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Aperiodic Sensing and Data-Driven System Identification of Nonlinear Systems Using Chebyshev Pseudospectral Approach

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Abstract— **This paper presents a novel aperiodic sensing scheme for reconstructing the dynamics of a nonlinear continuous-time system online using the Chebyshev pseudospectral (PS) method. Unlike traditional system identification via adaptive control techniques, where the sensor measures the system states periodically, this research employs an aperiodic sensing scheme for online data collection using the idea of Chebyshev nodes that guarantee an arbitrary approximation accuracy. A moving time window approach is introduced to determine the sensing time instances within each time window online. The number of nodes (sensing times) within a window is incremented or decremented adaptively until the desired approximation accuracy is reached. The least-square approach is employed to estimate the coefficients of the Chebyshev basis function for the time window. An adaptive identifier is also proposed to estimate the system states using the piecewise approximated system dynamics. The convergence of the state estimation and parameter estimation errors is ensured analytically using the Lyapunov stability theory. Numerical results are also included to show the efficacy of the sensing and identification scheme with a 2D example.**

I. Introduction

Over the past three decades, pseudospectral (PS) techniques have emerged as efficient tools in nonlinear function approximation and optimal control problems [1], [2]. Among the spectrum of PS methods, only the global orthogonal Legendre and Chebyshev polynomials have been rigorously validated to ensure approximation feasibility, consistency, and convergence [3]. The Chebyshev polynomials are found to be more advantageous than others primarily due to three reasons. Firstly, the Chebyshev approximation results in optimal polynomial approximation in the $\mathcal{L}_\infty$ norm [4]. Secondly, the Chebyshev polynomial nodes inherently cluster towards the interval's endpoints, a feature that mitigates the Runge phenomenon [5]. Thirdly, in contrast to Legendre nodes, Chebyshev nodes are amenable to closed-form evaluations. Consequently, the Chebyshev PS approach streamlines computations by obviating the necessity for intricate numerical linear algebra methods pivotal for Legendre node calculation [6].

In earlier works, Chebyshev polynomials are employed to solve optimal control problems by approximating the solution to the state differential equation offline [7], [8]. The state information spanning the entire state space is collected apriori, and the values corresponding to Chebyshev nodes are interpolated for approximation. Fast Chebyshev Transform (FCT) approach [9] is employed to compute the coefficients of the Chebyshev basis functions using the collected data. However, online approximation of the system dynamics directly in a forward-in-time manner is still an open problem.

Concurrent advancements in neural networks (NN) resulted in the employment of Chebyshev and Legendre polynomials in functional link artificial neural networks (FLANN) as the basis for approximating unknown nonlinear functions [10]. In FLANN, the weights are updated periodically with the measured system state/output and input information, leading to significant computational demand. Recently, to reduce the computational burden of the NN-based approaches, event-based weight tuning is introduced [11], [12]. Although these approaches reduce the sampling instants and computations while maintaining system stability, they do not guarantee the desired NN approximation accuracy. In addition, the adaptive event-triggering conditions used to determine the sampling instants trigger the system to sample the states frequently in the initial learning phase to ensure NN wight convergence, leading to higher computations [13].

Motivated by the above limitations, in this paper, we propose a novel online sensing and adaptive identifier design to approximate the unknown system dynamics of a nonlinear continuous-time system and estimate the system states using the Chebyshev interpolation polynomials. The system states are measured aperiodically to approximate the system dynamics using the Chebyshev polynomial as a basis. The aperiodic time instants are determined using the Chebyshev nodes (solution to the Chebyshev polynomial) within a moving time window. Since the data points use Chebyshev nodes for approximation, the desired approximation accuracy is ensured by selecting the appropriate number of nodes.

To achieve this, in the proposed approach, the number of nodes within a time window is selected dynamically using the average approximation error within respective windows and comparing it with the desired approximation error. If the average approximation error in the current time window exceeds the desired error threshold, nodes are incremented for the next approximation interval or vice versa. Next, an adaptive identifier is designed using the approximated system dynamics to estimate the continuous system states. The identifier uses the aperiodically available state measurements at the window transition time of each window to reset its initial condition for the next window. The adaptive identifier parameters are updated using the data collected within each time window at the end of each interval using the least square

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approach. The uniform boundedness of state and parameter estimation errors are shown using the extension of Lyapunov theory for hybrid dynamical systems.

The main contributions of the paper are 1) determining aperiodic sensing time instants online using Chebyshev nodes, 2) the introduction of a moving time window approach to adaptively determine the number of nodes or intra-sampling points within each window, and 3) ensuring the boundedness of the state and parameter estimation error. The next section provides background information on Chebyshev approximation and formulates the problem.

