Resolvent Estimates for the Stokes Operator in Bounded and Exterior C^1 Domains

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Abstract

We establish resolvent estimates in L^q spaces for the Stokes operator in a bounded C^1 domain Ω in $\mathbb{R}^d$. As a corollary, it follows that the Stokes operator generates a bounded analytic semigroup in $L^q(\Omega; \mathbb{C}^d)$ for any $1 < q < \infty$ and $d \geq 2$. The case of an exterior C^1 domain is also studied.

Keywords: Resolvent Estimate; Stokes Operator; C^1 Domain.

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1 Introduction

In this paper we study the resolvent problem for the Stokes operator with the Dirichlet condition,

$$\begin{cases} -\Delta u + \nabla p + \lambda u = F & \text{in } \Omega, \\ \operatorname{div}(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda \in \Sigma_\theta$ is a parameter and

$$\Sigma_\theta = \{ z \in \mathbb{C} \setminus \{0\} : \ |\arg(z)| < \pi - \theta \} \tag{1.2}$$

for $\theta \in (0, \pi/2)$. The following two theorems are the main results of the paper. The first one covers the case of bounded domains with C^1 boundaries, while the second treats the case of exterior C^1 domains.

Theorem 1.1. *Let Ω be a bounded C^1 domain in $\mathbb{R}^d$, $d \geq 2$. Let $1 < q < \infty$ and $\lambda \in \Sigma_\theta$. Then for any $F \in L^q(\Omega; \mathbb{C}^d)$, the Dirichlet problem (1.1) has a unique solution (u, p) in $W_0^{1,q}(\Omega; \mathbb{C}^d) \times L^q(\Omega; \mathbb{C})$ with $\int_\Omega p = 0$. Moreover, the solution satisfies the estimate,*

$$(|\lambda| + 1)^{1/2} \|\nabla u\|_{L^q(\Omega)} + (|\lambda| + 1) \|u\|_{L^q(\Omega)} \leq C \|F\|_{L^q(\Omega)}, \tag{1.3}$$

where C depends only on d, q, θ and Ω.

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Theorem 1.2. *Let Ω be an exterior domain with C^1 boundary in $\mathbb{R}^d$, $d \geq 2$. Let $1 < q < \infty$ and $\lambda \in \Sigma_\theta$ with $|\lambda| \geq \delta > 0$. Then for any $F \in L^q(\Omega; \mathbb{C}^d)$, the Dirichlet problem (1.1) has a unique solution (u, p) in $W_0^{1,q}(\Omega; \mathbb{C}^d) \times L^q_{\text{loc}}(\overline{\Omega}; \mathbb{C})$. Moreover, the solution satisfies the estimate,*

$$|\lambda|^{1/2} \|\nabla u\|_{L^q(\Omega)} + |\lambda| \|u\|_{L^q(\Omega)} \leq C \|F\|_{L^q(\Omega)}, \tag{1.4}$$

where C depends only on d, q, θ, δ and Ω. Furthermore, if $d \geq 3$, the estimate,

$$|\lambda| \|u\|_{L^q(\Omega)} \leq C \|F\|_{L^q(\Omega)}, \tag{1.5}$$

holds with C independent of δ.

Resolvent estimates for the Stokes operator play an essential role in the functional analytic approach of Fujita and Kato [11] to the nonlinear Navier-Stokes equations. The resolvent estimate (1.5) in domains with smooth boundaries has been studied extensively since 1980's. Under the assumption that Ω is a bounded or exterior domain with $C^{1,1}$ boundary, the estimate (1.5) holds for any $1 < q < \infty$ [21, 16, 2, 10]. We refer the reader to [10] for a review as well as a comprehensive list of references in the case of smooth domains. The recent work in this area focuses on domains with nonsmooth boundaries. If Ω is merely a bounded Lipschitz domain, it was proved by one of the present authors [20] that the resolvent estimate (1.5) holds if $d \geq 3$ and

$$\left| \frac{1}{q} - \frac{1}{2} \right| < \frac{1}{2d} + \varepsilon, \tag{1.6}$$

where $\varepsilon > 0$ depends on Ω. In particular, in the case $d = 3$, this shows that the estimate (1.5) holds for $(3/2) - \varepsilon < q < 3 + \varepsilon$ and gives an affirmative answer to a conjecture of M. Taylor [23]. For a two-dimensional bounded Lipschitz domain, F. Gabel and P. Tolksdorf [12] were able to establish the resolvent estimate (1.5) for $(4/3) - \varepsilon < q < 4 + \varepsilon$. It is not known whether the range in (1.6) is sharp for Lipschitz domains. In [6] P. Deuring constructed an interesting example of an unbounded Lipschitz domain for which the resolvent estimate fails for large q. For related work on the Stokes and Navier-Stokes equations in Lipschitz or C^1 domains, we refer to the reader to [8, 5, 7, 17, 18, 19, 14, 24, 25]

The main contribution of this paper lies in the smoothness assumption for the domain Ω. We are able to establish the resolvent estimates for the full range $1 < q < \infty$ under the assumption that $\partial\Omega$ is C^1. In view of the example by P. Deuring [6], this assumption is more or less optimal. As we mentioned earlier, the full range is known previously for $C^{1,1}$ domains [10]. A recent result of D. Breit [4] implies the resolvent estimates for a three-dimensional Lipschitz domain satisfying certain Besov-type conditions, which are weaker than $C^{1,1}$ and somewhat close to $C^{1,\alpha}$ for certain $\alpha > 0$. Note that in the case of smooth domains, in addition to the L^q estimates for u and ∇u in (1.3) and (1.4), one also obtains an estimate for $\nabla^2 u$,

$$\|\nabla^2 u\|_{L^q(\Omega)} \leq C \|F\|_{L^q(\Omega)}, \tag{1.7}$$

for $1 < q < \infty$, if Ω is bounded (some restrictions on q are needed if Ω is an exterior domain; see [10]). However, such $W^{2,q}$ estimates fail in C^1 domains, even for the Laplace operator.

Let $C^\infty_{0,\sigma}(\Omega) = \big\{ u \in C_0^\infty(\Omega; \mathbb{C}^d) : \text{div}(u) = 0 \big\}$ and

$$L^q_\sigma(\Omega) = \text{the closure of } C^\infty_{0,\sigma}(\Omega) \text{ in } L^q(\Omega; \mathbb{C}^d). \tag{1.8}$$

For $1 < q < \infty$, we define the Stokes operator A_q in $L^q_\sigma(\Omega)$ by

$$A_q(u) = -\Delta u + \nabla p, \tag{1.9}$$

with the domain

$$\begin{aligned} \mathcal{D}(A_q) = \Big\{ u \in W^{1,q}_0(\Omega; \mathbb{C}^d) : \operatorname{div}(u) = 0 \text{ in } \Omega \text{ and} \\ -\Delta u + \nabla p \in L^q_\sigma(\Omega) \text{ for some } p \in L^q_{\text{loc}}(\overline{\Omega}; \mathbb{C}) \Big\}. \end{aligned} \tag{1.10}$$

It follows from Theorems 1.1 and 1.2 that for $\lambda \in \Sigma_\theta$ and $1 < q < \infty$, the inverse operator $(\lambda + A_q)^{-1}$ exists as a bounded operator on $L^q_\sigma(\Omega)$. Moreover, the estimate,

$$\|(\lambda + A_q)^{-1} F\|_{L^q(\Omega)} \le C|\lambda|^{-1} \|F\|_{L^q(\Omega)}, \tag{1.11}$$

holds, where C depends only on d, q, θ and Ω, if Ω is a bounded C^1 domain in $\mathbb{R}^d$, $d \ge 2$ or an exterior C^1 domain in $\mathbb{R}^d$, $d \ge 3$. As a corollary, we obtain the following.

Corollary 1.3. *Let Ω be a bounded C^1 domain in $\mathbb{R}^d$, $d \ge 2$ or an exterior C^1 domain in $\mathbb{R}^d$, $d \ge 3$. Then the Stokes operator $-A_q$ generates a uniformly bounded analytic semigroup $\{e^{-tA_q}\}_{t\ge 0}$ in $L^q_\sigma(\Omega)$ for $1 < q < \infty$.*

The uniform boundedness of the semigroup in the case of two-dimensional exterior C^1 domains is left open by Corollary 1.3. We note that the uniform boundedness for the two-dimensional exterior C^2 domains was established in [3] by using the method of layer potentials for λ near 0.

We now describe our approach to Theorems 1.1 and 1.2, which is based on a perturbation argument of R. Farwig and H. Sohr [10]. The basic idea is to work out first the cases of the whole space $\mathbb{R}^d$ and the half-space $\mathbb{R}^d_+$. One then uses a perturbation argument to treat the case of a region above a graph,

$$\mathbb{H}_\psi = \left\{ (x', x_d) \in \mathbb{R}^d : \ x' \in \mathbb{R}^{d-1} \text{ and } x_d > \psi(x') \right\},$$

where $\psi : \mathbb{R}^{d-1} \to \mathbb{R}$. Finally, a localization procedure, together with some compactness argument, is performed to handle the cases of bounded or exterior domains. To establish the resolvent estimates for C^1 domains, the key step is to carry out the perturbation argument under the assumption that $\psi : \mathbb{R}^{d-1} \to \mathbb{R}$ is Lipschitz continuous and to show that the error terms are bounded by the Lipschitz norm $\|\nabla'\psi\|_\infty$, where ∇' denotes the gradient with respect to $x' = (x_1, \dots, x_{d-1})$.

To this end, we consider a more general Stokes resolvent problem,

$$\begin{cases} -\Delta u + \nabla p + \lambda u = F + \operatorname{div}(f), \\ \qquad\qquad \operatorname{div}(u) = g, \end{cases} \tag{1.12}$$

in $\mathbb{H}_\psi$ with the boundary condition $u = 0$ on $\partial\mathbb{H}_\psi$, where $F \in L^q(\mathbb{H}_\psi; \mathbb{C}^d)$ and $f \in L^q(\mathbb{H}_\psi; \mathbb{C}^{d\times d})$. We introduce two Banach spaces,

$$X^q_\psi = W^{1,q}_0(\mathbb{H}_\psi; \mathbb{C}^d) \times A^q_\psi \quad \text{and} \quad Y^q_\psi = W^{-1,q}(\mathbb{H}_\psi; \mathbb{C}^d) \times B^q_\psi, \tag{1.13}$$

where A^q_ψ and B^q_ψ are two spaces defined by (4.2). In comparison with the spaces used in [10] for $C^{1,1}$ domains, we point out that since we work with C^1 domains, no $W^{2,q}$ spaces can be used. Note that the scaling-invariant property of the Lipschitz norm $\|\nabla'\psi\|_\infty$ allows us to fix $\lambda \in \Sigma_\theta$ with $|\lambda| = 1$. Consider the linear operator

$$S^\lambda_\psi(u, p) = (-\Delta u + \nabla p + \lambda u, \operatorname{div}(u)). \tag{1.14}$$

We are able to show that $S^\lambda_\psi : X^q_\psi \to Y^q_\psi$ is a bijection and that

$$\|(S^\lambda_\psi)^{-1}\|_{Y^q_\psi \to X^q_\psi} \le C(d, q, \theta) \tag{1.15}$$

for $1 < q < \infty$, provided that $\|\nabla'\psi\|_\infty \le c_0$ and $c_0 = c_0(d, q, \theta) > 0$ is sufficiently small. See Theorem 4.3. To prove (1.15), one first considers the special case $\psi = 0$; i.e., $\mathbb{H}_\psi = \mathbb{R}^d_+$. The general case follows from the facts that

$$S^\lambda_\psi(u, p) = S^\lambda_0(\widetilde{u}, \widetilde{p}) \circ \Psi + R(\widetilde{u}, \widetilde{p}) \circ \Psi, \tag{1.16}$$

and that the operator norm of the second term in the right-hand side of (1.16) is bounded by $C\|\nabla'\psi\|_\infty$ if $\|\nabla'\psi\|_\infty \le 1$. As a by-product, we also obtain the resolvent estimate (1.4) in the case $\Omega = \mathbb{H}_\psi$ if $\|\nabla'\psi\|_\infty \le c_0(d, q, \theta)$. See Theorem 4.1.

The paper is organized as follows. We start with the case of the whole space $\mathbb{R}^d$ in Section 2. The case $\Omega = \mathbb{R}^d_+$ is studied in Section 3. In Section 4 we carry out the perturbation argument described above for the region above a Lipschitz graph. In Section 5 we consider the case of bounded C^1 domains and give the proof of Theorem 1.1. The case of exterior C^1 domains is studied in Section 6, where Theorem 1.2 is proved. Finally, we prove some useful uniqueness and regularity results for exterior C^1 domains in the Appendix.

We end this section with a few notations that will be used throughout the paper. Let Ω be a (bounded or unbounded) domain in $\mathbb{R}^d$. By $u \in L^q_{\text{loc}}(\overline{\Omega}; \mathbb{C}^m)$ we mean $u \in L^q(B \cap \Omega; \mathbb{C}^m)$ for any ball B in $\mathbb{R}^d$. For $1 < q < \infty$, let

$$W^{1,q}(\Omega; \mathbb{C}^m) = \big\{ u \in L^q(\Omega; \mathbb{C}^m) : \ \nabla u \in L^q(\Omega; \mathbb{C}^{d\times m}) \big\} \tag{1.17}$$

be the usual Sobolev space in Ω for functions with values in $\mathbb{C}^m$. By $W^{1,q}_0(\Omega; \mathbb{C}^m)$ we denote the closure of $C^\infty_0(\Omega; \mathbb{C}^m)$ in $W^{1,q}(\Omega; \mathbb{C}^m)$. We use $W^{-1,q}(\Omega; \mathbb{C}^m)$ to denote the dual of $W^{1,q'}_0(\Omega; \mathbb{C}^m)$ and $W^{-1,q}_0(\Omega; \mathbb{C}^m)$ the dual of $W^{1,q'}(\Omega; \mathbb{C}^m)$, where $q' = \frac{q}{q-1}$. For $1 < q < \infty$, we let

$$\mathring{W}^{1,q}(\Omega; \mathbb{C}^m) = \big\{ u \in L^q_{\text{loc}}(\overline{\Omega}; \mathbb{C}^m) : \ \nabla u \in L^q(\Omega; \mathbb{C}^{d\times m}) \big\} \tag{1.18}$$

denote the homogeneous $W^{1,q}$ space with the norm $\|\nabla u\|_{L^q(\Omega)}$. As usual, we identify two functions in $\mathring{W}^{1,q}(\Omega; \mathbb{C}^m)$ if they differ by a constant. Let $\mathring{W}^{-1,q}(\Omega; \mathbb{C}^m)$ be the dual of $\mathring{W}^{1,q'}(\Omega; \mathbb{C}^m)$. Elements Λ in $\mathring{W}^{-1,q}(\Omega; \mathbb{C}^m)$ may be represented by $\operatorname{div}(f)$, where $f = (f_{jk}) \in L^q(\Omega; \mathbb{C}^{d\times m})$, in the sense that

$$\Lambda(u) = -\int_\Omega \partial_j u_k \cdot f_{jk}$$

for any $u = (u_1, \dots, u_m) \in \mathring{W}^{1,q'}(\Omega; \mathbb{C}^m)$, where $\partial_j = \partial/\partial x_j$, the index j is summed from 1 to d and k from 1 to m.

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2 The whole space

In this section we study the resolvent problem for the Stokes equations in $\mathbb{R}^d$, $d \geq 2$. The results in Theorem 2.1 are more or less standard. Since the Stokes equations are considered with a more general data set, we provide a proof for the reader's convenience.

Theorem 2.1. *Let $1 < q < \infty$ and $\lambda \in \Sigma_\theta$. For any $F \in L^q(\mathbb{R}^d;\mathbb{C}^d)$, $f \in L^q(\mathbb{R}^d;\mathbb{C}^{d\times d})$, and $g \in L^q(\mathbb{R}^d;\mathbb{C}) \cap \mathring{W}^{-1,q}(\mathbb{R}^d;\mathbb{C})$, there exists a unique $u \in W^{1,q}(\mathbb{R}^d;\mathbb{C}^d)$ such that*

$$\begin{cases} -\Delta u + \nabla p + \lambda u = F + \operatorname{div}(f), \\ \qquad\qquad \operatorname{div}(u) = g \end{cases} \tag{2.1}$$

hold in $\mathbb{R}^d$ for some $p \in L^1_{\text{loc}}(\mathbb{R}^d;\mathbb{C})$ in the sense of distributions. Moreover, the solution satisfies the estimate,

$$\begin{cases} |\lambda|^{1/2}\|\nabla u\|_{L^q(\mathbb{R}^d)} \leq C\left\{\|F\|_{L^q(\mathbb{R}^d)} + |\lambda|^{1/2}\|f\|_{L^q(\mathbb{R}^d)} + |\lambda|^{1/2}\|g\|_{L^q(\mathbb{R}^d)}\right\}, \\ \qquad |\lambda|\|u\|_{L^q(\mathbb{R}^d)} \leq C\left\{\|F\|_{L^q(\mathbb{R}^d)} + |\lambda|^{1/2}\|f\|_{L^q(\mathbb{R}^d)} + |\lambda|\|g\|_{\mathring{W}^{-1,q}(\mathbb{R}^d)}\right\}, \end{cases} \tag{2.2}$$

and $p \in L^q(\mathbb{R}^d;\mathbb{C}) + \mathring{W}^{1,q}(\mathbb{R}^d;\mathbb{C})$, where C depends on d, q and θ.

