Control Barrier Functions for Linear Continuous-Time Input-Delay Systems with Limited-Horizon Previewable Disturbances

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Abstract— Cyber-physical and autonomous systems are often equipped with mechanisms that provide predictions/projections of future disturbances, e.g., road curvatures, commonly referred to as preview or lookahead, but this preview information is typically not leveraged in the context of deriving control barrier functions (CBFs) for safety. This paper proposes a novel limited preview control barrier function (LPrev-CBF) that avoids both ends of the spectrum, where on one end, the standard CBF approach treats the (previewable) disturbances simply as worst-case adversarial signals and on the other end, a recent Prev-CBF approach assumes that the disturbances are previewable and known for the entire future. Moreover, our approach applies to input-delay systems and has recursive feasibility guarantees since we explicitly take input constraints/bounds into consideration. Thus, our approach provides strong safety guarantees in a less conservative manner than standard CBF approaches while considering a more realistic setting with limited preview and input delays.

I. INTRODUCTION

Many cyber-physical and autonomous systems (e.g., self-driving cars and robot exoskeletons) possess forecasting tools such as forward-looking sensors (e.g., cameras and LIDAR, topographical maps), and predictive models to anticipate what lies ahead. This preview information for a window into the future, if used correctly, can significantly improve system performance and safety. However, many control strategies often neglect to use this information and opt instead to consider worst-case scenarios, especially in the context of safety when computing robust controlled invariant sets, e.g., [1] or control barrier functions (CBFs), e.g., [2]–[4]. On the other hand, optimal and model predictive control (MPC) methods, e.g., [5], [6], do use certain preview information but often lack recursive safety and feasibility properties.

Recent research underscores the advantages of using preview information in safety controls for discrete-time systems, including those with input delays [7], [8], leading to enhanced safety with increased preview time. For continuous-time systems, the predictive CBF method [9] proposed an approach that can leverage information about 'controllable' predicted trajectories, while our recent work in [10] introduced a preview CBF approach that can utilize information of previewable (but 'uncontrollable') disturbances such as road gradients or curvatures or predicted future motion of other agents. However, the latter assumes that the preview horizon is as long as needed (unlimited), which is not always

realistic, e.g., due to limited sensing range of cameras or LIDARs. Further, to our best knowledge, such techniques for continuous-time systems with input delay, e.g., due to network latency or hardware constraints, are still lacking. CBF tools have been explored for input-delay systems [11], [12], but they do not or cannot make use of preview information.

Contributions. Building on our prior design of Prev-CBFs with unlimited preview in [10], this paper presents *limited preview control barrier functions* (LPrev-CBFs) for linear continuous-time input-delay systems where the preview horizon for the previewable disturbances is limited and fixed, which better reflects real-world settings where sensing ranges are limited. This research relaxes the restrictive setting in [10] that assumes unlimited preview and also addresses safety concerns stemming from input delays.

In contrast to standard CBFs with or without input delays, e.g., [2], [3], [11], that (implicitly) enforce robust safety by considering the worst-case future disturbances, our approach leverages preview information about these previewable disturbances to mitigate the conservatism. On the other hand, in contrast to our prior Prev-CBF approach [10] that assumes all future disturbances are previewable, our LPrev-CBF only uses the preview information of the disturbances for a limited and fixed horizon and considers the worst-case unpreviewed disturbances beyond that horizon. This is particularly beneficial in minimizing the need for interventions while still ensuring safety and robustness from disturbances, In other words, the LPrev-CBF is designed for a more realistic scenario while taking advantage of available preview information to be more permissive and the associated safety controller is less likely to be activated (i.e., less interventions when used as a safety filter) when compared to standard CBFs.

Additionally, our closed-form LPrev-CBF explicitly incorporates input constraints into its design and consequently, it is naturally guaranteed to be recursively feasible (and safe) when incorporated into an optimization-based safety controller to modify any nominal (input-delay) controller. Further, the results in this paper are in themselves a novel contribution even in the absence of input delay, as an extension of our prior work in [10] to the setting when the preview horizon is limited and fixed.

The efficacy of the proposed LPrev-CBFs is demonstrated via practical examples—an assistive shoulder robot equipped with interaction torque preview capabilities and a vehicle lane-keeping system that utilizes road curvature preview.

This paper is structured as follows. The problem of interest is outlined in Section II. Next, In Section III, we

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elaborate on the our proposed LPrev-CBFs to solve this problem. Then, we illustrate the efficacy of our approach in Section IV using examples of an assistive shoulder robot and vehicle lane keeping, and also discuss the advantages of preview for safety of a linear continuous-time input-delay system. Finally, we conclude by presenting a summary of our contributions and some future work in Section V.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. Notations

$\mathbb{R}_+$ and $\mathbb{R}^n$ denote the sets of non-negative real numbers and n-dimensional Euclidean space, respectively. An identity matrix of size n is represented by I_n and an $m \times n$ matrix of zeros is represented by $\mathbf{0}_{m \times n}$. Additionally, all vector inequalities represent element-wise inequalities, while $|\cdot|$ and $\text{sgn}(\cdot)$ serve as element-wise absolute value and signum operators, respectively, and $\text{diag}(v)$ represents a diagonal matrix whose diagonal elements are elements of a vector v. Further, a class $\mathcal{K}_\infty$ function $\alpha : [0, \infty) \to [0, \infty)$ is strictly increasing and continuous with $\alpha(0) = 0$ and $\lim_{r \to \infty} \alpha(r) = \infty$.

B. Problem Formulation

Consider the continuous-time linear control system that includes a time-delayed control input, along with additive previewable disturbances. This system is denoted as Σ_{delay} and is described as follows:

$$\Sigma_{delay} : \dot{x}(t) = Ax(t) + Bu(t - T_i) + B_d d(t), \quad (1)$$

where $x(t) \in \mathcal{X} \subseteq \mathbb{R}^n$ represents the system states, $u(t) \in \mathcal{U} \subseteq \mathbb{R}^m$ is the control input subject to a (fixed) time delay T_i, $d(t) \in \mathcal{D} \subseteq \mathbb{R}^l$ denotes bounded and previewable disturbances (only for a *fixed* preview horizon T_p, beyond which they are not previewed/uncertain). Specifically, the disturbance set is $\mathcal{D} \triangleq \{d \mid |d| \leq d_m\}$ and the control input set is $\mathcal{U} \triangleq \{u \mid |u| \leq u_m\}$, where d_m and u_m represent the disturbance and actuation bounds[1], respectively.

The term "previewable disturbance" is used to refer to any exogenous inputs, signals, or parameters for which future values may be anticipated/known. Examples include a reference signal in tracking tasks or the predicted trajectories of other agents, as well as road conditions such as curvature, gradient, or friction coefficients that could be measured through limited-range sensing and perception modules.

