Perturbation theory for dark-bright solitons of the Manakov system

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Abstract

We present a direct perturbation method to study the dynamics of dark-bright solitons of the Manakov system under the influence of perturbations. Our methodology combines a multiscale expansion method, perturbed conservation laws, and a boundary layer approach, which breaks the problem into an inner region, pertinent to the soliton core, and an outer region, which evolves independently of the dark-bright soliton. We find that a shelf emerges around the dark soliton component, which propagates with a speed depending on the background intensity. Conserved quantities of the Manakov system are employed to determine the properties of the perturbed solutions. We focus on dissipative perturbations, such as diffusion, as well as linear and nonlinear loss, and show that the effect of the bright ("filling") soliton component is to partially stabilize "bare" dark solitons of the scalar case against perturbation-induced dissipation. Our analytical predictions are corroborated by results of direct numerical simulations.

1 Introduction

The mathematical modeling of physical phenomena often leads to a certain class of nonlinear partial differential equations (PDEs) known as integrable systems. One of the distinguished features of integrable systems is that they admit exact soliton solutions, i.e., stable, exponentially localized traveling waves which interact with one another non-destructively, preserving their shape and velocity in the interaction [1]. Moreover, integrable systems possess an infinite number of conserved quantities, as well as a Lax pair that allows one to linearize them using the Inverse Scattering Transform (IST), a nonlinear analog of the Fourier transform [2]. One of the prototypical integrable equations is the nonlinear Schrödinger (NLS) equation:

$$i q_t + q_{xx} - 2\nu |q|^2 q = 0, \tag{1}$$

with $\nu = \mp 1$ corresponding to the "focusing" and "defocusing" regimes, where bright or dark solitons can be supported, respectively. The NLS equation is a universal model for weakly dispersive nonlinear wave trains, and it has been derived in such diverse fields as deep water waves, plasmas, nonlinear fiber optics, Bose-Einstein condensates (BECs), spin waves, etc [3–10].

Vector generalizations of the scalar NLS equations (VNLS equations for short) arise as relevant physical models, under conditions similar to those described by NLS, whenever there are suitable multiple wavetrains moving with nearly the same group velocity. This may happen, e.g., in nonlinear optics, when two waves of different polarizations or two waves of different frequencies feature a nonlinear interaction [11]. In such situations, of particular relevance is the so-called Manakov system [12], which is a coupled NLS system:

$$i\mathbf{q}_t + \mathbf{q}_{xx} - 2\nu\|\mathbf{q}\|^2\mathbf{q} = 0\,, \tag{2}$$

where $\mathbf{q}(x,t)$ is a two-component, complex vector function. Notably, the Manakov system is completely integrable, and, like its scalar counterpart, it admits a Lax pair, soliton solutions, infinite number of conserved quantities, etc.

Menyuk showed in [13] that in optical fibers with constant birefringence, assuming certain nonlinear (four-wave mixing) terms are neglected, the two polarization components of the complex electromagnetic field envelope orthogonal to direction of propagation along a fiber satisfy asymptotically the following nondimensional equations:

$$i(u_t + \delta u_x) + d u_{xx} + (|u|^2 + \alpha|v|^2) = 0, \qquad i(v_t + \delta v_x) + d v_{xx} + (\alpha|u|^2 + |v|^2) = 0, \tag{3}$$

where δ represents the group velocity "mismatch" between the components, $2d$ is the group velocity dispersion (whose sign accounts for focusing vs defocusing regimes), and α is a constant depending on the polarization properties of the fiber. The physical phenomenon of birefringence implies that the phase and group velocities of the electromagnetic wave are different for each polarization component. When $\alpha \neq 1$ the above system is not integrable. However, averaging over the fast birefringence fluctuations that are normally observed in a communications environment [14] yields $\delta = 0$ and $\alpha = 1$ – that is, the system reduces to the VNLS (2).

Notice that, besides its relevance to nonlinear fiber optics, more recently, the Manakov system has attracted much attention in studies related to homogeneous multicomponent BECs, composed by, e.g., different spin states of the same atom species [15–18]; in this context, a physically relevant situation refers to the case where the repulsive inter- and intra-species interactions are of (approximately) equal strength, a fact rendering Eq. (2) the appropriate model. Vector solitons of the VNLS model have also attracted much attention, especially in the defocusing setting ($\nu = +1$). Of particular interest in this setting are the dark-bright (DB) soliton solutions; in these states, the bright soliton —which is not supported by the scalar defocusing NLS— only emerges because of an effective potential well created by the dark soliton through the inter-component interaction; as such, DB solitons are commonly referred to as "symbiotic" solitons. Predicted in the seminal work by Busch and Anglin [19], DB solitons were first experimentally realized by phase impriting method [20], followed by experimental observation of trains of DB solitons generated by counterflow of two superfluids [21].

While the above discussion refers to integrable systems, in most physically relevant settings the pertinent model PDEs are usually non-integrable. Nevertheless, the theoretical predictions for the soliton solutions in the corresponding integrable cases have proved to be an extremely valuable tool for the investigation of the non-integrable solitary waves in regimes that are reasonably close to the integrable ones. As such, in many works, perturbation-based techniques of nearly integrable systems have been employed for the investigation of the evolution of solitons in the presence of small perturbations, such as linear or nonlinear loss or gain, diffusion, higher-order dispersion or nonlinearity effects, etc. The goal of this work is to present a rigorous direct perturbation theory for the study of DB solitons of the defocusing Manakov system over a constant background under small perturbations, namely:

$$i\mathbf{q}_t + \mathbf{q}_{xx} - 2(\|\mathbf{q}\|^2 - q_o^2)\mathbf{q} = \varepsilon\mathbf{R}[\mathbf{q}], \tag{4}$$

where $\mathbf{q}(x,t) = (q_1, q_2)^T$, $\mathbf{R}[\mathbf{q}] = (R_1[\mathbf{q}], R_2[\mathbf{q}])^T$, $0 < \varepsilon \ll 1$ is a perturbation parameter, and

$$\mathbf{q}(x,t) \sim \mathbf{q}^{\pm}(t) = (q_o e^{i\theta_\pm}, 0)^T \qquad x \to \pm\infty, \tag{5}$$

are boundary conditions (BCs) that correspond to solutions which are "dark" in the first component and "bright" in the second one. The linear term proportional to the background amplitude q_o in (4) has been introduced to make the boundary conditions independent of t. This can be achieved by a gauge transformation replacing $\mathbf{q}$ by $\exp(2iq_o^2 t)\mathbf{q}$, or, more generally, by $\exp[2i\int_0^t q_o(s)ds]\mathbf{q}$ when the background amplitude q_o depends on t.

We will consider arbitrary perturbations satisfying $\mathbf{R}[\mathbf{0}] = \mathbf{0}$, which are asymptotically phase-invariant, namely, such that $\lim_{x\to\pm\infty} R_j[\mathbf{q}e^{i\phi}] = e^{i\phi_\pm}\left(R_j[\mathbf{q}^\pm]\right)$ for $j = 1,2$, for any phase $\phi(x,t)$, with $\phi_\pm := \lim_{x\to\pm\infty}\phi(x,t)$ (of course, $\phi_\pm \equiv \phi$ in the case of a constant phase). Note that these are fairly general assumptions, as a wide class of physically relevant perturbations satisfy these conditions; these include, e.g., diffusion: $R_j[\mathbf{q}] = i\gamma_j\partial_x^2 q_j$, linear loss: $R_j[\mathbf{q}] = i\gamma_j q_j$, nonlinear loss (pertinent to two-photon absorption in optics): $R_j[\mathbf{q}] = i\gamma_j q_j|q_j|^2$, etc. Note that the gauge transformation used above to remove the fast evolution of the background phase is also phase-invariant, justifying the use of Eq. (4) instead of perturbing (2).

As mentioned above, the unperturbed defocusing Manakov system, namely Eq. (4) with $\varepsilon = 0$, admits exact DB soliton solutions of the form:

$$q_1(x,t) = \left[\operatorname{sgn}(V)\sqrt{q_o^2 - A_d^2} - iA_d\tanh(\sqrt{A_d^2 - A_b^2}(x - Vt - x_o))\right]e^{i\sigma_o}, \tag{6a}$$

$$q_2(x,t) = A_b\exp\left[i\left(\tfrac{V}{2}x - \left(\tfrac{V^2}{4} - (A_d^2 - A_b^2)\right)t + \varphi_o\right)\right]\operatorname{sech}(\sqrt{A_d^2 - A_b^2}(x - Vt - x_o)), \tag{6b}$$

where A_b, A_d determine the amplitudes of the bright/dark component, respectively, q_o is the background, V the soliton velocity, x_o the soliton center, and σ_o, φ_o are arbitrary phases [22–28]. Note that the velocity V is related to the amplitudes by

$$V^2 A_d^2 = 4(q_o^2 - A_d^2)(A_d^2 - A_b^2), \tag{7}$$

and for a stationary DB soliton $V = 0$, $A_d = q_o$ and $A_b < q_o$ is arbitrary. The scalar dark soliton can be obtained by setting $A_b = 0$, i.e., $q_2 \equiv 0$.

While perturbation theory for solitons that decay rapidly at infinity has been widely studied since the late seventies —with a variety of methods ranging from multi-scale perturbation analysis, IST-based perturbation techniques, perturbation of conserved quantities, and direct numerical simulations [29–33]— the nonvanishing background of dark solitons introduces severe complications when applying the perturbative methods developed in the rapidly decaying case. For the scalar defocusing NLS equation, dark solitons are completely determined by the four parameters q_o, A_d, x_o, σ_o, and in this case Eq. (7) reduces to $V^2 = q_o^2 - A_d^2$. In some early works, the perturbation of "black" (i.e., stationary dark) solitons in lossy fibers was studied numerically [34] and analytically [35, 36]. The method developed in [36] was subsequently extended to "grey" (i.e., non-stationary or moving dark) solitons and to generic perturbations, but only two of the four main soliton parameters, q_o and A_d, were determined. In [37], it was shown that under a perturbation the background evolves independently of the soliton, and after separating the background amplitude from the soliton "core", it is possible to determine the dark soliton's amplitude and width upon using the Hamiltonian approach of the adiabatic perturbation theory (which is based on perturbed conservation laws). A similar approach was also used for DB solitons in BECs [18].

It should be noted, however, that for dark solitons the adiabatic evolution of the soliton parameters alone is not sufficient to fully characterize the perturbed solution. The reason is that, in many cases, the perturbation generates a moving "shelf", namely a linear wave emerging on either side of the soliton; the existence of this shelf, which was confirmed both numerically and analytically, was in fact used to explain observed discrepancies in the perturbed conservation laws [38], though without determining analytically the core soliton parameters.

An alternative approach to soliton perturbation theory in the rapidly decaying case was pioneered in [29], as a way to determine the effects of small perturbations on the evolution of the soliton spectral (and hence physical) parameters. The method, which might be referred to as "integrable perturbation theory", relies on the IST and on completeness of squared eigenfunctions, namely quadratic combinations of Jost eigenfunctions and their adjoints which satisfy the linearized version of the integrable PDE. There have been many attempts at generalizing the IST-based perturbation theory to the case of dark solitons since the early nineties. For instance, in Ref. [39], orthogonality conditions were derived from a set of squared Jost functions for the scalar defocusing NLS equation over a constant background, and from these conditions one can in principle obtain all of the soliton parameters. This early work, however, did not account at all for the evolution of the background induced by the perturbation. Subsequent works presented proofs of the completeness of the squared eigenfunctions using different approaches [40–45], but the results were not consistent with each other. As an example, the proof in [40] was claimed to be incorrect in [44, 45], based on the observation that the complete set should have two, not just one, continuous spectrum basis vectors, which resulted in different predictions for the soliton velocity and the first-order correction. In [46], the results of [40, 42] and [44, 45] were then declared to be "equivalent" under some kind of "transformation between two integral variables". On the other hand, in [43] squared eigenfunctions were used (though without explicitly referring to them, or to their completeness) to develop an IST-based perturbation theory for the defocusing NLS on a background. The main drawbacks of all these works is that none of them accounted for perturbative contributions from the shelf that develops around the dark soliton, or presented comparisons of the theoretical predictions with numerical simulations.