Proof. Step 1. We establish the existence of the solution and the estimates in (2.2).

By rescaling we may assume $|\lambda| = 1$. By linearity, it suffices to consider two cases: (I) $g = 0$; (II) $f = 0$ and $F = 0$.

Case I. Assume $g = 0$. Let $\mathcal{F}$ denote the Fourier transform defined by

$$\mathcal{F}(h)(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} h(x)\, dx,$$

where $i = \sqrt{-1}$ and $\xi \in \mathbb{R}^d$. Let $u = (u_1, u_2, \ldots, u_d)$, $F = (F_1, F_2, \ldots, F_d)$ and $f = (f_{jk})$. By applying $\mathcal{F}$ to (2.1) with $g = 0$, we obtain

$$\begin{cases} (|\xi|^2 + \lambda)\mathcal{F}(u_j) + i\xi_j\mathcal{F}(p) = \mathcal{F}(F_j) + i\xi_\ell\mathcal{F}(f_{\ell j}) & \text{in } \mathbb{R}^d, \\ \qquad\qquad \xi_\ell\mathcal{F}(u_\ell) = 0 & \text{in } \mathbb{R}^d, \end{cases} \tag{2.3}$$

where the repeated index ℓ is summed from 1 to d. A solution of (2.3) is given by

$$\begin{cases} \mathcal{F}(u_j) = (\lambda + |\xi|^2)^{-1}\left(\delta_{jk} - \dfrac{\xi_j\xi_k}{|\xi|^2}\right)\left(\mathcal{F}(F_k) + i\xi_\ell\mathcal{F}(f_{\ell k})\right), \\ \mathcal{F}(p) = \dfrac{-i\xi_k}{|\xi|^2}\left(\mathcal{F}(F_k) + i\xi_\ell\mathcal{F}(f_{\ell k})\right), \end{cases} \tag{2.4}$$

where the repeated indices k, ℓ are summed from 1 to d. Since $\lambda \in \Sigma_\theta$ and $|\lambda| = 1$, we have $|\lambda + |\xi|^2| \approx 1 + |\xi|^2$. Thus, by the Mikhlin multiplier theorem, there exist $u \in W^{1,q}(\mathbb{R}^d;\mathbb{C}^d)$ and $p \in L^q(\mathbb{R}^d;\mathbb{C}) + \mathring{W}^{1,q}(\mathbb{R}^d;\mathbb{C})$, satisfying (2.1) and

$$\|\nabla u\|_{L^q(\mathbb{R}^d)} + \|u\|_{L^q(\mathbb{R}^d)} \leq C\left\{\|f\|_{L^q(\mathbb{R}^d)} + \|F\|_{L^q(\mathbb{R}^d)}\right\}, \tag{2.5}$$

for $1 < q < \infty$, where C depends on d, q and θ.

Case II. Assume that $F = 0$ and $f = 0$. Since $g \in L^q(\mathbb{R}^d;\mathbb{C}) \cap \mathring{W}^{-1,q}(\mathbb{R}^d;\mathbb{C})$, there exists $G \in \mathring{W}^{1,q}(\mathbb{R}^d;\mathbb{C})$ such that $\nabla G \in W^{1,q}(\mathbb{R}^d;\mathbb{C}^d)$, $\Delta G = g$ in $\mathbb{R}^d$,

$$\|\nabla G\|_{L^q(\mathbb{R}^d)} \le C\|g\|_{\mathring{W}^{-1,q}(\mathbb{R}^d)} \quad \text{and} \quad \|\nabla^2 G\|_{L^q(\mathbb{R}^d)} \le C\|g\|_{L^q(\mathbb{R}^d)}.$$

Let $u = \nabla G$ and $p = g - \lambda G$. Then $u \in W^{1,q}(\mathbb{R}^d;\mathbb{C}^d)$, $p \in L^q(\mathbb{R}^d;\mathbb{C}) + \mathring{W}^{1,q}(\mathbb{R}^d;\mathbb{C})$, and (u,p) satisfies (2.1) with $F = 0$ and $f = 0$. Moreover,

$$\|\nabla u\|_{L^q(\mathbb{R}^d)} \le C\|g\|_{L^q(\mathbb{R}^d)} \quad \text{and} \quad \|u\|_{L^q(\mathbb{R}^d)} \le C\|g\|_{\mathring{W}^{-1,q}(\mathbb{R}^d)}.$$

Step 2. We establish the uniqueness of the solution.

Let $u \in W^{1,q}(\mathbb{R}^d;\mathbb{C}^d)$ be a solution of (2.1) in $\mathbb{R}^d$ with $F = 0$, $f = 0$ and $g = 0$. It follows that for any $w \in C^\infty_{0,\sigma}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \nabla u \cdot \nabla w + \lambda \int_{\mathbb{R}^d} u \cdot w = 0, \tag{2.6}$$

where $C^\infty_{0,\sigma}(\mathbb{R}^d) = \{w \in C^\infty_0(\mathbb{R}^d;\mathbb{C}^d) : \text{div}(w) = 0 \text{ in } \mathbb{R}^d\}$. Since $u \in W^{1,q}(\mathbb{R}^d;\mathbb{C}^d)$, by a density argument, we deduce that (2.6) holds for any $w \in W^{1,q'}(\mathbb{R}^d;\mathbb{C}^d)$ with $\text{div}(w) = 0$ in $\mathbb{R}^d$. Let w be a solution in $W^{1,q'}(\mathbb{R}^d;\mathbb{C}^d)$ of the Stokes equations,

$$\begin{cases} -\Delta w + \nabla\phi + \lambda w = |u|^{q-2}\overline{u}, \\ \text{div}(w) = 0 \end{cases} \tag{2.7}$$

in $\mathbb{R}^d$, where $\overline{u}$ denotes the complex conjugate of u. Since $|u|^{q-2}\overline{u} \in L^{q'}(\mathbb{R}^d;\mathbb{C}^d)$, such solution exists in $W^{1,q'}(\mathbb{R}^d;\mathbb{C}^d)$ by Step 1. Again by a density argument, we may deduce from (2.7) that

$$\int_{\mathbb{R}^d} \nabla w \cdot \nabla u + \lambda \int_{\mathbb{R}^d} w \cdot u = \int_{\mathbb{R}^d} |u|^q. \tag{2.8}$$

In view of (2.6) and (2.8), we obtain $\int_{\mathbb{R}^d} |u|^q = 0$ and thus $u = 0$ in $\mathbb{R}^d$. □

Remark 2.2. Let F, f, g, (u,p) be the same as in Theorem 2.1. Let $F = (F_1, F_2, \ldots, F_d)$ and $f = (f_{jk})$. The k component of $\text{div}(f)$ is given by $\sum_j \partial_j f_{jk}$, where ∂_j denotes $\partial/\partial x_j$. Let $x = (x', x_d)$, where $x' \in \mathbb{R}^{d-1}$. Suppose that

$$\begin{cases} F_j \text{ is even in } x_d \text{ for } 1 \le j \le d-1 \text{ and } F_d \text{ is odd}, \\ g \text{ is even in } x_d, \\ f_{jk} \text{ is even in } x_d \text{ for } 1 \le j,k \le d-1, \\ f_{dd} \text{ is even in } x_d, \\ f_{jd} \text{ and } f_{dj} \text{ are odd in } x_d \text{ for } 1 \le j \le d-1. \end{cases} \tag{2.9}$$

Define

$$\begin{cases} v(x', x_d) = (u_1(x', -x_d), \ldots, u_{d-1}(x', -x_d), -u_d(x', -x_d)), \\ \phi(x', x_d) = p(x', -x_d). \end{cases}$$

Then (v, ϕ) is a solution of (2.1) with the same data F, f and g. By the uniqueness in Theorem 2.1, it follows that $u = v$ in $\mathbb{R}^d$. In particular, this implies that $u_d(x', 0) = 0$ for $x' \in \mathbb{R}^{d-1}$.

Remark 2.3. Assume $\lambda \in \Sigma_\theta$ and $|\lambda| = 1$. Let (u, p) be the solution of (2.1), given by Theorem 2.1. An inspection of the proof of Theorem 2.1 shows that $p = p_1 + p_2$, where $p_1 \in L^q(\mathbb{R}^d; \mathbb{C})$, $p_2 \in \mathring{W}^{1,q}(\mathbb{R}^d; \mathbb{C})$, and

$$\|p_1\|_{L^q(\mathbb{R}^d)} + \|\nabla p_2\|_{L^q(\mathbb{R}^d)} \le C \left\{ \|F\|_{L^q(\mathbb{R}^d)} + \|f\|_{L^q(\mathbb{R}^d)} + \|g\|_{L^q(\mathbb{R}^d)} + \|g\|_{\mathring{W}^{-1,q}(\mathbb{R}^d)} \right\}.$$

The constant C depends only on d, q and θ.

Remark 2.4. Let $1 < q_1 < q_2 < \infty$ and $\lambda \in \Sigma_\theta$. Suppose that $F \in L^{q_j}(\mathbb{R}^d; \mathbb{C}^d)$, $f \in L^{q_j}(\mathbb{R}^d; \mathbb{C}^{d\times d})$ and $g \in L^{q_j}(\mathbb{R}^d; \mathbb{C}) \cap \mathring{W}^{-1,q_j}(\mathbb{R}^d; \mathbb{C})$ for $j = 1, 2$. Let (u^j, p^j) be the unique solution of (2.1) in $W^{1,q_j}(\mathbb{R}^d; \mathbb{C}^d) \times (L^{q_j}(\mathbb{R}^d; \mathbb{C}) + \mathring{W}^{1,q_j}(\mathbb{R}^d; \mathbb{C}))$, given by Theorem 2.1. Then $(u^1, p^1) = (u^2, p^2)$. This follows from the observation that the solutions constructed in the proof do not depend on q.

3 A half-space

In this section we consider the resolvent problem for the Stokes equations in the half-space $\mathbb{R}^d_+$. Recall that $\mathring{W}^{1,q}(\mathbb{R}^d_+; \mathbb{C})$ is the homogeneous $W^{1,q}$ space in $\mathbb{R}^d_+$ defined by (1.18), and $\mathring{W}^{-1,q}(\mathbb{R}^d_+; \mathbb{C})$ denotes the dual of $\mathring{W}^{1,q'}(\mathbb{R}^d_+; \mathbb{C})$.

Theorem 3.1. *Let $1 < q < \infty$ and $\lambda \in \Sigma_\theta$. Let $F \in L^q(\mathbb{R}^d_+; \mathbb{C}^d)$, $f \in L^q(\mathbb{R}^d_+; \mathbb{C}^{d\times d})$, and $g \in L^q(\mathbb{R}^d_+; \mathbb{C}) \cap \mathring{W}^{-1,q}(\mathbb{R}^d_+; \mathbb{C})$. Then there exists a unique $u \in W^{1,q}_0(\mathbb{R}^d_+; \mathbb{C}^d)$ such that*

$$\begin{cases} -\Delta u + \nabla p + \lambda u = F + \text{div}(f), \\ \text{div}(u) = g \end{cases} \tag{3.1}$$

hold in $\mathbb{R}^d_+$ for some $p \in L^1_{\text{loc}}(\mathbb{R}^d_+; \mathbb{C})$ in the sense of distributions. Moreover, the solution u satisfies the estimate,

$$\begin{aligned} &|\lambda|^{1/2}\|\nabla u\|_{L^q(\mathbb{R}^d_+)} + |\lambda|\|u\|_{L^q(\mathbb{R}^d_+)} \\ &\le C \left\{ \|F\|_{L^q(\mathbb{R}^d_+)} + |\lambda|^{1/2}\|f\|_{L^q(\mathbb{R}^d_+)} + |\lambda|^{1/2}\|g\|_{L^q(\mathbb{R}^d_+)} + |\lambda|\|g\|_{\mathring{W}^{-1,q}(\mathbb{R}^d_+)} \right\}, \end{aligned} \tag{3.2}$$

and $p \in L^q(\mathbb{R}^d_+; \mathbb{C}) + \mathring{W}^{1,q}(\mathbb{R}^d_+; \mathbb{C})$, where C depends on d, q and θ.

Our proof of Theorem 3.1 follows closely a line of argument in [10].

For a function h in $\mathbb{R}^{d-1}$, we use $\widehat{h}$ to denote the Fourier transform of h,

$$\widehat{h}(\xi') = \int_{\mathbb{R}^{d-1}} e^{-i\xi'\cdot x'} h(x')\, dx', \tag{3.3}$$

for $\xi' \in \mathbb{R}^{d-1}$.

Lemma 3.2. *Let T be a bounded linear operator on $L^2(\mathbb{R}^{d-1}; \mathbb{C}^m)$. Suppose that $\widehat{Tf}(\xi') = m(\xi')\widehat{f}(\xi')$ and that the multiplier $m(\xi')$ satisfies the estimate,*

$$|\xi'|^{|\alpha|}|D^\alpha m(\xi')| \le M, \tag{3.4}$$

for $|\alpha| \le \left[\frac{d-1}{2}\right] + 1$*, where* $\alpha = (\alpha_1, \ldots, \alpha_{d-1})$ *and* $D^\alpha = \partial_1^{\alpha_1} \cdots \partial_{d-1}^{\alpha_{d-1}}$*. Then*

$$\|Tf\|_{L^q(\mathbb{R}^{d-1})} \le CM\|f\|_{L^q(\mathbb{R}^{d-1})}$$

for $1 < q < \infty$*, where* C *depends on* d *and* q*.*

Proof. This is the well known Mikhlin multiplier theorem in $\mathbb{R}^{d-1}$. □

We use $W^{1-\frac{1}{q},q}(\mathbb{R}^{d-1};\mathbb{C}^m)$ to denote the trace space of $W^{1,q}(\mathbb{R}^d_+;\mathbb{C}^m)$ on $\mathbb{R}^{d-1}$.

Lemma 3.3. *Let* T *be a bounded linear operator from* $L^2(\mathbb{R}^{d-1})$ *to* $L^2(\mathbb{R}^d_+)$*. Suppose that*

$$\widehat{Tf}(\xi', x_d) = m(\xi', x_d)\widehat{f}(\xi')$$

and that $m(\xi', x_d)$ *satisfies the condition*

$$|\xi'|^{|\alpha|}|D^\alpha m(\xi', x_d)| + |\xi'|^{|\alpha|-1}|D^\alpha \partial_d m(\xi', x_d)| \le \frac{M_0 e^{-\delta|\xi'|x_d}}{1 + x_d} \tag{3.5}$$

for $x_d > 0$*,* $\xi' \in \mathbb{R}^{d-1}$ *and* $|\alpha| \le \left[\frac{d-1}{2}\right] + 1$*, where* $\delta > 0$*. Then*

$$\begin{cases} \|Tf\|_{L^q(\mathbb{R}^d_+)} \le C\|f\|_{L^q(\mathbb{R}^{d-1})}, \\ \|\nabla T(f)\|_{L^q(\mathbb{R}^d_+)} \le C\|f\|_{W^{1-\frac{1}{q},q}(\mathbb{R}^{d-1})}, \end{cases} \tag{3.6}$$

for $1 < q < \infty$*, where* C *depends on* d*,* q*,* δ *and* M_0*.*

Proof. Note that for each $x_d > 0$, $m(\xi', x_d)$ satisfies (3.4) with $M = M_0(1 + x_d)^{-1}$. It follows from Lemma 3.2 that

$$\begin{aligned} \int_{\mathbb{R}^d_+} |Tf(x', x_d)|^q \, dx' dx_d &\le CM_0^q \int_0^\infty \int_{\mathbb{R}^{d-1}} \frac{|f(x')|^q}{(1 + x_d)^q} \, dx' dx_d \\ &\le CM_0^q \int_{\mathbb{R}^{d-1}} |f|^q \, dx'. \end{aligned}$$

To prove the second inequality in (3.6), we write

$$\widehat{\partial_j Tf}(\xi', x_d) = e^{\delta_0 x_d|\xi'|} m(\xi', x_d) \cdot i\xi_j e^{-\delta_0 x_d|\xi'|}\widehat{f}(\xi')$$

for $1 \le j \le d - 1$, and

$$\widehat{\partial_d Tf}(\xi', x_d) = |\xi'|^{-1} e^{\delta_0 x_d|\xi'|} \partial_d m(\xi', x_d) \cdot |\xi'| e^{-\delta_0 x_d|\xi'|}\widehat{f}(\xi'),$$

where $\delta_0 = \delta/2$. Using (3.5), it is not hard to show that for each $x_d > 0$, both

$$e^{\delta_0 x_d|\xi'|} m(\xi', x_d) \quad \text{and} \quad |\xi'|^{-1} e^{\delta_0 x_d|\xi'|} \partial_d m(\xi', x_d)$$

satisfy the condition (3.4) with M independent of x_d. This implies that

$$\int_{\mathbb{R}^d_+} |\nabla Tf(x', x_d)|^q \, dx \le C \int_{\mathbb{R}^d_+} |\nabla v(x', x_d)|^q \, dx,$$

where v is defined by

$$\widehat{v}(\xi', x_d) = e^{-\delta_0 x_d |\xi'|} \widehat{f}(\xi').$$

Finally, we note that if $\delta_0 = 1$, v is a solution of the Dirichlet problem,

$$\begin{cases} \left(\partial_1^2 + \cdots + \partial_{d-1}^2 + \partial_d^2\right) v = 0 & \text{in } \mathbb{R}^d_+, \\ v = f & \text{on } \mathbb{R}^{d-1} \times \{0\}, \end{cases} \tag{3.7}$$

given by the Poisson integral of f. It is well known that v satisfies the estimate,

$$\|\nabla v\|_{L^q(\mathbb{R}^d_+)} \le C \|f\|_{W^{1-\frac{1}{q},q}(\mathbb{R}^{d-1})},$$

where C depends on d and q [22, Chapter V]. The general case follows from the case $\delta_0 = 1$ by a rescaling in x_d. As a result, we obtain the second inequality in (3.6). □

Proof of Theorem 3.1. By rescaling we may assume $|\lambda| = 1$.

