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Twists of graded Poisson algebras and related properties

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ABSTRACT

We introduce a Poisson version of the graded twist of a graded associative algebra and prove that every graded Poisson structure on a connected graded polynomial ring $A := \Bbbk[x_1, \dots, x_n]$ is a graded twist of a unimodular Poisson structure on A, namely, if π is a graded Poisson structure on A, then π has a decomposition

$$\pi = \pi_{unim} + \frac{1}{\sum_{i=1}^{n} \deg x_i} E \wedge \mathbf{m}$$

where E is the Euler derivation, π_{unim} is the unimodular graded Poisson structure on A corresponding to π, and $\mathbf{m}$ is the modular derivation of (A, π). This result is a generalization of the same one in the quadratic setting. The rigidity of graded twisting, PH^1-minimality, and H-ozoneness are studied. As an application, we compute the Poisson cohomologies of the quadratic Poisson structures on the polynomial ring of three variables when the potential is irreducible, but not necessarily having an isolated singularity.

Introduction

Poisson algebras have recently been studied extensively by many researchers, see e.g., [2,3,12–15,18,21,23–25], with topics related to (twisted) Poincaré duality and the modular derivation, Poisson Dixmier-Moeglin equivalences, Poisson enveloping algebras, and so on. Poisson algebras have been used in the representation theory of PI Sklyanin algebras [37,38]. The isomorphism problem and the cancellation problem in the Poisson setting have been investigated in [10,11]. Poisson valuations are utilized to address problems related to rigidity, automorphisms, Dixmier property, isomorphisms, and embeddings of Poisson algebras and fields [16,17].

Let $\Bbbk$ be a base field. Except for Sections 1 and 2 we further assume that $\Bbbk$ is of characteristic zero. Quadratic Poisson structures on $\Bbbk[x_1, \dots, x_n]$ with $\deg x_i = 1$ for all $i = 1, \cdots, n$ have played an important role in several other subjects, see papers [22] by Liu-Xu, [4] by Bondal, and [31] by Pym. Deformation quantizations of such Poisson structures are homogeneous coordinate rings of quantum $\mathbb{P}^{n-1}$s. In general, such a deformation quantization is skew Calabi-Yau; while it is Calabi-Yau if and only if the Poisson structure on $\Bbbk[x_1, \dots, x_n]$ is unimodular [6].

In addition to the quadratic case, we are interested in weighted Poisson structures on $\Bbbk[x_1, \dots, x_n]$ where $\deg x_i > 0$ for all $i = 1, \dots, n$. Note that deformation quantizations of weighted Poisson structures are homogeneous coordinate rings of

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weighted quantum $\mathbb{P}^{n-1}$s. If π is a graded Poisson structure on $\Bbbk[x_1,\dots,x_n]$ where $\sum_{i=1}^n \deg x_i \neq 0$ in the base field $\Bbbk$, we prove that π has a decomposition

$$\pi = \pi_{unim} + \frac{1}{\sum_{i=1}^n \deg x_i} E \wedge \mathbf{m} \tag{E0.0.1}$$

where E is the Euler derivation, π_{unim} is the unimodular graded Poisson structure on $\Bbbk[x_1,\dots,x_n]$ corresponding to π, and $\mathbf{m}$ is the modular derivation of $(\Bbbk[x_1,\dots,x_n],\pi)$. If $\deg x_i = 1$ for all i, (E0.0.1) was observed by Bondal [4], Liu-Xu [22], and in the book [20, Theorem 8.26]. Similar to the ideas in [31], to classify all graded Poisson structures on polynomial rings where $\deg x_i > 0$, it is a good idea to first classify unimodular ones.

To prove (E0.0.1), we will use a Poisson version of the graded twist [39]. Let A be a $\mathbb{Z}$-graded Poisson algebra such that both the commutative multiplication $\cdot$ and the Poisson bracket $\pi := \{-,-\}$ are graded of degree 0. If $a \in A$ is homogeneous, we use $|a|$ to denote its degree. Define the Euler derivation E of A by

$$E(a) = |a|\, a \tag{E0.0.2}$$

for all homogeneous elements $a \in A$. Let δ be a graded Poisson derivation of A. We define a new Poisson structure, denoted by $\pi_{new} := \{-,-\}_{new}$, to be

$$\{a,b\}_{new} := \{a,b\} + E(a)\delta(b) - \delta(a)E(b) \tag{E0.0.3}$$

for all homogeneous elements $a, b \in A$, or equivalently

$$\pi_{new} := \pi + E \wedge \delta.$$

We will show that $(A, \cdot, \{-,-\}_{new})$ (or (A, π_{new})) is a graded Poisson algebra in Section 2 and it is denoted by A^δ.

Now we state some results. Let A be a polynomial algebra $\Bbbk[x_1,\dots,x_n]$ and let δ be a derivation of A. By [20, (4.21)], the *divergence* of δ is defined to be

$$\operatorname{div}(\delta) := \sum_{i=1}^n \frac{\partial \delta(x_i)}{\partial x_i}, \tag{E0.0.4}$$

which is independent of the choices of generators $\{x_1,\dots,x_n\}$ [Definition 1.1 and Lemma 1.2]. For a more general Poisson algebra, the definition of $\operatorname{div}(\delta)$ will be given in Definition 1.1, which is dependent on the volume form. Recall a *Hamiltonian derivation* of a Poisson algebra A is given by $H_a := \{a,-\}$ for any $a \in A$. The *modular derivation* of A is defined by

$$\mathbf{m}(a) := -\operatorname{div}(H_a) \tag{E0.0.5}$$

for all $a \in A$ [Definition 1.3]. We need the following lemma that concerns the divergence of the modular derivation.

Lemma 0.1. *[36, Corollary 3.10] [20, Proposition 4.17] Let A be a Poisson algebra with volume form ν and $\mathbf{m}$ be the modular derivation of A corresponding to ν. Then* $\operatorname{div}(\mathbf{m}) = 0$.

Proof. Following the notation of [36, Theorem 3.5], we denote $\mathbf{m}$ by ϕ and ν by vol. By the proof of [36, Corollary 3.10], $\mathcal{L}_{\mathbf{m}}(\nu) = 0$. Then, by Definition 1.1, $\operatorname{div}(\mathbf{m}) = 0$. □

According to the ideas of Dolgushev [6], the modular derivation of a Poisson algebra corresponds to the Nakayama automorphism of a noetherian AS regular algebra. For more information on AS regular algebras, we refer the reader to [1] and the references therein. Hence the above lemma is a Poisson version of [33, Corollary 5.5] which says that the Nakayama automorphism of a noetherian AS regular algebra has homological determinant 1.

When A is a polynomial algebra $\Bbbk[x_1,\dots,x_n]$ with any Poisson structure, the definitions of the divergence div and the modular derivation $\mathbf{m}$ are independent of choices of the volume form. Here is one of the main results of this paper.

Theorem 0.2. *Let δ be a graded Poisson derivation of a $\mathbb{Z}$-graded Poisson polynomial algebra $A := \Bbbk[x_1,\dots,x_n]$. Let $\mathbf{n}$ be the modular derivation of A^δ. Then*

$$\mathbf{n} = \mathbf{m} + \left(\sum_{i=1}^n \deg x_i\right)\delta - \operatorname{div}(\delta)E.$$

Note that Theorem 0.2 holds even when $\operatorname{char} \Bbbk > 0$, see Remark 3.7.

If we consider the analogy between the modular derivation of a Poisson algebra and the Nakayama automorphism of a graded skew Calabi-Yau algebra [6], Theorem 0.2 is a Poisson version of [32, Theorem 0.3]. Combining Theorem 0.2 with Lemma 0.1, we obtain

Corollary 0.3. *Let A be a weighted graded Poisson algebra $\Bbbk[x_1, \dots, x_n]$ with $\deg x_i > 0$ for all i. Let $\delta = -\frac{1}{\mathfrak{l}}\mathbf{m}$ where $\mathfrak{l} = \sum_{i=1}^{n} \deg x_i$. Then A^{δ} is unimodular. As a consequence, (E0.0.1) holds.*

Let A be a $\mathbb{Z}$-graded Poisson algebra. Suppose δ is a derivation of A and a, b, c are homogeneous elements of A. Set

$$\begin{aligned} p(\{-,-\}, \delta; a, b, c) := & |a|a[\delta(\{b, c\}) - \{\delta(b), c\} - \{b, \delta(c)\}] \\ & - |b|b[\delta(\{a, c\}) - \{\delta(a), c\} - \{a, \delta(c)\}] \\ & + |c|c[\delta(\{a, b\}) - \{\delta(a), b\} - \{a, \delta(b)\}]. \end{aligned} \tag{E0.3.1}$$

Definition 0.4. Let A be a $\mathbb{Z}$-graded Poisson algebra. A derivation δ of A is called *semi-Poisson* if $p(\{-,-\}, \delta, a, b, c) = 0$ for all homogeneous elements a, b, c in A.

It is clear that

Poisson derivation $\Rightarrow$ semi-Poisson derivation $\Rightarrow$ derivation

and opposite implications are not true [Example 2.6]. Let $Gspd(A)$ (resp. $Gpd(A)$) be the set of graded semi-Poisson derivations (resp. graded Poisson derivations) of degree 0. We prove the following

Theorem 0.5. *Let A be a graded Poisson algebra $\Bbbk[x_1, \dots, x_n]$ with $\deg x_i > 0$ for all i.*

(1) *If A is unimodular, then $Gspd(A) = Gpd(A)$.*
(2) *If B is a twist of A, then $Gspd(A) = Gspd(B)$.*
(3) *$Gspd(A)$ is a finite-dimensional Lie algebra.*

Now we introduce the *rigidity of graded twisting* of A, denoted by $rgt(A)$ (see Definition 4.3), to measure the complexity/rigidity of a Poisson structure on A. We relate the rigidity with other properties. We say a Poisson derivation ϕ of A is *ozone* if $\phi(z) = 0$ for all z in the Poisson center of A. It is obvious that every Hamiltonian derivation is ozone, but the converse is not true in general.

Let M be a $\mathbb{Z}$-graded $\Bbbk$-vector space. The Hilbert series of M is defined to be

$$h_M(t) = \sum_{i \in \mathbb{Z}} (\dim M_i) t^i. \tag{E0.5.1}$$

Let $PH^i(A)$ denote the ith Poisson cohomology of A (E1.5.3). Recall that E denotes the Euler derivation. We have the following result.

Theorem 0.6. *Let $\Bbbk$ be algebraically closed and $A = \Bbbk[x_1, x_2, x_3]$ be a graded Poisson algebra with $\deg x_i = 1$ for $i = 1, 2, 3$. Denote by Z the Poisson center of A. Then the following are equivalent.*

(1) $rgt(A) = 0$.
(2) *Any graded twist of A is isomorphic to A.*
(3) *The Hilbert series of the graded vector space of Poisson derivations of A is $\frac{1}{(1-t)^3}$.*
(4) *$h_{PH^1(A)}(t)$ is $\frac{1}{1-t^3}$.*
(5) *$h_{PH^1(A)}(t)$ is equal to $h_Z(t)$.*
(6) *Every Poisson derivation ϕ has a decomposition*

$$\phi = zE + H_a$$

where $z \in Z$ and $a \in A$. Here z is unique and a is unique up to a central element.

(7) *Every ozone derivation is Hamiltonian.*
(8) *A is unimodular, and the potential is irreducible.*
(9) $h_{PH^3(A)}(t) - h_{PH^2(A)}(t) = t^{-3}$.

Some partial generalizations of the above theorem to the higher dimensional cases are given in Section 7. As an application, we have the following result.

Corollary 0.7. *Let $\Bbbk$ be algebraically closed. Let A be the unimodular quadratic Poisson structure on $\Bbbk[x, y, z]$ with irreducible potential Ω. Then*

(1) $h_{PH^0(A)}(t) = \dfrac{1}{1-t^3}$.

(2) $h_{PH^1(A)}(t) = \dfrac{1}{1-t^3}$.

(3) $h_{PH^2(A)}(t) = \dfrac{1}{t^3}\left(\dfrac{(1+t)^3}{1-t^3} - 1\right)$.

(4) $h_{PH^3(A)}(t) = \dfrac{(1+t)^3}{t^3(1-t^3)}$.

When the potential Ω has an isolated singularity, the Poisson cohomologies have been computed by several authors, see [27–30,35] and the references therein. The above corollary is probably the first computation of the Poisson cohomologies when Ω is irreducible, but does not have an isolated singularity.

The paper is organized as follows. Section 1 recalls some basic definitions, such as divergence and modular derivation. In Section 2, we introduce the Poisson version of a graded twist. The proofs of Theorem 0.2 and Corollary 0.3 are given in Section 3. The rigidity of graded twisting is introduced in Section 4, and Theorem 0.5 is proved there. In Sections 5 and 6, we compute the rigidity of some Poisson structures on polynomial rings. Theorem 0.6 and Corollary 0.7 are proved in Section 7.

1. Preliminaries

In this section, we recall several definitions, such as *divergence*, *modular derivation*, and *Poisson cohomology*. Other basic definitions of Poisson algebras can be found in the book [20]. Everything in this section is well-known.

