\right) = \int_{B_R}\rho^2\left|\nabla\underline{\sigma}\right|^2 dx + \int_{C_R} d\left(s\right)\lambda^{1-2s}\varphi^2\left|\nabla\sigma\right|^2 dxd\lambda.$$
We have
$$\begin{aligned}\int_{B_R}\rho^2\left|\nabla\underline{\sigma}\right|^2 dx &= \int_{B_R}\left[\nabla\cdot\left(\underline{\sigma}\rho^2\nabla\underline{\sigma}\right) - \underline{\sigma}\nabla\cdot\left(\rho^2\nabla\underline{\sigma}\right)\right] dx \\ &= \int_{\partial B_R}\underline{\sigma}\rho^2\partial_\nu\sigma dS_x - \int_{B_R}\underline{\sigma}\nabla\cdot\left(\rho^2\nabla\underline{\sigma}\right) dx \\ &\leq \left(\int_{\partial B_R}\left(\underline{\sigma}\rho\right)^2 dS_x\right)^{\frac{1}{2}}\left(\int_{\partial B_R}\rho^2\left|\nabla\underline{\sigma}\right|^2 dS_x\right)^{\frac{1}{2}} - \int_{B_R}\underline{\sigma}\nabla\cdot\left(\rho^2\nabla\underline{\sigma}\right) dx,\end{aligned} \tag{2.13}$$
and
$$\begin{aligned}&\int_{C_R}\lambda^{1-2s}\varphi^2\left|\nabla\sigma\right|^2 dxd\lambda \\ &\leq \int_{C_R}\nabla\cdot\left(\lambda^{1-2s}\sigma\varphi^2\nabla\sigma\right) dxd\lambda \\ &= \int_{\partial^+C_R}\lambda^{1-2s}\sigma\varphi^2\partial_\nu\sigma dS - \int_{B_R}\underline{\sigma}\underline{\varphi}^2\lim_{\lambda\to 0^+}\lambda^{1-2s}\partial_\lambda\sigma dx\end{aligned} \tag{2.14}$$

$$\le \left(\int_{\partial^+ C_R} \lambda^{1-2s} (\sigma\varphi)^2 \, dS\right)^{\frac{1}{2}} \left(\int_{\partial^+ C_R} \lambda^{1-2s} \varphi^2 (\partial_\nu \sigma)^2 \, dS\right)^{\frac{1}{2}}$$
$$- \int_{B_R} \underline{\sigma} \underline{\varphi}^2 \lim_{\lambda \to 0^+} \lambda^{1-2s} \partial_\lambda \sigma dx.$$

Here $\partial^+ C_R = \partial C_R \backslash \{\lambda = 0\}$ and $\nu$ is the unit outer normal on $\partial^+ C_R$. It follows from (2.11), (2.13) and (2.14) that

$$\begin{aligned}
H(R) &\le \left(\int_{\partial B_R} (\underline{\sigma}\rho)^2 \, dS_x\right)^{\frac{1}{2}} \left(\int_{\partial B_R} \rho^2 |\nabla \underline{\sigma}|^2 \, dS_x\right)^{\frac{1}{2}} \\
&\quad + d(s) \left(\int_{\partial^+ C_R} \lambda^{1-2s} (\sigma\varphi)^2 \, dS\right)^{\frac{1}{2}} \left(\int_{\partial^+ C_R} \lambda^{1-2s} \varphi^2 (\partial_\nu \sigma)^2 \, dS\right)^{\frac{1}{2}} \\
&\le \left(\int_{\partial^+ C_R} d(s) \lambda^{1-2s} (\sigma\varphi)^2 \, dS + \int_{\partial B_R} (\underline{\sigma}\rho)^2 \, dS_x\right)^{\frac{1}{2}} \\
&\quad \cdot \left(\int_{\partial^+ C_R} d(s) \lambda^{1-2s} \varphi^2 (\partial_\nu \sigma)^2 \, dS + \int_{\partial B_R} \rho^2 |\nabla \underline{\sigma}|^2 \, dS_x\right)^{\frac{1}{2}} \\
&\le (H'(R))^{\frac{1}{2}} \left(\int_{\partial^+ C_R} d(s) \lambda^{1-2s} (\sigma\varphi)^2 \, dS + \int_{\partial B_R} (\underline{\sigma}\rho)^2 \, dS_x\right)^{\frac{1}{2}}.
\end{aligned}$$

If $H(R) > 0$, then

$$\left(\int_{\partial^+ C_R} d(s) \lambda^{1-2s} (\sigma\varphi)^2 \, dS + \int_{\partial B_R} (\underline{\sigma}\rho)^2 \, dS_x\right)^{-1} \le \frac{H'(R)}{H^2(R)}. \tag{2.15}$$

Assume $\sigma$ were not constant. Then there exists $R_0 > 0$ such that $H(R) > 0$ for all $R \ge R_0$. For $r_2 > r_1 > R_0$, integrating (2.15) and applying Schwartz inequality, we yield

$$\begin{aligned}
&\frac{1}{H(r_1)} - \frac{1}{H(r_2)} \\
&\ge \int_{r_1}^{r_2} \left(\int_{\partial^+ C_R} d(s) \lambda^{1-2s} (\sigma\varphi)^2 \, dS + \int_{\partial B_R} (\underline{\sigma}\rho)^2 \, dS_x\right)^{-1} dR \\
&\ge (r_2 - r_1)^2 \left(\int_{r_1}^{r_2} \left(\int_{\partial^+ C_R} d(s) \lambda^{1-2s} (\sigma\varphi)^2 \, dS + \int_{\partial B_R} (\underline{\sigma}\rho)^2 \, dS_x\right) dR\right)^{-1} \\
&\ge (r_2 - r_1)^2 \left(\int_{C_{r_2} \backslash C_{r_1}} d(s) \lambda^{1-2s} (\sigma\varphi)^2 \, dS + \int_{B_{r_2} \backslash B_{r_1}} (\underline{\sigma}\rho)^2 \, dx\right)^{-1} \qquad (2.16)
\end{aligned}$$