II. Background and Problem Formulation

This section presents the background on function approximation using the Chebyshev polynomials [6], [14] and formulates the system identification problem with online sensing.

A. Background

Consider an autonomous continuous-time nonlinear system represented by

$$\dot{x}(t) = F(x(t)), \tag{1}$$

where $x : \mathbb{R}_{\geq 0} \to \Omega$ is the state vector, and the vector function $F(x) \in \mathcal{L}^{\infty}_{n_p}(\Omega)$ is the internal system dynamics, where $\Omega \subseteq \mathbb{R}^n$ is the domain of F and $\mathcal{L}^{\infty}_{n_p}(\Omega)$ is the space of $\mathbb{R}^{n_p}$-valued essentially bounded measurable functions over Ω. The function $F(x)$ is unknown but locally Lipschitz-continuous. Note that the system in (1) can be considered as the closed-loop dynamics of the system with a stabilizing feedback control input $u = \mu(x)$, i.e., the system $\dot{x} = f(x,u) = f(x,\mu(x)) = F(x)$. Note that the time argument in the state x are dropped for brevity. To formulate the identification problem, we assume the existence of a stabilizing control input $\mu(x)$ for the system (1).

The goal is to design an adaptive identifier by finding the best approximation polynomial $\hat{F}(x) \in \mathcal{P}_m$ of the function $F(x)$ with respect to the $\mathcal{L}_\infty$ norm, i.e. $\epsilon(x) = \inf_{\hat{F}\in\mathcal{P}_m} \left\| F(x) - \hat{F}(x) \right\|_\infty$, where $\epsilon(x)$ denotes the approximation error and $\mathcal{P}_m$ is the set of all polynomials of degree at most m.

PS approaches using Chebyshev polynomials as basis are widely employed to approximate nonlinear functions in an interval. The Chebyshev polynomial $U^*{}_N(z)$ of the second kind for a 1D system ($n_p = 1$) is a polynomial of degree N defined by $U_N(\cos(\theta)) = \frac{\sin((N+1)\theta)}{\sin(\theta)}$, where $\theta = \arccos(z)$, $\theta \in [0,\ \pi]$. Alternatively, the 1D Chebyshev polynomials can be expressed using the recurrence relations given by [6]

$$\begin{cases} U_0(z) = 1, \quad U_1(z) = 2z, \\ U_h(z) = 2zU_{h-1}(z) - U_{h-2}(z), \quad h = 2,3,...,N. \end{cases} \tag{2}$$

The roots of the Chebyshev polynomial $U_h(z)$, often referred to as *Chebyshev nodes*, can be calculated as $z_l = \cos\left(\frac{l\pi}{N+1}\right),\ l = 1,2,\ldots,N$ in the interval $[-1,\ 1]$.

For example, given a function $G(z)$ on the interval $[-1,1]$ and N points $z_1, z_2, \ldots, z_N$ in that interval, the interpolation polynomial is that unique polynomial $\hat{G}$ of degree at most $N-1$, which has the value $G(z_l)$ at each point z_l. When the interpolation nodes $z_1, z_2, \ldots, z_N$ are the roots of U_N, it can be verified that the approximation error at each point satisfies [15]

$$\epsilon(z) = G(z) - \hat{G}(z) = \frac{G^{(N)}(\xi)}{N!}\prod_{l=1}^{N}(z - z_l), \tag{3}$$

for some ξ (depending on z) in the interval $[-1,1]$. The term $\prod_{l=1}^{N}(z - z_l)$ causes oscillation outside the nodes. Therefore, the number of nodes is selected such that $\max_{z\in[-1,1]} \left|\prod_{l=1}^{N}(z - z_l)\right|$ is minimized. It has been proven in [16] that for the second kind Chebyshev polynomial, the maximal value of this term is bounded by $\frac{1}{2^N}$.