In Sections 1 and 2, let $\Bbbk$ be a base field of any characteristic and A be any commutative Poisson $\Bbbk$-algebra unless specified otherwise. Let $\Omega^1(A)$ be the module of Kähler differentials over A [20, Sect. 3.2.1]. For each $k \geq 0$, let $\Omega^k(A)$ be $\wedge_A^p \Omega^1(A)$ [20, Sect. 3.2.2]. Set $d = \operatorname{Kdim} A$ where Kdim denotes the Krull dimension. If A is smooth and $\Omega^d(A)$ is a free A-module with a generator ν, then ν is called a *volume form* of A. The differential $d : A \to \Omega^1(A)$ extends to a well-defined differential of the complex $\Omega^\bullet(A)$ and the complex $(\Omega^\bullet(A), d)$ is called the algebraic *de Rham complex* of A.

For each $p \geq 0$, let $\mathfrak{X}^p(A)$ be the set of skew-symmetric p-derivations of A. It is also true that

$$\mathfrak{X}^p(A) \cong \operatorname{Hom}_A(\Omega^p(A), A) \tag{E1.0.1}$$

for all $p \geq 0$ [20, (3.15)].

For every element $P \in \mathfrak{X}^p(A)$, the *internal product* with respect to P, denoted by ι_P, is an A-module map

$$\iota_P : \Omega^\bullet(A) \to \Omega^{\bullet - p}(A)$$

which is determined by

$$\iota_P(dF_1 \wedge dF_2 \wedge \cdots \wedge dF_k) = \begin{cases} 0 & k < p, \\ \sum_{\sigma \in \mathbb{S}_{p,k-p}} sgn(\sigma) P[F_{\sigma(1)}, \ldots, F_{\sigma(p)}] & \\ \qquad dF_{\sigma(p+1)} \wedge \cdots \wedge dF_{\sigma(k)} \in \Omega^{k-p}(A) & k \geq p \end{cases} \tag{E1.0.2}$$

for all $dF_1 \wedge dF_2 \wedge \cdots \wedge dF_k \in \Omega^k(A)$. Here $\mathbb{S}_{p,q} \subset \mathbb{S}_k$ is the set of (p,q)-shuffles with $p + q = k$.

For every $P \in \mathfrak{X}^p(A)$, the *Lie derivative* with respective to P is defined to be

$$\mathcal{L}_P = [\iota_P, d] : \Omega^\bullet(A) \to \Omega^{\bullet - p + 1}(A), \tag{E1.0.3}$$

see [20, (3.49)]. Below is the definition of the divergence of a derivation. In several definitions in this paper we assume that A is a smooth Poisson algebra with a fixed volume form ν.

Definition 1.1. [20, (4.20)] Let δ be a derivation of A, namely, $\delta \in \mathfrak{X}^1(A)$. The *divergence* of δ, denoted by $\operatorname{div}(\delta)$, is an element in A defined by the equation

$$\mathcal{L}_\delta(\nu) = \operatorname{div}(\delta)\nu. \tag{E1.1.1}$$

It is clear that the divergence of δ is dependent on the volume form ν, but independent of the Poisson structure of A. The definition of the divergence of a skew-symmetric k-derivation, for $k \geq 2$, can be found in [20, Sect. 4.3.3].

Let G be an abelian group (or semigroup). A G-graded algebra is a $\Bbbk$-algebra $A = \bigoplus_{g \in G} A_g$ where A_g is a $\Bbbk$-vector space for each $g \in G$ and $A_{g_1} \cdot A_{g_2} \subseteq A_{g_1 g_2}$ for any $g_1, g_2 \in G$. A $\Bbbk$-linear map $\varphi\colon A \longrightarrow A$ of a G-graded algebra A is called graded if $\varphi(A_g) \subseteq A_g$ for each $g \in G$. A Poisson structure $\{-,-\}$ on a G-graded commutative algebra A is called G-graded (of degree zero) if $\deg\{a, b\} = \deg a + \deg b$ for all homogeneous elements $a, b \in A$. Part (1) of the following lemma justifies the definition of the divergence given in (E0.0.4).

Lemma 1.2. *Let A be a Poisson polynomial algebra $\Bbbk[x_1, \dots, x_n]$ and δ be a derivation of A.*

(1)

$$\operatorname{div}(\delta) = \sum_{i=1}^{n} \frac{\partial \delta(x_i)}{\partial x_i}. \tag{E1.2.1}$$

(2) *If A is a $\mathbb{Z}$-graded Poisson polynomial algebra with x_i homogeneous for all i and if δ is graded (of degree 0), then* $\operatorname{div}(\delta) \in A_0$.

(3) *If A is a connected $\mathbb{N}$-graded Poisson polynomial algebra with* $\deg x_i > 0$ *for all i and if δ is graded (of degree 0), then* $\operatorname{div}(\delta) \in \Bbbk$.

(4) *Suppose, in addition to* (3), $\deg x_i = 1$ *for all i. Let δ be a graded derivation of A (of degree 0). Write*

$$\delta(x_i) = \sum_{j=1}^{n} c_{ij} x_j$$

where $c_{ij} \in \Bbbk$ for all $1 \le i, j \le n$. Then

$$\operatorname{div}(\delta) = \sum_{i=1}^{n} c_{ii}. \tag{E1.2.2}$$

(5) *Let A be a $\mathbb{Z}$-graded Poisson algebra with* $\deg x_i \in \mathbb{Z}$ *and let E be the Euler derivation of A as defined in* (E0.0.2). *Then* $\operatorname{div}(E) = \deg \nu$.

Proof. (1) Since $A = \Bbbk[x_1, \dots, x_n]$ is a polynomial algebra, $\nu := dx_1 \wedge \cdots \wedge dx_n$ is a volume form. By the definition of the Lie derivative $\mathcal{L}_\delta$,

$$\begin{aligned}
\mathcal{L}_\delta \nu &= d\left(\sum_{i=1}^{n} (-1)^{i-1} \delta(x_i) dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n\right) \\
&= \sum_{i=1}^{n} (-1)^{i-1} \left(\sum_{j=1}^{n} \frac{\partial \delta(x_i)}{\partial x_j} dx_j\right) \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\
&= \left(\sum_{i=1}^{n} \frac{\partial \delta(x_i)}{\partial x_i}\right) dx_1 \wedge \cdots \wedge dx_n \\
&= \left(\sum_{i=1}^{n} \frac{\partial \delta(x_i)}{\partial x_i}\right) \nu.
\end{aligned}$$

Then, the assertion follows.

(2) Since $\deg \delta = 0$, $\deg \delta(x_i) = \deg x_i$. As a consequence, $\deg \frac{\partial \delta(x_i)}{\partial x_i} = 0$. By part (1), $\deg \operatorname{div}(\delta) = 0$. The assertion follows.

(3) This follows from part (3) and the fact that $A_0 = \Bbbk$.

(4) This follows from part (1) and the fact that $\frac{\partial \delta(x_i)}{\partial x_i} = c_{ii}$ for all i.

(5) In this case, $\nu = dx_1 \wedge \cdots \wedge dx_n$ and $E(x_i) = (\deg x_i) x_i$. The assertion follows from (E1.2.1). □

We recall the following definition.

Definition 1.3. Let A be a Poisson algebra with volume form ν.

(1) [20, Definition 4.10] The *modular derivation* (or *modular vector field*) of A associated to ν is defined to be

$$\mathbf{m}(a) := -\operatorname{div}(H_a)$$

for all $a \in A$, or equivalently,

$$\mathcal{L}_{H_a}(\nu) = -\mathbf{m}(a)\nu.$$

(2) [20, Definition 4.12] If $\mathbf{m} = 0$ for some volume form ν, then A is called *unimodular*.

If $A = \Bbbk[x_1, \dots, x_n]$, then $\mathbf{m}$ is independent of the choice of the volume form ν.
Let us give an easy example.

Example 1.4. Let A be the Poisson polynomial algebra $\Bbbk[x_1, x_2]$ with $\{x_1, x_2\} = x_1^n$ for some integer $n \geq 0$. It is easy to check that $\mathbf{m}(x_1) = 0$ and $\mathbf{m}(x_2) = nx_1^{n-1}$. If char $\Bbbk = p > 0$ and $p \mid n$, then A is unimodular. Now suppose $n = 2$. Since $\{x_1x_2, x_2\} = x_1^2x_2$, $\mathbf{m}(\{x_1x_2, x_2\}) = \mathbf{m}(x_1^2x_2) = 2x_1^3$. As a consequence, div$([\delta_1, \delta_2])$ is in general nonzero for any two derivations δ_1, δ_2.

Next, we review the Poisson cohomology. Let (A, π) be a Poisson algebra. For each $k \geq 0$, $\mathfrak{X}^k(A)$ denotes the space of skew-symmetric k-derivations of A. The Poisson coboundary map $d_\pi : \mathfrak{X}^\bullet(A) \to \mathfrak{X}^{\bullet+1}(A)$ is defined as follows. For any $Q \in \mathfrak{X}^q(A)$, where $q \in \mathbb{N}$, we define

$$\begin{aligned} d_\pi^q(Q)[F_0, \dots, F_q] := &\sum_{i=0}^{q} (-1)^i \{F_i, Q[F_0, \dots, \widehat{F_i}, \dots, F_q]\} \\ &+ \sum_{0 \leq i \leq j \leq q} (-1)^{i+j} Q[\{F_i, F_j\}, F_0, \dots, \widehat{F_i}, \dots, \widehat{F_j}, \dots, F_q], \end{aligned} \tag{E1.5.1}$$

for all $F_0, \cdots, F_q \in A$. In particular,

$$\begin{aligned} d_\pi^0(Q)[F_0] &= \{F_0, Q\}, \\ d_\pi^1(Q)[F_0, F_1] &= \{F_0, Q[F_1]\} - \{F_1, Q[F_0]\} - Q[\{F_0, F_1\}], \\ d_\pi^2(Q)[F_0, F_1, F_2] &= \{F_0, Q[F_1, F_2]\} - \{F_1, Q[F_0, F_2]\} + \{F_2, Q[F_0, F_1]\} \\ &\quad - Q[\{F_0, F_1\}, F_2] + Q[\{F_0, F_2\}, F_1] - Q[\{F_1, F_2\}, F_0]. \end{aligned}$$

For $P \in \mathfrak{X}^p(A)$ and $Q \in \mathfrak{X}^q(A)$, the wedge product $P \wedge Q \in \mathfrak{X}^{p+q}(A)$ is the skew-symmetric $(p+q)$-derivation of A, defined by

$$\begin{aligned} &(P \wedge Q)[F_1, \cdots, F_{p+q}] := \\ &\qquad \sum_{\sigma \in \mathbb{S}_{p,q}} sgn(\sigma) P[F_{\sigma(1)}, \dots, F_{\sigma(p)}]\, Q[F_{\sigma(p+1)}, \dots, F_{\sigma(p+q)}], \end{aligned}$$

for all $F_1, \cdots, F_{p+q} \in A$. In particular, if $P \in \mathfrak{X}^1(A)$ and $Q \in \mathfrak{X}^2(A)$, then we have

$$(P \wedge Q)[F_1, F_2, F_3] = P[F_1]Q[F_2, F_3] - P[F_2]Q[F_1, F_3] + P[F_3]Q[F_1, F_2]. \tag{E1.5.2}$$

Therefore, $(\mathfrak{X}^*(A), \wedge, d)$ is a dga (differential graded algebra). For each $q \geq 0$, the *q-th Poisson cohomology* of A is defined to be

$$PH^q(A) := \frac{\ker d_\pi^q}{\operatorname{im} d_\pi^{q-1}}. \tag{E1.5.3}$$

A derivation $\delta\colon A \longrightarrow A$ of a Poisson algebra A is called a *Poisson derivation* if $\delta(\{a, b\}) = \{\delta(a), b\} + \{a, \delta(b)\}$ for any $a, b \in A$. It is clear from the definition that the first Poisson cohomology of A is

$$PH^1(A) := \frac{\text{the set of Poisson derivations}}{\text{the set of Hamiltonian derivations}}. \tag{E1.5.4}$$

If A is a quadratic Poisson algebra $\Bbbk[x, y, z]$ (with $\deg(x) = \deg(y) = \deg(z) = 1$), then the complex $(\mathfrak{X}^\bullet(A), d_\pi)$ is [29, (15)]

$$0 \to A \to (A[1])^{\oplus 3} \to (A[2])^{\oplus 3} \to A[3] \to 0.$$

By the additivity of the Hilbert series, we have

$$\sum_{i=0}^{3} (-1)^i h_{PH^i(A)}(t) = -t^{-3}. \tag{E1.5.5}$$

It is easy to check that

(a) the lowest degree of nonzero elements in $PH^0(A)$ is 0 and $PH^0(A)_0 = \Bbbk$;
(b) the lowest degree of nonzero elements in $PH^1(A)$ is ≥ -1;
(c) the lowest degree of nonzero elements in $PH^2(A)$ is ≥ -2;
(d) the lowest degree of nonzero elements in $PH^3(A)$ is -3 and $PH^3(A)_{-3} = \Bbbk$.

If A is further unimodular, then

(e) the lowest degree of nonzero elements in $PH^2(A)$ is -2 and $PH^2(A)_{-2} = \Bbbk^{\oplus 3}$.