Let $N_0$ be such that $2^{N_0} > R_0$. $r_2 = 2^{j+1}$, $r_1 = 2^j$ with $j \ge N_0$. Summing over $j$ from $N_0$ to $N$, (2.12) and (2.16) imply

$$\frac{1}{H(2^{N_0})} \ge \frac{1}{4C} \sum_{j=N^0}^{N} \frac{1}{G(2^{j+1})}.$$

Let $N \to \infty$, a contradiction to the assumption

$$\sum_{j=1}^{\infty} \frac{1}{G(2^{j+1})} = +\infty. \qquad \square$$

*Proof of Theorem 2.1.* Let $u$ be a layer solution of (1.1) and $\widetilde{u}$ be the $s-$extension of $F(u)$ to $\mathbb{R}_+^{n+1}$. For $i>1$, take $\sigma^i=\frac{\partial_i\widetilde{u}}{\partial_1\widetilde{u}}$, $\varphi=\partial_1\widetilde{u}$ and $\rho=\partial_1 u$. Then $\underline{\varphi}=F'(u)\partial_1 u$ and $\underline{\sigma^i}=\frac{\partial_i u}{\partial_1 u}$ on $\mathbb{R}^n$. We will show $\sigma^i$, $\varphi$, $\rho$, $\underline{\varphi}$, $\underline{\sigma^i}$ satisfy the assumptions of Theorem 2.11. Indeed, we have

$$\begin{aligned}\nabla\cdot\left(\lambda^{1-2s}\varphi^2\nabla\sigma^i\right)&=\nabla\cdot\left(\partial_1\widetilde{u}\partial_i\lambda^{1-2s}\nabla\widetilde{u}-\partial_i\widetilde{u}\partial_1\lambda^{1-2s}\nabla\widetilde{u}\right)\\&=\lambda^{1-2s}\left(\partial_1\nabla\widetilde{u}\cdot\partial_i\nabla\widetilde{u}-\partial_i\nabla\widetilde{u}\cdot\partial_1\nabla\widetilde{u}\right)\\&\quad+\partial_1\widetilde{u}\partial_i\nabla\cdot\left(\lambda^{1-2s}\nabla\widetilde{u}\right)-\partial_i\widetilde{u}\partial_1\nabla\cdot\left(\lambda^{1-2s}\nabla\widetilde{u}\right)\\&=0\quad\text{in }\mathbb{R}_+^{n+1}.\end{aligned}$$

For boundary flux, we have

$$\begin{aligned}&-\underline{\sigma^i}\underline{\varphi}^2\lim_{\lambda\to0^+}d(s)\lambda^{1-2s}\partial_\lambda\sigma^i(x,\lambda)-\underline{\sigma^i}\nabla_x\cdot\left(\rho^2\nabla_x\underline{\sigma^i}\right)\\&=-\underline{\sigma^i}\underline{\varphi}^2d(s)\lim_{\lambda\to0^+}\frac{\partial_1\widetilde{u}(x,\lambda)\partial_i\lambda^{1-2s}\partial_\lambda\widetilde{u}(x,\lambda)-\partial_i\widetilde{u}(x,\lambda)\partial_1\lambda^{1-2s}\partial_\lambda\widetilde{u}(x,\lambda)}{\left(\partial_1\widetilde{u}(x,\lambda)\right)^2}\\&\quad-\underline{\sigma^i}\nabla\cdot\left(\partial_1u\partial_i\nabla u-\partial_iu\partial_1\nabla u\right)\\&=\underline{\sigma^i}F'(u)\left(\partial_1u\partial_i-\partial_iu\partial_1\right)(-\Delta)^sF(u)+\underline{\sigma^i}\left(\partial_1u\partial_i-\partial_iu\partial_1\right)(-\Delta u)\\&=\underline{\sigma^i}\left(\partial_1u\partial_i-\partial_iu\partial_1\right)\left[F'(u)(-\Delta)^sF(u)\right]+\underline{\sigma^i}\left(\partial_1u\partial_i-\partial_iu\partial_1\right)(-\Delta u)\\&=\underline{\sigma^i}\left(\partial_1u\partial_i-\partial_iu\partial_1\right)G'(u)\\&=\underline{\sigma^i}G''(u)\left(\partial_1u\partial_iu-\partial_iu\partial_1u\right)=0\quad\text{on }\mathbb{R}^n.\end{aligned}$$

Moreover, by energy estimates when $n=2$, $s>0$ or $3$, $s\geq\frac{1}{2}$,

$$\int_{C_R}d(s)\lambda^{1-2s}\left(\varphi\sigma^i\right)^2dxd\lambda+\int_{B_R}\left(\rho\underline{\sigma^i}\right)^2dx\leq\widetilde{J}\left(u,\widetilde{u},B_R\right)\leq C\Phi_{n,s}(R)\leq CR^2\ln R.$$

Since $G(R)=\ln R$ satisfies

$$\sum_{j=1}^{\infty}\frac{1}{G\left(2^{j+1}\right)}=\infty.$$

We conclude from Theorem 2.11 that $\sigma^i$ is equal to a constant $a^i$ for $i>1$. Therefore $\nabla u=\left(1,a^2\right)\partial_1u$ when $n=2$ and $\nabla u=\left(1,a^2,a^3\right)\partial_1u$ when $n=3$, which is equivalent to the 1-d symmetry of $u$. □

3. **1-d symmetry of stable solutions of (1.1).** In this section, we study the 1-d symmetry of stable solutions of (1.1) under the additional assumption that $F\in C^3(\mathbb{R})$ is strictly monotone with $F'$ bounded away from zero. We first prove the following proposition.