To generate the nodes over an arbitrary interval $[a,\ b]$, an affine transformation can be used, expressed by [6]

$$z_l = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)\cos\left(\frac{l\pi}{N+1}\right), \tag{4}$$

for $l = 1,\ldots,N$. Therefore, in the interval $[a,b]$ containing N distinct nodes, the interpolation error is bounded as

$$\left|G(z) - \hat{G}(z)\right| \leq \frac{1}{2^N N!}\left(\frac{b-a}{2}\right)^N \max_{\xi\in[a,b]}\left|G^{(N)}(\xi)\right|. \tag{5}$$

B. Problem Statement

To formulize the approximation problem, consider the simple 1D case for $z \in \mathbb{R}$. The function $G(z) \in \mathbb{R}$ can be approximated in an interval $[a,b]$ using PS methods with Chebyshev polynomial basis as follows:

$$G(z) = \sum_{k=0}^{\infty}\theta_k^* U_k(z) = \sum_{k=0}^{N}\theta_k^* U_k(z) + \epsilon(z), \tag{6}$$

where $\theta_k^*,\ k = 0,1,\cdots,N$ are the unknown coefficients, and $\epsilon(z)$ is the approximation error due to truncation. Let z_l for $l = 1,2,\cdots,N$ determined by (4), are the N Chebyshev nodes (sample data points) to approximate $G(z)$. The approximation of the function $G(z)$ over the interval $[a,b]$ can be expressed as

$$\hat{G}(z) = \sum_{k=0}^{N}\theta_k U_k(z). \tag{7}$$

where θ_k's for $k = 0,1,\cdots,N$ are the estimates of θ_k^*.

To generalize the approximation for $z = [z^1, z^2, \cdots, z^{n_p}]^T \in \mathbb{R}^{n_p}$, the function $G(z) \in \mathbb{R}$ can be expressed using the one-dimensional Chebyshev polynomials in (2) as

$$\hat{G}(z) = \sum_{k_1=0}^{N}\cdots\sum_{k_{n_p}=0}^{N}\left(\theta_{k_1,k_2,\ldots,k_{n_p}} U_{k_1}(z^1)\cdots U_{k_{n_p}}(z^{n_p})\right). \tag{8}$$

In (8), we assume the same number of nodes N for each state $z^j,\ j = 1,2,\cdots,n_p$ for ease of exposition. However, one can choose a different number of nodes.

In a compact form, the Chebyshev polynomial basis can be expressed as the Kroneker product in each dimension as $\mathbb{U}_N(z) = \bigotimes_{j=1}^{n_p} \bar{U}_N(z^j)$ where $\bar{U}_N(z^j) = \left[U_0(z^j) \quad U_1(z^j) \quad \cdots \quad U_N(z^j)\right]^T$, with z^j being the j^{th} element of $z \in \mathbb{R}^{n_p}$ for $j = 1, 2, \cdots, n_p$ and $\bar{U}_N(z^j)$ forms a 1D Chebyshev vector and $\mathbb{U}_N(z) \in \mathbb{R}^{(N+1)^{n_p}}$. For example in the case of a 2D system ($n_p = 2$) with nodes $N = 3$, the Chebyshev basis vector $\mathbb{U}_3(z) = \bar{U}_3(z^1) \otimes \bar{U}_3(z^2) = [1 \quad 2z^1 \quad U_2(z^1) \quad U_3(z^1)]^T \otimes [1 \quad 2z^2 \quad U_2(z^2) \quad U_3(z^2)]^T \in \mathbb{R}^{16}$.

Equation (8) can be expressed in vectorized form as

$$\hat{G}(z) = \theta^T \mathbb{U}_N(z), \tag{9}$$

where $\theta \in \mathbb{R}^{(N+1)^{n_p}}$. For a minimum interpolation error defined in (5), we can measure the Chebyshev nodes z_l and the corresponding $G(z_l)$ for $l = 1, 2, .., N$ in the interval $[a, b]$ to compute the coefficient vector θ.

On the other hand, for a dynamical system in (1), to approximate the system dynamics $F(x(t)) \in \mathbb{R}^{n_p}$ using the Chebyshev PS method, i.e., estimating the coefficients, that results in minimum interpolation error as in (5), we need data consisting of Chebyshev nodes $x_l = [x_l^1, \cdots, x_l^{n_p}]^T$ and corresponding $F(x_l) \in \mathbb{R}^{n_p}$ for $l = 1, 2, \cdots, N$. Online measurement of the Chebyshev nodes, i.e., the states x_l^j for $l = 1, 2, \cdots, N$ and $j = 1, 2, \cdots, n_p$ computed using (4), and their derivatives forward-in-time is not feasible primarily for three reasons. First, to compute the node (state) vector $x_l \in \mathbb{R}^{n_p}$, we need the information about the interval of approximation of each element x^j of the state vector x, which is unknown. Second, even when the interval is known, determining the corresponding time instants t_l, for each x_l^j, $l = 1, 2, \ldots, N$ such that a smart sensor can actively measure the system states at these time instants and compute the state derivatives is not possible. Third, traditional Chebyshev-based approximation assumes a certain number of nodes N to determine the degree of the Chebyshev polynomial that guarantees the approximation accuracy. With the dynamics $F(x)$ unknown, it is impossible to determine the number of nodes N required to approximate the function to a desired accuracy without prior knowledge about the dynamics.