A natural operation on $\mathfrak{X}^\bullet(A)$ is the Schouten bracket

$$[\cdot,\cdot]_S : \mathfrak{X}^p(A) \times \mathfrak{X}^q(A) \to \mathfrak{X}^{p+q-1}(A)$$

for all $p, q \geq 0$. We refer to [20, Section 3.3.2] for the precise definition. By [20, (4.5)],

$$d_\pi(\cdot) = -[\cdot, \pi]_S.$$

By [20, Proposition 3.7], $(\mathfrak{X}^\bullet(A), \wedge, [\cdot,\cdot]_S)$ is a Gerstenhaber algebra.

Let $A = (A, \pi)$ be a Poisson algebra with Poisson bracket π. Let $\xi \in \Bbbk^\times$ be any nonzero scalar. We define a new Poisson bracket $\pi_\xi := \xi\pi$ or $\{-,-\}_\xi := \xi\{-,-\}$. Then it is easy to see that $A' := (A, \pi_\xi)$ is indeed a Poisson algebra. In general, A' is not isomorphic to A, but they are closely related as follows.

Lemma 1.5. *Retain the notations as above. Let d^q_π (resp. $d^q_{\pi'}$) be the differential of $\mathfrak{X}^\bullet(A)$ (resp. $\mathfrak{X}^\bullet(A')$) as defined in (E1.5.1). The following hold:*

(1) $d^q_{\pi'} = \xi d^q_\pi$ *for all* q.
(2) $\ker d^q_{\pi'} = \ker d^q_\pi$ *for all* q.
(3) $\mathrm{im} d^q_{\pi'} = \mathrm{im} d^q_\pi$ *for all* q.
(4) $PH^q(A) = PH^q(A')$ *for all* q.

2. Twists of graded Poisson algebras

Let G be an abelian group and A be a G-graded Poisson algebra (namely, both the multiplication $\cdot$ and the Poisson bracket $\{-,-\}$ of A are graded of degree 0). We use g for elements in G. If a is a homogeneous element in A, we use $|a|$ to denote its degree in G.

The aim of this section is to define a Poisson version of the graded twist of graded associative algebras [39].

Definition 2.1. Let $\delta := \{\delta_g \mid g \in G\}$ be a set of graded derivations of A (of degree 0). We say δ is a *Poisson twisting system* if it satisfies the following conditions:

(1) For all $g, h \in G$,

$$\delta_g \delta_h = \delta_h \delta_g. \tag{E2.1.1}$$

(2) For homogeneous elements $a, b \in A$,

$$\delta_{|ab|} = \delta_{|a|} + \delta_{|b|}. \tag{E2.1.2}$$

(3) For homogeneous elements $a, b, c, \in A$ and $g \in G$,

$$p(\{-,-\}, \delta_g; a, b, c) = 0. \tag{E2.1.3}$$

Remark 2.2.

(1) The definition of a Poisson twisting system is a "translation" of the twisting system in the setting of graded associative algebras given in [39, Definition 2.1].
(2) If δ is a Poisson derivation, it is automatic that $p(\{-,-\}, \delta; a, b, c) = 0$. The converse is not true; see Example 2.6.
(3) Suppose $G = \mathbb{Z}$ and let $\phi = \delta_1$. By (E2.1.2), $\delta_n = n\phi$. It is clear that

$$p(\{-,-\}, \phi; a, b, c) = (E \wedge d^1_\pi(\phi))(a, b, c),$$

which implies that (E2.1.3) is equivalent to $E \wedge d^1_\pi(\phi) = 0$. By [20, Sect. 4.3] and the fact that $d^1_\pi(E) = 0$, the equation $E \wedge d^1_\pi(\phi) = 0$ is equivalent to $d^2_\pi(E \wedge \phi) = 0$.
(4) Let $G = \mathbb{Z}$ and A be a $\mathbb{Z}$-graded Poisson algebra. A convenient Poisson twisting system is constructed as follows. Let ϕ be a graded Poisson derivation of A (namely, $d^1_\pi(\phi) = 0$). For each $n \in \mathbb{Z}$, let $\delta_n := n\phi$ and $\delta := \{\delta_n \mid n \in \mathbb{Z}\}$. Then (E2.1.1) and (E2.1.2) are obvious and (E2.1.3) follows from the fact that δ_n is a Poisson derivation; see part (2) or (3).

Example 2.3. Let $G = \mathbb{Z}/(n)$ for some positive integer n. Let A be a G-graded Poisson algebra and δ be a graded Poisson derivation of A. Suppose $p := \mathrm{char}\, \Bbbk$ is positive. If $p \mid n$, let $\delta_i = i\delta$ for all $i \in G$. Then $\{\delta_i \mid i \in G\}$ is a Poisson twisting system for A. If $p \nmid n$, there is no nontrivial Poisson twisting system for A. Suppose $\{\delta_i \mid i \in G\}$ is a Poisson twisting system for A. We have

$$0 = \delta_{\overline{0}} = \delta_{\overline{n}} = n\delta_{\overline{1}},$$

which implies that $\delta_{\overline{1}} = 0$ and hence $\delta_i = 0$ for all $i \in G$.

Let A be a G-graded Poisson algebra and let $\delta := \{\delta_g \mid g \in G\}$ be a system of derivations of A. We define

$$\langle a, b\rangle := \{a, b\} + a\delta_{|a|}(b) - b\delta_{|b|}(a) \tag{E2.3.1}$$

for all homogeneous elements $a, b \in A$.

Theorem 2.4. *Let $\delta := \{\delta_g \mid g \in G\}$ be a set of graded derivations of a G-graded Poisson algebra A satisfying (E2.1.1) and (E2.1.2). Then, the following hold.*

(1) *$\langle -, -\rangle$ is skew-symmetric.*
(2) *For every homogeneous element $a \in A$, $\langle a, -\rangle$ is a derivation of A*
(3) *$\langle -, -\rangle$ satisfies the Jacobian identity if and only if (E2.1.3) holds.*

In particular, if (E2.1.3) holds, then $(A, \langle -, -\rangle)$ is a Poisson algebra.

Proof. If $G = \mathbb{Z}$, there is a shorter proof using the Schouten bracket. We make the following direct computation for a general abelian group G.

(1) follows immediately from (E2.3.1).

(2) For homogeneous elements a, b, c in A, we have

$$\begin{aligned}
\langle a, bc\rangle &= \{a, bc\} + a\delta_{|a|}(bc) - bc\delta_{|bc|}(a) \\
&= \{a, b\}c + \{a, c\}b + a\left(b\delta_{|a|}(c) + \delta_{|a|}(b)c\right) - bc\delta_{|bc|}(a), \\
\langle a, b\rangle c &= \left(\{a, b\} + a\delta_{|a|}(b) - b\delta_{|b|}(a)\right)c, \\
b\langle a, c\rangle &= b\left(\{a, c\} + a\delta_{|a|}(c) - c\delta_{|c|}(a)\right).
\end{aligned}$$

By the above and (E2.1.2), we obtain that

$$\langle a, bc\rangle = \langle a, b\rangle c + b\langle a, c\rangle.$$

(3) For homogeneous elements a, b, c in A, we have

$$\begin{aligned}
\langle a, \langle b, c\rangle\rangle &= \{a, \langle b, c\rangle\} + a\delta_{|a|}(\langle b, c\rangle) - \langle b, c\rangle\delta_{|bc|}(a) \\
&= \{a, \left(\{b, c\} + b\delta_{|b|}(c) - c\delta_{|c|}(b)\right)\} + a\delta_{|a|}\left(\{b, c\} + b\delta_{|b|}(c) - c\delta_{|c|}(b)\right) \\
&\quad - \left(\{b, c\} + b\delta_{|b|}(c) - c\delta_{|c|}(b)\right)\delta_{|bc|}(a) \\
&= \{a, \{b, c\}\} + \{a, b\}\delta_{|b|}(c) + b\{a, \delta_{|b|}(c)\} - \{a, c\}\delta_{|c|}(b) \\
&\quad - c\{a, \delta_{|c|}(b)\} + a\delta_{|a|}(\{b, c\}) + a\delta_{|a|}(b)\delta_{|b|}(c) + ab\delta_{|a|}\delta_{|b|}(c) \\
&\quad - a\delta_{|a|}(c)\delta_{|c|}(b) - ac\delta_{|a|}\delta_{|c|}(b) - \{b, c\}\delta_{|bc|}(a) - b\delta_{|b|}(c)\delta_{|bc|}(a) \\
&\quad + c\delta_{|c|}(b)\delta_{|bc|}(a)
\end{aligned}$$

and

$$\begin{aligned}
\langle\langle a, b\rangle, c\rangle &= \langle c, \langle b, a\rangle\rangle \\
&= \{c, \{b, a\}\} + \{c, b\}\delta_{|b|}(a) + b\{c, \delta_{|b|}(a)\} - \{c, a\}\delta_{|a|}(b) \\
&\quad - a\{c, \delta_{|a|}(b)\} + c\delta_{|c|}(\{b, a\}) + c\delta_{|c|}(b)\delta_{|b|}(a) + cb\delta_{|c|}\delta_{|b|}(a) \\
&\quad - c\delta_{|c|}(a)\delta_{|a|}(b) - ca\delta_{|c|}\delta_{|a|}(b) - \{b, a\}\delta_{|ba|}(c) - b\delta_{|b|}(a)\delta_{|ba|}(c) \\
&\quad + a\delta_{|a|}(b)\delta_{|ba|}(c)
\end{aligned}$$

and

$$\begin{aligned}\langle b, \langle a, c\rangle\rangle = &\{b, \{a, c\}\} + \{b, a\}\delta_{|a|}(c) + a\{b, \delta_{|a|}(c)\} - \{b, c\}\delta_{|c|}(a)\\ &- c\{b, \delta_{|c|}(a)\} + b\delta_{|b|}(\{a, c\}) + b\delta_{|b|}(a)\delta_{|a|}(c) + ba\delta_{|b|}\delta_{|a|}(c)\\ &- b\delta_{|b|}(c)\delta_{|c|}(a) - bc\delta_{|b|}\delta_{|c|}(a) - \{a, c\}\delta_{|ac|}(b) - a\delta_{|a|}(c)\delta_{|ac|}(b)\\ &+ c\delta_{|c|}(a)\delta_{|ac|}(b).\end{aligned}$$

Using the Jacobi identity

$$-\{a, \{b, c\}\} + \{\{a, b\}, c\} + \{b, \{a, c\}\} = 0,$$

(E2.1.1) and (E2.1.2), we can simplify

$$-\langle a, \langle b, c\rangle\rangle + \langle\langle c, b\rangle, a\rangle + \langle b, \langle a, c\rangle\rangle$$

to

$$p(\{-, -\}, \delta; a, b, c).$$

Therefore, $\langle -, -\rangle$ satisfies the Jacobi identity when $p(\delta; a, b, c) = 0$. (3) follows. The consequence is clear. □

Definition 2.5. Let $\delta := \{\delta_g \mid g \in G\}$ be a Poisson twisting system of a G-graded Poisson algebra A. Then the new Poisson algebra $(A, \langle -, -\rangle)$ given in Theorem 2.4 is called the *twist of A by δ* and denoted by A^δ.

Example 2.6. Let $A = \Bbbk[x, y]$ be the $\mathbb{Z}$-graded Poisson algebra defined by $\{x, y\} = x^2$. Let ϕ be the derivation sending $x \to -x$ and $y \to y - x$. Let $\delta_n = n\phi$. It is easy to see that

$$\begin{aligned}d^1_\pi(\phi)(x, y) &= -\phi(\{x, y\}) + \{x, \phi(y)\} + \{\phi(x), y\}\\ &= -\phi(x^2) + \{x, y - x\} + \{-x, y\} = 2x^2 \neq 0,\end{aligned}$$

which implies that ϕ is not a Poisson derivation.

We claim that $\delta := \{\delta_n\}$ is a Poisson twisting system. Let f be the derivation of A determined by

$$f(x) = 0, \quad \text{and} \quad f(y) = -x.$$

It is easy to verify that f is a Poisson derivation. By Remark 2.2(4), $f' := \{nf \mid n \in \mathbb{Z}\}$ is a Poisson twisting system, and by Theorem 2.4, $A^{f'}$ is equipped with a Poisson structure such that

$$\langle x, y\rangle = \{x, y\} + xf(y) - yf(x) = x^2 - x^2 - 0 = 0.$$

Therefore $A^{f'}$ has trivial Poisson structure. Let g be the Poisson derivation of $A^{f'}$ determined by

$$g(x) = -x, \quad \text{and} \quad g(y) = y.$$

Let $g' = \{ng \mid n \in \mathbb{Z}\}$. By Remark 2.2(4), g' is a Poisson twisting system of $A^{f'}$ and the Poisson structure of $(A^{f'})^{g'}$ is determined by, for all homogeneous elements $a, b \in A$,

$$\begin{aligned}\{a, b\}_{new} := &\langle a, b\rangle + |a|ag(b) - |b|bg(a)\\ = &\{a, b\} + |a|af(b) - |b|bf(a) + |a|ag(b) - |b|bg(a)\\ = &\{a, b\} + ah_{|a|}(b) - bh_{|b|}(a)\end{aligned}$$

where $h_n = nf + ng$ for all $n \in \mathbb{Z}$. Since $f + g$ is a derivation of A, by Theorem 2.4, $h' := \{h_n \mid n \in \mathbb{Z}\}$ is a Poisson twisting system of A. It is clear that $\delta = h'$. So δ is a Poisson twisting system.

