A PERTURBATIVE APPROACH TO THE PARABOLIC OPTIMAL TRANSPORT PROBLEM

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ABSTRACT. Fix a pair of smooth source and target densities ρ and ρ^* of equal mass, supported on bounded domains $\Omega, \Omega^* \subset \mathbb{R}^n$. Also fix a cost function $c_0 \in C^{4,\alpha}(\overline{\Omega} \times \overline{\Omega^*})$ satisfying the weak regularity criterion of Ma, Trudinger, and Wang, and assume Ω and Ω^* are uniformly c_0- and c_0^*-convex with respect to each other. We consider a parabolic version of the optimal transport problem between (Ω, ρ) and (Ω^*, ρ^*) when the cost function c is a sufficiently small C^4 perturbation of c_0, and where the size of the perturbation depends on the given data. Our main result establishes global-in-time existence of a solution $u \in C_x^2 C_t^1(\overline{\Omega} \times [0, \infty))$ of this parabolic problem, and convergence of $u(\cdot, t)$ as $t \to \infty$ to a Kantorovich potential for the optimal transport map between (Ω, ρ) and (Ω^*, ρ^*) with cost function c. This is the first convergence result for the parabolic optimal transport problem when the cost function c fails to satisfy the weak Ma-Trudinger-Wang condition by a quantifiable amount.

1. INTRODUCTION

The optimal transport problem is intimately tied to the theory of second-order elliptic equations via a fully nonlinear PDE of Monge-Ampère type coupled with the so-called second boundary condition [29]. This connection suggests a natural method for proving the existence of optimal maps: solve a parabolic version of this PDE whose stationary state is a solution of the elliptic problem, and let the time variable tend to infinity. We refer to this as the *parabolic optimal transport problem*; see (Par OT) in Section 2 for a precise formulation. Such an asymptotic approach has been shown to work in the papers [24, 25, 1, 32] and provides a natural algorithm for approximating optimal maps owing to its fast rate of convergence to the steady state [24, 1].

Existing results for the parabolic optimal transport problem make crucial use of the so-called weak Ma-Trudinger-Wang (MTW) condition (see Definition 2.4). This is a sign condition on a fourth-order tensor related to the cost function that is necessary for the regularity of optimal maps even in the stationary case [27]. Due to the fact that the weak-MTW condition is difficult to verify and is not satisfied by many cost functions, it is a natural

question to determine if, given fixed source and target measures, one can solve the parabolic optimal transport problem when the cost function fails to satisfy (weak-MTW) by a quantifiable amount. Our results in this paper show that, given a fixed pair of source and target measures and a cost function c_0 that satisfies (weak-MTW), the solution to the parabolic problem (Par OT) exists for all time and converges to a solution of the elliptic problem when the cost function c is a sufficiently small C^4 perturbation of c_0; the smallness of this perturbation depends on the source and target measures and other structural quantities. We stress that c *does not* necessarily satisfy (weak-MTW).

Our main theorem is as follows. We refer to Section 2 for precise definitions and terminology.

Theorem 1.1. *Let $\Omega, \Omega^* \subset \mathbb{R}^n$ be smooth, bounded domains and let $\rho(x)\, dx$, $\rho^*(y)\, dy$, be absolutely continuous measures supported on Ω, Ω^* respectively satisfying* (Den Bds) *and* (Mass Bal). *Suppose there exists $r_0 > 0$ such that the cost function $c_0 \in C^{4,\alpha}(\overline{\Omega} \times \overline{\mathcal{N}_{r_0}(\Omega^*)})$ for some $\alpha \in (0,1]$ satisfies the conditions* (Bi-Twist), (Non-Deg), *and* (weak-MTW). *Assume, in addition that the domains Ω, Ω^* satisfy* (Dom c-Conv) *and* (Tar c^*-Conv) *with respect to c_0. Then there exists a constant $\hat{R} > 0$ depending only the structure (see Definition 2.1) such that if $c \in C^{4,\alpha}(\overline{\Omega} \times \overline{\mathcal{N}_{r_0}(\Omega^*)})$ satisfies $||c - c_0||_{C^4(\overline{\Omega} \times \overline{\mathcal{N}_{r_0}(\Omega^*)})} \leq \hat{R}$, then there exists a locally uniformly c-convex function $u_{initial} \in C^{4,\alpha}(\Omega)$ satisfying* (IC) *such that a solution $u \in C_x^2 C_t^1(\overline{\Omega} \times [0, \infty))$ of the flow* (Par OT) *exists for all time and $u(\cdot, t)$ converges in $C^2(\overline{\Omega})$ as $t \to \infty$ to a c-convex function u^∞, which is a Kantorovich potential for the optimal transport problem between (ρ, Ω) and (ρ^*, Ω^*) with cost c.*

Remark 1.2. We can actually obtain exponential convergence of the flow from our previous result [1], where the (weak-MTW) condition was not used.

1.1. **A Key Example.** Let $c_0(x,y) := \frac{1}{2}|x - y|^2$. Assume Ω, Ω^* are fixed uniformly convex domains such that $\text{dist}(\Omega, \Omega^*) > 0$. Let ρ, ρ^* be fixed measures supported on Ω, Ω^* respectively and satisfying (Den Bds) and (Mass Bal). For $p \in [-2, 2] \setminus \{0\}$, consider the cost $c(x,y) := \frac{1}{p}|x - y|^p$. It is known that c satisfies (Bi-Twist) and (Non-Deg) for $p \in [-2, 2] \setminus \{0, 1\}$ (provided $\text{dist}(\Omega, \Omega^*) > 0$) but fails to satisfy (weak-MTW) for any $p \notin [-2, 1) \cup \{2\}$; see [33, Section 8, Example 4].

As a consequence of Theorem 1.1, we have the following result for the flow (Par OT) with p-th power costs:

Proposition 1.3. *Let $c_0, \Omega, \Omega^*, \rho, \rho^*$ be as above. Suppose there exist constants $M_1, M_2 > 0$ such that*

$$0 < M_1 \leq |x - y| \leq M_2 \text{ for all } (x, y) \in \Omega \times \Omega^*.$$

Then there exist $\gamma_0, r_0 > 0$ depending only on M_1, M_2 and the structure of the problem such that if $|p \pm 2| \leq \gamma_0$, then $||c - c_0||_{C^4(\overline{\Omega} \times \overline{\mathcal{N}_{r_0}(\Omega^*)})} \leq \hat{R}$ (where $\hat{R}$ is as in Theorem 1.1). Consequently, there exists an initial condition for which the solution of the flow (Par OT) *corresponding to c exists for all time and converges in $C^2(\overline{\Omega})$ to a solution of the optimal transport problem between (ρ, Ω) and (ρ^*, Ω^*) for the p-th power cost with $|p \pm 2| \leq \gamma_0$.*

This is a parabolic counterpart (when perturbing around $p = 2$) of results obtained previously by Caffarelli-González-Nguyen [6] and Chen-Figalli [9] in the elliptic case under slightly different assumptions on the cost and densities. We refer the reader to Subsection 1.3 for more details on these and related works.

1.2. **Remarks on the main theorem and proof strategy.**
As stated previously, the condition (weak-MTW) is not necessarily preserved by C^4 perturbations. By a counterexample of Loeper in [27], it is known the condition is sharp in the sense that if a cost function fails to satisfy (weak-MTW), then there exist smooth source and target measures for which the optimal transport map is discontinuous. However, since the threshold value $\hat{R}$ in the statement of Theorem 1.1 depends on the structure of the problem (in particular on the particular choices of ρ and ρ^*), our results do not violate Loeper's counterexample. We also do not require any special structure on the source and target measures beyond (Mass Bal), (Den Bds), and sufficient smoothness.

Let us comment on the strategy behind the proof of Theorem 1.1. We note that the conclusion of the theorem does not immediately follow from an inverse function theorem type argument. In fact, such an approach would require a version of the inverse function theorem on Frechét manifolds, which is known to be false in general. As such, we must revisit the approach taken in [25] for proving infinite-time existence and convergence of the flow (Par OT), taking care to not make use of the (weak-MTW) condition. A number of a priori estimates from [25] hold true even in the non-MTW setting, but a missing essential ingredient is the global C^2 estimate. In [25], the interior C^2 estimate makes crucial use of (weak-MTW), while the boundary C^2 estimate relies on the validity of an appropriate interior estimate. To circumvent the role played by (weak-MTW), we quantify the failure of (weak-MTW) and establish a dichotomy (see Proposition 4.2) that allows us to exclude blow-up of the C^2 norm of the solution if the initial data is chosen appropriately. Such a dichotomy argument relies on the C^4 closeness of the cost function to one that satisfies (weak-MTW), and is in the spirit of [34]. The construction of appropriate initial data requires an implicit function theorem argument which also makes use of the C^4 closeness to an MTW cost.

1.3. **Comments on related literature.**

1.3.1. *Perturbation results for elliptic Monge-Ampère equations.* Only a limited number of works address the regularity of solutions to elliptic Monge-Ampère equations arising from optimal transport problems when the cost function does not satisfy (weak-MTW). The first such result was obtained by Caffarelli, González, and Nguyen [6], who consider the cost function $c(x,y) = \frac{1}{p}|x-y|^p$ with $p \approx 2$ and domains Ω, Ω^* that are uniformly convex and a positive distance apart. As mentioned previously, these cost functions do not satisfy (weak-MTW) unless $p = 2$. Global versions of the results in [6] were obtained by Chen and Figalli [9]. Their more recent work [10] investigates the regularity of optimal maps when the cost function is a C^2 perturbation of an MTW cost. The proof strategy in all the aforementioned works uses localization arguments originally due to Caffarelli [7] and shows that the optimal map for the perturbed cost inherits regularity from the optimal map for the quadratic cost (or some other fixed cost satisfying (weak-MTW)). We note that our approach in the present paper, aside from being in the parabolic setting, differs from that in [6, 9, 10] as we do not perform any localization arguments; however, we do require higher regularity of the cost function as well as the source and target measures. Finally, the work of Warren [34] establishes regularity results for optimal transport problems between log-concave mass distributions supported on small balls with cost functions that are sufficiently close to the quadratic cost. As mentioned in Subsection 1.2, our dichotomy argument for establishing the parabolic C^2 estimates is inspired by the corresponding elliptic estimates in [34]. However we require neither log-concavity of the measures nor smallness of their supports, both of which are essential ingredients in Warren's proof.

1.3.2. *Perturbation of distance-squared cost on Riemannian manifolds.* A different class of perturbative results exists in the optimal transport literature due to work of Figalli, Rifford, Villani, and Loeper [28, 18, 19, 20]. These papers consider C^4 perturbations of canonical metrics on Riemannian manifolds with cost function given by the Riemannian distance squared. The qualitative difference between these results and ours is that the perturbed cost functions considered in [28, 18, 19, 20] end up satisfying a form of (weak-MTW); in fact, this is one of the main contributions of the aforementioned papers. A separate collection of perturbative results is due to Delanoë and Ge [13, 15, 16, 14]. These works consider distance-square costs for Riemannian metrics whose Gauss curvature is almost constant, and prove regularity of the optimal map.