Inspired by the literature on time-delay control systems, we also represent Σ_{delay} using *predicted states* $z(t) \triangleq x(t + T_i)$ with the following predictor system:

$$\Sigma_{pred} : \begin{cases} \dot{z}(t) = Az(t) + Bu(t) + B_d d(t + T_i), \\ y(t) = Cz(t), \end{cases} \quad (2)$$

where $z(t) \in \mathcal{Z} = \mathcal{X} \subseteq \mathbb{R}^n$ serves as the new system states with the initial predicted state $z(0)$ as given in (4) with $u(\tau)$ for all $\tau \in [-T_i, 0]$, and $y(t) \in \mathbb{R}$ is the scalar output. The scalar output $y(t) \in \mathbb{R}$ represents the system variable of

interest that relates to the safety of the system. Specifically, we consider *system safety* as the satisfaction of a desired output constraint[1] given as:

$$|y(t)| = |Cz(t)| \leq y_m, \quad \forall t \geq 0, \quad (3)$$

with known output bounds $y_m \in \mathbb{R}$. We further assume that Σ_{pred} with control input $u(t)$ and disturbance (input) $d(t)$ as well as output $y(t)$ has a relative degree of 2, meaning that $CB = 0$, $CB_d = 0$, $CAB \neq 0$ and $CAB_d \neq 0$.

In contrast to the assumption in our prior work [10, Assumption 2] that the preview horizon is as long as needed (unlimited), this paper considers the (more realistic) setting where the preview horizon T_p is limited and fixed, beyond which the *unpreviewed* disturbance is unknown but bounded.

Assumption 1. *The delay time T_i and preview horizon T_p are known and fixed/constant and they are such that $T_p > T_i$.*

Assumption 2. *For a given time $t \in \mathbb{R}_+$ and known preview horizon T_p, the previewed disturbance $\mathbf{d}_p(t) \triangleq \{d(\tau) \in \mathcal{D}, t \leq \tau < t + T_p\}$ is known. Furthermore, beyond the preview horizon, the unpreviewed disturbance $\mathbf{d}_{np}(t) \triangleq \{d(\tau) \in \mathcal{D}, t + T_p \leq \tau < \infty\}$ is unknown but bounded with known bounds in $\mathcal{D}$.*

By assuming $T_p > T_i$, the *predicted state* $z(t)$ in (2), i.e.,

$$\begin{aligned} z(t) &= e^{AT_i} x(t) \\ &+ \int_0^{T_i} e^{A(T_i - \tau)} (Bu(t - T_i + \tau) + B_d d(t + \tau)) d\tau, \end{aligned} \quad (4)$$

is exactly known; thus, we can equivalently consider Σ_{pred} in lieu of Σ_{delay} and for simplicity, we shall also directly define the safe sets based on $z(t)$ under the assumption that the system is safe for all $0 \leq t < T_i$ such that $z(0)$ is within the controlled invariant set defined below.

Definition 1 (Safe Sets). *Let $S_z \subseteq \mathbb{R}^n$ be a* safe set *of Σ_{pred} that describes desirable/given safety constraints on the states, and let $S_{z,p} \subseteq \mathbb{R}^n \times \mathcal{D}^{[0,T_p)}$ be the T_p-augmented safe set of Σ_{pred}, defined as*

$$S_{z,p} \triangleq \{(z, \mathbf{d}_p) \mid z \in S_z, \mathbf{d}_p \in \mathcal{D}^{[0,T_p)}\},$$

where $\mathcal{D}^{[0,T_p)}$ is the set of all trajectories of $d(\tau)$ within the time interval of $[0, T_p] \triangleq \{\tau | 0 \leq \tau < T_p\}$, defined as,

$$\mathcal{D}^{[0,T_p)} \triangleq \{d(\tau), \forall \tau \in [0, T_p] \mid d(\tau) \in \mathcal{D}\}.$$

Definition 2 (Controlled Invariant Set). *A set $\mathcal{C} \subseteq S_z$ is a* robust controlled invariant set *of Σ_{pred} in a safe set $S_z \subseteq \mathbb{R}^n$ if for all $z(0) \in \mathcal{C}$, there exists some $u(t) \in \mathbb{R}^m$ such that $z(t) \in \mathcal{C} \subseteq S_z$, $\forall t \geq 0$, for all $d(t) \in \mathcal{D}$. $\mathcal{C}_{max}$ is the maximal robust controlled invariant set in S_z if $\mathcal{C}_{max}$ contains all robust controlled invariant sets in S_z.*

Further, a set $\mathcal{C}_p \in S_{z,p}$ is a limited preview controlled invariant set *of Σ_{pred} in an augmented safe set $S_{z,p}$ if for all $(z(0), \mathbf{d}_p(0)) \in \mathcal{C}_p$, there exists some $u(t) \in \mathbb{R}^m$ such that $(z(t), \mathbf{d}_p(t)) \in \mathcal{C}_p \subseteq S_{z,p}$, $\forall t \geq 0$, for all $\mathbf{d}_{np} \in \mathcal{D}^{[T_p, \infty)}$. $\mathcal{C}_{max,p}$ is the maximal limited preview controlled invariant set in $S_{z,p}$ if $\mathcal{C}_{max,p}$ contains all limited preview controlled invariant sets of Σ_{pred} in $S_{z,p}$.*

Additionally, the definitions presented herein can be interpreted as the continuous-time analogs of their discrete-

time counterparts delineated in [8]. Of particular significance is the proof provided in the aforementioned study that the maximal controlled invariant sets for systems without preview is a subset of the maximal controlled invariant sets for systems with preview in a discrete-time framework, even in the presence of input delays. Inspired by these findings, we postulate that preview may also offer similar advantages in continuous-time systems with input delays.

In contrast to the objective of identifying the maximal limited preview controlled invariant set for Σ_{pred}, this study shifts its focus towards the exploration of limited preview control barrier functions with preview capabilities for Σ_{pred}. Specifically, we aim to render some time-varying set $\mathcal{C}_{z,t} \subseteq S_z$ controlled invariant. To achieve this, we introduce a novel concept of a time-varying 'limited preview safe set', denoted as $\mathcal{C}_{z,p,t} \subseteq S_{z,p}$, which is not only controlled invariant but also serves to imply the existence of some $\mathcal{C}_{z,t} \subseteq S_z$ that is controlled invariant by contruction/design. It is worth noting that the efficacy of control barrier functions for systems without input delay, but with preview capabilities (for an "infinite"/unlimited horizon), has been previously established in our prior work [10].

Definition 3 (Limited Preview Safe Set). *Given a predictor system with preview Σ_{pred} (with known $\mathbf{d}_p \in \mathcal{D}^{[0,T_p)}$ and unknown $\mathbf{d}_{np} \in \mathcal{D}^{[T_p,\infty)}$), a super-level set $\mathcal{C}_{z,p,t}$ defined on a time-varying function $h : \mathcal{X} \times \mathcal{D}^{[0,T_p)} \times \mathbb{R}_+ \to \mathbb{R}$:*

$$\mathcal{C}_{z,p,t} \triangleq \{(z, \mathbf{d}_p, t) \mid h(z, \mathbf{d}_p, t) \geq 0, \}, \qquad (5)$$

based on another time-varying function h_{np} according to $h(z, \mathbf{d}_p, t) \triangleq \min_{\mathbf{d}_{np} \in \mathcal{D}^{[T_p,\infty)}} h_{np}(z, \mathbf{d}_p, \mathbf{d}_{np}, t)$, is a limited preview safe set for Σ_{pred} if $(z(t), \mathbf{d}_p(t), t) \in \mathcal{C}_{z,p,t}$ for all $t \geq 0$ implies that $z(t) \in S_z$ for all $t \geq 0$. $\mathcal{D}^{[T_p,\infty)}$ is the set of all trajectories of $d(t)$ starting from T_p, and the boundary $\partial \mathcal{C}_{z,p,t}$ and interior $Int(\mathcal{C}_{z,p,t})$ of $\mathcal{C}_{z,p,t}$ are similarly defined with the $\geq$ operator being replaced by $=$ and $>$, respectively.

