2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

Limited Preview Control Barrier Functions for Continuous-Time Nonlinear Systems with Input Delays

Tarun Pati, Seunghoon Hwang and Sze Zheng Yong

2024 IEEE 63rd Conference on Decision and Control (CDC) | 979-8-3503-1633-9/24/$31.00 ©2024 IEEE | DOI: 10.1109/CDC56724.2024.10886357

***Abstract*— Control systems can often forecast/predict future disturbances, such as road curvatures, yet this lookahead or preview data is seldom utilized for safety critical control when designing control barrier functions (CBFs). This paper extends the recent limited preview control barrier function for linear systems with input delays to a class of nonlinear input-delay systems, which similarly leverage preview information for a limited preview time horizon to provide less conservative safety guarantees than traditional CBF methods. To achieve this extension, we propose two algorithmic linearization methods, namely affine abstractions and approximate linear immersions, with rigorous approximation error characterization and then, we take this error into consideration in the proposed limited preview nonlinear CBF. Further, our approach explicitly incorporate input bounds; thus, recursive feasibility of its corresponding optimization-based safety controller is guaranteed.**

I. Introduction

Many control systems such as self-driving cars have forward looking sensors, e.g., cameras and Lidar, that can be used for providing information about upcoming conditions for improving performance and safety. While optimal and model predictive control approaches have leveraged such preview information for improved performance [1], [2], safety control methods have only recently begun to consider the problems associated with recursive assurances of safety and feasibility. Prior to this, most traditional safety control approaches, including using the control barrier function (CBF) framework (e.g., [3], [4]), consider worst-case robustness to guarantee recursive safety and are often overly conservative.

Recent studies have demonstrated that incorporating preview information into safety control of certain classes of discrete-time systems, including those with input delays, can yield significant advantages [5], [6]. For continuous-time systems, a predictive CBF has been proposed in [7] when the preview comes from changeable reference trajectories, while our recent work developed a preview CBF approach that utilizes information on previewable but uncontrollable disturbances, such as road gradients, curvatures, or the predicted future motion of other agents [8], [9]. Further, this work is extended to linear continuous-time input-delay systems, where the preview horizon for the previewable disturbances is limited and fixed, which better reflects real-world settings where sensing ranges are limited [9]. However, these techniques are only applicable to uncertain linear systems.

T. Pati and S.Z. Yong are with the Mechanical and Industrial Engineering Department, Northeastern University, Boston, MA ({pati.ta,s.yong}@northeastern.edu); S. Hwang is with the School for Engineering of Matter, Transport and Energy, Arizona State University, Tempe, AZ (shwang45@asu.edu). This work was supported in part by NSF grants CMMI-1925110, CNS-2312007 and CNS-2313814.

Our approach for handling this challenge is to over-approximate nonlinear systems as uncertain linear systems with bounded approximation errors. Thus, a relevant body of literature pertains to that of abstraction of nonlinear systems as (piecewise) affine systems, also known as hybridization [10] or affine abstractions [11], [12]. Yet another related approach is that of approximate state immersions [13], [14] that lift the nonlinear dynamics (without control inputs) to a higher dimensional approximate linear model, which we will build upon to allow control inputs.

Contributions. In this paper, we present a control barrier function for a class of nonlinear continuous-time systems with input delays and limited-horizon preview (LPrev-nCBF), as an extension of previous designs for linear systems in [8], [9]. One key challenge for this extension is our requirement of a closed-form LPrev-nCBF solution that in turn requires a closed-form expression for state solutions. Thus, we first propose two algorithmic linearization methods for over-approximating the nonlinear system dynamics with uncertain linear dynamics with a careful characterization of the approximation errors. Specifically, we propose to leverage affine abstraction methods in [11], [12] for this linearization process, as well as introduce an approximate linear immersion approach that extends the approach in [14] to lift the nonlinear dynamics with control based on its relative degree. By design, these approaches simultaneously compute and minimize the linearization error bounds.

Then, using this linearized system (with potentially higher dimension) and the linearization error bounds, we present a closed-form input-constrained LPrev-nCBF that is robust to the linearization errors. Since the proposed LPrev-nCBF incorporates the knowledge of input bounds, recursive feasibility and safety of the associated optimization-based safety controller can be guaranteed. The efficacy of the proposed LPrev-nCBF is demonstrated using a vehicle lane-keeping scenario with road curvature as the *previewable* disturbance.

II. Preliminaries and Problem Formulation

Notations. $\mathbb{R}^n$ and $\mathbb{R}_+$ refer to the n-dimensional Euclidean space and the set of non-negative real numbers. All vector inequalities represent element-wise inequalities. Further, $\text{sgn}(\cdot)$ and $|\cdot|$ are signum and element-wise absolute value operators, and a diagonal matrix $\text{diag}(v)$ has v as its diagonal elements. Additionally, a function $\alpha : [0, \infty) \to [0, \infty)$ belongs to the class $\mathcal{K}_\infty$ if it is continuous and strictly increasing, with $\lim_{r\to\infty} \alpha(r) = \infty$ and $\alpha(0) = 0$.

System Model. Consider a class of continuous-time nonlinear control system with a time-delayed and linear control input, along with previewable disturbances:

$$\Sigma_{delay}: \dot{x}(t) = f(x(t), d(t)) + Bu(t - T_i), \qquad (1)$$

with state $x(t) \in \mathcal{X} \subseteq \mathbb{R}^n$, input $u(t) \in \mathcal{U} \subseteq \mathbb{R}^m$ subject to constant time delay T_i, bounded and previewable (only for a constant and limited preview horizon T_p) disturbances $d(t) \in \mathcal{D} \subseteq \mathbb{R}^l$. $\mathcal{D} \triangleq \{d \mid |d| \leq d_m\}$ and $\mathcal{U} \triangleq \{u \mid |u| \leq u_m\}$ are bounded sets[1] for the disturbance and input, respectively.

The term "previewable disturbance" refers to external inputs, signals, or parameters whose future values can be predicted or measured by sensors or perception modules with limited range. This encompasses, for instance, a target signal for tracking, anticipated paths of other entities, and road characteristics like curvature, slope, or friction coefficients.

The system variable of interest is represented by a scalar output $y(t) = Cx(t) \in \mathbb{R}$ and its corresponding constraint $S_x = \{x \in \mathcal{X} \mid |Cx| \leq y_m\}$ represents desired *safety constraint*. Further, without loss of generality, we assume that the system Σ_{delay} with this output has a relative degree of 2 with respective to the input, i.e., $CB = 0$. Additionally, we assume that the relative degree r of $f(x, d)$ with respect to u is constant. Further, as in [9], we consider the setting where the preview horizon T_p is limited and fixed, beyond which the *unpreviewed* disturbance is unknown but bounded.

Assumption 1. *The constant preview horizon T_p satisfies $T_p > T_i$, where T_i represents a constant input-delay time.*

Assumption 2. *For a known and fixed preview horizon T_p and given time $t \in \mathbb{R}_+$, the previewed disturbance $\mathbf{d}_p(t) \triangleq \{d(\tau) \in \mathcal{D}, t \leq \tau < t + T_p\}$ is known and beyond this preview horizon, the* unpreviewed *disturbance $\mathbf{d}_{np}(t) \triangleq \{d(\tau) \in \mathcal{D}, t + T_p \leq \tau < \infty\}$, is unknown but bounded (with known bounds).*

Inspired by the literature on time-delay systems and our prior work on preview based safety critical control, we consider the problem in terms of a predictor system based on the predicted state $z(t) = x(t + T_i)$:

$$\Sigma_{pred}: \dot{z}(t) = f(z(t), d(t + T_i)) + Bu(t). \qquad (2)$$

Note that under Assumption 1 that $T_p > T_i$, the *predicted state* $z(t)$ at any time t can be computed from $x(t)$ by using ODE/DDE solvers (for linear systems, predictions can be found in closed-form as in [9]). Thus, we can equivalently consider Σ_{pred} in lieu of Σ_{delay} owing to exact knowledge of $z(t)$ from $x(t)$ and for ease of exposition, we will directly consider the safe sets in terms of the predicted state $z(t)$.