1.3.3. *Gradient flow of the cost functional.* The flow (Par OT) provides a natural algorithm for approximating optimal maps, and is one of only a handful of known asymptotic methods that have been proposed for solving the optimal transport problem [2, 17]. Among these, the one most relevant to the present paper originates in the work of Angenent, Haker, and Tannenbaum [2]. While the approach in [2] easily adapts to arbitrary cost

functions and densities, it has been most studied in the case

$$(1) \quad c(x,y) = |x-y|^2, \quad \rho = \mathcal{L}^n \llcorner \Omega, \quad \rho^* \text{ a.c. w.r.t } \mathcal{L}^n, \text{ supp}(\rho^*) = \Omega^*.$$

The key idea in [2] is to consider a one-parameter family of admissible maps T^t such that the function

$$t \mapsto \mathcal{C}(T^t) := \int_\Omega |x - T^t(x)|^2 \, dx$$

is decreasing. This can be achieved if T^t solves the nonlinear and non-local vectorial transport problem

$$(\text{AHT}) \qquad \begin{cases} \dot{T}^t + (\mathbf{v}^t \cdot \nabla)T^t = 0, \\ \mathbf{v}^t = \mathbb{P}T^t, \\ (T^0)_{\#}\rho = \rho^*, \end{cases}$$

where $\mathbb{P}$ is the so-called Leray projector onto divergence-free vector fields in Ω satisfying a no-flux condition on $\partial\Omega$. The short-time existence of the flow (AHT) and infinite-time existence of a regularized version is proved in [2]; see [31, Chapter 6, Section 6.2] for additional results in two dimensions. Note that any gradient is a fixed point of (AHT). It is an interesting and challenging problem to determine conditions on T^0 guaranteeing the infinite-time existence and convergence of (AHT) to an admissible map that is also the gradient of a convex function; by a celebrated theorem of Brenier [5], such a map is the unique optimal transport map between ρ and ρ^* for the quadratic cost. Some progress towards solving this problem has been made in the recent paper [30], whose authors show that if T^0 is sufficiently close in $H^s(\Omega)$ for some $s > 1 + \frac{n}{2}$ to the gradient of a uniformly convex function φ (where the closeness depends on φ), then T^t converges exponentially fast in $H^{s-1}(\Omega)$ as $t \to \infty$ to the optimal map. An appealing aspect of the result in [30] is that it applies to smooth domains Ω, Ω^* that are not necessarily convex; note that, in this case, the optimal map need not even be continuous, owing to counterexamples constructed by Caffarelli [8] (also see [22]). On the other hand, the conditions imposed on T^0 are rather stringent; moreover, each target density ρ^* will require an appropriate choice of T^0 to ensure the convergence of the flow. In comparison, the results in the present paper, combined with our previous works [25, 1], show that for the configuration (1) with Ω, Ω^* convex, the parabolic Monge-Ampère-type equation (Par OT) exhibits desirable asymptotic behavior when initialized using a single function u_{initial} for *any* target density ρ^* supported on Ω^*. The function u_{initial} can also be constructed explicitly by solving the optimal transport problem for the configuration (1) with $\rho^* := \frac{\text{Vol}(\Omega)}{\text{Vol}(\Omega^*)}\mathcal{L}^n \llcorner \Omega^*$.

1.3.4. *Application to numerics.* We mention that solving the parabolic optimal transport equation (Par OT) has potential applications to numerical approximations of optimal transport. By combining approaches for elliptic Monge-Ampère equations with second boundary value conditions (for

example, as in [3]) with a time-discretization, it seems feasible to implement solvers for the parabolic problem. As mentioned in Remark 1.2, the parabolic flow exhibits exponential convergence to the steady state solution (and the constants involved can be explicitly determined); hence such a solver would converge very quickly to the true solution. Our result in this paper indicates that such an approach may be fruitful for certain source and target measures even when the cost function does not satisfy (weak-MTW), a class for which available numerical methods are virtually nonexistent. A simple implementation using a finite difference scheme is available for the cost $c(x, y) = \frac{1}{2}|x - y|^2$ in one spatial dimension; good behavior of the error is shown in [4].

1.4. Additional Remarks.

Remark 1.4. The C^4 closeness of c to c_0 in $\Omega \times \mathcal{N}_{r_0}(\Omega^*)$ will guarantee that c satisfies (Bi-Twist). However, in certain situations, (Bi-Twist) can be verified independently (for example, in the case of the p-th power cost). For another class of such examples, let $c_0(x, y) := \frac{1}{2}|x - y|^2$. Consider the cost $c(x, y) = c_0(x, y) + \eta(x, y)$ for some smooth function η. Let us assume that η satisfies the following *anti-monotonicity* condition:

$$y \mapsto -\nabla_x \eta(x, y) \text{ is a monotone map} \quad \forall x \in \overline{\Omega},$$

(2) $$\qquad x \mapsto -\nabla_y \eta(x, y) \text{ is a monotone map} \quad \forall y \in \overline{\Omega^*}.$$

We claim if η satisfies (2), then c satisfies (Bi-Twist). Indeed,

$$\nabla_x c(x, y) = x - y + \nabla_x \eta(x, y).$$

Consequently, if there exist $x_0 \in \Omega$ and $y_1, y_2 \in \Omega^*$ such that $\nabla_x c(x_0, y_1) = \nabla_x c(x_0, y_1)$, then

$$y_2 - y_1 = \nabla_x \eta(x_0, y_2) - \nabla_x \eta(x_0, y_1).$$

This implies

$$|y_2 - y_1|^2 = \langle \nabla_x \eta(x_0, y_2) - \nabla_x \eta(x_0, y_1), y_2 - y_1 \rangle \leq 0,$$

where in the final inequality we have used (2). We thus conclude $y_2 = y_1$. Carrying out a similar argument in the x variables, we conclude that c satisfies (Bi-Twist) whenever η satisfies (2).

Remark 1.5. The C^4 closeness of c to c_0 also allows us to prove existence of appropriate initial conditions u_{initial} for the flow. As the arguments in Section 3 will show, it is possible to construct u_{initial} by solving a Poisson equation with an oblique boundary condition. In addition, the initial condition for the cost c will be close to that of c_0, and the initial condition for c_0 can be chosen to be the solution of the optimal transport problem for c_0 with constant densities; this function is guaranteed to be $C^{2,\alpha}$ smooth by regularity theory for cost functions satisfying (weak-MTW) [33].

1.5. **Outline of the paper.** The remainder of this paper is structured as follows. Section 2 sets up the necessary notation to precisely state the parabolic optimal transport problem and recalls a number of useful estimates from [25]. Section 3 is devoted to the construction of suitable initial data for the flow (Par OT) when the cost function is sufficiently close to one satisfying (weak-MTW). Section 4 establishes the crucial second derivative estimates necessary for the infinite-time existence and convergence to steady state of the flow (Par OT), which is proved in Section 5. Appendix A collects a number of useful facts about a certain n-th degree polynomial that will be used in the proof of the C^2 estimates in Section 4.

2. Preliminaries

2.1. **Notation and Setup.** We will assume from here onward that Ω, Ω^* are open, smooth, bounded domains in $\mathbb{R}^n$. The outward-pointing unit normal to $\partial\Omega$ (resp. $\partial\Omega^*$) will be denoted by ν (resp. ν^*). The function h^* will be a C^2, normalized defining function for Ω^*; i.e. $h^* = 0$ on $\partial\Omega^*$, $h^* < 0$ on Ω^*, and $\nabla h^* = \nu^*$ on $\partial\Omega^*$ (for existence of such a function, see [25, Appendix A]). The measures $\rho(x)dx, \rho^*(y)dy$ will be assumed to be absolutely continuous with respect to n-dimensional Lebesgue measure and satisfy

(Den Bds) $\quad 0 < \lambda \le \rho, \rho^* \le \lambda^{-1} < \infty$ for a constant λ, and

(Mass Bal) $$\int_\Omega \rho = \int_{\Omega^*} \rho^*.$$

We will also assume $c \in C^{4,\alpha}(\overline{\Omega} \times \overline{\Omega^*})$ for some $\alpha \in (0,1]$, and satisfies the bi-twist conditions:

$$y \mapsto -\nabla_x c(x,y) \text{ is injective } \forall x \in \overline{\Omega},$$

(Bi-Twist) $\qquad x \mapsto -\nabla_y c(x,y) \text{ is injective } \forall y \in \overline{\Omega^*}.$

In addition, we assume that

(Non-Deg) $\qquad\qquad \det D^2_{x,y} c(x,y) \ne 0.$

For any $p \in -\nabla_x c(x, \Omega^*)$ and $x \in \Omega$, (resp. $q \in -\nabla_y c(\Omega, y)$ and $y \in \Omega^*$), we denote by $\exp^c_x(p)$ (resp. $\exp^{c^*}_y(q)$) the unique element of Ω^* (resp. Ω) such that

(3) $\qquad -\nabla_x c(x, \exp^c_x(p)) = p, \quad -\nabla_y c(\exp^{c^*}_y(q), y) = q.$

We say Ω *is c-convex with respect to* Ω^* if the set $-\nabla_y c(\Omega, y)$ is a convex set for each $y \in \Omega^*$. Similarly, Ω^* *is c^*-convex with respect to* Ω if the set $-\nabla_x c(x, \Omega^*)$ is a convex set for each $x \in \Omega$. Analytically, these conditions are satisfied if
(Dom c-Conv)
$$\left[\nu^j_i(x) - c^{\ell,k} c_{ij,\ell}(x,y) \nu^k(x) \right] \tau^i \tau^j \ge \delta |\tau|^2 \quad \forall\, x \in \partial\Omega,\ y \in \overline{\Omega^*},\ \tau \in T_x(\partial\Omega)$$

and
(Tar c^*-Conv)
$$\left[(\nu^*)_i^j(y) - c^{k,\ell}c_{\ell,ij}(x,y)(\nu^*)^k(x)\right](\tau^*)^i(\tau^*)^j \geq \delta^*|\tau^*|^2 \quad \forall\, y \in \partial\Omega^*,\ x \in \overline{\Omega},\ \tau^* \in T_y(\partial\Omega^*)$$

for some constants δ, $\delta^* \geq 0$ respectively, where we will always sum over repeated indices; here subscripts before and after a comma indicate partial derivatives with respect to x and y respectively, and $c^{i,j}$ is the (i,j)-entry of the inverse of the matrix $D_{x,y}^2 c$. If δ (resp. δ^*) is strictly positive, we say that Ω *is uniformly c-convex with respect to Ω^** (resp. Ω^* *is uniformly c^*-convex with respect to Ω*).

For the remainder of the paper, we will fix a distinguished cost function c_0, along with domains Ω and Ω^* that are uniformly c_0- and c_0^*-convex with respect to each other. Additionally, we assume that c_0 satisfies (Bi-Twist) and (Non-Deg) on $\overline{\Omega} \times \overline{\mathcal{N}_{r_0}(\Omega^*)}$ for some fixed radius $r_0 > 0$ small enough so that $\mathcal{N}_{r_0}(\Omega^*)$ has C^1 boundary; such an r_0 always exists due to regularity of the distance function of a smooth domain in a small neighborhood of the boundary.

Definition 2.1. We will say that a constant *depends on the structure of the problem* if it depends only on the following quantities:

  - r_0, n, $\mathrm{diam}(\Omega)$, $\mathrm{diam}(\Omega^*)$;
  - $\|\rho\|_{C^2(\overline{\Omega})}$, $\|\rho^*\|_{C^2(\overline{\Omega^*})}$, and the constant λ in (Den Bds);
  - $\|c_0\|_{C^4(\overline{\Omega} \times \overline{\mathcal{N}_{r_0}(\Omega^*)})}$, the supremum of $\|(D_{x,y}^2 c_0)^{-1}\|$ over $\overline{\Omega} \times \overline{\mathcal{N}_{r_0}(\Omega^*)}$, and the constants δ and δ^* in (Dom c-Conv) and (Tar c^*-Conv).