Since δ is a Poisson twisting system, by Remark 2.2(3), ϕ is a graded semi-Poisson derivation. By the first paragraph, ϕ is not a Poisson derivation.

Lemma 2.7. *Suppose G is cyclic. Then, the set of Poisson twisting systems of A is a $\Bbbk$-vector space.*

Proof. Let δ and φ be two Poisson twisting systems. It is clear that $c\delta$ is a Poisson twisting system for all $c \in \Bbbk$. It remains to show $h := \delta + \varphi$ is a Poisson twisting system.

Since G is cyclic, $h_n = nh_1$. So (E2.1.1) is clear. Now (E2.1.2) and (E2.1.3) hold as these are “linear” in terms of δ. □

Remark 2.8. If G is $\mathbb{Z}^2$, then the set of Poisson twisting systems of $(\Bbbk[x_1, x_2, x_3], 0)$ with $\deg x_1 = \deg x_2 = (1, 0)$ and $\deg x_3 = (0, 1)$ is not a $\Bbbk$-vector space. To see this, we consider two graded Poisson derivations δ_1 and ϕ_1 that do not commute (for example, $\delta_1 : x_1 \to x_1, x_2 \to 0, x_3 \to 0$ and $\phi_1 : x_1 \to x_2, x_2 \to 0, x_3 \to 0$). Let $\delta_{(n,m)} = n\delta_1$ and $\phi_{(n,m)} = m\phi_1$. It is easy to see that both δ and ϕ are twisting systems of the G-graded Poisson algebra $(\Bbbk[x_1, x_2, x_3], 0)$. We define $\delta + \phi$ by $(\delta + \phi)_{(n,m)} = n\delta_1 + m\phi_1$ for all $(n, m) \in \mathbb{Z}^2$. Since δ_1 and ϕ_1 do not commute, we see that (E2.1.1) fails for $\delta + \phi$.

As noted before, a derivation δ of A is Poisson if and only if $d^1_\pi(\delta) = 0$. By Definition 0.4, a graded derivation δ of a $\mathbb{Z}$-graded Poisson algebra A is *semi-Poisson* if $E \wedge d^1_\pi(\delta) = 0$.

Next, we show that the Poisson twisting systems induce an equivalence relation. Let A be a G-graded commutative algebra. Two graded Poisson structures π and π' on A are called equivalent if (A, π') is a graded twist of (A, π). In this case we write $(A, \pi) \sim (A, \pi')$.

Proposition 2.9. *Suppose G is cyclic and A is a G-graded commutative algebra. Then, $\sim$ is an equivalence relation.*

Proof. It is clear that $(A, \pi) \sim (A, \pi)$ by taking the trivial Poisson twisting system δ. So, $\sim$ is reflexive.

To prove the symmetry of $\sim$, we suppose that (A, π') is a graded twist of (A, π) by δ. We claim that $-\delta$ is a Poisson twisting system of (A, π'). Once proved, then it is obvious that $(A, \pi')^{-\delta} = (A, \pi)$ as desired. It remains to show that $-\delta$ satisfies (E2.1.1), (E2.1.2) and (E2.1.3). The first two are easy. For the last one, we compute

$$\begin{aligned}
&\delta_{|a|}(\langle b, c\rangle) - \langle \delta_{|a|}(b), c\rangle - \langle b, \delta_{|a|}(c)\rangle \\
&\quad= \delta_{|a|}(\{b, c\} + b\delta_{|b|}(c) - c\delta_{|c|}(b)) \\
&\qquad - [\{\delta_{|a|}(b), c\} + \delta_{|a|}(b)\delta_{|b|}(c) - c\delta_{|c|}\delta_{|a|}(b)] \\
&\qquad - [\{b, \delta_{|a|}(c)\} + b\delta_{|b|}\delta_{|a|}(c) - \delta_{|a|}(c)\delta_{|c|}(b)] \\
&\quad= \delta_{|a|}(\{b, c\}) - \{\delta_{|a|}(b), c\} - \{b, \delta_{|a|}(c)\} \\
&\qquad + \delta_{|a|}(b)\delta_{|b|}(c) + b\delta_{|a|}\delta_{|b|}(c) - \delta_{|a|}(c)\delta_{|c|}(b) - c\delta_{|a|}\delta_{|c|}(b) \\
&\qquad - \delta_{|a|}(b)\delta_{|b|}(c) + c\delta_{|c|}\delta_{|a|}(b) - b\delta_{|b|}\delta_{|a|}(c) + \delta_{|a|}(c)\delta_{|c|}(b) \\
&\quad= \delta_{|a|}(\{b, c\}) - \{\delta_{|a|}(b), c\} - \{b, \delta_{|a|}(c)\}
\end{aligned}$$

which implies that

$$p(\langle -, -\rangle, -\delta; a, b, c) = p(\{-, -\}, -\delta; a, b, c) = 0.$$

Therefore $-\delta$ is a Poisson twisting system of A^δ and $(A^\delta)^{-\delta} = A$. So, $\sim$ is symmetric.

To prove the transitivity of $\sim$, we use the idea given in Example 2.6. Suppose δ is a Poisson twisting system of A and ϕ a Poisson twisting system of A^δ. Let $\sigma := \{\sigma_g := \delta_g + \phi_g \mid g \in G\}$.

Since G is cyclic, $\sigma_n = n\sigma_1$ by definition for all $n \in G$. Therefore (E2.1.1)-(E2.1.2) are obvious. Define

$$\{a, b\}_{new} = \{a, b\} + a\sigma_{|a|}(b) - b\sigma_{|b|}(a) = \langle a, b\rangle + a\phi_{|a|}(b) - b\phi_{|b|}(a).$$

Then $\{-, -\}_{new}$ is the Poisson bracket of $(A^\delta)^\phi$. By Theorem 2.4, σ is a Poisson twisting system of A and $A^\sigma = (A^\delta)^\phi$. Therefore $\sim$ is transitive. □

Remark 2.10. Let G be $\mathbb{Z}^2$ and $A = \Bbbk[x_0, x_1, x_2, x_3]$ with $\deg x_i = (1, 0)$ for $i = 0, 1, 2$ and $\deg x_3 = (0, 1)$. We claim that $\sim$ is not an equivalence relation among the Poisson structures on A. We use Y for the $\mathbb{Z}^2$-graded Poisson algebra with trivial Poisson structure.

Let δ_1 be the Poisson derivation of Y sending $x_0 \to 0$, $x_1 \to x_2$, $x_2 \to 0$, and $x_3 \to 0$. Let ϕ_1 be the Poisson derivation of Y sending $x_0 \to 0$, $x_1 \to x_1$, $x_2 \to 0$ and $x_3 \to 0$. We define two Poisson twisting systems as follows. Let $\delta := \{\delta_{(n,m)} = n\delta_1\}$ and $\phi := \{\phi_{(n,m)} = m\phi_1\}$. Since δ_1 and ϕ_1 are Poisson derivations, it is easy to verify that δ, $-\delta$ and ϕ are Poisson twisting systems. By Theorem 2.4, $X := Y^{-\delta}$ is a Poisson algebra and by the first part of the proof of Proposition 2.9, δ is a Poisson twisting system of X and $Y = X^\delta$. So we have $X \sim Y$. Let $Z = Y^\phi$. Then $Y \sim Z$. We claim that $X \not\sim Z$. Suppose, on the contrary, that $X \sim Z$. Then $Z = X^h$ for some Poisson twisting system h of X. Applying (E2.3.1) to pairs of elements of the form (x_i, x_j) for all $0 \le i, j \le 3$, we see that $h = \delta + \phi$. But it is clear that (E2.1.1) fails for $\delta + \phi$ as $\delta_1\phi_1 \ne \phi_1\delta_1$. Therefore, there is no Poisson twisting system h such that $Z = X^h$ as desired.

This example suggests that there should be a more general definition of twisting systems that induce an equivalence relation $\sim$.

We conclude this section with more examples.

Example 2.11. Here are two examples of twists of graded Poisson algebras.

(1) Let A be the Poisson polynomial ring $\Bbbk[x_1, \dots, x_n]$ with trivial Poisson bracket. Consider A as a $\mathbb{Z}^n$-graded algebra with $\deg x_i = e_i$ where

$$e_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^n$$

with 1 in the ith position. Let $\{p_{ij} \mid 1 \le i < j \le n\}$ be a subset of $\Bbbk$. For each i, define a $\mathbb{Z}^n$-graded Poisson derivation δ_i by

$$\delta_i(x_j) = \begin{cases} p_{ij} x_j & j > i, \\ 0 & j \le i. \end{cases}$$

For each $(a_1, \dots, a_n) \in \mathbb{Z}^n$, let $\delta_{(a_1,\dots,a_n)} = \sum_{i=1}^n a_i \delta_i$. Since each $\delta_{(a_1,\dots,a_n)}$ is a graded Poisson derivation of A, it is easy to see that

$$\delta := \{\delta_{(a_1,\dots,a_n)} \mid (a_1, \dots, a_n) \in \mathbb{Z}^n\}$$

is a twisting system of A. By (E2.3.1), the Poisson bracket of the new Poisson algebra A^δ is determined by

$$\langle x_i, x_j \rangle = x_i \delta_i(x_j) - x_j \delta_j(x_i) = p_{ij} x_i x_j \quad \text{for all } i < j.$$

(2) Let A be the Poisson polynomial ring $\Bbbk[x_1, \dots, x_n]$ with trivial Poisson bracket. Consider A as a $\mathbb{Z}$-graded algebra with $\deg x_i = 1$ for all i. Let δ_1 be a Poisson derivation of A determined by

$$\delta_1(x_i) = \begin{cases} -x_{i-1} & i > 1, \\ 0 & i = 1. \end{cases}$$

Let $\delta := \{\delta_d := d\delta_1 \mid \forall\, d \in \mathbb{Z}\}$. Since δ_1 is a graded Poisson derivation, δ is a twisting system of A. By (E2.3.1), the Poisson bracket of the new Poisson algebra A^δ is determined by

$$\langle x_i, x_j \rangle = x_i \delta(x_j) - x_j \delta(x_i) = -x_i x_{j-1} + x_j x_{i-1} \quad \text{for all } i < j.$$

When $n = 2$, the Poisson bracket of A^δ is determined by

$$\langle x_2, x_1 \rangle = x_1^2.$$

3. Proofs of Theorem 0.2 and Corollary 0.3

For the rest of the paper we assume that $\operatorname{char} \Bbbk = 0$. Let A be a commutative Poisson $\Bbbk$-algebra of Krull dimension d. First, we recall the definition of *divergence* of a skew-symmetric k-derivation for $k \ge 0$ [20, Sect. 4.4.3]. A special case is given in Definition 1.1. For $P \in \mathfrak{X}^p(A)$, the *internal product* ι_P is defined at the beginning of Section 1.

Let ν be a volume form of A. Then $\nu \in \Omega^d(A)$, where $\Omega^d(A) \neq 0$ and $\Omega^k(A) = 0$ for $k > d$. The form ν is also called a d-form. We define the *star operator*

$$\star_A : \mathfrak{X}^\bullet(A) \to \Omega^{d-\bullet}(A)$$

as follows: for each $k \ge 0$ and $Q \in \mathfrak{X}^k(A)$, we set

$$\star_A Q := \iota_Q \nu.$$

So $\star_A$ is a $\Bbbk$-linear map from $\mathfrak{X}^k(A)$ to $\Omega^{d-k}(A)$ for each k. It follows from (E1.0.2) that $\star_A$ is an A-linear map. We simply write $\star_A$ as $\star$ if no confusion arises.

Lemma 3.1. *Let B be a smooth affine domain of dimension n with volume form ν. Then, $\star_B$ is an isomorphism.*

Proof. To prove that $\star$ is an isomorphism, it suffices to show that $\star_B \otimes_B B_{\mathfrak{m}}$ is an isomorphism for all maximal ideals $\mathfrak{m}$ of B. Let A be the local ring $B_{\mathfrak{m}}$. Then A is a regular local ring of global dimension, Krull dimension, and transcendence degree n. Since all the operations commute with the localization,

$$\star_B \otimes_B B_{\mathfrak{m}} = \star_B \otimes_B A = \star_A.$$

Now we assume that A is local with maximal ideal $\mathfrak{m}$ generated by $\{x_1, \dots, x_n\}$. Then $\Omega^1(A)$ is a free module of rank n. Write $\Omega^1(A) = \oplus_{i=1}^n A dx_i$. Then for each $k \ge 0$, $\Omega^k(A)$ is a free A-module with basis

$$\{d_{i_1,\cdots,i_k} := dx_{i_1} \wedge \cdots \wedge dx_{i_k} \mid 1 \le i_1 < \cdots < i_k \le n\}$$

and that, via (E1.0.1), $\mathfrak{X}^k(A)$ is a free A-module with basis as in (E1.0.1)

$$\{\partial_{i_1,\ldots,i_k} := \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_k}} \mid 1 \le i_1 < \cdots < i_k \le n\}.$$

Recall that $\nu = a\,d_{1,2,\ldots,n}$ for an invertible element $a \in A$. By definition,

$$\begin{aligned}\star_A \partial_{i_1,\ldots,i_k} &= \iota_{\partial_{i_1,\ldots,i_k}}(\nu) = a\iota_{d_{i_1,\ldots,i_k}}(d_{1,2,\cdots,n}) \\ &= \sum_{\sigma\in\mathbb{S}_{k,n-k}} sgn(\sigma) a\partial_{i_1,\ldots,i_k}[x_{\sigma(1)},\ldots,x_{\sigma(k)}]d_{1,\ldots,\widehat{\sigma(1)},\ldots,\widehat{\sigma(k)},\ldots,n} \\ &= \pm a\, d_{1,\ldots,\widehat{i_1},\cdots,\widehat{i_k},\ldots,n}\end{aligned}$$

where $\pm 1 = sgn(\{i_1,\ldots,i_k,1,\ldots,\widehat{i_1},\ldots,\widehat{i_k},\ldots,n\})$. Therefore, $\star_A$ is an isomorphism as desired. □

Definition 3.2. We say A is a *standard* Poisson algebra if A is an affine smooth $\mathbb{Z}$-graded Poisson domain with a homogeneous volume form ν (with $\deg \nu$ not necessarily zero) and $\star$ is an isomorphism.