Step 1. We establish the existence and the estimate (3.2).

Let $F \in L^q(\mathbb{R}^d_+; \mathbb{C}^d)$, $f \in L^q(\mathbb{R}^d_+; \mathbb{C}^{d\times d})$ and $g \in L^q(\mathbb{R}^d_+; \mathbb{C}) \cap \mathring{W}^{-1,q}(\mathbb{R}^d_+; \mathbb{C})$. We extend F, f, g to $\mathbb{R}^d$ by either the even or odd reflection in such a way that the extensions satisfy the condition (2.9). Let $\widetilde{F}$, $\widetilde{f}$, $\widetilde{g}$ denote the extensions of F, f, g, respectively. Note that $\widetilde{g} \in \mathring{W}^{-1,q}(\mathbb{R}^d)$ and

$$\|\widetilde{g}\|_{\mathring{W}^{-1,q}(\mathbb{R}^d)} \le 2 \|g\|_{\mathring{W}^{-1,q}(\mathbb{R}^d_+)}.$$

Let $(\widetilde{u}, \widetilde{p})$ denote the solution of (2.1) in $\mathbb{R}^d$, given by Theorem 2.1, with data $\widetilde{F}, \widetilde{f}, \widetilde{g}$. By Remark 2.2, we have $\widetilde{u}_d(x', 0) = 0$ for any $x' \in \mathbb{R}^{d-1}$. By subtracting $(\widetilde{u}, \widetilde{p})$ from solutions of (3.1), we reduce the problem to the Dirichlet problem,

$$\begin{cases} -\Delta u + \nabla p + \lambda u = 0 & \text{in } \mathbb{R}^d_+, \\ \operatorname{div}(u) = 0 & \text{in } \mathbb{R}^d_+, \\ u_j = h_j & \text{on } \mathbb{R}^{d-1} \times \{0\} \text{ for } 1 \le j \le d-1, \\ u_d = 0 & \text{on } \mathbb{R}^{d-1} \times \{0\}, \end{cases} \tag{3.8}$$

where $h_j = -\widetilde{u}_j$ for $1 \le j \le d-1$. We will show that there exist $u \in W^{1,q}(\mathbb{R}^d_+; \mathbb{C}^d)$ and $p \in L^q(\mathbb{R}^d_+; \mathbb{C})$ such that (u, p) satisfies (3.8) and the estimate,

$$\|\nabla u\|_{L^q(\mathbb{R}^d_+)} + \|u\|_{L^q(\mathbb{R}^d_+)} + \|p\|_{L^q(\mathbb{R}^d_+)} \le C \|h\|_{W^{1-\frac{1}{q},q}(\mathbb{R}^{d-1})}. \tag{3.9}$$

Since $h_j = -\widetilde{u}_j$ on $\mathbb{R}^{d-1} \times \{0\}$ and

$$\begin{aligned} \|\widetilde{u}\|_{W^{1-\frac{1}{q},q}(\mathbb{R}^{d-1})} &\le C \left\{ \|\nabla \widetilde{u}\|_{L^q(\mathbb{R}^d)} + \|\widetilde{u}\|_{L^q(\mathbb{R}^d)} \right\} \\ &\le C \left\{ \|\widetilde{F}\|_{L^q(\mathbb{R}^d)} + \|\widetilde{f}\|_{L^q(\mathbb{R}^d)} + \|\widetilde{g}\|_{L^q(\mathbb{R}^d)} + \|\widetilde{g}\|_{\mathring{W}^{-1,q}(\mathbb{R}^d)} \right\} \\ &\le C \left\{ \|F\|_{L^q(\mathbb{R}^d_+)} + \|f\|_{L^q(\mathbb{R}^d_+)} + \|g\|_{L^q(\mathbb{R}^d_+)} + \|g\|_{\mathring{W}^{-1,q}(\mathbb{R}^d_+)} \right\}, \end{aligned}$$

the desired estimate (3.2) follows from (3.9).

To solve (3.8), we use the partial Fourier transform in $x' = (x_1, \ldots, x_{d-1})$, defined by (3.3). Let

$$m_0(s, x_d) = \frac{e^{-\sqrt{\lambda+s^2}x_d} - e^{-sx_d}}{\sqrt{\lambda + s^2} - s},$$

where $s = |\xi'|$. It follows from [10] that a solution of (3.8) in the partial Fourier transform is given by

$$\begin{cases} \widehat{u}_j(\xi', x_d) = -\partial_d m_0(s, x_d)\dfrac{\xi_j\xi_k}{s^2}\widehat{h}_k(\xi') + \left(\delta_{jk} - \dfrac{\xi_j\xi_k}{s^2}\right) e^{-\sqrt{\lambda+s^2}x_d}\widehat{h}_k(\xi') \\ \widehat{u}_d(\xi', x_d) = im_0(s, x_d)\xi_k\widehat{h}_k(\xi') \end{cases} \tag{3.10}$$

for $1 \leq j \leq d-1$, and

$$\widehat{p}(\xi', x_d) = -s^{-2}(\lambda + s^2 - \partial_d^2)\partial_d\widehat{u}_d, \tag{3.11}$$

where the repeated index k is summed from 1 to $d-1$. Write

$$\widehat{u}_j(\xi', x_d) = m_{jk}(\xi', x_d)\widehat{h}_k(\xi)$$

for $1 \leq j \leq d$. Note that m_{jk} satisfies the condition (3.5) (see [10, Lemma 2.5]). By Lemma 3.3, we obtain

$$\|u\|_{L^q(\mathbb{R}^d_+)} + \|\nabla u\|_{L^q(\mathbb{R}^d_+)} \leq C\|h\|_{W^{1-\frac{1}{p},p}(\mathbb{R}^{d-1})}$$

for $1 < q < \infty$. Using the fact that

$$s^{-2}(\lambda + s^2 - \partial_d^2)\partial_d m_0(s, x_d) = s^{-1}(\sqrt{\lambda + s^2} + s)e^{-sx_d},$$

and that $s^{-1}(\sqrt{\lambda + s^2} + s)e^{-sx_d}$ satisfies the condition (3.5), it follows again by Lemma 3.3 that

$$\int_{\mathbb{R}^d_+} |p|^q \, dx \leq C\|h\|^q_{W^{1-\frac{1}{q},q}(\mathbb{R}^{d-1})}.$$

As a result, we have proved (3.9).

Step 2. With the existence established in Step 1 at our disposal, the uniqueness may be proved by using the same argument as in the proof of Theorem 2.1. We omit the details. □

Remark 3.4. Let $\lambda \in \Sigma_\theta$ and $|\lambda| = 1$. Let (u, p) be the solution of (3.1), given by Theorem 3.1. It follows from the proof of Theorem 3.1 that $p = p_1 + p_2$, where $p_1 \in L^q(\mathbb{R}^d_+; \mathbb{C})$, $p_2 \in \mathring{W}^{1,q}(\mathbb{R}^d_+; \mathbb{C})$, and

$$\|p_1\|_{L^q(\mathbb{R}^d_+)} + \|\nabla p_2\|_{L^q(\mathbb{R}^d_+)} \leq C\left\{\|F\|_{L^q(\mathbb{R}^d_+)} + \|f\|_{L^q(\mathbb{R}^d_+)} + \|g\|_{L^q(\mathbb{R}^d_+)} + \|g\|_{\mathring{W}^{-1,q}(\mathbb{R}^d_+)}\right\},$$

where C depends only on d, q and θ.

Remark 3.5. Let $1 < q_1 < q_2 < \infty$. Suppose that $F \in L^{q_j}(\mathbb{R}^d_+; \mathbb{C}^d)$, $f \in L^{q_j}(\mathbb{R}^d_+; \mathbb{C}^{d\times d})$, and $g \in L^{q_j}(\mathbb{R}^d_+; \mathbb{C}) \cap \mathring{W}^{-1,q_j}(\mathbb{R}^d_+; \mathbb{C})$ for $j = 1, 2$. Let $u^j \in W^{1,q_j}_0(\mathbb{R}^d_+; \mathbb{C}^d)$ be the solution of (3.1), given by Theorem 3.1, with the same data F, f, g, for $j = 1, 2$. Since the solutions constructed in $W^{1,q}_0(\mathbb{R}^d_+; \mathbb{C}^d)$ for the existence part of the proof do not depend on q, it follows that $u^1 = u^2$ in $\mathbb{R}^d_+$. As a result, we obtain $u^1 = u^2 \in W^{1,q_1}_0(\mathbb{R}^d_+; \mathbb{C}^d) \cap W^{1,q_2}_0(\mathbb{R}^d_+; \mathbb{C}^d)$.

4 The region above a Lipschitz graph

Let

$$\mathbb{H}_\psi = \left\{ (x', x_d) \in \mathbb{R}^d : \; x' \in \mathbb{R}^{d-1} \text{ and } x_d > \psi(x') \right\},$$

where $\psi : \mathbb{R}^{d-1} \to \mathbb{R}$ is a Lipschitz function. Note that if $\psi = 0$, we have $\mathbb{H}_0 = \mathbb{R}^d_+$. In this section we study the resolvent problem for the Stokes equations,

$$\begin{cases} -\Delta u + \nabla p + \lambda u = F + \operatorname{div}(f) & \text{in } \mathbb{H}_\psi, \\ \operatorname{div}(u) = g & \text{in } \mathbb{H}_\psi, \\ u = 0 & \text{on } \partial\mathbb{H}_\psi, \end{cases} \tag{4.1}$$

where $\lambda \in \Sigma_\theta$. For $1 < q < \infty$, define

$$A^q_\psi = L^q(\mathbb{H}_\psi; \mathbb{C}) + \mathring{W}^{1,q}(\mathbb{H}_\psi; \mathbb{C}) \quad \text{ and } \quad B^q_\psi = L^q(\mathbb{H}_\psi; \mathbb{C}) \cap \mathring{W}^{-1,q}(\mathbb{H}_\psi; \mathbb{C}), \tag{4.2}$$

where, as in the case $\mathbb{R}^d$ and $\mathbb{R}^d_+$,

$$\mathring{W}^{1,q}(\mathbb{H}_\psi; \mathbb{C}) = \left\{ u \in L^q_{\text{loc}}(\mathbb{H}_\psi; \mathbb{C}) : \; \nabla u \in L^q(\mathbb{H}_\psi; \mathbb{C}^d) \right\},$$

with the norm $\|\nabla u\|_{L^q(\mathbb{H}_\psi)}$, and $\mathring{W}^{-1,q}(\mathbb{H}_\psi; \mathbb{C})$ denotes the dual of $\mathring{W}^{1,q'}(\mathbb{H}_\psi; \mathbb{C})$. Note that A^q_ψ and B^q_ψ are Banach spaces with the usual norms,

$$\|p\|_{A^q_\psi} = \inf \left\{ \|p_1\|_{L^q(\mathbb{H}_\psi)} + \|\nabla p_2\|_{L^q(\mathbb{H}_\psi)} : \quad p = p_1 + p_2 \text{ in } \mathbb{H}_\psi \right\}$$

and

$$\|g\|_{B^q_\psi} = \|g\|_{L^q(\mathbb{H}_\psi)} + \|g\|_{\mathring{W}^{-1,q}(\mathbb{H}_\psi)}.$$

The goal of this section is to prove the following.

Theorem 4.1. *Let $\lambda \in \Sigma_\theta$ and $1 < q < \infty$. There exists $c_0 \in (0, 1)$, depending only on d, q and θ, such that if $\|\nabla'\psi\|_\infty \leq c_0$, then for any $F \in L^q(\mathbb{H}_\psi; \mathbb{C}^d)$, $f \in L^q(\mathbb{H}_\psi; \mathbb{C}^{d\times d})$ and $g \in B^q_\psi$, there exists a unique (u, p) such that $u \in W^{1,q}_0(\mathbb{H}_\psi; \mathbb{C}^d)$, $p \in A^q_\psi$, and (4.1) holds. Moreover, the solution satisfies*

$$\begin{aligned} &|\lambda|^{1/2} \|\nabla u\|_{L^q(\mathbb{H}_\psi)} + |\lambda| \|u\|_{L^q(\mathbb{H}_\psi)} \\ &\quad \leq C \left\{ \|F\|_{L^q(\mathbb{H}_\psi)} + |\lambda|^{1/2} \|f\|_{L^q(\mathbb{H}_\psi)} + |\lambda|^{1/2} \|g\|_{L^q(\mathbb{H}_\psi)} + |\lambda| \|g\|_{\mathring{W}^{-1,q}(\mathbb{H}_\psi)} \right\}, \end{aligned} \tag{4.3}$$

where C depends only on d, q and θ.

To prove Theorem 4.1, we introduce two Banach spaces,

$$X^q_\psi = W^{1,q}_0(\mathbb{H}_\psi; \mathbb{C}^d) \times A^q_\psi \quad \text{ and } \quad Y^q_\psi = W^{-1,q}(\mathbb{H}_\psi; \mathbb{C}^d) \times B^q_\psi, \tag{4.4}$$

with the usual product norms. For $\lambda \in \Sigma_\theta$ with $|\lambda| = 1$, consider the operator

$$S^\lambda_\psi(u, p) = \big(-\Delta u + \nabla p + \lambda u, \operatorname{div}(u) \big). \tag{4.5}$$

It is not hard to see that S^λ_ψ is a bounded linear operator from X^q_ψ to Y^q_ψ for any $1 < q < \infty$ and that

$$\|S^\lambda_\psi(u, p)\|_{Y^q_\psi} \leq C \|(u, p)\|_{X^q_\psi}, \tag{4.6}$$

where C depends only on d and q. Using Theorem 3.1 and a perturbation argument, we will show that S^λ_ψ is invertible if $\|\nabla'\psi\|_\infty$ is sufficiently small.

Lemma 4.2. *Let $\lambda \in \Sigma_\theta$ with $|\lambda| = 1$ and $1 < q < \infty$. Assume that $\psi = 0$. Then $S_0^\lambda : X_0^q \to Y_0^q$ is a bijection and*

$$\|(S_0^\lambda)^{-1}\|_{Y_0^q \to X_0^q} \leq C, \tag{4.7}$$

where C depends only on d, q and θ.

Proof. In the case $\psi = 0$, we have $\mathbb{H}_\psi = \mathbb{R}_+^d$. The lemma follows readily from Theorem 3.1 and the estimate for p in Remark 3.4. Indeed, note that for any $\Lambda \in W^{-1,q}(\Omega; \mathbb{C}^d)$, there exist $F \in L^q(\Omega; \mathbb{C}^d)$ and $f \in L^q(\Omega; \mathbb{C}^{d\times d})$ such that $\Lambda = F + \operatorname{div}(f)$ and $\|\Lambda\|_{W^{-1,q}(\Omega)} \approx \|F\|_{L^q(\Omega)} + \|f\|_{L^q(\Omega)}$. □

Theorem 4.3. *Let $\lambda \in \Sigma_\theta$ with $|\lambda| = 1$. Let $1 < q < \infty$. There exists $c_0 \in (0, 1)$, depending only on d, q and θ, such that if $\|\nabla'\psi\|_\infty \leq c_0$, then $S_\psi^\lambda : X_\psi^q \to Y_\psi^q$ is a bijection and*

$$\|(S_\psi^\lambda)^{-1}\|_{Y_\psi^q \to X_\psi^q} \leq C, \tag{4.8}$$

where C depends only on d, q and θ.