Definition 1 (Safe Sets)**.** *Let $S_z \subseteq \mathbb{R}^n$ be a* safe set *of Σ_{pred} that describes desirable/given safety constraints on the states, and let $S_{z,p} \subseteq \mathbb{R}^n \times \mathcal{D}^{[0,T_p)}$ be the T_p*-augmented safe set *of Σ_{pred}, defined as*

$$S_{z,p} \triangleq \{(z, \mathbf{d}_p) \mid z \in S_z, \mathbf{d}_p \in \mathcal{D}^{[0,T_p)}\},$$

where $\mathcal{D}^{[0,T_p)}$ is the set of all trajectories of $d(\tau)$ within the time interval of $[0, T_p] \triangleq \{\tau | 0 \leq \tau < T_p\}$, defined as,

$$\mathcal{D}^{[0,T_p)} \triangleq \{d(\tau), \forall \tau \in [0, T_p] \mid d(\tau) \in \mathcal{D}\}.$$

[1] For ease of exposition, symmetric bounds are assumed. Any asymmetric bounds can be handled by taking their midpoints as known signals and deviations from these midpoints as signals with symmetric bounds.

Definition 2 (Controlled Invariant Set)**.** *A set $\mathcal{C} \subseteq S_z$ is a* robust controlled invariant set *of Σ_{pred} in a safe set $S_z \subseteq \mathbb{R}^n$ if for all $z(0) \in \mathcal{C}$, there exists some $u(t) \in \mathbb{R}^m$ such that $z(t) \in \mathcal{C} \subseteq S_z$, $\forall t \geq 0$ for all $d(t) \in \mathcal{D}$. $\mathcal{C}_{max}$ is* the maximal robust controlled invariant set *in S_z if $\mathcal{C}_{max}$ contains all robust controlled invariant sets in S_z.*

Further, a set $\mathcal{C}_p \in S_{z,p}$ is a limited preview controlled invariant set *of Σ_{pred} in an augmented safe set $S_{z,p}$ if for all $(z(0), \mathbf{d}_p(0)) \in \mathcal{C}_p$, there exists some $u(t) \in \mathbb{R}^m$ such that $(z(t), \mathbf{d}_p(t)) \in \mathcal{C}_p \subseteq S_{z,p}$, $\forall t \geq 0$ for all $\mathbf{d}_{np} \in \mathcal{D}^{[T_p,\infty)}$. $\mathcal{C}_{max,p}$ is* the maximal limited preview controlled invariant set *in $S_{z,p}$ if $\mathcal{C}_{max,p}$ contains all limited preview controlled invariant sets of Σ_{pred} in $S_{z,p}$.*

Specifically, we aim to find limited preview control barrier functions with preview capabilities for Σ_{pred} that render some time-varying set $\mathcal{C}_{z,t} \subseteq S_z$ controlled invariant. To achieve this, we introduce a time-varying 'limited preview safe set', denoted as $\mathcal{C}_{z,p,t} \subseteq S_{z,p}$, which is not only controlled invariant but also implies the existence of some $\mathcal{C}_{z,t} \subseteq S_z$ that is controlled invariant by construction/design.

Definition 3 (Limited Preview Safe Set)**.** *Given a predictive system with preview Σ_{pred} (with known $\mathbf{d}_p \in \mathcal{D}^{[0,T_p)}$ and unknown $\mathbf{d}_{np} \in \mathcal{D}^{[T_p,\infty)}$), a super-level set $\mathcal{C}_{z,p,t}$ defined on a time-varying function $h : \mathcal{X} \times \mathcal{D}^{[0,T_p)} \times \mathbb{R}_+ \to \mathbb{R}$:*

$$\mathcal{C}_{z,p,t} \triangleq \{(z, \mathbf{d}_p, t) \mid h(z, \mathbf{d}_p, t) \geq 0, \}, \qquad (3)$$

which is defined based on another function h_{np} according to $h(z, \mathbf{d}_p, t) \triangleq \min_{\mathbf{d}_{np} \in \mathcal{D}^{[T_p,\infty)}} h_{np}(z, \mathbf{d}_p, \mathbf{d}_{np}, t)$ with $\mathcal{D}^{[T_p,\infty)}$ being the set of all trajectories of $d(t)$ starting from T_p, is a limited preview safe set *for Σ_{pred} if $(z(t), \mathbf{d}_p(t), t) \in \mathcal{C}_{z,p,t}$ for all $t \geq 0$ implies that $z(t) \in S_z$ for all $t \geq 0$..*

Note that, by design, the limited preview robust safe set in the above definition needs to be defined or chosen such that its controlled invariance implies the existence of some $\mathcal{C}_{z,t} \subseteq S_z$ that is controlled invariant. Further, note that while $z(t)$ can be computed from $x(t)$ using (2), predictions of $z(\tau)$ for $\tau \geq T_p - T_i$ are unknown and cannot generally be obtained in closed-form unlike in linear systems as in [9]; thus, in order to leverage the tools developed in [9], our first problem of interest is to over-approximate the nonlinear system by a linearized model with known error bounds.

Problem 1 (Algorithmic Linearization)**.** *Given an input-delay system with preview Σ_{delay} in (1) satisfying Assumptions 1–2, its corresponding equivalent predictor system Σ_{pred} in (2) and a safe set S_z (cf. Definition 1), design linearization algorithms that return A, B_L, B_d^i, $i \in \{0, \ldots, r\}$ and e_m to obtain a linearized (immersion) model:*

$$\Sigma_{lin}: \dot{z}_L(t) = A_L z_L(t) + \textstyle\sum_{i=0}^{r} B_d^i d^{(i)}(t + T_i) + B_L u(t) + e(t), \qquad (4)$$

that minimizes the linearization error $J(e_m)$, where $z_L(t)$ is the lifted predicted state, r is the relative degree of $f(x, d)$ with respect to u, $d^{(i)}$ is the i-th time derivative of d and e is the linearization error such that $|e| \leq e_m \in \mathbb{R}^n$. In the special case without lifting, $z_L(t) = z(t)$ and $r = 0$.

Then, our second problem is to construct the input-constrained limited preview (nonlinear) control barrier function (LPrev-nCBF) that is robust against linearization errors:

Problem 2 (Safety with Limited Preview)**.** *Given an input-delay system with preview Σ_{delay} in* (1) *satisfying Assumptions 1–2, its corresponding linearized system Σ_{lin} in* (4) *and a safe set S_z (cf. Definition 1), construct a limited preview (nonlinear) control barrier function (LPrev-nCBF) corresponding to $\mathcal{C}_{z,p,t}$ in* (3) *that guarantees limited preview controlled invariance of Σ_{lin} in S_z (and thus, safety of Σ_{delay} under Assumption 1 and the over-approximation/simulation property of Σ_{lin}).*

III. Main results

Section III-A introduces two algorithmic linearization methods to address Problem 1, while Section III-B presents the LPrev-nCBF that solves Problem 2 using the linearized models and their linearization error bounds from Problem 1.