Note that, by design, the limited preview robust safe set in the above definition needs to be defined or chosen such that its controlled invariance implies the existence of some $\mathcal{C}_{z,t} \subseteq S_z$ that is controlled invariant.

Then, the problem of interest can be formally written as:

Problem 1 (Safety with Limited Preview). *Given an input-delay system with preview Σ_{delay} in (1) satisfying Assumptions 1–2, its corresponding equivalent predictor system Σ_{pred} in (2) and a safe set S_z (cf. Definition 1), construct a limited preview control barrier function (LPrev-CBF) corresponding to $\mathcal{C}_{z,p,t}$ in (5) that guarantees limited preview controlled invariance of Σ_{pred} in S_z (and thus, safety of Σ_{delay}, since it is equivalent to Σ_{pred} under Assumption 1).*

III. MAIN RESULTS

We now describe our proposed method to address Problem 1, where we introduce a novel input-constrained limited preview CBF in closed-form and describe how it can be implemented computationally to guarantee safety.

A. Limited Preview Control Barrier Functions (LPrev-CBFs)

First, we present the definition of Limited Preview Control Barrier Function (LPrev-CBF) as an extension of Preview Control Barrier Function (Prev-CBF) in [10].

Definition 4 (Limited Preview CBF). *Given an input-delay system Σ_{delay}, its corresponding predictor system Σ_{pred} with a fixed-horizon previewable disturbance that satisfies Assumptions 1-2 and a safe set S_z (cf. Definition 1), then a continuously differentiable function $h : \mathcal{X} \times \mathcal{D}^{[0,T_p)} \times \mathbb{R}_+ \to \mathbb{R}$ is a limited preview CBF corresponding to uncertainty dependent safe set $\mathcal{C}_{z,p,t}$ in (5), if there exist a control input $u \in \mathcal{U}$ and a class $\mathcal{K}_\infty$ function α such that:*

$$\dot{h}(z, u, \mathbf{d}_p, t) \geq -\alpha(h(z, \mathbf{d}_p, t)), \qquad (6)$$

for all $z \in \mathcal{X}$ and $t \in \mathbb{R}_+$. Further, for any $t \in \mathbb{R}_+$, $z \in \mathcal{X}$ and $\mathbf{d}_p \in \mathcal{D}^{[0,T_p)}$, an associated safe input set is defined as:

$$K_\mathcal{C}(z, \mathbf{d}_p, t) = \{u \in \mathcal{U} \mid (6) \text{ holds}\}. \qquad (7)$$

Theorem 1 (Safety with Limited Preview). *Given a predictor system Σ_{pred} with a fixed-horizon previewable disturbance that satisfies Assumptions 1–2 and a safe set S_z (cf. Definition 1), if h is a LPrev-CBF corresponding to the limited preview safety set $\mathcal{C}_{z,p,t}$ from (5), then for the predictor system Σ_{pred} with $z(0) \in S_z$, any Lipschitz continuous controller $u(x, \mathbf{d}_p, t) \in K_\mathcal{C}(z, \mathbf{d}_p, t)$ with z in (4) ensures the controlled invariance of the limited preview safety set $\mathcal{C}_{z,p,t}$. Consequently, there exists some set $\mathcal{C}_{z,t} \subset S_z$ for the system Σ_{pred} for which $u(x, \mathbf{d}_p, t)$ also ensures its controlled invariance. Thus, the predictor system Σ_{pred} with preview is guaranteed to be safe, i.e., $z(t) \in S_z, \forall t \geq 0$.*

Proof. If h is a LPrev-CBF associated with the limited preview safe set $\mathcal{C}_{z,p,t}$, then any controller $u(x, \mathbf{d}_p, t) \in K_\mathcal{C}(z, p, t)$ enforces (6) for all $z \in \mathcal{X}$, $\forall t \geq 0$; hence $\mathcal{C}_{z,p,t}$ as defined in (5) is forward control invariant, i.e., $h(z, \mathbf{d}_p, t) \geq 0$, $\forall t \geq 0$. Consequently, the predictor system Σ_{pred} and in turn the input-delay system Σ_{delay} are safe with preview for all $t \geq 0$ with respect to the set $\mathcal{C}_{z,t} \subseteq S_z$, where $\mathcal{C}_{z,t}$ exists by construction. □

B. Closed-Form Candidate Limited Preview CBF

We now introduce a framework to formulate a closed-form candidate limited preview CBF (LPrev-CBF) to derive a limited preview safe set (cf. Definition 3) for the predictor system Σ_{pred}, given desired output constraints in (3).

The framework is inspired by our prior work [10], where a predictor-based method [11], [13] is considered, with the predicted state z projected T seconds into the future as:

$$z(t+T) = \phi(t, T) + \epsilon(t, T) + \int_0^T e^{A(T-\tau)} Bu(t+\tau) d\tau, \quad (8)$$

where $\phi(t, T) \triangleq e^{AT} z(t) + \int_0^{T_\delta} e^{A(T-\tau)} B_d d(t + T_i + \tau) d\tau$, $T_\delta \triangleq \min(T_p - T_i, T)$ and $\epsilon(t, T) \triangleq \int_{T_\delta}^T e^{A(T-\tau)} B_d d(t + T_i + \tau) d\tau$. Note that at any given t and given preview of the disturbance $T_p > T_i$ seconds into the future, $\phi(t, T)$ can always be pre-computed, while $\epsilon(t, T)$ contains unpreviewed disturbances that the LPrev-CBF must be robust with respect to

to their worst-case realizations. Then, to guarantee the satisfaction of the output bounds in (3), $\forall t \geq 0$, we enforce that the (immediate) future minima or maxima of worst-case output trajectories under maximum acceleration or deceleration inputs, respectively, always satisfies the output bounds, $t \geq 0$. Imposing these minima or maxima constraints also ensures the constraint feasibility at all times from the current time t to the time associated with the minima or maxima, which we call the worst-case (minimum) *stopping time*, as defined below:

