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Multisections with divides and Weinstein 4-manifolds

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We introduce a new decomposition of Weinstein 4-manifolds called multisections with divides and show these can be encoded diagrammatically by a sequence of cut systems on a surface, together with a separating collection of curves. We give two algorithms to construct a multisection with divides for a Weinstein 4-manifold, one starting with a Kirby-Weinstein handle decomposition and the other starting with a positive, allowable Lefschetz fibration (PALF). Through the connections with PALFs, we define a monodromy of a multisection and show how to symplectically carry out monodromy substitution on multisections with divides.

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1. Introduction

There are various methods to diagrammatically encode a 4-dimensional manifold, each of which is based on a decomposition theorem which breaks up the manifold into simple pieces such that the diagram encodes the way these pieces glue together (e.g. handle decompositions, Lefschetz pencils/fibrations). The most recent such method is the development of trisections by Gay and Kirby [6], generalized to multisections in [20]. Symplectic structures have played a key role in 4-dimensional topology, due to connections with gauge theory. Compatibility between symplectic topology and handle decompositions arose from Weinstein's construction [31] and in the 4-dimensional case was diagrammatically encoded through a Legendrian surgery diagram by Gompf [14]. Similarly, compatibility between symplectic manifolds and Lefschetz pencils and fibrations was established [2, 15, 24], so a symplectic manifold can be encoded by the fiber and base surfaces, pencil points, and ordered vanishing cycles. A notion of compatibility between trisections and symplectic manifolds was proposed in [22] and shown to exist in [23], but this compatibility did not yield a simple diagrammatic way to encode a symplectic structure (rather it was motivated by attempts to obtain genus bounds). Earlier, Gay gave a construction of a trisection from a Lefschetz pencil structure [7], but this also did not yield a diagrammatic characterization (see [7, Remark 9]).

In this article we define a stronger compatibility between a multisection and a symplectic structure, which can be diagrammatically encoded by collections of curves on a surface. In addition to the diagrammatic data of the smooth multisection, we keep track of another multi-curve representing the *dividing set* of convex surfaces in contact manifolds. Thus, we call our decomposition of a symplectic manifold a *multisection with divides*. Our main result is that every 4-dimensional Weinstein domain admits a multisection with divides.

Definition 1.1. A **multisection with divides** of a symplectic filling (W, ω) with contact boundary $(\partial W, \xi)$ is a decomposition $W = W_1 \cup \cdots \cup W_n$, such that

- $W_i \cong \natural_{k_i} S^1 \times D^3$.
- $W_i \cap W_{i+1} =: H_{i+1} \cong \natural_g S^1 \times D^2$ for $i = 1, \ldots, n-1$.
- $\Sigma := W_1 \cap \cdots \cap W_n = \partial H_i$ for all i
- Each $(W_i, \omega|_{W_i})$ is a symplectic filling of $(\partial W_i, \xi_i)$.

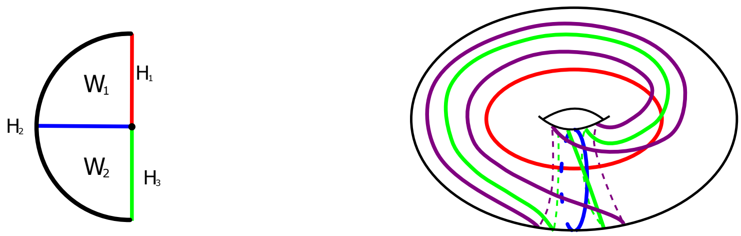

Figure 1: Left: A schematic of a bisection with divides. Each W_i is a 4-dimensional Weinstein 1-handlebody and each H_i is a 3-dimensional 1-handlebody obtained as a neighbourhood of a Legendrian graph. Right: A bisection diagram with divides of the unit cotangent bundle on S^2. The red, blue, and green curves represent curves bounding compressing disks in the respective handlebodies and the purple curves are the dividing set for the surface.

- $H_i \cup H_{i+1}$ is a contact Heegaard splitting of $(\partial W_i, \xi_i)$
- $H_1 \cup H_{n+1}$ is a contact Heegaard splitting of $(\partial W, \xi)$
- The contact structure on each H_i induces the same dividing set on Σ

A **bisection with divides** is a multisection with divides with $n = 2$.

The advantage of using contact Heegaard splittings is that the handlebodies each carry a standard positive *and* a standard negative contact structure, which are contactomorphic. This is important because for $i = 2, \ldots, n$, H_i needs a contact structure as a subset of ∂W_{i-1} and ∂W_i, which induce opposite orientations.

Remark 1.2. In the fourth bullet point of Definition 1.1, we ask for $(W_i, \omega|_{W_i})$ to be a symplectic filling. Note that in our setting weak, strong, Liouville, and Weinstein fillability are all equivalent, because there is a unique weak symplectic filling of $\#_{k_i} S^1 \times S^2$ up to symplectic deformation, and this filling is actually Weinstein (thus strong and Liouville) [26].

An essential feature of these multisections with divides is that they can be encoded as a sequence of cut systems together with a fixed dividing set on a closed surface. An example of such a diagrammatic representation together with a schematic of what this encodes can be seen in Figure 1.

We are able to encode symplectic geometric data diagrammatically because our multisection with divides are geometrically restrictive by asking each Heegaard splitting to be a *contact Heegaard splitting* (see section 2.1 for the definition). A typical multisection of a symplectic manifold would be unlikely to satisfy this condition, even if it were a "Weinstein multisection" as in [22, 23]. Therefore, it is surprising that these geometrically restrictive multisection decompositions actually exist quite generally. Our main theorem is the following.

Theorem. *Every compact 4-dimensional Weinstein domain admits a multisection with divides.*

We give two proofs of this theorem each with distinct advantages. Both proofs also yield algorithmic methods to produce a diagram for the multisection with divides. The first proof takes as input a Kirby-Weinstein diagram, and produces a bisection with divides. The disadvantage of this algorithm is that the core surface will generally have high genus, which is typically inefficient. On the other hand, the output only has two sectors, instead of arbitrarily many.

The second proof takes as input a positive allowable Lefschetz fibration (PALF) and produces a multisection with divides. In this case, the genus is potentially more controlled, being determined by the topology of the fiber of the PALF, however there may be many sectors (potentially one for each Lefschetz singularity). More specifically we prove the following.

Theorem. *Let $f : W^4 \to D^2$ be a PALF whose regular fiber is a genus g surface with b boundary components and n singular fibers. Then W^4 admits a genus $2g + b - 1$ n-section with divides.*

One can compare these results and Definition 1.1 to the definition of Weinstein trisection for closed symplectic manifolds in [22, 23]. The main difference is that those prior definitions do not require any compatibility between the contact structure induced on the boundary of each sector and the Heegaard splitting of the boundary induced by the trisection. As a consequence, there is not easy diagrammatic data that encodes the contact and symplectic topology in these prior definitions. (The most likely candidate for such diagrammatic data is a weighted foliation for each handlebody as in [21], but the data of a weighted foliation is not discrete.) By contrast, in our more restrictive notion of multisection with divides, the symplectic and contact geometry can be diagrammatically encoded by a single dividing set on the core surface.

Remark 1.3. Smooth multisections are compatible with both closed manifolds and manifolds with boundary. In this article, we have given the definition of a multisection for divides in the case that our symplectic manifold has contact boundary, because that is where we can show they exist. However, by results of Donaldson [1] and Giroux [11, 13], every closed symplectic 4-manifold can be realized as the union of a Weinstein domain with a neighborhood of a surface ("Donaldson divisor"). Therefore, our diagrammatic encodings of Weinstein 4-manifolds capture the "complicated part" of a closed symplectic 4-manifold.

The monodromy of a Lefschetz fibration is a product of right-handed Dehn twists. In general, there can be multiple ways to write the same mapping class element as a product of right-handed Dehn twists. Swapping out one of these with another is called a *monodromy substitution.* A number of important symplectic cut and paste operations like rational blow-down can be seen as a monodromy substitution operation on a Lefschetz fibration [4] [5]. By tracking the change induced by a monodromy substitution on a PALF through our algorithm, we are able to realize these cut and paste operations on multisections with divides. In Figure 17, we demonstrate this explicitly for the monodromy substitution coming from the lantern relation, which induces a C_2-rational blowdown.

We conclude the paper with a classification of genus-1 multisections with divides. The smooth genus-1 multisections with boundary were previously classified in [19] to only support linear plumbings of 2-spheres. Our requirement that the multisections be compatible with genus-1 contact Heegaard splittings restricts this significantly more, as in the following theorem.

Theorem. *Genus-1 n-sections with divides correspond to plumbings of $n-1$ disk bundles over 2-spheres, each of Euler number -2.*

The organization of this paper is as follows. Section 2 discusses the compatibility requirements between contact structures and Heegaard splittings. Section 3 gives our first proof of our main theorem, showing how to turn a Kirby-Weinstein handlebody diagram into a multisection with divides. Section 4 gives the second proof of our main theorem, showing how to turn a PALF into a multisection with divides. Section 5 classifies genus-1 multisections with divides. Section 6 shows how multisection diagrams with divides can be stabilized to increase the genus of the surface. Finally, in Section 7 we discuss some questions for future research.

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2. Contact geometry and Heegaard splittings

In this section, we explain the compatibility condition between contact structures and Heegaard splittings which we will require on the boundary of each sector. We also give a diagrammatic formulation of this compatibility. We begin with some background on surfaces in contact 3-manifolds.

A surface Σ embedded in a contact 3-manifold (Y, ξ) is said to be **convex** if there exists a contact vector field v for (Y, ξ) such that v is transverse to Σ. The purpose of the contact vector field is to define a collared neighborhood of the surface, which is needed for cut-and-paste constructions. Convex surfaces are generic, meaning every smoothly embedded surface has a C^∞-small isotopy to a convex surface [10, 17]. Given a contact vector field v transverse to Σ we obtain a multicurve called the **dividing set**, denoted Γ_Σ. This multicurve is defined by $\Gamma_\Sigma = \{x \in \Sigma | v_x \subset \xi\}$ and the isotopy class of this curve is an invariant of the embedding of Σ up to isotopy through convex surfaces.

The dividing set captures all of the contact geometric information of Σ in a neighbourhood $\Sigma \times [-\epsilon, \epsilon]$ obtained by flowing by the contact vector field. More precisely we have the Giroux flexibility theorem.

Theorem 2.1. *([10]) Let Σ be a closed orientable surface, and $f_0 : \Sigma \to (Y, \xi)$ and $g : \Sigma \to (Y', \xi')$ be convex embeddings of Σ. Suppose v is a contact vector field transverse to (Y, ξ). If the oriented multicurves $f_0^{-1}(\Gamma_{f_0(\Sigma)})$ and $g^{-1}(\Gamma_{g(\Sigma)})$ are isotopic, then there exists an isotopy f_t for $t \in [0,1]$ such that $f_1^*(\xi|_{f_1(\Sigma)}) = g^*(\xi|_{f_1(\Sigma)})$.*

As a consequence, if two contact 3-manifolds with boundary have diffeomorphic convex surfaces on their boundary, and (under this diffeomorphism) the dividing sets are isotopic, the contact manifolds with boundary can be glued together to a global contact manifold.

Given a handlebody, H, a **spine** for H is a graph, G, such that H retracts onto G. If H additionally carries a contact structure, then a spine, G, is said to be a **Legendrian spine** if each edge is a Legendrian arc or knot. By combining Darboux's theorem with the standard neighborhood for Legendrians [9, Theorem 2.5.8], we see that Legendrian spines have a standard tight contact neighborhood, determined by the ribbon neighborhood of the spine tangent to the contact planes.

2.1. Contact Heegaard splittings

Definition 2.2. A *contact Heegaard splitting* of a contact manifold (Y, ξ) is a Heegaard splitting $Y = H_1 \cup_\Sigma H_2$ such that H_1 and H_2 are contactomorphic to standard neighborhoods of Legendrian spines L_1 and L_2.

We will call the handlebodies which are standard neighborhoods of Legendrian graphs *standard contact handlebodies*. Note that a smooth handlebody of a given genus typically has multiple different “standard” contact structures, which are differentiated by the number of components of the dividing set on the boundary. The different options come from the fact that there are generally multiple different surfaces with boundary with the same Euler characteristic (so $F \times I$ and $F' \times I$ are diffeomorphic). The number of boundary components of F suffices to distinguish these from each other.

This notion of contact Heegaard splittings originated with Giroux [12] and was also developed by Torisu [28], and can be equivalently formulated as follows.