A. Algorithmic Linearization Methods

As above-mentioned, closed-form predicted states are non-trivial to obtain for nonlinear systems; hence, this section presents two linearization algorithms to solve Problem 1 that compute and minimize linearization error bounds.

1) Affine Abstraction: This first method builds upon the affine abstraction approach from [11], [12] to abstract/overapproximate the nonlinear function $f(z,d)$ in (2) within a given domain of $z \in \mathcal{X}$ and $d \in \mathcal{D}$ by a pair of affine hyperplanes/functions $\underline{f}$ and $\overline{f}$ such that $\underline{f}(z,d) \leq f(z,d) \leq \overline{f}(z,d)$ for all $z \in \mathcal{X}$ and $d \in \mathcal{D}$ with:

$$\begin{aligned} \underline{f}(z,d) &= Az + B_d d + \underline{e}_\ell, \\ \overline{f}(z,d) &= Az + B_d d + \overline{e}_\ell, \end{aligned} \tag{5}$$

and to-be-determined matrices A, B_d and vectors $\underline{e}_\ell$, $\overline{e}_\ell$ of appropriate dimensions. The following algorithm allows us to find these matrices and vectors that minimizes the magnitude of the linearization error bound $e_{\ell,m} = \frac{1}{2}(\overline{e}_\ell - \underline{e}_\ell)$ given by $\|e_{\ell,m}\|_\infty = \max_i e_{\ell,m,i}$.

Proposition 1 (Affine Abstraction [11])**.** *Given the function $f : \mathcal{X} \times \mathcal{D} \subset \mathbb{R}^{(n+l)} \to \mathbb{R}^n$ and the set $\mathcal{M}$ of (finite) mesh/grid points of $\mathcal{X} \times \mathcal{D}$. Suppose $A, B_d, \overline{e}, \underline{e}, \theta$ are solutions to the following linear program (LP):*

$$\begin{aligned} &\min_{\theta, A, B_d, \overline{e}, \underline{e}} \theta \\ &s.t\ Az_s + B_d d_s + \underline{e}_\ell + \sigma \leq f(z_s, d_s) \leq Az_s + B_d d_s + \overline{e}_\ell - \sigma, \\ &\quad \overline{e}_\ell - \underline{e}_\ell - 2\sigma \leq \theta \mathbf{1}_n,\ \forall (z_s, d_s) \in \mathcal{M}, \end{aligned} \tag{6}$$

where $\mathbf{1}_n \in \mathbb{R}^n$ is a vector of ones and σ can be computed via [11, Proposition 1] for different function classes. Then,

$$f(z,d) = Az_s + B_d d + e_c + e_\ell, \forall (z,d) \in \mathcal{X} \times \mathcal{D},$$

with a constant $e_c = \frac{1}{2}(\overline{e}_\ell + \underline{e}_\ell)$ and a linearization error e_ℓ satisfying $\|e_\ell\| \leq e_{\ell,m} = \frac{1}{2}(\overline{e}_\ell - \underline{e}_\ell)$.

Proof. The proof is a slight modification of the one in [11, Theorem 1] with $\overline{A} = \underline{A} = A$ and $\overline{B} = \underline{B} = B_d$, as well as with the interpolation error σ directly incorporated into the linear program. □

From Proposition 1, we obtain an affine abstraction-based linearized model of the nonlinear predictor system Σ_{pred}:

$$\Sigma_{abs}\colon \dot{z}(t) = Az(t) + Bu(t) + B_d d(t+T_i) + e_c + e_\ell(t), \tag{7}$$

with a constant $e_c = \frac{1}{2}(\overline{e}_\ell + \underline{e}_\ell)$ and a linearization error satisfying $|e_\ell(t)| \leq e_{\ell,m} = \frac{1}{2}(\overline{e}_\ell - \underline{e}_\ell)$.

2) Approximate Linear Immersion: The affine abstraction-based linearization method may sometimes lead to poor over-approximation of the nonlinear system and, in turn, a conservative LPrev-nCBF. Thus, we propose a second method for linearization that is an extension of the approximate linear immersion from [13], [14] to systems with control inputs. This approach involves lifting the nonlinear system to higher dimensions and applying linearization to the higher order derivative of the nonlinear function $f(x,d)$ in (2).

Specifically, inspired by [14], our approach involves finding an approximate linear immersion of the nonlinear function $f(x,d)$ in (2) with an order that is equal to the relative degree r of f with respect to u, i.e., $\frac{\partial}{\partial x}(L_f^i f)B = 0$ for all $i < r-1$ and $\frac{\partial}{\partial x}(L_f^i f)B \neq 0$ for $i = r-1$ with L_f being the Lie derivative with respect to f. The r-th order approximate linear immersion is defined as

$$\begin{aligned} f^{(r)}(z, \mathbf{q}^r, u) = \textstyle\sum_{l=0}^{r-1} \Gamma_l f^{(l)}(z, \mathbf{q}^{r-1}) + A_\ell z \\ + B_{d,\ell}\mathbf{q}^r + B_{u,\ell} u + e_c + e_\ell, \end{aligned} \tag{8}$$

with $\mathbf{q}^i = [d, \dot{d}, \ddot{d}, \ldots, d^{(i)}]$, $i \in \{1, \ldots, r\}$, $|e_\ell| \leq e_{\ell,m}$, and to-be-determined matrices $\Gamma_1, \ldots, \Gamma_{r-1}, A_\ell, B_{d,\ell}, B_{u,\ell}$ and vectors e_c, $e_{\ell,m}$ of appropriate dimensions. The following algorithm allows us to find these matrices and vectors that minimize the magnitude of the linearization error bound $e_{\ell,m}$ given by $\|e_{\ell,m}\|_\infty = \max_i e_{\ell,m,i}$.

Proposition 2 (Approximate Linear Immersion)**.** *Given the function $f : \mathcal{X} \times \mathcal{Q} \to \mathbb{R}^n$ and the set $\mathcal{M}_I$ of (finite) mesh/grid points of the domain $\mathcal{D} \triangleq \mathcal{X} \times \mathcal{Q} \times \ldots \times \mathcal{Q}^{(r)} \times \mathcal{U}$. Suppose $\Gamma_1, \ldots, \Gamma_{r-1}, A_\ell, B_{d,\ell}, B_{u,\ell}, e_c, e_{\ell,m}, \theta_I$ are solutions to the following linear program (LP):*

$$\begin{aligned} &\min_{\substack{\Gamma_1, \ldots, \Gamma_{r-1}, \Gamma_1^\oplus, \ldots, \Gamma_{r-1}^\oplus, \Gamma_1^\ominus, \ldots, \Gamma_{r-1}^\ominus, \\ A_\ell, B_{d,\ell}, B_{u,\ell}, e_c, e_{\ell,m}, \theta_I}} \theta_I \\ &s.t \quad -e_{\ell,m} + \sigma_I \leq f^{(r)}(z_s, \mathbf{q}_s^r, u_s) - B_{d,\ell}\mathbf{q}_s^r - B_{u,\ell} u_s \\ &\qquad - \textstyle\sum_{l=0}^{r-1} \Gamma_l f^{(l)}(z_s, \mathbf{q}_s^{r-1}) - A_\ell z_s - e_c \leq e_{\ell,m} - \sigma_I, \\ &\quad \sigma_I = \sigma_r + \textstyle\sum_{l=0}^{r-1} (\Gamma_l^\oplus + \Gamma_l^\ominus)\sigma_l, \\ &\quad \Gamma_l = \Gamma_l^\oplus - \Gamma_l^\ominus, \Gamma_l^\oplus \geq 0, \Gamma_l^\ominus \geq 0, \\ &\quad \overline{e}_\ell - \underline{e}_\ell - 2\sigma_I \leq \theta_I \mathbf{1}_n,\ \forall (z_s, \mathbf{q}_s^r, u_s) \in \mathcal{M}_I, \end{aligned} \tag{9}$$

where $\mathbf{1}_n \in \mathbb{R}^n$ is a vector of ones and σ_r can be computed via [11, Proposition 1] for different function classes for each nonlinear term $f^{(r)}$. Then, (8) *holds for all $(z, \mathbf{q}^r, u) \in \mathcal{D}$.*