**Proposition 3.1.** *Assuming $F\in C^3(\mathbb{R})$ is a strictly monotone function with $|F'|\geq\delta_0>0$ for some $\delta_0$. Then a solution $u$ to (1.1) is stable iff the linearized equation of (1.1) has a positive solution. Here $v$ is a solution of the linearized equation of (1.1) at $u$ if*

$$-\Delta v+F''(u)v(-\Delta)^sF(u)+F'(u)(-\Delta)^s[F'(u)v]+G''(u)v=0\text{ in }\mathbb{R}^n.\qquad(3.1)$$

*Proof.* Assume (3.1) has a positive solution $v$. Given any $\phi\in C_0^2(\mathbb{R}^n)$, multiply $\frac{\phi^2}{v}$ to (3.1), we get

$$0=\int_{\mathbb{R}^n}\left[-\Delta v\frac{\phi^2}{v}+F''(u)\phi^2(-\Delta)^sF(u)\right]$$

$$+\int_{\mathbb{R}^n}\left[F'(u)\frac{\phi^2}{v}(-\Delta)^s(F'(u)v)+G''(u)\phi^2\right]. \tag{3.2}$$

When $F'(u)\geq 0$ or $F'(u)\leq 0$ for all $u$, we can bound the third term on the right hand side of (3.2) by

$$\begin{aligned}
&\int_{\mathbb{R}^n} F'(u)\frac{\phi^2}{v}(-\Delta)^s(F'(u)v)dx\\
&=\frac{c_n(s)}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{(F'(u)(x)v(x)-F'(u)(y)v(y))\left(\frac{F'(u)(x)\phi^2(x)}{v(x)}-\frac{F'(u)(y)\phi^2(y)}{v(y)}\right)}{|x-y|^{n+2s}}dydx\\
&=\frac{c_n(s)}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{(F'(u)(x)\phi(x))^2+(F'(u)(y)\phi(y))^2}{|x-y|^{n+2s}}dydx\\
&\quad-\frac{c_n(s)}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{F'(u)(x)F'(u)(y)}{|x-y|^{n+2s}}\left(\frac{\phi^2(x)v(y)}{v(x)}+\frac{\phi^2(y)v(x)}{v(y)}\right)dydx\\
&\leq\frac{c_n(s)}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{(F'(u)(x)\phi(x))^2-2F'(u)(x)F'(u)(y)\phi(x)\phi(y)}{|x-y|^{n+2s}}dydx\\
&\quad+\frac{c_n(s)}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{(F'(u)(y)\phi(y))^2}{|x-y|^{n+2s}}dydx \qquad (3.3)\\
&=\int_{\mathbb{R}^n}F'(u)\phi(-\Delta)^s(F'(u)\phi)dx,
\end{aligned}$$

together with integration by parts of the first term in (3.2), we conclude

$$\begin{aligned}
\mathrm{Hess}(u)(\phi,\phi)&=\int_{\mathbb{R}^n}\left[|\nabla\phi|^2+G''(u)\phi^2\right]dx\\
&\quad+\int_{\mathbb{R}^n}\left[\phi^2F''(u)(-\Delta)^sF(u)+F'(u)\phi(-\Delta)^s(F'(u)\phi)\right]dx\\
&\geq\int_{\mathbb{R}^n}\left[(-\Delta v)\frac{\phi^2}{v}+\left|\frac{\phi}{v}\nabla v-\nabla\phi\right|^2+G''(u)\phi^2\right]dx\\
&\quad+\int_{\mathbb{R}^n}\left[F''(u)\phi^2(-\Delta)^sF(u)+F'(u)\frac{\phi^2}{v}(-\Delta)^sF'(u)v)\right]dx\\
&\geq\int_{\mathbb{R}^n}\left|\frac{\phi}{v}\nabla v-\nabla\phi\right|^2dx\geq 0.
\end{aligned}$$

Conversely, assume $u$ is a stable solution on (1.1). Let

$$h(x)=F''(u(x))(-\Delta)^sF(u)+G''(u(x)).$$

For every $R>0$, let $\lambda_R$ be the infimum of the quadratic form

$$\begin{aligned}
Q_R(\xi)&=\int_{B_R}\frac{1}{2}\left[|\nabla\xi(x)|^2+h(x)\xi^2(x)\right]dx\\
&\quad+\frac{c_n(s)}{4}\iint_{\mathbb{R}^n\times\mathbb{R}^n\setminus B_R\times B_R}\frac{(F'(u(x))\xi(x)-F'(u(y))\xi(y))^2}{|x-y|^{n+2s}}dydx
\end{aligned}$$

among functions in the class

$$A_R=\left\{\xi\in H_0^1(B_R):\int_{B_R}\xi^2=1\right\}.$$

By stability assumption, we conclude that $\lambda_R \geq 0$ for each $R$ and is a nonincreasing function of $R$. Since $h(x)$ is a bounded function, the functional $Q_R$ is bounded from below in the class $A_R$ and any minimizing sequence $\{\xi_k\}$ has $\{|\nabla \xi_k\}$ uniformly bounded in $L^2(B_R)$. By compact embedding of $H_0^1(B_R) \subset\subset L^2(B_R)$, we conclude that the infimum of $Q_R$ is achieved by a function $\phi_R \in A_R$. Moreover, we can take $\phi_R \geq 0$ since $|\phi_R|$ is a minimizer when $\phi_R$ is a minimizer from the following inequalities:

$$\int_{B_R} \left[|\nabla \xi(x)|^2 + h(x)\xi(x)^2\right] dx \geq \int_{B_R} \left[|\nabla |\xi|(x)|^2 + h(x)\xi(x)^2\right] dx,$$

and

$$\left(F'(u(x))\xi(x) - F'(u(y))\xi(y)\right)^2 \geq \left(F'(u(x))|\xi|(x) - F'(u(y))|\xi|(y)\right)^2$$

when $F$ is monotone.