In addition, using Chebyshev nodes to approximate the dynamics leads to aperiodic sensing of the system states since the distance between two nodes is not constant within an interval. Therefore, an adaptive identifier that can simultaneously approximate the system dynamics $F(x)$ and estimate the system states x must use the available aperiodic measurements. This introduces an additional measurement error in the adaptive identifier dynamics compared to the traditional extended Luenberger observer, making the identification challenging.

Therefore, the problem at hand is threefold: 1) developing an online aperiodic sensing scheme without the knowledge of the state values at the Chebyshev nodes in the current approximation interval, 2) introducing an online and forward-in-time approach to compute the number of nodes to achieve a desired approximation accuracy, and 3) developing an adaptive identifier that uses these aperiodic state measurements for estimating the system state and approximating the dynamics simultaneously. In the following sections, a solution to the above challenges is presented.

III. Adaptive Identifier Design

In this section, we present an online solution to the system identification problem by introducing a moving window approach for sensing the states online and approximating the dynamics.

A. Proposed Solution

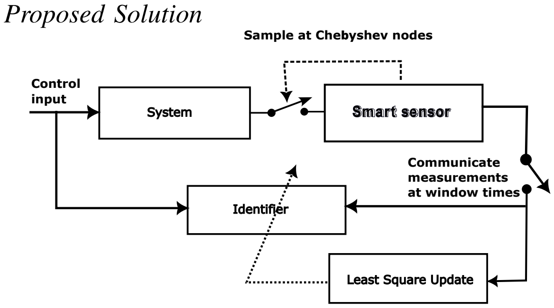

Fig. 1. Proposed aperiodic sensing for system identification.

The architecture of the proposed solution to the identification problem is shown in Figure 1. A smart sensor is connected to the system to sense the states actively at aperiodic sampling instants. The sampling times t_l^w, $l = 1, 2, \cdots, N_w$ are computed using (4), where N_w is the number of time-nodes in a moving time window $(t^{w-1}, t^w]$, $w = 1, 2, \cdots$. The smart sensor measures the state vector $x(t_l^w)$ and $x(t_l^w - \Delta t)$, where Δt is a small time interval. In addition, the smart sensor samples $x(t^w)$ as an initial condition for the identifier. The sensor stores all the measured states at time-nodes, i.e., $x(t_l^w)$ and $x(t_l^w - \Delta t)$ for $l = 1, 2, \ldots, N_w$ and sends them as one packet to the identifier at the end of each window. This reduces the number of transmissions between the sensor and the identifier. The identifier uses the least square approach to compute the coefficients of the Chebyshev basis and resets its states with the received system states at each t^w.

B. Moving Window Scheme for Online Aperiodic Sensing

As discussed in previous section, the time instants $\{t_l\}_{l=1}^N \subset \{t\}$ corresponding to the Chebyshev nodes $x_l^j \in [a, b]$ for $l = 1, 2, \ldots, N$, $j = 1, 2, \cdots, n_p$ in (4) for approximating the function $F(x)$ could not be computed directly. Since the dynamics $F(x(t)) = F(t)$ is an implicit function of time t, we propose to use time instants t_l in an interval computed using (4) as nodes (referred to as time-nodes) for sampling the states. The corresponding states $x(t_l)$ are used to formulate the Chebyshev basis function for approximation.

To compute the time-nodes, the initial and final time must be known. However, in an online approximation scheme the final time is not known apriori. To address this challenge, we propose a moving time window scheme to compute the time-nodes dynamically within each window using (4). Define a sequence of time instants $\{t^w\}_{w=0}^\infty$ with $t^0 = 0$ and $t^w > t^{w-1}$. We denote the interval $[t^{w-1}, t^w]$, $w = 1, 2, \ldots$ as the moving window which is employed to compute time-nodes

t_l^w using (4). The width of the w^{th} window $\tau^w = t^w - t^{w-1}$ can be fixed or time-varying. For clarity of the exposition, we assume $\tau^w \in \mathbb{R} \geq 0$ is constant in this paper and can be determined by the designer.

Remark 1: The selection of a variable width time window τ^w can be effectively guided by the principles of event-triggered or self-triggered control approaches developed in the literature [17].