Proof. The first constraint in the optimization problem (9) ensures that the upper and lower hyperplanes, $A_\ell z_s + B_{d,\ell}\mathbf{q}_s^r + e_c + e_{\ell,m} - \sigma_I$ and $A_\ell z_s + B_{d,\ell}\mathbf{q}_s^r + e_c + e_{\ell,m} + \sigma_I$, respectively, are always above and below $f_\ell(\cdot) \triangleq f^{(r)}(z_s, \mathbf{q}_s^r, u_s) - \sum_{l=0}^{r-1} \Gamma_l f^{(l)}(z_s, \mathbf{q}_s^{r-1})$, for all $(z_s, \mathbf{q}_s^r, u_s) \in \mathcal{M}_I$, as in Proposition 1. However, unlike $f(z,d)$ in Proposition 1, the function $f_\ell(\cdot)$ involves decision variables Γ_l's, which necessitates the use of triangle inequality to bound the interpolation error $\sigma_I = \sigma_r + \sum_{l=0}^{r-1} |\Gamma_i|\sigma_l$ in the second constraint, where $|\Gamma_i|$ denotes element-wise absolute value. Then, since $|\Gamma_i| = \Gamma_l^{\oplus} + \Gamma_l^{\ominus}$ if $\Gamma_l^{\oplus} = \max(\Gamma_l, 0)$ and $\Gamma^{\ominus} = \Gamma_l^{\oplus} - \Gamma_l$, we employ an optimization trick to avoid requiring an absolute value (that would lead to binary variables and hence, an NP-hard mixed-integer linear program) by constraining $\Gamma_l^{\oplus}, \Gamma_l^{\ominus} \geq 0$ using the third constraint and allowing the optimization solver to enforce the equivalence.

Further, we want to make the abstraction/over-approximation as tight as possible by minimizing the distance between the two abstractions $\theta = \max_{(z,d)} ||\overline{e}_l - \underline{e}_l - 2\sigma_I||_\infty$ and due to the linear nature of the upper and lower hyperplanes (the difference is either decreasing or increasing in each of the dimensions) the maximum difference happens at the grid points which leads to the final constraint in the optimization problem (8), thus completing the proof. □

Then, augmenting the predictor system Σ_{pred} with additional states $\eta_1, \eta_2, \ldots, \eta_{r-1}$ comprising $\eta_1 = f(x, d)$ and its higher-order time derivatives $\eta_i = \dot{\eta}_{i-1}$ for all $i = \{2, 3, \ldots, r-1\}$, we obtain the lifted system:

$$\dot{z}_L = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} \\ A_\ell & \Gamma_0 & \Gamma_1 & \cdots & \Gamma_{r-1} \end{bmatrix} z_L + \begin{bmatrix} B \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ B_{u,\ell} \end{bmatrix} u + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ B_{d,\ell} \end{bmatrix} \mathbf{q} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ e_{c\ell} \end{bmatrix}, \tag{10}$$

where $e_{c\ell} = e_c + e_\ell$, $\mathbf{q} = [d, \dot{d}, \ddot{d}, \ldots, d^{(r)}] \in \mathcal{Q} \triangleq \mathcal{D} \times \mathcal{D}_1 \times \ldots \times \mathcal{D}_r$ is treated as a previewable disturbance and $z_L = [z^\top, \eta_1^\top, \ldots, \eta_{r-1}^\top]^\top$ is the lifted state. Thus, we obtain an immersion-based linearized model of the nonlinear predictor system Σ_{pred} given by:

$$\Sigma_{im} : \dot{z}_L(t) = A_L z_L(t) + B_L u(t) + B_{d,L}\mathbf{q}(t + T_i) + e_{c,L} + e_{\ell,L}(t), \tag{11}$$

with a constant vector $e_{c,L} = [0^\top \; e_c^\top]^\top$ and a linearization error satisfying $|e_{\ell,L}(t)| \leq e_{\ell,L,m} = [0^\top \; e_{\ell,m}^\top]^\top$.

Finally, to compare the approximation tightness of both algorithmic linearization approaches in this section, we applied the techniques to the simulation example in Section IV. Figure 1 shows that both algorithmic linearization approaches result in upper and lower bounds that truly bound the true position e and velocity $\dot{e}$, as desired. Further, we observe that the error bounds are tighter when using the approximate linear immersion approach than with affine abstraction, demonstrating the benefits of the proposed approximate linear immersion method.

B. Limited-Horizon Preview Control Barrier Functions

In this section, we present the idea of Limited-horizon Preview Control Barrier Function for nonlinear input-delay systems (LPrev-nCBF) as an extension of LPrev-CBF in [9] to account for robustness to linearization errors.

Definition 4 (Limited-Horizon Preview Nonlinear CBF)**.** *Given the input-delay system Σ_{delay} and its corresponding predictive system Σ_{pred} with a limited-horizon previewable disturbances that satisfy Assumptions 1–2 and a safe set S_z (cf. Definition 1), the continuously differentiable function $h : \mathcal{X} \times \mathcal{D}^{[0,T_p)} \times \mathbb{R}_+ \to \mathbb{R}$ is a limited horizon preview nonlinear CBF (LPrev-nCBF) for systems Σ_{delay} and Σ_{pred} corresponding to time-varying limited preview safe set $\mathcal{C}_{z,p,t}$ in* (3)*, if $\exists\, u \in \mathcal{U}$ and $\alpha \in \mathcal{K}_\infty$ such that:*

$$\dot{h}(z, u, \mathbf{d}_p, t) \geq \alpha(h(z, \mathbf{d}_p, t)), \tag{12}$$

for all $t \geq 0$ and $z \in \mathcal{X}$. Further, for any $t \geq 0$, $z \in \mathcal{X}$ and $\mathbf{d}_p \in \mathcal{D}^{[0,T_p)}$, a corresponding safe input set is defined as:

$$K_{\mathcal{C}}(z, \mathbf{d}_p, t) = \{u \in \mathcal{U} \mid (12) \text{ holds}\}. \tag{13}$$

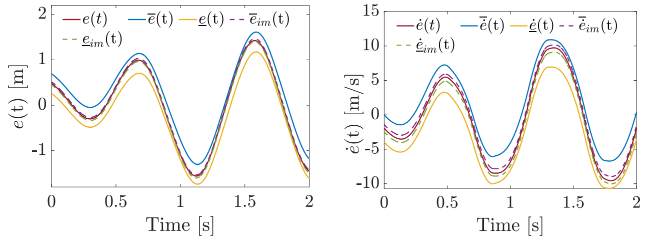

Fig. 1: The predicted states trajectories of true system, affine abstraction system, and approximate linear immersion system for e(t) (**left**) and $\dot{e}$(t) (**right**) in the simulation example in Section IV.