Note $\phi_R \geq 0$ is a solution, not identically zero, of

$$\begin{cases} L\phi_R = -\Delta\phi_R + h(x)\phi_R + F'(u)(-\Delta)^s\left(F'(u)\phi_R\right) = \lambda_R\phi_R & \text{on } B_R, \\ \phi_R = 0 & \text{on } \partial B_R. \end{cases}$$

We claim that $\phi_R > 0$ on $B_R$. In fact, let

$$g(x) = (h(x) - \lambda_R)\phi_R + F'(u)(-\Delta)^s(F'(u)\phi_R)),$$

since $F'(u)\phi_R \in W_0^{1,2}(B_R)$, we have $F'(u)(-\Delta)^s(F'(u)\phi_R)) \in W^{1-2s,2}(B_R)$, thus $g(x) \in L^2(B_R)$ when $0 < s < 1/2$ and $g(x) \in W^{1-2s,2}(B_R)$ when $s \geq 1/2$. From standard elliptic estimates and iteration, we conclude $\phi_R \in C^{2,\mu}$ for some $\mu > 0$. Assuming $F' > 0$, if $\phi_R(x_0) = 0$ for some $x_0 \in B_R$, we must have $-\Delta\phi_R(x_0) \leq 0$ and

$$(-\Delta)^s\left(F'(u)\phi_R\right)(x_0) = c(n,s)P.V.\int \frac{F'(u(x_0))\phi_R(x_0) - F'(u(y))\phi_R(y)}{|x-y|^{n+2s}}dy < 0.$$

which leads to

$$\begin{aligned} &L\phi_R(x_0) - \lambda_R\phi_R(x_0) \\ &= -\Delta\phi_R(x_0) + d(x_0)\phi_R(x_0) + F'(u)(-\Delta)^s\left(F'(u)\phi_R\right)(x_0) - \lambda_R\phi_R(x_0) < 0, \end{aligned}$$

a contradiction. $F' < 0$ case can be proved similarly.

Next we prove that $\lambda_R$ is decreasing in $R$. Indeed, assume there exists $R_1 < R_2$ such that $\lambda_{R_1} = \lambda_{R_2}$. Multiply $\phi_{R_1}$ to

$$L\phi_{R_1} - \lambda_{R_1}\phi_{R_1} = 0$$

and integrate by parts, we have

$$0 = \int_{B_{R_1}} |\nabla\phi_{R_1}|^2 + \left(h(x) - \lambda_{R_1}\right)\phi_{R_1}^2 + F'(u)\phi_{R_1}(-\Delta)^s(F'(u)\phi_{R_1}). \tag{3.4}$$

Multiply $\frac{\phi_{R_1}^2}{\phi_{R_2}}$ to

$$L\phi_{R_2} - \lambda_{R_2}\phi_{R_2} = 0,$$

and integrate by parts, we obtain

$$\begin{aligned} 0 = \int_{B_{R_1}} &-\left|\frac{\phi_{R_1}}{\phi_{R_2}}\nabla\phi_{R_2} - \nabla\phi_{R_1}\right|^2 + |\nabla\phi_{R_1}|^2 + \left(h(x) - \lambda_{R_2}\right)\phi_{R_1}^2 \\ &+ \int_{B_{R_1}} F'(u)\frac{\phi_{R_1}^2}{\phi_{R_2}}(-\Delta)^s(F'(u)\phi_{R_2}). \end{aligned} \tag{3.5}$$

When $F$ is monotone, following the same calculation in (3.3), we get

$$\int_{B_{R_1}} F'(u)\frac{\phi_{R_1}^2}{\phi_{R_2}}(-\Delta)^s(F'(u)\phi_{R_2}) \le \int_{B_{R_1}} F'(u)\phi_{R_1}(-\Delta)^s(F'(u)\phi_{R_1}). \tag{3.6}$$

Since $\lambda_{R_1} = \lambda_{R_2}$ and $\phi_{R_1} \neq \phi_{R_2}$, (3.5) and (3.6) yield

$$\int_{B_{R_1}} |\nabla\phi_{R_1}|^2 + (h(x) - \lambda_{R_1})\,\phi_{R_1}^2 + F'(u)\phi_{R_1}(-\Delta)^s(F'(u)\phi_{R_1}) > 0,$$

contradiction to (3.4).

Next, using $\lambda_R > 0$, we obtain

$$\begin{aligned} Q_R(\xi) &= \int_{B_R} |\nabla\xi|^2 + h(x)\xi^2 + F'(u)\xi(-\Delta)^s(F'(u)\xi) \ge \lambda_R \int_{B_R} \xi^2 \\ &\ge -\delta_R \int_{B_R} d(x)\xi^2 \end{aligned}$$

for all $\xi \in H_0^1(B_R)$ and $\delta_R$ chosen such that $0 < \delta_R \le \lambda_R/||d||_{L^\infty}$. From this, we deduce that

$$Q_R(\xi) \ge \frac{\delta_R}{1+\delta_R}\int_{B_R} |\nabla\xi|^2 \tag{3.7}$$

for all $\xi \in H_0^1(B_R)$.