Lemma 2.3 ([12, 28]). *Let $H_1 \cup_\Sigma H_2$ be a Heegaard splitting of (Y, ξ) with Σ a convex surface with dividing set d splitting the surface as $\Sigma = \Sigma^+ \cup_d \Sigma^-$. The following are equivalent*

1) *H_1 and H_2 are standard neighborhoods of Legendrian graphs*
2) *H_1 and H_2 are two halves of an open book decomposition supporting (Y, ξ)*
3) *For each handlebody H_i, there exists a system of compression disks cutting H_i into a ball, such that the intersection of the compression disk system with $\Sigma^\pm$ is an arc system (a collection of arcs which cuts $\Sigma^\pm$ into a disk).*

Though these equivalences are known, we review how to pass between them. If we start with H_1 and H_2 as standard neighborhoods of Legendrian graphs, we can see the page F of the corresponding open book decomposition as a contact framed ribbon of the Legendrian. The contact planes are tangent to F along the Legendrian, so in a standard neighborhood, $d\alpha$ is a positive area form when restricted to this page. Moreover, in the standard model, we can identify $H_i = F \times I / \sim$ (where $(x,t) \sim (x,t')$ for $x \in \partial F$), such that $d\alpha$

is positive on each $F \times \{t\}$. In this way, we see that the first characterization gives rise to the second characterization. Conversely, given a supporting open book decomposition, we can Legendrian realize a spine on two pages. Restricting the contact structure defined on an abstract open book to half of the pages, we see that this is a standard neighborhood of this Legendrian spine. The dividing set on the boundary of a standard neighborhood of a Legendrian graph is isotopic to the intersection of the contact framed ribbon with $\Sigma = \partial H_i$. The meridians of the edges of the graph thus intersect $\Sigma^\pm$ in an arc system. Similarly, if H_1 and H_2 are two halves of an open book decomposition, the dividing set on $\Sigma = \partial H_1 = \partial H_2$ is the binding of the open book ∂F. A system of compressing disks for $F \times I/\sim$ is given by $a_j \times I/\sim$ where $\{a_j\}$ is a collection of arcs on F which cut F into a disk. This system intersects $\Sigma^+ = F \times \{1\}$ and $\Sigma^- = F \times \{0\}$ in the arc systems $\{a_j \times \{1\}\}$ and $\{a_j \times \{0\}\}$. Conversely, suppose there is a cut system C for Σ where $C \cap \Sigma^\pm$ is an arc system. Since any two arc systems with the same boundaries on a surface are related by a diffeomorphism which fixes the boundary, up to diffeomorphism, we see that C is the double of an arc system on Σ^+. Thus the handlebody is identified with $\Sigma^+ \times I/\sim$, where the dividing set is $\partial\Sigma^+$. The contact structure induced by the half open book with page Σ^+ is one tight contact structure which induces this dividing set on the boundary. By the following lemma, this is the unique tight contact structure inducing this dividing set on the boundary.

Lemma 2.4. *Let H be a 3-dimensional handlebody with boundary Σ. Let d be a multicurve which separates Σ into homeomorphic surfaces with boundary $\Sigma^+ \cup_d \Sigma^-$. Let $D_1, \dots, D_g$ be a system of cut disks for H such that $|D_j \cap d| = 2$ for all j. Then up to contact isotopy there exists at most one tight contact structure on H such that the boundary Σ is a convex surface with dividing set isotopic to d.*

Proof. Let ξ be a tight contact structure on H. We will show that ξ is uniquely determined. We can cut the handlebody along the disks D_j in the cut system. Since each disk intersects the dividing set in exactly two points, since ξ is tight, there is a unique possible dividing set (up to isotopy) on each cut disk which is a single arc. This dividing set determines the contact structure in a neighborhood of each cut disk. After removing the neighborhoods of the D_j the result is a ball. We can round corners to determine the dividing set on the 2-sphere bounding this ball. Since ξ is tight, this dividing set must be connected and there is a unique tight contact structure on the 3-ball filling in this convex 2-sphere [3]. □

In particular, if we know H has a tight contact structure with convex boundary, and there is a system of cut disks intersecting the dividing set in exactly two points each, then the contact structure must be that of a standard contact handlebody. However, as pointed out to us by Nickolas Castro, one can find examples of a surface Σ, a cut system, and a multicurve one could propose as a dividing set, such that each cut curve intersects the dividing set in two points as in Lemma 2.4, but the restriction of the cut system to Σ^+ or Σ^- fails to be an arc system. In this case, it can happen that there is no tight contact structure on the handlebody inducing the specified multicurve as the dividing set. In an explicit example, after cutting along the compressing disks and rounding corners, one would obtain a dividing set on the 2-sphere with disconnected dividing set which cannot occur in a tight contact manifold.

There is a fundamental challenge in obtaining compatibility between a multisection and a symplectic structure. Each interior 3-dimensional handlebody in a multisection appears in the boundary of two 4-dimensional sectors, but the boundary orientations are opposite to each other. Viewing a sector as a symplectic filling of its boundary induces a contact structure on the boundary which is a *positive* contact structure with respect to the *boundary* orientation. The sign of a contact structure (with respect to a fixed orientation) is an inherent property of the contact planes which measures the direction/handedness of the twisting of the contact planes. Note, this is not the same as the co-orientation of the contact structure which depends only on the contact form, not the contact structure. Therefore, we cannot have identical contact structures on a fixed manifold realize both positive and negative contact structures with respect to a fixed orientation. In general this suggests that we would need two different contact structures on each interior H_i of a multisection. However, as we show in the following lemma, there are both positive and negative *standard contact handlebodies* which are orientation reversing contactomorphic to each other. Both the positive and negative contact structures are supported by the same half open book.

Lemma 2.5. *Let F be an oriented surface with non-empty boundary, and let $H = F \times I/\sim$ where $(x,t) \sim (x,t')$ whenever $x \in \partial F$. Consider the half open book on H with pages $F \times \{t\}$. There exists a positive contact structure ξ^+ and a negative contact structure ξ^- on H such that both $\xi^\pm$ are supported by the half open book. Moreover, there is an (orientation-reversing) contactomorphism between (H,ξ^+) and (H,ξ^-).*

Proof. Choose a 1-form η on F which evaluates positively on the oriented boundary of F such that $d\eta$ is a positive area form on F. (This is always possible when F has non-empty boundary.)

Choose an annular collared neighborhood of each boundary component of F, and coordinates $(s,\varphi) \in [0,1] \times S^1$ on each annular collar, such that $s = 0$ corresponds to the boundary of F. Let $\rho : F \to \mathbb{R}$ be a smooth function such that

- In each annular collar, $\rho(s,\varphi)$ depends only on s
- In each annular collar, $\rho(s,\varphi) = s^2$ for $s \in [0,1/2]$
- In each annular collar, $\rho, \frac{\partial\rho}{\partial s} \geq 0$
- ρ is the constant function 1 outside of the $[0,1) \times S^1$ open collars.

Define

$$\alpha^{\pm} = \pm\rho dt + \eta$$

and let $\xi^{\pm} = \ker(\alpha^{\pm})$. Then

$$d\alpha^{\pm} = \pm d\rho \wedge dt + d\eta.$$

Note that $\alpha^{\pm}$ and $d\alpha^{\pm}$ are well-defined on the quotient $F \times I/\sim$. This is because near each boundary component $\rho(s,\phi) = s^2$, so $\rho dt = s^2 dt$ and $d\rho \wedge dt = 2sds \wedge dt$. Just as $r^2 d\theta$ and $2rdr \wedge d\theta$ are a well-defined forms which extend over the origin when we use (r,θ) polar coordinates, so are $s^2 dt$ and $2sds \wedge dt$ well-defined forms on the collars $(s,\rho,t) \in I \times S^1 \times I/\sim$ when the t coordinate is collapsed along $s = 0$.

First we verify that $\xi^{\pm}$ is a $\pm$ contact structure.

$$\alpha^{\pm} \wedge d\alpha^{\pm} = \pm(\eta \wedge d\rho \wedge dt + \rho dt \wedge d\eta)$$

By definition, $\rho dt \wedge d\eta$ is a positive volume form except at the boundary of F where it vanishes. η evaluates positively on each boundary component of F and thus in each (sufficiently small) collared neighborhood, $\eta = f d\varphi + g ds + h dt$ for some positive function f and some arbitrary functions g and h. Near ∂F, $d\rho \wedge dt = 2sds \wedge dt$, which is a positive area form on $I \times I/\sim$ (convert the (s,t) "polar coordinates" to Cartesian coordinates $x = s\cos t$, $y = s\sin t$ to see $sds \wedge dt = dx \wedge dy$). Away from ∂F, $d\rho \wedge dt = \frac{d\rho}{dt} ds \wedge dt$. Since the s coordinate points inward in F along the boundary, $\eta \wedge d\rho \wedge dt$ is a positive volume form near ∂F and is a non-negative multiple of a volume form over all of H. Thus $\eta \wedge d\rho \wedge dt + \rho dt \wedge d\eta$ is a positive volume form so $\alpha^{\pm} \wedge d\alpha^{\pm}$ is a $\pm$ volume form on H.

To check that it is supported by the open book, we need to verify that $\alpha^{\pm}$ evaluates positively on the binding and $d\alpha^{\pm}$ is a positive area form on the pages. Indeed if T is a vector positively tangent to ∂F, $\alpha^{\pm}(T) = \eta(T) > 0$. The restriction of $d\alpha^{\pm}$ to a page is $d\eta$, which is a positive area form on F by assumption.

Finally, to see that (H, ξ^+) and (H, ξ^-) are contactomorphic, consider the diffeomorphism $\Psi : H \to H$ which sends t to $1 - t$. Then $\Psi^*\alpha^+ = \alpha^-$. $\square$

Remark 2.6. We will implicitly use this lemma throughout this article as follows. In a multisection, for $2 \leq i \leq n$ each H_i is a subset of the boundary of X_{i-1} and X_i, but with opposite boundary orientations. In a multisection with divides, ∂X_{i-1} and ∂X_i each inherit contact structures which are positive with respect to the appropriate boundary orientations. Thus, if we fix an orientation on H_i, it will need to support one positive and one negative contact structure as a subset of the boundaries of X_{i-1} and X_i. Lemma 2.5 shows that there exist both positive and negative contact structures compatible with the same half open book. Thus (up to contact isotopy), the two contact structures on H_i are specified by the page F, which in turn is determined by H_i and the dividing set ∂F.

2.2. Contact Heegaard diagrams

Motivated by the third characterization in Lemma 2.3, we define a diagrammatic version of contact Heegaard splittings.

Definition 2.7. A **contact Heegaard diagram** is a quadruple (Σ, d, C_1, C_2) such that:

- Σ is a closed oriented surface.
- d is a multicurve which separates Σ into two homeomorphic surfaces with boundary Σ^+ and Σ^-.
- C_i $i \in \{1, 2\}$ is a cut system for Σ such that $C_i \cap \Sigma^{\pm}$ is an arc system for $\Sigma^{\pm}$.

Given a contact Heegaard diagram, we can reconstruct a contact manifold together with a contact Heegaard splitting. In particular, if F is Σ^+, then we endow each of the handlebodies with the standard contact structures coming from $F \times I$. This induces an open book with page F and binding d,

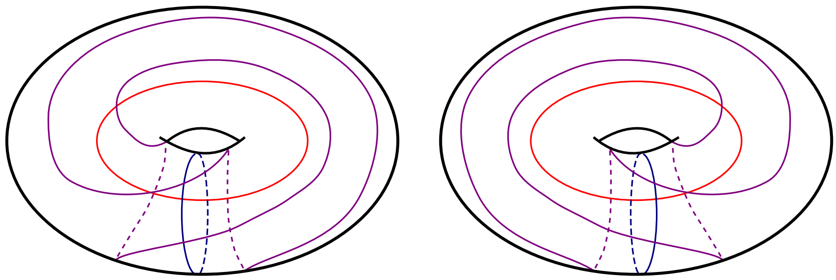

Figure 2: The two genus-1 convex Heegaard diagrams of S^3. The one on the left is tight whereas the one on the right is overtwisted.

which in turn induces a contact structure. Conversely, every contact Heegaard splitting has a contact Heegaard diagram by characterization (3) of Lemma 2.3.

This correspondence allows us to give a classification of genus-1 contact Heegaard splittings of S^3. In particular, by Euler characteristic considerations, genus-1 contact Heegaard splittings correspond to open book decompositions of S^3 with an annular page. The monodromy then consists of a left-handed or right-handed Dehn twist about the core of this annulus. The right-handed Dehn twist gives the tight contact structure on S^3 whereas the left-handed Dehn twist gives an overtwisted structure. These lead to the contact Heegaard diagrams shown in Figure 2.