Theorem 1 (Safety with Limited Preview)**.** *Given the input-delay system Σ_{delay} and its corresponding predictive system Σ_{pred} with a limited-horizon previewable disturbances that satisfy Assumptions 1–2 and a safe set S_z (cf. Definition 1), if h is a LPrev-nCBF and $\mathcal{C}_{z,p,t}$ is the corresponding limited preview safety set from* (3)*, then for the nonlinear predictive system Σ_{pred} with $z(0) \in S_z$, any Lipschitz continuous controller $u(x, \mathbf{d}_p, t) \in K_{\mathcal{C}}(z, \mathbf{d}_p, t)$ with known $z(t)$ (since $T_p > T_i$) ensures the controlled invariance of the limited preview safety set $\mathcal{C}_{z,p,t}$. Consequently, there exists some set $\mathcal{C}_{z,t} \subset S_z$ for the system Σ_{pred} for which $u(x, \mathbf{d}_p, t)$ also ensures its controlled invariance. Thus, the nonlinear system with input-delay Σ_{delay} is guaranteed to be safe, i.e., $x(t) \in S_x, \forall t \geq 0$.*

Proof. If h is a LPrev-nCBF corresponding to $\mathcal{C}_{z,p,t}$ (cf. Definition 3), then any controller $u \in K_{\mathcal{C}}(z, \mathbf{d}_p, t)$ ensures the feasibility of (12) for all $z \in \mathcal{X}$, $\forall t \geq 0$; hence, $\mathcal{C}_{z,p,t}$ is forward control invariant, i.e., $h(z, \mathbf{d}_p, t) \geq 0$, $\forall t \geq 0$. Consequently, the nonlinear predictive system Σ_{pred} and the corresponding nonlinear input-delay system Σ_{delay} with preview are safe for all $t \in \mathbb{R}_{\geq 0}$ with respect to the safe set $\mathcal{C}_{z,t} \subseteq S_z$ ($\mathcal{C}_{z,t}$ exists by construction). □

1) Closed-Form Candidate Limited Preview CBF: Next, we formulate a closed-form candidate LPrev-nCBF and a corresponding limited horizon preview safe set (cf. Definition 3), by projecting z into the future by a horizon T seconds using the linear system approximation in (7) (or (11)).

$$z(t+T) = \phi(t, T) + \epsilon(t, T) + \int_0^T e^{A(T-\tau)} B u(t+\tau) d\tau, \tag{14}$$

with $\phi(t, T) \triangleq e^{AT} z(t) + \int_0^{T_\delta} e^{A(T-\tau)} B_d d(t + T_i + \tau) d\tau + \int_0^T e^{A(T-\tau)} e_c d\tau$, $T_\delta \triangleq \min(T_p - T_i, T)$ and $\epsilon(t, T) \triangleq \int_{T_\delta}^T e^{A(T-\tau)} B_d d(t + T_i + \tau) d\tau + \int_0^T e^{A(T-\tau)} e_\ell(t) d\tau$. Note that $\phi(t, T)$ can be computed at any given t, since $T_p > T_i$. Further, the proposed LPrev-nCBF is robust to the linearization errors in $\epsilon(t,T)$, and the desired safety is enforced by enforcing that the (immediate) future minima or maxima of output under maximal acceleration or deceleration, respectively, as well as worst case linearization errors, similar to [9]. The time associated with this minima or maxima, is defined as the worst-case *stopping time* T_s.

Definition 5 (Worst-Case Stopping Time)**.** *Given* $t \geq 0$ *for the predictive system* Σ_{pred} *and its corresponding approximate linear realization with fixed-horizon preview in* (7) *or* (11)*, the worst-case stopping time* $T_s(t)$ *is defined as the minimum* $T_s(t)$ *such that the worst-case output velocity* $\dot{y}_w(t+T_s(t)) = C\dot{z}_w(t+T_s(t)) = CAz_w(t+T_s(t)) = 0$ *under maximum control input acceleration and disturbance-induced deceleration when* $\dot{y}(t) = C\dot{z}(t) \leq 0$ *or maximum control input deceleration and disturbance-induced acceleration when* $\dot{y}(t) = C\dot{z}(t) \geq 0$.

From (7) (or (11)), under the relative degree 2 assumption,

$$\ddot{y}(t) = CA(Az(t)+Bu(t)+B_d d(t+T_i)+e_c+e_\ell(t)), \quad (15)$$

from which we can infer that $\dot{y}(t) \leq 0$ [2], the maximum possible acceleration is with input $u(t) = \text{diag}(\text{sgn}(CAB))u_m$ under worst-case linearization errors $e_\ell(t) = -\text{diag}(\text{sgn}(CA))e_{\ell,m}$ and $d(t) = -\text{diag}(\text{sgn}(CAB_d))d_m$, and for $\dot{y}(t) \geq 0$, the maximum possible deceleration is obtained with input $u(t) = -\text{diag}(\text{sgn}(CAB))u_m$ under worst-case linearization errors $e_\ell(t) = \text{diag}(\text{sgn}(CA))e_{\ell,m}$ and $d(t) = \text{diag}(\text{sgn}(CAB_d))d_m$. Consequently, the worst-case output $y_w(t+T_s(t)$ and worst-case output velocity $\dot{y}_w(t+T_s(t))$ and computed by applying $u(\tau) = \hat{u}(t)$ with

$$\hat{u}(t) \triangleq -\text{sgn}(\dot{y}(t))\text{diag}(\text{sgn}(CAB))u_m \quad (16)$$

under worst-case linearization error $e_\ell(\tau) = \hat{e}_\ell(t)$ and $d(\tau) = \hat{d}(t)$ and with

$$\hat{d}(t) \triangleq \text{sgn}(\dot{y}(t))\text{diag}(\text{sgn}(CAB_d))d_m, \quad (17)$$

$$\hat{e}_\ell(t) \triangleq \text{sgn}(\dot{y}(t))\text{diag}(\text{sgn}(CA))e_{\ell,m}, \quad (18)$$

for all $\tau \in [t, T_s(t)]$, resulting in, $\ddot{y}_w(\tau) = CA^2 z_w(\tau) + CAe_c - sgn(\dot{y}(t))|CAB|u_m + sgn(\dot{y}(t))|CAB_d|d_m + sgn(\dot{y}(t))|CA|e_{\ell,m}$, which is computed from (15) with $u(t)$ in (16), $d(t)$ in (17) and $h(t)$ in (18). Further, $T_s(t)$ is the solution to $\dot{y}_w(t+T_s(t)) = CAz_w(t+T_s(t)) = 0$.

Note that at given time $t \geq 0$ for a computed (known) time-varying worst-case stopping time $T_s(t)$, under Assumption 1–2 with a fixed input delay T_i and fixed preview horizon T_p, when implemented to the predictor system Σ_{pred} framework to forecast $z(t+T_s(t))$ results in two distinct cases: (i) When $T_s(t) < T_p - T_i$ (i.e., when the available preview T_p exceeds the stopping time for $z(t)$), the disturbances $d(t+T_i)$ in (2) are known/previewed for the time interval up to $t+T_s(t)+T_i$, and (ii) when $T_s(t) \geq T_p - T_i$, the previewable disturbances $d(t+T_i)$ within the time interval $t+T_p \leq \tau \leq t+T_s(t)+T_i$ is unpreviewable but bounded.