Proof. Suppose $\|\nabla'\psi\|_\infty \leq 1$. Define a bi-Lipschitz map $\Psi : \mathbb{H}_\psi \to \mathbb{R}_+^d$ by

$$\Psi(x', x_d) = (x', x_d - \psi(x')).$$

Note that $\Psi^{-1}(x', x_d) = (x', x_d + \psi(x'))$. For a function u in $\mathbb{H}_\psi$, let $\widetilde{u} = u \circ \Psi^{-1}$, defined in $\mathbb{R}_+^d$. Thus, $u = \widetilde{u} \circ \Psi$ and

$$\begin{cases} \partial_j u = \partial_j \widetilde{u} \circ \Psi - \partial_d(\widetilde{u}\partial_j\psi) \circ \Psi & \text{for } 1 \leq j \leq d-1, \\ \partial_d u = \partial_d \widetilde{u} \circ \Psi. \end{cases}$$

A computation shows that

$$\Delta u = \Delta\widetilde{u} \circ \Psi - \partial_d(\partial_k\widetilde{u}\partial_k\psi) \circ \Psi - \partial_k(\partial_d\widetilde{u}\partial_k\psi) \circ \Psi + \partial_d(\partial_d\widetilde{u}|\nabla'\psi|^2) \circ \Psi,$$

where $|\nabla'\psi|^2 = |\partial_1\psi|^2 + \cdots + |\partial_{d-1}\psi|^2$ and the repeated index k is summed from 1 to $d-1$. For $(u, p) \in X_\psi^q$, let $\widetilde{u} = u \circ \Psi^{-1}$ and $\widetilde{p} = p \circ \Psi^{-1}$. Then

$$\begin{aligned} -\Delta u_j + \partial_j p + \lambda u_j = {} & -\Delta\widetilde{u}_j \circ \Psi + \partial_j\widetilde{p} \circ \Psi + \lambda\widetilde{u}_j \circ \Psi \\ & + \partial_d(\partial_k\widetilde{u}_j\partial_k\psi) \circ \Psi + \partial_k(\partial_d\widetilde{u}_j\partial_k\psi) \circ \Psi \\ & - \partial_d(\partial_d\widetilde{u}_j|\nabla'\psi|^2) \circ \Psi - \partial_d(\widetilde{p}\partial_j\psi) \circ \Psi \end{aligned} \tag{4.9}$$

for $1 \leq j \leq d-1$, and

$$\begin{aligned} -\Delta u_d + \partial_d p + \lambda u_d = {} & - \Delta\widetilde{u}_d \circ \Psi + \partial_d\widetilde{p} \circ \Psi + \lambda\widetilde{u}_d \circ \Psi \\ & + \partial_d(\partial_k\widetilde{u}_d\partial_k\psi) \circ \Psi + \partial_k(\partial_d\widetilde{u}_d\partial_k\psi) \circ \Psi \\ & - \partial_d(\partial_d\widetilde{u}_d|\nabla'\psi|^2) \circ \Psi, \end{aligned} \tag{4.10}$$

where the repeated index k is summed from 1 to $d-1$. Also, note that

$$\operatorname{div}(u) = \operatorname{div}(\widetilde{u}) \circ \Psi - \partial_d(\widetilde{u}_k\partial_k\psi) \circ \Psi. \tag{4.11}$$

In view of (4.9), (4.10) and (4.11), we obtain

$$S^\lambda_\psi(u,p) = S^\lambda_0(\widetilde{u},\widetilde{p}) \circ \Psi + R(\widetilde{u},\widetilde{p}) \circ \Psi, \tag{4.12}$$

where $R(\widetilde{u},\widetilde{p}) = (R_1(\widetilde{u},\widetilde{p}), \ldots, R_d(\widetilde{u},\widetilde{p}), R_{d+1}(\widetilde{u},\widetilde{p}))$ with

$$R_j(\widetilde{u},\widetilde{p}) = \partial_d(\partial_k\widetilde{u}_j\partial_k\psi) + \partial_k(\partial_d\widetilde{u}_j\partial_k\psi) - \partial_d(\partial_d\widetilde{u}_j|\nabla'\psi|^2) - \partial_d(\widetilde{p}\partial_j\psi) \tag{4.13}$$

for $1 \le j \le d-1$, and

$$\begin{cases} R_d(\widetilde{u},\widetilde{p}) = \partial_d(\partial_k\widetilde{u}_d\partial_k\psi) + \partial_k(\partial_d\widetilde{u}_d\partial_k\psi) - \partial_d(\partial_d\widetilde{u}_d|\nabla'\psi|^2), \\ R_{d+1}(\widetilde{u},\widetilde{p}) = -\partial_d(\widetilde{u}_k\partial_k\psi). \end{cases} \tag{4.14}$$

We claim that for any $(\widetilde{u},\widetilde{p}) \in X^q_0$,

$$\|R(\widetilde{u},\widetilde{p})\|_{Y^q_0} \le C\|\nabla'\psi\|_\infty \|(\widetilde{u},\widetilde{p})\|_{X^q_0}, \tag{4.15}$$

where C depends only on d and q. To show (4.15), we note that

$$\|R_j(\widetilde{u},\widetilde{p})\|_{W^{-1,q}(\mathbb{R}^d_+)} \le C\|\nabla'\psi\|_\infty \|\nabla\widetilde{u}\|_{L^q(\mathbb{R}^d_+)} + \|\partial_d(\widetilde{p}\partial_j\psi)\|_{W^{-1,q}(\mathbb{R}^d_+)} \tag{4.16}$$

for $1 \le j \le d-1$, and

$$\|R_d(\widetilde{u},\widetilde{p})\|_{W^{-1,q}(\mathbb{R}^d_+)} \le C\|\nabla'\psi\|_\infty \|\nabla\widetilde{u}\|_{L^q(\mathbb{R}^d_+)}, \tag{4.17}$$

where we have used the assumption $\|\nabla'\psi\|_\infty \le 1$. To bound the second term in the right-hand side of (4.16), we let

$$\widetilde{p} = \widetilde{p}_1 + \widetilde{p}_2 \in L^q(\mathbb{R}^d_+;\mathbb{C}) + \mathring{W}^{1,q}(\mathbb{R}^d_+;\mathbb{C}) = A^q_0.$$

Then

$$\begin{aligned} \|\partial_d(\widetilde{p}\partial_j\psi)\|_{W^{-1,q}(\mathbb{R}^d_+)} &\le \|\partial_d(\widetilde{p}_1\partial_j\psi)\|_{W^{-1,q}(\mathbb{R}^d_+)} + \|\partial_d(\widetilde{p}_2\partial_j\psi)\|_{W^{-1,q}(\mathbb{R}^d_+)} \\ &\le C\|\nabla'\psi\|_\infty \|\widetilde{p}_1\|_{L^q(\mathbb{R}^d_+)} + C\|\nabla'\psi\|_\infty \|\partial_d\widetilde{p}_2\|_{L^q(\mathbb{R}^d_+)}. \end{aligned}$$

This shows that

$$\|\partial_d(\widetilde{p}\partial_j\psi)\|_{W^{-1,q}(\mathbb{R}^d_+)} \le C\|\nabla'\psi\|_\infty \|\widetilde{p}\|_{A^q_0}.$$

As a result, we have proved that

$$\|R_j(\widetilde{u},\widetilde{p})\|_{W^{-1,q}(\mathbb{R}^d_+)} \le C\|\nabla'\psi\|_\infty \|(\widetilde{u},\widetilde{p})\|_{X^q_0} \tag{4.18}$$

for $1 \le j \le d$. This, together with the estimates,

$$\|R_{d+1}(\widetilde{u},\widetilde{p})\|_{L^q(\mathbb{R}^d_+)} + \|R_{d+1}(\widetilde{u},\widetilde{p})\|_{\mathring{W}^{-1,q}(\mathbb{R}^d_+)} \le C\|\nabla'\psi\|_\infty \left(\|\nabla\widetilde{u}\|_{L^q(\mathbb{R}^d_+)} + \|\widetilde{u}\|_{L^q(\mathbb{R}^d_+)} \right),$$

gives (4.15).

By Lemma 4.2, $S^\lambda_0 : X^q_0 \to Y^q_0$ is bounded and invertible for $1 < q < \infty$. It follows by a standard perturbation argument that $S^\lambda_0 + R : X^q_0 \to Y^q_0$ is bounded and invertible if

$$\|R(S^\lambda_0)^{-1}\|_{Y^q_0 \to Y^q_0} < 1.$$

Moreover, we have

$$\|(S_0^\lambda + R)^{-1}\|_{Y_0^q \to X_0^q} \le \frac{\|(S_0^\lambda)^{-1}\|_{Y_0^q \to X_0^q}}{1 - \|R(S_0^\lambda)^{-1}\|_{Y_0^q \to Y_0^q}}.$$

By (4.7) and (4.15),

$$\begin{aligned}\|R(S_0^\lambda)^{-1}\|_{Y_0^q \to Y_0^q} &\le \|R\|_{X_0^q \to Y_0^q} \|(S_0^\lambda)^{-1}\|_{Y_0^q \to X_0^q} \\ &\le C_0 \|\nabla' \psi\|_\infty,\end{aligned}$$

where C_0 depends only on d, q and θ. As a result, we have proved that if $\|\nabla'\psi\|_\infty \le (2C_0)^{-1}$, then $S_0^\lambda + R : X_0^q \to Y_0^q$ is invertible and

$$\|(S_0^\lambda + R)^{-1}\|_{Y_0^q \to X_0^q} \le C$$

for some C depending on d, q and θ. Finally, we note that

$$\|(u \circ \Psi^{-1}, p \circ \Psi^{-1})\|_{X_0^q} \approx \|(u, p)\|_{X_\psi^q}$$

for any $(u, p) \in X_\psi^q$, and

$$\|(\Lambda \circ \Psi^{-1}, g \circ \Psi^{-1})\|_{Y_0^q} \approx \|(\Lambda, g)\|_{Y_\psi^q}$$

for any $(\Lambda, g) \in Y_\psi^q$. By (4.12), we deduce that if $\|\nabla'\psi\|_\infty \le c_0(d, q, \theta)$, then $S_\psi^\lambda : X_\psi^q \to Y_\psi^q$ is invertible and (4.8) holds. This completes the proof. □

Proof of Theorem 4.1. The case $|\lambda| = 1$ follows readily from Theorem 4.3. The general case can be reduced to the case $|\lambda| = 1$ by rescaling. Indeed, let (u, p) be a solution of (4.1) in $\mathbb{H}_\psi$. Let $v(x) = u(|\lambda|^{-1/2}x)$ and $\phi(x) = |\lambda|^{-1/2}p(|\lambda|^{-1/2}x)$. Then (v, ϕ) is a solution of the resolvent problem for the Stokes equations in the graph domain $\mathbb{H}_{\psi_\lambda}$ with the parameter $\lambda|\lambda|^{-1} \in \Sigma_\theta$, where $\psi_\lambda(x') = |\lambda|^{1/2}\psi(|\lambda|^{-1/2}x')$. Moreover, we have $\|\nabla'\psi_\lambda\|_\infty = \|\nabla'\psi\|_\infty$. As a result, the general case follows from the case $|\lambda| = 1$. □

Remark 4.4. Let $1 < q_1 < q_2 < \infty$. Let $\lambda \in \Sigma_\theta$ and $|\lambda| = 1$. It follows from Lemma 4.2 and Remark 3.5 that $S_0^\lambda : X_0^{q_1} \cap X_0^{q_2} \to Y_0^{q_1} \cap Y_0^{q_2}$ is a bijection and

$$\|(S_0^\lambda)^{-1}\|_{Y_0^{q_1} \cap Y_0^{q_2} \to X_0^{q_1} \cap X_0^{q_2}} \le C,$$

where C depends only on d, q_1, q_2 and θ. By the same perturbation argument as in the proof of Theorem 4.3, we deduce that $S_\psi^\lambda : X_\psi^{q_1} \cap X_\psi^{q_2} \to Y_\psi^{q_1} \cap Y_\psi^{q_2}$ is a bijection and

$$\|(S_\psi^\lambda)^{-1}\|_{Y_\psi^{q_1} \cap Y_\psi^{q_2} \to X_\psi^{q_1} \cap X_\psi^{q_2}} \le C,$$

if $\|\nabla'\psi\|_\infty \le c_0(d, q_1, q_2, \theta)$, where C depends only on d, q_1, q_2 and θ. Consequently, if $F \in L^{q_1}(\mathbb{H}_\psi; \mathbb{C}^d) \cap L^{q_2}(\mathbb{H}_\psi; \mathbb{C}^d)$, $f \in L^{q_1}(\mathbb{H}_\psi; \mathbb{C}^{d\times d}) \cap L^{q_2}(\mathbb{H}_\psi; \mathbb{C}^{d\times d})$ and $g \in B_\psi^{q_1} \cap B_\psi^{q_2}$, then the solution u of (4.1), given by Theorem 4.1, belongs to $W_0^{1,q_1}(\mathbb{H}_\psi; \mathbb{C}^d) \cap W_0^{1,q_2}(H_\psi; \mathbb{C}^d)$, provided that $\|\nabla'\psi\|_\infty$ is sufficiently small.

Remark 4.5. Let (u, p) be a solution of the resolvent problem for the Stokes equations in $\mathbb{H}_\psi$. Let $v(x) = O^T u(Ox)$ and $\phi(x) = p(Ox)$, where O is a $d \times d$ orthogonal matrix. Then

$$\begin{cases} (-\Delta v + \nabla \phi + \lambda v)(x) = O^T(-\Delta u + \nabla p + \lambda u)(Ox), \\ \qquad\qquad \text{div}(v)(x) = \text{div}(u)(Ox). \end{cases}$$

Consequently, Theorem 4.1 continues to hold if the domain $\mathbb{H}_\psi$ is replaced by

$$O\mathbb{H}_\psi = \{y \in \mathbb{R}^d : y = Ox \text{ for some } x \in \mathbb{H}_\psi\}$$

for any $d \times d$ orthogonal matrix.

5 A bounded C^1 domain and the proof of Theorem 1.1

Throughout this section we assume that Ω is a bounded C^1 domain in $\mathbb{R}^d$. This implies that for any $c_0 > 0$, there exists some $r_0 > 0$ such that for each $z = (z', z_d) \in \partial\Omega$,

$$\Omega \cap B(z, 2r_0) = D \cap B(z, 2r_0) \quad \text{and} \quad \partial\Omega \cap B(z, 2r_0) = \partial D \cap B(z, 2r_0), \tag{5.1}$$

where D is given by

$$D = O\mathbb{H}_\psi \quad \text{for some orthogonal matrix } O \text{ and some } C^1 \text{ function } \psi \text{ in } \mathbb{R}^{d-1} \tag{5.2}$$

with $\nabla'\psi(z') = 0$ and $\|\nabla'\psi\|_\infty \le c_0$. Recall that ∇' denotes the gradient with respect to $x' = (x_1, \ldots, x_{d-1})$. We will use $L^q_0(\Omega; \mathbb{C})$ to denote the subspace of $L^q(\Omega; \mathbb{C})$ of functions p with $\int_\Omega p = 0$.

The goal of this section is to prove the following theorem, which contains Theorem 1.1 as a special case with $f = 0$ and $g = 0$.

Theorem 5.1. *Let Ω be a bounded C^1 domain in $\mathbb{R}^d$, $d \ge 2$. Let $1 < q < \infty$ and $\lambda \in \Sigma_\theta$. For any $F \in L^q(\Omega; \mathbb{C}^d)$, $f \in L^q(\Omega; \mathbb{C}^{d\times d})$ and $g \in L^q_0(\Omega; \mathbb{C})$, there exists a unique $u \in W^{1,q}_0(\Omega; \mathbb{C}^d)$ such that*

$$\begin{cases} -\Delta u + \nabla p + \lambda u = F + \text{div}(f), \\ \qquad\qquad \text{div}(u) = g \end{cases} \tag{5.3}$$

hold in Ω for some $p \in L^1_{\text{loc}}(\Omega; \mathbb{C})$ in the sense of distributions. Moreover, the solution u satisfies the estimate,

$$\begin{aligned} &(|\lambda| + 1)^{1/2}\|\nabla u\|_{L^q(\Omega)} + (|\lambda| + 1)\|u\|_{L^q(\Omega)} \\ &\qquad \le C\left\{\|F\|_{L^q(\Omega)} + (|\lambda| + 1)^{1/2}\|f\|_{L^q(\Omega)} + (|\lambda| + 1)\|g\|_{L^q(\Omega)}\right\}, \end{aligned} \tag{5.4}$$

and $p \in L^q(\Omega; \mathbb{C})$, where C depends only on d, q, θ and Ω.

Theorem 5.1 follows from Theorems 2.1 and 4.1 by a localization argument.

Lemma 5.2. *Let $u \in W^{1,q}_0(\Omega; \mathbb{C}^d)$ for some $1 < q < \infty$. Suppose $\text{div}(u) = 0$ in Ω. Then*

$$\|\text{div}(u\varphi)\|_{\mathring{W}^{-1,q}(\mathbb{R}^d)} \le C(\|\nabla\varphi\|_\infty + \|\nabla^2\varphi\|_\infty)\|u\|_{W^{-1,q}(\Omega)}, \tag{5.5}$$

where $\varphi \in C^\infty_0(\Omega; \mathbb{R})$ and C depends on d, q, diam(Ω) and the Lipschitz character of Ω.