2.3. Dividing sets on standard neighborhoods from Legendrian front projections

In Section 3, we will be looking at Heegaard splittings of S^3, or more generally $\#_{k_1} S^1 \times S^2$ where one handlebody is a standard neighborhood of a Legendrian graph $\widetilde{\Lambda}$ described via a front projection, and the other handlebody is the complement. In order to verify that the complement is a standard contact handlebody, it will be useful to know exactly how to draw the dividing set on the boundary of the standard neighborhood of an explicitly embedded Legendrian graph in terms of the front projection.

We will mainly focus on trivalent Legendrian graphs. Higher valence vertices can be split into trivalent vertices by growing additional Legendrian edges via a Legendrian deformation which preserves the standard neighborhood and thus, the dividing set on its boundary. Given a front projection representing a Legendrian embedding in $\mathbb{R}^3$, we can draw the corresponding dividing set on the boundary of a standard neighborhood. Recall that the dividing set on the boundary of a standard contact handlebody is the boundary of the page of the compatible open book decomposition. A page of the open book is given by the contact framed ribbon of the Legendrian knot. Considering how the contact framing wraps around at left and right cusps and at left and right trivalent vertices, we obtain the local models for the dividing set as shown in Figure 3. The first five models cover the generic front projections of a trivalent Legendrian graph. The last model includes a Legendrian arc which degenerately projects to a single point, whose two end points are trivalent vertices. We can isotope this model to a generic front projection as in Figure 4, and thus derive its local model from the previous models. We include this last "compound" model for convenience as we will use it extensively in implementing our algorithm of Section 3.1.

3. Kirby-Weinstein handlebody diagrams and multisections with divides

In this section we will show how to use a Kirby-Weinstein handlebody diagram to produce a bisection with divides. A consequence of our proof is an algorithm to obtain a multisection diagram with divides (defined in Section 3.2) from a Kirby-Weinstein diagram.

3.1. Existence of bisections with divides from Kirby-Weinstein handlebody diagrams

Theorem 3.1. *Every compact* 4*-dimensional Weinstein domain admits a bisection with divides.*

Proof. By definition, a Weinstein 4-manifold has a Weinstein handle structure. By [14], this handle structure can be represented in a standard form by a Legendrian front projection with 1-handles (which we will call a Kirby-Weinstein diagram). We will give an algorithm to convert a Kirby-Weinstein diagram in Gompf standard form into a multisection diagram with divides. If we ignore the symplectic and contact structure, the smooth part of this algorithm agrees with the methods in [6] and [25] used for converting a Kirby diagram into a trisection.

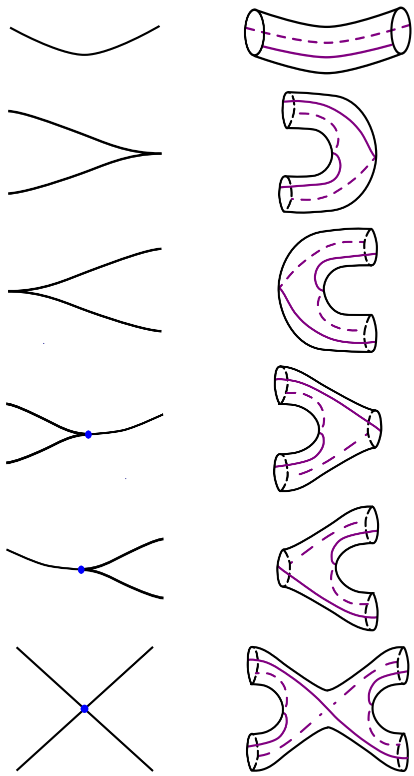

Figure 3: Left: Local models for a trivalent Legendrian graph. Right: Local models for the dividing set of the boundary of a regular neighbourhood of the graph. To obtain the last model from the previous ones, see Figure 4.

The union of the Weinstein 0- and 1-handles will be W_1. This is diffeomorphic to $\natural_{k_i} S^1 \times D^3$ and with the Weinstein structure of W restricted to W_1, it is a Weinstein filling of its boundary $(\partial W_1, \xi_1)$.

We will construct a contact Heegaard splitting $H_1 \cup_\Sigma H_2$ of $(\partial W_1, \xi_1)$ such that the Legendrian attaching spheres for the 2-handles of W are an embedded subset of the Legendrian spine of H_2. Then we will define W_2 to be a collar of H_2 together with the 2-handles of W. Because the attaching spheres of the 2-handles are a subset of the Legendrian spine of H_2, we will see that W_2 will be diffeomorphic to $\natural_{k_2} S^1 \times D^3$. There is a naturally induced Heegaard splitting of ∂W_2 given by $\overline{H}_2 \cup H_3$ where H_3 is obtained from H_2 by doing Legendrian surgery on the attaching spheres of the 2-handles. We will then show that this is also a contact Heegaard splitting, and that $(W_2, \omega|_{W_2})$ is a symplectic filling of this contact manifold.

Let Λ be the Legendrian attaching link for the 2-handles of W in $\#_{k_1} S^1 \times S^2$. To construct the appropriate contact Heegaard splitting $H_1 \cup H_2$ of $(\partial W_1, \xi_1)$, we will add Legendrian tunnels to Λ, yielding a Legendrian graph $\widetilde{\Lambda}$ containing Λ. A standard contact neighborhood of $\widetilde{\Lambda}$ will be H_2 and its complement will be H_1. The first purpose of the tunnels is to ensure that the complement of the neighborhood of $\widetilde{\Lambda}$ is a smooth handlebody. Additional tunnels will be added to ensure this handlebody has a standard contact structure.

The construction of $\widetilde{\Lambda}$ is as follows.

1) Start with Λ in Gompf standard form.
2) If there is any 1-handle of W whose belt sphere is disjoint from Λ, add a Legendrian circle which passes through that 1-handle once.
3) For each 1-handle add Legendrian arcs to connect all the strands that pass through that 1-handle on the left and right as shown in Figure 8.
4) At each crossing in the diagram, add a Legendrian arc which projects to the crossing point connecting the over- and under-strands. See Figure 4 for generic front projections for a Legendrian isotopic graph.
5) If necessary, add Legendrian arcs to connect disconnected components until the graph is connected.
6) The resulting front projection divides up the plane into regions, such that the boundary of each region is a Legendrian unknot. Add further Legendrian arcs to cut up each region as in Figure 5 so that at the end, each region is bounded by a Legendrian unknot with $tb = -1$. Namely, every region in the front projection should have a unique "right cusp" and a unique "left cusp" where a vertex is a right (resp. left) cusp of a region if the two edges on the boundary of the region which meet in the vertex both approach the vertex from the left (resp. right). Note that here we can treat each crossing as a single vertex by shrinking the Legendrian arc connecting the two strands by a Legendrian deformation as in the rightmost part of Figure 4.

Now if H_2 is a standard contact neighborhood of $\widetilde{\Lambda}$, and H_1 is the complement, we want to show that $H_1 \cup H_2$ is a contact Heegaard splitting. First we recall why it is a smooth Heegaard splitting. Consider the case where there are no 1-handles in the Kirby-Weinstein diagram. Observe that H_2 is isotopic to a neighborhood of the diagram of $\widetilde{\Lambda}$ (since we put tunnels at each

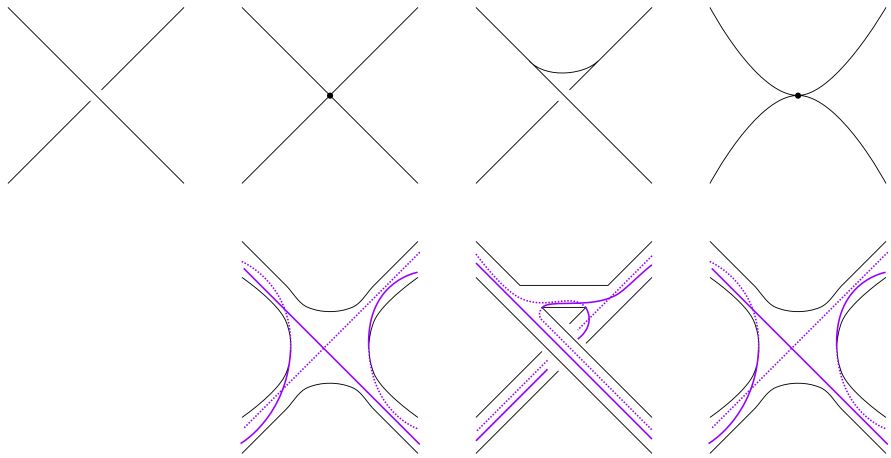

Figure 4: Adding a tunnel at each crossing which projects to the crossing point is Legendrian isotopic to the generic front projection shown on the top row center-right. By shrinking the tunnel to length zero, we obtain a Legendrian deformation of the graph as shown on the far right. The dividing set on the boundary of a neighborhood can be determined for the second graph using the standard models. Isotoping this yields the dividing sets on the boundaries of the neighborhoods of the other graphs.

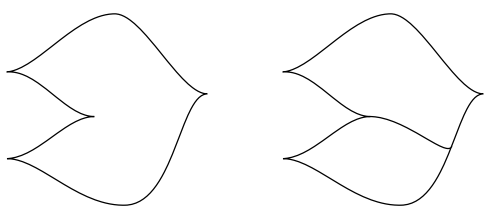

Figure 5: To make each region a Legendrian unknot with $tb = -1$ we add tunnels to regions with additional cusps.

crossing) and the diagram lies on a disk in S^3. The complement of a neighborhood of this disk is a 3-ball, and the complement of the neighborhood of the knot diagram is obtained from this by attaching a 1-handle for each bounded planar region in the diagram of $\widetilde{\Lambda}$. When there are n 1-handles in the Kirby-Weinstein diagram, note that there is a standard Heegaard splitting for $\#_n S^1 \times S^2$ where each of the handlebodies has one 3-dimensional 1-handles through each of the 4-dimensional 1-handles. As in Figure 6, we

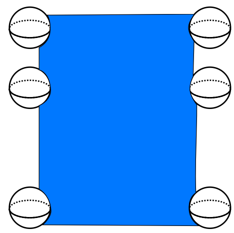

Figure 6: We may assume our diagram for $\widetilde{\Lambda}$ is contained in the shaded blue surface which passes through each of the 1-handles. A thickened neighborhood of this surface gives a handlebody whose complement in $\#_n S^1 \times S^2$ is also a handlebody.

can arrange this so that the diagram of $\widetilde{\Lambda}$ lies on a surface with boundary S such that one of the handlebodies in this standard Heegaard splitting is $S \times [-\varepsilon, \varepsilon]$. To modify this to our chosen Heegaard splitting, where H_2 is a neighborhood of the graph $\widetilde{\Lambda}$, for each bounded region D_j in the complement of the diagram we need to cut out $\{D_j \times [-\varepsilon, \varepsilon]$ from $S \times [-\varepsilon, \varepsilon]$ and add it into the other handlebody. Each time, this adds a 1-handle to the complementary handlebody, so H_1 remains a handlebody. H_2 is a standard contact handlebody by definition, because it is the neighborhood of a Legendrian graph.

Next, we consider the contact structure on H_1. Since H_1 is a subset of $\#_k S^1 \times S^2$ with its tight contact structure, the contact structure on H_1 is tight. By Lemma 2.4, to see that H_1 has a standard contact structure, it suffices to show that there is a set of compressing curves for H_1 on Σ such that each curve intersects the dividing set in two points.

Using the models from Figure 3, we can draw the dividing set d on the boundary Σ in terms of the front projection. Let $\{D'_j\}$ be the bounded regions of the complement of the front projection of $\widetilde{\Lambda}$, and let $D_j = D'_j \cap H_1$. If there are no 1-handles in the handle diagram for W, then $\{D_j\}$ form a collection of compressing disks for H_1, which cut H_1 into a ball. If there are 1-handles, there is also a compressing disk as in Figure 7 for each 1-handle. After performing handleslides over the regions that pass through the 1-handle, we can realize this compressing curve as in Figure 8 so that it intersects the dividing set in exactly two points. To see that that boundary of each D_j intersects the dividing set in two points, we use the property from the last step of the construction of $\widetilde{\Lambda}$, that each region is bounded by a Legendrian unknot with $tb = -1$, meaning there is a unique left cusp (which

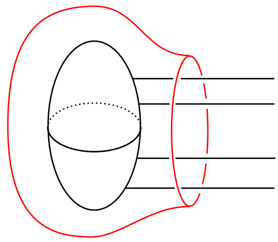

Figure 7: A compressing disk for the exterior of $\widetilde{\Lambda}$ corresponding to a 1-handle.

may be a vertex) and a unique right cusp (which may be a vertex). Any other vertices along the boundary of the region have the two edges on the boundary of this region entering the vertex from different sides. Examining all of the ways that our local models in Figure 3 may appear as the boundary of a region, we see that each cusp (either a standard cusp or a vertex cusp) contributes one intersection point between the dividing set ∂D_j, and the remaining edges and vertices in the boundary of the region do not contribute any intersections between the dividing set and ∂D_j. Thus we see that H_1 is a standard contact handlebody.