Definition 2.2. (i) A function $\varphi : \Omega \to \mathbb{R}$ is said to be *c-convex* if for any $x_0 \in \Omega$, there exists $y_0 \in \Omega^*$ and $z_0 \in \mathbb{R}$ such that

$$\varphi(x_0) = -c(x_0, y_0) + z_0,$$
$$\varphi(x) \geq -c(x, y_0) + z_0, \quad \forall x \in \Omega.$$

 (ii) We say φ is *strictly c-convex* if it is c-convex, and the second inequality above is strict whenever $x \neq x_0$.
 (iii) A function $\varphi \in C^2(\overline{\Omega})$ is said to be *locally, uniformly c-convex* if $D^2\varphi(x) - A^c(x, \nabla\varphi(x)) > 0$ as a matrix for every $x \in \overline{\Omega}$, where A^c is defined below.

Remark 2.3. We note here that the sign convention on the cost function c is negative that of [33, 25, 1]. We have made this choice in order to express the associated optimal transport problem as a minimization, while maintaining that our potential functions are c-convex (supported from below by functions constructed from c). However, due to our choice of sign for the matrix function A, all expressions in terms of A will be the same as in the above mentioned references.

For a function $u \in C_x^2 C_t^1(\overline{\Omega} \times [0, \infty))$ (which, in the sequel, will be the solution to a parabolic optimal transportation problem), we will employ the

following notation (if the cost function c and the function u are clear from context, we may suppress them from the notation at times):

(i) $A^c(x,p) := -D_x^2 c(x,y)|_{y=\exp_x^c(p)}$

(ii) $T^{c,u}(x,t) = \exp_x^c(\nabla u(x,t))$

(iii) $B^c(x,p) = |\det (D_{x,y}^2 c)(x,y)|_{y=\exp_x^c(p)}| \cdot \frac{\rho(x)}{\rho^*(\exp_x^c(p))}$

(iv) $G^c(x,p) = h^*(\exp_x^c(p))$

(v) $\beta^{c,u}(x,t) = \nabla_p G^c(x,p)|_{p=\nabla u(x,t)}$

(vi) $W^{c,u}(x,t) = D^2 u(x,t) - A^c(x,\nabla u(x,t))$

The components of the matrix $W^{c,u}(x,t)$ will be denoted by $w_{ij}^{c,u}$, while the components of the inverse matrix will be denoted by $w_{c,u}^{ij}$.

Finally, we recall the weak Ma-Trudinger-Wang condition, first introduced in [29] in a stronger form.

Definition 2.4. The cost function $c(x,y)$ satisfies the *weak Ma-Trudinger-Wang* condition if

(weak-MTW)
$$D_{p_i p_j}^2 A_{k\ell}^c(x,p) V^i V^j \eta^k \eta^\ell \geq 0 \quad \text{for all } x \in \overline{\Omega}, \ p \in -\nabla_x c(x,\overline{\Omega^*}), \ V \perp \eta.$$

2.2. **The Parabolic Optimal Transport Problem.** Using the above notation, we can now precisely state the parabolic optimal transportation problem. Given a cost function c satisfying (Bi-Twist) and (Non-Deg), domains Ω, Ω^* satisfying the convexity conditions (Dom c-Conv) and (Tar c^*-Conv), absolutely continuous measures ρ, ρ^* supported on Ω, Ω^* respectively and satisfying (Mass Bal) and (Den Bds), and a locally, uniformly c-convex (as in Definition 2.2, (iii)) function u_{initial} satisfying the compatibility conditions (IC) below, we seek to find a function $u \in C_x^2 C_t^1(\overline{\Omega} \times [0, \infty))$ that solves the evolution equation

(Par OT)
$$\begin{cases} \dot{u}(x,t) = \log \det(W^{c,u}(x,t)) - \log B^c(x, \nabla u(x,t)), & x \in \Omega, \ t > 0 \\ G^c(x, \nabla u(x,t)) = 0, & x \in \partial\Omega, \ t > 0 \\ u(x,0) = u_{\text{initial}}(x), & x \in \Omega. \end{cases}$$

Here, a dot indicates differentiation in the time variable. The function u_{initial} must satisfy the compatibility conditions

(IC)
$$\begin{cases} u_{\text{initial}} \in C^{4,\alpha}(\overline{\Omega}) \text{ for some } \alpha \in (0,1] \\ G^c(x, \nabla u_{\text{initial}}(x)) = 0 \text{ on } \partial\Omega \\ T_{\text{initial}}^c(\Omega) = \Omega^*, \text{ where } T_{\text{initial}}^c(x) := \exp_x^c(\nabla u_{\text{initial}}(x)), \end{cases}$$

It follows from the short-time existence result [25, Theorem 4.4] that for any given $c, \Omega, \Omega^*, \rho, \rho^*$ satisfying the conditions outlined in Subsection 2.1, there is a time $t_{\max} > 0$ depending on $c, \Omega, \Omega^*, \rho, \rho^*$, and u_{initial} such that the solution u of (Par OT) exists on $\overline{\Omega} \times [0, t_{\max})$. Since in this paper we will fix $\Omega, \Omega^*, \rho, \rho^*$ but vary the cost function c in a neighborhood of the distinguished cost c_0, and construct initial data u_{initial} corresponding to

each choice of c, we will often highlight the dependence of $t_{\max}$ on c. If, in addition, the cost function c satisfies the (weak-MTW) condition, then the main result of [25] shows that $t_{\max}(c) = +\infty$. As mentioned in the introduction, one of the goals of this paper is to show that $t_{\max}(c) = +\infty$ for certain cost functions c that fail to satisfy the (weak-MTW) condition, given ρ and ρ^*.

We end this section by collecting those uniform estimates established in [25] that do not require the (weak-MTW) condition.

Proposition 2.5. *Suppose $c \in C^{4,\alpha}(\overline{\Omega} \times \mathcal{N}_{r_0}(\overline{\Omega^*}))$ satisfies (Bi-Twist), and Ω and Ω^* are respectively uniformly c- and c^*-convex with respect to each other. Let $u \in C_x^4 C_t^2(\overline{\Omega} \times [0, t_{\max}))$ be a solution of (Par OT) for some $t_{\max} > 0$. Then there are constants $K_1, K_2 > 0$ depending only on the structure of the problem and $\|u_{initial}\|_{C^2(\overline{\Omega})}$ such that,*

$$\max\left\{ \sup_{(x,t)\in\overline{\Omega}\times[0,t_{\max})} |\nabla u(x,t)|, \sup_{(x,t)\in\overline{\Omega}\times[0,t_{\max})} |\dot{u}(x,t)| \right\} \leq K_1,$$
$$\inf_{(x,t)\in\partial\Omega\times[0,t_{\max})} \langle \beta^{c,u}(x,t), \nu(x) \rangle \geq K_2.$$

Proof. This follows immediately from [25, Theorems 6.1, 8.1, 9.2] noting that none of the above estimates rely on the condition (weak-MTW). For those results which rely on [25, Lemma 5.7] (whose proof makes use of (weak-MTW)) we may instead use Proposition 2.6 below. $\square$

In [25, Section 5], the condition (weak-MTW) is used to show that the map T^c is injective and the inverse is related to a parabolic PDE corresponding to reversing the roles of the domains Ω and Ω^*; this "dual" problem is used to obtain the uniform obliqueness estimates in [25, Section 9]. Using recent partial regularity results established in [11], we can circumvent the use of (weak-MTW); for completeness, we provide an alternative proof of the relevant results of [25, Section 5] in the proposition below.

Proposition 2.6. *If u is a solution of (Par OT), then $T^c(\cdot, t)$ is injective for each $t \in [0, t_{\max}(c))$. Moreover, the function $u^* : \overline{\Omega^*} \times [0, t_{\max}(c)) \to \mathbb{R}$ defined by $u^*(y,t) := -c((T^c)^{-1}(y,t), t) - u((T^c)^{-1}(y,t), t)$ satisfies the (dual) boundary value problem*

(4)
$$\begin{cases} \dot{u}^*(y,t) = \log\det(W^{c^*,u^*}(y,t)) - \log B^{c^*}(y, \nabla u^*(y,t)), & x \in \Omega^*, \ t > 0 \\ G^{c^*}(y, \nabla u^*(y,t)) = 0, & y \in \partial\Omega^*, \ t > 0 \\ u^*(y,0) = -c((T_{initial}^c)^{-1}(y), y) - u_{initial}((T_{initial}^c)^{-1}(y)), & y \in \Omega^*, \end{cases}$$

where

(i) $B^{c^*}(y,q) = |\det (D_{y,x}^2 c)(x,y)_{x=\exp_y^{c^*}(q)}| \cdot \dfrac{\rho^*(y)}{\rho(\exp_y^{c^*}(q))}$

(ii) $G^{c^*}(y,q) = h(\exp_y^{c^*}(q))$

(iii) $W^{c^*,u^*}(y,t) = D^2 u^*(y,t) + D_y^2 c(x,y)_{x=\exp_y^{c^*}(\nabla u^*(y,t))}$

and h is a normalized defining function for Ω.

Proof. Since c is fixed, we supress the cost in the notation T^c for this proof. Fix $t \in [0, t_{\max}(c))$. By [11, Theorem 1.3], there exist closed sets $\Sigma \subset \Omega$ and $\Sigma^* \subset \Omega^*$, both of zero Lebesgue measure, such that $T(\cdot, t) : \Omega \setminus \Sigma \to \Omega^* \setminus \Sigma^*$ is a homeomorphism. Now suppose that $y := T(x_1, t) = T(x_2, t)$ for some $x_1 \neq x_2 \in \Omega$. Then we must have $x_1, x_2 \in \Sigma$ and $y \in \Sigma^*$. Since(Par OT) implies

$$\det D_x T = \det W^{c,u} |\det (D^2_{x,y}c)(x,T)|^{-1} = \frac{e^{\mathring{u}}\rho}{\rho^* \circ T} \neq 0$$

on Ω, by the inverse function theorem we can see there are open sets U_1, $U_2 \subset \Omega$ and $V \subset \Omega^*$ with $x_1 \in U_1$, $x_2 \in U_2$, and $y \in V$ where there exist local inverses $S_i : V \to U_i$ of $T(\cdot, t)$, $i = 1, 2$; without loss we may assume $U_1 \cap U_2 = \emptyset$. Since Σ^* has measure zero, there exists a point $y' \in V \setminus \Sigma^*$, and $S_i(y') \in U_i \setminus \Sigma$. However, this would imply that $T(S_1(y'), t) = y' = T(S_2(y'), t)$ which is a contradiction as $S_1(y') \neq S_2(y') \in \Omega \setminus \Sigma$. Thus $T(\cdot, t)$ is injective on Ω.

By the inverse function theorem, the map T^{-1} is differentiable on $\overline{\Omega^*} \times [0, t_{\max}(c))$, thus by differentiating the relation $u^*(y, t) = -c((T^c)^{-1}(y, t), t) - u((T^c)^{-1}(y, t), t)$ we can easily see u^* satisfies (4). $\qquad\square$

3. Construction of Initial Data

In this section we will work toward showing the existence of suitable initial conditions for (Par OT) corresponding to a cost function c sufficiently close to c_0. First we show that if c is sufficiently close to c_0 in $C^4(\overline{\Omega} \times \overline{\mathcal{N}_{r_0}(\Omega^*)})$, then c and the domains involved inherit various structural properties from c_0.