For every given constant $C_R > 0$, consider the functional

$$\tilde{Q}_R(\xi) = Q_R(\xi) + \int_{B_R} c_R h(x)\xi$$

for $\xi \in H_0^1(B_R)$. (3.7) implies $\tilde{Q}_R$ is bounded below and coercive. The existence of a minimizer $\psi_R \in H_0^1(B_R)$ of $\tilde{Q}_R$ then follows from the compact embedding $H_0^1(B_R) \subset\subset L^2(B_R)$. Moreover, following the same proof as $\phi_R$, we conclude that $\psi_R \in C^{2,\nu}$ for some $\nu > 0$. Setting $\varphi_R = \psi_R + c_R$, then $\varphi_R$ is a solution of

$$\begin{cases} -\Delta\varphi_R + h(x)\varphi_R + F'(u)(-\Delta)^s(F'(u)\varphi_R) = 0 & \text{in } B_R, \\ \varphi_R = c_R & \text{on } \partial B_R. \end{cases} \tag{3.8}$$

Next, we claim that $\varphi_R > 0$ in $B_R$. By assumption, the negative part $\varphi_R^-$ of $\varphi_R$ vanises on $\partial B_R$. Multiplying $\varphi_R^-$ to (3.8) and integrating by parts, we obtain

$$0 = Q_R(\varphi_R^-) + \int_{B_R} F'(u)\varphi_R^-(-\Delta)^s(F'(u)\varphi_R^+). \tag{3.9}$$

Since $F$ is monotone, direct calculation using the integral expression (1.3) for $(-\Delta)^s$ gives

$$\int_{B_R} F'(u)\varphi_R^-(-\Delta)^s(F'(u)\varphi_R^+) \ge 0,$$

thus $Q_R(\varphi_R^-) \le 0$ by (3.9). On the other hand, since $u$ is a stable solution, we have $Q_R(\varphi_R^-) \ge 0$. Therefore $Q_R(\varphi_R^-) = 0$. By definition of the first eigenvalue $\lambda_R$ and the fact that $\lambda_R > 0$, this implies that $\varphi_R^- \equiv 0$, i.e. $\varphi_R > 0$ in $B_R$.

Picking $c_R > 0$ in (3.8) so that $\varphi_R(0,0) = 1$. Let $\zeta_R = F'(u)\varphi_R$, then $\zeta_R$ satisfyies

$$-\nabla\cdot\left(\frac{1}{[F'(u)]^2}\nabla\zeta_R\right) + g(x)\zeta_R + (-\Delta)^s\zeta_R = 0,$$

where

$$g(x) = \frac{h(x)}{[F'(u)]^2} - \frac{1}{F'(u)}\Delta\frac{1}{F'(u)}.$$

Apply Harnack inequality to $\zeta_R$ [5], we have

$$\sup_{B_R} \delta_0 \varphi_S \leq \sup_{B_R} \zeta_S \leq C_R \text{ for } S > 4R.$$

Standard elliptic estimates and Sobolev embedding theorem give uniform $C^\beta(\overline{B_{R/2}})$ bound on $\varphi_S$. Thus a subsequence converges locally to a positive solution $\varphi$. $\square$

As a direct corollary of Proposition 3.1, we conclude that a layer solution of (1.1) is a stable solution when $F$ is monotone.

**Corollary 3.2.** *If $F \in C^3(\mathbb{R})$ is a strictly monotone function with $F'$ bounded away from zero, then any layer solution of (1.1) is a stable solution.*

*Proof.* Let $u$ be a layer solution of (1.1), then $u_{x_1}$ is a positive solution of (3.1), conclusion follows from Proposition 3.1. $\square$

**Theorem 3.3.** *If $F \in C^3(\mathbb{R})$ is a strictly monotone function with $F'$ bounded away from zero and $u \in C^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ is a stable solution to (1.1) in $\mathbb{R}^2$, then $u$ is 1-d.*

*Proof.* Assume $u$ is a stable solution of (1.1) and $\phi$ is a positive solution to the linearlized equation (3.1). Let $\tilde{v}^i$ be the $s-$extension of $F'(u)u_{x_i}$ to $\mathbb{R}^{n+1}_+$ and $\tilde{\phi}$ be the $s-$extension of $F'(u)\phi$ to $\mathbb{R}^{n+1}_+$. Set $\sigma^i = \frac{\tilde{v}^i}{\tilde{\phi}}$, $\varphi = \tilde{\phi}$ and $\rho = \phi$, then $\underline{\varphi} = F'(u)\phi$ and $\underline{\sigma^i} = \frac{u_{x_i}}{\phi}$ on $\mathbb{R}^n$. We show that $\sigma^i, \varphi, \rho, \underline{\varphi}, \underline{\sigma^i}$ satisfy the assumptions of Theorem 2.11. Indeed,

$$\begin{aligned} \nabla \cdot (\lambda^{1-2s} \tilde{\phi}^2 \nabla \sigma^i) &= \nabla \cdot \lambda^{1-2s} (\tilde{\phi} \nabla \tilde{v}^i - \tilde{v}^i \nabla \tilde{\phi}) \\ &= \tilde{\phi} \nabla \cdot (\lambda^{1-2s} \nabla \tilde{v}^i) - \tilde{v}^i \nabla \cdot (\lambda^{1-2s} \nabla \tilde{\phi}) \\ &= 0 \text{ in } \mathbb{R}^{n+1}_+, \end{aligned}$$

and

$$\begin{aligned} &- \underline{\sigma^i} \underline{\tilde{\phi}}^2 \lim_{\lambda \to 0^+} d(s) \lambda^{1-2s} \partial_\lambda \sigma^i(x, \lambda) - \underline{\sigma^i} \nabla_x \cdot (\rho^2 \nabla_x \underline{\sigma^i}) \\ &= -\underline{\sigma^i} \underline{\tilde{\phi}}^2 \lim_{\lambda \to 0^+} d(s) \frac{\lambda^{1-2s} (\partial_\lambda \tilde{v}^i \tilde{\phi} - \tilde{v}^i \partial_\lambda \tilde{\phi})}{\tilde{\phi}^2} - \underline{\sigma^i} \nabla_x \cdot (\phi \nabla_x u_{x_i} - u_{x_i} \nabla_x \phi) \\ &= \underline{\sigma^i} \left[ F'(u)\phi(-\Delta)^s (F'(u)u_{x_i}) - F'(u)u_{x_i}(-\Delta)^s (F'(u)\phi) \right] - \underline{\sigma^i} \left[ \phi \Delta u_{x_i} - u_{x_i} \Delta \phi \right] \\ &= -\underline{\sigma^i} \left[ \phi h(x) u_{x_i} - u_{x_i} h(x) \phi \right] = 0 \text{ on } \mathbb{R}^n. \end{aligned}$$