To date, the most comprehensive analysis of dark-soliton perturbation for the scalar defocusing NLS is found in [47], where a multiscale expansion method and perturbed conservation laws were used to find both the growth in magnitude and phase of the shelf, and the adiabatic evolution of all soliton parameters. This work also highlighted the emergence of a moving boundary layer connecting the inner soliton core to the outer background.

There are many papers available in the literature on bright soliton perturbation theory, but, as we mentioned above, less so for dark soliton, and only a handful that addressed vector/multicomponent problems on a nontrivial background. In light of the difficulties that scalar problems for dark solitons have presented, it is not surprising that few attempts have been made so far to develop perturbative approaches for multicomponent integrable systems on a background. In particular, to the best of our knowledge the only works on soliton perturbation theory for the defocusing Manakov system on a nontrivial background are Refs. [18, 48, 49]. The approach in these papers relies on the adiabatic approximation, and evolution equations for the soliton parameters of a DB soliton are derived by expanding the solution into a set of complete eigenfunctions of the linearized operator. We should mention, however, that the completeness result for the eigenfunctions appears to be an unsettled issue even in the scalar case, and in the Manakov system the defect of analyticity of the scattering eigenfunctions further complicates the problem. Furthermore, it is not clear how or even if the soliton shelf can be incorporated into the description, how the results in these papers compare with the scalar reduction on this account, and no comparisons with direct numerical simulations are offered to corroborate the results. Arguably, the perturbation theory for DB Manakov solitons is to a large extent still an open problem, and the present work aims at filling this gap. We want to stress the great practical importance of this problem: besides the obvious relevance of being able to include, at least perturbatively, physical effects such as dissipation or loss, in optical fibers described by Eq. (3) one would also be able to account for more general polarization properties of the fiber (e.g., for values of $\alpha \neq 1$ but reasonably close to it), and/or include four-wave mixing effects as perturbations of the integrable case. Also, in the recent applications to BECs, the harmonic trap necessary to achieve confinement of the atoms in multicomponent repulsive condensates is typically devised in such way that the ratio of longitudinal vs transverse trapping frequencies is very small ($\Omega \sim 10^{-2}$). In these cigar-shaped geometry for the condensates, the BEC dynamics for the longitudinal part of the wave function is indeed described as a perturbation of the integrable defocusing Manakov system. Moreover, in [50, 51] the coupling coefficients for "symmetric" spin-independent and "antisymmetric" spin-dependent interaction terms are also such that their ratio is a small parameter up to which the model equation can be considered a small perturbation of a 3- component Manakov system. These are just some examples of the practical applicative relevance of this work. Specifically, the goal of this paper is a highly non-trivial generalization of the methods developed in [47] to describe perturbations of DB solitons.

The plan of the paper is the following. In Sec. 2 we discuss the evolution of the background, and in Sec. 3 we use multiple scales in time to formulate the perturbation problem for a stationary DB soliton. It is shown that, similarly to the scalar case, a shelf emerges, propagating with a speed determined by the background intensity. The problem is then broken into an inner region, where the core of the soliton resides, and an outer region which evolves independently of the soliton. Here it is noted that, even for stationary DB solitons the amplitude of the bright soliton, A_b, is an additional physical parameter one needs to determine. Therefore, the perturbed conservation laws (for Hamiltonian, energy and momentum) used in the scalar case are not sufficient to determine all the soliton parameters. We are able to suitably augment the set of conserved quantities of the unperturbed Manakov system, and use them to determine the properties of the moving shelf, as well as the adiabatic evolution of the soliton parameters. Our results for a stationary DB soliton coincide with the ones in [47] in the reduction to the scalar case, and show good agreement with direct numerical simulations. In Sec. 4, we generalize the multiscale perturbation theory to a moving DB soliton. Although in this case we are not able to obtain a complete solution for the amplitudes and phases of the DB soliton at $O(\varepsilon)$, we use the augmented perturbed conservation laws to obtain a nonlinear system of adiabatic evolution equations for the soliton and shelf parameters (8 coupled ODEs in the slow time $T_1 = \varepsilon t$ for the DB soliton parameters q_o, A_d, A_b, σ_o, and for the asymptotic amplitudes and phases of the shelf as $x \to \pm\infty$). In Sec. 5, we discuss the boundary layer region. In Sec. 6, we present the solution of the system of equations for the adiabatic evolution of the soliton parameters for specific perturbations (diffusion, linear and nonlinear loss in both components). This is a highly nontrivial result, since, a priori, one would have no guarantee that the system of equations for the above mentioned DB soliton parameters could be solved in closed form. Furthermore, we show that our results are consistent with [47] in the scalar reduction, and we also compare them in the general vector case with some direct numerical simulations, with excellent agreement. Finally, Sec. 7 is devoted to some concluding remarks and a discussion for future work.

2 The boundary at infinity

Let us consider the perturbed Manakov system (4) with BCs (5), consistent with a DB soliton. We seek the solution in the form of the following asymptotic expansions in ε:

$$q_1 = q_{10} + \varepsilon q_{11} + O(\varepsilon^2), \quad q_2 = q_{20} + \varepsilon q_{21} + O(\varepsilon^2), \tag{8}$$

and introduce two time-scales, $T_o = t$ and $T_1 = \varepsilon t$, so that $\frac{\partial}{\partial t} \to \frac{\partial}{\partial T_o} + \varepsilon \frac{\partial}{\partial T_1} + O(\varepsilon^2)$. The $O(1)$ equations are of course the unperturbed Manakov system (4) with $\varepsilon = 0$ for q_{10} and q_{20}. Collecting $O(\varepsilon)$ terms yields the first order correction system of equations:

$$i\frac{\partial q_{11}}{\partial T_o} + i\frac{\partial q_{10}}{\partial T_1} + \frac{\partial^2 q_{11}}{\partial x^2} - 2(|q_{10}|^2 + |q_{20}|^2 - q_o^2)q_{11} - 4q_{10}\mathrm{Re}(q_{10}q_{11}^* + q_{20}q_{21}^*) = R_1[q_{10}], \tag{9a}$$

$$i\frac{\partial q_{21}}{\partial T_o} + i\frac{\partial q_{20}}{\partial T_1} + \frac{\partial^2 q_{21}}{\partial x^2} - 2(|q_{10}|^2 + |q_{20}|^2 - q_o^2)q_{21} - 4q_{20}\mathrm{Re}(q_{10}q_{11}^* + q_{20}q_{21}^*) = R_2[q_{20}], \tag{9b}$$

where * denotes complex conjugate, and where we have assumed that $R_j[\mathbf{q}] = R_j[q_j]$, namely that the perturbation of each component only depends on the corresponding component. Note that this assumption has only the purpose of simplifying the description and the notation, but it is not essential in any of the following. We also assume $\partial_x^2 q_{11}, \partial_x^2 q_{21} \to 0$ as $|x| \to \infty$ (continuous wave background), and the perturbative corrections q_{11}, q_{21} to be functions of T_1 only. Under these assumptions, taking the limit as $x \to \pm\infty$ in (9) yields

$$i\frac{\partial q_{10}^{\pm}}{\partial T_1} = R_1[q_{10}^{\pm}], \quad i\frac{\partial q_{20}^{\pm}}{\partial T_1} = R_2[q_{20}^{\pm}], \tag{10}$$

where the superscripts $^{\pm}$ denote limits as $x \to \pm\infty$, and we note that since q_1, q_2 have to satisfy the boundary conditions to all orders, $q_{11}^{\pm}, q_{21}^{\pm}$ are both zero, together with all higher order terms.

Let us express the asymptotic behavior of the dark component as $q_{10}^{\pm} = q_o e^{i\phi^{\pm}}$, and take into account that in the DB soliton case $q_{20}^{\pm} = 0$. Then, separating real and imaginary parts in the first of Eqs. (10) yields

$$\frac{\partial q_o}{\partial T_1} = \mathrm{Im}(R_1[q_o e^{i\phi^{\pm}}]e^{-i\phi^{\pm}}), \quad q_o\frac{\partial \phi^{\pm}}{\partial T_1} = \mathrm{Re}(R_1[q_o e^{i\phi^{\pm}}]e^{-i\phi^{\pm}}), \tag{11}$$

i.e., using the asymptotic phase invariance of the perturbation:

$$\frac{\partial q_o}{\partial T_1} = \mathrm{Im}(R_1[q_o]), \quad \frac{\partial \Delta\phi^{\infty}}{\partial T_1} = 0. \tag{12}$$

Here, $\Delta\phi^{\infty} = \phi^+ - \phi^-$ is the asymptotic phase difference of the background. Since the second component is assumed to be rapidly decaying, not surprisingly, Eqs. (12) for the boundary at infinity are the same as in the scalar case, and are independent of the perturbation in the bright component.

3 The first order correction for a stationary DB soliton

For a stationary DB soliton of the unperturbed Manakov system, in (6) we set $A_d = q_o$ and $V = 0$:

$$q_d = -iq_o \tanh(\sqrt{q_o^2 - A_b^2}(x - x_o))e^{i\sigma_o}, \quad q_b = A_b e^{i(q_o^2 - A_b^2)t}\mathrm{sech}(\sqrt{q_o^2 - A_b^2}(x - x_o))e^{i\varphi_o}. \tag{13}$$

Note that, unlike black solitons in the scalar NLS, here the amplitude of the bright soliton, $A_b < q_o$, is an additional free parameter.

Let us use Madelung coordinates for both components, namely write $q_1 = ue^{i\phi}$ for the dark component, and $q_2 = ve^{i\psi}$ for the bright one, with $u, v, \phi, \psi \in \mathbb{R}$. Thus the system (4) becomes

$$-u\phi_t + u_{xx} - \phi_x^2 u - 2(u^2 + v^2 - q_o^2)u = \varepsilon\mathrm{Re}(R_1[ue^{i\phi}]e^{-i\phi}), \tag{14a}$$

$$u_t + u\phi_{xx} + 2u_x\phi_x = \varepsilon\mathrm{Im}(R_1[ue^{i\phi}]e^{-i\phi}), \tag{14b}$$

$$-v\psi_t + v_{xx} - \psi_x^2 v - 2(u^2+v^2-q_o^2)v = \varepsilon \mathrm{Re}(R_2[ve^{i\psi}]e^{-i\psi}), \tag{14c}$$

$$v_t + v\psi_{xx} + 2v_x\psi_x = \varepsilon \mathrm{Im}(R_2[ve^{i\psi}]e^{-i\psi}). \tag{14d}$$

As before, we introduce multiple scales in time: $T_o = t$, $T_1 = \varepsilon t$, and expand u, v, ϕ, ψ as:

$$u = u_o + \varepsilon u_1 + O(\varepsilon^2), \quad v = v_o + \varepsilon v_1 + O(\varepsilon^2), \quad \phi = \phi_o + \varepsilon\phi_1 + O(\varepsilon^2), \quad \psi = \psi_o + \varepsilon\psi_1 + O(\varepsilon^2). \tag{15}$$