The Chebyshev time-nodes within each time window $(t^{w-1} \;\; t^w]$, $w = 1, 2, \cdots$ can be computed using (4) as

$$t_l^w = \frac{1}{2}\left(t^{w-1} + t^w\right) + \frac{1}{2}\left(\tau^w\right)\cos\left(\frac{l\pi}{N_w + 1}\right) \tag{10}$$

for $l = 1, 2, \ldots, N_w$, where N_w denotes the number of nodes at the w^{th} time window. Note that N_w governs the degree of the Chebyshev polynomial and approximation accuracy and is not known apriori. In Section III-E, we propose an online node selection algorithm to address this challenge. Before introducing it, the identifier design is presented next.

C. Nonlinear Online Identifier Design

The Chebyshev basis for each moving window $(t^{w-1},\ t^w]$ can be rewritten as $\mathbb{U}_{N_w}^w(x) = \bigotimes_{j=1}^{n_p} \bar{U}_{N_w}^w\left(x^j\right)$, where $\bar{U}_{N_w}^w(x^j) = \begin{bmatrix}1 & x^j & U_2^w\left(x^j\right) & \cdots & U_{N_w}^w\left(x^j\right)\end{bmatrix}^T$, with x^j is the j^{th} element of x for $j = 1, 2, \cdots, n_p$ and $\mathbb{U}_{N(w)}^w(x) \in \mathbb{R}^{(N_w+1)^{n_p}}$.

The system dynamics $F(x) \in \mathbb{R}^{n_p}$ within each window using Chebyshev approximation can be expressed as

$$F^w(x) = \Theta^{w^*T}\mathbb{U}_{N_w}^w(x) + \epsilon^w(x), t^{w-1} < t \leq t^w, \tag{11}$$

where $\Theta^{w^*} = \begin{bmatrix}\theta_1^{w^*} & \theta_2^{w^*} & \cdots & \theta_{n_p}^{w^*}\end{bmatrix} \in \mathbb{R}^{(N_w+1)^{n_p} \times n_p}$ with $\theta_j^{w^*} \in \mathbb{R}^{(N_w+1)^{n_p}}$ is the parameter matrix for the truncated Chebyshev polynomial with N_w nodes and $\epsilon^w(x) \in \mathbb{R}^{n_p}$ is the truncation error. The estimate of the system dynamics in w^{th} time window can be expressed as

$$\hat{F}^w(x) = \Theta^{w^T}\mathbb{U}_{N_w}^w(x),\ t^{w-1} < t \leq t^w \tag{12}$$

where $\hat{F}^w(x) \in \mathbb{R}^{n_p}$ is the approximated dynamics and Θ^w denotes the estimated parameter matrix.

The system dynamics in (1) using the Chebyshev approximation within $(t^{w-1}, t^w]$ in (11) can be written as

$$\dot{x} = \Theta^{w^*T}\mathbb{U}_{N_w}^w(x) + \epsilon^w(x),\ t^{w-1} < t \leq t^w \tag{13}$$

for $w = 1, 2, \cdots$. Since Θ^w is estimated only at the end of the interval $(t^{w-1}, t^w]$, the adaptive identifier dynamics using (12) with previously updated parameter Θ^{w-1} can be expressed as

$$\dot{\hat{x}} = \Theta^{w-1^T}\mathbb{U}_{N_w}^{w-1}\left(\hat{x}\right) - K^w\left(x^{w-1} - \hat{x}\right),\quad t^{w-1} < t \leq t^w \tag{14}$$

$$\hat{x}(t) = \begin{cases}x(t^{w-1}) & t = t^{w-1}\\ \hat{x}(t) & t^{w-1} < t \leq t^w\end{cases} \tag{15}$$

where $x^{w-1} = x(t^{w-1})$ for $w = 1, 2, \cdots$ is the state measurement received at the identifier from the smart sensor at the time t^{w-1}, and K^w is the observer gain matrix to be designed in each time window $(t^{w-1}, t^w]$. Upon the arrival of the new packet, the identifier state $\hat{x}$ gets updated using (15) with the state $x(t^{w-1})$ at the time of the last node in w^{th} window.

Remark 2: Although the system states are measured at each time-node t_l^w within a window, the sensor transmits all the states measured within a window as a single packet. Therefore, we propose to update the identifier state $\hat{x}$ at the latest state information, i.e., the state measurement at the window transition time instant t^w only. Further, resetting the identifier state vector $\hat{x}$ with the lastest received system state vector x^{w-1} serves as initializing the identifier states at each window. This forces the identification error to reset to zero.

D. Parameter Update Law for Coefficient Estimation

To estimate the coefficients Θ^w in (12), in addition to the state measurements at the time-nodes, we also need the state derivative $\dot{x}$ information. We compute the state derivatives at each time-node.