Definition 5 (Worst-Case Stopping Time). *At any given time $t \in \mathbb{R}_+$ for the predictor system Σ_{pred} with fixed-horizon preview in (2), we define the worst-case (minimum) stopping time $T_s(t)$ as the minimum $T_s(t)$ such that the worst-case output velocity $\dot{y}_w(t+T_s(t)) = C\dot{z}_w(t+T_s(t)) = CAz_w(t+T_s(t)) = 0$ under maximum control input acceleration and disturbance-induced deceleration when $\dot{y}(t) = C\dot{z}(t) \leq 0$ or maximum control input deceleration and disturbance-induced acceleration when $\dot{y}(t) = C\dot{z}(t) \geq 0$.*

From (2), under the relative degree 2 assumption,

$$\ddot{y}(t) = CA^2 z(t) + CABu(t) + CAB_d d(t), \qquad (9)$$

from which we can observe that when $\dot{y}(t) \leq 0$ [2], the maximum obtainable acceleration is with $\text{diag}(\text{sgn}(CAB))u_m$ under worst-case disturbance $-\text{diag}(\text{sgn}(CAB_d))d_m$, while when $\dot{y}(t) \geq 0$, the maximum deceleration is obtained with $-\text{diag}(\text{sgn}(CAB))u_m$ under worst-case disturbance $\text{diag}(\text{sgn}(CAB_d))d_m$. In other words, the worst-case output velocity $\dot{y}_w(t+T_s(t))$ and worst-case output $y_w(t+T_s(t))$ can be found by applying $u(\tau) = \hat{u}(t)$ with

$$\hat{u}(t) \triangleq -\text{sgn}(\dot{y}(t))\text{diag}(\text{sgn}(CAB))u_m \qquad (10)$$

under worst-case disturbance $d(\tau) = \hat{d}(t)$ with

$$\hat{d}(t) \triangleq sgn(\dot{y}(t))\text{diag}(\text{sgn}(CAB_d))d_m \qquad (11)$$

for all $\tau \in [t, T_s(t)]$, resulting in,

$$\ddot{y}_w(\tau) = CAz_w(\tau) - sgn(\dot{y}(t))|CAB|u_m + sgn(\dot{y}(t))|CAB_d|d_m,$$

which is obtained from (9) with $u(t)$ in (10) and $d(t)$ in (11), and the worst-case stopping time is the $T_s(t)$ that is the solution to $\dot{y}_w(t+T_s(t)) = CAz_w(t+T_s(t)) = 0$.

It is worth noting that for a known/computed time-varying worst-case stopping time $T_s(t)$ at a given time $t \in \mathbb{R}_+$, Assumption 2 when applied to the predictor system Σ_{pred}, under Assumption 1 with a fixed preview horizon T_p and fixed input delay T_i, translates to two distinct cases: (I) When $T_p - T_i > T_s(t)$ (i.e., when the constant time horizon T_p exceeds the stopping time for $z(t)$), the disturbances $d(t+T_i)$ in (2) are previewed/known for the entire time interval up to $t + T_s(t) + T_i$, and (II) when $T_p - T_i \leq T_s(t)$, the previewable disturbances $d(t+T_i)$ within the time interval $t+T_p \leq \tau \leq t+T_s(t)+T_i$ remain indeterminate/unpreviewed but bounded by $\mathcal{D}$.

Further, note that the idea of worst-case (minimum) stopping time is akin to and an extension of the (minimum) stopping time in [10], while the concept of utilizing the

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[2]Note that per Assumption 1, $\dot{y}(t) = C\dot{z}(t) = CAz(t)$ is exactly known.

immediate future minima or maxima of the output trajectory is inspired by the idea of minimizers/maximizers in [9], [10] for a "predicted" desired/reference trajectory. In particular, by enforcing that the worst-case predicted outputs $T_s(t)$ seconds into the future, i.e.,

$$|Cz_w(t+T_s(t))| \leq y_m, \quad \forall t \geq 0, \qquad (12)$$

with $Cz_w(t + T_s(t))$ corresponding to a minimizer/maximizer, we are essentially ensuring the satisfaction of the output constraints for a future moving time horizon that includes the current time. Hence, ensuring the robust controlled invariance of (12) corresponding to $\mathcal{C}_{z,p,t}$ implies the robust controlled invariance of (3) corresponding to $\mathcal{S}_z$.

Next, we present a closed-form candidate LPrev-CBF to compute a controlled invariant limited preview safe set. Note that for the remainder of this manuscript, the (explicit) dependence of T_s and other terms on the current time t is omitted for the sake of better readability.

Lemma 1 (Closed-Form Candidate Limited Preview CBF). *Suppose Assumptions 1–2 hold. Then,*

$$\begin{aligned} h(z, \mathbf{d}_p, t) &= y_m - \text{sgn}(\dot{y}(t))y_w(t+T_s) \\ &= y_m - \text{sgn}(\dot{y}(t))Cz_w(t+T_s) \qquad (13) \\ &\geq 0, \end{aligned}$$

is a valid candidate LPrev-CBF that guarantee the satisfaction of the safety bounds in (12) corresponding to $\mathcal{C}_{z,p,t}$ and, consequently, the safety bounds in (3) associated with $\mathcal{S}_z$. The worst-case predicted state $z_w(t + T_s) = \phi(t, T_s) + \hat{\epsilon}(t, T_s) + (\int_0^{T_s} e^{A(T_s-\tau)}d\tau)B\hat{u}$ is derived from (8), with $\phi(t, T_s)$ as defined below (8) (when $T = T_s$) and $\hat{\epsilon}(t, T_s) \triangleq (\int_{T_\delta}^{T_s} e^{A(T_s-\tau)}d\tau)B_d\hat{d}$ with $\hat{d}(\tau)$ defined in (11) and $T_\delta = \min(T_p - T_i, T_s(t))$, as well as $\hat{u}$ defined in (10).

Proof. First, when $\dot{y}(t) \leq 0$ under maximum acceleration input $\hat{u}$, the desired worst-case safety constraint is $y_w(t+T_s(t)) \geq -y_m$, where $y_w(t+T_s(t))$ is the smallest possible y when the system comes to a stop (before changing directions) under the worst-case unpreviewed disturbance $\hat{d}$. Similarly, when $\dot{y}_w(t) \geq 0$ under maximum deceleration input $\hat{u}$, the desired worst-case safety constraint is $y_w(t + T_s) \leq y_m$. Combining these two constraints yields

$$y_m - \text{sgn}(\dot{y}(t))y_w(t + T_s) \geq 0. \qquad (14)$$

Further, $y_w(t+T_s) = Cz_w(t+T_s)$, where as described above (10), $z_w(t+T_s) = Cy_w(t+T_s)$ is the worst-case predictive output that can be derived from (8) by considering $T = T_s$ as well as $u(\tau) = \hat{u}(t)$ and $d(\tau) = \hat{d}(\tau)$, $\forall \tau \in [t+T_p, t+T_s]$ with $\hat{u}$ and $\hat{d}$ as defined in (10) and (11), respectively.