Next, we look at the second sector W_2. We need to show that $W_2 \cong \natural_{k_2} S^1 \times D^3$, and that there is a contact structure ξ on ∂W_2 with a contact Heegaard splitting $\overline{H_2} \cup_\Sigma H_3$ such that $(W_2, \omega|_{W_2})$ is a symplectic filling of $(\partial W_2, \xi)$. Recall that W_2 is obtained from $H_2 \times I$ by attaching 2-handles along the Legendrian link $\Lambda \subset \widetilde{\Lambda}$ with framing $(ct-1)$ where ct is the framing induced by the contact planes. Since Λ is embedded in $\widetilde{\Lambda}$ (the spine of H_2) W_2 is smoothly diffeomorphic to $\natural_{k_2} S^1 \times D^3$, since each 2-handle cancels a 1-handle of $H_2 \times I$. A natural Heegaard splitting of ∂W_2 is given by $\overline{H_2} \cup_\Sigma H_3$ where $H_3 = (H_2)_{(ct-1)}(\Lambda)$ ($(ct-1)$ surgery of H_2 along Λ). There is a well-defined *tight* contact structure obtained by $(ct-1)$ Legendrian surgery, which agrees with the contact structure on H_2 near $\partial H_3 = \Sigma$ (since the surgery is performed on the interior). Therefore the dividing set on $\partial H_3 = \Sigma$ can be viewed as the same as the dividing set on $\partial H_2 = \Sigma$, but the compressing curves for H_3 change based on the surgery. Namely, for each component Λ_k of Λ, the compressing curve changes from a meridian of Λ_k to a $(ct-1)$-framed copy of Λ_k on Σ. Since the dividing set is parallel to the ct framing of Λ_k, the $(ct-1)$ framing of Λ_k intersects the dividing set exactly twice. Therefore, by Lemma 2.4, H_3 is a standard contact handlebody with

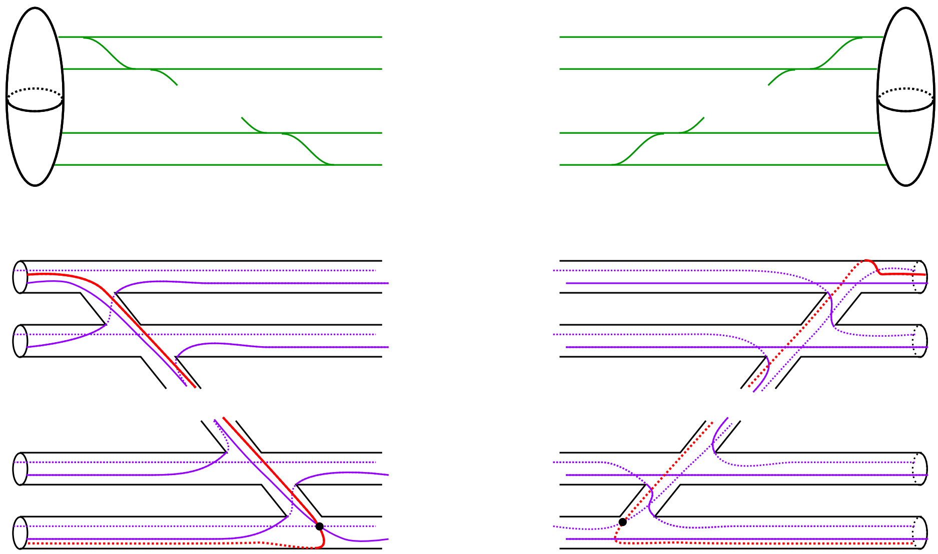

Figure 8: Adding tunnels between parallel strands passing through the same 1-handle allows us to locate a curve on the boundary of the neighborhood of the Legendrian graph $\widetilde{\Lambda}$ bounding a disk on the exterior coming from the surgery induced by the 1-handle. Choosing the arcs in precisely the manner shown ensures that we can find such a curve intersecting the dividing set in exactly two points. The lower part of the figure shows the relevant portion of the Heegaard surface with the dividing set (in purple) and the compressing curve (in red).

the contact structure induced by Legendrian surgery on H_2. Note that $\overline{H_2}$ is also a standard contact handlebody, using the negative contact structure on H_2 from Lemma 2.5. Putting these together, we get a contact Heegaard splitting of ∂W_2.

It suffices to show that $(W_2, \omega|_{W_2})$ is a Weinstein filling of ∂W_2 where the contact structure on ∂W_2 is given by the contact Heegaard splitting $\overline{H_2} \cup_\Sigma H_3$. For this, notice that $H_2 \times I$ is a 1-handlebody, and if we restrict ω to this subset of W_2, up to shrinking I, the symplectic structure must be a standard neighborhood of the isotropic spine of H_2. In other words, $H_2 \times I$ with the symplectic structure ω is symplectomorphic to a Weinstein 1-handlebody. Moreover, the induced unique tight contact structure on $\partial(H_2 \times I) = \#_{k_1} S^1 \times S^2$ is supported by the open book with page F and trivial monodromy. This open book with trivial monodromy gives rise exactly to the contact Heegaard splitting $\overline{H_2} \cup H_2$. Since W_2, (with the contact Heegaard splitting $\overline{H_2} \cup_\Sigma H_3$) is obtained from the Weinstein 1-handlebody $H_2 \times I$ (with the contact Heegaard splitting $\overline{H_2} \cup_\Sigma H_2$) by attaching Weinstein 2-handles along Legendrian knots in H_2, we have that W_2 is a Weinstein filling of the contact Heegaard splitting $\overline{H_2} \cup_\Sigma H_3$. □

3.2. Multisection diagrams with divides

A fundamental feature of multisections with divides is that they can be encoded by curves on a surface. In this section we define these diagrams and show how the previous proof gives an algorithm for obtaining a bisection diagram with divides from a Kirby-Weinstein handlebody diagram.

Definition 3.2. A **multisection diagram with divides** is a closed orientable surface, together with a set of dividing curves d splitting Σ as $\Sigma^+ \cup_d \Sigma^-$ and cut systems $C_1, C_2, ..., C_n$ such that

- For all $i \in \{1, 2, ..., n\}$, $\{C_i\} \cap \Sigma^\pm$ is an arc system for $\Sigma^\pm$
- for all $i \in \{1, 2, ..., n-1\}$, $(\Sigma, d, C_i, C_{i+1})$ is a contact Heegaard splitting of the *tight* contact structure on $\#_{k_i} S^1 \times S^2$ for some $k_i \in \mathbb{N}$.

Remark 3.3. It is fairly straightforward to check whether $\{C_i\} \cap \Sigma^\pm$ gives an arc system. However, it is potentially difficult to check whether the union of two consecutive contact handlebodies forms the *tight* contact structure on $\#_{k_i} S^1 \times S^2$.

The proof of Theorem 3.1 gives an algorithmic method to obtain a multisection diagram with divides as follows.

Starting from a Kirby-Weinstein handlebody diagram, construct the Legendrian graph $\widetilde{\Lambda}$ as in the proof. Use the models from Figure 3 to draw the dividing set d on Σ as the boundary of the neighborhood of $\widetilde{\Lambda}$ (H_2) in terms of the front projection.

We can describe cut systems C_i for the handlebodies H_i for $i = 1, 2, 3$ as follows. The cut system C_1 is given by taking the regions of the planar diagram together with an extra curve for each Weinstein 1-handle, as in Figure 8–note the curve in this figure intersects the dividing set in two points. As the regions are diagrams of $tb = -1$ unknots, each of the curves given by the boundary of a region intersects the dividing set twice. Since we know that H_1 has a tight contact structure, this suffices to check that it is a standard contact handlebody, but if we want to check directly that intersecting C_1 with $\Sigma^{\pm}$ gives an arc system for $\Sigma^{\pm}$, we use the following argument.

Let $\{D'_j\}$ be the bounded regions of the complement of the front projection of $\widetilde{\Lambda}$, and let $D_j = D'_j \cap H_1$. If there are no 1-handles in the handle diagram for W, then $\{D_j\}$ form a collection of compressing disks for H_1, which cut H_1 into a ball. If there are 1-handles, there are also compressing curves as in Figure 8. For a bounded region, the two cusps yield the end points of arcs $a_j^{\pm}$ given by $\partial D_j \cap \Sigma^{\pm}$. We want to see that this collection of arcs, $\{a_j^{\pm}\}$, cuts $\Sigma^{\pm}$ into a disk. We will argue this in the case of Σ^+, the portion of the surface in the lower half of the tubes along the boundaries of the local models of Figure 3 (the Σ^- case is similar).

Note that Σ^+ is homeomorphic to a contact ribbon neighborhood of the Legendrian graph $\widetilde{\Lambda}$. For each region D'_j in the complement of the front projection, let U_j denote the upper portion of the boundary of D_j between the left and right cusp. Let U'_j denote the edge in the graph $\widetilde{\Lambda}$ containing U_j. Choose j_1 such that the entirety of U_{j_1} is adjacent to the outer (unbounded) region (this may not be unique, but any choice will do). Let $\widetilde{\Lambda}_1 = \widetilde{\Lambda} \setminus U'_{j_1}$. By Figure 9 which shows Σ near U_{j_1}, we see that cutting Σ^+ along $a_{j_1}^+$ results in a surface Σ_1^+ which is homeomorphic to a contact ribbon neighborhood of $\widetilde{\Lambda}_1$. Thus, $b_1(\Sigma_1^+) < b_1(\Sigma^+)$.

We will repeat this procedure, now with $\widetilde{\Lambda}_1$. We take a j_2 with all of U_{j_2} adjacent to the outer region (when considered as a subset of $\widetilde{\Lambda}_1$). Cutting along $a_{j_2}^+$ will then result in a surface Σ_2^+ with $b_1(\Sigma_2^+) < b_1(\Sigma_1^+)$. If the Kirby-Weinstein diagram had no 1-handles, repeating this g times, we will eventually cut along all the a_j^+, reducing the first Betti number of the surface each time to reduce $b_1(\Sigma^+)$ from g to 0. Thus $\{a_j^+\}$ gives an arc system for Σ^+.

In the case that the handlebody diagram has 1-handles, we will perform the same argument as above, for all the bounded regions (bounded after connecting through the 1-handles), and then at the end, for each 1-handle cut along the arc corresponding to the cut curve shown in Figure 8. Cutting along this arc yields a surface homeomorphic to a neighborhood of $\widetilde{\Lambda}_k := \widetilde{\Lambda}_{k-1} \setminus T_k$ where T_k is the lowest edge of $\widetilde{\Lambda}$ that passes through that 1-handle. The argument for Σ^- is similar, but in this case, one uses the lower portion of the boundary of D_j between the left and right cusp, and orders the regions starting at the bottom instead of the top (we still cut along the arcs from the 1-handles last, but in the Σ^- case in this step, we delete from the Legendrian graph the highest edge of $\widetilde{\Lambda}$ that passes through the 1-handle).

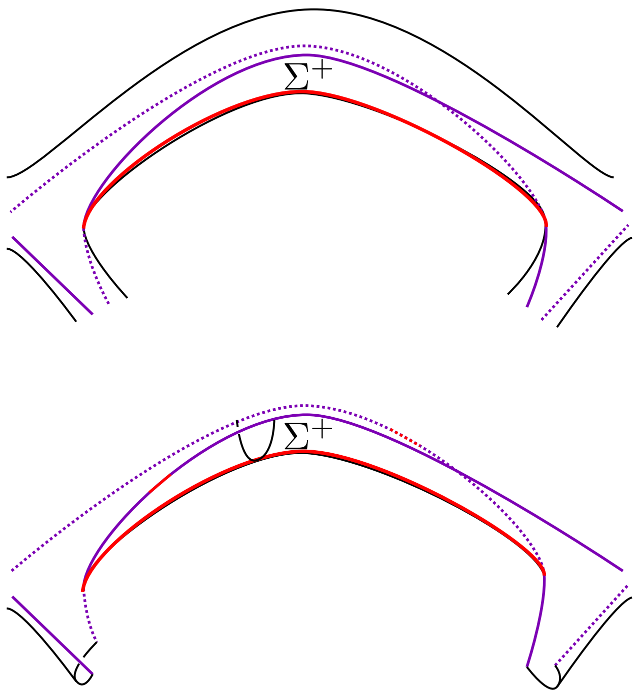

Figure 9: Cutting Σ^+_{k-1} along the red arc a_{j_k} results in a surface which deformation retracts to $\widetilde{\Lambda}_k$, so it has strictly smaller b_1. The top figure shows Σ near U_{j_1}. The bottom figure shows Σ^+_{k-1} near U_k, noting that portions of the boundary of Σ^+_{k-1} may contain arcs which were cut along in an earlier step.