The state vector derivative at l^{th} Chebyshev time-node in w^{th} window can be computed as $\dot{x}_l^w \approx \frac{x(t_l^w) - x(t_l^w - \Delta t)}{\Delta t}$, where $x\left(t_l^w - \Delta t\right)$ denotes the value of state variable picked out in close proximity of the Chebyshev time-node t_l^w.

Using the least-square method that minimizes the performance index $J = \sum_{i=1}^{N_w} \tilde{x}_i^{w^T}\tilde{x}_i^w + \theta_0^{w^T}R_0^w\theta_0^w$, the coefficients θ_j^w within the w^{th} window can be estimated as

$$\theta_j^w = \left(\bar{\mathbb{U}}_{N_w}^w\left(x^w\right)\bar{\mathbb{U}}_{N_w}^{w^T}\left(x^w\right) + R_0^w\right)^{-1}\left(R_0^w\theta_0^w + \bar{\mathbb{U}}_{N_w}^w\left(x^w\right)X^{j,w}\right), \tag{16}$$

for $w = 1, 2, \cdots$ and $j = 1, 2, \cdots n_p$ where $\tilde{x}_i$ is the state estimation error, θ_0^w is any arbitrary initial parameter, and R_0^w is a positive definite matrix of appropriate dimension to ensure the inverse in (17) exists. The regressor vector $\bar{\mathbb{U}}_{N_w}^w(x^w) = \begin{bmatrix}\mathbb{U}_{N_w}^w(x_1^w) & \mathbb{U}_{N_w}^w(x_2^w) & \cdots & \mathbb{U}_{N_w}^w(x_{N_w}^w)\end{bmatrix} \in \mathbb{R}^{(N_w+1)^{n_p} \times N_w}$ and $X^{j,w} = \begin{bmatrix}\dot{x}_1^{j,w} & \dot{x}_2^{j,w} & \cdots & \dot{x}_{Nw}^{j,w}\end{bmatrix}^T \in \mathbb{R}^{N_w}$ are the concatenated regression and j^{th} element of the state derivative vectors.

In a matrix form, the coefficient matrix

$$\Theta^w = \left(\bar{\mathbb{U}}_{N_w}^w\left(x^w\right)\bar{\mathbb{U}}_{N_w}^{w^T}\left(x^w\right) + R_0^w\right)^{-1}\left(R_0^w\Theta_0 + \bar{\mathbb{U}}_{N_w}^w\left(x^w\right)X^w\right), \tag{17}$$

where $X^w = \begin{bmatrix}X^{1,w} & X^{2,w} & \cdots & X^{n_p,w}\end{bmatrix} \in \mathbb{R}^{N_w \times n_p}$ and Θ_0 is the augmented θ_0.

E. Dynamic Node Selection

From (3), it is clear that the optimal number of nodes for the 1D system can be determined by minimizing the term $\prod_{l=1}^{N}(z - z_l)$. Therefore, to select the number of nodes for n_p-dimension that can guarantee a desired approximation accuracy ϵ_{th}, we define the average approximation error within a window as

$$e_w = \frac{1}{\tau^w}\int_{t^{w-1}}^{t^w}\left(\prod_{l=1}^{N_w}\left\|(\hat{x}(t) - x(t_l^w))\right\|\right)dt. \tag{18}$$

Using (3), the number of nodes N_w within a window satisfies $N_w! = F^{(N)}(\xi)\frac{\prod_{l=1}^{N_w}(z - z_l)}{\epsilon(z)}$.

Obtaining a closed-form solution for the number of nodes from the above expression is not possible. Taking natural log on both sides and approximating $N_w!$ using Stirling's approximation [18], i.e., $\ln(N_w!) \approx N_w \ln(N_w) - N_w$, one can develop a recursive solution to update the N_w. We propose Algorithm 1 for adaptively updating the number of nodes in each window. Here ϵ_{th} denotes the desired approximation error threshold, $\alpha > 0$

is the step size, and $\kappa > 1$ is the hysteresis factor to create a dead zone around the error tolerance threshold.