Lemma 3.1. *There is a constant $R_0 > 0$ depending only on the structure of the problem such that:*

    (1) *If $\|c - c_0\|_{C^4(\overline{\Omega} \times \overline{\mathcal{N}_{r_0}(\Omega^*)})} < R_0$, then Ω and Ω^* are uniformly c- and c^*-convex with respect to each other, with constants $\delta/2$ and $\delta^*/2$ in (Dom c-Conv) and (Tar c^*-Conv).*

    (2) *If $\|c - c_0\|_{C^4(\overline{\Omega} \times \overline{\mathcal{N}_{r_0}(\Omega^*)})} < R_0$, c satisfies conditions (Bi-Twist) and (Non-Deg) on $\overline{\Omega} \times \overline{\mathcal{N}_{r_0}(\Omega^*)}$.*

    (3) *Under the same conditions on the cost functions c, c_0 as in (2), if $u_0 \in C^{2,\alpha}(\overline{\Omega})$ is a c_0-convex function such that T^{c_0, u_0} is a homeomorphism on $\overline{\Omega}$ with $\det DT^{c_0, u_0}(x) \neq 0$ for all $x \in \overline{\Omega}$, and $\|u - u_0\|_{C^{2,\alpha}(\overline{\Omega})} < R_0$ for some $\alpha > 0$, then $T^{c,u}$ is also a homeomorphism on $\overline{\Omega}$; here R_0 may also depend on u_0.*

Proof. Claim (1) above and the fact that c will satisfy (Non-Deg) are clear if r_0 is sufficiently small, depending only on the structure of the problem.

Suppose by contradiction that some cost fails (Bi-Twist) for R_0 arbitrarily small. Then there exists a sequence of cost functions c_k converging to c_0 in

$C^4(\overline{\Omega} \times \overline{\mathcal{N}_{r_0}(\Omega^*)})$, and sequences of points $x_k \in \overline{\Omega}$, $y_{1,k} \neq y_{2,k} \in \overline{\mathcal{N}_{r_0}(\Omega^*)}$ such that $-\nabla_x c_k(x_k, y_{1,k}) = -\nabla_x c_k(x_k, y_{2,k})$. By compactness, we may pass to subsequences and assume that x_k, $y_{1,k}$, $y_{2,k}$ converge respectively to points x_∞, $y_{1,\infty}$, and $y_{2,\infty}$. This implies $-\nabla_x c_0(x_\infty, y_{1,\infty}) = -\nabla_x c_0(x_\infty, y_{2,\infty})$, and since c_0 satisfies (Bi-Twist), we must have $y_{1,\infty} = y_{2,\infty} =: y_\infty$. Since for k large we may assume c_k satisfies (Non-Deg), we see $D^2_{x,y} c_k(x_k, y)$ is invertible for any $y \in \overline{\mathcal{N}_{r_0}(\Omega^*)}$, moreover we can see that the operator norms of these inverses and the Lipschitz constants of the mappings $y \mapsto D^2_{x,y} c_k(x_k, y)$ are bounded by constants depending only the structure of the problem. Since Ω^* is c^*-convex with respect to Ω by the first claim in the lemma, it has Lipschitz boundary; we may thus combine [12, Theorem 1] with [26, Chapter XIV, Lemma 1.3] to see there is a neighborhood around y_∞ on which the mappings $y \mapsto -\nabla_x c_k(x_k, y)$ are invertible for all k sufficiently large. This is a contradiction for k large enough that both $y_{1,k}$ and $y_{2,k}$ belong to this neighborhood, hence we obtain claim (2).

Claim (3) follows in a similar manner, allowing for the extra dependency on u_0. Suppose for contradiction there exists a sequence of cost functions c_k converging to c_0 in $C^4(\overline{\Omega} \times \overline{\mathcal{N}_{r_0}(\Omega^*)})$ and a sequence of functions u_k converging to u_0 in $C^{2,\alpha}(\overline{\Omega})$ such that the maps $T_k := T^{c_k, u_k}$ are not homeomorphisms on $\overline{\Omega}$. Then there exist sequences of points $x_{1,k}, x_{2,k} \in \overline{\Omega}$ with $x_{1,k} \neq x_{2,k}$ such that $T_k(x_{1,k}) = T_k(x_{2,k})$ for each k. By compactness, we may pass to subsequences and assume that $x_{1,k}, x_{2,k}$ converge respectively to points $x_{1,\infty}, x_{2,\infty} \in \overline{\Omega}$. By the C^4 convergence of the c_k to c_0 and $C^{2,\alpha}$ convergence of the u_k to u_0, we have $C^{1,\alpha}$ convergence of T_k to $T_0 := T^{c_0, u_0}$, in particular we obtain $T_0(x_{1,\infty}) = T_0(x_{2,\infty})$. Since T_0 is a homeomorphism, this implies $x_{1,\infty} = x_{2,\infty} = x_\infty$. Now by assumption $DT_0(x_\infty)$ is invertible, and since u_k converges to u_0 in $C^{2,\alpha}(\overline{\Omega})$, for k sufficiently large we can make $\|DT_0(x_\infty) - DT_k(x_\infty)\|$ arbitrarily small. Thus we can use a combination of [12, Theorem 1] and [26, Chapter XIV, Lemma 1.3] to deduce that there exists $\delta_0 > 0$ depending only on T_0 such that, for all k sufficiently large, the maps T_k are invertible on $B_{\delta_0}(x_\infty)$. However, this results in a contradiction, as $x_{1,k}, x_{2,k} \in B_{\delta_0}(x_\infty)$ for k sufficiently large. $\square$

Lemma 3.2. *Suppose $c \in C^4(\overline{\Omega} \times \overline{\mathcal{N}_{r_0}(\Omega^*)})$ with $\|c - c_0\|_{C^4(\overline{\Omega} \times \overline{\mathcal{N}_{r_0}(\Omega^*)})} < \min\{R_0, \frac{r_0}{6 \sup \|(D^2_{x,y} c_0)^{-1}\|}\}$ where R_0 is the constant from Lemma 3.1. Then*

$$\mathcal{N}_{\frac{r_0}{2 \sup \|(D^2_{x,y} c_0)^{-1}\|}}(-\nabla_x c_0(x, \overline{\Omega^*})) \subset -\nabla_x c(x, \overline{\mathcal{N}_{r_0}(\Omega^*)}) \quad \forall\, x \in \overline{\Omega}.$$

Proof. Fix $x \in \overline{\Omega}$, and let us write

$$S_0(y) := -\nabla_x c_0(x, y), \quad S(y) := -\nabla_x c(x, y).$$

By Lemma 3.1 (2), S and S_0 are both homeomorphisms on $\overline{\mathcal{N}_{r_0}(\Omega^*)}$.

Suppose that $q \in \mathcal{N}_{\frac{r_0}{2\sup\|(D^2_{x,y}c_0)^{-1}\|}}(S_0(\overline{\Omega^*})) \setminus S(\overline{\mathcal{N}_{r_0}(\Omega^*)})$. Then there exists $y \in \overline{\Omega^*}$ such that $|q - S_0(y)| < \frac{r_0}{2\sup\|(D^2_{x,y}c_0)^{-1}\|}$. Also since $\mathcal{N}_{r_0}(\Omega^*)$ has C^1 boundary and S is a homeomorphism, there is an $s \in [0,1)$ such that $q_s := (1-s)S(y) + sq \in \partial S(\mathcal{N}_{r_0}(\Omega^*)) = S(\partial \mathcal{N}_{r_0}(\Omega^*))$. In particular, there exists $y_s \in \partial\mathcal{N}_{r_0}(\Omega^*)$ such that $q_s = S(y_s)$. Then,

$$
\begin{aligned}
\frac{r_0}{2\sup\|(D^2_{x,y}c_0)^{-1}\|} + \|S_0 - S\|_{C^0(\overline{\Omega})} &\geq |q - S(y)| > |q_s - S(y)| \\
&= |S(y_s) - S(y)| \\
&\geq |S_0(y) - S_0(y_s)| - |S_0(y_s) - S(y_s)| - |S_0(y) - S(y)| \\
&\geq [S_0^{-1}]^{-1}_{C^{0,1}(\overline{S_0(\mathcal{N}_{r_0}(\Omega^*))})}|y - y_s| - 2\|S_0 - S\|_{C^0(\overline{\Omega})} \\
&\geq [S_0^{-1}]^{-1}_{C^{0,1}(\overline{S_0(\mathcal{N}_{r_0}(\Omega^*))})}r_0 - 2\|S_0 - S\|_{C^0(\overline{\Omega})},
\end{aligned}
$$

but this is a contradiction after rearranging since $[S_0^{-1}]_{C^{0,1}(\overline{S_0(\mathcal{N}_{r_0}(\Omega^*))})} \leq \sup\|(D^2_{x,y}c_0)^{-1}\|$ and $\|S_0 - S\|_{C^0(\overline{\Omega})} \leq \|c - c_0\|_{C^4(\overline{\Omega}\times\overline{\mathcal{N}_{r_0}(\Omega^*)})}$. $\square$

Finally, we use an implicit function theorem argument to show suitable initial conditions for a cost c exist given a "good" initial condition for c_0.

Lemma 3.3. *Suppose $u_0 \in C^{2,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1]$, is strictly c_0-convex, is locally, uniformly c_0-convex, and satisfies $\int_\Omega u_0(x)\,dx = 0$. Assume, in addition, that T^{c_0,u_0} is a homeomorphism between $\overline{\Omega}$ and $\overline{\Omega^*}$ and $\det DT^{c_0,u_0} \neq 0$ on $\overline{\Omega}$. Then there exists $R_1 > 0$ depending on the structure of the problem and u_0, and a continuous mapping $\Psi : B_{R_1}^{C^4(\overline{\Omega}\times\overline{\mathcal{N}_{r_0}(\Omega^*)})}(c_0) \to C^{2,\alpha}(\overline{\Omega})$ such that for any $c \in B_{R_1}^{C^4(\overline{\Omega}\times\overline{\mathcal{N}_{r_0}(\Omega^*)})}(c_0)$,*

(5) $\Psi(c)$ *is strictly c-convex and locally, uniformly c-convex.*

(6) $T^{c,\Psi(c)}$ *is a homeomorphism between $\overline{\Omega}$ and $\overline{\Omega^*}$.*

Proof. For ease of notation, during this proof we will supress the first variable in the notation for G^c. Define $\mathcal{B}_1 := \overline{B_{\tilde{r}_0}^{C^4(\overline{\Omega}\times\overline{\mathcal{N}_r(\Omega^*)})}(c_0)}$, where $\tilde{r}_0 < \min\{R_0, \frac{r_0}{6\sup\|(D^2_{x,y}c_0)^{-1}\|}\}$ and R_0 is the radius obtained from (1)-(3) of Lemma 3.1, hence depends on the structure of the problem and u_0.