Proof. Let $h \in \mathring{W}^{1,q'}(\mathbb{R}^d;\mathbb{C})$. Note that

$$\int_{\mathbb{R}^d} \text{div}(u\varphi)\cdot h = \int_\Omega (u\cdot\nabla\varphi)\left(h - \fint_\Omega h\right),$$

where we have used the assumption $\text{div}(u) = 0$ in Ω. It follows that

$$\begin{aligned}\left|\int_{\mathbb{R}^d} \text{div}(u\varphi)\cdot h\right| &\leq \|u\|_{W^{-1,q}(\Omega)} \|\nabla\varphi(h - \fint_\Omega h)\|_{W_0^{1,q'}(\Omega)}\\ &\leq C(\|\nabla\varphi\|_\infty + \|\nabla^2\varphi\|_\infty)\|u\|_{W^{-1,q}(\Omega)}\|\nabla h\|_{L^{q'}(\mathbb{R}^d)},\end{aligned}$$

where we have used a Poincaré inequality in Ω. This gives (5.5). □

Remark 5.3. Let u be the same as in Lemma 5.2. Suppose $\varphi \in C_0^\infty(B(z,2r_0);\mathbb{R})$, where $z \in \partial\Omega$ and $\Omega \cap B(z,2r_0)$ satisfies (5.1)-(5.2). Let $W_0^{-1,q}(\Omega;\mathbb{C}^d)$ denote the dual of $W^{1,q'}(\Omega;\mathbb{C}^d)$. Then

$$\|\text{div}(u\varphi)\|_{\mathring{W}^{-1,q}(D)} \leq C(\|\nabla\varphi\|_\infty + \|\nabla^2\varphi\|_\infty)\|u\|_{W_0^{-1,q}(\Omega)}, \tag{5.6}$$

where D is given by (5.1)-(5.2). To see this, we note that for any $h \in \mathring{W}^{1,q'}(D;\mathbb{C})$,

$$\int_D \text{div}(u\varphi)\cdot h = \int_\Omega (u\cdot\nabla\varphi)\left(h - \fint_\Omega h\right),$$

where we have used the assumptions that $\text{div}(u) = 0$ in Ω and $u = 0$ on $\partial\Omega$.

Lemma 5.4. *Let $1 < q < \infty$. Then for any $p \in L_0^q(\Omega;\mathbb{C})$,*

$$\|p\|_{L^q(\Omega)} \leq C\|\nabla p\|_{W^{-1,q}(\Omega)}, \tag{5.7}$$

where C depends on d, q, diam(Ω) and the Lipschitz character of Ω.

Proof. Since Ω is a bounded Lipschitz domain and $\overline{p}|p|^{q-2} \in L^{q'}(\Omega;\mathbb{C})$, there exists $v \in W_0^{1,q'}(\Omega;\mathbb{C}^d)$ such that

$$\text{div}(v) = \overline{p}|p|^{q-2} - \fint_\Omega \overline{p}|p|^{q-2} \quad \text{in } \Omega$$

(see [13, Theorem III.3.1]). Moreover, the function v satisfies

$$\|v\|_{W^{1,q'}(\Omega)} \leq C\|\overline{p}|p|^{q-2}\|_{L^{q'}(\Omega)} = C\|p\|_{L^q(\Omega)}^{q-1}. \tag{5.8}$$

Using

$$\int_\Omega |p|^q = \int_\Omega p\cdot\text{div}(v),$$

we obtain

$$\begin{aligned}\|p\|_{L^q(\Omega)}^q &\leq \|\nabla p\|_{W^{-1,q}(\Omega)}\|v\|_{W_0^{1,q'}(\Omega)}\\ &\leq C\|\nabla p\|_{W^{-1,q}(\Omega)}\|p\|_{L^q(\Omega)}^{q-1},\end{aligned}$$

where we have used (5.8) for the last inequality. This yields (5.7). □

The following lemma contains a key a priori estimate. Recall that $W_0^{-1,q}(\Omega;\mathbb{C}^d)$ denotes the dual of $W^{1,q'}(\Omega;\mathbb{C}^d)$.

Lemma 5.5. *Let $1 < q < \infty$ and $\lambda \in \Sigma_\theta$. Let $(u,p) \in W_0^{1,q}(\Omega;\mathbb{C}^d) \times L_0^q(\Omega;\mathbb{C})$ be a solution of (5.3) with $F \in L^q(\Omega;\mathbb{C}^d)$, $f \in L^q(\Omega;\mathbb{C}^{d\times d})$ and $g = 0$. There exist $\lambda_0 > 1$ and $C > 0$, depending only on d, q, θ, diam(Ω) and the C^1 character of Ω, such that if $|\lambda| \geq \lambda_0$, then*

$$|\lambda|^{1/2}\|\nabla u\|_{L^q(\Omega)} + |\lambda|\|u\|_{L^q(\Omega)} \leq C\left\{\|F\|_{L^q(\Omega)} + |\lambda|^{1/2}\|f\|_{L^q(\Omega)} + |\lambda|\|u\|_{W_0^{-1,q}(\Omega)}\right\}. \tag{5.9}$$

Proof. Let $z \in \overline{\Omega}$ and $r_0 > 0$ be small. Let $\varphi \in C_0^\infty(B(z,2r_0);\mathbb{R})$ such that $\varphi = 1$ in $B(z,r_0)$ and $|\nabla\varphi| \leq Cr_0^{-1}$, $|\nabla^2\varphi| \leq Cr_0^{-2}$. A computation shows that

$$\begin{cases} -\Delta(u\varphi) + \nabla(p\varphi) + \lambda u\varphi = F\varphi + \operatorname{div}(f\varphi) - f(\nabla\varphi) + p\nabla\varphi - 2\operatorname{div}(u\otimes\nabla\varphi) + u\Delta\varphi, \\ \operatorname{div}(u\varphi) = u\cdot\nabla\varphi. \end{cases} \tag{5.10}$$

We consider two cases: (1) $B(z,2r_0) \subset \Omega$ and (2) $z \in \partial\Omega$.

Case (1). Suppose $B(z,2r_0) \subset \Omega$. Then the Stokes equations in (5.10) hold in $\mathbb{R}^d$. Since $u\varphi \in W^{1,q}(\mathbb{R}^d;\mathbb{C}^d)$ and $p\varphi \in L^q(\mathbb{R}^d;\mathbb{C})$, it follows by Theorem 2.1 that

$$\begin{aligned} &|\lambda|^{1/2}\|\nabla(u\varphi)\|_{L^q(\mathbb{R}^d)} + |\lambda|\|u\varphi\|_{L^q(\mathbb{R}^d)} \\ &\leq C\Big\{\|F\varphi\|_{L^q(\mathbb{R}^d)} + |\lambda|^{1/2}\|f\varphi\|_{L^q(\mathbb{R}^d)} + \|f\nabla\varphi\|_{L^q(\mathbb{R}^d)} + \|p\nabla\varphi\|_{L^q(\mathbb{R}^d)} \\ &\qquad + \|u\Delta\varphi\|_{L^q(\mathbb{R}^d)} + |\lambda|^{1/2}\|u\nabla\varphi\|_{L^q(\mathbb{R}^d)} + |\lambda|\|\operatorname{div}(u\varphi)\|_{\mathring{W}^{-1,q}(\mathbb{R}^d)}\Big\}. \end{aligned}$$

This leads to

$$\begin{aligned} &|\lambda|^{1/2}\|\nabla u\|_{L^q(B(z,r_0))} + |\lambda|\|u\|_{L^q(B(z,r_0))} \\ &\leq Cr_0^{-2}\Big\{\|F\|_{L^q(\Omega)} + (1+|\lambda|^{1/2})\|f\|_{L^q(\Omega)} \\ &\qquad + \|p\|_{L^q(\Omega)} + (1+|\lambda|^{1/2})\|u\|_{L^q(\Omega)} + |\lambda|\|u\|_{W^{-1,q}(\Omega)}\Big\}, \end{aligned} \tag{5.11}$$

where we have used Lemma 5.2 and the fact $\varphi = 1$ in $B(z,r_0)$.

Case (2). Suppose $z \in \partial\Omega$. Let D be given by (5.1)-(5.2). We assume r_0 is sufficiently small so that $\|\nabla'\psi\|_\infty < c_0$, where $c_0 = c_0(d,q,\theta) > 0$ is given by Theorem 4.1. Note that $u\varphi \in W_0^{1,q}(D;\mathbb{C}^d)$, $p\varphi \in L^q(D;\mathbb{C})$, and (5.10) holds in D. It follows by Theorem 4.1 and Remark 4.5 that

$$\begin{aligned} &|\lambda|^{1/2}\|\nabla(u\varphi)\|_{L^q(D)} + |\lambda|\|u\varphi\|_{L^q(D)} \\ &\leq C\Big\{\|F\varphi\|_{L^q(D)} + |\lambda|^{1/2}\|f\varphi\|_{L^q(D)} + \|f\nabla\varphi\|_{L^q(D)} + \|p\nabla\varphi\|_{L^q(D)} + \|u\Delta\varphi\|_{L^q(D)} \\ &\qquad + |\lambda|^{1/2}\|u\nabla\varphi\|_{L^q(D)} + |\lambda|\|\operatorname{div}(u\varphi)\|_{\mathring{W}^{-1,q}(D)}\Big\}, \end{aligned}$$

which yields

$$\begin{aligned} &|\lambda|^{1/2}\|\nabla u\|_{L^q(\Omega\cap B(z,r_0))} + |\lambda|\|u\|_{L^q(\Omega\cap B(z,r_0))} \\ &\leq Cr_0^{-2}\Big\{\|F\|_{L^q(\Omega)} + (1+|\lambda|^{1/2})\|f\|_{L^q(\Omega)} + \|p\|_{L^q(\Omega)} \\ &\qquad + (1+|\lambda|^{1/2})\|u\|_{L^q(\Omega)} + |\lambda|\|u\|_{W_0^{-1,q}(\Omega)}\Big\}, \end{aligned} \tag{5.12}$$

where we have used the estimate in Remark 5.3 and the fact $\varphi = 1$ in $B(z, r_0)$.

We now cover Ω by a finite number of balls $\{B(z_k, r_0)\}$ with the properties that either $B(z_k, 2r_0) \subset \Omega$ or $z_k \in \partial\Omega$. In view of (5.11) and (5.12), by summation, we deduce that

$$
\begin{aligned}
&|\lambda|^{1/2}\|\nabla u\|_{L^q(\Omega)} + |\lambda|\|u\|_{L^q(\Omega)} \\
&\le C\left\{\|F\|_{L^q(\Omega)} + (1+|\lambda|^{1/2})\|f\|_{L^q(\Omega)} + \|p\|_{L^q(\Omega)} + (1+|\lambda|^{1/2})\|u\|_{L^q(\Omega)} + |\lambda|\|u\|_{W_0^{-1,q}(\Omega)}\right\} \\
&\le C\left\{\|F\|_{L^q(\Omega)} + (1+|\lambda|^{1/2})\|f\|_{L^q(\Omega)} + \|\nabla u\|_{L^q(\Omega)} + (1+|\lambda|^{1/2})\|u\|_{L^q(\Omega)} + |\lambda|\|u\|_{W_0^{-1,q}(\Omega)}\right\},
\end{aligned} \tag{5.13}
$$

where we have used Lemma 5.4 and the equation $\nabla p = \Delta u - \lambda u + F + \mathrm{div}(f)$ for the last inequality. The constant C in (5.13) depends only on d, q, θ and Ω. We obtain (5.9) by choosing $\lambda_0 > 1$ so large that $|\lambda| \ge 4C|\lambda|^{1/2}$ for $|\lambda| \ge \lambda_0$. □

Lemma 5.6. *Let $2 \le q < \infty$ and $\lambda \in \Sigma_\theta$. Let $(u, p) \in W_0^{1,q}(\Omega; \mathbb{C}^d) \times L_0^q(\Omega; \mathbb{C})$ be a solution of (5.3) with $F \in L^q(\Omega; \mathbb{C}^d)$, $f \in L^q(\Omega; \mathbb{C}^{d\times d})$ and $g = 0$. Then,*

$$
(|\lambda|+1)^{1/2}\|\nabla u\|_{L^q(\Omega)} + (|\lambda|+1)\|u\|_{L^q(\Omega)} \le C\left\{\|F\|_{L^q(\Omega)} + (|\lambda|+1)^{1/2}\|f\|_{L^q(\Omega)}\right\}, \tag{5.14}
$$

where $C > 0$ depends on d, q, θ, diam(Ω) and the C^1 character of Ω.

Proof. The case $q = 2$ is well known and follows from the energy estimates. For $q > 2$, we first consider the case $|\lambda| \ge \lambda_0$, where $\lambda_0 > 1$ is given by Lemma 5.5. Since Ω is bounded, by Lemma 5.5, the estimate

$$
|\lambda|^{1/2}\|\nabla u\|_{L^s(\Omega)} + |\lambda|\|u\|_{L^s(\Omega)} \le C\left\{\|F\|_{L^s(\Omega)} + |\lambda|^{1/2}\|f\|_{L^s(\Omega)} + |\lambda|\|u\|_{W_0^{-1,s}(\Omega)}\right\} \tag{5.15}
$$

holds for any $s \in [2, q]$. By Sobolev imbedding, $L^t(\Omega; \mathbb{C}^d) \subset W_0^{-1,s}(\Omega; \mathbb{C}^d)$, where $1 < t < d$ and $\frac{1}{t} = \frac{1}{s} + \frac{1}{d}$. In particular, if $2 < s \le \frac{2d}{d-2}$, then $L^2(\Omega; \mathbb{C}^d) \subset W_0^{-1,s}(\Omega; \mathbb{C}^d)$ and

$$
\begin{aligned}
|\lambda|\|u\|_{W_0^{-1,s}(\Omega)} &\le C|\lambda|\|u\|_{L^2(\Omega)} \\
&\le C\left\{\|F\|_{L^2(\Omega)} + |\lambda|^{1/2}\|f\|_{L^2(\Omega)}\right\} \\
&\le C\left\{\|F\|_{L^s(\Omega)} + |\lambda|^{1/2}\|f\|_{L^s(\Omega)}\right\}.
\end{aligned}
$$

This, together with (5.15), gives (5.14) for $2 < q \le \frac{2d}{d-2}$. By a bootstrapping argument, one may show that the estimate (5.14) holds for any $2 < q < \infty$ in a finite number of steps.

We now consider the case $|\lambda| < \lambda_0$. We rewrite the Stokes equations as

$$
\begin{cases}
-\Delta u + \nabla p + (\lambda + 2\lambda_0)u = F + \mathrm{div}(f) + 2\lambda_0 u, \\
\mathrm{div}(u) = 0.
\end{cases} \tag{5.16}
$$

Since $\lambda + 2\lambda_0 \in \Sigma_\theta$ and $|\lambda + 2\lambda_0| > \lambda_0$, it follows from the previous case that

$$
\|\nabla u\|_{L^q(\Omega)} \le C\left\{\|F\|_{L^q(\Omega)} + \|f\|_{L^q(\Omega)} + \|u\|_{L^q(\Omega)}\right\}. \tag{5.17}
$$

Since $W_0^{1,2}(\Omega; \mathbb{C}^d) \subset L^s(\Omega; \mathbb{C}^d)$ for $s = \frac{2d}{d-2}$, we obtain

$$
\begin{aligned}
\|\nabla u\|_{L^q(\Omega)} &\le C\left\{\|F\|_{L^q(\Omega)} + \|f\|_{L^q(\Omega)} + \|\nabla u\|_{L^2(\Omega)}\right\} \\
&\le C\left\{\|F\|_{L^q(\Omega)} + \|f\|_{L^q(\Omega)}\right\}
\end{aligned}
$$

for $2 < q \le \frac{2d}{d-2}$, where we have used the estimate (5.14) for $q = 2$ for the last inequality. As before, a bootstrapping argument, using (5.17), gives

$$\|\nabla u\|_{L^q(\Omega)} \le C \left\{ \|F\|_{L^q(\Omega)} + \|f\|_{L^q(\Omega)} \right\}$$

for $2 < q < \infty$ in a finite number of steps. This, together with a Poincaré inequality, yields (5.14) for the case $|\lambda| < \lambda_0$. $\square$

We are now in a position to give the proof of Theorem 5.1.

Proof of Theorem 5.1. Step 1. Consider the case $2 < q < \infty$ and $g = 0$.

The uniqueness follows from the case $q = 2$. To show the existence and the estimate (5.4), let $F \in L^q(\Omega; \mathbb{C}^d)$ and $f \in L^q(\Omega; \mathbb{C}^{d\times d})$. Note that the constant C in (5.14) depends only on d, q, θ, the diameter of Ω as well as the C^1 character of $\partial\Omega$. As a result, we may construct a sequence of smooth domains $\{\Omega_k\}$ such that $\Omega_k \subset \Omega$ and the estimate (5.14) holds in Ω_k with a constant C independent of k. Let (u^k, p^k) be the unique solution in $W_0^{1,2}(\Omega_k; \mathbb{C}^d) \times L_0^2(\Omega_k; \mathbb{C})$ of the Stokes system (5.3) in Ω_k with $g = 0$, F^k in the place of F and f^k in the place of f, where $F^k \in C_0^\infty(\Omega_k; \mathbb{C}^d)$, $f^k \in C_0^\infty(\Omega_k; \mathbb{C}^{d\times d})$ and $\|F^k - F\|_{L^q(\Omega)} + \|f^k - f\|_{L^q(\Omega)} \to 0$. Since Ω_k and F^k, f^k are smooth, it is well known that $(u^k, p^k) \in W_0^{1,q}(\Omega_k; \mathbb{C}^d) \times L_0^q(\Omega_k; \mathbb{C})$ [13]. We extend (u^k, p^k) to Ω by zero and still denote the extension by (u^k, p^k). It follows by Lemma 5.6 that

$$(|\lambda|+1)^{1/2}\|\nabla u^k\|_{L^q(\Omega)} + (|\lambda|+1)\|u^k\|_{L^q(\Omega)} \le C \left\{ \|F^k\|_{L^q(\Omega_k)} + (|\lambda|+1)^{1/2}\|f^k\|_{L^q(\Omega_k)} \right\}, \quad (5.18)$$

where C depends only on d, q, θ and Ω. Note that by Lemma 5.4, $\{p^k\}$ is bounded in $L^q(\Omega; \mathbb{C})$. By passing to a subsequence, we may assume that $u^k \to u$ weakly in $W_0^{1,q}(\Omega; \mathbb{C}^d)$ and $p^k \to p$ weakly in $L^q(\Omega; \mathbb{C})$. It is not hard to see that (u, p) is a solution of (5.3) in Ω with data (F, f) and $g = 0$. By letting $k \to \infty$ in (5.18), it follows that u satisfies the estimate (5.4).