At $O(1)$, the system (14) is satisfied by the stationary DB soliton (13), with:

$$u_o = q_o \tanh(\sqrt{q_o^2 - A_b^2}(x-x_o)), \quad \phi_o = \sigma_o, \tag{16a}$$

$$v_o = A_b \operatorname{sech}(\sqrt{q_o^2 - A_b^2}(x-x_o)), \quad \psi_o = \varphi_o + (q_o^2 - A_b^2)T_o. \tag{16b}$$

At $O(\varepsilon)$, the system (14) yields:

$$-u_o\phi_{oT_1} - u_1\phi_{oT_o} - u_o\phi_{1T_o} + u_{1xx} - 2\phi_{ox}\phi_{1x}u_o - \phi_{ox}^2 u_1$$
$$-4u_o^2 u_1 - 4u_o v_o v_1 - 2(u_o^2+v_o^2-q_o^2)u_1 = \mathrm{Re}(R_1[u_o e^{i\phi_o}]e^{-i\phi_o}), \tag{17a}$$

$$u_{1T_o} + u_{oT_1} + u_o\phi_{1xx} + u_1\phi_{oxx} + 2u_{1x}\phi_{ox} + 2u_{ox}\phi_{1x} = \mathrm{Im}(R_2[u_o e^{i\phi_o}]e^{-i\phi_o}), \tag{17b}$$

$$-v_o\psi_{oT_1} - v_1\psi_{oT_o} - \psi_{1T_o}v_o + v_{1xx} - 2\psi_{ox}\psi_{1x}v_o - \psi_{ox}^2 v_1$$
$$-4v_o^2 v_1 - 4u_o v_o u_1 - 2(u_o^2+v_o^2-q_o^2)v_1 = \mathrm{Re}(R_1[v_o e^{i\psi_o}]e^{-i\psi_o}), \tag{17c}$$

$$v_{1T_o} + v_{oT_1} + v_o\psi_{1xx} + v_1\psi_{oxx} + 2v_{1x}\psi_{0x} + 2v_{ox}\psi_{1x} = \mathrm{Re}(R_2[v_o e^{i\psi_o}]e^{-i\psi_o}). \tag{17d}$$

Here and in the following subscripts T_o and T_1 denote derivatives with respect to the corresponding time-scales. Note that (16) imply

$$\phi_{oT_o} = 0, \quad \psi_{oT_o} = A_b^2 - q_o^2, \quad \phi_{ox} = \phi_{oxx} = \psi_{ox} = \psi_{oxx} = 0. \tag{18}$$

Furthermore, seeking a stationary solution, we assume

$$u_{1T_o} = \phi_{1T_o} = v_{1T_o} = \psi_{1T_o} = 0.$$

In addition, we assume that q_o and A_b are independent of T_o, but both in general have $O(\varepsilon)$ terms depending on T_1. Then we can write the system (17) in matrix form as:

$$\begin{bmatrix} \partial_x^2 - 6u_o^2 - 2v_o^2 + 2q_o^2 & -4u_o v_o \\ -4u_o v_o & \partial_x^2 - 6v_o^2 - 2u_o^2 + q_o^2 + A_b^2 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} u_o \frac{d\sigma_o}{dT_1} + \mathrm{Re}(R_1[u_o e^{i\phi_o}]e^{-i\phi_o}) \\ v_o \frac{d\varphi_o}{dT_1} + \mathrm{Re}(R_2[v_o e^{i\psi_o}]e^{-i\psi_o}) \end{bmatrix}, \tag{19a}$$

$$\begin{bmatrix} u_o\partial_x^2 + 2u_{ox}\partial_x & 0 \\ 0 & v_o\partial_x^2 + 2v_{ox}\partial_x \end{bmatrix} \begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix} = \begin{bmatrix} -u_{oT_1} + \mathrm{Im}(R_1[u_o e^{i\phi_o}]e^{-i\phi_o}) \\ -v_{oT_1} + \mathrm{Im}(R_2[v_o e^{i\psi_o}]e^{-i\psi_o}) \end{bmatrix}. \tag{19b}$$

The expressions for u_{oT_1}, v_{oT_1} which contribute to the forcing in the equations (19b) for the phases can be easily computed from (16):

$$u_{oT_1} = -u_{ox}\left(\frac{dx_o}{dT_1} - \frac{q_o\frac{dq_o}{dT_1} - A_b\frac{dA_b}{dT_1}}{q_o^2 - A_b^2}(x-x_o)\right) + \frac{u_o}{q_o}\frac{dq_o}{dT_1}, \tag{20a}$$

$$v_{oT_1} = -v_{ox}\left(\frac{dx_o}{dT_1} - \frac{q_o\frac{dq_o}{dT_1} - A_b\frac{dA_b}{dT_1}}{q_o^2 - A_b^2}(x-x_o)\right) + \frac{v_o}{A_b}\frac{dA_b}{dT_1}. \tag{20b}$$

In turn, the T_1-dependence of q_o is given by (11), while dA_b/dT_1 is related to dq_o/dT_1 via

$$A_b(2q_o^2 - A_b^2)\frac{dA_b}{dT_1} - q_o(4q_o^2 - 3A_b^2)\frac{dq_o}{dT_1} = \\ = \sqrt{q_o^2 - A_b^2}\left(E_o\frac{dq_o}{dT_1} + (q_o^2 - A_b^2)\mathrm{Im}\int_{-\infty}^{\infty} R_2[v_o e^{i\psi_o}]v_o e^{-i\psi_o}dx\right), \quad (21)$$

which is obtained from the perturbed conservation laws, and specifically Eq. (66c) in Sec. 6. In Table 1 below, expressions for dq_o/dT_1 and dA_b/dT_1 are given for various types of perturbations, which are then used to compute particular solutions to Eqs. (19b) for the phases in each case.

	Diffusion	Linear loss	Nonlinear loss
$R_1[q_1]$	$i\gamma_1\partial_x^2 q_1$	$i\gamma_1 q_1$	$i\gamma_1\lvert q_1\rvert^2 q_1$
$R_2[q_2]$	$i\gamma_2\partial_x^2 q_2$	$i\gamma_2 q_2$	$i\gamma_2\lvert q_2\rvert^2 q_2$
$\frac{dq_o}{dT_1}$	0	$\gamma_1 q_o$	$\gamma_1 q_o^3$
$\frac{dA_b}{dT_1}$	$-\frac{2}{3}\gamma_2\frac{A_b(A_b^2 - q_o^2)^2}{(2q_o^2 - A_b^2)}$	$\frac{A_b(\gamma_1 q_o^2 + 2\gamma_2(q_o^2 - A_b^2))}{2q_o^2 - A_b^2}$	$\frac{A_b(3\gamma_1 q_o^4 + 4\gamma_2(q_o^2 - A_b^2))}{3(2q_o^2 - A_b^2)}$

Table 1: dq_o/dT_1 and dA_b/dT_1 for perturbations corresponding to diffusion, linear loss, and nonlinear loss.

Ideally, one should determine the general solution of the linear, non-homogeneous system (19) for the first order corrections to amplitudes and phases of the stationary DB soliton. The situation in the Manakov system, however, is significantly more complicated than in the scalar case. Indeed, while the equations (19b) for the phases are fully decoupled, and one can obtain the general homogeneous solution, as well as a particular solution, the system (19a) for the amplitudes remains fully coupled, and we are only able to obtain explicit expression for the particular solutions, as well as certain asymptotic information on the general homogeneous solution.

3.1 First order corrections for the phases

The system (19b) for the phases ϕ_1, ψ_1 is decoupled, and has homogeneous solutions:

$$\phi_{1h} = c_1 + c_2\left(x - \frac{\coth\left(\sqrt{q_o^2 - A_b^2}(x - x_o)\right)}{\sqrt{q_o^2 - A_b^2}}\right), \quad (22a)$$

$$\psi_{1h} = d_1 + d_2\left(x + \frac{\sinh\left(2\sqrt{q_o^2 - A_b^2}(x - x_o)\right)}{2\sqrt{q_o^2 - A_b^2}}\right), \quad (22b)$$

where c_1, c_2, d_1, d_2 are arbitrary constants. It is clear that in order to avoid exponential growth in the phase one needs to choose $d_2 = 0$. Also, the coth term in the expression of ϕ_{1h} becomes singular at $x = x_o$, and at first glance one might assume that this requires $c_2 = 0$ as well. However, the phase term in the dark component is multiplied by $\tanh\left(\sqrt{q_o^2 - A_b^2}(x - x_o)\right)$ [see Eq. (16)], which vanishes for $x = x_o$; this indicates that more general solutions could exist with $c_2 \neq 0$. On the other hand, particular solutions are determined by the forcing, and the right-hand sides (RHSs) of Eqs. (19b) for the phases for various choices of perturbations are given in the table below.

Perturbation	RHS of (19b) for ϕ_1, ψ_1
Diffusion	$u_{ox}\left(\frac{dx_o}{dT_1} - \gamma_2 \frac{2A_b^2(q_o^2-A_b^2)}{3(2q_o^2-A_b^2)}(x-x_o)\right) + \gamma_1 u_{oxx}$ $v_{ox}\left(x_{oT_1} - \gamma_2 \frac{2A_b^2(q_o^2-A_b^2)}{3(2q_o^2-A_b^2)}(x-x_o)\right) + \gamma_2 \frac{2(q_o^2-A_b^2)^2}{3(2q_o^2-A_b^2)} v_o + \gamma_2 v_{oxx}$
Linear loss	$u_{ox}\left(\frac{dx_o}{dT_1} - \frac{2(\gamma_1 q_o^2-\gamma_2 A_b^2)}{2q_o^2-A_b^2}(x-x_o)\right)$ $v_{ox}\left(\frac{dx_o}{dT_1} - \frac{2(\gamma_1 q_o^2-\gamma_2 A_b^2)}{2q_o^2-A_b^2}(x-x_o)\right) + \frac{\gamma_2 A_b^2-\gamma_1 q_o^2}{2q_o^2-A_b^2} v_o$
Nonlinear loss	$u_{ox}\left(\frac{dx_o}{dT_1} - \frac{2(3\gamma_1 q_o^4-2\gamma_2 A_b^4)}{3(2q_o^2-A_b^2)}(x-x_o)\right) - \gamma_1 q_o^2 u_o + \gamma_1 u_o^3$ $v_{ox}\left(\frac{dx_o}{dT_1} - \frac{2(3\gamma_1 q_o^4-2\gamma_2 A_b^4)}{3(2q_o^2-A_b^2)}(x-x_o)\right) - \frac{3\gamma_1 q_o^4+4\gamma_2 A_b^2(q_o^2-A_b^2)}{3(2q_o^2-A_b^2)} v_o + \gamma_2 v_o^3$

Table 2: RHSs of Eqs. (19b) for perturbations of the form of diffusion, linear loss, and nonlinear loss (cf Table 1).

From the explicit expressions of the forcing terms in Table 2, one can compute particular solutions for the Eqs. (19b). It is important to point out that for all three perturbations considered here as examples (diffusion, linear loss, and nonlinear loss), phase invariance can be used to simplify the RHSs of (19b). Indeed, linear and nonlinear loss are phase-invariant for all x, not just asymptotically; in the case of the diffusion perturbation, in general first and second order derivatives of the phase of the bright component would appear, but these are zero in the case of a stationary DB soliton according to (18).