Moreover, let $\eta$ be a cut off function which is supported in $B_{2R}$ and equals to 1 on $B_R$. Multiplying $u\eta^2$ to (1.1) and integrating by parts, we obtain

$$\begin{aligned} &\int_{B_{2R}} |\nabla u|^2 \eta^2 dx \\ &\leq C \int_{B_{2R} \setminus B_R} u^2 |\nabla \eta|^2 dx + \left| \int_{B_{2R}} G'(u) u \eta^2 \right| + \left| \int_{B_{2R}} F'(u) u \eta^2 (-\Delta)^s F(u) \right| \\ &\leq C R^2. \end{aligned}$$

We conclude from theorem 2.11 that $\sigma^i$ is equal to a constant $a^i$. Therefore $u$ is 1-d. $\square$

4. **1-d symmetry of stable solutions of** (1.13)**.** In this section, we prove the 1-d symmetry of stable solutions of (1.13), which is (1.1) without the Laplace term. We shall use a different approach here, mainly following ideas from [11, 24, 29] to obtain suitable BV estimates for stable solutions. We introduce the following notations.

$$\mathcal{J}^s(u,\Omega)=\int\int_{\mathbb{R}^n\times\mathbb{R}^n\setminus(\Omega^c\times\Omega^c)}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dydx,$$

and

$$\begin{aligned}L(u,\Omega)&=\int_\Omega G(u)dx+\frac{c_n(s)}{2}\int\int_{\mathbb{R}^n\times\mathbb{R}^n\setminus(\Omega^c\times\Omega^c)}\frac{|F(u(x))-F(u(y))|^2}{|x-y|^{n+2s}}dydx\\&=J^l(u,\Omega)+\frac{c_n(s)}{2}J^s(u,\Omega).\end{aligned}$$

Fix a unit vector $\mathbf{v}\in\mathbb{R}^n$, following [11, 24, 29], we construct suitable variations of energy with respect to $\mathbf{v}$ as follows. Let $R\geq 1$ and

$$\psi_{t,\mathbf{v}}(x):=x+t\tau(x)\mathbf{v},$$

where

$$\tau(x)=\begin{cases}1, & |x|\leq\frac{R}{2}\\ 2-2\frac{|x|}{R}, & \frac{R}{2}\leq|x|\leq R\\ 0, & |x|\geq R.\end{cases}\tag{4.1}$$

For $|t|$ small, $\psi_{t,\mathbf{v}}$ is a Lipschitz diffeomorphism and we define

$$P_{t,\mathbf{v}}u(x):=u\left(\psi_{t,\mathbf{v}}^{-1}(x)\right).$$

Then $P_{t,\mathbf{v}}u(x):=u(\psi_{t,\mathbf{v}}^{-1}(x))=u(x-t\mathbf{v})$ for $x\in B_{1/2}$ and $|t|$ small. Following the notations in [29] (see also [8, 24]), we define the second variation operator $\Delta^t_{\mathbf{vv}}$ with respect to $\mathbf{v}$ on any functional $\mathcal{F}$ as

$$\Delta^t_{\mathbf{vv}}\mathcal{F}(u,\Omega):=\mathcal{F}(P_{t,\mathbf{v}}u,\Omega)+\mathcal{F}(P_{-t,\mathbf{v}}u,\Omega)-2\mathcal{F}(u,\Omega).$$

We first quote the following Lemma from [29].

**Lemma 4.1** (Lemma 3.2 from [29], see also Lemma 4.3 from [8] and Lemma 2.1 from [24])**.** *There exists a universal constant $C(n,s)$ such that*

$$\Delta^t_{\mathbf{vv}}\mathcal{J}^s(u,B_R)\leq C(n,s)\frac{\mathcal{J}^s(u,B_R)}{R^2,}\quad\forall R\geq 1.$$

Following the same proof of the above lemma, we can get the following estimates on the second variations of fractional energy for

**Lemma 4.2.** *Given any $F$ with $|F'|\leq M_0$, we have*

$$\Delta^t_{\mathbf{vv}}L(u,B_R)=\Delta^t_{\mathbf{vv}}J^s(u,B_R)\leq C(n,s)M_0^2\frac{J^s(u,B_R)}{R^2,}\quad\forall R\geq 1.$$

*Proof.* Let $\eta$ be a smooth vector field supported in $B_R$, we consider the map

$$F_t(x)=x+t\eta(x),$$

and set

$$P_tu(x):=u(F_t^{-1}(x)).$$

We use $\tilde{B}_R$ to denote $\mathbb{R}^n\times\mathbb{R}^n\setminus(B_R\times B_R)$ and estimate

$$\Delta^tJ^s(u,B_R):=J^s(P_tu,B_R)+J^s(P_{-t}u,B_R)-2J^s(u,B_R).$$

Direct calculation shows

$$\begin{aligned}
&\Delta^t J^s(u,B_R)\\
&=\iint_{\tilde{B}_R} |F(u(x))-F(u(y))|^2\,[K(z+t\varepsilon|z|)JF_t(x)JF_t(y)\\
&\quad+K(z-t\varepsilon|z|)JF_{-t}(x)JF_{-t}(y)-2K(z)]\,dydx\\
&\le M_0^2\iint_{\tilde{B}_R} |u(x)-u(y)|^2\,[K(z+t\varepsilon|z|)JF_t(x)JF_t(y)\\
&\quad+K(z-t\varepsilon|z|)JF_{-t}(x)JF_{-t}(y)-2K(z)]\,dydx\\
&={M_0}^2\Delta^t\mathcal{J}^s(u,B_R)\le M_0^2C(n,s)||\nabla\eta||^2_{L^\infty(B_R)}K(z)t^2.
\end{aligned}$$