Now let $\mathcal{B}_2 := \{u \in \overline{B_{\tilde{r}_0}^{C^{2,\alpha}(\overline{\Omega})}(u_0)} \mid \int_\Omega u = 0\}$, and define the map $\Phi : \mathcal{B}_1 \times \mathcal{B}_2 \to C^{0,\alpha}(\overline{\Omega}) \times C^{1,\alpha}(\partial\Omega)$ by

$$\Phi(c,u) := (\Delta(u - u_0), G^c(\nabla u)|_{\partial\Omega}).$$

Note that $\Phi(c_0, u_0) = (0,0)$. Since T^{c_0,u_0} is a homeomorphism between $\overline{\Omega}$ and $\overline{\Omega^*}$, by Lemma 3.2 we have for any $x \in \overline{\Omega}$ and $u \in \mathcal{B}_2$,

$$\nabla u(x) \in \mathcal{N}_{\tilde{r}_0}(\nabla u_0(x)) = \mathcal{N}_{\tilde{r}_0}(-\nabla_x c_0(x, T^{c_0,u_0}(x))) \subset -\nabla_x c(x, \overline{\mathcal{N}_{r_0}(\Omega^*)}).$$

In particular, $G^c(x, \nabla u(x))$ is well-defined for all $x \in \overline{\Omega}$. Denoting the
Fréchet derivative of the map $u \mapsto \Phi(c_0, u)$ at u_0 by $D_u\Phi(c_0, u_0)$, we claim
that $D_u\Phi(c_0, u_0)$ is injective; here we consider $\mathcal{B}_2$ as a closed subset of the
Banach space $\mathcal{B} := \{u \in C^{2,\alpha}(\overline{\Omega}) \mid \int_\Omega u = 0\}$ with the $C^{2,\alpha}$ norm, then u_0 is
an interior point of $\mathcal{B}_2$. First note that an explicit calculation of the Fréchet
derivative of Φ (this requires C^4 smoothness of the cost c_0) shows that for
any $\phi \in \mathcal{B}$,

$$(7) \qquad D_u\Phi(c_0, u_0)(\phi) = (\Delta\phi, \langle D_p G^{c_0}(\nabla u_0), \nabla\phi\rangle|_{\partial\Omega}).$$

By Proposition 2.5 $G^{c_0}(\cdot, \nabla u_0)$ is uniformly oblique on $\partial\Omega$, hence by Hopf's
lemma combined with standard existence theory for Poisson's equation (for
example, [21, Theorem 6.31]) we see $D_u\Phi(c_0, u_0)$ is a bijection between $\mathcal{B}$ and
$C^{0,\alpha}(\overline{\Omega}) \times C^{1,\alpha}(\partial\Omega)$. Thus by the implicit mapping theorem [35, Theorem
4.B], there is a bijective, continuous mapping $\Psi : B_{R_1}^{C^4(\overline{\Omega}\times\overline{\mathcal{N}_r(\Omega^*)})}(c_0) \to \mathcal{V}$
for some radius $R_1 > 0$ and neighborhood $\mathcal{V}$ of u_0 in $\mathcal{B}$ such that $\Psi(c_0) = u_0$
and $\Phi(c, \Psi(c)) = (0, 0)$ for all $c \in B_{R_1}^{C^4(\overline{\Omega}\times\overline{\mathcal{N}_r(\Omega^*)})}(c_0)$. We may further
shrink R_1, but it will always be in a manner that depends only on the
structure of the problem and u_0, hence we can also assume that the size
of the neighborhood $\mathcal{V}$ also depends only on these parameters. We re-
mark here this implies that the quantities $\|c\|_{C^4(\overline{\Omega}\times\overline{\mathcal{N}_{r_0}(\Omega^*)})}$, $\|u\|_{C^{2,\alpha}(\overline{\Omega})}$, and
$\|(D_{x,y}^2 c)^{-1}\|_{C^4(\overline{\Omega}\times\overline{\mathcal{N}_{r_0}(\Omega^*)})}$ have upper bounds that also depend only on the
structure of the problem and u_0, provided R_1 is taken small enough.

Now, for any $c \in B_{R_1}^{C^4(\overline{\Omega}\times\overline{\mathcal{N}_{r_0}(\Omega^*)})}(c_0)$, we have $G^c(\cdot, \nabla\Psi(c)) \equiv 0$ on $\partial\Omega$;
this implies $T^{c,\Psi(c)}(\partial\Omega) \subset \partial\Omega^*$. Possibly taking R_1 even smaller, by Lemma
3.1 (3) we may assume that $T^{c,\Psi(c)}$ is a homeomorphism from $\overline{\Omega}$ to $T^{c,\Psi(c)}(\overline{\Omega})$,
hence a homeomorphism from $\partial\Omega$ to $T^{c,\Psi(c)}(\partial\Omega)$. By Lemma 3.1 (1), Ω is
uniformly c-convex with respect to Ω^* and Ω^* is uniformly c^*-convex with
respect to Ω, hence Ω and Ω^* are both homeomorphic to balls, in particular
$\partial\Omega$ and $\partial\Omega^*$ are connected. Thus we must have $T^{c,\Psi(c)}(\partial\Omega) = \partial\Omega^*$. Now
suppose for some $x \in \Omega$ we have $T^{c,\Psi(c)}(x) \notin \Omega^*$. Then writing $c_s :=
(1-s)c_0 + sc$, by continuity of the map Ψ the curve $\{T^{c_s,\Psi(c_s)}(x) \mid s \in [0,1]\}$
is continuous with $T^{c_0,\Psi(c_0)}(x) \in \Omega^*$. Thus for some value of s, we will
have $T^{c_s,\Psi(c_s)}(x) \in \partial\Omega^*$ contradicting that $T^{c_s,\Psi(c_s)}$ is injective on $\overline{\Omega}$ and
$T^{c,\Psi(c)}(\partial\Omega) = \partial\Omega^*$. Thus $T^{c,\Psi(c)}(\Omega) = \Omega^*$ proving the claim (6).

We will now show (5). Fix $c \in B_{R_1}^{C^4(\overline{\Omega}\times\overline{\mathcal{N}_{r_0}(\Omega^*)})}(c_0)$ and write $u := \Psi(c)$.
Since u_0 is locally, uniformly c_0-convex, by continuity and compactness the
smallest eigenvalues of the matrices $D^2 u_0(x) - A^{c_0}(x, \nabla u_0(x))$ have a strictly
positive lower bound independent of $x \in \overline{\Omega}$. Then there exists some $R > 0$
depending only on the structure of the problem and u_0 such that $D^2 u_0(x) -
A^{c_0}(x, \nabla u_0(x_0))$ also has a strictly positive bound independent of x and x_0,
whenever $|x - x_0| < R$. Next, possibly shrinking R_1, we can assume there
exists some $\epsilon_0 > 0$ depending only on the structure of the problem and u_0

such that

$$(8) \qquad D^2 u(x) - A^c(x, \nabla u(x_0)) \geq \epsilon_0 \mathrm{Id}, \quad \forall \, x_0, x \in \overline{\Omega}, \ |x - x_0| < R.$$

In particular, u is locally, uniformly c-convex. Now fix x, $x_0 \in \overline{\Omega}$ and let $y_0 := \exp^c_{x_0}(\nabla u(x_0))$ which belongs to $\overline{\Omega^*}$ by above. First suppose $x \neq x_0 \in \overline{\Omega}$ is such that $|-\nabla_x c(x, y_0) + \nabla_x c(x_0, y_0)| < \frac{R}{M_1}$ for an M_1 depending only the structure of the problem such that $\sup \|(D^2_{x,y} c)^{-1}\| < M_1$. Then write for $s \in [0, 1]$,

$$p_0 := -\nabla_y c(x_0, y_0) = \nabla u(x_0),$$
$$p := -\nabla_y c(x, y_0),$$
$$x(s) := \exp^{c^*}_{y_0}((1 - s)p_0 + sp).$$

Note by Lemma 3.1 (1), the set $-\nabla_y c(\overline{\Omega}, y_0)$ is uniformly convex, hence $x(s) \in \overline{\Omega}$. A quick calculation yields

$$\dot{x}_i(s) = -c^{k,i}(p - p_0)_k$$
$$\ddot{x}_i(s) = -c^{j,i} c_{kl,j} c^{m,k} c^{n,l} (p - p_0)_m (p - p_0)_n$$

where all terms involving c are evaluated at $(x(s), y_0)$. Using this we find that for any $s \in [0, 1]$,

$$|x(s) - x_0| \leq \int_0^s |\dot{x}(s)| ds \leq \sup \|(D^2_{x,y} c)^{-1}\| |p - p_0| < \frac{R \sup \|(D^2_{x,y} c)^{-1}\|}{M_1} < R.$$

Using Taylor expansion, we thus find some $\tilde{s} \in [0, 1]$ such that

$$u(x) - (-c(x, y_0) + c(x_0, y_0) + u(x_0))$$
$$= \langle \nabla u(x_0) + \nabla_x c(x_0, y_0), \dot{x}(0) \rangle$$
$$\quad + \langle [D^2 u(x(\tilde{s})) + D^2_x c(x(\tilde{s}), y_0)] \dot{x}(\tilde{s}), \dot{x}(\tilde{s}) \rangle + \langle \nabla u(x(\tilde{s})) + \nabla_x c(x(\tilde{s}), y_0), \ddot{x}(\tilde{s}) \rangle$$
$$\geq \epsilon_0 |((-D^2_{x,y} c(x_0, y_0))^T)^{-1}(p - p_0)|^2$$
$$\quad - \sup \|((D^2_{x,y} c)^T)^{-1}\| \|c\|_{C^4(\overline{\Omega} \times \overline{\mathcal{N}_r(\Omega^*)})} |\nabla u(x(\tilde{s})) + \nabla_x c(x(\tilde{s}), y_0)| |((-D^2_{x,y} c(x(\tilde{s}), y_0))^T)^{-1}(p - p_0)|^2$$

(9)

where we have used that $\nabla u(x_0) + \nabla_x c(x_0, y_0) = 0$. We can also calculate

$$|\nabla u(x(\tilde{s})) + \nabla_x c(x(\tilde{s}), y_0)| = |\nabla u(x(\tilde{s})) - [(1 - \tilde{s})p_0 + \tilde{s}p]|$$
$$\leq |\nabla u(x(\tilde{s})) - \nabla u(x_0)| + \tilde{s}|p - p_0|$$
$$\leq \sup \|D^2 u\| \sup |\dot{x}| + \tilde{s}|p - p_0|$$
$$\leq (1 + \sup \|D^2 u\| \sup \|((D^2_{x,y} c)^T)^{-1}\|) |p - p_0|$$
$$\leq \frac{R(1 + \sup \|D^2 u\| \sup \|((D^2_{x,y} c)^T)^{-1}\|)}{M_1}.$$

Thus by taking M_1 large enough and combining with (9), we obtain for some $\epsilon_1 > 0$ depending only on the structure of the problem and u_0 (note

here that $\sup \|D^2 u\|$ is controlled by the C^2 norm of u_0 and the size of the neighborhood $\mathcal{V}$)

$$(10) \qquad u(x) - (-c(x, y_0) + c(x_0, y_0) + u(x_0)) \geq \epsilon_1 |p - p_0|^2 > 0.$$

Now suppose $|-\nabla_x c(x, y_0) + \nabla_x c(x_0, y_0)| \geq \frac{R}{M_1}$. Since u_0 is strictly c_0-convex, the function

$$\underline{u}(z, z_0) := u_0(z) - (-c_0(z, T^{c_0, u_0}(z_0)) + c_0(z_0, T^{c_0, u_0}(z_0)) + u_0(z_0))$$

is strictly positive on the set $\mathcal{D}_0 := \{(z, z_0) \in \overline{\Omega} \times \overline{\Omega} \mid |-\nabla_x c_0(z, T^{c_0, u_0}(z_0)) + \nabla_x c_0(z_0, T^{c_0, u_0}(z_0))| \geq \frac{R}{2M_1}\}$, which is seen to be compact by the continuity of T^{c_0, u_0} and boundedness of Ω. In particular, $\inf_{\mathcal{D}_0} \underline{u} > 0$. Possibly shrinking R_1, we can ensure that $(x, x_0) \in \mathcal{D}_0$, hence

$$u(x) - (-c(x, y_0) + c(x_0, y_0) + u(x_0)) \geq \underline{u}(x, x_0) - 2(\|u - u_0\|_{C^0(\overline{\Omega})} + \|c - c_0\|_{C^0(\overline{\Omega})}) > 0,$$

possibly shrinking R_1 further. This shows that $\Psi(c)$ is strictly c-convex, finishing the proof of (5). $\qquad\square$

4. C^2 ESTIMATES

Our goal in this section is to establish global C^2 estimates for solutions of (Par OT) independent of the time of existence $t_{\max}(c)$. In previous work of the second author, [25], such uniform estimates were obtained by making crucial use of the assumption that the cost function satisfies the (weak-MTW) condition. However, since this condition is not preserved under C^4 perturbations of the cost function, we must revisit the proof in [25, Section 10]. Our strategy is to establish a dichotomy for the operator norm of the matrix $W^{c,u}$. This argument is similar in spirit to one carried out in the elliptic case by Warren in [34]. However, we note that the approach of [34] makes heavy use of the log-concavity of the source and target measures in various barrier constructions and maximum-principle type arguments. Additionally, we mention that since our condition (12) below is a perturbation of (weak-MTW), we must necessarily take an approach à la Pogorelov, using an auxiliary function similar to the one used in [33]. This is in sharp contrast to the elliptic C^2 estimates in [29] and the parabolic C^2 estimates in [24], both of which rely on a stronger version of the MTW condition. In particular, we note that we are not able to prove the necessary estimates simply by choosing $w_{ij} \xi^i \xi^j$ as the auxiliary function in the proof of Proposition 4.2 below. As our argument will illustrate, additional lower order terms need to be introduced in the auxiliary function to establish the desired polynomial inequality for the quantity $\sup \|W^{c,u}\|$.