Step 2. We establish the existence and estimate (5.4) for $1 < q < 2$ and $g = 0$.

For $F, G \in C_0^\infty(\Omega; \mathbb{C}^d)$ and $f, h \in C_0^\infty(\Omega; \mathbb{C}^{d\times d})$, let $(u, p), (v, \phi) \in W_0^{1,2}(\Omega; \mathbb{C}^d) \times L_0^2(\Omega; \mathbb{C})$ be weak solutions of (5.3) in Ω with data $(F, f), (G, h)$, respectively; i.e.,

$$\begin{cases} -\Delta u + \nabla p + \lambda u = F + \operatorname{div}(f) & \text{in } \Omega, \\ \operatorname{div}(u) = 0 & \text{in } \Omega, \end{cases}$$

$$\begin{cases} -\Delta v + \nabla \phi + \lambda v = G + \operatorname{div}(h) & \text{in } \Omega, \\ \operatorname{div}(v) = 0 & \text{in } \Omega. \end{cases}$$

Note that

$$\int_\Omega F \cdot v - \int_\Omega f \cdot \nabla v = \int_\Omega \nabla u \cdot \nabla v + \lambda \int_\Omega u \cdot v = \int_\Omega G \cdot u - \int_\Omega h \cdot \nabla u.$$

It follows that

$$\begin{aligned} &\left| \int_\Omega G \cdot u - \int_\Omega h \cdot \nabla u \right| \le \|F\|_{L^q(\Omega)} \|v\|_{L^{q'}(\Omega)} + \|f\|_{L^q(\Omega)} \|\nabla v\|_{L^{q'}(\Omega)} \\ &\le C(|\lambda|+1)^{-1} \left\{ \|F\|_{L^q(\Omega)} + (|\lambda|+1)^{1/2}\|f\|_{L^q(\Omega)} \right\} \left\{ \|G\|_{L^{q'}(\Omega)} + (|\lambda|+1)^{1/2}\|h\|_{L^{q'}(\Omega)} \right\}, \end{aligned}$$

where we have used the estimate,

$$(|\lambda|+1)^{1/2}\|\nabla v\|_{L^{q'}(\Omega)} + (|\lambda|+1)\|v\|_{L^{q'}(\Omega)} \le C\left\{\|G\|_{L^{q'}(\Omega)} + (|\lambda|+1)^{1/2}\|h\|_{L^{q'}(\Omega)}\right\},$$

obtained in Step 1 for $q' > 2$. By duality this gives

$$(|\lambda|+1)^{1/2}\|\nabla u\|_{L^{q}(\Omega)} + (|\lambda|+1)\|u\|_{L^{q}(\Omega)} \le C\left\{\|F\|_{L^{q}(\Omega)} + (|\lambda|+1)^{1/2}\|f\|_{L^{q}(\Omega)}\right\}.$$

As a result, we have proved the existence and the estimate (5.4) for $F \in C_0^\infty(\Omega;\mathbb{C}^d)$ and $f \in C_0^\infty(\Omega;\mathbb{C}^{d\times d})$. The general case, where $F \in L^q(\Omega;\mathbb{C}^d)$, $f \in L^q(\Omega;\mathbb{C}^{d\times d})$ and $g = 0$, for $1 < q < 2$, follows readily by a density argument.

Step 3. We establish the uniqueness.

The uniqueness for $q > 2$ follows from the uniqueness for $q = 2$. To handle the case $1 < q < 2$, let $u \in W_0^{1,q}(\Omega;\mathbb{C}^d)$ be a solution of (5.3) in Ω with $F = 0$, $f = 0$ and $g = 0$. Since $\overline{u}|u|^{q-2} \in L^{q'}(\Omega;\mathbb{C}^d)$, by Step 1, there exists $(v,\phi) \in W_0^{1,q'}(\Omega;\mathbb{C}^d) \times L_0^{q'}(\Omega;\mathbb{C})$ such that

$$\begin{cases} -\Delta v + \nabla \phi + \lambda v = |u|^{q-2}\overline{u} & \text{in } \Omega,\\ \operatorname{div}(v) = 0 & \text{in } \Omega. \end{cases}$$

As in the case $\Omega = \mathbb{R}^d$, this leads to $\int_\Omega |u|^q = 0$. Hence, $u = 0$ in Ω.

Step 4. The case $g \ne 0$.

Let $g \in L_0^q(\Omega;\mathbb{C})$. Since Ω is a bounded Lipschitz domain, there exists $w \in W_0^{1,q}(\Omega;\mathbb{C}^d)$ such that

$$\operatorname{div}(w) = g \quad \text{in } \Omega \quad \text{and} \quad \|w\|_{L^q(\Omega)} + \|\nabla w\|_{L^q(\Omega)} \le C\|g\|_{L^q(\Omega)}. \tag{5.19}$$

By considering $\widetilde{u} = u - w$, we reduce the problem to the case $g = 0$. Indeed, let $\widetilde{u}$ be a solution of

$$\begin{cases} -\Delta \widetilde{u} + \nabla p + \lambda \widetilde{u} = F + \operatorname{div}(f + \nabla w) - \lambda w,\\ \operatorname{div}(\widetilde{u}) = 0 \end{cases}$$

in Ω. Then $u = \widetilde{u} + w$ is a solution of (5.3). □

Remark 5.7. Let $1 < q < \infty$ and Ω be a bounded C^1 domain in $\mathbb{R}^d$. By letting $\lambda \in \mathbb{R}_+$ and $\lambda \to 0$ in Theorem 5.1, one may show that for any $F \in L^q(\Omega;\mathbb{C}^d)$, $f \in L^q(\Omega;\mathbb{C}^{d\times d})$ and $g \in L_0^q(\Omega;\mathbb{C})$, there exists a unique $(u,p) \in W_0^{1,q}(\Omega;\mathbb{C}^d) \times L_0^q(\Omega;\mathbb{C})$ such that

$$\begin{cases} -\Delta u + \nabla p = F + \operatorname{div}(f),\\ \operatorname{div}(u) = g \end{cases} \tag{5.20}$$

in Ω. Moreover, the solution (u,p) satisfies the estimate

$$\|\nabla u\|_{L^q(\Omega)} + \|p\|_{L^q(\Omega)} \le C\left\{\|F\|_{L^q(\Omega)} + \|f\|_{L^q(\Omega)} + \|g\|_{L^q(\Omega)}\right\}, \tag{5.21}$$

where C depends on d, q and Ω. The $W^{1,q}$ estimate (5.21) is known for C^1 domains [7]. If Ω is a bounded Lipschitz domain, the estimate (5.21) holds for $(3/2) - \varepsilon < q < 3 + \varepsilon$ if $d = 3$, and for $(4/3) - \varepsilon < q < 4 + \varepsilon$ if $d = 2$, where ε depends on Ω [5]. If $d \ge 4$, some partial results are known [14]. We point out that the results in [5, 7, 14] rely on the estimates for a non-homogeneous Dirichlet problem, which is solved by using the methods of layer potentials. The approach used in this paper, which is based on a perturbation argument, seems to be more accessible. However, it does not work for a general Lipschitz domain.

We end this section with a localized $W^{1,q}$ estimate that will be used in the next section.

Theorem 5.8. *Let Ω be a bounded C^1 domain and $2 < q < \infty$. Let $B = B(x_0, r_0)$, where $x_0 \in \partial\Omega$ and $r_0 > 0$ is small. Suppose that $u \in W^{1,2}(2B \cap \Omega; \mathbb{C}^d)$, $p \in L^2(2B \cap \Omega; \mathbb{C})$, and*

$$\begin{cases} -\Delta u + \nabla p = F + \operatorname{div}(f) & \text{in } 2B \cap \Omega, \\ \operatorname{div}(u) = g & \text{in } 2B \cap \Omega, \\ u = 0 & \text{on } 2B \cap \partial\Omega, \end{cases} \tag{5.22}$$

where $F \in L^q(2B \cap \Omega; \mathbb{C}^d)$, $f \in L^q(2B \cap \Omega; \mathbb{C}^{d \times d})$ and $g \in L^q(2B \cap \Omega; \mathbb{C})$. Then $u \in W^{1,q}(B \cap \Omega; \mathbb{C}^d)$, $p \in L^q(B \cap \Omega; \mathbb{C})$, and

$$\begin{aligned} &\|\nabla u\|_{L^q(B\cap\Omega)} + \|p - \fint_{B\cap\Omega} p\|_{L^q(B\cap\Omega)} \\ &\qquad \le C \left\{ \|F\|_{L^q(2B\cap\Omega)} + \|f\|_{L^q(2B\cap\Omega)} + \|g\|_{L^q(2B\cap\Omega)} + \|u\|_{L^2(2B\cap\Omega)} \right\}, \end{aligned} \tag{5.23}$$

where C depends on d, q, r_0 and Ω.

Proof. Theorem 5.8 follows from the estimate (5.21) by a localization argument. However, some cares are needed to handle the error term $p(\nabla\varphi)$, introduced by the pressure p, where φ is a cut-off function.

Consider the Stokes equations (5.20) with $F = 0$ and $g = 0$; i.e.,

$$-\Delta u + \nabla p = \operatorname{div}(f) \quad \text{and} \quad \operatorname{div}(u) = 0$$

in Ω. It follows from (5.21) that $\|\nabla u\|_{L^q(\Omega)} \le C\|f\|_{L^q(\Omega)}$. By Sobolev imbedding, we obtain

$$\|u\|_{L^s(\Omega)} \le C\|f\|_{L^q(\Omega)},$$

where $\frac{1}{s} = \frac{1}{q} - \frac{1}{d}$ and $1 < q < d$. By a duality argument, as in Step 2 in the proof of Theorem 5.1, this implies that the solution of

$$-\Delta u + \nabla p = F \quad \text{and} \quad \operatorname{div}(u) = 0$$

in Ω satisfies the estimate,

$$\|\nabla u\|_{L^q(\Omega)} + \|p\|_{L^q(\Omega)} \le C\|F\|_{L^s(\Omega)},$$

where $\frac{1}{s} = \frac{1}{q} + \frac{1}{d}$ and $1 < s < d$. This observation allows us to improve the estimate (5.21) to

$$\|\nabla u\|_{L^q(\Omega)} + \|p\|_{L^q(\Omega)} \le C \left\{ \|F\|_{L^{s_*}(\Omega)} + \|f\|_{L^q(\Omega)} + \|g\|_{L^q(\Omega)} \right\}, \tag{5.24}$$

where $s_* = \max\{2, s\} < q$ and $\frac{1}{s} = \frac{1}{q} + \frac{1}{d}$. Using (5.24), a standard localization procedure, together with a bootstrapping argument, yields (5.23). We omit the details. □

6 An exterior C^1 domain and the proof of Theorem 1.2

In this section we consider the case of an exterior C^1 domain Ω; i.e., Ω is a connected open subset of $\mathbb{R}^d$ with compact complement and C^1 boundary. Let $F \in L^2(\Omega; \mathbb{C}^d)$, $f \in L^2(\Omega; \mathbb{C}^{d\times d})$ and $\lambda \in \Sigma_\theta$. By the Lax-Milgram Theorem, there exists a unique $u \in W_0^{1,2}(\Omega; \mathbb{C}^d)$ such that

$$\begin{cases} -\Delta u + \nabla p + \lambda u = F + \operatorname{div}(f), \\ \qquad\qquad \operatorname{div}(u) = 0 \end{cases} \tag{6.1}$$

holds in Ω for some $p \in L^2_{\text{loc}}(\overline{\Omega}; \mathbb{C})$ in the sense of distributions. Moreover, the solution satisfies

$$|\lambda|^{1/2} \|\nabla u\|_{L^2(\Omega)} + |\lambda| \|u\|_{L^2(\Omega)} \le C \left\{ \|F\|_{L^2(\Omega)} + |\lambda|^{1/2} \|f\|_{L^2(\Omega)} \right\}, \tag{6.2}$$

where C depends only on d and θ. We will call u the energy solution of (6.1). Note that, if $F \in L^q(\Omega; \mathbb{C}^d) \cap L^2(\Omega; \mathbb{C}^d)$ and $f \in L^2(\Omega; \mathbb{C}^{d\times d}) \cap L^q(\Omega; \mathbb{C}^{d\times d})$ for some $q > 2$, then $(u, p) \in W^{1,q}(\Omega \cap B; \mathbb{C}^d) \times L^q(\Omega \cap B; \mathbb{C})$ for any ball B in $\mathbb{R}^d$. This follows from the regularity theory for the Stokes equations (5.20) in bounded C^1 domains. See Theorem 5.8.

Let

$$\Sigma_{\theta,\delta} = \{ z \in \mathbb{C} : |z| > \delta \text{ and } |\arg(z)| < \pi - \theta \}, \tag{6.3}$$

where $\theta \in (0, \pi/2)$ and $\delta \in (0, 1)$. The goal of this section is to prove the following.

Theorem 6.1. *Let Ω be an exterior C^1 domain in $\mathbb{R}^d$, $d \ge 2$. Let $1 < q < \infty$ and $\lambda \in \Sigma_{\theta,\delta}$. For any $F \in L^q(\Omega; \mathbb{C}^d)$ and $f \in L^q(\Omega; \mathbb{C}^{d\times d})$, there exists a unique $u \in W_0^{1,q}(\Omega; \mathbb{C}^d)$ such that (6.1) holds in Ω for some $p \in L^1_{\text{loc}}(\Omega; \mathbb{C})$. Moreover, the solution satisfies the estimate,*

$$|\lambda|^{1/2} \|\nabla u\|_{L^q(\Omega)} + |\lambda| \|u\|_{L^q(\Omega)} \le C \left\{ \|F\|_{L^q(\Omega)} + |\lambda|^{1/2} \|f\|_{L^q(\Omega)} \right\}, \tag{6.4}$$

where C depends on d, q, θ, δ and Ω.

Fix a large ball $B_0 = B(0, 2R_0)$ such that $\Omega \setminus B(0, R_0) = \mathbb{R}^d \setminus B(0, R_0)$ and $B_0 \cap \Omega$ is a bounded C^1 domain.

Lemma 6.2. *Let $1 < q < \infty$ and $\lambda \in \Sigma_\theta$. Let $u \in W_0^{1,2}(\Omega; \mathbb{C}^d)$ be an energy solution of (6.1) with $F \in L^q(\Omega; \mathbb{C}^d) \cap L^2(\Omega; \mathbb{C}^d)$ and $f \in L^2(\Omega; \mathbb{C}^{d\times d}) \cap L^q(\Omega; \mathbb{C}^{d\times d})$. Then $u \in W_0^{1,q}(\Omega; \mathbb{C}^d)$. Moreover, if $|\lambda| \ge \lambda_0$,*

$$|\lambda|^{1/2} \|\nabla u\|_{L^q(\Omega)} + |\lambda| \|u\|_{L^q(\Omega)} \le C \left\{ \|F\|_{L^q(\Omega)} + |\lambda|^{1/2} \|f\|_{L^q(\Omega)} + |\lambda| \|u\|_{W_0^{-1,q}(\Omega \cap 2B_0)} \right\}, \tag{6.5}$$

where $\lambda_0 > 1$ and C depend on d, q, θ and Ω.