The cut system C_2 is given by taking a meridian of each tunnel together with a meridian of each knot in the Kirby-Weinstein diagram. Using the local model at the top of Figure 3 and Figure 10 we see that these curves are doubles of an arc system for Σ^-. Because (Σ, C_1, C_2, d) represents a contact Heegaard splitting of the boundary of the 0- and 1-handles of the Kirby-Weinstein handlebody diagram, it is symplectically fillable and thus must support the tight contact structure on a connected sum of copies of $S^1 \times S^2$.

To obtain C_3, we start by taking a Legendrian push off of each knot component K_i in the diagram. Each such component intersects its chosen meridian in C_2 once and does not intersect the dividing set at all. C_3 is

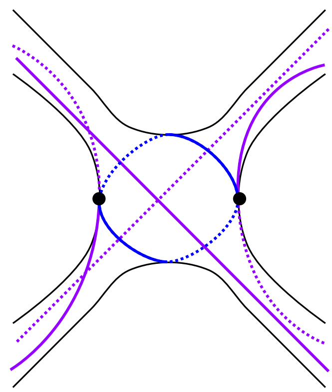

Figure 10: The meridian of the tunnel at a crossing is the blue curve in the figure. Note this curve intersects the dividing set at two points.

obtained from C_2 by replacing each meridian with K_i', the image of the meridian under a right-handed Dehn twist about K_i. Up to isotopy C_2 and C_3 intersect Σ^- in the same arc system. In $\Sigma+$, $C_3 \cap \Sigma^+$ is the image of these arcs after Dehn twisting along the K_i. Since the image of an arc system under a diffeomorphism is still an arc system, $C_3 \cap \Sigma^+$ is an arc system for Σ^+. The diagram (Σ, d, C_2, C_3) represents a contact Heegaard splitting of a connected sum of copies of $S^1 \times S^2$ with the tight contact structure (as in the proof of Theorem 3.1. Therefore $(\Sigma, d, C_1, C_2, C_3)$ is a bisection diagram with divides of the given manifold.

This algorithm is carried out in Figures 11 and 12 for the result of attaching a Weinstein handle to the max *tb* right-handed trefoil and in Figures 13 and 14 for a Weinstein domain which is a disk bundle over $\mathbb{RP}^2$.

Note that the cut systems C_2 and C_3 which are output from our algorithm have a very particular form. More specifically, each component γ of C_3 either agrees with or is dual to a component β of C_2. In the latter case, there exists a curve V in Σ which is disjoint from the dividing set, dual to the component β with respect to C_2, such that $\gamma = \tau_V(\beta)$ where τ_A is a right-handed Dehn twist about A with respect to the orientation on Σ induced as the boundary of H_2. Let's call two cut systems related in this way *standard Weinstein cobordant*. By isotoping each V into the Legendrian core of the handlebody H_2, we find a Legendrian link in H_2 such that the $ct-1$ framing for this link is represented by the corresponding γ curves in C_3, (V represents the contact framing, and β is a meridian of the

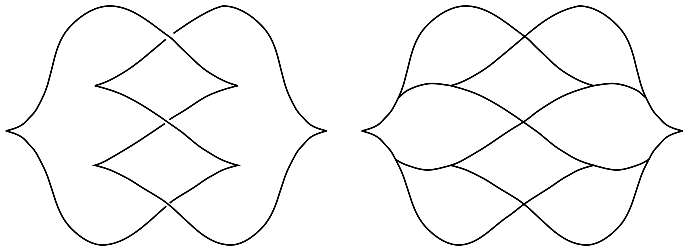

Figure 11: Adding tunnels to the right-handed Legendrian trefoil on the left results in the graph on the right. Note that the exterior is a handlebody and that each bounded region in the diagram is a $tb = -1$ unknot.

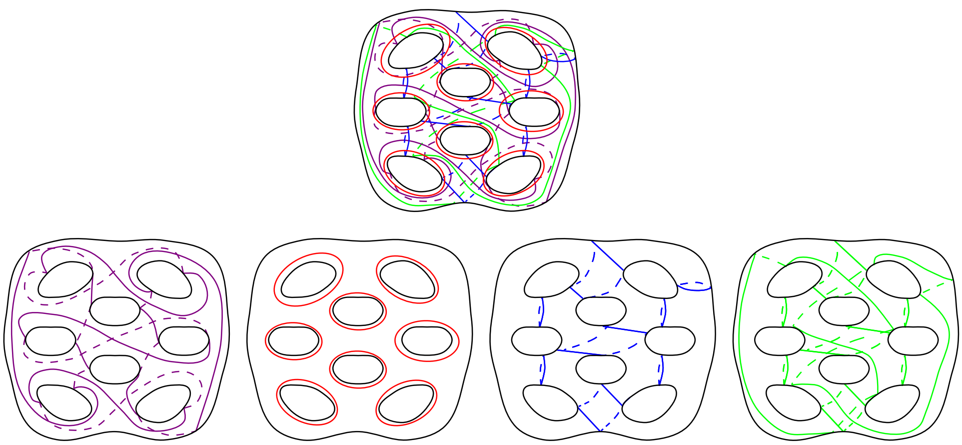

Figure 12: Top: A bisection diagram of the manifold obtained by attaching a Weinstein handle to the max tb trefoil. On this diagram, many green curves which are parallel to blue curves are omitted for visual clarity. Bottom: The dividing set, followed by each of the cut systems in order are drawn out individually.

surgery torus so $\tau_V(\beta) = \tau_\beta^{-1}(V)$ represents the contact framing -1). Thus we see that if two cut systems with a dividing set are standard Weinstein cobordant, the corresponding sector can be endowed with the structure of a Weinstein cobordism from H_i to H_{i+1} obtained by attaching Weinstein 2-handles to $H_i \times I$. We can always endow the first sector with the structure of a Weinstein 1-handlebody (since by definition (Σ, d, C_1, C_2) is a contact Heegaard splitting of $\#_{k_1} S^1 \times S^2$ with the tight (fillable) contact structure).

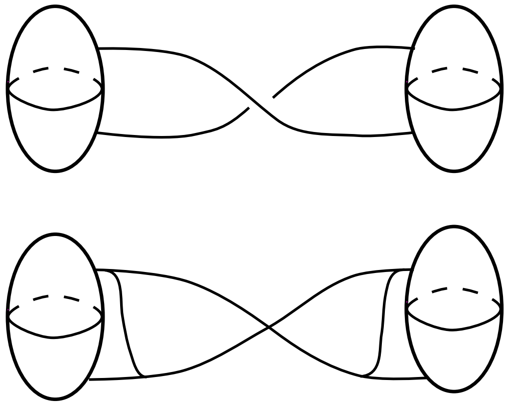

Figure 13: Top: A Kirby-Weinstein diagram for a Weinstein 4-manifold with a single 1-handle and a single 2-handle. Bottom: Following the proof of Theorem 3.1 we add Legendrian tunnels to the diagram above so that the exterior of the tunnels drawn is a standard contact handlebody.

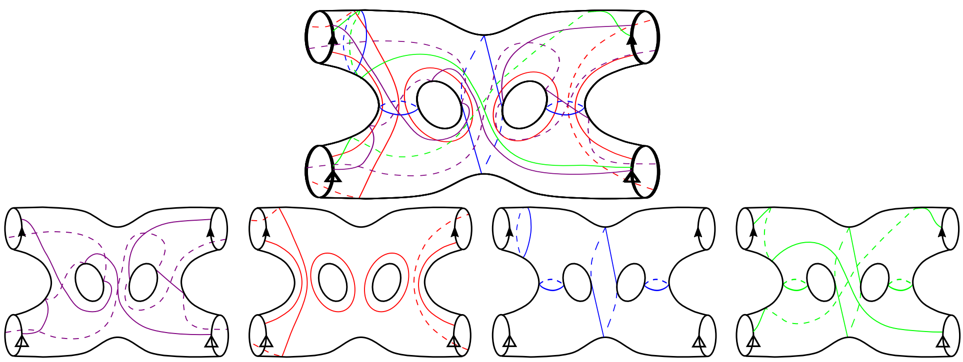

Figure 14: Top: A bisection diagram for a disk bundle over $\mathbb{R}P^2$. On this diagram, many green curves which are parallel to blue curves are omitted for visual clarity. Bottom: The dividing set, followed by each of the cut systems in order are drawn out individually.

Therefore, we have an (overly strong) condition that ensures a multisection diagram with divides corresponds to a Weinstein 4-manifold.

Proposition 3.4. *Let* $(\Sigma, C_1, \ldots, C_n, d)$ *be a multisection diagram with divides such that* (C_i, C_{i+1}, d) *are standard Weinstein cobordant for* $1 < i < n$. *Then* $(\Sigma, C_1, \ldots, C_n, d)$ *corresponds to a Weinstein* 4*-manifold.*

In Section 4, we will see another condition that ensures a multisection diagram with divides corresponds to a Weinstein 4-manifold. A priori, a multisection with divides may exist on a symplectic 4-manifold which does not admit any global Weinstein structure. Thus, it is an interesting question to ask whether there is a general characterization of the multisection diagrams with divides which correspond to Weinstein 4-manifolds.

4. PALFs, monodromy substitution and multisections with divides

4.1. PALFS and monodromy factorizations

Fibration structures on symplectic manifolds have a long history of study, dating to Donaldson's work in [2] where it was shown that every closed symplectic 4-manifold admits a Lefschetz pencil. Conversely, Gompf proved that every 4-manifold with a Lefschetz pencil admits a symplectic structure [15]. The corresponding objects in the Weinstein category are positive allowable Lefschetz fibrations.

Definition 4.1. A *Lefschetz fibration* on X is a map $\pi : X \to B$ to a surface B such that near each critical point of π, there are local orientation preserving coordinates such that π is modeled on $(z_1, z_2) \mapsto z_1 z_2$. A **positive allowable Lefschetz fibration** (PALF) is a Lefschetz fibration whose base B is D^2 and whose regular fiber is a compact surface with boundary such that every vanishing cycle is homologically essential (allowable).

Following Loi and Piergallini [24], every Weinstein domain admits a PALF. Conversely, every PALF supports a Weinstein structure [15].

In this section, we will show how to use the PALF structure to obtain a decomposition of a Weinstein domain as a multisection with divides. As in Figure 15, we cut the disk D^2 into closed subdisks, $D_1, \dots, D_n$, such that

- Each D_i contains a unique critical value of π,
- $D_i \cap D_{i+1}$ is diffeomorphic to an interval for $i = 1, \cdots, n-1$,
- $\partial D^2 \subset X_1 \cup X_n$, and
- $D_1 \cap \cdots \cap D_n = \{(0,1), (0,-1)\}$.

Then let $X_i = \pi^{-1}(D_i)$, $H_1 = \pi^{-1}(\partial D^2 \cap D_1)$, $H_i = \pi^{-1}(D_{i-1} \cap D_i)$ for $i = 2, \dots, n$, $H_{n+1} = \pi^{-1}(\partial D^2 \cap D_n)$. Then $H_i = X_{i-1} \cap X_i$ and $X_1 \cap \cdots \cap$

$X_n = \partial H_i$ (after quotienting by the D^2 factor along points in $\partial F \times D^2$ which is a Weinstein homotopic domain).

Theorem 4.2. *The decomposition* $(X_1, \ldots, X_n)$ *is a multisection with divides for* X*.*

Proof. Each H_i is a 3-dimensional handlebody since it is diffeomorphic to $F \times I$ and $\Sigma := X_1 \cap \cdots \cap X_n = (F \times \{0\}) \cup (F \times \{1\}) \cup ((\partial F \times D^2)/\sim)$ so $\Sigma = \partial H_i$ for all i. Thus to check this is a multisection with divides it suffices to check that (1) each X_i is diffeomorphic to $\natural_{k_i} S^1 \times D^3$, (2) $(X_i, \omega|_{X_i})$ a symplectic filling of its boundary, (3) $H_i \cup H_{i+1}$ is a contact Heegaard splitting of ∂X_i with the induced tight contact structure, and (4) $H_1 \cup H_{n+1}$ is a contact Heegaard splitting of ∂X with the contact strucure induced as the boundary of a PALF.

First we look at each X_i. We will use $F := \pi^{-1}((0,1))$ as the regular fiber. The vanishing cycles are curves $(c_1, \ldots, c_n)$ in F which collapse to the critical point under parallel transport from $(0,1)$ to the critical value. The model for Lefschetz singularities shows that X_i is diffeomorphic to the manifold obtained from $H_i \times I$ by attaching a 2-handle along c_i with framing given by one less than the page framing. $H_i \times I$ is certainly a 1-handlebody, so the result will still be a 4-dimensional 1-handlebody if this 2-handle cancels with one of the 1-handles of $H_i \times I$. Thus to see that X_i is diffeomorphic to a 1-handlebody it suffices to check that there is a meridional disk in H_i which intersects c_i exactly once.