Note that every polynomial Poisson algebra $\Bbbk[x_1,\ldots,x_n]$ is standard (even when $\mathrm{char}\,\Bbbk > 0$). Lemma 3.1 provides another class of such algebras. Now we assume that A is standard of dimension n. By [20, Sect. 4.4.3], the *divergence* operator with respect to the volume form ν is a graded $\Bbbk$-linear map of degree -1,

$$\mathrm{div} : \mathfrak{X}^\bullet(A) \to \mathfrak{X}^{\bullet-1}(A),$$

which makes the following diagram commutes

$$\begin{array}{ccc} \mathfrak{X}^\bullet(A) & \xrightarrow{\star} & \Omega^{n-\bullet}(A) \\ {\scriptstyle\mathrm{div}}\downarrow & & \downarrow{\scriptstyle d} \\ \mathfrak{X}^{\bullet-1}(A) & \xrightarrow{\star} & \Omega^{n-\bullet+1}(A). \end{array}$$

Since A is standard, the star operator $\star$ is an A-linear isomorphism. Now we have the following lemmas proved in [20, Sect. 4.4.3].

Lemma 3.3. *[20, Proposition 4.16] Suppose (A,π) is standard with volume form ν. Let δ and ϕ be two derivations of A. Then*

$$\mathrm{div}(\delta\wedge\phi) = \mathrm{div}(\phi)\delta - \mathrm{div}(\delta)\phi - [\delta,\phi].$$

The following lemma gives another proof of Lemma 0.1.

Lemma 3.4. *[20, Proposition 4.17] Suppose (A,π) is standard with volume form ν. Let $\mathbf{m}$ be the modular derivation of A. Then*

$$\mathbf{m} = -\mathrm{div}(\pi).$$

Consequently, the divergence of $\mathbf{m}$ is zero.

Question 3.5. It is not clear how to handle nonaffine smooth domain A as the proof of Lemma 3.1 uses the fact A is affine.

Theorem 3.6. *Suppose (A,π) is standard with volume form ν. Let δ be a graded semi-Poisson derivation of A. Let $\mathbf{m}$ (resp. $\mathbf{n}$) be the modular derivation of A (resp. A^δ). Then*

$$\mathbf{n} = \mathbf{m} + (\mathrm{div}\,E)\delta - (\mathrm{div}\,\delta)E.$$

Proof. Let π' be the Poisson structure of A^δ. By (E2.3.1),

$$\pi' = \pi + E\wedge\delta.$$

By Lemmas 3.3 and 3.4, we have

$$\begin{aligned}\mathbf{n} &= -\mathrm{div}(\pi') = -\mathrm{div}(\pi) - \mathrm{div}(E\wedge\delta) \\ &= \mathbf{m} + \mathrm{div}(E)\delta - \mathrm{div}(\delta)E - [\delta, E].\end{aligned}$$

The assertion follows as $[\delta,E]=0$ for each graded derivation δ. □

Proof of Theorem 0.2. Let A be a $\mathbb{Z}$-graded Poisson algebra $\Bbbk[x_1,\ldots,x_n]$ with each x_i homogeneous. By Lemma 1.2(5), $\operatorname{div}(E)=\deg\nu=\sum_{i=1}^n \deg x_i =: \mathfrak{l}$. Now the assertions follow from Theorem 3.6. □

Remark 3.7. There is a different proof of Theorem 0.2 without using Theorem 3.6 (details are omitted). In fact, the different proof does not use the hypothesis that $\operatorname{char}\Bbbk=0$.

The following result is an immediate consequence of Theorem 3.6.

Theorem 3.8. *Suppose (A,π) is standard with a volume form $\nu\in\Omega^d(A)$. Assume that $\operatorname{div}(E)\in\Bbbk$ is nonzero. Let $\mathbf{m}$ be the modular derivation of A and let $\delta=-\frac{1}{\operatorname{div}(E)}\mathbf{m}$. Then (A^δ,π') is unimodular and*

$$\pi=\pi'+\frac{1}{\operatorname{div}(E)}E\wedge\mathbf{m}.$$

In particular, we have $\mathcal{L}_\delta(\alpha)=0$ where $\alpha=\star\pi'$ is the closed differential $(d-2)$-form associated with the unimodular Poisson structure π' on A.

Proof. Let $\mathbf{n}$ be the modular derivation of (A^δ,π'). By Theorem 3.6 and the fact $\delta=-\frac{1}{\operatorname{div}(E)}\mathbf{m}$,

$$\mathbf{n}=\mathbf{m}+\operatorname{div}(E)\delta-\operatorname{div}(\delta)E=-\operatorname{div}(\delta)E=0$$

where the last equation follows from $\operatorname{div}(\mathbf{m})=0$ [Lemma 0.1]. Therefore (A^δ,π') is unimodular. By (E2.3.1), for all $a,b\in A$,

$$\begin{aligned}\pi'(a,b)&=\langle a,b\rangle=\{a,b\}+a\delta_{|a|}(b)-b\delta_{|b|}(a)\\&=\pi(a,b)+|a|a\delta(b)-|b|b\delta(a)=\pi(a,b)+E(a)\delta(b)-\delta(a)E(b).\end{aligned}$$

As a result, we have that

$$\pi'=\pi+E\wedge\delta,$$

which is equivalent to the above assertion. Finally by [20, Proposition 3.11(2)] we have

$$\begin{aligned}\mathcal{L}_\delta(\alpha)=&\mathcal{L}_\delta\left(\iota_{\pi'}(\nu)\right)=\iota_{\pi'}\left(\mathcal{L}_\delta(\nu)\right)+\iota_{[\delta,\pi']_S}(\nu)\\=&\iota_{\pi'}\left(\operatorname{div}(\delta)\nu\right)+\iota_{[\delta,\pi']_S}(\nu)=\iota_{[\delta,\pi']_S}(\nu).\end{aligned}$$

One can easily check that δ is also a Poisson derivation of (A^δ,π'). So $[\delta,\pi']_S=-d_{\pi'}(\delta)=0$ and we get $\mathcal{L}_\delta(\alpha)=0$. □

Proof of Corollary 0.3. Let A be a $\mathbb{Z}$-graded Poisson algebra $\Bbbk[x_1,\cdots,x_n]$ with each x_i homogeneous. By Lemma 1.2(5), $\mathfrak{l}=\sum_{i=1}^n\deg x_i$. Now the assertions follow from Theorem 3.8. □

4. Rigidity of graded twisting

Let A be a $\mathbb{Z}$-graded Poisson algebra. Recall that the set of graded semi-Poisson derivations (resp. graded Poisson derivations) of A with degree 0 is denoted by $Gspd(A)$ (resp. $Gpd(A)$). We first prove Theorem 0.5.

Lemma 4.1. *Let A be a $\mathbb{Z}$-graded Poisson algebra $\Bbbk[x_1,\ldots,x_n]$ with $\deg x_i>0$ for every i. Suppose that A is unimodular and that $\mathfrak{l}:=\sum_{i=1}^n\deg x_i$ is a nonzero element in $\Bbbk$. If δ is a graded semi-Poisson derivation of A, then δ is a Poisson derivation of A. Namely, $Gspd(A)=Gpd(A)$.*

Proof. Since $\deg x_i>0$ for all i, by Lemma 1.2(1,3), both $\operatorname{div}(\delta)$ and $\mathfrak{l}=\operatorname{div}(E)$ are in $\Bbbk$. Let B be the twist A^δ with modular derivation $\mathbf{n}$. By Theorem 0.2,

$$\mathbf{n}=\mathbf{m}+\mathfrak{l}\delta-\operatorname{div}(\delta)E=\mathfrak{l}\delta-\operatorname{div}(\delta)E.$$

Since $\mathbf{n}$ and E (and $\operatorname{div}(\delta)E$) are Poisson derivations of B, we have that $\mathfrak{l}\delta$ (and hence δ) is a Poisson derivation of B. Let $\langle -,-\rangle'$ (resp. $\{-,-\}$) be the Poisson structure of B (resp. A). By (E2.3.1), we have

$$\{-,-\}=\langle -,-\rangle-E\wedge\delta.$$

Then, for all homogeneous elements $a,b\in A$,

$$\begin{aligned}
\delta(\{a,b\}) &- \{\delta(a),b\} - \{a,\delta(b)\} \\
&= \delta(\langle a,b\rangle) - \langle \delta(a),b\rangle - \langle a,\delta(b)\rangle \\
&\quad - \delta((E\wedge\delta)[a,b]) + E\wedge\delta[\delta(a),b] + E\wedge\delta[a,\delta(b)] \\
&= 0 - \delta(|a|a\delta(b) - |b|b\delta(a)) \\
&\quad + \left(|a|\delta(a)\delta(b) - |b|b\delta^2(a)\right) + \left(|a|a\delta^2(b) - |b|\delta(b)\delta(a)\right) \\
&= -|a|\delta(a)\delta(b) - |a|a\delta^2(b) + |b|\delta(b)\delta(a) + |b|b\delta^2(a) \\
&\quad + \left(|a|\delta(a)\delta(b) - |b|b\delta^2(a)\right) + \left(|a|a\delta^2(b) - |b|\delta(b)\delta(a)\right) \\
&= 0.
\end{aligned}$$

Therefore δ is a Poisson derivation of A. □

Lemma 4.2. *Let B be a twist of A. Then $Gspd(A) = Gspd(B)$.*

Proof. Write $B = A^{\delta}$ for some graded semi-Poisson derivation δ of A. So $B = A$ as a commutative algebra. Let π (resp. π') be the Poisson bracket of A (resp. B).

Let ϕ be a graded derivation of A with degree zero. Then we have $[E,\phi]_S = [E,\phi] = 0$. For any two graded derivations ϕ_1, ϕ_2 of A with degree zero, we have

$$\begin{aligned}
[E\wedge\phi_1, E\wedge\phi_2]_S &= \pm[E, E\wedge\phi_2]_S\wedge\phi_1 \pm E\wedge[\phi_1, E\wedge\phi_2]_S \\
&= \pm([E,E]_S\wedge\phi_2 \pm [E,\phi_2]_S\wedge E)\wedge\phi_1 \\
&\qquad \pm E\wedge([\phi_1,E]_S\wedge\phi_2 \pm [\phi_1,\phi_2]_S\wedge E) \\
&= 0.
\end{aligned}$$

Let ϕ be a graded semi-Poisson derivation of A with degree zero. By definition,

$$[E\wedge\phi, \pi]_S = 0.$$

Then

$$\begin{aligned}
[E\wedge\phi, \pi']_S &= [E\wedge\phi, \pi + E\wedge\delta]_S \\
&= [E\wedge\phi, \pi]_S + [E\wedge\phi, E\wedge\delta]_S \\
&= 0.
\end{aligned}$$

Therefore, ϕ is a graded semi-Poisson derivation of B with degree zero. □

Proof of Theorem 0.5. Part (1) is Lemma 4.1 and part (2) is Lemma 4.2.

(3) By Corollary 0.3 and part (2), we may assume that A is unimodular. By part (1), $Gspd(A)$ is the $\Bbbk$-vector space of graded Poisson derivations A. It is well-known that it is a Lie algebra. Let ϕ be any Poisson derivation of A. It is clear that ϕ is determined by $\{\phi(x_i)\}_{i=1}^n$. Therefore $Gspd(A)$ is finite-dimensional. □

One of the main definitions in this paper is the following.

Definition 4.3. Let A be a $\mathbb{Z}$-graded Poisson algebra.

(1) The *rigidity of graded twisting* (or simply *rigidity*) of A is defined to be

$$rgt(A) = 1 - \dim_{\Bbbk} Gspd(A).$$

(2) We say A is *rigid* if $rgt(A) = 0$.
(3) We say A is *(−1)-rigid* if $rgt(A) = -1$.

Note that this notion of rigidity is different from the rigidity defined in [9, Definition 0.1] and other papers.

It follows from Lemma 4.2 that

$$rgt(A) = rgt(A^{\delta}) \tag{E4.3.1}$$

for every graded semi-Poisson derivation δ of A.

Other basic facts about $rgt(A)$ are listed in the following lemma.