Proof. The proof, which uses a localization argument, is similar to that of Lemma 5.5 for the bounded domain. However, we need to add another case to handle the neighborhood of ∞. Choose $\varphi \in C^\infty(\mathbb{R}^d; \mathbb{R})$ such that $\varphi = 1$ in $\mathbb{R}^d \setminus B(0, 2R_0)$ and $\varphi = 0$ in $B(0, R_0)$. Then the Stokes equations in (5.10) hold in $\mathbb{R}^d$. Since $p \in L^q(\Omega \cap 2B_0)$, where $B_0 = B(0, 2R_0)$, it follows by Theorem 2.1 and Remark 2.4 that $u\varphi \in W_0^{1,q}(\mathbb{R}^d; \mathbb{C}^d)$ and

$$\begin{aligned} &|\lambda|^{1/2} \|\nabla(u\varphi)\|_{L^q(\mathbb{R}^d)} + |\lambda| \|u\varphi\|_{L^q(\mathbb{R}^d)} \\ &\le C \Big\{ \|F\varphi\|_{L^q(\mathbb{R}^d)} + |\lambda|^{1/2} \|f\varphi\|_{L^q(\mathbb{R}^d)} + \|f\nabla\varphi\|_{L^q(\mathbb{R}^d)} + \|p\nabla\varphi\|_{L^q(\mathbb{R}^d)} \\ &\qquad + \|u\Delta\varphi\|_{L^q(\mathbb{R}^d)} + |\lambda|^{1/2} \|u\nabla\varphi\|_{L^q(\mathbb{R}^d)} + |\lambda| \|\operatorname{div}(u\varphi)\|_{\mathring{W}^{-1,q}(\mathbb{R}^d)} \Big\}. \end{aligned}$$

Note that the same argument as in the proof of Lemma 5.2 also yields

$$\|\mathrm{div}(u\varphi)\|_{\mathring{W}^{-1,q}(\mathbb{R}^d)} \le C\|u\|_{W_0^{-1,q}(\Omega\cap B_0)}.$$

Hence,

$$\begin{aligned}
&|\lambda|^{1/2}\|\nabla u\|_{L^q(\Omega\setminus B_0)} + |\lambda|\|u\|_{L^q(\Omega\setminus B_0)}\\
&\le C\Big\{\|F\|_{L^q(\Omega)} + |\lambda|^{1/2}\|f\|_{L^q(\Omega)} + \|f\|_{L^q(\Omega\cap B_0)} + \|p\|_{L^q(\Omega\cap B_0)}\\
&\qquad + (1+|\lambda|^{1/2})\|u\|_{L^q(\Omega\cap B_0)} + |\lambda|\|u\|_{W_0^{-1,q}(\Omega\cap B_0)}\Big\}.
\end{aligned} \tag{6.6}$$

Since $\Omega\cap B_0$ is a bounded C^1 domain, it follows from the proof of Lemma 5.5 that

$$\begin{aligned}
&|\lambda|^{1/2}\|\nabla u\|_{L^q(\Omega\cap B_0)} + |\lambda|\|u\|_{L^q(\Omega\cap B_0)}\\
&\le C\Big\{\|F\|_{L^q(\Omega)} + |\lambda|^{1/2}\|f\|_{L^q(\Omega)} + \|f\|_{L^q(\Omega\cap 2B_0)} + \|p\|_{L^q(\Omega\cap 2B_0)}\\
&\qquad + (1+|\lambda|^{1/2})\|u\|_{L^q(\Omega\cap 2B_0)} + |\lambda|\|u\|_{W_0^{-1,q}(\Omega\cap 2B_0)}\Big\}.
\end{aligned}$$

This, together with (6.6), gives

$$\begin{aligned}
&|\lambda|^{1/2}\|\nabla u\|_{L^q(\Omega)} + |\lambda|\|u\|_{L^q(\Omega)}\\
&\le C\Big\{\|F\|_{L^q(\Omega)} + (|\lambda|^{1/2}+1)\|f\|_{L^q(\Omega)} + \|p\|_{L^q(\Omega\cap 2B_0)}\\
&\qquad + (|\lambda|^{1/2}+1)\|u\|_{L^q(\Omega\cap 2B_0)} + |\lambda|\|u\|_{W_0^{-1,q}(\Omega\cap 2B_0)}\Big\}\\
&\le C\Big\{\|F\|_{L^q(\Omega)} + (|\lambda|^{1/2}+1)\|f\|_{L^q(\Omega)} + \|\nabla u\|_{L^q(\Omega\cap 2B_0)}\\
&\qquad + (|\lambda|^{1/2}+1)\|u\|_{L^q(\Omega\cap 2B_0)} + |\lambda|\|u\|_{W_0^{-1,q}(\Omega\cap 2B_0)}\Big\},
\end{aligned} \tag{6.7}$$

where we have assumed $\int_{\Omega\cap 2B_0} p = 0$ and used Lemma 5.4 for the last inequality. As a result, we have proved that $u\in W_0^{1,q}(\Omega;\mathbb{C}^d)$. Moreover, we obtain (6.5) if $|\lambda|\ge\lambda_0$ and $\lambda_0>1$ is sufficiently large. □

Remark 6.3. Suppose that $\lambda\in\Sigma_\theta$ and $|\lambda|\le\lambda_0$. Let $2<q<\infty$. It follows from (6.7) and Theorem 5.8 as well as the interior estimates for the Stokes equations with $\lambda=0$ that

$$|\lambda|^{1/2}\|\nabla u\|_{L^q(\Omega)} + |\lambda|\|u\|_{L^q(\Omega)} \le C\left\{\|F\|_{L^q(\Omega)} + \|f\|_{L^q(\Omega)} + \|u\|_{L^q(\Omega\cap 3B_0)}\right\}, \tag{6.8}$$

where C depends on d, q, θ and Ω.

The next lemma gives the uniqueness for $q>2$.

Lemma 6.4. *Let $2\le q<\infty$ and $\lambda\in\Sigma_\theta$. Let $u\in W_0^{1,q}(\Omega;\mathbb{C}^d)$ be a solution of (6.1) in Ω with $F=0$ and $f=0$. Then $u=0$ in Ω.*

Proof. The case $q=2$ is well known. To handle the case $q>2$, we choose $\varphi\in C^\infty(\mathbb{R}^d;\mathbb{R})$ such that $\varphi=1$ in $\Omega\setminus B(0,2R_0)$ and $\varphi=0$ in $B(0,R_0)$, as in the proof of Lemma 6.2. Then the Stokes equations in (5.10) hold in $\mathbb{R}^d$ with $F=0$ and $f=0$. Since the right-hand sides of (5.10) have compact support and thus are in $L^2(\mathbb{R}^d;\mathbb{C}^d)$, it follows from Remark 2.4 that $u\varphi\in W^{1,2}(\mathbb{R}^d;\mathbb{C}^d)$. As a result, $u\in W_0^{1,2}(\Omega;\mathbb{C}^d)$. By the uniqueness for $q=2$, we conclude that $u=0$ in Ω. □

Lemma 6.5. *Let $2 \le q < \infty$ and $\lambda \in \Sigma_{\theta,\delta}$. Let $u \in W_0^{1,2}(\Omega;\mathbb{C}^d)$ be an energy solution of (6.1) with $F \in L^q(\Omega;\mathbb{C}^d) \cap L^2(\Omega;\mathbb{C}^d)$ and $f \in L^q(\Omega;\mathbb{C}^{d\times d}) \cap L^2(\Omega;\mathbb{C}^{d\times d})$. Then*

$$|\lambda|^{1/2}\|\nabla u\|_{L^q(\Omega)} + |\lambda|\|u\|_{L^q(\Omega)} \le C\left\{\|F\|_{L^q(\Omega)} + |\lambda|^{1/2}\|f\|_{L^q(\Omega)}\right\}, \tag{6.9}$$

where C depends on d, q, θ, δ and Ω.

Proof. The case $q = 2$ is the well known energy estimate. To handle the case $q > 2$, we argue by contradiction. Note that by Lemma 6.2, $u \in W_0^{1,q}(\Omega;\mathbb{C}^d)$. Suppose the estimate (6.9) is not true. Then there exist sequences $\{u^\ell\} \subset W_0^{1,q}(\Omega;\mathbb{C}^d)$, $\{F^\ell\} \subset L^q(\Omega;\mathbb{C}^d) \cap L^2(\Omega;\mathbb{C}^d)$, $\{f^\ell\} \subset L^q(\Omega;\mathbb{C}^{d\times d}) \cap L^2(\Omega;\mathbb{C}^{d\times d})$ and $\{\lambda^\ell\} \subset \Sigma_{\theta,\delta}$ such that

$$\begin{cases} -\Delta u^\ell + \nabla p^\ell + \lambda^\ell u^\ell = F^\ell + \operatorname{div}(f^\ell) & \text{in } \Omega, \\ \operatorname{div}(u^\ell) = 0 & \text{in } \Omega, \end{cases} \tag{6.10}$$

for some $p^\ell \in L^2_{\text{loc}}(\overline{\Omega};\mathbb{C})$,

$$|\lambda^\ell|^{1/2}\|\nabla u^\ell\|_{L^q(\Omega)} + |\lambda^\ell|\|u^\ell\|_{L^q(\Omega)} = 1, \tag{6.11}$$

and

$$\|F^\ell\|_{L^q(\Omega)} + |\lambda^\ell|^{1/2}\|f^\ell\|_{L^q(\Omega)} \to 0 \quad \text{as } \ell \to \infty. \tag{6.12}$$

Since $|\lambda^\ell| \ge \delta$, it follows from (6.11) that $\|u^\ell\|_{W_0^{1,q}(\Omega)} \le C$. By passing to a subsequence, we may assume that $u^\ell \to u$ weakly in $W_0^{1,q}(\Omega;\mathbb{C}^d)$. We may also assume that either $|\lambda^\ell| \to \infty$ or $\lambda^\ell \to \lambda \in \mathbb{C}$.

We consider three cases: (1) $\lambda^\ell \to \lambda \in \mathbb{C}$ and $|\lambda| > 2\lambda_0$, where $\lambda_0 > 1$ is given by Lemma 6.2; (2) $\lambda^\ell \to \lambda$ and $|\lambda| \le 2\lambda_0$; and (3) $|\lambda^\ell| \to \infty$.

Case (1). Suppose $\lambda^\ell \to \lambda \in \mathbb{C}$ and $|\lambda| > 2\lambda_0$. It follows that $u \in W_0^{1,q}(\Omega;\mathbb{C}^d)$ is a solution of (6.1) in Ω with $F = 0$ and $f = 0$. By Lemma 6.4, we obtain $u = 0$ in Ω. Thus, $u^\ell \to 0$ weakly in $W_0^{1,q}(\Omega;\mathbb{C}^d)$. This implies that $u^\ell \to 0$ strongly in $W_0^{-1,q}(\Omega \cap 2B_0;\mathbb{C}^d)$. However, by (6.5) and (6.11)-(6.12), we have

$$\begin{aligned} 1 =& |\lambda^\ell|^{1/2}\|\nabla u^\ell\|_{L^q(\Omega)} + |\lambda^\ell|\|u^\ell\|_{L^q(\Omega)} \\ &\le C\left\{\|F^\ell\|_{L^q(\Omega)} + |\lambda^\ell|^{1/2}\|f^\ell\|_{L^q(\Omega)} + |\lambda^\ell|\|u^\ell\|_{W_0^{-1,q}(\Omega\cap 2B_0)}\right\} \to 0, \end{aligned} \tag{6.13}$$

which yields a contradiction.

Case (2). Suppose $\lambda^\ell \to \lambda$ and $|\lambda| \le 2\lambda_0$. As in case (1), $u^\ell \to 0$ weakly in $W_0^{1,q}(\Omega;\mathbb{C}^d)$. It follows from (6.8) that

$$\begin{aligned} 1 = |\lambda^\ell|^{1/2}\|\nabla u^\ell\|_{L^q(\Omega)} + |\lambda^\ell|\|u^\ell\|_{L^q(\Omega)} \\ \le C\left\{\|F^\ell\|_{L^q(\Omega)} + \|f^\ell\|_{L^q(\Omega)} + \|u^\ell\|_{L^q(\Omega\cap 3B_0)}\right\}. \end{aligned}$$

This gives us a contradiction, as $u^\ell \to 0$ strongly in $L^q(\Omega \cap 3B_0;\mathbb{C}^d)$.

Case (3). Suppose that $|\lambda^\ell| \to \infty$. In view of (6.11), we have $u^\ell \to 0$ strongly in $L^q(\Omega;\mathbb{C}^d)$. By passing to a subsequence, we assume that $\lambda^\ell u^\ell \to v$ weakly in $L^q(\Omega;\mathbb{C}^d)$. Note that if $w \in C_0^\infty(\Omega;\mathbb{C}^d)$ and $\operatorname{div}(w) = 0$ in Ω, then

$$-\int_\Omega u^\ell \cdot \Delta w + \int_\Omega \lambda^\ell u^\ell \cdot w = \int_\Omega F^\ell \cdot w - \int_\Omega f^\ell \cdot \nabla w.$$

By letting $\ell \to \infty$, we obtain $\int_\Omega v \cdot w = 0$. This implies that $v = \nabla \phi$ for some $\phi \in \mathring{W}^{1,q}(\Omega; \mathbb{C})$. Since $\lambda^\ell u^\ell \in W_0^{1,q}(\Omega; \mathbb{C}^d)$ and $\mathrm{div}(\lambda^\ell u^\ell) = 0$ in Ω, we also have $\int_\Omega v \cdot \nabla \varphi = 0$ for any $\varphi \in C_0^\infty(\mathbb{R}^d; \mathbb{C})$. It follows that $\phi \in \mathring{W}^{1,q}(\Omega; \mathbb{C})$ is a solution of the Neumann problem: $\Delta \phi = 0$ in Ω and $\frac{\partial \phi}{\partial n} = 0$ on $\partial\Omega$. Since $\nabla \phi \in L^q(\Omega; \mathbb{C}^d)$, we conclude that $v = \nabla \phi = 0$ in Ω. See Lemma 7.1 in Appendix. Thus, $\lambda^\ell u^\ell \to 0$ weakly in $L^q(\Omega; \mathbb{C}^d)$ and thus strongly in $W^{-1,q}(\Omega \cap 2B_0; \mathbb{C}^d)$. Consequently, (6.13) holds and gives us a contradiction. This completes the proof. □

Proof of Theorem 6.1. Step 1. Assume $2 \le q < \infty$. The uniqueness is given by Lemma 6.4. Since $L^2(\Omega; \mathbb{C}^d) \cap L^q(\Omega; \mathbb{C}^d)$ is dense in $L^q(\Omega; \mathbb{C}^d)$, the existence as well as the estimate (6.4) follows from Lemma 6.5 by a standard density argument.

Step 2. Assume $1 < q < 2$. As in the cases of $\mathbb{R}^d$ and $\mathbb{R}^d_+$, the uniqueness follows from the existence for $q' > 2$, proved in Step 1. By a duality argument, similar to that in the proof of Theorem 5.1, one may show that if $F \in C_0^\infty(\Omega; \mathbb{C}^d)$ and $f \in C_0^\infty(\Omega; \mathbb{C}^{d\times d})$, the energy solutions of (6.1) satisfy the estimate (6.4). As before, the existence and the estimate (6.4) for $F \in L^q(\Omega; \mathbb{C}^d)$ and $f \in L^q(\Omega; \mathbb{C}^{d\times d})$ follow by a density argument. □

Proof of Theorem 1.2. The estimate (1.4) with C depending on δ is contained in Theorem 6.1. To establish the estimate (1.5) with C independent of δ for $d \ge 3$, we first consider the case $q < (d/2)$ and argue by contradiction. Suppose (1.5) is not true. Then there exist sequences $\{F^\ell\} \subset L^q(\Omega; \mathbb{C}^d)$, $\{u^\ell\} \subset W_0^{1,q}(\Omega; \mathbb{C}^d)$, $\{\lambda^\ell\} \subset \Sigma_\theta$ such that $\lambda^\ell \to 0$,

$$\begin{cases} -\Delta u^\ell + \nabla p^\ell + \lambda^\ell u^\ell = F^\ell, \\ \mathrm{div}(u^\ell) = 0, \end{cases} \tag{6.14}$$

in Ω,

$$|\lambda^\ell| \|u^\ell\|_{L^q(\Omega)} = 1, \tag{6.15}$$

and $\|F^\ell\|_{L^q(\Omega)} \to 0$ as $\ell \to \infty$. By Theorem 7.3 in the Appendix,

$$\|\nabla u^\ell\|_{L^s(\Omega)} \le C \left\{ \|F^\ell\|_{L^q(\Omega)} + \|\lambda^\ell u^\ell\|_{L^q(\Omega)} \right\},$$

where $\frac{1}{s} = \frac{1}{q} - \frac{1}{d}$. As a result, $\{\nabla u^\ell\}$ is bounded in $L^s(\Omega; \mathbb{C}^{d\times d})$ and by Sobolev imbedding, $\{u^\ell\}$ is bounded in $L^{s_*}(\Omega; \mathbb{C}^d)$, where $\frac{1}{s_*} = \frac{1}{s} - \frac{1}{d} = \frac{1}{q} - \frac{2}{d}$ and we have used the fact $u^\ell \in L^q(\Omega; \mathbb{C}^d)$. By passing to a subsequence, we may assume that $\lambda^\ell u^\ell \to v$ weakly in $L^q(\Omega; \mathbb{C}^d)$, $u^\ell \to u$ weakly in $L^{s_*}(\Omega; \mathbb{C}^d)$, and $\nabla u^\ell \to \nabla u$ weakly in $L^s(\Omega; \mathbb{C}^{d\times d})$. Since $\lambda^\ell \to 0$, we obtain $v = 0$. It then follows from (6.14) that $-\Delta u + \nabla p = 0$, $\mathrm{div}(u) = 0$ in Ω and $u = 0$ on $\partial\Omega$. Since $u \in L^{s_*}(\Omega; \mathbb{C}^d)$, $\nabla u \in L^s(\Omega; \mathbb{C}^{d\times d})$ and $s < d$, we deduce from Lemma 7.2 that $u = 0$ in Ω. This implies that $u^\ell \to 0$ strongly in $L^q(\Omega \cap B; \mathbb{C}^d)$ for any ball B. However, by (6.8) and (6.15), we have

$$1 = |\lambda^\ell| \|u^\ell\|_{L^q(\Omega)} \le C \left\{ \|F^\ell\|_{L^q(\Omega)} + \|u^\ell\|_{L^q(\Omega \cap 3B_0)} \right\},$$

which yields a contradiction.