The idea of worst-case stopping time and the corresponding immediate future minima or maxima are inspired by [9] and [7], respectively. By enforcing safety for the worst-case predicted outputs $T_s(t)$ seconds into the future, i.e.,

$$|Cz_w(t+T_s(t))| \leq y_m, \quad \forall t \geq 0, \quad (19)$$

with $Cz_w(t+T_s(t))$ being a minimizer/maximizer, we are guaranteeing the satisfaction of the safety constraints for a future moving time horizon including the current time.

Next, a closed-form candidate LPrev-nCBF and its corresponding controlled invariant limited preview safe set are presented. Note that for brevity, the (explicit) dependence on the current time t is omitted for the rest of this manuscript.

Lemma 1 (Closed-Form Candidate Limited Preview CBF)**.** *Given Assumptions 1–2 hold. Then, a valid candidate LPrev-nCBF can be given as*

$$h(z, \mathbf{d}_p, t) = y_m - \text{sgn}(\dot{y}(t))Cz_w(t+T_s) \geq 0, \quad (20)$$

with the worst case predicted state $z_w(t+T_s) = \phi(t,T_s) + \hat{\epsilon}(t,T_s) + (\int_0^{T_s} e^{A(T_s-\tau)}d\tau)B\hat{u}$ *from* (14)*,* $\phi(t,T_s)$ *as defined below* (14) *(with* $T = T_s$*) and* $\hat{\epsilon}(t,T_s) \triangleq (\int_{T_\delta}^{T_s} e^{A(T_s-\tau)}d\tau)B_d\hat{d} + (\int_0^{T_s} e^{A(T_s-\tau)}d\tau)\hat{e}_\ell$ *with* $\hat{d}(\tau)$ *and* $\hat{e}_\ell(\tau)$ *defined in* (17) *and* (18)*,* $T_\delta = \min(T_p - T_i, T_s(t))$*.*

Proof. We begin the proof by considering the immediate smallest (worst-case under disturbance $\hat{d}$) possible output $y_w(t+T_s(t))$ (when the system changes directions) under maximum acceleration input $\hat{u}$ when $\dot{y}(t) \leq 0$, and enforce the desired safety condition $y_w(t+T_s(t)) \geq -y_m$. Similarly, when $\dot{y}_w(t) \geq 0$ with maximum deceleration input $\hat{u}$, the desired safety condition is $y_w(t+T_s) \leq y_m$. Consequently, the two conditions can be combined as

$$y_m - \text{sgn}(\dot{y}(t))y_w(t+T_s) \geq 0. \quad (21)$$

Further, $y_w(t+T_s) = Cz_w(t+T_s)$ (as described above (16)) with $z_w(t+T_s)$ as defined below (20) is the worst-case predicted y that is derived from (14) by substituting $T = T_s$, $u(\tau) = \hat{u}(t)$ and $d(\tau) = \hat{d}(\tau)$, $\forall \tau \in [t+T_p, t+T_s]$, where $\hat{d}$ and $\hat{u}$ are defined in (17) and (16), Further, $y_w(t+T_s) = Cz_w(t+T_s)$ (as described above (16)), where y_w is the worst-case predicted y that is derived from (14) by substituting $T = T_s$, $u(\tau) = \hat{u}(t)$ and $d(\tau) = \hat{d}(\tau)$, $\forall \tau \in [t+T_p, t+T_s]$ with $\hat{d}$ and $\hat{u}$ defined in (17) and (16), respectively.

Thus, (19) can be enforced by enforcing (21). Hence, h is a valid LPrev-nCBF, i.e., there exists a piece-wise constant input $u(\tau) = \hat{u}(t) = -\text{sgn}(\dot{y}(t))\text{diag}(\text{sgn}(CAB))u_m$, $\forall \tau \in [t, t+T_s]$ that enforces $h(z, \mathbf{d}_p, t) \geq 0$, $\forall t \geq 0$. Consequently, because of the guaranteed feasibility of (19) at minima or maxima (i.e., when $\dot{y}_w(t+T_s) = C\dot{z}_w(t+T_s) = 0$), the safety condition is also feasible for the horizon from $t+T_i$ to $t+T_i+T_s$, hence safety is guaranteed. □

2) Worst-Case Stopping Time: The construction of the candidate LPrev-nCBF in (20) depends on the stopping time T_s (cf. Lemma 1), which is computed next.

Lemma 2 (Worst-Case Stopping Time)**.** *For a given time t, the worst-case stopping time* $T_s(t)$ *is a solution to* $CAz_w(t+T_s) = 0$*, specifically* $T_s(t)$ *is the the smallest positive solution, with* $z_w(t+T_s)$ *given as* (20)*, i.e.,*

$$C(A\phi(t,T_s) + e^{AT_s}(B\hat{u}+\hat{e}_\ell) + e^{A(T_s-T_\delta)}B_d\hat{d}) = C\hat{e}_\ell \quad (22)$$

[2]Note that per Assumption 1, $\dot{y}(t) = C\dot{z}(t) = CAz(t)$ is exactly known.

with $\hat{u}(t), \hat{d}(t)$, $\hat{e}_\ell(t)$ defined in (16), (17), (18)*, respectively, and $\phi(t, T_s)$ and T_δ defined below* (14) *(with $T = T_s$).*

Proof. At any given $t \geq 0$, the predicted output velocity T_s seconds into the future as per the predictor system in (4) is $\dot{y}(t+T_s) = C\dot{z}(t+T_s) = CAz(t+T_s)$ (relative degree 2 of the system with respect to both the disturbance and input implies $CB = CB_d = 0$). Consequently, with $z_w(t+T_s)$ below (20), $\hat{u}(t)$, $\hat{d}(t)$, and $\hat{e}_\ell(t)$, defined in (16)-(18), for all $\tau \in [t, t+T_s(t)]$, the worst-case output velocity is given by

$$\begin{aligned}
&\dot{y}_w(t+T_s) = CAz_w(t+T_s)\\
&= CA\phi(t,T_s) + C\int_0^{T_s} Ae^{A(T_s-\tau)}d\tau(B\hat{u}+\hat{e}_\ell)\\
&\quad + C\int_{T_\delta}^{T_s} Ae^{A(T_s-\tau)}B_d\hat{d}(\tau)d\tau\\
&= CA\phi(t,T_s) + C(e^{AT_s} - I)(B\hat{u}+\hat{e}_\ell)\\
&+ C(e^{A(T_s-T_\delta)} - I)B_d\hat{d}\\
&= CA\phi(t,T_s) + Ce^{AT_s}(B\hat{u}+\hat{e}_\ell) + Ce^{A(T_s-T_\delta)}B_d\hat{d} - C\hat{e}_\ell,
\end{aligned}$$

where the final equality is the consequence of relative degree 2 assumption ($CB = CB_d = 0$). ☐

C. Closed-Form Limited Preview Control Barrier Function

In this section, we show that the LPrev-nCBF in Lemma 1 is a valid limited preview nonlinear CBF per Definition 4.