Throughout this section, the cost function c will remain fixed, and u will be the solution of (Par OT) corresponding to this fixed cost function c that exists up to $t_{\max}(c)$; thus we omit the dependence on c from various pieces of notation. We will make use of the following linearization of (Par OT):

$$(11) \qquad \mathcal{L}\theta := w^{ij} (\theta_{ij} - D_{p_k} A_{ij} \theta_k) - D_{p_k}(\log B)\theta_k - \dot{\theta}$$

where in the coefficients, $p = \nabla u(x,t)$; note that differentiating (Par OT) in time shows $\mathcal{L}\dot{u} = 0$. We will also assume the key condition which replaces the (weak-MTW) condition, namely that for some $\sigma_{\mathrm{MTW}} > 0$,

$$(12) \qquad D^2_{p_k p_\ell} A_{ij} V^i V^j \eta^k \eta^l \geq -\sigma_{\mathrm{MTW}} |V|^2 |\eta|^2, \quad \forall \, V \perp \eta.$$

Note that (12) makes no assumption on the quantity appearing in the left-hand side when V, η are not orthogonal.

We begin with a barrier construction for the linear operator (11). This is similar to the construction of an elliptic barrier in [23, Lemma 2.2], but instead of using the condition (weak-MTW), we will utilize the bound (12).

Lemma 4.1. *There exists a constant $K_0 > 0$ depending on the structure of the problem so that, as long as $\sigma_{MTW} < K_0$, there exists a function $\psi \in C^2_x C^1_t (\overline{\Omega} \times [0, t_{\max}))$ such that for all $(x,t) \in \overline{\Omega} \times [0, t_{\max})$,*

$$\mathcal{L}\psi(x,t) \geq C_1 \mathrm{tr}(W^{-1}(x,t)) - C_2,$$

$$0 < C_3^{-1} \leq \psi(x,t) \leq C_3,$$

for some constants $C_1, C_2, C_3 > 0$ depending on the structure of the problem, and $\|u_{initial}\|_{C^2(\overline{\Omega})}$ but independent of $t_{\max}$.

Proof. Let $\bar{u}(x)$ be the function constructed in [23, Lemma 2.1] with the choice $g(x,y,z) = -c(x,y) - z$; note that the proof there does not require the (weak-MTW) condition. Also, let

$$U(t) := \frac{1}{|\Omega|} \int_\Omega u(x,t) \, dx,$$

which belongs to $C^1([0, t_{\max}))$ by Proposition 2.5. For any fixed $x_0 \in \Omega$, define the function $\bar{u}_\epsilon(x) := \bar{u}(x) - \frac{\epsilon}{2}|x - x_0|^2$. Reasoning similarly to [23, Lemma 2.2], we have for any sufficiently small $\epsilon > 0$ a constant $a_1 \in \mathbb{R}$ independent of $t_{\max}$ such that

$$(13) \qquad \log(\det D^2 \bar{u}_\epsilon - A(x, \nabla \bar{u}_\epsilon)) \geq a_1,$$

and that $\nabla \bar{u}_\epsilon(x) \in -\nabla_x c(x, \Omega^*)$ for any $x \in \Omega$. Then, for some $K > 0$ to be determined, we define

$$\psi(x,t) := e^{K(U(t) - u(x,t) + \bar{u}(x))}.$$

We claim ψ satisfies uniform bounds above and below independent of $t_{\max}$. To show this, it suffices to establish a bound on the term $U(t) - u(x_0, t)$ for any $x_0 \in \Omega$ and $t \in [0, t_{\max})$. So we fix such an x_0 and t, and let $y_0 := \exp^c_{x_0}(\nabla u(x_0, t))$. Let $x \in \Omega$ be another arbitrary point. Then for $s \in [0,1]$, we let

$$p_0 := -\nabla_y c(x_0, y_0) = \nabla u(x_0, t),$$

$$p := -\nabla_y c(x, y_0),$$

$$x(s) := \exp^{c^*}_{y_0}((1-s)p_0 + sp).$$

Since $\dot{x}_i(s) = -c^{k,i}\big|_{(x(s),y_0)}(p - p_0)_k$, we have

$$u(x,t) - u(x_0,t) = \int_0^1 \frac{d}{ds} u(x(s),t) \; ds$$

$$= \int_0^1 \langle \nabla u(x(s),t), \dot{x}(s) \rangle \; ds$$

$$\leq \sup_{\overline{\Omega} \times [0,t_{\max})} |\nabla u| \sup_{\overline{\Omega} \times \overline{\Omega}^*} \|(D_{x,y}^2 c)^{-1}\| |p - p_0|$$

$$\leq \sup_{\overline{\Omega} \times [0,t_{\max})} |\nabla u| \sup_{\overline{\Omega} \times \overline{\Omega}^*} \|(D_{x,y}^2 c)^{-1}\| 2 \sup_{\overline{\Omega} \times \overline{\Omega}^*} |\nabla_y c|.$$

Consequently

$$|U(t) - u(x_0,t)| \leq \frac{1}{|\Omega|} \int_\Omega |u(x,t) - u(x_0,t)| \; dx$$

$$\leq \frac{2}{|\Omega|} \sup_{\overline{\Omega} \times [0,t_{\max})} |\nabla u| \sup_{\overline{\Omega} \times \overline{\Omega}^*} \|(D_{x,y}^2 c)^{-1}\| \sup_{\overline{\Omega} \times \overline{\Omega}^*} |\nabla_y c| \quad \text{for any } t \in [0,t_{\max}).$$

Since $\bar{u}$ is uniformly bounded on Ω, using Proposition 2.5 there exists a constant $\Lambda > 0$ depending only on the structure of the problem such that

(14) $\qquad e^{-K\Lambda} \leq \psi(x,t) \leq e^{K\Lambda} \quad \text{for all } (x,t) \in \overline{\Omega} \times [0,t_{\max}).$

Recall the linearized operator $\mathcal{L}$ defined in (11). Denoting

$$\tilde{\mathcal{L}}v := w^{ij} \left(v_{ij} - D_{p_k} A_{ij}(\cdot, \nabla u) v_k \right),$$

we have for a constant $C > 0$ depending only on the structure of the problem and $\|u_{\text{initial}}\|_{C^2(\overline{\Omega})}$,

$$\mathcal{L}\psi = \tilde{\mathcal{L}}\psi - K\psi(\dot{U} - \dot{u} + D_{p_k} B(x, \nabla u)(\bar{u}_k - u_k))$$

(15) $\qquad\qquad \geq \tilde{\mathcal{L}}\psi - CKe^{K\Lambda},$

where we have used Proposition 2.5. Writing $\eta(x,t) = U(t) - u(x,t) + \bar{u}(x)$ for ease of notation, we calculate

(16) $\qquad\qquad\qquad \tilde{\mathcal{L}}\psi = K\psi(Kw^{ij}\eta_i \eta_j + \tilde{\mathcal{L}}\eta).$

Then,

$$\tilde{\mathcal{L}}\eta = \tilde{\mathcal{L}}\left(\frac{\epsilon}{2}|x - x_0|^2\right) + \tilde{\mathcal{L}}\bar{u}_\epsilon + \tilde{\mathcal{L}}(U(t) - u(x,t))$$

$$= \epsilon \operatorname{tr} W^{-1} - \epsilon w^{ij} D_{p_k} A_{ij}(x, \nabla u)(x - x_0)_k$$

$$\quad + w^{ij}((\bar{u}_\epsilon)_{ij} - D_{p_k} A_{ij}(x, \nabla u)(\bar{u}_\epsilon)_k) - w^{ij}(u_{ij} - D_{p_k} A_{ij}(x, \nabla u)u_k)$$

$$= \epsilon \operatorname{tr} W^{-1} - \epsilon w^{ij} D_{p_k} A_{ij}(x, \nabla u)(x - x_0)_k$$

$$\quad + w^{ij}([(\bar{u}_\epsilon)_{ij} - A_{ij}(x, \nabla \bar{u}_\epsilon)] - [u_{ij} - A_{ij}(x, \nabla u)])$$

(17) $\quad + w^{ij}(A_{ij}(x, \nabla \bar{u}_\epsilon) - A_{ij}(x, \nabla u) - D_{p_k} A_{ij}(x, \nabla u)((\bar{u}_\epsilon)_k - u_k)).$

Let us write $\boxed{\text{I}} := -\epsilon w^{ij} D_{p_k} A_{ij}(x, \nabla u)(x - x_0)_k$, and $\boxed{\text{II}}$ and $\boxed{\text{III}}$ respectively for the final two lines above. By concavity of $\log \det$, we obtain using (13) and (Par OT),

$$\boxed{\text{II}} \geq \log \det(D^2 \bar{u}_\epsilon - A(x, \nabla \bar{u}_\epsilon)) - \log \det(D^2 u - A(x, \nabla u))$$

$$(18) \qquad \geq a_1 - \dot{u}(x,t) - \log B(x, \nabla u) \geq -C$$

for some constant by Proposition 2.5. By the c^*-convexity of Ω^* with respect to Ω, the point $q_s := (1-s)\nabla \bar{u}_\epsilon + s\nabla u$ belongs to $-\nabla_x c(x, \Omega^*)$ for any $s \in [0,1]$. Then by Taylor expanding, for some $s \in [0,1]$ we obtain

$$\boxed{\text{III}} = \frac{w^{ij}}{2} D^2_{p_k p_\ell} A_{ij}(x, q_s)((\bar{u}_\epsilon)_k - u_k)((\bar{u}_\epsilon)_\ell - u_\ell).$$

By choosing $x = x_0$, we then obtain

$$\boxed{\text{III}} = \frac{w^{ij}}{2} D^2_{p_k p_\ell} A_{ij}(x_0, q_s)(\bar{u}_k - u_k)(\bar{u}_\ell - u_\ell),$$

note for this choice, we also have $\boxed{\text{I}} = 0$.