Specifically, for a perturbation of the form of diffusion in both components, we find:

$$\phi_{1p} = \frac{1}{2}x\frac{dx_o}{dT_1} - \gamma_2(x-x_o)\frac{A_b^2\sqrt{q_o^2-A_b^2}}{3(2q_o^2-A_b^2)}\coth\left(\sqrt{q_o^2-A_b^2}(x-x_o)\right) - \frac{2}{3}\gamma_1 \ln\cosh\left(\sqrt{q_o^2-A_b^2}(x-x_o)\right), \tag{23a}$$

$$\psi_{1p} = \frac{1}{2}x\frac{dx_o}{dT_1} - \gamma_2\frac{x(x-2x_o)A_b^2(q_o^2-A_b^2)}{6(2q_o^2-A_b^2)} - \frac{2}{3}\gamma_2 \ln\cosh\left(\sqrt{q_o^2-A_b^2}(x-x_o)\right), \tag{23b}$$

for linear loss we obtain:

$$\phi_{1p} = \frac{1}{2}x\frac{dx_o}{dT_1} - (x-x_o)\frac{\gamma_1 q_o^2-\gamma_2 A_b^2}{(2q_o^2-A_b^2)\sqrt{q_o^2-A_b^2}}\coth\left(\sqrt{q_o^2-A_b^2}(x-x_o)\right), \tag{24a}$$

$$\psi_{1p} = \frac{1}{2}x\frac{dx_o}{dT_1} - \frac{x(x-2x_o)(\gamma_1 q_o^2-\gamma_2 A_b^2)}{2(2q_o^2-A_b^2)}, \tag{24b}$$

and, finally, in the case of nonlinear loss we find:

$$\phi_{1p} = \frac{1}{2}x\frac{dx_o}{dT_1} + (x-x_o)\frac{3\gamma_1 q_o^6+2\gamma_2 A_b^4(q_o^2-A_b^2)}{3(2q_o^4-3A_b^2q_o^2+A_b^4)\sqrt{q_o^2-A_b^2}}\coth\left(\sqrt{q_o^2-A_b^2}(x-x_o)\right), \tag{25a}$$

$$\psi_{1p} = \frac{1}{2}x\frac{dx_o}{dT_1} - \frac{x(x-2x_o)(3\gamma_1 q_o^4-2\gamma_2 A_b^4)}{6(2q_o^2-A_b^2)} + \frac{1}{3}\gamma_2\frac{A_b^2}{q_o^2-A_b^2}\ln\cosh\left(\sqrt{q_o^2-A_b^2}(x-x_o)\right). \tag{25b}$$

One can then write the full $O(\varepsilon)$ contributions to the phases by adding the homogeneous solutions in (22) and the perturbation-dependent particular solutions in (23), (24) or (25). Note that in all cases the phase of the bright component, ψ, exhibits a chirp-like quadratic growth as $x \to \pm\infty$. However, we should point out that the limit $|x| \to \infty$ is generically outside the range of validity of the asymptotic expansion in small time scales, since for large x the $O(\varepsilon)$ terms are no longer necessarily smaller than the $O(1)$ terms. Also, the asymptotic behavior of the x-derivative of the $O(\varepsilon)$ contributions to the phase of the dark soliton, ϕ_{1x}, as $x \to \pm\infty$ is given by Table 3 below. Furthermore, a π jump as $x \to \pm\infty$ in the $O(1)$ term phase of the dark component has to be included, on account of the limit of $\tanh\left(\sqrt{q_o^2 - A_b^2}(x - x_o)\right)$ in (16).

Perturbation	$(\phi_1)_x^{\pm}$
Diffusion	$\frac{1}{2}\frac{dx_o}{dT_1} + c_2 \mp \gamma_2 \frac{A_b^2\sqrt{q_o^2-A_b^2}}{3(2q_o^2-A_b^2)} \mp \frac{2}{3}\gamma_1\sqrt{q_o^2 - A_b^2}$
Linear loss	$\frac{1}{2}\frac{dx_o}{dT_1} + c_2 \mp \frac{\gamma_1 q_o^2 - \gamma_2 A_b^2}{(2q_o^2-A_b^2)\sqrt{q_o^2-A_b^2}}$
Nonlinear loss	$\frac{1}{2}\frac{dx_o}{dT_1} + c_2 \mp \frac{\gamma_1 q_o^2(5q_o^2-A_b^2)-2\gamma_2 A_b^4}{(2q_o^2-A_b^2)\sqrt{q_o^2-A_b^2}}$

Table 3: Asymptotic behavior of the x-derivative of the $O(\varepsilon)$ solutions for the phases of dark component as $x \to \pm\infty$ for the various perturbations considered above.

3.2 First order corrections for the amplitudes

We now proceed to obtaining information on the $O(\varepsilon)$ perturbative terms for the amplitudes of dark and bright components. As mentioned above, the system (19a) for the amplitudes is fully coupled, and the complete set of homogeneous solutions is not available. Note, however, that as $|x| \to \infty$ the homogeneous system reduces to:

$$\partial_x^2 u_{1h} \sim 4q_o^2 u_{1h}, \qquad \partial_x^2 v_{1h} \sim (q_o^2 - A_b^2) v_{1h}, \tag{26}$$

and since both RHSs are positive (recall that $0 \le A_b < q_o$), the homogeneous solutions for the amplitudes decay exponentially (exponentially growing terms would be unphysical), and their contributions can therefore be neglected compared to the particular solutions.

The particular solutions at $O(\varepsilon)$ for the system (19a) for u_1, v_1 can be obtained as follows. Assume that in the RHS of (19a) one has $\mathrm{Re}(R_1[u_o e^{i\phi_o}]e^{-i\phi_o}) = \mathrm{Re}(R_2[v_o e^{i\psi_o}]e^{-i\psi_o}) = 0$. This condition is satisfied by all the perturbations considered here, namely, diffusion as well as linear and nonlinear loss, in the case of a stationary DB soliton. Now recall the $O(1)$ equations for u_o, v_o, and write the corresponding homogeneous system in the matrix form:

$$\begin{bmatrix} \partial_x^2 - 2u_o^2 + 2q_o^2 & -2u_o v_o \\ -2u_o v_o & \partial_x^2 - 2v_o^2 + q_o^2 + A_b^2 \end{bmatrix} \begin{bmatrix} u_o \\ v_o \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{27}$$

Taking derivatives of this system with respect to q_o and A_b, we obtain two systems

$$\begin{bmatrix} \partial_x^2 - 6u_o^2 - 2v_o^2 + 2q_o^2 & -4u_o v_o \\ -4u_o v_o & \partial_x^2 - 6v_o^2 - 2u_o^2 + q_o^2 + A_b^2 \end{bmatrix} \begin{bmatrix} u_{oq_o} \\ v_{oq_o} \end{bmatrix} = \begin{bmatrix} -4q_o u_o \\ -2q_o v_o \end{bmatrix}, \tag{28}$$

$$\begin{bmatrix} \partial_x^2 - 6u_o^2 - 2v_o^2 + 2q_o^2 & -4u_o v_o \\ -4u_o v_o & \partial_x^2 - 6v_o^2 - 2u_o^2 + q_o^2 + A_b^2 \end{bmatrix} \begin{bmatrix} u_{oA_b} \\ v_{oA_b} \end{bmatrix} = \begin{bmatrix} 0 \\ -2A_b v_o \end{bmatrix}. \tag{29}$$

Notice that both systems have exactly the same matrix as in (19a), and since q_o, A_b are space-independent, we can seek solutions u_1, v_1 to (19a) in the following form:

$$u_1 = A_1 u_{oq_o} + A_2 u_{oA_b}, \quad v_1 = A_1 v_{oq_o} + A_2 v_{oA_b}, \tag{30}$$

where A_1, A_2, A_3, A_4 are varying coefficients. Taking linear combinations of the two aforementioned systems, we can match the RHS of (19a) and obtain the u_1, v_1 solution in terms of derivatives of unperturbed solutions with respect to their parameters, namely:

$$u_1 = -\left(\frac{\sigma_{oT_1}}{4q_o}\right)\frac{\partial u_o}{\partial q_o} + \left(\frac{\sigma_{oT_1} - 2\varphi_{oT_1}}{4A_b}\right)\frac{\partial u_o}{\partial A_b}, \quad v_1 = -\left(\frac{\sigma_{oT_1}}{4q_o}\right)\frac{\partial v_o}{\partial q_o} + \left(\frac{\sigma_{oT_1} - 2\varphi_{oT_1}}{4A_b}\right)\frac{\partial v_o}{\partial A_b}. \tag{31}$$

We should note that in order to completely determine the $O(\varepsilon)$ contributions to the amplitudes u_1, v_1 one needs to obtain the dependence on the slow time T_1 of the phase parameters of the dark and bright components of the soliton, namely $d\sigma_o/dT_1$ and $d\varphi_o/dT_1$. The former will be obtained from the $O(\varepsilon)$ perturbed conservation laws in Sec. 6, but $d\varphi_o/dT_1$ is not determined at this order. On the other hand, the terms involving $d\varphi_o/dT_1$ do not contribute to u_1, v_1 in the limit as $x \to \pm\infty$, and one can obtain from the above equations the asymptotic behavior:

$$u_1 \to \mp\frac{\sigma_{oT1}}{4q_o}, \quad v_1 \to 0 \text{ as } x \to \pm\infty. \tag{32}$$

Notice that the particular solutions above are obtained under the assumption that the real parts of the perturbations in both components are zero. If one has $\mathrm{Re}(R_1[u_o]) = \alpha_1 u_o$, $\mathrm{Re}(R_2[v_o]) = \alpha_2 v_o$, where α_1, α_2 are x-independent, then a particular solution can be built in a similar way.

4 The first order correction for a moving DB soliton

For a moving DB soliton, we have $0 < A_d < q_o$, $0 < A_b < A_d$ and $V \neq 0$, with all parameters being, in general, functions of the slow time T_1. In a co-moving reference frame: $X = x - \int_0^\tau V(\varepsilon s)ds - x_o$, $\tau = t$, the DB soliton solution is written as:

$$q_d = \left(\mathrm{sgn}(V)\sqrt{q_o^2 - A_d^2} - iA_d \tanh(X\sqrt{A_d^2 - A_b^2})\right)e^{i\sigma_o}, \tag{33}$$

$$q_b = A_b e^{i(\varphi_o + \frac{V}{2}(X + \int_0^\tau V(\varepsilon s)ds + x_o) - (\frac{V^2}{4} - (A_d^2 - A_b^2))\tau)} \mathrm{sech}(X\sqrt{A_d^2 - A_b^2}), \tag{34}$$

and the perturbed Manakov system (4) takes the form:

$$i\mathbf{q}_\tau - i(V + x_{o\tau})\mathbf{q}_X + \mathbf{q}_{XX} - 2(\|\mathbf{q}\|^2 - q_o^2)\mathbf{q} = \varepsilon\mathbf{R}[\mathbf{q}]. \tag{35}$$

To avoid dealing with singularities in the derivatives with respect to the soliton parameters, in the following we will assume that $V(0) > 0$, and that V remains sign definite as a function of T_1.