Proposition 3 (Closed-Form LPrev-nCBF)**.** *Given an input-delay system with preview Σ_{delay}, its predictive system Σ_{pred} and its linear approximations* (7) *or* (11) *that satisfies Assumptions 1–2, with $T_s(t)$ calculated based on Lemma 2. The continuously differentiable mapping $h : \mathbb{R}^n \times \mathcal{D}^{[0,T_p)} \times \mathbb{R}_+ \to \mathbb{R}$ in Lemma 1 is a limited preview horizon nonlinear control barrier function for the input-delay system Σ_{delay} and its predictor system Σ_{pred}, if $\exists \;\; u \in \mathcal{U}$ and $\exists \; \alpha \in \mathcal{K}_\infty$. such that inequality* (12) *is satisfied with*

$$\begin{aligned}\dot{h}(z,u,\mathbf{d}_p,t) = -\mathrm{sgn}(\dot{y}(t))[Ce^{AT_s}(Az(t)+Bu(t)\\ +B_d d(t+T_i) + e_c + e_\ell(t)) + \psi(t,T_s)],\end{aligned} \tag{23}$$

with $\psi(t,T_s) \triangleq \int_0^{T_\delta} Ce^{A(T_s-\tau)}B_d\dot{d}(t+T_i+\tau)d\tau$, where T_δ is defined below (14) *(with $T = T_s$). Further,* (19) *holds and consequently, the output constraint in $|y(t)| \leq y_m$ holds.*

Proof. First, we consider the closed-form candidate LPrev-nCBF h from (20) in Lemma 1. Next, we compute $\dot{h}$ the derivative of h with respect to the current time t and apply Theorem 1 with the LPrev-nCBF condition in (23). Consequently, $\dot{h}(z,u,\mathbf{d}_p,t) = -\mathrm{sgn}(\dot{y}(t))\frac{d}{dt}y_w(t+T_s)$, with $\frac{d}{dt}y_w(t+T_s)$ computed from $y_w(t+T_s)$ as defined below (21), which is derived by employing Leibniz integration rule and the fact that $CB = CB_d = 0$ (by the relative degree 2 assumption) as follows:

$$\begin{aligned}
&\tfrac{d}{dt}y_w(t+T_s) = Ce^{AT_s}\dot{z}(t) + \psi(t,T_s)\\
&\quad + CA(\phi(t,T_s) + \hat{\epsilon}(t,T_s) + (\textstyle\int_0^{T_s} e^{A(T_s-\tau)}d\tau)(B\hat{u}+\hat{e}_\ell)\dot{T}_s\\
&\quad + Ce^{A(T_s-T_\delta)}B_d(d(t+T_i+T_\delta)+\hat{d})\dot{T}_\delta\\
&= Ce^{AT_s}\dot{z} + L(t) + \psi(t,T_s) + CAz_w(t+T_s)\dot{T}_s\\
&\quad + Ce^{A(T_s-T_\delta)}B_d(d(t+T_i+T_\delta)+\hat{d})\dot{T}_\delta,
\end{aligned} \tag{24}$$

with $\psi(t,T_s)$ defined below (23), $\hat{\epsilon}(t,T_s)$ and $\phi(t,T_s)$ defined below (14) and (20), respectively, and $z_w(t+T_s)$ defined below (20) in the second equality.

Next, $CAz_w(t+T_s) = 0$, by Lemma 2, i.e., the third term in the last equality above in (24) becomes 0. Further, as a consequence of definition $T_\delta = \min(T_p - T_i, T_s)$, when $T_\delta = T_p - T_i$ we have $\dot{T}_\delta = 0$ (T_p and T_i are fixed constants) and when $T_\delta = T_s$, by relative degree 2 assumption $Ce^{A(T_s-T_\delta)}B_d = CB_d = 0$; consequently, the final term in (24) that contains $\dot{T}_\delta$ is also equal to 0. This further simplifies the expression for $\dot{h}$ to $\dot{h}(z,u,\mathbf{d}_p,t) = -\mathrm{sgn}(\dot{y}(t))\frac{d}{dt}y_w(t+T_s) = -\mathrm{sgn}(\dot{y}(t))(Ce^{AT_s}\dot{z}(t) + \psi(t,T_s))$. Finally, (23) is obtained by substituting $\dot{z}(t)$ from the linearized (immersion) system in (4). ☐

D. Optimization-Based Safety Control

Next, using the proposed LPrev-nCBF we design a safety critical controller by minimally modifying an existing nominal or legacy controller.

Proposition 4 (Optimization-Based Safety Control)**.** *At any given time $t > 0$, any stabilizing nominal controller $u = k(x,z,t)$ with known $z(t)$ (since $T_p > T_i$), for the input-delay system Σ_{delay} in* (1)*, can be minimally modified to compute a safety critical controller $u(x,\mathbf{d}_p,t)$ that guarantees safety by solving the quadratic program (QP):*

$$\begin{aligned}u(x,\mathbf{d}_p,t) = \arg\min_{u\in\mathcal{U}} \tfrac{1}{2}\|u - k(x,z,t)\|\\ s.t.\; P(t)u \leq q(t),\end{aligned} \tag{25}$$

with z, computed numerically and $h(z,\mathbf{d}_p,t), T_s$ and $\psi(t,T_s)$ from Lemma 1, Lemma 2, and Proposition 3 (as defined below (23)*), respectively, and $\alpha \in \mathcal{K}_\infty$, such that:*

$$\begin{aligned}
P(t) &\triangleq \mathrm{sgn}(\dot{y}(t))Ce^{AT_s(t)}B,\\
q(t) &\triangleq \alpha(h(z,\mathbf{d}_p,t)) - \mathrm{sgn}(\dot{y}(t))(\psi(t,T_s)\\
&\quad + Ce^{AT_s(t)}(Az(t) + B_d d(t+T_i) + e_c) + \Delta(t),\\
\Delta(t) &\triangleq \;\min\{\mathrm{sgn}(\dot{y}(t)Ce^{AT_s(t)}e_{\ell,m},\\
&\qquad -\mathrm{sgn}(\dot{y}(t)Ce^{AT_s(t)}e_{\ell,m}\}.
\end{aligned}$$

Proof. The LPrev-nCBF constraint in (12) in Definition 4 and Theorem 1, $\dot{h}(z,u,\mathbf{d}_p,t) \geq -\alpha(h(z,\mathbf{d}_p,t))$, with closed-from $h(z,\mathbf{d}_p,t)$ from Lemma 1 and $\dot{h}(z,u,\mathbf{d}_p,t)$ given in (23) Proposition 3 implies:

$$\begin{aligned}-\alpha(h(z,\mathbf{d}_p,t)) \leq -\mathrm{sgn}(\dot{y}(t))(Ce^{AT_s}(Az(t)+Bu(t)\\ B_d d(t+T_i)) + \psi(t,T_s)),\end{aligned}$$

which can be rearranged by defining $P(t)$ and $q(t)$ as shown above to recover the constraint $P(t)u \leq q(t)$ in (25). ☐

Note that analytically computing $T_s(t)$ by solving (22) for implementing the proposed safety critical controller in Proposition 4 is non-trivial, but it can be numerically computed, e.g., using MATLAB functions `vpasolve`, `fzero` or `fsolve`. Our future work will consider ways to side-step such computationally expensive numerical solvers.

IV. Illustrative Examples

In this section, we apply the proposed method to the lane-keeping example for lateral positioning of a vehicle when limited preview of the road curvature is available using the nonlinear global frame vehicle dynamics and the road-centric model in [15, Section 2.3, 2.5]:

$$\Sigma_{delay} : \dot{x}(t) = Ax + Bu(t-T_i)) + B_f\theta_f + B_r\theta_r, \tag{26}$$

with system parameters from [15, Section 3.1][3], as well as

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{2C_f}{M} \\ 0 \\ \frac{2C_f l_f}{I_z} \end{bmatrix},$$

$$B_d = \begin{bmatrix} 0 & -v_0 & 0 & 0 \end{bmatrix}^\top, C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},$$

$$B_f = \begin{bmatrix} 0 & \frac{-2C_f}{M} & 0 & \frac{-2C_f l_f}{I_z} \end{bmatrix}^\top, B_r = \begin{bmatrix} 0 & \frac{-2C_f}{M} & 0 & \frac{2C_f l_f}{I_z} \end{bmatrix}^\top,$$

and state $x \triangleq [e_1, \dot{e}_1, e_2, \dot{e}_2]^\top$, where e_1 represents the distance of the center of gravity of the vehicle from the center of the lane and e_2 represents the orientation error of the vehicle with respective to the road. The nonlinearity arises from the front tire velocity angle $\theta_f = \tan^{-1}(\frac{\dot{e}_1 + l_f \dot{e}_2}{v_0} - e_2 + l_f r_d)$ and the rear tire velocity angle $\theta_r = \tan^{-1}(\frac{\dot{e}_1 - l_r \dot{e}_2}{v_0} - e_2 - l_r r_d)$, where $r_d(t) = \frac{1}{R(t)} = 0.001 sin(t)$ represents the road curvature as previewable disturbance with radius of curvature $R(t)$, the input u denotes the front steering angle, and $v_0 = 30\,m/s$ represents a constant longitudinal velocity.