Now for ease of notation, let us write

$$\text{MTW}(V, W, \eta, \xi) := D^2_{p_k p_\ell} A_{ij}(x_0, q_s) V^i W^j \eta^k \xi^\ell,$$

and after fixing $t \in [0, t_{\max})$ make a rotation of coordinates to diagonalize the matrix $W(x_0, t)$. Letting $\{e_i\}_{i=1}^n$ be the standard basis in $\mathbb{R}^n$ and using the above calculations, we thus obtain

$$\tilde{\mathcal{L}}\eta(x_0, t) \geq \epsilon \text{tr}(W^{-1}(x_0, t)) - C$$

$$(19) \qquad\qquad + \frac{1}{2} \sum_{i=1}^n w^{ii} \text{MTW}(e_i, e_i, \nabla(\bar{u} - u), \nabla(\bar{u} - u)).$$

Denoting $\nabla_i^\perp(\bar u - u) := \nabla(\bar u - u) - (\bar u_i - u_i)e_i$, and using (12) we calculate

$$\sum_{i=1}^n w^{ii} \operatorname{MTW}(e_i, e_i, \nabla(\bar u - u), \nabla(\bar u - u))$$

$$= \sum_{i=1}^n w^{ii} \Big[\operatorname{MTW}(e_i, e_i, \nabla_i^\perp(\bar u - u), \nabla_i^\perp(\bar u - u))$$

$$+ 2(\bar u - u)_i \operatorname{MTW}(e_i, e_i, e_i, \nabla_i^\perp(\bar u - u)) + (\bar u_i - u_i)^2 \operatorname{MTW}(e_i, e_i, e_i, e_i) \Big]$$

$$\geq \sum_{i=1}^n w^{ii} \Big[-\sigma_{\operatorname{MTW}}|\nabla_i^\perp(\bar u - u)|^2 + (\bar u_i - u_i)^2 \operatorname{MTW}(e_i, e_i, e_i, e_i)$$

$$+ 2 \sum_{j \neq i} (\bar u_i - u_i)(\bar u_j - u_j) \operatorname{MTW}(e_i, e_i, e_i, e_j) \Big]$$

$$\geq \sum_{i=1}^n w^{ii} \left[-\sigma_{\operatorname{MTW}}\|\nabla(\bar u - u)\|^2 - (\bar u_i - u_i)^2\|D_p^2 A\| - 2\|D_p^2 A\| \sum_{j \neq i} \left(\tilde\epsilon(\bar u_j - u_j)^2 + \frac{1}{4\tilde\epsilon}(\bar u_i - u_i)^2 \right) \right]$$

$$\geq -(\sigma_{\operatorname{MTW}} + 2\tilde\epsilon\|D_p^2 A\|)\|\nabla(\bar u - u)\|^2 \operatorname{tr}(W^{-1}) - \|D_p^2 A\|(1 + \frac{n-1}{2\tilde\epsilon}) \sum_{i=1}^n w^{ii}(\bar u_i - u_i)^2$$

for $\tilde\epsilon > 0$ to be determined. It follows from (19) that

$$\tilde{\mathcal L}\eta(x_0, t) \geq \operatorname{tr}(W^{-1}(x_0, t)) \left[\epsilon - (\sigma_{\operatorname{MTW}} + 2\tilde\epsilon\|D_p^2 A\|)\|\nabla(\bar u - u)\|^2 \right]$$

$$- \|D_p^2 A\| \left(1 + \frac{n-1}{2\tilde\epsilon} \right) \sum_{i=1}^n w^{ii}(\bar u_i - u_i)^2.$$

We choose $\sigma_{\operatorname{MTW}}$ and $\tilde\epsilon$ small enough so that

$$\epsilon - (\sigma_{\operatorname{MTW}} + 2\tilde\epsilon\|D_p^2 A\|)\|\nabla(\bar u - u)\|^2 \geq \epsilon/2.$$

Therefore, by (16), we have

$$\tilde{\mathcal L}\psi(x_0, t) \geq K\psi(x_0, t) \left(\sum_{i=1}^n w^{ii} \left[K - \|D_p^2 A\| \left(1 + \frac{n-1}{2\tilde\epsilon} \right) \right] (\bar u_i - u_i)^2 + \frac{\epsilon}{2}\operatorname{tr}(W^{-1}(x_0, t)) \right)$$

We then choose K large enough so that

$$K - \|D_p^2 A\| \left(1 + \frac{n-1}{2\tilde\epsilon} \right) \geq 0.$$

Consequently,

$$\tilde{\mathcal L}\psi(x_0, t) \geq K\psi(x_0, t)\frac{\epsilon}{2}\operatorname{tr}(W^{-1}(x_0, t)).$$

Finally, using (15), and then (14), we conclude that

$$\mathcal{L}\psi(x_0, t) \geq K\psi(x_0, t)\frac{\epsilon}{2}\mathrm{tr}(W^{-1}(x_0, t)) - CKe^{K\Lambda}$$

$$\geq Ke^{-K\Lambda}\frac{\epsilon}{2}\mathrm{tr}(W^{-1}(x_0, t)) - CKe^{K\Lambda}.$$

This finishes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We are now ready to carry out the aforementioned dichotomy argument. The following proposition shows that if the constant in (12) is sufficiently small (i.e., the cost function is "sufficiently close to satisfying (weak-MTW)") then the operator norm of $W(x, t)$ either has a uniform upper bound independent of $t_{\max}$, or must be larger than a specific value somewhere; both of these threshold values are explicit.

Proposition 4.2. *There exists a constant $\tilde{\sigma}_0 > 0$ depending only on the structure of the problem and $\|u_{initial}\|_{C^2(\overline{\Omega})}$ such that, as long as $0 < \sigma_{MTW} < \tilde{\sigma}_0$, exactly one of the two following alternatives hold: for any $0 \leq T < t_{\max}$, either*

(i)

$$\max_{(x,t)\in\overline{\Omega}\times[0,T]} ||W(x, t)|| \geq \frac{1}{n}\left(\frac{1}{n\sigma_{MTW}}\right)^{\frac{1}{n-1}}, \quad or$$

(ii)

$$\max_{(x,t)\in\overline{\Omega}\times[0,T]} ||W(x, t)|| \leq \frac{1}{2n}\left(\frac{1}{n\sigma_{MTW}}\right)^{\frac{1}{n-1}}.$$

Proof. Define the auxiliary function $v : \Omega \times [0, t_{\max}) \times \mathbb{S}^{n-1} \to \mathbb{R}$ by

$$v(x, t, \xi) := \log w_{\xi\xi}(x, t) + a|\nabla u(x, t)| + \tilde{C}a\psi(x, t),$$

for constants $a, \tilde{C} > 0$ to be determined, where ψ is the function from Lemma 4.1. Suppose v achieves a maximum at (x_0, t_0, ξ_0). Let us first assume x_0 belongs to the interior of Ω. By following the calculations in [25, Theorem 10.1], we obtain (note the second displayed block of equations on p.148 of [25, Theorem 10.1] is missing a term $2au_l w^{ij}D_{x_l}A_{ij}$, which yields the term $-\tilde{\tilde{C}}a\mathrm{tr}(W^{-1})$ below)

$$0 \geq \mathcal{L}v(x_0, t_0, \xi_0)$$

$$\geq \frac{w^{ij}(D_{p_k p_\ell}A_{ij})w_{k\xi_0}w_{l\xi_0}}{w_{\xi_0\xi_0}} + (2a - C_1)\mathrm{tr}(W) - C_2 a - (C_3 + \tilde{\tilde{C}}a - \tilde{C}a)\mathrm{tr}(W^{-1})$$

$$\geq \frac{w^{ij}(D_{p_k p_\ell}A_{ij})w_{k\xi_0}w_{l\xi_0}}{w_{\xi_0\xi_0}} + (2a - C_1)\mathrm{tr}(W) - C_2 a$$

(20)

where the last inequality is obtained by taking $\tilde{C}$ large enough that $C_3 + \tilde{\tilde{C}}a - \tilde{C}a \leq 0$, and we assume $2a - C_1 > 0$. Here, it can be seen that the

constants C_1, C_2, and C_3 depend on upper bounds on ∇u, and on up to second order derivatives of A_{ij} and $\log B$. From the definitions of A_{ij} and B, and Proposition 2.5 it can thus be seen that these constants along with $\tilde{C}$ and a can be chosen to have upper bounds depending only on the structure of the problem.

Now diagonalize $(w_{ij}(x_0, t_0))$ and let $\lambda_1 \leq \ldots \leq \lambda_n$ be the associated eigenvalues; note that, by choice, ξ_0 is an eigenvector corresponding to λ_n. We then estimate,

$$(21) \qquad \operatorname{tr}(W^{-1}) = \sum_{i=1}^{n} \lambda_i^{-1} = \frac{\sum_{i=1}^{n} \prod_{j \neq i} \lambda_j}{\lambda_1 \cdots \lambda_n} \leq \frac{n \lambda_n^{n-1}}{\lambda_1 \cdots \lambda_n} \leq \frac{n \operatorname{tr}(W)^{n-1}}{\det W}.$$

Taking $V = \lambda_i^{-1/2} e_i$ for $1 \leq i \leq n-1$ and $\eta = e_n$ in (12) and using (21) yields

$$\frac{w^{ij}(D_{p_k p_\ell} A_{ij}) w_{k\xi_0} w_{l\xi_0}}{w_{\xi_0 \xi_0}} = \lambda_n \sum_{i=1}^{n} \lambda_i^{-1} D_{p_n p_n} A_{ii}$$

$$\geq -\sigma_{\mathrm{MTW}} \lambda_n \sum_{i=1}^{n-1} \lambda_i^{-1} + D_{p_n p_n}^2 A_{nn}$$

$$\geq -\sigma_{\mathrm{MTW}} \lambda_n \sum_{i=1}^{n} \lambda_i^{-1} - \|D_p^2 A\|$$

$$\geq -\frac{n \sigma_{\mathrm{MTW}} \lambda_n \operatorname{tr}(W)^{n-1}}{\det W} - \|D_p^2 A\|$$

$$\geq -C_4 \sigma_{\mathrm{MTW}} \operatorname{tr}(W)^n - \|D_p^2 A\|$$

where C_4 depends on u_0 and the structure of the problem. Combining with (20) and rearranging yields

$$\operatorname{tr}(W) - \frac{C_4 \sigma_{\mathrm{MTW}}}{2a - C_1} \operatorname{tr}(W)^n \leq \frac{C_2 a + \|D_p^2 A\|}{2a - C_1}.$$

Taking a large enough that $\frac{C_4}{2a - C_1} \leq 1$, we then obtain

$$(22) \qquad \operatorname{tr}(W) - \sigma_{\mathrm{MTW}} \operatorname{tr}(W)^n - C_5 \leq 0$$

for $C_5 > 0$ depending only on the structure of the problem.

Let us now assume $\sigma_{\mathrm{MTW}} \in (0, \sigma_0]$, where $\sigma_0 > 0$ is a fixed constant satisfying

$$(23) \qquad \sigma_0 \leq K_0 \quad \text{and} \quad \frac{1}{n\sigma_0} \geq \left(\frac{2n^2 C_5}{n-1} \right)^{n-1},$$

where K_0 is the constant appearing in Lemma 4.1. Examining the roots of the polynomial $s \mapsto s - \sigma_{\mathrm{MTW}} s^n - C_5$ on the half-line $\{s \geq 0\}$ (see (28) and (29) in Appendix A) yields two possibilities.

Case 1: $\operatorname{tr}(W(x_0, t_0)) \geq \left(\frac{1}{n\sigma_{\mathrm{MTW}}} \right)^{\frac{1}{n-1}}$.

In this case, we find that

$$\left(\frac{1}{n\sigma_{\text{MTW}}}\right)^{\frac{1}{n-1}} \leq \text{tr}(W(x_0, t_0)) \leq n \sup_{\overline{\Omega}\times[0,T]} \|W\|.$$

Case 2: $\text{tr}(W(x_0, t_0)) \leq \frac{1}{2n}\left(\frac{1}{n\sigma_0}\right)^{\frac{1}{n-1}}$.

In this case, for any (x, t, ξ) we obtain (recall ξ_0 is an eigenvector of $(w_{ij}(x_0, t_0))$ corresponding to λ_n)

$$\log w_{\xi\xi}(x, t) \leq v(x_0, t_0, \xi_0)$$

$$\leq \log\left(\frac{1}{2n}\left(\frac{1}{n\sigma_0}\right)^{\frac{1}{n-1}}\right) + a|\nabla u(x_0, t_0)| + \tilde{C}a\psi(x_0, t_0)$$

$$\leq C_6 + \log\left(\frac{1}{2n}\left(\frac{1}{n\sigma_0}\right)^{\frac{1}{n-1}}\right)$$

Taking exponentials and then a supremum over (x, t, ξ) yields

$$\sup_{\overline{\Omega}\times[0,T]} \|W\| \leq \frac{e^{C_6}}{2n}\left(\frac{1}{n\sigma_0}\right)^{\frac{1}{n-1}} \leq \frac{1}{2n}\left(\frac{1}{n\sigma_{\text{MTW}}}\right)^{\frac{1}{n-1}},$$

so long as we choose $\sigma_{\text{MTW}} < \frac{1}{e^{(n-1)C_6}}\sigma_0$.