Lemma 4.4. *Let A be a $\mathbb{Z}$-graded Poisson algebra with $A_i \neq 0$ for some $i \neq 0$. In parts (2)-(6), we further assume that A is $\Bbbk[x_1, \dots, x_n]$ with $\deg x_i > 0$ for all i.*

(1) *Suppose that $rgt(A) = 0$. Then every graded twist of A is isomorphic to A.*
(2) *If A is not unimodular, then $rgt(A) \leq -1$.*
(3) *If $rgt(A) = 0$, then A is unimodular.*
(4) *If $rgt(A) \neq 0$, then $\dim_\Bbbk Gspd(A) \geq \dim_\Bbbk Gpd(A) \geq 2$.*
(5) *If $rgt(A) = -1$, then $\dim_\Bbbk Gspd(A) = \dim_\Bbbk Gpd(A) = 2$.*
(6) *Let $\{A(a)\}_{a \in \Bbbk}$ be a family of Poisson polynomial algebras such that (i) $A(a)$ is a Poisson twist of $A(a')$ for all $a, a' \in \Bbbk$ and that (ii) there is an a_0 such that $A(a_0)$ is unimodular. If $\dim_\Bbbk Gpd(A(a_0)) = 2$, then $rgt(A(a)) = -1$ for all a.*
(7) *If $rgt(A) = 0$ and A is a connected graded domain, then every Poisson normal element of A is Poisson central.*
(8) *If $\dim_\Bbbk Gpd(A) = 1$ and $\operatorname{div}(E) \neq 0$, then $rgt(A) = 0$.*

Proof. (1) Since $A_i \neq 0$ for some i, the Euler derivation E is not zero. Since $rgt(A) = 0$, $Gspd(A) = \Bbbk E$. Let B be a graded twist of A. Then $B = A^\delta$ where $\delta \in Gspd(A)$. Let $\langle -, - \rangle$ be the Poisson bracket of B. Since $\delta = \alpha E$ for some $\alpha \in \Bbbk$, one sees from (E2.3.1) that $\langle a, b \rangle = \{a, b\}$ where $\{a, b\}$ is the original Poisson bracket of A. The assertion follows.

(2) Since A is not unimodular, the modular derivation $\mathbf{m}$ is not in $\Bbbk E$, as $\operatorname{div}(E) = \sum_{i=1}^n \deg x_i \neq 0$ and $\operatorname{div}(\mathbf{m}) = 0$ [Lemma 0.1]. Therefore $\dim_\Bbbk Gspd(A) \geq 2$. The assertion follows.

(3) This is equivalent to (2).

(4) By definition, it is clear that $\dim_\Bbbk Gspd(A) \geq \dim_\Bbbk Gpd(A)$. It remains to show $\dim_\Bbbk Gpd(A) \geq 2$. If A is unimodular, then, by Lemma 4.1,

$$\dim_\Bbbk Gpd(A) = \dim_\Bbbk Gspd(A) = 1 - rgt(A) \geq 2.$$

Now we assume that A is not unimodular with nonzero modular derivation $\mathbf{m}$. Since $\operatorname{div}(E) \neq 0$ and $\operatorname{div}(\mathbf{m}) = 0$, the $\Bbbk$-dimension of $Gpd(A)$ is at least 2 as desired.

(5) By definition, $\dim_\Bbbk Gspd(A) = 1 - rgt(A) = 2$. The assertion follows from part (4).

(6) It follows from Lemmas 4.1 and 4.2 that we have

$$rgt(A(a)) = rgt(A(a_0)) = 1 - \dim_\Bbbk Gspd(A(a_0)) = 1 - \dim_\Bbbk Gpd(A(a_0)) = -1.$$

The assertion follows from (4).

(7) We only need to consider a homogeneous Poisson normal element f of positive degree. Note the log-Hamiltonian derivation $LH_f := f^{-1}\{f, -\}$ is a Poisson derivation of degree 0. Suppose f is not central. Then $LH_f(f) \neq 0$. Note that LH_f is clearly not the Euler derivation. Therefore $rgt(A) \leq -1$, yielding a contradiction. The assertion follows.

(8) We know $Gpd(A)$ is spanned by the Euler derivation E, which is not the modular derivation by Lemma 0.1. This implies that A is unimodular and the result follows by Lemma 4.2. □

Examples of $rgt(A)$ will be given in the next 2 sections.

5. Examples and comments

Note that the graded twists in the associative algebra setting has an important property, namely, a graded algebra R and its twist have isomorphic corresponding graded module categories [39]. The following example shows that a similar result does not hold in the Poisson setting.

Example 5.1. Let A be the Poisson algebra $\Bbbk[x_1, x_2]$ with trivial Poisson structure. Let δ be the Poisson derivation of A determined by

$$\delta(x_1) = 0 \quad \text{and} \quad \delta(x_2) = x_2.$$

Let B be the graded twist of A by δ, namely, $B = A^\delta$. By definition, B is a Poisson algebra $\Bbbk[x_1, x_2]$ with Poisson bracket determined by

$$\{x_1, x_2\} = x_1 x_2.$$

Let $U(A)$ denote the Poisson enveloping algebra of A [3]. Since A has the trivial Poisson structure, $U(A)$ is the commutative polynomial ring $\Bbbk[x_1, x_2, y_1, y_2]$. Let $U(B)$ be the Poisson enveloping algebra of B. We claim that $U(B)$ is not a graded twist of $U(A)$ in the sense of [39].

Suppose on the contrary that $U(B)$ is a graded twist of $U(A)$ in the sense of [39]. Then, by [39, Theorem 1.1],

$$\text{GrMod-}U(A) \cong \text{GrMod-}U(B). \tag{E5.1.1}$$

Let $D(A)$ (resp. $D(B)$) be the degree zero part of the graded quotient ring of $U(A)$ (resp., $U(B)$). Then it follows from (E5.1.1) that $D(A) \cong D(B)$. Since $U(A)$ is commutative, $D(A)$ is commutative. Thus $D(B)$ is commutative. Next we prove that $D(B)$ is not commutative, so we obtain a contradiction. By [3, Theorem 2.2], $U(B)$ is generated by four elements $x_1, x_2, \delta_1, \delta_2$ and subject to 6 relations

$$x_1x_2 = x_2x_1,$$
$$\delta_1x_1 = x_1\delta_1,$$
$$\delta_1x_2 = x_2\delta_1 + x_1x_2,$$
$$\delta_2x_1 = x_1\delta_2 - x_1x_2,$$
$$\delta_2x_2 = x_2\delta_2,$$
$$\delta_2\delta_1 = \delta_1\delta_2 + x_2\delta_1 + x_1\delta_2.$$

Let $a = x_1x_2^{-1}$ and $b = \delta_2x_2^{-1}$ which are elements in $D(B)$. It follows from the six relations that

$$ba = ab - a.$$

So $D(B)$ is not commutative, yielding a contradiction. Therefore $U(B)$ is not a graded twist of $U(A)$.

As a consequence, the category of graded Poisson modules over A^δ, denoted by GrPMod-A^δ is not equivalent to the category of graded Poisson modules over A, denoted by GrPMod-A. That is,

$$\text{GrPMod-}A^\delta \ncong \text{GrPMod-}A.$$

Remark 5.2. When A is a connected graded Poisson algebra with $A_i \neq 0$ for some $i > 0$, $PH^1(A)$ is also graded. Since $(PH^1(A))_0 \cong Gpd(A)$, we have $rgt(A) \leq 1 - \dim_{\Bbbk}(PH^1(A))_0$. If $\dim_{\Bbbk}(PH^1(A))_0 = 1$, then $rgt(A) = 0$ by Lemma 4.4(8). Therefore we can obtain information about $rgt(A)$ from $PH^1(A)$.

Remark 5.3. Let $A = \Bbbk[x_0, x_1, x_2, x_3]$ be the polynomial algebra with the Poisson bracket defined by

$$\{x_i, x_{i+1}\} = x_{i+2}x_{i+3} - \lambda^2 x_i x_{i+1},$$
$$\{x_i, x_{i+1}\} = \lambda(x_{i+1}^2 - x_{i+3}^2),$$

for some $\lambda \in \Bbbk$ with indices $i = 0, 1, 2, 3$ (modulo 4). Then A can be considered the semiclassical limit of the 4-dimensional Sklyanin algebra. The Poisson (co)homologies of A have been computed in [27, p.1154]. By the Poincaré duality, both $PH^0(A)$ and $PH^1(A)$ have Hilbert series $\frac{1}{(1-t^2)^2}$. By Remark 5.2, $rgt(A) = 0$. Further, since $h_{PH^1(A)}(t) = h_{PH^0(A)}(t) = h_Z(t) = \frac{1}{(1-t^2)^2}$, where Z denotes the Poisson center of A, A is PH^1-minimal in the sense of Definition 7.3(1).

Note that the Poisson (co)homologies of the quadratic Poisson algebra $A = \Bbbk[x, y, z]$ of Sklyanin type were computed in [28,29,35]. An argument similar to the above shows that $rgt(A) = 0$ and A is PH^1-minimal. We will give an elementary and direct computation of this $rgt(A)$ in Example 6.6(Case 3).

Remark 5.4. For $n \geq 2$ and $a \in \mathbb{C}$. Let $A(n, a) = \mathbb{C}[x_1, \dots, x_n]$ be the family of Poisson polynomial algebras studied in [21]. The Poisson bracket on $A(n, a)$ is defined as follows:

$$\{x_i, x_j\} = z_j x_{i-1} x_j - z_i x_{j-1} x_i$$

where $z_j = a + j$ and $x_{-1} = 0$. One can check for each fixed $n \geq 2$ the family $\{A(n, a) | a \in \mathbb{C}\}$ satisfies all the assumptions stated in Lemma 4.4(6) with $a_0 = \frac{(n+2)(1-n)}{2(n+1)}$ such that $rgt(A(n, a)) = -1$. Computations are omitted. In particular for any $a, a' \in \mathbb{C}$, $A(n, a)$ is a Poisson twist of $A(n, a')$, which is a Poisson version of [21, Theorem 4.2].

We will compute rgt for some classes of Poisson algebras. Here is a warm-up.

Example 5.5. Let $A = \Bbbk[x, y, z]$ with $\deg(x) = 1, \deg(y) = 2$ and $\deg(z) = 3$. Let $\Omega = x^6 + y^3 + z^2 + \lambda xyz$ where $\lambda \in \Bbbk$. Define a Poisson structure on $A := \Bbbk[x, y, z]$ by

$$\{f, g\} := \det \begin{pmatrix} \Omega_x & f_x & g_x \\ \Omega_y & f_y & g_y \\ \Omega_z & f_z & g_z \end{pmatrix}$$

for all $f, g \in A$. It is easy to see that

$$\{x, y\} = \Omega_z = 2z + \lambda xy, \tag{E5.5.1}$$
$$\{x, z\} = -\Omega_y = -(3y^2 + \lambda xz), \tag{E5.5.2}$$
$$\{y, z\} = \Omega_x = 6x^5 + \lambda yz \tag{E5.5.3}$$

and that A is unimodular. If $\lambda^6 \neq 6^3$, then $A_{sing} := A/(\Omega_x, \Omega_y, \Omega_z)$ is finite dimensional. In this case, Ω has an isolated singularity. Let δ be a graded Poisson derivation of degree zero. Then

$$\delta(x) = c_1 x,$$
$$\delta(y) = c_2 y + c_3 x^2,$$
$$\delta(z) = c_4 z + c_5 x^3 + c_6 xy.$$

Subtracting by $c_1 E$, we may assume that $c_1 = 0$. Applying δ (with $c_1 = 0$) to (E5.5.1), we obtain that

$$c_2(2z + \lambda xy) = 2(c_4 z + c_5 x^3 + c_6 xy) + \lambda x(c_2 y + c_3 x^2),$$

which implies that $c_2 = c_4$, $c_6 = 0$, and $2c_5 + \lambda c_3 = 0$. Applying δ (with $c_1 = 0$ and $c_6 = 0$) to (E5.5.2), we obtain that

$$-c_4(3y^2 + \lambda xz) = -6y(c_2 y + c_3 x^2) - \lambda x(c_4 z + c_5 x^3),$$

which implies that $c_2 = c_3 = c_4 = c_5 = 0$. Therefore $\delta = 0$. This means that $rgt(A) = 0$. By Lemma 4.4(1), A has no non-trivial twists.

In general, when Ω has an isolated singularity, the fact that $rgt(A) = 0$ also follows from the Poisson cohomology computation given in [29, Proposition 4.5] (after matching up the notations). The same idea applies to the algebra in Example 6.6(Case 3).

6. Some computations of *rgt*

In this section, we compute *rgt* for all quadratic Poisson structures on $A = \Bbbk[x, y, z]$ with $\deg(x) = \deg(y) = \deg(z) = 1$. Some of the computations have been done by other researchers in different language (for example, some are hidden inside in Poisson cohomology computation), but we provide all details of computations of *rgt* for completeness. The classification of all quadratic Poisson structures on $\Bbbk[x, y, z]$ were given in [8,7,22].