As π is a PALF, $c_i \subset F$ is homologically essential. Thus, by Poincare-Lefschetz duality, there exists an arc $a_i \subset F$ such that $|a_i \cap c_i| = 1$. Parallel transport defines diffeomorphisms $\Psi_i : F \times I \to H_i$ which identify $F \times \{0\}$ with $F \subset H_i$. Then $\Psi_i(a \times I)$ is a meridional disk of H_i which intersects $c_i \subset F = \Psi_i(F \times \{0\})$ at a single point. Thus X_i is a 4-dimensional 1-handlebody.

Since $X_i = \pi^{-1}(D_i)$ and D_i is a disk, X_i has the structure of a Weinstein manifold induced by the (restricted) PALF. This agrees with the symplectic structure on X since both are compatible with the Lefschetz fibration. Thus X_i is a symplectic filling of ∂X_i.

To see that $H_i \cup H_{i+1}$ gives a contact Heegaard splitting of ∂X_i with the contact structure induced by the Weinstein structure on X_i, we use the open book construction of contact Heegaard splittings. Restricting π to ∂X_i gives an open book decomposition of ∂X_i which supports the contact structure induced by the Weinstein structure on X_i since the Weinstein structure comes from the Lefschetz fibration structure. H_i and H_{i+1} are precisely the two halves of the open book which give a contact Heegaard splitting.

Similarly, $H_1 \cup H_{n+1}$ gives a contact Heegaard splitting of ∂X because H_1 and H_{n+1} are the two halves of the open book decomposition induced on the boundary of the PALF on X. □

Remark 4.3. Using the fact that all Weinstein manifolds are supported by Lefschetz fibrations, Theorem 4.2 gives a similar result to Theorem 3.1. The main difference is that Theorem 3.1 yields a *bisection*, whereas Theorem 4.2 will usually have many more than two sectors.

4.2. Multisection diagrams with divides from a PALF

A PALF can be encoded combinatorially through the fiber surface F and the ordered set of vanishing cycles $(c_1, \ldots, c_n)$. In this section we show how to use the combinatorial data of a PALF to obtain the combinatorial data of the multisection diagram with divides corresponding to the decomposition from Theorem 4.2.

The monodromy about a Lefschetz critical value with vanishing cycle c is a right-handed Dehn twist about c, which we denote by τ_c. Thus a PALF can equivalently be encoded by an ordered sequence of right-handed Dehn twists about the vanishing cycles called a **monodromy factorization**.

The core surface of the multisection Σ is diffeomorphic to the union of two copies of F glued together along their boundary. The dividing set on Σ is given by the boundary of F. More precisely, if $\Psi_i : F \times I \to H_i$ is the diffeomorphism defined by parallel transport, $\Sigma = \Psi_i(F \times \{0\} \cup F \times \{1\}) = F \cup \Psi_i(F \times \{1\})$ (note we are suppressing the quotient of the I direction at points in ∂F).

To understand a multisection diagram with divides, we want to fix an identification of Σ, and then draw cut systems for each handlebody H_i. We will use Ψ_1 to identify $\Sigma = \partial H_1$ as $F_0 \cup F_1$. Then the restriction of Ψ_i to $F_0 \cup F_1$ gives a diffeomorphism from Σ to ∂H_i.

Let $\{a_1, \ldots, a_k\}$ be a complete arc system for F i.e. a collection of properly embedded arcs which cut F into a disk. Then $\{\Psi_i(a_1 \times I), \ldots, \Psi_i(a_k \times I)\}$ gives a cut system of disks for H_i. We want to see the boundaries of these disks on our fixed identification of Σ. Namely, we want to describe the curves $\Psi_i^{-1}(a_j \times \{0\} \cup a_j \times \{1\})$ in $F_0 \cup F_1$. Each Ψ_i is the identity on F_0, and defines parallel transport along the arc from $(0, 1)$ to $(0, -1)$ over which H_i lies. Since the Ψ_i define parallel transport, and we are using Ψ_1 to identify Σ with $F_0 \cup F_1$, we see that $\Psi_i(a_j \times \{1\})$ in Σ is the image of a_j under the monodromy around the curve which goes from $(0, -1)$ to $(0, 1)$ along the H_i curve and then goes from $(0, 1)$ to $(0, -1)$ along the H_1 curve.

This monodromy is $\tau_{c_1}, ..., \tau_{c_{i-1}}$. Thus, the cut system for H_i is obtained from the cut system for H_{i-1} by applying the right-handed Dehn twist $\tau_{c_{i-1}}$ where $c_{i-1} \subset F_1$. Any choice of complete arc system defines a cut system for H_1 by gluing together the same arcs on F_0 and F_1.

To summarize, given a PALF with fiber F and ordered vanishing cycles $\{c_1, \ldots, c_n\}$, the corresponding multisection with divides is given by

- $\Sigma = F_0 \cup_\partial F_1$ where F_0 and F_1 are copies of F (F_0 is oppositely oriented).
- The dividing set d is $\partial F = F_0 \cap F_1$.
- The cut system for H_1 is $\{a_1 \cup a_1, \ldots, a_g \cup a_g\}$ where $\{a_1, \ldots, a_g\}$ is a complete arc system for F.
- For $i > 1$, the cut system for H_i is obtained from the cut system for H_{i-1} by applying a right-handed Dehn twist about $c_{i-1} \subset F_1$.

Note that these cut systems intersect $\Sigma^- = F \times \{0\}$ in precisely the arc system $\{a_1, \ldots, a_g\}$, and intersect Σ^+ in the image of the arc system $\{a_1, \ldots, a_g\}$ under a product of Dehn twists. Since the image of an arc system under a diffeomorphism is still an arc system, these cut systems satisfy the requirement for a multisection diagram with divides. (The tightness condition is ensured by the fact that each contact Heegaard splitting corresponds to an open book whose monodromy is a product of right-handed Dehn twists.)

Remark 4.4. Note that the output of a PALF yields a slightly more general condition which ensures that a multisection with divides corresponds to a Weinstein manifold. In this case, every consecutive pair of cut systems differs by a right-handed Dehn twist about a curve which lies entirely in the Σ_+ side of the dividing set. Let's call such (C_i, C_{i+1}, d) *generalized standard Weinstein cobordant* (gsWc). (This generalizes the notion of standard Weinstein cobordant from Proposition 3.4.) If consecutive cut systems C_i and C_{i+1} are gsWc where the curve defining the Dehn twist relating them is dual to one of the components of C_i, they define a smooth multisection by Lemma 4.5. Furthermore, whenever we have gsWc cut systems representing handlebodies H_i and H_{i+1} forming the Heegaard splitting on the boundary of a sector W_i, we can again interpret W_i as a Weinstein cobordism from H_i to H_{i+1}. This is because we can use the Lefschetz fibration interpretation of W_i and then view the Lefschetz critical point as an attachment of a Weinstein 2-handle to $H_i \times I$. Since $H_i \times I$ can either be interpreted as a filling

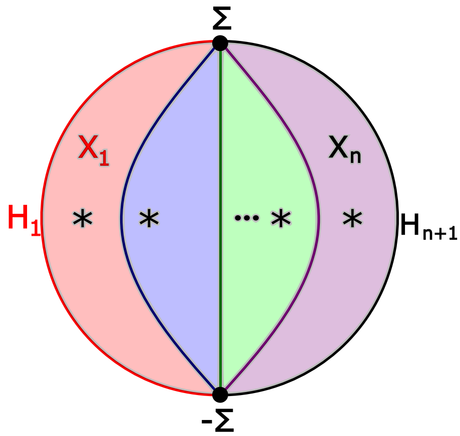

Figure 15: Decomposing a PALF into pieces containing exactly one Lefschetz singularity yields a multisection. The multisection surface is two fiber surfaces glued along their binding and each handlebody is a product region between these two surfaces.

or a Weinstein cobordism, the sector W_i also supports a Weinstein structure making it a filling of its boundary, and a Weinstein structure making it a cobordism from H_i to H_{i+1}. This shows that Proposition 3.4 holds under the more general assumption that consecutive pairs of cut systems are gsWc, and the curve defining the Dehn twist for a pair of cut systems is dual to one of the curves from the first cut system.

The previous construction suggests that multisections can also be encoded by a monodromy factorization, and here, we show that this is indeed the case. Through some care, one can determine a particular monodromy factorization from a multisection, however for the present paper it will be sufficient to have a family of possible factorizations.

Fixing a surface Σ, a system of cut curves defines a handlebody with boundary Σ. Given a handlebody, we can choose any system of cut curves which bound disks in the handlebody to encode it. If we specify a diffeomorphism Φ of Σ, we can apply that diffeomorphism to a cut curve system, to produce a new cut curve system. If the original cut curve system defines a handlebody H, we let H_Φ denote the handlebody defined by the image of a cut-system for H under the map Φ. Note this is independent of the choice of cut-system for H, because two different cut systems $\{\alpha_i\}$ and $\{\alpha_i'\}$ for H will be related by some sequence of handle slides, and thus $\{\Phi(\alpha_i)\}$ will be related to $\{\Phi(\alpha_i')\}$ by a sequence of handle slides as well. We begin with a lemma which yields a sufficient condition for H and H_Φ to form a Heegaard splitting of $\#_{g-1}S^1 \times S^2$ when Φ is a Dehn twist.

Lemma 4.5. *Let Σ be a closed genus g surface, c be a simple closed curve on Σ, and ϕ be a right or left-handed Dehn twist about a curve c. Let H be a handlebody with boundary Σ. Suppose there exists a properly embedded disk $D \subset H$ whose boundary, $\partial D = E_1$, is non-separating on Σ such that $|E_1 \cap c| = 1$. Then $H \cup_\Sigma H_\phi$ is a Heegaard splitting of $\#_{g-1}S^1 \times S^2$.*

Proof. We will produce a Heegaard diagram of $H \cup_\Sigma H_\phi$ which consists of $g-1$ pairs of parallel curves and one pair of geometrically dual curves, which proves the lemma. Let N be the boundary of a tubular neighborhood of $E_1 \cup c$. Then N bounds a separating disk in H which splits off a genus 1 summand. A cut system for the other summand thus extends E_1 to a cut system, E, for H disjoint from N. We may then obtain a Heegaard diagram for $H \cup_\Sigma H_\phi$ as $(\Sigma, E, \phi(E))$. As c is disjoint from all of the curves other than E_1 there are $g-1$ curves which are unchanged by ϕ and are therefore parallel. On the other hand E_1 and $\phi(E_1)$ intersect once, providing the desired Heegaard diagram. $\square$

Using the above lemma, we obtain a sufficient criteria for Dehn twists on cut systems to yield two sequential handlebodies in a multisection. Conversely, the following proposition shows that the sequential handlebodies in all multisections can be obtained in the fashion.

Proposition 4.6. *Let $\mathfrak{M} = X_1 \cup X_2 \cdots \cup X_n$ be a genus-g multisection with multisection surface Σ, and with $X_i \cong \natural_{k_i} S^1 \times D^3$. Let $H^1, H^2, \ldots, H^n, H^{n+1}$ be the 3-dimensional handlebodies lying at the boundaries of the X_i. Then there exist curves*

$$c_1^1, c_2^1, \ldots, c_{g-k_1}^1, c_1^2, c_2^2, \ldots, c_{g-k_2}^2, \ldots, c_1^n, c_2^n, \ldots, c_{g-k_n}^n$$

such that $H^i_{\phi_i} = H^{i+1}$ where $\phi_i = \tau_{c_1^i} \circ \tau_{c_2^i} \cdots \circ \tau_{c_{g-k_i}^i}$

Proof. For each handlebody, H^i, we will show how to produce the curves $c_i^1 \ldots c_i^{g-k_i}$. Recall that the handlebodies H_i and H_{i+1} meet at the multisection surface Σ to form a a Heegaard splitting of $\#^{k_i} S^1 \times S^2$. Consider a Heegaard diagram of $H_i \cup H_{i+1}$. By Waldhausen's theorem [29], after a sequence of handle slides there is a cut system of curves $a_1, \ldots, a_g$ for H_i and $b_1 \ldots b_g$ for H_{i+1} such that $a_i = b_i$ for $0 \leq i \leq k_i$ and $|a_n \cap b_m| = \delta_{n,m}$ for for $k_{i+1} \leq n, m \leq g$.