Algorithm 1: Node Selection for Next Time Window

Input: Initialize number of node N_w, prediction error e_w, error threshold ϵ_{th}, scale factor α, parameter κ
if $e_w > \epsilon_{th}$ **then**
  $N_{w+1} \leftarrow N_w + \left\lfloor \alpha \ln\left(\frac{e_w}{\epsilon_{\text{th}}}\right)\right\rfloor$
else if $e_w < \frac{\epsilon_{th}}{\kappa}$ **then**
  $N_{w+1} \leftarrow N_w + \left\lfloor \alpha \ln\left(\frac{e_w}{\epsilon_{\text{th}}}\right)\right\rfloor$
  if $N_{w+1} < 2$ **then**
    $N_{w+1} \leftarrow 2$
  end
end
if $\frac{\epsilon_{th}}{\kappa} \leq e_w \leq \epsilon_{th}$ **then**
  $N_{w+1} \leftarrow N_w$
end

Remark 3: From the recursive node update N_w, it is routine to check that the number of the nodes increases when the average error $e_w > \epsilon_{th}$ and decreases as the average error $e_w < \frac{\epsilon_{th}}{\kappa}$. The number of nodes remains constant for $\frac{\epsilon_{\text{th}}}{\kappa} \leq e_w \leq \epsilon_{\text{th}}$.

The dead zone prevents frequent adjustments of the number of nodes due to small fluctuations around the error threshold.

IV. Main Results

Before we present the main results, we define the following three errors. The state estimation error for the w^{th} window is given by

$$\tilde{x} = x - \hat{x}, \quad t^{w-1} < t \leq t^w, \tag{19}$$

The state measurement error due to the aperiodic availability of the state at the identifier is defined as

$$e_s^w = x^{w-1} - x, \quad t^{w-1} < t \leq t^w, \tag{20}$$

and the parameter estimation error is defined as

$$\tilde{\Theta}^w = \Theta^{w^*} - \Theta^w, \quad t^{w-1} < t \leq t^w. \tag{21}$$

Remark 4: In event-triggered control [13], the measurement error e_s^w is used to design the triggering condition that determines the sampling instants. For identification of a stable system, the error e_s^w is bounded, i.e., $\|e_s^w\| < e_{s,M}^w$.

The following lemma guarantees the convergence of the parameter estimation error to its ultimate bound.

Lemma 1: Consider the system in (1) in a parametric form (13) and the identifier in (14) and (15) over the time window $(t^{w-1}, t^w]$ for $w = 1, 2, \cdots$. Let the parameter Θ^w be updated using (17) in w^{th} time window. Then, the parameter estimation error $\tilde{\Theta}^w$ is uniformly bounded (UB).

Proof. The proof uses a discrete-time Lyapunov function to show the boundedness of the parameter estimation error within each time window. The complete proof is omitted due to space constraints.

Next, the boundedness of the state and parameter estimation errors are presented in the following theorem.

Theorem 1: Consider the system dynamics in (1), approximated using PS method in (13), and adaptive identifier dynamics in (14) and (15). Let the adaptive identifier parameters be updated using (17). Then the state $\tilde{x}$ and parameter estimation errors $\tilde{\Theta}^w$ are UB, provided the gain matrix K^w in each time window $(t^{w-1}, t^w]$ satisfies the Lyapunov equation

$$P^w K^w + K^{w^T} P^w = -Q^w \tag{22}$$

where P^w and Q^w are symmetric positive definite matrices satisfying $\lambda_{\min}(Q^w) > 4$.

Proof. The proof uses the Lyapunov approach for hybrid systems to ensure UB of the parameter and state estimation errors within the moving windows and window transition times. A Lyapunov function candidate that is a function of both $\tilde{\Theta}^w$ and $\tilde{x}$ in continuous time within each moving window is used. The boundedness of both errors during the window transitions are shown by combining the results with Lemma 1. The complete proof is omitted due to space constraints.

V. Simulation Results

Consider the following continuous-time 2D system to be identified using the Chebyshev approximation

$$\dot{x}_1 = -x_1 \sin(x_1) - x_2^3 \cos(x_1), \; \dot{x}_2 = -x_1 x_2^2 - x_2. \tag{23}$$

The following parameters were selected for the simulation. The time window $\tau_w = 0.1$ was selected as a constant, and the desired approximation error was chosen as $\epsilon_{\text{th}} = 10^{-4}$. The simulation was conducted over a total duration of $T =$ 3s with a fixed time step of 0.001s. For implementing the proposed approximation method, the initial nodes were set as two. The initial condition of the identifier in (14) was selected as $[1\ 1]^T$. In addition, the initial value of Θ^w for $w = 0$ was selected as $\begin{bmatrix}0.0005 \times \mathbf{1}_{9\times1} & -0.0005 \times \mathbf{1}_{9\times1}\end{bmatrix}$ where $\mathbf{1}$ is the vector with all element 1. The initial condition of the true system was selected as $[2\ 2]^T$. The node selection parameters in Algorithm 1 were considered as $\kappa = 100$ and $\alpha = 0.5$. To compute the identifier gain K^w using (22) within each window, the matrices P^w and Q^w were selected as $P^w = \text{diag}(10, 10)$, $Q^w = \text{diag}(5, 4.5)$, $R_0 = 0.001 * I_{(N_w+1)^2}$, and $\theta_0 = I$ where I is an identity matrix of the appropriate dimension for a respective window. ThThis resulted in a constant Hurwitz matrix for each window w. The simulation results are shown in Figures 2 to 4.