Thus, enforcing (14) in turn enforces (12). Hence, h is a valid candidate limited preview control barrier function (LPrev-CBF), i.e., there always exists a piece-wise constant input $u(\tau) = \hat{u}(t) = -\text{sgn}(\dot{y}(t))\text{diag}(\text{sgn}(CAB))u_m$, for all $\tau \in [t, t+T_s]$ that enforces $h(z, \mathbf{d}_p, t) \geq 0$ for all $t \geq 0$. Consequently, due to the guaranteed feasibility of (12) at maxima or minima (i.e., when $\dot{y}_w(t+T_s) = C\dot{z}_w(t+T_s) = 0$), the safety constraints is also feasible for the whole time horizon from $t + T_i$ to $t + T_i + T_s$; hence, (3) holds. $\square$

C. Worst-Case Stopping Time

As seen from (13) in Lemma 1, the design of the candidate LPrev-CBF depends on the worst-case stopping time T_s that will be derived next.

Lemma 2 (Worst-Case Stopping Time)**.** *At any given time t the worst-case stopping time is the smallest positive solution $T_s(t)$ to the equality $CAz_w(t + T_s) = 0$ with $z_w(t + T_s)$ given below (13), i.e.,*

$$CA\phi(t, T_s) + Ce^{AT_s}B\hat{u}(t) + Ce^{A(T_s - T_\delta)}B_d\hat{d}(t) = 0, \tag{15}$$

where $\hat{u}(t)$, $\hat{d}(t)$ are defined in (10) and (11), respectively, while T_δ and $\phi(t, T_s)$ are defined below (8) (with $T = T_s$).

Proof. At any given time t as per the predictor state dynamics in (2), the output velocity T_s seconds into the future is given as $\dot{y}(t + T_s) = C\dot{z}(t + T_s) = CAz(t + T_s)$ (since the system has relative degree 2 with respect to both the input $u(\cdot)$ and the disturbance $d(\cdot)$, $CB = CB_d = 0$). Then, with $z_w(t + T_s)$ below (13), the worst-case output velocity with $\hat{u}(t)$ and $\hat{d}(t)$ for all $\tau \in [t, t + T_s(t)]$ is given by

$$
\begin{aligned}
\dot{y}_w(t + T_s) &= CAz_w(t + T_s) \\
&= CA\phi(t, T_s) + C\int_0^{T_s} Ae^{A(T_s - \tau)}d\tau B\hat{u} \\
&\quad + C\int_{T_\delta}^{T_s} Ae^{A(T_s - \tau)}B_d\hat{d}(\tau)d\tau \\
&= CA\phi(t, T_s) + C(e^{AT_s} - I)B\hat{u} + C(e^{A(T_s - T_\delta)} - I)B_d\hat{d} \\
&= CA\phi(t, T_s) + Ce^{AT_s}B\hat{u} + Ce^{A(T_s - T_\delta)}B_d\hat{d},
\end{aligned}
$$

where the final equality holds since $CB = 0$ and $CB_d = 0$ by the relative degree 2 assumption. $\qquad\square$

D. Closed-Form Limited Preview Control Barrier Function

Now that we have a candidate LPrev-CBF from Lemma 1 and an expression for the worst-case stopping time in Lemma 2, we can prove that the candidate LPrev-CBF satisfies the definition of limited preview CBF in Definition 4.

Proposition 1 (Closed-Form LPrev-CBF)**.** *Given a input-delay system with preview Σ_{delay} and a corresponding predictor system Σ_{pred} that satisfies Assumptions 1–2 with worst-case stopping time $T_s(t)$ computed based on Lemma 2, the continuously mapping $h : \mathbb{R}^n \times \mathcal{D}^{[0,T_p]} \times \mathbb{R}_+ \to \mathbb{R}$ in Lemma 1 is a limited preview control barrier function (LPrev-CBF) for Σ_{pred}, if there exist a control input $u \in \mathcal{U}$ and a class $\mathcal{K}_\infty$ function α that satisfy (6) with*

$$
\begin{aligned}
\dot{h}(z, u, \mathbf{d}_p, t) = -\mathrm{sgn}(\dot{y}(t))[Ce^{AT_s}(Az(t) + Bu(t) \\
+ B_dd(t + T_i)) + \psi(t, T_s)], \tag{16}
\end{aligned}
$$

with $\psi(t, T_s) \triangleq \int_0^{T_\delta} Ce^{A(T_s - \tau)}B_d\dot{d}(t + T_i + \tau)d\tau$, where T_δ is defined below (8) (with $T = T_s$). Further, (12) holds and consequently, the output constraint in (3) holds.

Proof. We begin the proof by considering the closed-form candidate LPrev-CBF h from (13) in Lemma 1. Next, by applying Theorem 1 to h, a closed form expression for $\dot{h}$ in (6) is obtained by computing the derivative of h with respective to current time t. Consequently, $\dot{h}(z, u, \mathbf{d}_p, t) = -\mathrm{sgn}(\dot{y}(t))\frac{d}{dt}y_w(t + T_s)$, where $\frac{d}{dt}y_w(t + T_s)$ is the time derivative of $y_w(t + T_s) = Cz_w(t + T_s)$ with $z_w(t + T_s)$ defined below (13), which can be derived by employing

Leibnitz integral rule and leveraging the fact that $CB = 0$ and $CB_d = 0$ (relative degree 2 assumption) to obtain

$$
\begin{aligned}
\tfrac{d}{dt}y_w(t + T_s) &= Ce^{AT_s}\dot{z}(t) + \psi(t, T_s) \\
&\quad + CA(\phi(t, T_s) + \hat{\epsilon}(t, T_s) + (\int_0^{T_s} e^{A(T_s - \tau)}d\tau)B\hat{u})\dot{T}_s \\
&\quad + Ce^{A(T_s - T_\delta)}B_d(d(t + T_i + T_\delta) + \hat{d})\dot{T}_\delta \\
&= Ce^{AT_s}\dot{z}(t) + \psi(t, T_s) + CAz_w(t + T_s)\dot{T}_s \\
&\quad + Ce^{A(T_s - T_\delta)}B_d(d(t + T_i + T_\delta) + \hat{d})\dot{T}_\delta, \tag{17}
\end{aligned}
$$

where we defined $\psi(t, T_s)$ below (16), with $\phi(t, T_s)$ and $\hat{\epsilon}(t, T_s)$ defined below (8) and (13), respectively, and applied the definition of $z_w(t + T_s)$ below (13) in the second equality.

Next, by Lemma 2, $CAz_w(t + T_s) = 0$, i.e., the third term in the above becomes 0. Additionally, since $T_\delta = \min(T_p - T_i, T_s)$, we have that when $T_\delta = T_p - T_i$, $\dot{T}_\delta = 0$ (T_p and T_i are fixed constants) and when $T_\delta = T_s$, $Ce^{A(T_s - T_\delta)}B_d = CB_d = 0$ (by relative degree 2 assumption); consequently, the final term in the above that multiplies $\dot{T}_\delta$ is also equal to 0. Thus, the expression for $\dot{h}$ simplifies to $\dot{h}(z, u, \mathbf{d}_p, t) = -\mathrm{sgn}(\dot{y}(t))\frac{d}{dt}y_w(t + T_s) = -\mathrm{sgn}(\dot{y}(t))(Ce^{AT_s}\dot{z}(t) + \psi(t, T_s))$. Finally, we obtain (16) by substituting the expression for $\dot{z}(t)$ from the predictor state dynamics in (2). $\qquad\square$

E. Optimization-Based Safety Control

Next, we show how the proposed LPrev-CBF can be coupled with any nominal or legacy controller to minimally modify it to guarantee recursive safety.