After applying Madelung ansatz for each component, $q_1 = ue^{i\phi}$, $q_2 = ve^{i\psi}$, and separating real and imaginary parts, Eq. (35) yields:

$$-u\phi_\tau + Vu\phi_X + u_{XX} - \phi_X^2 u - 2(u^2 + v^2 - q_o^2)u = \varepsilon\mathrm{Re}(R_1[ue^{i\phi}]e^{-i\phi}), \tag{36a}$$

$$u_\tau - Vu_X + u\phi_{XX} + 2u_X\phi_X = \varepsilon\mathrm{Im}(R_1[ue^{i\phi}]e^{-i\phi}), \tag{36b}$$

$$-v\psi_\tau + Vv\psi_X + v_{XX} - \psi_X^2 v - 2(u^2 + v^2 - q_o^2)v = \varepsilon\mathrm{Re}(R_2[ve^{i\psi}]e^{-i\psi}), \tag{36c}$$

$$v_\tau - Vv_X + v\psi_{XX} + 2v_X\psi_X = \varepsilon\mathrm{Im}(R_2[ve^{i\psi}]e^{-i\psi}). \tag{36d}$$

Expanding u, v, ϕ, ψ in powers of ε as in (15), and letting as before $\partial_\tau = \partial_{T_o} + \varepsilon\partial_{T_1} + O(\varepsilon^2)$, we find for the $O(1)$ terms in the co-moving coordinates:

$$u_o = \sqrt{q_o^2 - A_d^2 \operatorname{sech}^2\left(X\sqrt{A_d^2 - A_b^2}\right)}, \quad \phi_o = \sigma_o + \arctan\left(-\frac{A_d}{\sqrt{q_o^2 - A_d^2}} \tanh\left(X\sqrt{A_d^2 - A_b^2}\right)\right), \tag{37a}$$

$$v_o = A_b \operatorname{sech}(X\sqrt{A_d^2 - A_b^2}), \quad \psi_o = \varphi_o + \frac{V}{2}\left(X + \int_0^{T_o} V\,ds + x_o\right) - \left(\frac{V^2}{4} - (A_d^2 - A_b^2)\right)T_o. \tag{37b}$$

Then, stationary solutions at $O(\varepsilon)$ are governed by the following, fully coupled system of ODEs:

$$\begin{aligned} -u_o\phi_{oT_1} + Vu_1\phi_{oX} + Vu_o\phi_{1X} + u_{1XX} - 2\phi_{oX}\phi_{1X}u_o - \phi_{oX}^2 u_1 \\ -4u_o^2 u_1 - 4u_o v_o v_1 - 2(u_o^2 + v_o^2 - q_o^2)u_1 = \operatorname{Re}(R_1[u_o e^{i\phi_o}]e^{-i\phi_o}), \end{aligned} \tag{38a}$$

$$u_{oT_1} - Vu_{1X} + u_o\phi_{1XX} + u_1\phi_{oXX} + 2u_{1X}\phi_{oX} + 2u_{oX}\phi_{1X} = \operatorname{Im}(R_1[u_o e^{i\phi_o}]e^{-i\phi_o}), \tag{38b}$$

$$\begin{aligned} -v_o\psi_{oT_1} + Vv_1\psi_{oX} + Vv_o\psi_{1X} + v_{1XX} - 2\psi_{oX}\psi_{1X}v_o - \psi_{oX}^2 v_1 \\ -4v_o^2 v_1 - 4u_o v_o u_1 - 2(u_o^2 + v_o^2 - q_o^2)v_1 = \operatorname{Re}(R_2[v_o e^{i\psi_o}]e^{-i\psi_o}), \end{aligned} \tag{38c}$$

$$v_{oT_1} - Vv_{1X} + v_o\psi_{1XX} + v_1\psi_{oXX} + 2v_{1X}\psi_{0X} + 2v_{oX}\psi_{1X} = \operatorname{Im}(R_2[v_o e^{i\psi_o}]e^{-i\psi_o}). \tag{38d}$$

In the scalar case, for which $v_o = \psi_o = 0$, the above system reduces to:

$$\begin{bmatrix} \partial_X^2 + V\phi_{oX} - \phi_{oX}^2 - 6u_o^2 + 2q_o^2 & u_o(V - 2\phi_{oX})\partial_X \\ (-V + 2\phi_{oX})\partial_X + \phi_{oXX} & u_o\partial_X^2 + 2u_{oX}\partial_X \end{bmatrix} \begin{bmatrix} u_1 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} J_1[u_o, \phi_o] \\ J_2[u_o, \phi_o] \end{bmatrix}, \tag{39}$$
$$J_1[u_o, \phi_o] = \operatorname{Re}(R_1[u_o e^{i\phi_o}]e^{-i\phi_o}) + u_o\phi_{oT_1}, \quad J_2[u_o, \phi_o] = \operatorname{Im}(R_1[u_o e^{i\phi_o}]e^{-i\phi_o}) - u_{oT_1}.$$

Now, from Eq. (37) we easily find:

$$\phi_{oT_o} = -\phi_{oX}V, \quad \psi_{oT_o} = -\frac{V^2}{4} + A_d^2 - A_b^2, \tag{40}$$

$$\phi_{oX} = \frac{V}{2}\,\frac{A_d^2 \operatorname{sech}^2(X\sqrt{q_o^2 - A_b^2})}{A_d^2 \operatorname{sech}^2(X\sqrt{q_o^2 - A_b^2}) - q_o^2}, \quad \psi_{oX} = \frac{V}{2}, \tag{41}$$

as well as

$$u_{oT_1} = u_{oX}\left(X\frac{A_d A_{dT_1} - A_b A_{bT_1}}{A_d^2 - A_b^2} - x_{oT_1}\right) - \frac{A_{dT_1}}{A_d}\frac{q_o^2 - u_o^2}{u_o} + \frac{q_o q_{oT_1}}{u_o}, \tag{42a}$$

$$v_{oT_1} = v_{oX}\left(X\frac{A_d A_{dT_1} - A_b A_{bT_1}}{A_d^2 - A_b^2} - x_{oT_1}\right) + \frac{A_{bT_1}}{A_b}v_o, \tag{42b}$$

$$\begin{aligned} \phi_{oT_1} &= \sigma_{oT_1} + \\ &+\phi_{oX}\left(X\frac{A_d A_{dT_1} - A_b A_{bT_1}}{A_d^2 - A_b^2} - x_{oT_1} + \frac{q_o}{2A_d}\frac{\sinh\left(2X\sqrt{A_d^2 - A_b^2}\right)}{\sqrt{A_d^2 - A_b^2}}\frac{q_o A_{dT_1} - A_d q_{oT_1}}{q_o^2 - A_d^2}\right), \end{aligned} \tag{42c}$$

$$\psi_{oT_1} = \varphi_{oT_1} + \frac{V}{2}x_{oT_1} + \frac{1}{2}V_{T_1}\left(X + \int_0^{T_o} V\,ds + x_o\right) - \left(\frac{V}{2}V_{T_1} - 2(A_d A_{dT_1} - A_b A_{bT_1})\right)T_o. \tag{42d}$$

To this end, in the limit $X \to \pm\infty$, Eqs. (38) yield:

$$-q_o\left(\sigma_{oT_1} \mp \frac{1}{q_o}\frac{q_o A_{dT_1} - A_d q_{oT_1}}{\sqrt{q_o^2 - A_d^2}}\right) + Vq_o\phi_{1X}^\pm - 4q_o^2 u_1^\pm = \operatorname{Re}(R_1[u_o e^{i\phi_o}]e^{-\phi_o})^\pm, \tag{43a}$$

$$-Vu_{1X}^{\pm} + q_o\phi_{1XX}^{\pm} + q_{oT_1} = \mathrm{Im}(R_1[u_o e^{i\phi_o}]e^{-\phi_o})^{\pm}, \tag{43b}$$

$$\frac{V^2}{4}v_1^{\pm} = \mathrm{Re}(R_2[v_o e^{i\psi_o}]e^{-i\psi_o})^{\pm}, \tag{43c}$$

$$0 = \mathrm{Im}(R_2[v_o e^{i\psi_o^{\pm}}]e^{-i\psi_o})^{\pm}, \tag{43d}$$

where we have used $u_{1XX}^{\pm} = v_{1XX}^{\pm} = 0$ for a continuous wave background. Using (12), the second equation simplifies down to $Vu_{1X}^{\pm} = q_o\phi_{1XX}^{\pm}$. Assuming that $u_{1X} \to 0$ as $X \to \pm\infty$, then both u_1 and ϕ_{1X} tend to constants as $X \to \pm\infty$, which corresponds to the shelf developing around the soliton. In this case, the second equation in the system is automatically satisfied, and does not yield any additional information.

5 Boundary layer

Since u_1, ϕ_1 do not vanish as $x \to \pm\infty$, the solution to order ε does not match the boundary conditions at infinity. Thus, our problem is now broken into two regions: the region that matches imposed non-decaying boundary conditions at infinity, which is unaffected by the soliton, and the region in which the $O(\varepsilon)$ correction term is valid and the solution is quasi-stationary. To resolve this mismatch, we introduce a boundary layer in which there is a transition from the non-zero value in the perturbation term to the boundary conditions at infinity.

In the boundary layer region, we can write:

$$u = q_o + \varepsilon w + O(\varepsilon^2), \quad v = \varepsilon p + O(\varepsilon^2), \quad \phi = \phi^{\pm} + \varepsilon\theta + O(\varepsilon^2), \quad \psi = \psi^{\pm} + \varepsilon\mu + O(\varepsilon^2), \tag{44}$$

where w, p, θ, μ are real-valued functions of x and t. As before, utilizing two time scales, with $\partial_t \to \partial_{T_o} + \varepsilon\partial_{T_1} + O(\varepsilon^2)$, we can write

$$\frac{dq_o}{dT_o} = 0, \quad \frac{d\phi^{\pm}}{dT_o} = 0, \quad \frac{dq_o}{dT_1} = \mathrm{Im}(R_1[q_o]), \quad -q_o\frac{d\psi^{\pm}}{dT_1} = -\mathrm{Re}(R_2[q_o]). \tag{45}$$

At $O(1)$, the system (14) is satisfied exactly, and at $O(\varepsilon)$ we have

$$-q_o\frac{\partial\phi^{\pm}}{\partial T_1} - w\frac{\partial\phi^{\pm}}{\partial T_o} - \frac{\partial\theta}{\partial T_o}q_o + \frac{\partial^2 w}{\partial x^2} - 2\frac{\partial\phi^{\pm}}{\partial x}\frac{\partial\theta}{\partial x}q_o - \left(\frac{\partial\phi^{\pm}}{\partial x}\right)^2 w - 4q_o^2 w = \mathrm{Re}R_1[q_o + \varepsilon w], \tag{46a}$$

$$\frac{\partial w}{\partial T_o} + \frac{dq_o}{dT_1} + q_o\frac{\partial^2\theta}{\partial x^2} + w\frac{\partial^2\phi^{\pm}}{\partial x^2} + 2\frac{\partial w}{\partial x}\frac{\partial\phi^{\pm}}{\partial x} + 2\frac{\partial q_o}{\partial x}\frac{\partial\theta}{\partial x} = \mathrm{Im}R_1[q_o + \varepsilon w], \tag{46b}$$

$$-p\frac{d\psi^{\pm}}{dT_o} + \frac{\partial^2 p}{\partial x^2} - \left(\frac{\partial\psi^{\pm}}{\partial x}\right)^2 p = \mathrm{Re}R_2[\varepsilon p], \tag{46c}$$

$$\frac{\partial p}{\partial T_o} + p\frac{\partial^2\psi^{\pm}}{\partial x^2} + 2\frac{\partial p}{\partial x}\frac{\partial\psi^{\pm}}{\partial x} = \mathrm{Im}R_2[\varepsilon p]. \tag{46d}$$

Using (45), we obtain

$$\mathrm{Re}R_1[q_o + \varepsilon w] + q_o\frac{\partial\phi^{\pm}}{\partial T_1} = \mathrm{Re}R_1[q_o + \varepsilon w] - \mathrm{Re}R[q_o] = O(\varepsilon), \tag{47a}$$

$$\mathrm{Im}R_1[q_o + \varepsilon w] - \frac{dq_o}{dT_1} = \mathrm{Im}R_1[q_o + \varepsilon w] - \mathrm{Im}R_1[q_o] = O(\varepsilon), \tag{47b}$$

therefore these terms go to higher order. In addition, using the fact that $\phi^\pm, \psi^\pm$ only depend on the slow time T_1, we obtain the system of equations:

$$-\frac{\partial\theta}{\partial T_o} q_o + \frac{\partial^2 w}{\partial x^2} - 4q_o^2 w = 0, \quad \frac{\partial w}{\partial T_o} + q_o \frac{\partial^2\theta}{\partial x^2} = 0, \quad -2pq_o^2 + \frac{\partial^2 p}{\partial x^2} = 0, \quad \frac{\partial p}{\partial T_o} = 0 \tag{48}$$

from which it follows that $p = C_1 e^{\sqrt{2}q_o x} + C_2 e^{-\sqrt{2}q_o x}$, with C_1, C_2 independent of T_o. In turn, w, θ satisfy:

$$\frac{\partial\theta}{\partial T_o} = -4q_o^2 w + \frac{\partial^2 w}{\partial x^2}, \qquad \frac{\partial w}{\partial T_o} = -q_o \frac{\partial^2\theta}{\partial x^2}. \tag{49}$$