From Figure 2, it can be seen that the approximated system dynamics converge close to the actual dynamics within 0.1 s. The error in the first moving window is large due to arbitrary intial parameter Θ^0. With the increased number of nodes

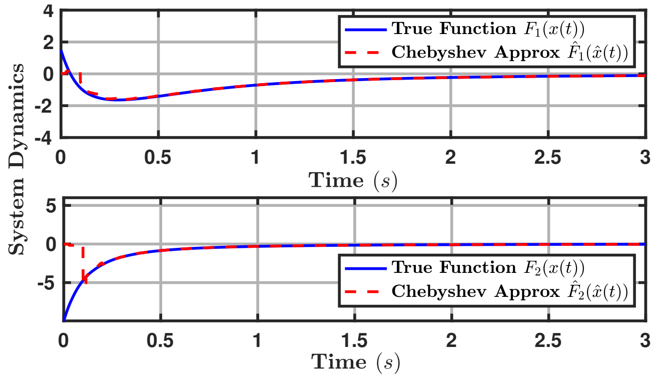

Fig. 2. Convergence of Chebyshev approximated dynamics $\hat{F}(\hat{x})$ to actual dynamics $F(x)$.

in the following time windows, the approximation error is reduced.

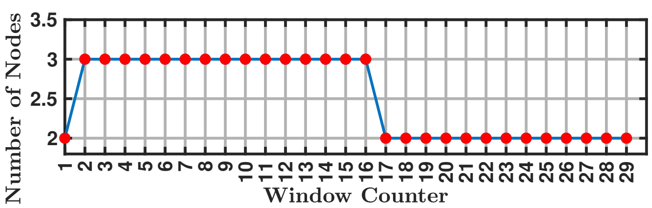

Fig. 3. Plot of the number of nodes at each time window

Figure 3 depicts the increment and decrement of nodes in each moving time window. At the 5^{th} window, the estimation error reaches 8.6×10^{-5}, which is less than the error threshold. But as κ is selected 100, the nodes aren't decreased until the 16^{th} interval, in which the estimation error is 8.8×10^{-7} which is less than $\varepsilon_{th} \times \frac{1}{\kappa} = 10^{-6}$. After 16^{th} interval, the number of nodes is dropped to 2 at the steady state. This demonstrates the efficacy of the proposed online approximation with minimal and aperiodic measurements of the states.

The state estimation errors, plotted in Figure 4, reset to zero at the start of each moving window since the estimated states are reset to the system.

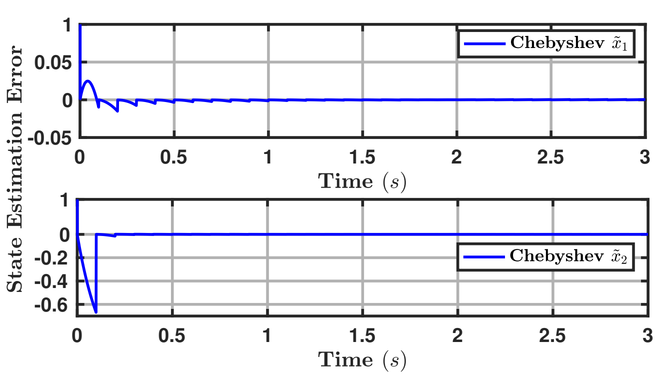

Fig. 4. Convergence of the state estimation errors $\tilde{x}$.

From the simulation results, it can be concluded that the adaptive identifier effectively approximates the system dynamics and state with a reduced number of measurements and desired approximation accuracy.

VI. Conculsion and Future Directions

This paper presented an online aperiodic sensing method for approximating the unknown nonlinear dynamics of a continuous-time nonlinear system using the Chebyshev polynomial of the second kind. The proposed adaptive identifier was able to approximate the system dynamics to achieve the desired approximation accuracy using a reduced number of data collected aperiodically at the time nodes. The numerical results also validated the analytical design. We plan to investigate the impact of time-varying windows on approximation in our future research.

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