Proposition 2 (Optimization-Based Safety Control)**.** *For the input-delay system Σ_{delay} in (1), any (stabilizing) nominal (input-delay) controller $u = k(x, z, t)$ at time t with $z(t)$ in (4) can be minimally modified to guarantee safety by computing a new safe control input $u(x, \mathbf{d}_p, t)$ that is a solution to the following quadratic program (QP):*

$$
\begin{aligned}
u(x, \mathbf{d}_p, t) &= \arg\min_{u \in \mathcal{U}} \tfrac{1}{2}\|u - k(x, z, t)\| \\
&\quad s.t.\ P(t)u \leq q(t), \tag{18}
\end{aligned}
$$

with z, $\psi(t, T_s)$, $h(z, \mathbf{d}_p, t)$ and T_s from (4), Proposition 1 (as defined below (16)), Lemma 1 and Lemma 2, respectively, and a class $\mathcal{K}_\infty$ function α, where we defined

$$
\begin{aligned}
P(t) &\triangleq \mathrm{sgn}(\dot{y}(t))Ce^{AT_s(t)}B, \\
q(t) &\triangleq \alpha(h(z, \mathbf{d}_p, t)) - \mathrm{sgn}(\dot{y}(t))(\psi(t, T_s) \\
&\quad + Ce^{AT_s(t)}(Az(t) + B_dd(t + T_i))).
\end{aligned}
$$

Further, the optimization problem (18) is recursively feasible.

Proof. The LPrev-CBF constraint in Theorem 1 and Definition 4, $\dot{h}(z, u, \mathbf{d}_p, t) \geq -\alpha(h(z, \mathbf{d}_p, t))$, with $h(z, \mathbf{d}_p, t)$ from Lemma 1 and $\dot{h}(z, u, \mathbf{d}_p, t)$ from Proposition 1 can be written as:

$$
\begin{aligned}
-\alpha(h(z, \mathbf{d}_p, t)) \leq -\mathrm{sgn}(\dot{y}(t))(Ce^{AT_s}(Az(t) + Bu(t) \\
B_dd(t + T_i)) + \psi(t, T_s)),
\end{aligned}
$$

which can be rearranged as $P(t)u \leq q(t)$ with the above $P(t)$ and $q(t)$. Finally, recursive feasibility is guaranteed since our design explicitly considers input constraints. $\qquad\square$

In the above, solving (15) analytically to find $T_s(t)$ for the application of Proposition 2 is non-trivial, but it can be found numerically, e.g., using MATLAB functions `fsolve`,

`fzero` or `vpasolve`. Further, note that in the absence of input delay (i.e., when $T_i = 0$), the results in this paper are in themselves a novel contribution for when the preview horizon is limited and fixed, in contrast to our prior work in [10] that assumed unlimited preview.

IV. ILLUSTRATIVE EXAMPLES

A. Assistive Shoulder Exoskeleton Robot

Consider the dynamics of an industrial shoulder exoskeleton robot system [14] with a fixed input delay given by:

$$I_j \ddot{e}(t) + B_j \dot{e}(t) + K_j e(t) = \tau_e(t) + u(t - T_i), \quad (19)$$

where $e(t) = \theta(t) - \theta_d(t)$ is the angular displacement error, with $\dot{e}$ and $\ddot{e}$ as its velocity and acceleration. The terms $I_j = I_h + I_r$, $B_j = B_h + B_r$, and $K_j = K_h + K_r$ represent the combined inertia, damping, and stiffness of the human-exoskeleton system. Here, subscripts h and r denote human and robot components, respectively.

Assuming the robot-human shoulder joint is aligned and the interaction torque τ_e is previewable, satisfying Assumption 2, the control input u functions as either a spring or damping force to keep system states within safety boundaries, specifically ensuring $|e(t)| \le \delta$. When the human-robot system (19) is transformed into the state space form of the predictor system[3] as described in (2), its state is represented by $z(t) = \begin{bmatrix} e^T(t + T_i) & \dot{e}^T(t + T_i) \end{bmatrix}^T$. The previewable disturbance for this system is given by $d(t + T_i) = \tau_e(t + T_i)$ and the matrices for this system are:

$$A = \begin{bmatrix} 0 & 1 \\ -I_j^{-1} K_j & -I_j^{-1} B_j \end{bmatrix}, B = \begin{bmatrix} 0 \\ I_j^{-1} \end{bmatrix}, B_d = \begin{bmatrix} 0 \\ I_j^{-1} \end{bmatrix},$$

$$B_w = \begin{bmatrix} 0 \\ I_j^{-1} \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The output $y(t) = e(t + T_i)$ must satisfy $|e(t + T_i)| \le \delta$, and the control input is bounded as $|u(t)| \le u_m$.

In this study, the simulation parameters are: $I_j = 1\,Nms^2/rad$, $B_j = 2\,Nms/rad$, $K_j = 2\,Nm/rad$, $\tau_e(t) = 0.43\sin(0.2\pi t)\,Nm$, and $\delta = 0.2\,rad$. The constant or fixed preview horizon and input delay times are set at $T_p = 10\,ms$ and $T_i = 8\,ms$, respectively. Input delays in robotic systems are typically in the order of milliseconds [15], but since our shoulder robot system in [16] is not commercial and operates at a sample rate of $4\,ms$, for this simulation, we have chosen a fixed time delay of $8\,ms$, which is double the sample rate of our system. The control input bound u_m is varied between 1.119 to 2.0 to analyze its effect on proposed LPrev-CBF approach in Section III, in comparison to the standard CBF approach in [2], and our prior Prev-CBF [10], which does not impose limitations on the preview horizon. For simplicity, the nominal controller is zero, meaning that $u(t)$ also indicates the safety controller's intervention. The rationale behind selecting u_m from 1.119 is to illustrate an instance in which the standard CBF fails to be safe.

[3]Note that we directly use the predictor system since $z(t)$ is exactly known under Assumption 2 and also such that we can compare with other related approaches in the literature that do not consider input delays. Moreover, using this system for both examples allows us to illustrate the benefits of preview even in the absence of input delays.

1) LPrev-CBFs: The LPrev-CBF, described in Section III, ensures safety and controlled invariance for the shoulder robot, requiring $|y(t)| = |Cz(t)| \le y_m = \delta$. The control input $u(t)$ adheres to the constraint $|u(t)| \le u_m$. We specifically employ the optimization-based safety controller from Proposition 2, with the nominal control input set to zero, denoted as $k(z, t) = 0$. The closed-form LPrev-CBF is further elaborated in Lemmas 1 and 2, and Proposition 1.