For $k_i \leq j \leq g$ we let $c_i^j = \tau_{b_j}(a_j)$. Then, $\tau_{c_i^j}(a_j) = b_j$. Moreover, since c_j^i does not intersect any of the other a_k for $k \neq j$, $\tau_{c_j^i}(a_k) = a_k$ for $k \neq j$. Then the product $\Pi_{j=1}^{k_i} \tau_{c_j^i}(a_k)$ takes a cut system for H_i to a cut system for H_{i+1}. □

We call the product $\Pi_{i=1}^{n} \Pi_{j=1}^{g-k_i} \tau_{c_j^i}$ a **monodromy factorization** for $\mathfrak{M}$. By following through our construction in Theorem 4.2, we can track the monodromy of a PALF onto the monodromy of a multisection which immediately yields the following.

Corollary 4.7. *Suppose that (X^4, ω) is a Weinstein manifold which admits a PALF with fiber surface F and monodromy factorization $P = \Pi_{i=1}^{n} \tau_{c_i}$. Then (X^4, ω) admits an n-section with divides with multisection surface $\Sigma = F \cup -F$, dividing set ∂F, and monodromy factorization P' obtained by applying the Dehn twists of P to $F \subset \Sigma$.*

4.3. Monodromy Substitution

In this section we will demonstrate how a monodromy substitution affects a multisection with divides. We begin with the analogous construction for PALFs.

Definition 4.8. Let $f : M^4 \to D^2$ be a PALF with fiber surface Σ and monodromy factorization $\Pi_{i=1}^{n} \tau_{c_i}$. Suppose that for some k, l, m, n and curves $c'_m \ldots c'_n$ we have that, as mapping classes, $\Pi_{i=k}^{l} \tau_{c_k} = \Pi_{j=m}^{n} \tau_{c'_j}$. Then we may obtain a new Lefschetz fibration with monodromy factorization given by

$$\Pi_{i=1}^{k} \tau_{c_i} \Pi_{j=m}^{n} \tau_{c'_j} \Pi_{i=l+1}^{n} \tau_{c_i}.$$

We say that the new Lefschetz fibration is obtained by a **monodromy substitution** on f.

Monodromy substitution has been used extensively to produce new symplectic manifolds from existing ones. In particular, in [4], the authors show that the lantern relation, pictured in Figure 16, can be used to perform a rational blowdown on the configuration C_2 (see [16] Section 8.5 for an exposition on these operations). This was later generalized in [5] to realize an infinite family of rational blowdowns as monodromy substitutions using daisy relations. In general, any monodromy substitution can be thought of as some symplectic cut-and-paste operation.

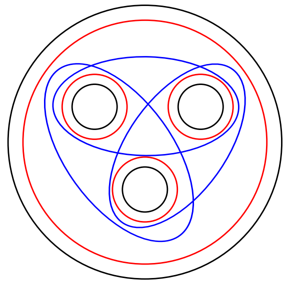

Figure 16: The lantern relation in the mapping class group in a 4-holed sphere states that right-handed Dehn twists about the red curves gives the same mapping class as the right-handed Dehn twists about the blue curves.

There is an analogous process of monodromy substitution on a multisection.

Definition 4.9. Let $\mathfrak{M}_1$ be a multisection (not necessarily with boundary) starting at the handlebody H with monodromy factorization given by $\Pi_{i=1}^n \tau_{c_i}$. Let H^k be the handlebody $H_{\tau_{c_1} \cdot \tau_{c_2} \cdot \dots \cdot \tau_{c_k}}$ and suppose that $c'_{k+1}, c'_{k+2}, \dots, c'_{j'}$ is a sequence of curves such that for all $l \in \{k+1, k+2, \dots, j'\}$ we have that c'_l is dual to some disk in $H^k_{\tau_{c'_{k+1}} \cdot \dots \cdot \tau_{c'_{l-1}}}$ (this will guarantee that the assumptions of Lemma 4.5 hold). Suppose further that $H^k_{\tau_{c_{k+1}} \cdot \tau_{c_{k+2}} \cdot \dots \cdot \tau_{c_j}} = H^k_{\tau_{c'_{k+1}} \cdot \tau_{c'_{k+2}} \cdot \dots \cdot \tau_{c'_{j'}}}$. Then we may obtain a new multisection $\mathfrak{M}_2$ starting at the handlebody H and specified by the monodromy factorization $\Pi_{i=1}^k \tau_{c_i} \Pi_{i=k+1}^{j'} \tau_{c'_i} \Pi_{i=j}^n \tau_{c_i}$. We call $\mathfrak{M}_2$ a **monodromy substitution** of $\mathfrak{M}_1$.

It follows immediately from Corollary 4.7 that we can find monodromy substitutions by doubling a PALF and a monodromy substitution of that PALF. Carrying this out for the lantern relation gives us a monodromy substitution on a multisection with divides yielding the operation outlined in Figure 17.

5. Genus-1 multisections

In this section we will provide a characterization of genus-1 multisections with divides. For examples of the diagrams for the unique 2- and 3-sections with divides, see Figure 18. Smooth genus-1 multisections are well characterized by their diagrams, which consist of sequences $(\alpha_1, \dots \alpha_n)$ with

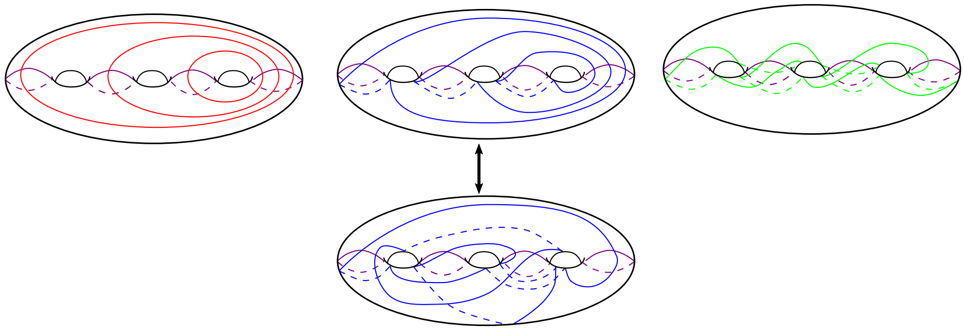

Figure 17: The three handlebodies on the top row yield a bisection whose monodromy is the double of the right-handed Dehn twists about the red curves in Figure 16. Replacing the middle handlebody by the one below it yields a multisection whose monodromy is the double of the right-handed Dehn twists about the blue curves in Figure 16. The overall change in a bisection containing these handlebodies is a C_2 rational blowdown.

$|\alpha_i \cap \alpha_{i+1}| = 1$. In [19, Proposition 5.5], the authors show that smooth genus-1 n-sections with boundary correspond to linear plumbings of $(n-1)$ disk bundles over the sphere. Moreover, given the oriented sequence of cut curves $(\alpha_1, \alpha_2, ...\alpha_n)$ defining a genus-1 multisection diagram, the Euler number of the i^{th} disk bundle is given by the algebraic intersection $\langle \alpha_{i-2}, \alpha_i \rangle$. Note that the boundary of such a linear plumbing is a lens space $L(p,q)$ (where the Euler numbers determine a continued fraction expansion of $-p/q$).

As long as the Euler numbers of the disk bundles are at most -2, any such linear plumbing supports Weinstein structures (actually multiple different Weinstein structures when any of the Euler numbers are strictly less than -2).

Proposition 5.1. *There is a unique genus-1 n-section with divides for each $n \geq 2$. These correspond to the linear plumbing of $(n-1)$ disk bundles of Euler number -2 over the sphere (T^*S^2's).*

Proof. First we observe that the linear plumbing of $(n-1)$ disk bundles of Euler number -2 supports a PALF structure whose fiber is an annulus with n vanishing cycles all parallel to the core circle of the annulus. By the algorithm in Theorem 4.2, these Weinstein domains have genus 1 multisections.

Now we show that these are the only genus 1 multisections with divides.

For a contact Heegaard splitting of S^3, the dividing set consists of two parallel curves. Fixing coordinates (a,b) for $H_1(T^2)$, we may assume, after

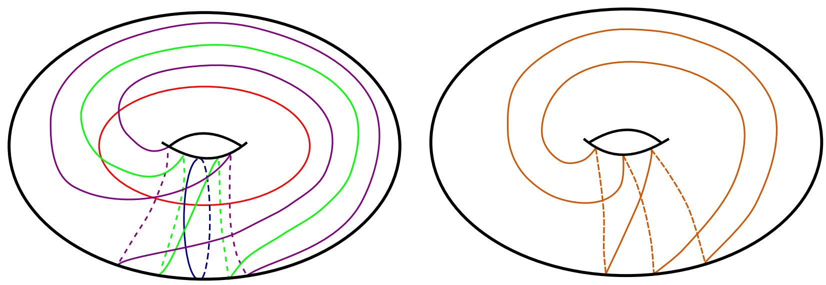

Figure 18: Left: The unique genus-1 bisection with divides corresponding to the disk bundle over the sphere of Euler number -2. Right: Adding this curve to the decomposition yields the unique genus-1 3-section with divides.

an orientation preserving homeomorphism that $\alpha_1 = (0,1)$ and $\alpha_2 = (1,0)$. The dividing set will be two parallel curves of slope d. Note that α_i intersects the dividing set twice if and only if $|\alpha_i \cap d| = 1$. Therefore $d = (1, \pm 1)$. Since $d = (1,1)$ corresponds to a contact Heegaard splitting for an overtwisted contact structure on S^3, we must have $d = (1, -1)$.

We first treat the case $n = 2$, and then proceed inductively. In this case, we seek to find the possible slopes for α_3. As $|\alpha_2 \cap d| = 1$, all curves which intersect α_2 once are given by Dehn twists of α_2 about d. In addition the requirement that $|\alpha_3 \cap d| = 1$ means that α_3 is a single Dehn twist of α_2 about d. If this Dehn twist is left-handed, then the quadruple $(\Sigma, d, \alpha_2, \alpha_3)$ gives a diagram for the overtwisted S^3, so the Dehn twist must be right-handed. Therefore $\alpha_3 = (1, -2)$ and by the classification of smooth genus 1 multisections $(\Sigma, \alpha_1, \alpha_2, \alpha_3, d)$ gives a bisection with divides of the disk bundle of Euler number -2 over the sphere.

In general suppose that $(\alpha_1 ... \alpha_{n-1})$ is a sequence of curves defining a $(n-1)$-section with divides. Then, as in the base case, α_{n-1} is a right-handed Dehn twist of α_{n-2} about d and α_n is a right-handed Dehn twist of α_{n-1} about d. Therefore, $\langle \alpha_{n-2}, \alpha_n \rangle = -2$ so that we have indeed plumbed an additional -2-sphere. $\square$

6. Stabilization

Here, we will introduce an operation on multisections with divides which takes a genus-g n-section and produces a genus-$(g+1)$ $(n+1)$-section. An

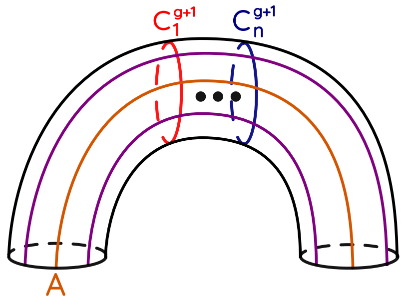

Figure 19: The annulus used to perform a stabilization of a multisection with divides. The existing cut systems $C_1 \ldots C_n$ each receive a new curve c_i^{g+1} to form cut systems C_i' for $i \in \{1, \ldots, n\}$. The arc A glues with an arc in the existing multisection which is disjoint from the dividing set to yield a curve c. Performing a Dehn twist about c to each curve in the cut system C_n' yields a new handlebody C_{n+1}' so that the sequence of cut systems $(C_1', \ldots, C_{n+1}')$ is a new multisection diagram.

explicit example of this process applied to the genus-1 bisection of T^*S^2 can be seen in Figure 20.

This stabilization operation can be seen from both the perspective of a handle decomposition, as in Section 3.1 or from the perspective of a PALF. We will focus on the second perspective, and we recall the definition of a stabilization of a PALF.

Definition 6.1. Let $f : M^4 \to D^2$ be a PALF with fiber surface Σ and monodromy factorization $\Pi_{i=1}^n \tau_{c_i}$. Let Σ' is obtained by attaching a 2-dimensional 1-handle to Σ and let c_{n+1} be a curve on Σ' intersecting the belt sphere of the attached 1-handle geometrically once. Then a **stabilization along $\boldsymbol{c_{n+1}}$** of f is the PALF with fiber surface Σ' and monodromy factorization $\Pi_{i=1}^{n+1} \tau_{c_i}$.