Finally, we note that by duality, the estimate (1.5) holds for $\frac{d}{d-2} < q < \infty$. This gives the estimate for $1 < q < \infty$ in the case $d \ge 4$. If $d = 3$, the range $(3/2) \le q \le 3$ follows by using the Riesz-Thorin Interpolation Theorem. □

7 Appendix

In this Appendix we prove several uniqueness and regularity results in exterior C^1 domains, which are used in the previous sections. In the case of exterior domains with C^2 boundaries, the proofs may be found in [13].

Lemma 7.1. *Let Ω be an exterior C^1 domain in $\mathbb{R}^d$, $d \geq 2$ and $1 < q < \infty$. Suppose that $\phi \in \mathring{W}^{1,q}(\Omega;\mathbb{C})$, $\Delta\phi = 0$ in Ω and $n \cdot \nabla\phi = 0$ on $\partial\Omega$. Then ϕ is constant in Ω.*

Proof. By using the mean value property for harmonic functions and $|\nabla\phi| \in L^q(\Omega)$, we obtain $\nabla\phi(x) = o(1)$ as $|x| \to \infty$. By the expansion theorem at ∞ for harmonic functions [1], we deduce that $\nabla\phi(x) = O(|x|^{-1})$ for $d = 2$. In the case $d \geq 3$, we obtain $\nabla\phi(x) = O(|x|^{2-d})$. It follows that $\phi(x) = O(\log|x|)$ for $d = 3$ and $\phi(x) = O(1)$ for $d \geq 4$. Since ϕ is harmonic, by the expansion theorem, this implies that $\phi(x) = L + O(|x|^{2-d})$ for some $L \in \mathbb{C}$ and that $\nabla\phi(x) = O(|x|^{1-d})$ as $|x| \to \infty$ for $d \geq 3$. As a result, we have proved that $\nabla\phi(x) = O(|x|^{1-d})$ as $|x| \to \infty$ for $d \geq 2$.

Next, note that since $\partial\Omega$ is C^1 and $n \cdot \nabla\phi = 0$ on $\partial\Omega$, we have $\nabla\phi \in L^2(\Omega \cap B(0,R);\mathbb{C}^d)$ for any $R > 1$. Moreover, for R sufficiently large,

$$\begin{aligned}\int_{\Omega\cap B(0,R)} |\nabla\phi|^2 &= \int_{\partial B(0,R)} \frac{\partial\phi}{\partial n}(\phi - \beta)\\ &\leq \|\nabla\phi\|_{L^2(\partial B(0,R))}\|\phi - \beta\|_{L^2(\partial B(0,R))} \leq CR\|\nabla\phi\|^2_{L^2(\partial B(0,R))},\end{aligned}$$

where $\beta = \fint_{\partial B(0,R)} \phi$ and we have used a Poincaré inequality on $\partial B(0,R)$. By letting $R \to \infty$ and using $\nabla\phi(x) = O(|x|^{1-d})$ as $|x| \to \infty$ for $d \geq 2$, we see that $\|\nabla\phi\|_{L^2(\Omega)} = 0$ if $d \geq 3$ and $\|\nabla\phi\|_{L^2(\Omega)} < \infty$ if $d = 2$. As a result, $\nabla\phi = 0$ and ϕ is constant in Ω for $d \geq 3$. Finally, to handle the case $d = 2$, we use the Caccioppoli inequality,

$$\begin{aligned}\int_{\Omega\cap B(0,R)} |\nabla\phi|^2 &\leq \frac{C}{R^2}\int_{B(0,2R)\setminus B(0,R)} |\phi - \alpha|^2\\ &\leq C_0 \int_{B(0,2R)\setminus B(0,R)} |\nabla\phi|^2,\end{aligned} \tag{7.1}$$

for R large, where $\alpha = \fint_{B(0,2R)\setminus B(0,R)} \phi$ and we have used a Poincaré inequality. It follows that

$$\int_{\Omega\cap B(0,R)} |\nabla\phi|^2 \leq \frac{C_0}{C_0 + 1}\int_{\Omega\cap B(0,2R)} |\nabla\phi|^2.$$

By letting $R \to \infty$, we obtain $\|\nabla\phi\|_{L^2(\Omega)} \leq c_0\|\nabla\phi\|_{L^2(\Omega)}$ for some $c_0 < 1$. This implies that $\|\nabla\phi\|_{L^2(\Omega)} = 0$ if $\|\nabla\phi\|_{L^2(\Omega)} < \infty$. Consequently, we conclude that $\nabla\phi = 0$ and ϕ is constant in Ω for $d \geq 2$. □

Lemma 7.2. *Let Ω be an exterior C^1 domain in $\mathbb{R}^d$, $d \geq 2$. Let $1 < q < d$ and $\frac{1}{q_*} = \frac{1}{q} - \frac{1}{d}$. Suppose that $u \in L^{q_*}(\Omega;\mathbb{C}^d)$, $\nabla u \in L^q(\Omega;\mathbb{C}^{d\times d})$, $u = 0$ on $\partial\Omega$, and*

$$-\Delta u + \nabla p = 0 \quad \textit{and} \quad \operatorname{div}(u) = 0 \tag{7.2}$$

hold in Ω in the sense of distributions. Then $u = 0$ in Ω.

Proof. The proof is similar to that of Lemma 7.1 for the case $d \geq 3$. By the interior estimates for the Stokes equations,

$$|x||\nabla^2 u(x)| + |\nabla u(x)| \leq C \left(\fint_{B(x,R/4)} |\nabla u|^q \right)^{1/q}, \qquad (7.3)$$

where $R = |x|$ is sufficiently large. It follows from $|\nabla u| \in L^q(\Omega)$ that $\nabla u(x) = o(|x|^{-\gamma})$ as $|x| \to \infty$, where $\gamma = (d/q)$. Since $\gamma > 1$, this implies that $\lim_{|x|\to\infty} u(x)$ exists. Using $u \in L^{q_*}(\Omega; \mathbb{C}^d)$, we deduce that $u(x) = o(1)$ as $|x| \to \infty$. Also note that by the interior estimates, $\nabla^2 u(x) = o(|x|^{-\gamma-1})$ as $|x| \to \infty$. Thus, $\nabla p(x) = o(|x|^{-\gamma-1})$. It follows that $\lim_{|x|\to\infty} p(x)$ exists. By subtracting a constant, we may assume that $\lim_{|x|\to\infty} p(x) = 0$. As a result, we obtain $p(x) = o(|x|^{-\gamma})$ as $|x| \to \infty$.

Next, assume $d \geq 3$. We use the Green representation formula for the Stokes equations in the domain $D_R = \{x : R_0 < |x| < R\}$ to write $(u(x), p(x))$ as a sum of layer potentials on $\partial D_R = \partial B(0, R) \cup \partial B(0, R_0)$. Since $|\nabla u(x)| + |p(x)| = o(|x|^{-\gamma})$, where $\gamma > 1$, and $|u(x)| = o(1)$ as $|x| \to \infty$, it is not hard to see that the layer potentials on $\partial B(0, R)$ converge to 0 as $R \to \infty$. This allows to upgrade the decay of (u, p) at ∞ to

$$|x|^{-1}|u(x)| + |\nabla u(x)| + |p(x)| = O(|x|^{1-d}) \quad \text{as } |x| \to \infty \qquad (7.4)$$

for $d \geq 3$.

Finally, we note that since $\partial\Omega$ is C^1 and $u = 0$ on $\partial\Omega$, we have $u \in W^{1,2}(\Omega \cap B(0, R); \mathbb{C}^d)$ for any $R > 1$. Moreover, for $R > 1$ large,

$$\int_{\Omega\cap B(0,R)} |\nabla u|^2 = \int_{\partial B(0,R)} \left(\frac{\partial u}{\partial n} - np \right) \cdot u.$$

In view of (7.4) for $d \geq 3$ as well as the decay estimates, $u(x) = o(1)$ and $|\nabla u(x)| + |p(x)| = o(|x|^{-\gamma})$ for $d = 2$, by letting $R \to \infty$, we obtain $\|\nabla u\|_{L^2(\Omega)} = 0$. Since $u = 0$ on $\partial\Omega$, it follows that $u = 0$ in Ω. □

The following theorem is used in the proof of the estimate (1.7) for small $|\lambda|$.

Theorem 7.3. *Let Ω be an exterior C^1 domain in $\mathbb{R}^d$, $d \geq 3$ and $1 < q < (d/2)$. Let $u \in W_0^{1,q}(\Omega; \mathbb{C}^d)$ be a solution of*

$$-\Delta u + \nabla p = F \quad \text{and} \quad \text{div}(u) = 0 \qquad (7.5)$$

in Ω, where $F \in L^q(\Omega; \mathbb{C}^d)$. Then $u \in W_0^{1,s}(\Omega; \mathbb{C}^d)$ and

$$\|\nabla u\|_{L^s(\Omega)} \leq C\|F\|_{L^q(\Omega)}, \qquad (7.6)$$

where $\frac{1}{s} = \frac{1}{q} - \frac{1}{d}$ and C depends on d, q and Ω.

Proof. Since $W_0^{1,q}(\Omega; \mathbb{C}^d) \subset L^s(\Omega; \mathbb{C}^d)$. It suffices to prove (7.6). We divide the proof into two steps.

Step 1. We show that the solution u satisfies the estimate,

$$\|\nabla u\|_{L^s(\Omega)} \leq C \left\{ \|F\|_{L^q(\Omega)} + \|u\|_{L^q(\Omega\cap B_0)} \right\}, \qquad (7.7)$$

where $B_0 = B(0, 2R_0)$ and $R_0 > 1$ is sufficiently large. To this end, we choose $R_0 > 1$ such that $\Omega \setminus B(0, R_0) = \mathbb{R}^d \setminus B(0, R_0)$ and $\Omega \cap B(0, 2R_0)$ is a bounded C^1 domain. Choose $\varphi_1 \in C_0^\infty(\mathbb{R}^d; \mathbb{R})$ such that $\varphi_1 = 1$ in $\Omega \setminus B(0, (3/2)R_0)$ and $\varphi_1 = 0$ in $B(0, (5/4)R_0)$. Let $\varphi_2 = 1 - \varphi_1$. Then

$$\begin{cases} -\Delta(u\varphi_1) + \nabla(p\varphi_1) = F\varphi_1 - 2(\nabla u)(\nabla\varphi_1) - u\Delta\varphi_1 + p\nabla\varphi_1, \\ \qquad\qquad \operatorname{div}(u\varphi_1) = u \cdot \nabla\varphi_1 \end{cases}$$

in $\mathbb{R}^d$. It follows from the $W^{2,q}$ estimates [13] for the Stokes equations (with $\lambda = 0$) in $\mathbb{R}^d$ that

$$\begin{aligned} \|\nabla(u\varphi_1)\|_{L^s(\mathbb{R}^d)} \le C\Big\{ &\|F\varphi_1\|_{L^q(\mathbb{R}^d)} + \|(\nabla u)(\nabla\varphi_1)\|_{L^q(\mathbb{R}^d)} \\ &+ \|u\Delta\varphi_1\|_{L^q(\mathbb{R}^d)} + \|p\nabla\varphi_1\|_{L^q(\mathbb{R}^d)} + \|\nabla(u\nabla\varphi_1)\|_{L^q(\mathbb{R}^d)} \Big\}. \end{aligned} \tag{7.8}$$

Let $\Omega_0 = \Omega \cap B(0, 2R_0)$. Note that $u\varphi_2 = 0$ on $\partial\Omega_0 = \partial\Omega \cup \partial B(0, 2R_0)$ and

$$\begin{cases} -\Delta(u\varphi_2) + \nabla(p\varphi_2) = F\varphi_2 - 2(\nabla u)(\nabla\varphi_2) - u\Delta\varphi_2 + p\nabla\varphi_2, \\ \qquad\qquad \operatorname{div}(u\varphi_2) = u \cdot \nabla\varphi_2 \end{cases}$$

in Ω_0. It follows from the $W^{1,q}$ estimates for the Stokes equations (with $\lambda = 0$) in the C^1 domain Ω_0 that

$$\begin{aligned} \|\nabla(u\varphi_2)\|_{L^s(\Omega_0)} \le C\Big\{ &\|F\varphi_2\|_{L^q(\Omega_0)} + \|(\nabla u)(\nabla\varphi_2)\|_{L^q(\Omega_0)} + \|u\Delta\varphi_2\|_{L^q(\Omega_0)} \\ &+ \|p\nabla\varphi_2\|_{L^q(\Omega_0)} + \|u\nabla\varphi_2\|_{L^q(\Omega_0)} \Big\}. \end{aligned} \tag{7.9}$$

See Remark 5.7. The estimate (7.7) follows from (7.8) and (7.9) as well as the interior estimates for the Stokes equations.

Step 2. We establish the estimate (7.6) by a compactness argument.

Suppose (7.6) is not true. Then there exist sequences $\{F^\ell\} \subset L^q(\Omega; \mathbb{C}^d)$, $\{u^\ell\} \subset W_0^{1,q}(\Omega; \mathbb{C}^d) \cap W_0^{1,s}(\Omega; \mathbb{C}^d)$, such that

$$\begin{cases} -\Delta u^\ell + \nabla p^\ell = F^\ell, \\ \qquad \operatorname{div}(u^\ell) = 0 \end{cases} \tag{7.10}$$

hold in Ω for some $p^\ell \in L^1_{\text{loc}}(\Omega; \mathbb{C})$,

$$\|\nabla u^\ell\|_{L^s(\Omega)} = 1, \tag{7.11}$$

and $\|F^\ell\|_{L^q(\Omega)} \to 0$, as $\ell \to \infty$. Since $\|u^\ell\|_{L^{s_*}(\Omega)} \le C\|\nabla u^\ell\|_{L^s(\Omega)} = C$, where $\frac{1}{s_*} = \frac{1}{s} - \frac{1}{d}$, by passing to a subsequence, we may assume $u^\ell \to u$ weakly in $L^{s_*}(\Omega; \mathbb{C}^d)$ and $\nabla u^\ell \to \nabla u$ weakly in $L^s(\Omega; \mathbb{C}^{d\times d})$. It follows that u is a solution of (7.5) with $F = 0$. Note that $q < (d/2)$ implies $s = q_* < d$. Thus, by Lemma 7.2, $u = 0$ in Ω. This implies that $u^\ell \to 0$ strongly in $L^q(\Omega \cap B_0; \mathbb{C}^d)$. However, by (7.7),

$$\|\nabla u^\ell\|_{L^s(\Omega)} \le C\left\{ \|F^\ell\|_{L^q(\Omega)} + \|u^\ell\|_{L^q(\Omega\cap B_0)} \right\},$$

which leads to a contradiction with (7.11) if we let $\ell \to \infty$. □

Recall that $C_{0,\sigma}^\infty(\Omega) = \left\{u \in C_0^\infty(\Omega; \mathbb{C}^d) : \text{div}(u) = 0 \text{ in } \Omega\right\}$. Let $L^q_\sigma(\Omega)$ denote the closure of $C_{0,\sigma}^\infty(\Omega)$ in $L^q(\Omega; \mathbb{C}^d)$ and

$$G_q(\Omega) = \left\{u : u = \nabla p \text{ for some } p \in \mathring{W}^{1,q}(\Omega; \mathbb{C})\right\}.$$

Theorem 7.4. *Let Ω be a bounded or exterior domain with C^1 boundary in $\mathbb{R}^d$, $d \geq 2$. Then*

$$L^q(\Omega; \mathbb{C}^d) = L^q_\sigma(\Omega) \oplus G_q(\Omega) \tag{7.12}$$

for $1 < q < \infty$. That is, for any $u \in L^q(\Omega; \mathbb{C}^d)$, there exists a unique $(v, w) \in L^q_\sigma(\Omega) \times G_q(\Omega)$ such that $u = v + w$ in Ω and

$$\|v\|_{L^q(\Omega)} + \|w\|_{L^q(\Omega)} \leq C\|u\|_{L^q(\Omega)}, \tag{7.13}$$

where C depends on d, q and Ω.

The formula (7.12) is referred to as the Helmholtz decomposition, which is well known in the case of bounded or exterior domains with smooth boundaries (see [10] for references). In the case of bounded or exterior domains with C^1 boundaries, a sketch of the proof for (7.12) may be found in [10]. Also see [9]. The decomposition also holds for $1 < q < \infty$ if Ω is a bounded convex domain [15]. If Ω is a bounded or exterior domain with Lipschitz boundaries, the Helmholtz decomposition (7.12) holds if

$$\begin{cases} (3/2) - \varepsilon < q < 3 + \varepsilon & \text{for } d \geq 3, \\ (4/3) - \varepsilon < q < 4 + \varepsilon & \text{for } d = 2, \end{cases} \tag{7.14}$$

where $\varepsilon > 0$ depends on Ω. The ranges in (7.14) are known to be sharp. See [9]. We remark that Theorem 7.4 is not used in this paper.

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