2) Standard CBFs: We then compare our approach with the standard CBF from [2], specifically from the lane keeping example in [2, Section V-B]. Besides the output constraint:

$$|y(t)| = |e(t + T_i)| = |Cz(t)| \le y_m,$$

it assumes a bounded output acceleration:

$$|\ddot{y}(t)| = |\ddot{e}(t + T_i)| = |C\ddot{z}(t)| \le a_{\max},$$

which, given input constraints $|u| \le u_m$, is inherently bounded by the dynamics in (19):

$$u = I_d \ddot{e} + B_d \dot{e} + K_d e - \tau_e.$$

Then, using worst-case bounds on $\ddot{e}$, $\dot{e}$, e, and τ_e given by $\ddot{e}_{\max} = a_{\max}$, $\dot{e}_{\max}$, $e_{\max} = y_m$, and $\tau_{e,\max}$ (from Assumption 2), respectively, the triangle inequality gives:

$$|u| \le I_d a_{\max} + B_d \dot{e}_{\max} + K_d e_{\max} + \tau_{e,\max} \triangleq u_m.$$

The output constant acceleration bound $a_{\max}$ is then:

$$a_{\max} = I_d^{-1}(u_m - B_d \dot{e}_{\max} - K_d e_{\max} - \tau_{e,\max}). \quad (20)$$

To meet this bound, the standard CBF method's control input must be:

$$u(t) \in [-I_d a_{\max} + F_0(t), I_d a_{\max} + F_0(t)], \quad (21)$$

with $F_0(t) \triangleq B_d \dot{e}(t) + K_d e(t) - \tau_e(t)$.

The standard lane keeping CBF approach from [2, Section V-B] proposes the following CBF:

$$h(z) = (y_m - sgn(\dot{y}(t))y(t)) - \frac{\dot{y}(t)^2}{2a_{\max}}. \quad (22)$$

In our simulation, parameters are set as: $\dot{e}_{\max} = 0.1326$, $e_{\max} = 0.2$, and $d_m = 0.43$. By adjusting the input bounds u_m between 1.119 and 2.0, we proportionally modify a_{max} within the range of 0.0238 to 0.9048 to meet the conditions of (20).

3) Prev-CBFs [10]: We further compared between the proposed LPrev-CBF with $T_p = 10\,ms$ and our prior Prev-CBF in [10], where the preview horizon T_p is unconstrained and unlimited, implying full knowledge of previewable disturbances throughout the entire future horizon.

Figure 1 shows the simulated output and input trajectories under various conditions: without a safety controller, with the standard CBF in [2], with Prev-CBF in [10], and with LPrev-CBF, for $u_m = 1.119$ and $u_m = 1.8$. Without safety measures, the safety constraint (depicted by black dashed lines) is violated. By contrast, the standard CBF, Prev-CBF, and LPrev-CBF ensure safety. Notably, with a smaller u_m, the standard CBF deviates significantly from the trajectory without safety, while Prev-CBF and LPrev-CBF remain closer. This distinction is also evident in the (intervention) input trajectories. The standard CBF intervenes earlier and more aggressively, while Prev-CBF and LPrev-
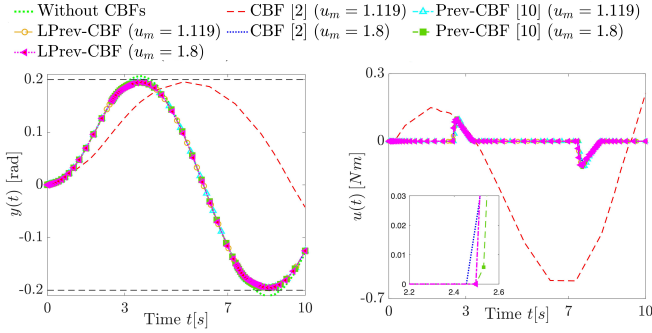
Fig. 1: Angular error (**left**) and input (**right**) trajectories: (i) Without CBFs (exceeds black dashed bounds), (ii) standard CBF [2] with $u_m = 1.119$, (iii) Prev-CBF [10] with $u_m = 1.119$, (iv) LPrev CBF with $u_m = 1.119$, (v) standard CBF [2] with $u_m = 1.8$, (vi) Prev-CBF [10] with $u_m = 1.8$, and (vii) LPrev CBF with $u_m = 1.8$. Furthermore, an enlarged segment in the (**right**) plot elucidates the initial intervention disparities for all three conditions when $u_m = 1.8$.
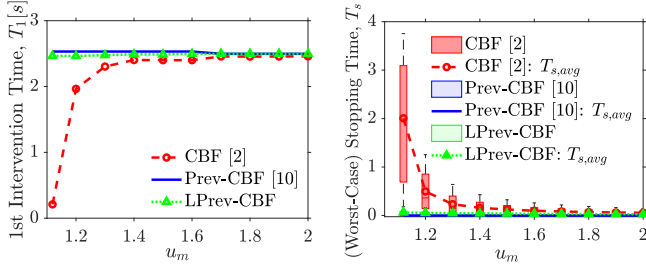

Fig. 2: Intervention times (**left**) and stopping times (**right**, shown as box plots) vary with different values of u_m between 1.119 and 2. The standard CBF [2] typically acts sooner and has longer stopping times compared to LPrev-CBF. Conversely, Prev-CBF [10] shows late intervention and exhibits shorter stopping times than LPrev-CBF. However, these differences become less pronounced as u_m is increased.

CBF operates primarily near the safety limits. For $u_m = 1.8$, interventions from all methods are minimal, although the standard CBF still intervenes sooner than Prev-CBF and LPrev-CBF.

We also assessed the effect of varying u_m on the initial intervention time T_1, i.e., the first time instance with non-zero input. A T_1 closer to 3.1643 s (the violation time without CBFs) indicates later safety intervention, implying a less conservative safety controller. Additionally, we examined the influence of u_m on the (worst-case) stopping time T_s. As seen in Figure 2, LPrev-CBFs intervene later and have lower (worst-case) stopping times than standard CBFs, suggesting their superior utilization of preview information, resulting in less conservatism. However, when compared with Prev-CBFs, the LPrev-CBFs intervene earlier and have a little larger (worst-case) stopping times than Prev-CBFs, which is as expected since the preview information is more limited with LPrev-CBFs. In summary, the results affirm that even with a limited preview horizon, our proposed LPrev-CBF ensures system safety with reduced conservatism.