The above system can be decoupled by taking two derivatives in x, which yields two identical equations for w and θ:

$$\frac{\partial^2 w}{\partial T_o^2} = 4q_o^2 \frac{\partial^2 w}{\partial x^2} - \frac{\partial^4 w}{\partial x^4}, \qquad \frac{\partial^2\theta}{\partial T_o^2} = 4q_o^2 \frac{\partial^2\theta}{\partial x^2} - \frac{\partial^4\theta}{\partial x^4}. \tag{50}$$

Note that the $O(\epsilon)$ perturbation μ of $\psi^\pm$ (cf (44)) is not determined in this order. Assuming a long-wave approximation, the above equations become

$$\frac{\partial^2 w}{\partial T_o^2} = 4q_o^2 \frac{\partial^2 w}{\partial x^2}, \qquad \frac{\partial^2\theta}{\partial T_o^2} = 4q_o^2 \frac{\partial^2\theta}{\partial x^2}, \tag{51}$$

namely, the $O(\epsilon)$ perturbative terms of amplitude and phase of the dark soliton satisfy the wave equation. This allows us to conclude that, similarly to the scalar case, the edges of the shelf move with the velocities $W = \pm 2q_o$.

6 Perturbed conservation laws

6.1 Stationary DB soliton

In order to find the slow time evolution of a stationary DB soliton parameters, A_b, σ_o and x_o, we employ the first three conservation laws for the unperturbed Manakov system, involving:

$$E = \int_{-\infty}^{\infty} \left[\|\mathbf{q}\|^2 - q_o^2\right] dx, \qquad \text{(Energy)}, \tag{52a}$$

$$I = 2\,\mathrm{Im} \int_{-\infty}^{\infty} \left[\mathbf{q}\cdot\mathbf{q}_x\right] dx - q_o^2 \Delta\phi_o, \qquad \text{(Renormalized Momentum)}, \tag{52b}$$

$$H = -\int_{-\infty}^{\infty} \left[\|\mathbf{q}_x\|^2 + (\|\mathbf{q}\|^2 - q_o^2)^2\right] dx, \qquad \text{(Hamiltonian)}, \tag{52c}$$

where $\cdot$ denotes the usual complex vector dot product: $(a,b)^T \cdot (c,d)^T = ac^* + bd^*$, and $\Delta\phi_o$ is the phase difference across the soliton. The evolution equations for these integrals of motion take the form:

$$\frac{dE}{dt} = 2\epsilon\left(-q_o \frac{dq_o}{dT_1} + \mathrm{Im}\left[\int_{-\infty}^{\infty} \mathbf{R}[\mathbf{q}]\cdot\mathbf{q}\,dx\right]\right), \tag{53a}$$

$$\frac{dI}{dt} = -4\epsilon\mathrm{Re}\int_{-\infty}^{\infty} \mathbf{R}[\mathbf{q}]\cdot\mathbf{q}_x dx - \epsilon\Delta\phi_o \frac{dq_o^2}{dT_1} - \epsilon q_o^2 \frac{d\Delta\phi_o}{dT_1}, \tag{53b}$$

$$\frac{dH}{dt} = 2\epsilon\left(E\frac{d}{dT_1} q_o^2 + \mathrm{Re}\int_{-\infty}^{\infty} \mathbf{R}[\mathbf{q}]\cdot\mathbf{q}_t dx\right). \tag{53c}$$

Matching $O(\varepsilon)$ terms in dE/dt and (53a), we obtain

$$\frac{d}{dt}\int_{-\infty}^{\infty} 2[u_o u_1 + 2v_o v_1]\,dx = -2q_o\frac{dq_o}{dT_1} + 2\mathrm{Im}\left[\int_{-\infty}^{\infty}\mathbf{R}[\mathbf{q}]\cdot\mathbf{q}\,dx\right]. \tag{54}$$

Since the edges of the shelf are moving with velocity $\pm 2q_o$ (see end of the previous section), we change the interval of integration in the LHS to $(-2q_o t, 2q_o t)$, and the Fundamental Theorem of Calculus yields

$$4q_o\left[u_o(2q_o t)u_1(2q_o t) + u_o(-2q_o t)u_1(-2q_o t)\right] = -2q_o\frac{dq_o}{dT_1} + 2\mathrm{Im}\left[\int_{-\infty}^{\infty}\mathbf{R}[\mathbf{q}]\cdot\mathbf{q}\,dx\right]. \tag{55}$$

In the limit $x \to \pm\infty$, using $u_o \to \pm q_o$, $u_1 \to u_1^{\pm}$, the above equation reduces to:

$$4q_o^2(u_1^+ - u_1^-) = -2q_o\frac{dq_o}{dT_1} + 2\mathrm{Im}\left[\int_{-\infty}^{\infty}\mathbf{R}[\mathbf{q}]\cdot\mathbf{q}\,dx\right]. \tag{56}$$

Recalling the definitions of $u_1^{\pm}$ in (32), we finally get

$$\frac{d\sigma_o}{dT_1} = \frac{dq_o}{dT_1} - \frac{1}{q_o}\mathrm{Im}\left[\int_{-\infty}^{\infty}\mathbf{R}[\mathbf{q}]\cdot\mathbf{q}\,dx\right]. \tag{57}$$

Matching $O(\varepsilon)$ terms in dI/dt and (53b), and proceeding in a similar way we obtain:

$$\phi_{1x}^+ + \phi_{1x}^- = 0, \tag{58}$$

which in turn, in light of the asymptotic behavior of ϕ_{1x} in Table 3, implies

$$\frac{dx_o}{dT_1} + 2c_2 = 0. \tag{59}$$

The last equation relates the arbitrary phase constant c_2 for a stationary DB soliton to the slow-time evolution of the soliton center. Although in this work we do not determine $x_o(T_1)$, our numerical simulations indicate that for all three types of perturbations considered in Table 1 the soliton center does not move, which indicates that $c_2 = 0$. We note that the phase constant can be independently determined by analyzing the radiation field associated with the soliton, as in [52–54] in the context of instability-induced dynamics of dark solitons.

Finally, matching $O(\varepsilon)$ terms in dH/dt and (53c), yields:

$$-\frac{2}{3}\frac{d}{dT_1}\left(\sqrt{q_o^2 - A_b^2}(4q_o^2 - A_b^2)\right) = -8q_o\sqrt{q_o^2 - A_b^2}\frac{dq_o}{dT_1} + 2\mathrm{Im}\int_{-\infty}^{\infty} R_2[v_o e^{i\psi_o}]\,v_o e^{-i\psi_o}\,dx, \tag{60}$$

from where, knowing dq_o/dT_1 (see Eq. (12)), and the form of the perturbation $\mathbf{R}[\mathbf{q}]$, we can find dA_b/dT_1 (cf Table 1 for some specific perturbations).

Next, we proceed to compare the above findings with direct numerical simulations. We present here the case of nonlinear loss as an example, but we note that for the other considered perturbations the results —and the agreement between the analytical predictions and simulations— are similar. For this example, we evolve Eqs. (4) using Eqs. (6a) as initial conditions for the case of a stationary DB soliton, with $A_d = q_o$, using a fourth order Runge-Kutta method in time. In the left panel of Fig. 1 we depict the complete evolution of the black soliton, so as to demonstrate the evolution of the shelf, under the nonlinear loss. The evolution of the DB soliton parameters A_d and q_o are depicted in Fig. 2. It is observed that the analytical predictions are in very good agreement with the results of the direct numerical simulations.

Below we will consider the perturbed conservation laws for the general case of a moving DB soliton, and obtain from them explicit equations for the adiabatic evolution of the soliton parameters for the various perturbations. The reductions to stationary DB solitons and to the scalar case will also be given.

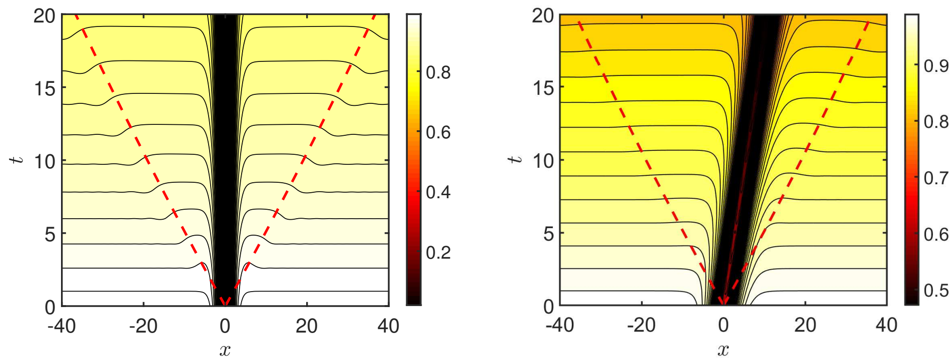

Figure 1: Left: Contour plot of the modulus of the dark component of a stationary DB soliton under nonlinear loss. Here, $q_o(0) = 1$, $A_b(0) = 1/\sqrt{2}$, $A_d = q_o$, $x_0 = 0$ and $\epsilon\gamma_1 = \epsilon\gamma_2 = -0.01$. Right: Contour plot of the modulus of the dark component of a moving DB soliton under linear loss. Here, parameter values are: $q_o(0) = 1$, $A_b(0) = 1/\sqrt{2}$, $A_d = 0.8$, $x_0 = 0$ and $\epsilon\gamma_1 = \epsilon\gamma_2 = -0.01$. In both plots the dashed red lines correspond to the analytical prediction for the evolution of the shelf's edge.

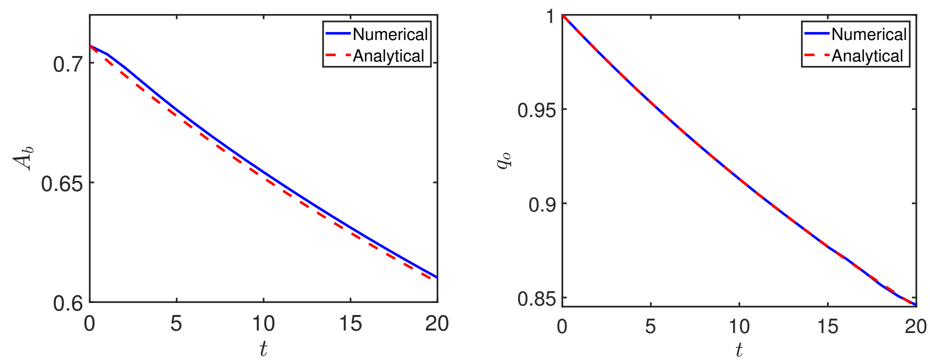

Figure 2: The evolution of the stationary DB soliton parameters A_d and q_o under nonlinear loss. The solid blue lines correspond to the numerical simulations results, whereas the dashed red lines to the analytical predictions. Here, $q_o(0) = 1$, $A_b(0) = 1/\sqrt{2}$, $A_d = q_o$, $x_0 = 0$ and $\epsilon\gamma_1 = \epsilon\gamma_2 = -0.01$.