First we fix some notations. Let $\Bbbk$ be an algebraically closed field of characteristic zero (one might assume $\Bbbk = \mathbb{C}$ if necessary). Let $V = A_1 = \Bbbk x + \Bbbk y + \Bbbk z$ and let $\{-, -\}$ be a quadratic Poisson bracket of $A := \Bbbk[x, y, z] = \Bbbk[V]$. Let f be a graded Poisson derivation of $(A, \{-, -\})$. Let $W = \{V, V\}$. It is clear that

$$f(W) = f(\{V, V\}) \subseteq \{f(V), V\} + \{V, f(V)\} \subseteq \{V, V\} = W. \tag{E6.0.1}$$

Write

$$f(x) = a_1 x + a_2 y + a_3 z,\ f(y) = b_1 x + b_2 y + b_3 z,\ f(z) = c_1 x + c_2 y + c_3 z. \tag{E6.0.2}$$

After replacing f by $f - a_1 E$, we can further assume that

$$a_1 \text{ in (E6.0.2) is zero.} \tag{E6.0.3}$$

Note that, for any given polynomial $\Omega \in A$, one can define a Poisson bracket on A as follows:

$$\{x, y\} = \frac{\partial \Omega}{\partial z}, \quad \{y, z\} = \frac{\partial \Omega}{\partial x}, \quad \{z, x\} = \frac{\partial \Omega}{\partial y}.$$

Such a bracket is called a *Jacobian Poisson bracket* and Ω is called a *potential*. If $\{-, -\}$ is unimodular, it comes from a potential $\Omega \in A_3$. One can classify cubic Ω as follows: (a): Ω is a product of three linear terms, (b): Ω is a product of a linear term and an irreducible polynomial of degree 2, and (c): Ω is irreducible of degree 3. This classification is well-known (e.g., [5,19,34]); we list them below for the reader's convenience (Table 1).

Table 1
Classification of cubic potential Ω in $\Bbbk[x, y, z]$.

	Reducible Ω	Irreducible Ω
(a)	x^3, x^2y, xyz $xy(x+y)$	
(b)	$xyz + x^3$, $xy^2 + x^2z$	
(c)		$x^3 + y^2z$, $x^3 + x^2z + y^2z$ $\frac{1}{3}(x^3 + y^3 + z^3) + \lambda xyz$, $\lambda^3 \neq -1$

The following four examples deal with the first case, namely, Ω is a product of three linear terms. Define A_{sing} to be $A/(\Omega_x, \Omega_y, \Omega_z)$.

Example 6.1. Let $\Omega = x^3$. Then

$$\begin{aligned} \{x, y\} &= \Omega_z = 0, \\ \{z, x\} &= \Omega_y = 0, \\ \{y, z\} &= \Omega_x = 3x^2. \end{aligned}$$

It is clear that $\text{Kdim}(A_{sing}) = 2$. Let f be a graded Poisson derivation of A. By (E6.0.1)

$$2xf(x) = f(x^2) \in f(W) \subseteq W = \Bbbk x^2.$$

Then $f(x) = ax$ for some $a \in \Bbbk$. By (E6.0.3), we may assume that $f(x) = 0$. Retain the notations in (E6.0.2). Applying f to $\{y, z\} = 3x^2$ implies that $b_2 + c_3 = 0$ with b_1, b_3, c_1, c_2 free. Therefore $rgt(A) = -5$. One can check that every Poisson normal element of A is Poisson central.

Example 6.2. Let $\Omega = x^2 y$. Then

$$\begin{aligned} \{x, y\} &= \Omega_z = 0, \\ \{z, x\} &= \Omega_y = x^2, \\ \{y, z\} &= \Omega_x = 2xy. \end{aligned} \tag{E6.2.1}$$

It is clear that $\text{Kdim}(A_{sing}) = 2$. Let f be a graded Poisson derivation of A. By (E6.0.1), we have

$$\Bbbk x^2 + \Bbbk xy = W \supseteq f(W) = \Bbbk(2xf(x)) + \Bbbk(f(x)y + xf(y)).$$

Then $f(x)y$ does not have terms y^2 and yz. So $f(x) = ax$ for some $a \in \Bbbk$, and by (E6.0.3), we may assume that $f(x) = 0$. Using the notations in (E6.0.2), then (E6.2.1) implies that $b_1 = b_3 = c_3 = 0$ with b_2, c_1, c_2 free. Therefore $rgt(A) = -3$.

Example 6.3. Let $\Omega = xyz$. Then

$$\begin{aligned} \{x, y\} &= \Omega_z = xy, \\ \{z, x\} &= \Omega_y = xz, \\ \{y, z\} &= \Omega_x = yz. \end{aligned} \tag{E6.3.1}$$

As before we assume that $a_1 = 0$. Note that $W := \Bbbk xy + \Bbbk yz + \Bbbk xz$ which does not contain term x^2 and y^2. By (E6.0.1), we have

$$(a_1 x + a_2 y + a_3 z)y + (b_1 x + b_2 y + b_3 z)x = f(x)y + xf(y) = f(xy) \in W$$

which implies that $a_2 = b_1 = 0$. Similarly, using $f(xz), f(yz) \in W$, we obtain that $a_3 = c_1 = b_3 = c_2 = 0$. Thus $f(x) = 0$, $f(y) = b_2 y$ and $f(z) = c_3 z$. Therefore $rgt(A) = -2$.

One can check that $\text{Kdim}(A_{sing}) = 1$.

Example 6.4. Let $\Omega = xy(x + y)$. Then

$$\begin{aligned} \{x, y\} &= \Omega_z = 0, \\ \{z, x\} &= \Omega_y = x^2 + 2xy, \\ \{y, z\} &= \Omega_x = 2xy + y^2. \end{aligned} \tag{E6.4.1}$$

Again we may assume that $a_1 = 0$. By (E6.0.1), we have

$$\begin{aligned} f(x^2 + 2xy) &= 2xf(x) + 2xf(y) + 2yf(x) \in \Bbbk(x^2 + 2xy) + \Bbbk(2xy + y^2) =: W, \\ f(2xy + y^2) &= 2xf(y) + 2yf(x) + 2yf(y) \in \Bbbk(x^2 + 2xy) + \Bbbk(2xy + y^2). \end{aligned}$$

As a consequence, both $f(x)$ and $f(y)$ do not have a z term, namely, $a_3 = b_3 = 0$. Furthermore, by the above and a little bit of linear algebra, we have

$$b_1 = b_2 = -a_2.$$

Now we can write $f(x) = a_2 y$ and $f(y) = -a_2 x - a_2 y$. Applying f to the second equation of (E6.4.1), we obtain that $c_3 = 0$ and $a_2 = 0$ with c_1 and c_2 free. Therefore $rgt(A) = -2$.

One can check that $\text{Kdim}(A_{sing}) = 1$.

Next we consider the second case. Some linear algebra details will be omitted in the next two examples.

Example 6.5. Case 1: $\Omega = xyz + x^3$. In this case, the Poisson bracket of A is determined by

$$\begin{aligned} \{x, y\} &= \Omega_z = xy, \\ \{z, x\} &= \Omega_y = xz, \\ \{y, z\} &= \Omega_x = yz + 3x^2. \end{aligned} \tag{E6.5.1}$$

One can check that $\mathrm{Kdim}(A_{sing}) = 1$. Recall that W is $\{V, V\} = \Bbbk xy + \Bbbk xz + \Bbbk(yz + 3x^2)$ which does not involve either y^2 or z^2. By the second equation of (E6.5.1), we have

$$f(x)z + xf(z) \in W$$

which implies that $f(x)$ does not have the z term, or $a_3 = 0$. Similarly, $b_3 = 0$ by the third equation of (E6.5.1). By using the first equation of (E6.5.1), we obtain that $a_2 = 0$. By (E6.0.3), we can assume that $f(x) = 0$. Now the first equation of (E6.5.1) implies that $\{x, f(y)\} = xf(y)$. So $f(y) \in \Bbbk y$ or $f(y) = b_2 y$.

Using the second equation of (E6.5.1), one can show that $c_1 = c_2 = 0$. By using the third equation of (E6.5.1) and the fact that W does not contain the term y^2, we obtain that $c_2 = 0$. So $f(z) = c_3 z$. From this we can derive that $b_2 + c_3 = 0$. Therefore $rgt(A) = -1$.

Case 2: $\Omega = xy^2 + x^2 z$. Then we have

$$\begin{aligned} \{x, y\} &= \Omega_z = x^2, \\ \{z, x\} &= \Omega_y = 2xy, \\ \{y, z\} &= \Omega_x = y^2 + 2xz. \end{aligned} \tag{E6.5.2}$$

One can check that $\mathrm{Kdim}(A_{sing}) = 1$. By definition $W = \Bbbk x^2 + \Bbbk xy + \Bbbk(y^2 + 2xz)$, and it does not contain terms z^2 and yz. Using the third equation of (E6.5.2), we have

$$f(y^2 + 2xz) = 2yf(y) + 2xf(z) + 2zf(x) \in W.$$

Therefore $f(x)z$ does not contain a z^2 term. So $f(x) = a_1 x + a_2 y$ and with (E6.0.3) we can assume that $f(x) = a_2 y$. Now we apply f to the first equation of (E6.5.2), we obtain that $b_2 = 0$ and $b_3 = -a_2$. (Some calculations are omitted.) Applying f to the second equation of (E6.5.2), we obtain that $a_2 = 0$, $c_3 = 0$ and $c_2 = -2b_1$. Finally applying f to the third equation of (E6.5.2), we obtain that $c_1 = 0$ with b_1 free. Therefore $rgt(A) = -1$.

The final example deals with the irreducible cubic Ω.

Example 6.6. Suppose Ω is an irreducible cubic function in x, y, z. By classification (see, for example, [19, Theorems 1 and 2] and [5, Theorem 2.12]), there are following two singular ones and one non-singular.

Case 1: $\Omega = x^3 + y^2 z$. Then we have

$$\begin{aligned} \{x, y\} &= \Omega_z = y^2, \\ \{z, x\} &= \Omega_y = 2yz, \\ \{y, z\} &= \Omega_x = 3x^2. \end{aligned} \tag{E6.6.1}$$

So $W := \Bbbk y^2 + \Bbbk yz + \Bbbk x^2$ does not have terms z^2, xy and xz. Then $f(x^2) = 2xf(x) \in W$ implies that $f(x) \in \Bbbk x$. By (E6.0.3), we have $f(x) = 0$. Applying f to the first equation of (E6.6.1), we obtain that $f(y) = 0$. Applying f to the last two equations of (E6.6.1), we obtain that $f(z) = 0$. So $rgt(A) = 0$.

One can check that $\mathrm{Kdim}(A_{sing}) = 1$. By a Gröbner Basis argument, one sees that the Hilbert series of A_{sing} is $\frac{2}{(1-t)} + t^2 + t - 1$.

Case 2: $\Omega = x^3 + x^2 z + y^2 z$. Then we have

$$\begin{aligned} \{x, y\} &= \Omega_z = x^2 + y^2, \\ \{z, x\} &= \Omega_y = 2yz, \\ \{y, z\} &= \Omega_x = 3x^2 + 2xz. \end{aligned} \tag{E6.6.2}$$

So W does not have terms z^2 and xy. By the second equation of (E6.6.2), we have

$$f(yz) = yf(z) + zf(y) \in W$$

which implies that $f(y)$ has no z term and $f(z)$ has no x term. By the third relation of (E6.6.2), we obtain that $f(x)$ has no z term. By (E6.0.3), one can assume that $f(x)=a_2y$. Applying f to the first equation, we obtain that $b_2=0$ and $b_1=-a_2$ (namely, $f(x)=a_2y$ and $f(y)=-a_2x$). Applying f to the second equation, we obtain that $c_2=a_2=0$ (so $f(x)=f(y)=0$). Then applying f to the third equation of (E6.6.2) yields that $f(z)=0$. Therefore $rgt(A)=0$.

One can check that $\mathrm{Kdim}(A_{sing})=1$. By a Gröbner Basis argument, one sees that the Hilbert series of A_{sing} is $\frac{2}{(1-t)}+t^2+t-1$.