B. Lane Keeping with Road Curvature Preview

Next, we consider the lane-keeping example of lateral positioning of a vehicle when limited preview of the road curvature is available. Specifically, we are inspired by the
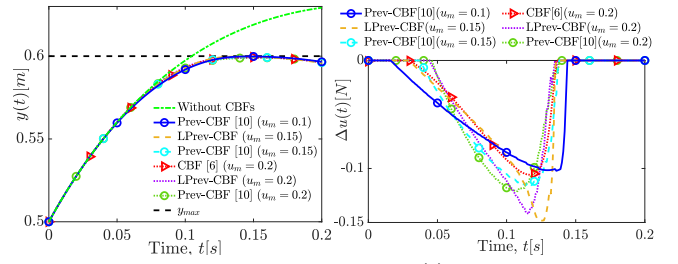

Fig. 3: Lateral displacement trajectories $y(t)$ (**left**) and trajectories of CBF intervention of $u(t)$ (**right**) given by $\Delta u(t) \triangleq u(t) - k(x(t), t)$, where $k(x(t), t)$ is the legacy controller.

lane-keeping problem in [2, Section V-B], and the relevant predictor state dynamics[3] can be written as in (2) with

$$A = \begin{bmatrix} 0 & 1 & v_0 & 0 \\ 0 & -\frac{C_f + C_r}{M v_0} & 0 & \frac{bC_r - aC_f}{M v_0} - v_0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{bC_r - aC_f}{I_z v_0} & 0 & -\frac{a^2 C_f + b^2 C_r}{I_z v_0} \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{C_f}{M} \\ 0 \\ a\frac{C_f}{I_z} \end{bmatrix},$$

$$B_d = \begin{bmatrix} 0 & 0 & -1 & 0 \end{bmatrix}^\top, C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},$$

with states $x \triangleq [y, \nu, \psi, r]^\top$, representing lateral velocity ν, lateral displacement y, yaw rate r and error yaw angle ψ. In this specific setup, the steering angle of the front tire serves as the input u to our system, whereas the desired yaw rate, $r_d = \frac{v_0}{R}$, is the disturbance with constant longitudinal velocity v_0 and (unknown but previewable) road curvature R. In this example, we considered a sinusoidal r_d as disturbance, and the known signals and system parameters are: Vehicle mass $M = 1650 \, kg$, distances of rear and front wheels from center of mass $b = 1.59 \, m$ and $a = 1.11 \, m$, respectively, rear and front tire stiffness parameters $C_r = 133000 \, N/rad$ and $C_f = 98800 \, N/rad$, respectively, and the vehicle moment of inertial with respective its center of mass $I_z = 2315.3 \, kgm^2$, taken from [2, Section V-B]. Further, the system has a constant input delay of $T_i = 10 \, ms$ and a preview of road curvature for a constant preview time $T_p = 20 \, ms$ along with the initial predicted state $z(0) = [0.5, 1.2, 0, 0]^\top$.

Next, for stabilizing the vehicle in the center of the lane, we employ a nominal controller $k(z, t) = K(z_{ff} - z)$, with $z_{ff} = [0 \ 0 \ 0 \ r_d]^\top$ and $K = [0.091 \ 0.022 \ 1.419 \ 0.078]$. Safety here constitutes adhering to the lane boundary constraint $|y| \le y_m$, where y_m is chosen as 0.6 m in this example. Further, the input constraint is $|u| \le u_m$, where we consider three distinct values of u_m for comparison.

1) Limited Preview CBFs: We apply the proposed LPrev-CBF in (13) in Lemma 1 within an optimization-based framework in Proposition 2 to this problem and also compare its performance with our prior design, Prev-CBF [10], with unlimited preview and with the standard CBF [2] that does not consider preview.

2) Preview CBFs: For comparison, we consider Prev-CBF that we previously proposed in [10], where a sufficiently large or "infinite" preview horizon T_p is assumed and the rest of the simulation parameters for Prev-CBF are kept the same as for the proposed LPrev-CBF.

3) Standard CBF: Lastly, we also consider the Standard Lane-Keeping CBF proposed in [2, Section V-B] given in

(22), where y_m represents half the lane width and $\dot{y}$ the lateral velocity, calculated from $\dot{y}(t) = \nu + \psi v_0$ derived from vehicle dynamics. Using the relationship between $\ddot{y}(t)$ and u derived from the lane-keeping dynamics,

$$\ddot{y} = \frac{C_f u - F_0}{M},$$

where $F_0 \triangleq C_f \frac{\nu + ar}{v_0} + C_r \frac{\nu - br}{v_0} + M v_0 r_d$ such that $|F_0| \leq F_{0,\max}$ with a known $F_{0,\max}$, we can formulate the acceleration limits $a_{\max}$ as

$$a_{\max} = \tfrac{1}{M}(C_f u_m - F_{0,\max})$$

and with the control input that satisfies:

$$u(t) \in [\tfrac{1}{C_f}(-M a_{\max} + F_0(t)), \tfrac{1}{C_f}(M a_{\max} + F_0(t))]. \quad (23)$$

As evident from Figure 3 (left), in the absence of any CBFs, the vehicle with just the nominal controller violates the lateral safety condition, whereas the proposed LPrev-CBF under input constraints $u_m \in \{0.2, 0.15\}$, Prev-CBF [10] under input constraints $u_m \in \{0.2, 0.15, 0.1\}$ and the Standard CBF [2] under input constraints $u_m = 0.2$ ensure that the vehicle stays within its lane. Moreover, from our simulations, we observe that the approach with the longest preview, i.e., Prev-CBF that has unlimited preview, can remain safe with the least control limit, $u_m \geq 0.09$, while the standard CBF that does not utilize preview requires the highest control authority, $u_m \geq 0.18$, to remain safe; thus, preview is clearly advantageous. On the other hand, the proposed LPrev-CBF provides a middle ground where preview is available but limited, and can maintain safety with a smaller input bound, $u_m \geq 0.14$, than the standard CBF. In other words, there is some form of partial ordering of the 3 approaches: Prev-CBF > LPrev-CBF > Standard CBF in terms of minimum control authority needed for safety.

Further, from Figure 3 (right), it can be observed that for a given controller (e.g., Prev-CBF with $u_m \in \{0.2, 0.15, 0.1\}$), the greater the actuation/control authority is, the lesser the safety controller needs to intervene against the nominal controller, i.e., the intervention time is reduced. Similarly, for a fixed actuation limit (e.g., $u_m = 0.2$), Prev-CBF interferes less than the proposed LPrev-CBF that in turn interferes less than the standard CBF.

Discussion of Results. To summarize, the examples presented along with the proposed approaches emphasize the value of preview information even if only limited preview is available and also when there is input delay. This provides broader actuation authority range for other control goals, e.g., for maximizing performance, and causes less overall interference against the nominal controller for safety when compared to the cases where preview information is not utilized. Further, with the increase in actuation limit u_m, the value or advantages of the preview information decreases since the input range becomes large enough to counter effects of relatively smaller worst-case disturbances. The finding of this work confirms findings of [7] about the value of preview for discrete-time systems and also the findings of [10] about the value of preview for continuous-time linear systems.

V. Conclusion

In this paper, we introduced a limited preview control barrier function (LPrev-CBF) for linear continuous-time input-delay systems where the preview horizon for the previewable disturbances is limited and fixed, e.g., due to limited sensing ranges. In contrast to the standard CBF approach that simply considers worst-case disturbances, our approach can leverage preview information to reduce conservatism, while avoiding the assumption in the Prev-CBF approach that the disturbances are previewable for an infinite horizon. Further, our LPrev-CBF explicitly takes input constraints/bounds into consideration and thus, it naturally has recursive feasibility/safety guarantees. Future directions include the extensions of limited preview CBFs to consider preview horizons that may be state- or time-dependent as well as the presence of non-previewable uncertainties/disturbances.

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