This motivates a definition for a stabilization of a multisection with divides, by doubling as follows. Let $\mathfrak{M}$ be a diagram for a multisection with divides given by $(\Sigma, d, C_1, \ldots, C_n)$ and let C_i be made up of components $(c_i^1, \ldots, c_i^g)$. Let p_1 and p_2 be two points on d with neighborhoods $N(p_1)$ and $N(p_2)$ and let A' be an arc in Σ^+ connecting p_1 and p_2. Let T be the cylinder shown in Figure 19.

Let $g : \partial T \to \partial N(p_1) \cup \partial N(p_2)$ be a gluing map which sends the end-points of the purple arcs in T to the intersection points of d with $\partial N(p_1)$ and $\partial N(p_2)$ and sends the end-points of A to the intersection of A' with $\partial N(p_1)$ and $\partial N(p_2)$.

Definition 6.2. The *stabilization* of $\mathfrak{M}$ along A' is the diagram $\mathfrak{M}_S = (\Sigma', d', C'_1, \dots, C'_{n+1})$ where

- $\Sigma' = (\Sigma \setminus (N(p_1) \cup N(p_2))) \cup_g T$,
- d' is the union of $d \cap (\Sigma \setminus (N(p_1) \cup N(p_2)))$ with the purple arcs in T,
- $C'_i = (c_i^1, \dots, c_i^g, c_i^{g+1})$ for $1 \leq i \leq n$, and
- $C'_{n+1} = (\tau_c(c_n^1), \dots, \tau_c(c_n^g), \tau_c(c_n^{g+1})$

where $c = (A' \setminus (N(p_1) \cup N(p_2)) \cup A$ and c_i^{g+1} is a meridian of T.

We next show that, when a multisection diagram with divides represents a Weinstein domain, its stabilization is a multisection diagram with divides representing a Weinstein homotopic domain.

Proposition 6.3. *Let $\mathfrak{M}$ be a diagram for a multisection with divides given by $(\Sigma, d, C_1, ..., C_n)$. Let $\mathfrak{M}_S$ be a stabilization $(\Sigma', d, C'_1, ..., C'_{n+1})$. Then, $\mathfrak{M}_S$ is a multisection diagram with divides, and the Weinstein manifolds encoded by $\mathfrak{M}$ and $\mathfrak{M}_S$ are Weinstein homotopic.*

Proof. We first verify that $\mathfrak{M}_S$ is still a multisection diagram with divides. That $\mathfrak{M}_S$ still represents a multisection smoothly follows from Lemma 4.5. Next, we check the condition that each cut system intersects each of $(\Sigma')^+$ and $(\Sigma')^-$ in an arc system. Note that as in Figure 19, $(\Sigma')^\pm$ is obtained from $\Sigma^\pm$ by attaching a single 2-dimensional 1-handle. Furthermore, $c_1^{g+1}, \dots, c_n^{g+1}$ intersect the $+$ and $-$ 1-handles as co-cores. Thus, since $\mathfrak{M}$ was a multisection diagram with divides, the cut systems $C'_1, \dots, C'_n$ intersect $(\Sigma')^\pm$ as arc systems. For the cut system C'_{n+1}, we note that it is obtained from C'_n by applying a Dehn twist about a curve c which is disjoint from the dividing set. Therefore, $C'_n \cap \Sigma^\pm$ is the image of an arc system under a diffeomorphism of $\Sigma^\pm$, and thus is an arc system. Finally, we need to check that (C'_i, C'_{i+1}, d') represents a contact Heegaard splitting of the *tight* contact structure on $\#_{g-1} S^1 \times S^2$. For $i = 1, \dots, n-1$, this holds because there is a Weinstein cobordism from the contact Heegaard splitting (C_i, C_{i+1}, d) to the contact Heegaard splitting (C'_i, C'_{i+1}, d') built from attaching a single 1-handle where the attaching S^0 is $\{p_1, p_2\}$. Thus,

(C_i', C_{i+1}', d') is a contact Heegaard splitting for $(Y, \xi)\#(S^1 \times S^2, \xi_{std})$ where (Y, ξ) is the contact manifold with contact Heegaard diagram (C_i, C_{i+1}, d), which is $(\#_k S^1 \times S^2, \xi_{std})$ by the assumption that $\mathfrak{M}$ is a multisection diagram with divides. That (C_n', C_{n+1}', d') reprsents the tight contact structure follows from the fact that we can obtain this contact Heegaard splitting as the boundary of a PALF sector as in the proof of Theorem 4.2.

Observe that the manifold represented by $(\Sigma', d, C_1 \cup c_1^{g+1}, C_2 \cup c_1^{g+1} \dots, C_n' \cup c_1^{g+1})$ is related to the manifold represented by $(\Sigma, d, C_1, C_2, \dots, C_n)$ by attaching a single 1-handle (whose attaching sphere is an S^0). Thus if $(\Sigma, d, C_1, \dots, C_n)$ represents a Weinstein domain W, then, since any embedding of S^0 is isotropic so there exists a unique embedding of S^0 up to isotropic isotopy, $(\Sigma', d, C_1 \cup c_1^{g+1}, C_2 \cup c_1^{g+1} \dots, C_n' \cup c_1^{g+1})$ represents a the Weinstein domain $W \natural S^1 \times D^3$. By Remark 4.4, the new sector W_n amounts to attaching a Weinstein cobordism to $W'' = W_1' \cup \cdots \cup W_{n-1}'$. Therefore, the stabilized diagram $(\Sigma', d, C_1', \dots, C_n', C_{n+1}')$ also represents a Weinstein domain. Furthermore, the Weinstein cobordism from the sector W_n attaches a Weinstein 2-handle which cancels with the added 1-handle, as the attaching sphere of the 2-handle intersects the belt sphere of the 1-handle in one point. Therefore, in total, we have added a trivial Weinstein cobordism (one which is Weinstein homotopic to a trivial cobordism). Namely, if before stabilization, the multisection diagram with divides represented a Weinstein domain, then after stabilization, the multisection diagram with divides represents the same Weinstein domain up to Weinstein homotopy as desired. □

7. Questions

Donaldson [1] and Giroux [11, 13] proved that every symplectic manifold admits a symplectic divisor such that the complement of a standard neighborhood of the divisor is a Weinstein domain. In search of a diagrammatic theory for closed symplectic manifolds, one strategy would be to find a suitable structure on the neighborhood of the divisor,and glue as in [20] to a multisection with divides for the Weinstein complement. This leads us to the following questions.

Question 7.1. Can we construct a generalization of multisections with divides for closed symplectic 4-manifolds or those with *concave* boundary? Do we need different diagrammatic information to encode concave boundary?

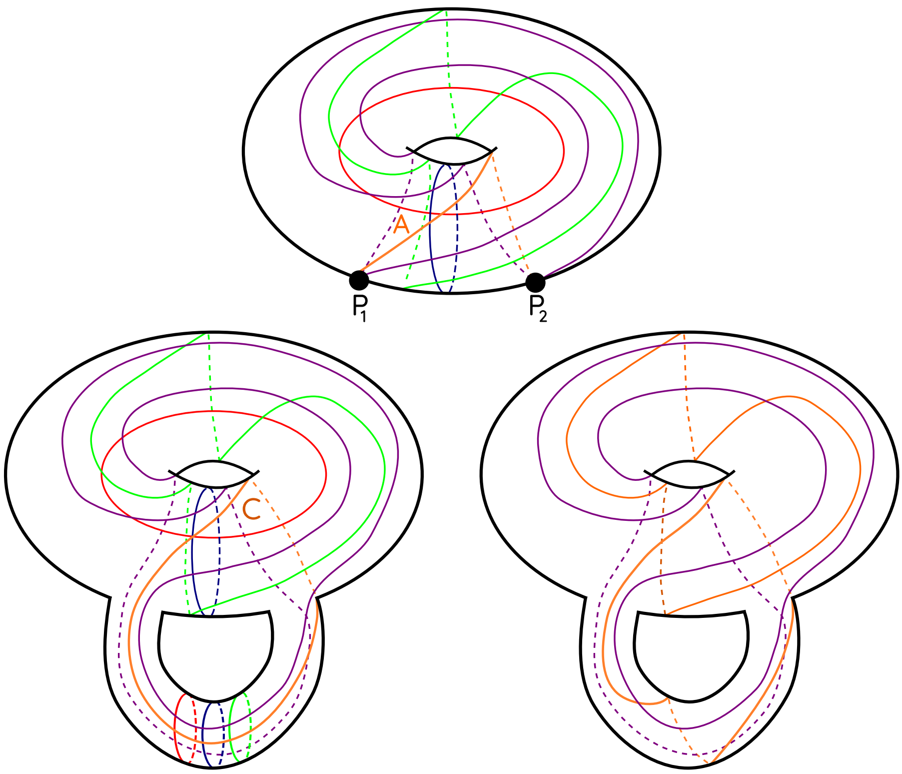

Figure 20: This figure gives the process for stabilizing the genus-1 multisection diagram for T^*S^2. Top: The genus-1 multisection diagram for T^*S^2 together with two points on the dividing set and an arc A whose interior is disjoint from the dividing set. Bottom left: Gluing the stabilizing annulus in Figure 19 yields the first three cut systems for the stabilization, together with a curve C. Bottom Right: Dehn twisting the green cut system about the curve C yields the final cut system for the stabilized multisection.

How do we specify in a multisection diagram how to symplectically glue convex pieces to concave pieces?

The results of Section 4 primarily consisted of using PALFs to obtain information about multisections with divides, but in favorable conditions (see Remark 4.4), this construction can be reversed to obtain a PALF from a multisection with divides. It is an open question as to whether two PALFs corresponding to the same Weinstein 4-manifold are related by stabilization, Hurwitz equivalence, and an overall conjugation. We have translated the stabilization move in Section 6, and using a similar approach, the other two moves can readily be translated into moves on multisections with divides. Here, techniques used in the stable equivalence of trisections in [6] could prove fruitful in addressing the following question.

Question 7.2. By passing to the related multisection with divides, can we show that any two PALFS corresponding to the same Weinstein 4-manifold are related by stabilization, Hurwitz equivalence, and an overall conjugation?

A preliminary result one would need in order to answer the previous question would be a uniqueness result for multisections with divides. We note that the stable uniqueness of trisections [6] could provide a useful outline for such a result. In addition, the operations turning closed multisections into trisections (see [20] and [18]) used in proving the stable equivalence of multisections could be adapted to this setting.

Question 7.3. Let W_1 and W_2 be multisections with divides for the same underlying Weinstein manifold W. What is a sufficient set of moves relating W_1 and W_2.

In [8] and [18] the authors show that every multisection with multisection surface Σ_g can be realized as a generic path of smooth real-valued functions on Σ_g. Much of the data of these functions is discarded in this process, so that, in the end, the smooth topology is determined by the level sets of the regular times. By keeping track of more information, it is likely that one provide an answer to the following question which would link symplectic topology with the theory of smooth functions on surfaces.

Question 7.4. Which generic paths of smooth functions $f : \Sigma \times [0, 1] \to \mathbb{R}$ yield multisections with divides?

In Proposition 3.4 and Remark 4.4, we gave a sufficient condition for a multisection diagram with divides to correspond to a Weinstein manifold. This condition is likely not necessary and is, in practice, difficult to check. We therefore pose the following question.

Question 7.5. Which multisection diagrams with divides correspond to Weinstein manifolds?

One of the conditions we require for a multisection with divides is that the contact structure on the boundary of each sector is tight. While checking tightness is generally quite difficult, we are dealing with the restricted class of connected sums of $S^1 \times S^2$ where the problem is possibly more tractable. As these Heegaard splittings correspond to open book decompositions, Wand's criterion for tightness [30] coming from the monodromy of an open book could be translated into our context. As an alternative approach, the non-vanishing of contact invariant in Heegaard-Floer homology is equivalent to tightness for connected sums of $S^1 \times S^2$ [27]. Moreover, the Heegaard-Floer homology of this manifold is particularly simple. Since each sector in a multisection has a contact Heegaard splitting on its boundary, we posit the following question.

Question 7.6. Is there a simple algorithm for computing the contact invariant of a contact structure on $\#_k S^1 \times S^2$ from a contact Heegaard diagram?

By analyzing a PALF filling of an open book supporting a contact structure Oszváth and Szabó show that the contact invariant invariant is non-vanishing for Weinstein fillable manifolds [27]. A multisection with divides seems to encode similar information to a PALF, but with Heegaard splittings playing the role of open book decompositions. In light of this, it is possible that one can give a more direct proof of this non-vanishing result.

Question 7.7. Given a multisection with divides, locate the contact element of the three manifold given by the contact Heegaard splitting of the boundary. Show that if the multisection consists of standard Weinstein cobordisms (as in Proposition 3.4 or the generalization in Remark 4.4) this element is non-vanishing.

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