Here $z=x-y$, $K(z)=\frac{1}{|z|^{n+2s}}$, $\varepsilon(x,y):=\frac{\eta(x)-\eta(y)}{|x-y|}$ and

$$JF_t(x)=1+t\mathrm{div}\eta(x)+A(\eta(x))t^2,$$

with $A(\eta)=\frac{(\mathrm{div}\eta)^2-tr(\nabla\eta)^2}{2}$. The last step follows from estimates in Lemma 3.2 [29] and

$$|F(u(x))-F(u(y))|\le M_0|u(x)-u(y)|.$$

Choosing $\eta=\tau\mathbf{v}$ where $\tau(x)$ is defined by (4.1) and observing

$$\Delta^t_{\mathbf{vv}}J^l(u,B_R)=0,$$

the conclusion follows. □

**Lemma 4.3** (Lemma 2.2 [29]). *Assume $|u|\le 1$ and $||\nabla u||_{L^\infty(B_1)}\le L_0$, where $L_0\ge 2$, then for $s\in[1/2,1)$,*

$$\begin{aligned}
&\int_{B_1}\int_{B_1}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy\\
&\le\frac{1}{1-s}|S^{n-1}|L_0|^{2s-1}\left((2-2s)\log(2L_0)+1\right)\int_{B_1}|\nabla u(x)|dx\\
&=C(n,s)L_0^{2s-1}\log(L_0)\int_{B_1}|\nabla u(x)|dx.
\end{aligned}$$

**Lemma 4.4** (Lemma 1.8 [29]). *For any ball $B_R\subset\mathbb{R}^n$ and $u$ which belongs to appropriate space with $|u|\le 1$, and let $s\in(0,\frac{1}{2})$, there exists universal constant $C=C(n,s)>0$ such that for any $R\ge 1$,*

$$\mathcal{J}^s(u,B_R)\le C\left(\int_{B_{2R}}|\nabla u|dx+R^{n-2s}+R^n\right).$$

*If $\frac{1}{2}\le s<1$ and $u$ is assumed to be a Lipschitz function with $||\nabla u||_{L^\infty(B_R)}\le L_0$, $L_0\ge 2$. then there exists $C=C(n,s)>0$ such that*

$$\mathcal{J}^s(u,B_R)\le C\left(R^{n-2s}+L_0^{2s-1}\log(2L_0R)\int_{B_{2R}}|\nabla u|\right).$$

When $F$ is monotone, we have the following counterparts of Lemma 3.3 and Remark 3.4 in [29].

**Lemma 4.5.** *Let $\Omega\subset\mathbb{R}^n$. Assume $F(u)$ is monotone function in $u$. For any functions $u,v$ in appropriate space, let $u\wedge v=\max\{u,v\}$, $u\vee v:=\min\{u,v\}$, we have the following identity*

$$J^s(u,\Omega)+J^s(v,\Omega)-J^s(u\wedge v,\Omega)-J^s(u\vee v,\Omega)$$

$$= 2\iint_{\mathbb{R}^n\times\mathbb{R}^n\setminus(\Omega^c\times\Omega^c)} (F(v)-F(u))_+(x)(F(v)-F(u))_-(y)K(x-y)dydx,$$

*where* $K(z) = \frac{1}{|z|^{n+2s}}$, $(F(v)-F(u))_+ = (F(u)-F(v))\vee 0$ *and* $(F(v)-F(u))_- = (F(u)-F(v))\wedge 0$.

*Proof.* Set

$$A := \{x\in\mathbb{R}^n : v(x) > u(x)\}$$

and

$$\tilde\Omega := \mathbb{R}^n\times\mathbb{R}^n\setminus(\Omega^c\times\Omega^c).$$

Direct calculation gives

$$\begin{aligned}
&J^s(u,\Omega)+J^s(v,\Omega)-J^s(u\wedge v,\Omega)-J^s(u\vee v,\Omega)\\
&= \iint_{(A\times A^c)\cap\tilde\Omega}\left(|F(u(x))-F(u(y))|^2-|F(u(x))-F(v(y))|^2\right)K(x-y)dydx\\
&\quad+\iint_{(A\times A^c)\cap\tilde\Omega}\left(|F(v(x))-F(v(y))|^2-|F(v(x))-F(u(y))|^2\right)K(x-y)dydx\\
&\quad+\iint_{(A^c\times A)\cap\tilde\Omega}\left(|F(u(x))-F(u(y))|^2-|F(v(x))-F(u(y))|^2\right)K(x-y)dydx\\
&\quad+\iint_{(A^c\times A)\cap\tilde\Omega}\left(|F(v(x))-F(v(y))|^2-|F(u(x))-F(v(y))|^2\right)K(x-y)dydx\\
&= 2\iint_{(A\times A^c)\cap\tilde\Omega}\left(F(v(x))-F(u(x))\right)\left(F(u(y))-F(v(y))\right)K(x-y)dydx\\
&\quad+2\iint_{(A^c\times A)\cap\tilde\Omega}\left(F(u(x))-F(v(x))\right)\left(F(v(y)-F(u(y))\right)K(x-y)dydx\\
&= 2\iint_{\mathbb{R}^n\times\mathbb{R}^n\setminus(\Omega^c\times\Omega^c)}(F(v)-F(u))_+(x)(F(v)-F(u))_-(y)K(x-y)dydx,
\end{aligned}$$

where we used the monotonicity of $F$ in the last step. $\square$

**Remark 4.6.** The above lemma implies

$$J^s(u,\Omega)+J^s(v,\Omega)\geq J^s(u\wedge v,\Omega)+J^s(u\vee v,\Omega)$$

if $F$ is a montone function and "=" holds only if either $v\leq u$ or $u\leq v$ in $\Omega$.