6.2 Moving DB soliton

For a moving DB soliton, we have eight independent parameters: $u_1^{\pm}$, $\phi_{1X}^{\pm}$, and slow evolution of variables A_b, A_d, q_o, and σ_o. Other parameters of the first order correction of the solution, namely $\psi_{1X}^{\pm}$ are not determined at this order of perturbation. Determining the slow time evolution of x_o requires the explicit expression of the first order corrections for amplitudes and phases, and will be the subject of future investigation.

As in the case of stationary DB soliton discussed in Sec. 5, the edge of the shelf still propagates with velocity $W(T_1) = 2q_o(T_1)$, and the velocity V of the soliton now may also depend on the slow time T_1. In the moving frame of reference $X = x - \int_0^t V(\epsilon s)ds - x_o$, the boundaries of the shelf are given by:

$$S_L(t) = -\int_0^t [2q_o(\epsilon s) + V(\epsilon s)]ds, \quad S_R(t) = \int_0^t [2q_o(\epsilon s) - V(\epsilon s)]ds, \tag{61}$$

where S_L and S_R give the position in X of the left and right boundaries of the shelf, respectively.

In addition to E, I and H we used above, here we will also employ the conserved quantity:

$$F = \mathrm{Im}\int_{-\infty}^{\infty}\left[\|\mathbf{q}\|^2(\mathbf{q}\cdot\mathbf{q}_X)+\frac{1}{3}\mathbf{q}_{XX}\cdot\mathbf{q}_X\right]dx, \tag{62}$$

which was found through symbolical software [55, 56]; the time evolution of F is given by

$$\frac{dF}{dt} = -2\varepsilon\mathrm{Re}\int_{-\infty}^{\infty}\mathbf{R}[\mathbf{q}]\cdot\left((\|\mathbf{q}\|^2+\mathbf{q}_X\cdot\mathbf{q})\mathbf{q}_X-\frac{1}{3}\mathbf{q}_{XXX}\right)dX. \tag{63}$$

Note that each term in the integrand of the conserved quantity (62) involves spatial derivatives of the field $\mathbf{q}$, which decay at infinity, thus ensuring the convergence of the integral. Furthermore, unlike what happens for the momentum (52b) (see, e.g. [53]), since there is no straightforward physical quantity associated with this higher moment F, there is no underlying physical motivation that requires a renormalization of the integral.

Furthermore, we will use the so-called center of energy:

$$\mathscr{R} = \int_{-\infty}^{\infty}X(\|\mathbf{q}\|^2-q_o^2)\,dX, \quad \text{(Center of Energy)}, \tag{64}$$

whose time evolution is related to the momentum I:

$$\frac{d\mathscr{R}}{dt} = I+2\varepsilon\mathrm{Im}\int_{-\infty}^{\infty}X(\mathbf{R}[\mathbf{q}]\cdot\mathbf{q}-q_oR[q_o])\,dX. \tag{65}$$

Repeating the same steps as in the previous subsection while restricting the interval of integration to $X\in[S_R,S_L]$ for the d/dT_o terms, we obtain four evolution equations:

$$\frac{dE_o}{dT_1} = -2q_o((2q_o+V)u_1^-+(2q_o-V)u_1^+)-2q_o\frac{dq_o}{dT_1}+2\mathrm{Im}\int_{-\infty}^{\infty}\mathbf{R}[\mathbf{q}_o]\cdot\mathbf{q}_o\,dX, \tag{66a}$$

$$\frac{dI_o}{dT_1} = 2q_o^2((2q_o+V)\phi_{1X}^-+(2q_o-V)\phi_{1X}^+)-\frac{d(q_o^2\Delta\phi)}{dT_1}-4\mathrm{Re}\int_{-\infty}^{\infty}\mathbf{R}[\mathbf{q}_o]\cdot\frac{\partial\mathbf{q}_o}{\partial X}dX, \tag{66b}$$

$$\frac{dH_o}{dT_1} = 4E_oq_o\frac{dq_o}{dT_1}+2\mathrm{Re}\int_{-\infty}^{\infty}\mathbf{R}[\mathbf{q}_o]\cdot\frac{\partial\mathbf{q}_o}{\partial T_o}dX, \tag{66c}$$

$$\begin{aligned}\frac{dF_o}{dT_1} &= -q_o^4\left(\phi_{1X}^+(2q_o-V)+\phi_{1X}^-(2q_o+V)\right)\\ &\quad -2\mathrm{Re}\int_{-\infty}^{\infty}\mathbf{R}[\mathbf{q}_o]\cdot\left(\left(\|\mathbf{q}_o\|^2+\frac{\partial\mathbf{q}_o}{\partial X}\cdot\mathbf{q}_o\right)\frac{\partial\mathbf{q}_o}{\partial X}-\frac{1}{3}\frac{\partial^3\mathbf{q}_o}{\partial X^3}\right)dX,\end{aligned} \tag{66d}$$

where E_o, I_o, H_o, F_o are the values of the corresponding conservation laws integrals computed for $\varepsilon=0$:

$$E_o = -2\sqrt{A_d^2-A_b^2}, \tag{67a}$$

$$I_o = \frac{4}{A_d}(A_d^2-A_b^2)\sqrt{q_o^2-A_d^2}, \tag{67b}$$

$$H_o = -\frac{2\sqrt{A_d^2-A_b^2}}{3A_d^2}\left(4A_d^2(A_d^2-A_b^2)+3q_o^2A_b^2\right), \tag{67c}$$

$$F_o = \frac{2\left(A_d^2-A_b^2\right)\sqrt{q_o^2-A_d^2}\left(A_b^2\left(q_o^2-2A_d^2\right)-3q_o^2A_d^2+2A_d^4\right)}{3A_d^3}. \tag{67d}$$

In addition, we obtain one non-evolution equation coming from leading order of $d\mathscr{R}/dt$:

$$(2q_o - V)(2u_1^+ + \phi_{1X}^+) - (2q_o + V)(2u_1^- - \phi_{1X}^-) = 0. \tag{68}$$

Putting everything together, Eqs. (66), along with (68), (43a), (12) give us a linear system of equations for eight unknowns: $u_1^\pm$, $\phi_{1X}^\pm$, and slow time derivatives of σ_o, A_b, A_d and q_o. In what follows, we provide solutions of these equations for each of the three perturbations considered in the paper.

6.3 DB soliton parameters

Diffusion perturbations: $R_j = i\gamma_j q_{jxx}$, $j = 1,2$. In this case, the adiabatic evolution of dA_b/dT_1, dA_d/dT_1, $d\sigma_o/dT_1$ is described by the following equations:

$$\frac{dA_b}{dT_1} = -\frac{2\gamma_2 A_b\left(A_b^2 - A_d^2\right)\left(A_b^2\left(7q_o^2A_d^4 - 4A_d^6\right) + A_b^4\left(2q_o^2A_d^2 - 3q_o^4\right) - 6q_o^2A_d^6 + 4A_d^8\right)}{3A_d^2\left(-4A_b^2A_d^4 + q_o^2A_b^4 + 4A_d^6\right)}, \tag{69a}$$

$$\frac{dA_d}{dT_1} = \frac{2\gamma_2 A_b^2\left(A_b^2 - A_d^2\right)\left(A_d^2 - q_o^2\right)\left(A_b^2\left(4A_d^2 - 3q_o^2\right) - 4A_d^4\right)}{3\left(q_o^2A_b^4A_d - 4A_b^2A_d^5 + 4A_d^7\right)}, \tag{69b}$$

$$\frac{d\sigma_o}{dT_1} = 2\sqrt{A_d^2 - A_b^2}\left[3q_o\left(-4A_b^2A_d^4 + q_o^2A_b^4 + 4A_d^6\right)\right]^{-1} \times \tag{69c}$$
$$\times\left[A_b^4\left(8\gamma_2A_d^4 + 2\left(\gamma_1 - 6\gamma_2\right)q_o^2A_d^2 + 3\gamma_2q_o^4\right) - 2A_b^2\left(4\left(\gamma_1+\gamma_2\right)A_d^6 - 5\gamma_2q_o^2A_d^4\right) + 8\gamma_1A_d^8\right].$$

Notice that in the scalar limit ($A_b = 0$), the full system of equations reduces to:

$$\frac{d\sigma_o}{dT_1} = \frac{4\gamma_1A_d^3}{3q_o}, \quad \frac{dA_d}{dT_1} = \frac{dq_o}{dT_1} = \frac{dV}{dT_1} = 0, \quad u_1^\pm = -\frac{1}{2}\frac{d\sigma_o}{dT_1}\frac{1}{2q_o \mp V}, \quad \phi_{1X}^\pm = \pm 2u_1^\pm, \tag{70}$$

consistently with the results of Ref. [47], whereas in the stationary limit ($V = 0$, $A_d = q_o$) we find:

$$\frac{d\sigma_o}{dT_1} = \frac{2q_o\sqrt{q_o^2 - A_b^2}(4\gamma_1q_o^2 - (2\gamma_1 - \gamma_2)A_b^2)}{3(2q_o^2 - A_b^2)}, \quad \frac{dA_b}{dT_1} = -\frac{2\gamma_2A_b(q_o^2 - A_b^2)^2}{3(2q_o^2 - A_b^2)}, \tag{71a}$$

$$\frac{dV}{dT_1} = \frac{dq_o}{dT_1} = \frac{dA_d}{dT_1} = 0, \quad u_1^\pm = -\frac{1}{2}\frac{d\sigma_o}{dT_1}\frac{1}{2q_o}, \quad \phi_{1X}^\pm = \pm 2u_1^\pm. \tag{71b}$$

Linear loss perturbations: $R_j = i\gamma_j q_j$, $j = 1,2$. The adiabatic evolution of dA_b/dT_1, dA_d/dT_1, $d\sigma_o/dT_1$ now reads:

$$\frac{dA_b}{dT_1} = \frac{2\left(\gamma_1 + 2\gamma_2\right)A_bA_d^6 - A_b^3\left(6\gamma_2A_d^4 + \gamma_1q_o^2A_d^2\right) + 2\gamma_2q_o^2A_b^5}{-4A_b^2A_d^4 + q_o^2A_b^4 + 4A_d^6}, \tag{72a}$$

$$\frac{dA_d}{dT_1} = \frac{-2\gamma_1A_b^2A_d^3\left(A_d^2 + q_o^2\right) + A_b^4\left(\left(\gamma_1 + 2\gamma_2\right)q_o^2A_d - 2\gamma_2A_d^3\right) + 4\gamma_1A_d^7}{-4A_b^2A_d^4 + q_o^2A_b^4 + 4A_d^6}, \tag{72b}$$

$$\frac{d\sigma_o}{dT_1} = \frac{2A_d^2\left(\gamma_1A_d^2 - \gamma_2A_b^2\right)\left(A_b^2\left(q_o^2 - 2A_d^2\right) + 2A_d^4\right)}{q_o\sqrt{A_d^2 - A_b^2}\left(-4A_b^2A_d^4 + q_o^2A_b^4 + 4A_d^6\right)}. \tag{72c}$$

In the scalar limit, this gives:

$$\frac{d\sigma_o}{dT_1} = \frac{\gamma_1A_d}{q_o}, \quad \frac{dA_d}{dT_1} = \gamma_1A_d, \quad \frac{dV}{dT_1} = \gamma_1V, \tag{73a}$$