Let us now assume v achieves a maximum at (x_0, t_0, ξ_0) where $x_0 \in \partial\Omega$. Then for any (x, t, ξ), we have

$$\log w_{\xi\xi}(x, t) \leq v(x_0, t_0, \xi_0) = \log w_{\xi_0\xi_0}(x_0, t_0) + a|\nabla u(x_0, t_0)| + \tilde{C}a\psi(x_0, t_0)$$

$$\leq C_6 + \log\left(\sup_{\partial\Omega\times[0,T]} \|W\|\right)$$

Consequently, there exists a constant $C_7 > 0$ depending only on the structure of the problem such that

$$(24) \qquad \sup_{\overline{\Omega}\times[0,T]} \|W\| \leq C_7 \sup_{\partial\Omega\times[0,T]} \|W\|.$$

Upon inspection of [25, Section 11], we find that the bound (24) allows us to show that

$$\sup_{\overline{\Omega}\times[0,T]} \|W\| \leq C_8,$$

where C_8 depends only on the structure of the problem and the C^2 norm of u_{initial}. The dependency of C_8 on u_{initial} is through the constant α used in [25, Equation (9.9)]. By the final displayed inequality on [25, pg. 143], it can be seen that α depends only on $\|u_{\text{initial}}\|_{C^2(\overline{\Omega})}$.

Choosing $\tilde{\sigma}_0 > 0$ to be a constant satisfying the inequalities

$$\tilde{\sigma}_0 \leq \frac{1}{e^{(n-1)C_6}}\sigma_0 \quad \text{and} \quad C_8 \leq \frac{1}{2n}\left(\frac{1}{n\tilde{\sigma}_0}\right)^{\frac{1}{n-1}},$$

(and recalling that σ_0 already satisfies (23)), we obtain the conclusion of the Proposition.

$\square$

Before concluding this section, we note the following consequence of Proposition 4.2.

Corollary 4.3. *Suppose* (12) *is satisfied with* $0 < \sigma_{MTW} \leq \tilde{\sigma}_0$, *where* $\tilde{\sigma}_0$ *is the constant appearing in Proposition 4.2. For* $\tau \in [0, t_{\max})$, *consider the quantity*

$$\omega(\tau) := \max_{(x,t)\in\overline{\Omega}\times[0,\tau]} ||W^{c,u}(x,t)||.$$

If $\omega(0) \leq \frac{1}{2n}\left(\frac{1}{n\sigma_{MTW}}\right)^{\frac{1}{n-1}}$, *then* $\omega(\tau) \leq \frac{1}{2n}\left(\frac{1}{n\sigma_{MTW}}\right)^{\frac{1}{n-1}}$ *for all* $\tau \in [0, t_{\max})$.

Proof. This follows from the compactness of $\overline{\Omega}$ coupled with the joint continuity of $A^c, \nabla u, D^2 u$ (and consequently of $W^{c,u}$) in (x,t). Indeed, since $\omega(\tau)$ is an increasing function of τ, we may suppose by contradiction that there exists $\tau^* \in [0, t_{\max})$ such that $\omega(\tau^*) > \frac{1}{2n}\left(\frac{1}{n\sigma_{\mathrm{MTW}}}\right)^{\frac{1}{n-1}}$ but $\omega(\tau) \leq \frac{1}{2n}\left(\frac{1}{n\sigma_{\mathrm{MTW}}}\right)^{\frac{1}{n-1}}$ for all $0 \leq \tau < \tau^*$. By the assumption on $\omega(0)$, we know that $\tau^* > 0$. Then there must exist $x^* \in \overline{\Omega}$ such that $W^{c,u}(x^*, \tau^*) = \omega(\tau^*)$. The dichotomy imposed by Proposition 4.2 implies $W^{c,u}(x^*, \tau^*) \geq \frac{1}{n}\left(\frac{1}{n\sigma_{\mathrm{MTW}}}\right)^{\frac{1}{n-1}}$. On the other hand, since $W^{c,u}(x,t)$ is jointly continuous in (x,t) whenever $(x,t) \in \Omega \times (0, t_{\max})$, it follows that

$$\frac{1}{n}\left(\frac{1}{n\sigma_{\mathrm{MTW}}}\right)^{\frac{1}{n-1}} \leq W^{c,u}(x^*, \tau^*) = \lim_{\epsilon\to 0^+} W^{c,u}(x^*, \tau^*-\epsilon) \leq \frac{1}{2n}\left(\frac{1}{n\sigma_{\mathrm{MTW}}}\right)^{\frac{1}{n-1}},$$

which is a contradiction. $\square$

5. Proof of Theorem 1.1

We begin by letting u_0 be the unique solution of the steady state problem

$$(25) \quad \begin{cases} \det(W^{c_0,u_0}(x)) = B^{c_0}(x, \nabla u_0(x)), & x \in \Omega, \\ G^{c_0}(x, \nabla u_0(x)) = 0, & x \in \partial\Omega, \\ \int_\Omega u_0(x)\, dx = 0. \end{cases}$$

Since c_0 satisfies the (weak-MTW) condition, the results of [33] imply $u_0 \in C^{2,\alpha}(\overline{\Omega})$, is strictly c_0-convex, and is locally uniformly c_0-convex.

We assume $||c - c_0||_{C^4(\Omega\times\mathcal{N}_{r_0}(\Omega^*))} < \hat{R}$ for some $\hat{R}$ to be determined. We first choose $\hat{R} \leq R_0$ in Lemma 3.1 (1)-(3). This ensures c satisfies the necessary structural conditions in Subsection 2.1. Next, we choose $\hat{R} \leq R_1$ in Lemma 3.3; this ensures the map $\Psi : B_{R_1}^{C^4(\overline{\Omega}\times\overline{\mathcal{N}_{r_0}(\Omega^*)})}(c_0) \to C^{2,\alpha}(\overline{\Omega})$ is well-defined. Recalling that $\Psi(c_0) = u_0$, we next let $R_3 > 0$ be such

that if $\hat{R} \leq R_3$, then $||\Psi(c)||_{C^{2,\alpha}(\overline{\Omega})} \leq 2||u_0||_{C^{2,\alpha}(\overline{\Omega})}$ and $\max_{\overline{\Omega}}||W^{c,\Psi(c)}|| \leq 2\max_{\overline{\Omega}}||W^{c_0,u_0}||$; this follows from the continuity of the map Ψ at $c = c_0$ and the compactness of $\overline{\Omega}$. Now let u be the solution of (Par OT) with $u_{\text{initial}} = \Psi(c)$. Then u satisfies the estimates in Proposition 2.5 (which do not require the (weak-MTW) condition) with constants K_1, K_2 depending only on c_0 and u_0. Finally, we let $R_4 > 0$ be such that if $\hat{R} \leq R_4$, then c satisfies (12) with a constant $\sigma_{\text{MTW}} \leq \tilde{\sigma}_0$, where $\tilde{\sigma}_0$ is the constant appearing in Proposition 4.2, and $\max_{\overline{\Omega}}||W^{c_0,u_0}|| \leq \frac{1}{4n}\left(\frac{1}{n\sigma_{\text{MTW}}}\right)^{\frac{1}{n-1}}$. Then Corollary 4.3 implies $\sup_{\overline{\Omega}\times[0,t_{\max}(c))}||W^{c,u}|| \leq \frac{1}{2n}\left(\frac{1}{n\sigma_{\text{MTW}}}\right)^{\frac{1}{n-1}}$. Consequently, u satisfies uniform second derivative estimates independent of $t_{\max}(c)$. We can thus invoke the arguments from [25, Section 12] to conclude the infinite-time existence of the solution u of (Par OT) and the convergence to a potential function for the optimal transport problem between (Ω, ρ) and (Ω^*, ρ^*) with cost function c.

APPENDIX A. PROPERTIES OF AN n-TH DEGREE POLYNOMIAL

Fix $C \geq 0$ and consider the polynomial

$$p_\sigma(s) := s - \sigma s^n - C \quad \text{for any } \sigma \in (0, \sigma_0],$$

where $\sigma_0 > 0$ is a constant satisfying the inequality

$$(26) \qquad \frac{1}{n\sigma_0} \geq \left(\frac{2n^2C}{n-1}\right)^{n-1}.$$

On the half-line $\{s \geq 0\}$, p_σ has a single critical point at $s = \hat{s}(\sigma)$ where

$$(27) \qquad \hat{s}(\sigma)^{n-1} = \frac{1}{n\sigma}.$$

Since $\sigma \leq \sigma_0$, (26) implies

$$\hat{s}(\sigma)^{n-1} = \frac{1}{n\sigma} \geq \frac{1}{n\sigma_0} \geq \left(\frac{2n^2C}{n-1}\right)^{n-1}.$$

Consequently, $\hat{s}(\sigma) \geq \frac{2n^2C}{n-1}$. This, combined with (27), implies

$$p_\sigma(\hat{s}(\sigma)) = \hat{s}(\sigma)(1 - \sigma\hat{s}(\sigma)^{n-1}) - C = \hat{s}(\sigma)\left(\frac{n-1}{n}\right) - C \geq (2n-1)C > 0$$

whenever $\sigma \in (0, \sigma_0]$. Since $p_\sigma(0) = -C < 0$, it follows that there exists a root $s_1(\sigma) \in (0, \hat{s}(\sigma))$ of p_σ. Similarly, since $\lim_{s \to +\infty} p_\sigma(s) = -\infty$ whenever $\sigma > 0$, there exists a second root $s_2(\sigma) \in (\hat{s}(\sigma), \infty)$ of p_σ. By Descartes' rules of signs, these are the only two positive, real roots of p_σ.

Clearly, $s_1(\sigma) = \sigma s_1(\sigma)^n + C \geq C$ for all $\sigma \in (0, \sigma_0]$. Also, since $s_1(\sigma) < \hat{s}(\sigma)$, we have by (27)

$$s_1(\sigma)^{n-1} < \hat{s}(\sigma)^{n-1} = \frac{1}{n\sigma}.$$

This implies

$$1 - \sigma s_1(\sigma)^{n-1} > \frac{n-1}{n}.$$

Since $p_\sigma(s_1(\sigma)) = 0$, we have $s_1(\sigma)(1 - \sigma s_1(\sigma)^{n-1}) = C$, and so by (26)

$$s_1(\sigma) = \frac{C}{1 - \sigma s_1(\sigma)^{n-1}} < \frac{nC}{n-1} \leq \frac{1}{2n}\left(\frac{1}{n\sigma_0}\right)^{\frac{1}{n-1}} \quad \text{for all } \sigma \in (0, \sigma_0].$$

On the other hand, since $s_2(\sigma) > \hat{s}(\sigma)$, we have

$$s_2(\sigma)^{n-1} > \hat{s}(\sigma)^{n-1} = \frac{1}{n\sigma}.$$

Thus,

$$s_2(\sigma) \geq \left(\frac{1}{n\sigma}\right)^{\frac{1}{n-1}} \quad \text{for all } \sigma \in (0, \sigma_0].$$

We conclude that if $p_\sigma(s) \leq 0$ for some $s \geq 0$, then either

$$(28) \qquad\qquad 0 \leq s \leq \frac{1}{2n}\left(\frac{1}{n\sigma_0}\right)^{\frac{1}{n-1}}, \text{ or}$$

$$(29) \qquad\qquad s \geq \left(\frac{1}{n\sigma}\right)^{\frac{1}{n-1}}.$$

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