We can prove the following lemma.

**Lemma 4.7.** *Let $F$ be a monotone function with $|F'|\geq\alpha>0$. If $u$ is a stable solution to (1.13), then there exists a universal constant $C=C(n,s)$ such that for any $R\geq 1$,*

$$\left(\int_{B_{1/2}}(\partial_{\mathbf{v}}u(y))_+\right)\left(\int_{B_{1/2}}(\partial_{\mathbf{v}}u(y))_-\right)\leq\frac{C}{\alpha^2R^2}J^s(u,B_R). \tag{4.2}$$

*and*

$$\int_{B_{1/2}}|\nabla u(x)|dx\leq C\left(1+\sqrt{J^s(u,B_R)}\right). \tag{4.3}$$

*Proof.* We follow the proof of Lemma 3.6 in [29]. Let $\overline{u}=\max\{P_{t,\mathbf{v}}u,u\}$, $\underline{u}=\min\{P_{t,\mathbf{v}}u,u\}$. According to Lemma 4.5 and Remark 4.6, we have

$$J^s(\overline{u},B_R)+J^s(\underline{u},B_R)$$

$$+ 2\int_{B_{1/2}}\int_{B_{1/2}} \frac{(F(u(x-t\mathbf{v}))-F(u(x)))_+ (F(u(y-t\mathbf{v}))-F(u(y)))_-}{|x-y|^{n-2s}} dydx$$
$$\leq J^s(P_{t,\mathbf{v}}u, B_R) + J^s(u, B_R).$$

Since $|x-y| < 1$ when $x, y \in B_{1/2}$ and

$$J^l(P_{t,\mathbf{v}}u, B_R) + J^l(u, B_R) = J^l(\overline{u}, B_R) + J^l(\underline{u}, B_R),$$

we have

$$\begin{aligned} & L(\overline{u}, B_R) + L(\underline{u}, B_R) \\ & \quad + 2\alpha^2 \int_{B_{1/2}}\int_{B_{1/2}} (u(x-t\mathbf{v})-u(x))_+ (u(y-t\mathbf{v})-u(y))_- \, dydx \\ & \leq L(P_{t,\mathbf{v}}u, B_R) + L(u, B_R). \end{aligned} \tag{4.4}$$

Adding $L(P_{-t,\mathbf{v}}u, B_R) - 3L(u, B_R)$ to both sides of (4.4) and using the stability condition of $u$ and Lemma 4.2, we have

$$\begin{aligned} & 2\alpha^2 \int_{B_{1/2}}\int_{B_{1/2}} (u(x-t\mathbf{v})-u(x))_+ (u(y-t\mathbf{v})-u(y))_- \, dydx \\ & \leq o(t^2) + \Delta^t_{\mathbf{vv}} L(u, B_R) \leq Ct^2 J^s(u, B_R)/R^2. \end{aligned}$$

Dividing $t^2$ on both sides and pass to the limit as $t \to 0$, (4.2) follows. (4.3) can be derived from (4.4) by the same proof in Lemma 3.6 [29]. □

The following BV and energy estimates for stable solutions of (1.13) can be proved by following the proof of Proposition 1.7 in [29].

**Proposition 4.8.** *Let $|F'| \leq M_0$ and $u \in C^2(\mathbb{R}^n)$ be a stable solution to (1.13) in $\mathbb{R}^n$ with $|u| \leq 1$, then there exists constants $C_1 = C_1(n, s, M_0)$ and $C_2 = C_2(n, G, M_0, s)$ such that for any ball $B_R \subset \mathbb{R}^n$, $R \geq 1$, we have*

$$\int_{B_R} |\nabla u| \leq \begin{cases} C_1 R^{n-1} & 0 < s < \frac{1}{2} \\ C_2 R^{n+2s-2} & \frac{1}{2} \leq s < 1 \end{cases}$$

*and*

$$J^s(u, B_R) \leq \begin{cases} C_1 R^{n-2s} & 0 < s < \frac{1}{2} \\ C_2 R^{n+2s-2}\log^2(K_0 R) & \frac{1}{2} \leq s < 1, \end{cases}$$

*where $K_0 \geq 2$ is an upper bound for $||G||_{L^\infty}$.*

*Proof.* Since

$$J^s(u, B_R) \leq M_0^2 \mathcal{J}^s(u, B_R),$$

Lemma 4.7 yields

$$\int_{B_{1/2}} |\nabla u(x)| dx \leq C(n, M_0, s)\left(1 + \sqrt{\mathcal{J}^s(u, B_R)}\right).$$

We can then repeat the proof of Proposition 1.7 in [29] line by line to finish the proof. □

As a direct corollary of Lemma 4.7 and Proposition 4.8, we can prove the following theorem.

**Theorem 4.9.** *If $F \in C^3(\mathbb{R})$ is a strictly monotone function with $F'$ bounded away from zero and $u \in C^2(\mathbb{R}^2)$ is a stable solution to (1.13) in $\mathbb{R}^2$ and $|u| \leq 1$, then $u$ is 1-d.*

*Proof.* Letting $R \to \infty$ in (4.2), by Proposition 4.8, we have

$$\left(\int_{B_{1/2}} (\partial_{\mathbf{v}} u(y))_+\right)\left(\int_{B_{1/2}} (\partial_{\mathbf{v}} u(y))_-\right) = 0. \tag{4.5}$$

Hence $u$ is monotone in $B_{1/2}$ along direction $\mathbf{v}$. Since (4.5) holds for any $\mathbf{v}$ and any half ball, we conclude $u$ is 1-d by continuity of $u$. □

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