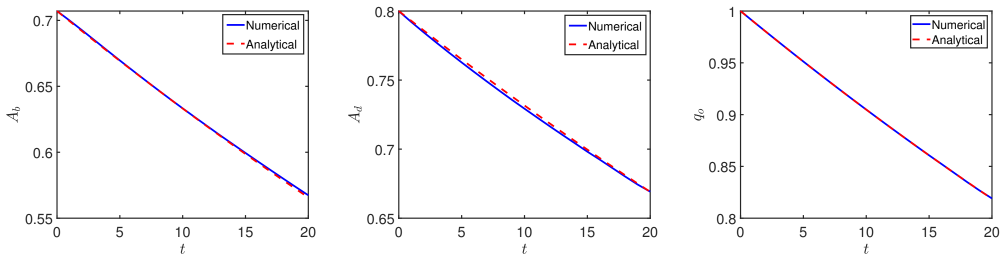

Figure 3: The evolution of the moving DB soliton parameters under linear loss. The solid blue lines correspond to the numerical simulations results, whereas the dashed red lines, to the analytical predictions. Here, $q_o(0) = 1$, $A_b(0) = 1/\sqrt{2}$, $A_d = 0.8$, $x_0 = 0$ and $\epsilon\gamma_1 = \epsilon\gamma_2 = -0.01$.

$$\frac{dq_o}{dT_1} = \gamma_1 q_o, \quad u_1^{\pm} = -\frac{1}{2}\frac{d\sigma_o}{dT_1}\frac{1}{2q_o \mp V}, \quad \phi_{1X}^{\pm} = \pm 2u_1^{\pm}, \tag{73b}$$

again consistently with [47], and in the stationary limit, we get:

$$\frac{d\sigma_o}{dT_1} = \frac{2q_o(\gamma_1 q_o^2 - \gamma_2 A_b^2)}{\sqrt{q_o^2 - A_b^2}(2q_o^2 - A_b^2)}, \quad \frac{dA_b}{dT_1} = \frac{A_b(\gamma_1 q_o^2 + 2\gamma_2(q_o^2 - A_b^2))}{2q_o^2 - A_b^2}, \tag{74a}$$

$$\frac{dV}{dT_1} = 0, \quad \frac{dA_d}{dT_1} = \frac{dq_o}{dT_1} = \gamma_1 q_o, \quad u_1^{\pm} = -\frac{1}{2}\frac{d\sigma_o}{dT_1}\frac{1}{2q_o}, \quad \phi_{1X}^{\pm} = \pm 2u_1^{\pm}. \tag{74b}$$

Nonlinear loss perturbations: $R_j = i\gamma_j |q_j|^2 q_j$**.** In this case, the adiabatic evolution of dA_b/dT_1, dA_d/dT_1, $d\sigma_o/dT_1$ is given by:

$$\frac{dA_b}{dT_1} = \frac{A_b\left(-12\gamma_2 A_b^4 A_d^4 + 2A_d^6\left(4\gamma_2 A_b^2 + 5\gamma_1 q_o^2\right) - 3\gamma_1 q_o^4 A_b^2 A_d^2 + 4\gamma_2 q_o^2 A_b^6 - 4\gamma_1 A_d^8\right)}{3\left(-4A_b^2 A_d^4 + q_o^2 A_b^4 + 4A_d^6\right)}, \tag{75a}$$

$$\frac{dA_d}{dT_1} = \frac{A_d\left(4\gamma_2 A_b^6\left(q_o^2 - A_d^2\right) + 2\gamma_1 A_b^2 A_d^2\left(-5q_o^2 A_d^2 + 2A_d^4 - 3q_o^4\right) + 3\gamma_1 q_o^4 A_b^4 + 4\gamma_1 A_d^6\left(5q_o^2 - 2A_d^2\right)\right)}{3\left(-4A_b^2 A_d^4 + q_o^2 A_b^4 + 4A_d^6\right)}, \tag{75b}$$

$$\begin{aligned}\frac{d\sigma_o}{dT_1} =\, & 2A_d^2\left[3q_o\sqrt{A_d^2 - A_b^2}\left(-4A_b^2 A_d^4 + q_o^2 A_b^4 + 4A_d^6\right)\right]^{-1}\left[-2\gamma_2 A_b^4\left(A_b^2\left(q_o^2 - 2A_d^2\right) + 2A_d^4\right) + \right. \\ & \left. + \gamma_1\left(-2q_o^2 A_d^2\left(7A_b^2 A_d^2 + A_b^4 - 7A_d^4\right) + 3q_o^4 A_b^2\left(A_b^2 + A_d^2\right) + 4A_d^6\left(A_b^2 - A_d^2\right)\right)\right].\end{aligned} \tag{75c}$$

In the scalar limit, this yields:

$$\frac{d\sigma_o}{dT_1} = \frac{\gamma_1 A_d}{3q_o}\left(\frac{1}{2}V^2 + 5q_o^2\right), \quad \frac{dA_d}{dT_1} = \gamma_1 A_d\left(\frac{1}{6}V^2 + q_o^2\right), \quad \frac{dV}{dT_1} = \frac{\gamma_1 V}{6}(V^2 + 2q_o^2), \tag{76a}$$

$$\frac{dq_o}{dT_1} = \gamma_1 q_o^3, \quad u_1^{\pm} = -\frac{1}{2}\frac{\left(\frac{d\sigma_o}{dT_1} \pm \frac{d\Delta\phi}{dT_1}\right)}{2q_o \mp V}, \quad \phi_{1X}^{\pm} = \pm 2u_1^{\pm} \mp \frac{2}{3}\gamma_1 A_d, \tag{76b}$$

where

$$\frac{d\Delta\phi_o}{dT_1} = \frac{1}{q_o}\frac{q_o A_{dT_1} - A_d q_{oT_1}}{\sqrt{q_o^2 - A_d^2}} \tag{77}$$

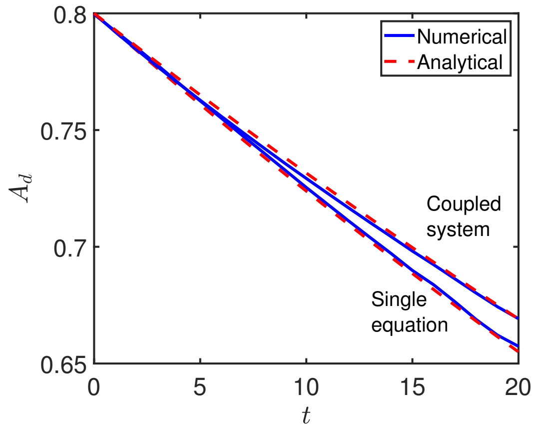

Figure 4: The evolution of the solitons' parameters under linear loss. The solid lines correspond to the numerical simulations results, whereas the dashed lines, to the analytical predictions. Here, $q_o(0) = 1$, $A_b(0) = 1/\sqrt{2}$, $A_d = 0.8$, $x_0 = 0$ and $\epsilon\gamma_1 = \epsilon\gamma_2 = -0.01$.

with $\Delta\phi_o$ being the phase difference across the soliton (which is different from asymptotic phase difference of the background, $\Delta\phi^\infty$). Finally, in the stationary limit, we have:

$$\frac{d\sigma_o}{dT_1} = \frac{2q_o(\gamma_1 q_o^2(5q_o^2 - A_b^2) - 2\gamma_2 A_b^4)}{3(2q_o^2 - A_b^2)\sqrt{q_o^2 - A_b^2}}, \quad \frac{dA_b}{dT_1} = \frac{A_b(3\gamma_1 q_o^4 + 4\gamma_2 A_b^2(q_o^2 - A_b^2))}{3(2q_o^2 - A_b^2)}, \tag{78a}$$

$$\frac{dV}{dT_1} = 0, \quad \frac{dA_d}{dT_1} = \frac{dq_o}{dT_1} = \gamma_1 q_o^3, \quad u_1^\pm = -\frac{1}{2}\frac{d\sigma_o}{dT_1}\frac{1}{2q_o}, \quad \phi_{1X}^\pm = \pm 2u_1^\pm. \tag{78b}$$

As in the stationary case, it is relevant to compare the above findings with results of direct numerical simulations. Here, as an example, we consider the case of linear loss (for the other considered perturbations, the results are similar). For the present example, we again evolve Eqs. (4), using Eqs. (6a) as initial conditions, employing a fourth order Runge-Kutta method in time. In the right panel of Fig. 1, we depict the complete evolution of the gray component of the moving DB soliton, so as to demonstrate the emergence and evolution of the shelf in the presence of linear loss. Furthermore, the evolution of the DB solitons amplitudes A_d and A_b, and of the background parameter q_o, are depicted in Fig. 3. Once again, the analytical results are found to be in very good agreement with the numerical simulations.

Finally, it is relevant to compare the evolution of the grey soliton component of a moving DB soliton with that of a grey soliton in the scalar case. An important conclusion stemming from such a comparison is that the coupling (or the "symbiosis") between the grey and the bright soliton has a profound effect. Indeed, it seems that the bright soliton tends to reduce the decaying effects of the dissipative perturbations as compared to the scalar equation. While the dark soliton will eventually decay to the background, and the background will vanish in both cases, in the Manakov system this is delayed significantly as shown in Fig. 4. This result is in a qualitative agreement with the analysis of Ref. [57], where it was found that the effect of the bright ("filling") soliton component is to partially stabilize "bare" dark solitons against temperature-induced dissipation in BECs, thus providing longer lifetimes.

7 Conclusion

In conclusion, we have presented a direct perturbation theory to study the evolution of the dark-bright solitons of the Manakov system under the action of perturbations. Our approach relies on the combination of a multiscale expansion method and a boundary layer theory, such that the problem is broken into an inner region —pertinent to the soliton core— and a "shelf", namely a linear wave emerging due to the perturbation, which matches the boundary conditions at infinity. Our analysis was first performed for stationary dark-bright solitons, and then was generalized for moving ones. Various typical perturbations were considered, and in particular physically relevant dissipative ones, namely diffusion, as well as linear and nonlinear loss (or gain, depending on the sign of the relevant coefficients).

Our methodology is similar to the one that was used for the study of perturbed dark solitons in the scalar defocusing NLS equation [47]. Our results, however, extend beyond the ones presented in that work. For example, in the simpler case of stationary DB solitons, we were able to completely determine the asymptotic phases of the DB solitons (whereas in [47] the particular solution of the linear, nonhomogeneous equations for the $O(\varepsilon)$ terms in the phases had not been obtained). Furthermore, the additional effect of the perturbation in the bright component on the asymptotic phases of both dark and bright soliton components was studied.

For the typical dissipative perturbations we considered, the asymptotic approximations showed very good agreement with results of direct numerical simulations. The relevant numerical computations confirmed the emergence of the analytically predicted shelf and, in all cases, were found to fully support our analytical predictions. Our analysis and computations also revealed an important difference between scalar grey solitons and the grey solitons of the Manakov system: it was found that the presence of the bright ("filling") component hinders the perturbation- induced dissipation associated with the grey soliton, thus offering a partial stabilization —i.e. a longer lifetime— to the corresponding Manakov DB soliton structure, in comparison to its "bare" scalar dark soliton counterpart.

Our analysis and results suggest further interesting studies. Indeed, first we note that it would be relevant to further extend our analysis in order to determine the evolution of the center x_o of the DB soliton. To do this, one needs to employ the $O(\varepsilon)$ correction terms for amplitudes and phases explicitly. In principle, one could follow a similar strategy as in [47], but in the Manakov system the ODEs for the $O(\varepsilon)$ ODEs for the amplitudes and phases do not decouple, and the solution of this problem will require finding homogeneous and particular solutions for a fourth-order linear, non-homogeneous ODE, with coefficients given by the $O(1)$ dark and bright soliton solutions and their derivatives, while the non-homogeneous term depends on the perturbation. In any case, the determination of the evolution of the soliton center would also be relevant for studies involving conservative perturbations, as, e.g., is the case with external potentials which are particularly relevant to the physics of BECs. Solution of the relevant problem for the soliton center could then bridge our analysis with other perturbative studies of the Manakov system, relying on the adiabatic approximation [18].

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