Additionally, the system has a constant input delay of $T_i = 0.175\,s$ and a preview of road curvature for a constant preview time $T_p = 0.3\,s$ along with the initial predicted state $z(0) = [0.5481, 1.0032, 0.0338, 0.0364]^\top$. Moreover, for stabilizing the vehicle in the center of the lane, we employ a nominal controller $k(z,t) = -Kz + \delta_{ff}$, where the design of the feedback gain K and feedforward correction term δ_{ff} is described in [15, Section 3.1] and [15, Section 3.2], respectively. Safety here constitutes the vehicle's distance to the lane center adhering to the constraint $|e_1| \leq y_m$, where y_m is chosen as 1.25 m in this example. Further, the input constraint is $|u| \leq u_m$, where we chose $u_m = 0.5\ rad$.

We apply the proposed LPrev-nCBF in (20) in Lemma 1 within the optimization-based framework in Proposition 4 and compare its performance with the two linearized models. Figure 2 (left) shows that the nominal controller fails to satisfy the safety requirements without the LPrev-nCBF (WO CBF), while the vehicle remains within the lane with the LPrev-nCBF based on both linearized models. with the immersion model being allowed closer to the safety boundaries compared to the affine abstraction model due to the lower approximation error, as elucidated by Figure 1. Further, Figure 2 (right) shows that the input with the immersion model (u_{im}) intervenes later and with a reduced maximum magnitude when compared to using the affine abstraction model (u_{abs}), indicating a less conservative solution.

V. Conclusion

This paper introduced a limited preview nonlinear control barrier function (LPrev-nCBF) for nonlinear systems with input delays that leverage two algorithmic linearization methods with linearization error characterization to guarantee recursive safety. Specifically, the proposed approximate linear immersion approach is found to produce a more accurate approximation of the nonlinear system behavior and thus, tighter bounds on the predicted states than the affine

[3] $M = 1573\,kg$, $l_f = 1.1\,m$, $l_r = 1.58\,m$, $C_r = 80000\,N/rad$, $C_f = 80000\,N/rad$, and $I_z = 2873\,kgm^2$.

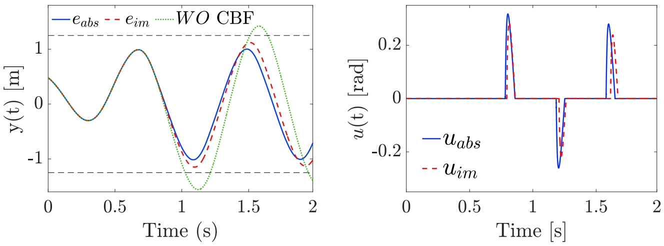

Fig. 2: Lateral displacement from center trajectories $y(t)$ (**left**) and trajectories of CBF intervention of $u(t)$ (**right**) given by $\Delta u(t) \triangleq u(t) - k(x(t), t)$, where $k(x(t), t)$ is the nominal controller. The black dotted lines in (**left**) are $\pm y_{max}$, and the T_p and T_i are 0.3 and 0.175 s, respectively (subscripts abs: Abstraction, im: Immersion).

abstraction approach, which in turn led to a less conservative solution in terms of less intervention (i.e., modifications of the nominal controller) for enforcing safety. Our method will help to balance performance and robustness for safer cyber-physical systems in the real world, e.g., self-driving cars, by taking advantage of preview information. Future directions include investigating the control sharing property when there is more than one (scalar) safety constraint and the scenario when the preview horizon is state- or time-dependent.

References

[1] M. Tomizuka and D. E. Whitney, "Optimal discrete finite preview problems (Why and how is future information important?)," *Journal of Dynamic Systems, Measurement, and Control*, vol. 97, no. 4, pp. 319–325, 12 1975.

[2] C. E. Garcia, D. M. Prett, and M. Morari, "Model predictive control: Theory and practice–A survey," *Automatica*, vol. 25, no. 3, pp. 335–348, 1989.

[3] A. D. Ames, S. Coogan, M. Egerstedt, G. Notomista, K. Sreenath, and P. Tabuada, "Control barrier functions: Theory and applications," in *European Control Conference (ECC)*. IEEE, 2019, pp. 3420–3431.

[4] T. Pati and S. Z. Yong, "Robust control barrier functions for control affine systems with time-varying parametric uncertainties," *IFAC-PapersOnLine*, pp. 11 356–11 362, 2023.

[5] Z. Liu and N. Ozay, "On the value of preview information for safety control," in *American Control Conference (ACC)*. IEEE, 2021, pp. 2348–2354.

[6] Z. Liu, L. Yang, and N. Ozay, "Scalable computation of controlled invariant sets for discrete-time linear systems with input delays," in *American Control Conference (ACC)*. IEEE, 2020, pp. 4722–4728.

[7] J. Breeden and D. Panagou, "Predictive control barrier functions for online safety critical control," in *IEEE Conference on Decision and Control (CDC)*. IEEE, 2022, pp. 924–931.

[8] T. Pati, S. Hwang, and S. Z. Yong, "Preview control barrier functions for linear continuous-time systems with previewable disturbances," in *2023 European Control Conference (ECC)*. IEEE, 2023, pp. 1–7.

[9] ——, "Control barrier functions for linear continuous-time input-delay systems with limited-horizon previewable disturbances," in *American Control Conference*. IEEE, 2024, pp. 2412–2419.

[10] A. Girard and S. Martin, "Synthesis for constrained nonlinear systems using hybridization and robust controllers on simplices," *IEEE Trans. on Automatic Control*, vol. 57, no. 4, pp. 1046–1051, 2011.

[11] K. R. Singh, Q. Shen, and S. Z. Yong, "Mesh-based affine abstraction of nonlinear systems with tighter bounds," in *2018 IEEE Conference on Decision and Control (CDC)*. IEEE, 2018, pp. 3056–3061.

[12] Z. Jin, Q. Shen, and S. Z. Yong, "Mesh-based piecewise affine abstraction with polytopic partitions for nonlinear systems," *IEEE Control Systems Letters*, vol. 5, no. 5, pp. 1543–1548, 2020.

[13] Z. Wang, R. M. Jungers, and C. J. Ong, "Computation of invariant sets via immersion for discrete-time nonlinear systems," *Automatica*, vol. 147, p. 110686, 2023.

[14] L. Menini and A. Tornambè, "Linearization through state immersion of nonlinear systems admitting Lie symmetries," *Automatica*, vol. 45, no. 8, pp. 1873–1878, 2009.

[15] R. Rajamani, *Vehicle dynamics and control*. Springer Science & Business Media, 2011.