Case 3: $\Omega=\frac{1}{3}(x^3+y^3+z^3)+\lambda xyz$ where $\lambda^3\neq -1$ (which is the Hesse normal form given in [5, Theorem, 2.12]). One can check that A_{sing} is finite dimensional or $\mathrm{Kdim}(A_{sing})=0$. Consequently, Ω has an isolated singularity at zero. As mentioned at the end of Example 5.5, we have $rgt(A)=0$ which follows from the Poisson cohomology computation given in [29, Proposition 4.5] (and [35, Theorem 5.1]). Here we will give a direct computation. By definition,

$$\begin{aligned} \{x,y\}&=\Omega_z=z^2+\lambda xy,\\ \{z,x\}&=\Omega_y=y^2+\lambda xz,\\ \{y,z\}&=\Omega_x=x^2+\lambda yz. \end{aligned} \tag{E6.6.3}$$

Note that $W=\Bbbk(z^2+\lambda xy)+\Bbbk(y^2+\lambda xz)+\Bbbk(x^2+\lambda yz)$. This means that in W, z^2 (respectively, y^2 and x^2) appears together with λxy (respectively, xz and yz). By the first equation of (E6.6.3), we have $f(z^2+\lambda xy)\in W$. Using the notation in (E6.0.2), we compute

$$\begin{aligned} f(z^2+\lambda xy)&=2zf(z)+\lambda\,(xf(y)+yf(x))\\ &=2z(c_1x+c_2y+c_3z)+\lambda[x(b_1x+b_2y+b_3z)+y(a_1x+a_2y+a_3z)]\\ &\equiv 2c_1xz+2c_2yz+2c_3(-\lambda xy)+\lambda[b_1(-\lambda yz)+b_2xy+b_3xz\\ &\qquad +a_1xy+a_2(-\lambda xz)+a_3yz] \mod W\\ &\equiv (2c_1+\lambda b_3-\lambda^2a_2)xz+(2c_2-\lambda^2b_1+\lambda a_3)yz\\ &\qquad +(-2\lambda c_3+\lambda b_2+\lambda a_1)xy \mod W. \end{aligned}$$

So we have

$$\begin{aligned} 2c_1+\lambda b_3-\lambda^2a_2&=0,\\ 2c_2+\lambda a_3-\lambda^2b_1&=0,\\ -2\lambda c_3+\lambda b_2+\lambda a_1&=0. \end{aligned}$$

From now on we assume that $\lambda\neq 0$ (if $\lambda=0$, the proof is slightly simpler and is omitted to save the space). With this assumption, we can remove λ from the third equation of the above system. Similarly, by working with the last two equations in (E6.6.3), we obtain the following

$$\begin{aligned} 2b_1+\lambda c_2-\lambda^2a_3&=0,\\ 2b_3+\lambda a_2-\lambda^2c_1&=0,\\ -2b_2+c_3+a_1&=0,\\ 2a_2+\lambda c_1-\lambda^2b_3&=0,\\ 2a_3+\lambda b_1-\lambda^2c_2&=0,\\ -2a_1+b_2+c_3&=0. \end{aligned}$$

By (E6.0.3), we may assume that $a_1=0$. Then by the three equations involving a_1, we obtain that $b_2=c_3=0$. Applying f to three equations in (E6.6.3) with some linear algebra computations, we obtain that $f(x)=f(y)=f(z)=0$ (but this is true only if $\lambda^3\neq -1$). Therefore $rgt(A)=0$.

Since $\{\Omega_x,\Omega_y,\Omega_z\}$ is a regular sequence, the Hilbert series of A_{sing} is $(1+t)^3$.

Based on the above examples, we have the following classification.

Corollary 6.7. *Let A be a quadratic Poisson polynomial ring $\Bbbk[x,y,z]$.*

(1) *Suppose A is unimodular. Then $(A,\Omega,rgt(A))$ is listed as follows up to isomorphisms.*

Ω	0	x^3	x^2y	xyz	$xy(x+y)$	$xyz+x^3$	xy^2+x^2z
$rgt(A)$	-8	-5	-3	-2	-2	-1	-1
		Ex. 6.1	*Ex. 6.2*	*Ex. 6.3*	*Ex. 6.4*	*Ex. 6.5(1)*	*Ex. 6.5(2)*

Ω	x^3+y^2z	$x^3+x^2z+y^2z$	$\frac{1}{3}(x^3+y^3+z^3)+\lambda xyz, \lambda^3\neq -1$
$rgt(A)$	0	0	0
	Ex. 6.6(1)	*Ex. 6.6(2)*	*Ex. 6.6(3)*

(2) *$Gpd(A)$ is one-dimensional if and only if A is unimodular with Ω in the second table.*

Definition 6.8. A Poisson derivation ϕ of a Poisson algebra A is called *ozone* if $\phi(z)=0$ for all z in the Poisson center of A.

By Definition 1.3(1), the modular derivation $\mathbf{m}$ is always ozone.

Lemma 6.9. *Let A be the quadratic Poisson algebra in Example 6.6(1) with $\Omega=x^3+y^2z$. Then every ozone derivation of A is Hamiltonian.*

Proof. By definition,

$$\{x,y\}=y^2, \tag{E6.9.1}$$
$$\{z,x\}=2yz, \tag{E6.9.2}$$
$$\{y,z\}=3x^2. \tag{E6.9.3}$$

We define a new grading on the polynomial ring $A=\Bbbk[x,y,z]$. Let G be $\mathbb{Z}$ and define $\deg_G x=0$, $\deg_G y=1$ and $\deg_G z=-2$. For example, $\deg_G \Omega=0$. Every element $f\in A$ can be written as $\sum_{i\in\mathbb{Z}} f_{(i)}$ where $f_{(i)}$ is homogeneous of G-degree i. Then $f=f_{(i)}$ if and only if f is homogeneous of G-degree i. By (E6.9.1) and (E6.9.2) the Hamiltonian derivation H_x has G-degree 1.

Claim 1: If f is homogeneous of G-degree i, then $H_x(f)=ify$.

Proof: Let f be a linear combination of monomials $x^ay^bz^c$. Since $\deg_G f=i$, we have $b-2c=i$. Then

$$\begin{aligned} H_x(x^ay^bz^c) &= bx^ay^{b-1}y^2z^c+cx^ay^bz^{c-1}(-2yz)\\ &=(b-2c)(x^ay^bz^c)y=i(x^ay^bz^c)y. \end{aligned}$$

So the claim follows.

Let ϕ denote an ozone Poisson derivation of A.

Claim 2: Up to a Hamiltonian derivation, $\phi(x)=yw_{(0)}$ where $\deg_G w_{(0)}=0$.

Proof: Since Ω is Poisson central, $\phi(\Omega)=0$. Then

$$0=\phi(\Omega)=3x^2\phi(x)+2yz\phi(y)+y^2\phi(z). \tag{E6.9.4}$$

This implies that $y\mid\phi(x)$. Let $\phi(x)=yw$ where $w=\sum_{i\in\mathbb{Z}} w_{(i)}$ where $\deg_G w_{(i)}=i$. By **Claim 1**, $H_{\sum_{i\neq 0}\frac{-1}{i}w_{(i)}}(x)=H_x(\sum_{i\neq 0}\frac{1}{i}w_{(i)})=\sum_{i\neq 0} w_{(i)}y$. After replacing ϕ by $\phi-H_{\sum_{i\neq 0}\frac{-1}{i}w_{(i)}}$, we obtain that $\phi(x)=yw_{(0)}$ as required.

Claim 3: If $\deg_G w=0$, then w is a polynomial in x and Ω.

Proof: Since w has G-degree 0, $w=\sum_{i,j\geq 0}\alpha_{i,j}x^iy^{2j}z^j$. The assertion follows after replacing y^2z by $\Omega-x^3$.

From now on, we assume that $\phi(x)=yw_{(0)}$.

Claim 4: $\deg_G\{\phi(x),y\}=3$ or $\{\phi(x),y\}=\{\phi(x),y\}_{(3)}$.

Proof: Write $\phi(x)=y\sum_{i,k\geq 0}\alpha_{i,k}x^i\Omega^k$. We compute

$$\begin{aligned} \{\phi(x),y\} &= \left\{y\left(\sum_{i,k\geq 0}\alpha_{i,k}x^i\Omega^k\right),y\right\}=y\left(\sum_{i,k\geq 0}\alpha_{i,k}ix^{i-1}y^2\Omega^k\right)\\ &=y^3\left(\sum_{i,k\geq 0}\alpha_{i,k}ix^{i-1}\Omega^k\right), \end{aligned}$$

which has G-degree 3.

Claim 5: $y^3\mid\phi(x)$.

Proof: By **Claim 2**, $\phi(x) = y\sum_{i,k\geq 0}\alpha_{i,k}x^i y^{2k}z^k$. If $\alpha_{i,0} \neq 0$ for some i, we have a nonzero term yx^{i+2} in $3x^2\phi(x)$. But yx^{i+2} cannot appear in $2yz\phi(y) + y^2\phi(z)$ for any i, which contradicts (E6.9.4). Therefore $\alpha_{i,0} = 0$ for all i and $y^3 \mid \phi(x)$.

Claim 6: $y \mid \phi(y)$.

Proof: This follows from (E6.9.4) and **Claim 5**.

It follows from (E6.9.4), **Claim 5** and **Claim 6** that

$$\phi(x) = yw_{(0)} = y^3 zv_{(0)}, \tag{E6.9.5}$$

$$\phi(y) = yf, \tag{E6.9.6}$$

$$\phi(z) = -2zf - 3x^2 yzv_{(0)} \tag{E6.9.7}$$

where v_0 has G-degree 0 and $f \in A$. Next we will apply ϕ to the relations given in (E6.9.1) and (E6.9.2). We compute

$$\begin{aligned}
0 = \phi\left(\{x, y\} - y^2\right) &= \{\phi(x), y\} + \{x, \phi(y)\} - 2y\phi(y) \\
&= \{yw_{(0)}, y\} + \{x, yf\} - 2y\phi(y) \\
&= y\{w_{(0)}, y\}_{(2)} + y^2 f + y\{x, f\} - 2y^2 f \\
&= y\{w_{(0)}, y\}_{(2)} - y^2\left(\sum_{i\in\mathbb{Z}} f_{(i)}\right) + y\{x, \sum_{i\in\mathbb{Z}} f_{(i)}\} \\
&= y\{w_{(0)}, y\}_{(2)} - y^2\left(\sum_{i\in\mathbb{Z}} f_{(i)}\right) + y^2\left(\sum_{i\in\mathbb{Z}} if_{(i)}\right) \\
&= y\{w_{(0)}, y\}_{(2)} + y^2\left(\sum_{i\in\mathbb{Z}}(i-1)f_{(i)}\right).
\end{aligned}$$

Therefore

$$f = f_{(1)} \quad \text{and} \quad \{w_{(0)}, y\} = 0. \tag{E6.9.8}$$

As a consequence, $f = yq_{(0)}$ where $\deg_G q_{(0)} = 0$. So we have $\phi(y) = y^2 q_{(0)}$ and $\phi(z) = -2yzq_{(0)} - 3x^2 yzv_{(0)}$.

Applying ϕ to (E6.9.2), we have

$$\begin{aligned}
0 = \phi(\{z, x\} - 2yz) &= \{\phi(z), x\} + \{z, \phi(x)\} - 2\phi(y)z - 2y\phi(z) \\
&= \left(-2q_{(0)} - 3x^2 v_{(0)}\right)\{yz, x\} + zv_{(0)}\{z, y^3\} - 2y^2 q_{(0)}z - 2y\left(-2yzq_{(0)} - 3x^2 yzv_{(0)}\right) \\
&= \left(-2q_{(0)} - 3x^2 v_{(0)}\right) y^2 z + zv_{(0)}(-9x^2y^2) - 2y^2 q_{(0)}z + 4y^2 zq_{(0)} + 6x^2y^2 zv_{(0)} \\
&= -6x^2 y^2 zv_{(0)}.
\end{aligned}$$

Therefore $v_{(0)} = 0$ and

$$\phi(x) = 0, \tag{E6.9.9}$$

$$\phi(y) = y^2 q_{(0)}, \tag{E6.9.10}$$

$$\phi(z) = -2yzq_{(0)}. \tag{E6.9.11}$$

Write $q_{(0)} = \sum_{i,k\geq 0}\alpha_{i,k}x^i\Omega^k$. Let $q'_{(0)} = \sum_{i,k\geq 0}\beta_{i,k}x^i\Omega^k$ where $\beta_{i,k} := \frac{\alpha_{i,k}}{i+1}$. It is easy to check that $H_{xq'_{(0)}} = \phi$. Therefore ϕ is Hamiltonian as desired. □

Lemma 6.10. *Let A be the quadratic Poisson algebra in Example 6.6(2) with potential $\Omega = x^3 + x^2 z + y^2 z$. Then every ozone derivation of A is Hamiltonian.*

Proof. The Jacobian Poisson structure on $A = \Bbbk[x, y, z]$ is explicitly given by

$$\{x, y\} = x^2 + y^2,\ \{y, z\} = 3x^2 + 2xz,\ \{z, x\} = 2yz.$$

We show that every ozone derivation of A is Hamiltonian, which is based on a tedious computation. Since A is graded, it suffices to check every graded Poisson derivation ϕ of degree n vanishing on the Poisson center Z is Hamiltonian. So we can write

$$\phi(x)=\sum_{i=0}^{n+1}\phi_i x^i\in\Bbbk[x,y,z] \text{ of degree } n+1 \text{ with } \phi_i\in\Bbbk[y,z]_{n+1-i}.$$

Claim 1: By subtracting a Hamiltonian derivation $H_g=\{g,-\}$ from ϕ for some suitable $g\in\Bbbk[x,y,z]_n$, we can assume that $\phi_i\in\Bbbk[z]\bigoplus y^3z(\Bbbk[z,y^2z])$ for all $0\le i\le n-1$.

Proof: For simplicity, we denote the $\Bbbk$-linear map $T_m:\Bbbk[y,z]_m\to\Bbbk[y,z]_{m+1}$ by

$$T_m(f):=2yz\frac{\partial f}{\partial z}-y^2\frac{\partial f}{\partial y} \text{ for any } m\ge 0.$$

It is clear to check that

- $\ker(T_m)=0$ if $3\nmid m$ and $\ker(T_m)=\Bbbk(y^2z)^{\frac{m}{3}}$ if $3\mid m$;
- $\mathrm{img}(T_m)\bigoplus y(\ker(T_m))=y(\Bbbk[y,z]_m)$. In particular $y(y^2z)^{\frac{m}{3}}\notin\mathrm{img}(T_m)$ if $3\mid m$.

For any homogeneous polynomial $g=\sum g_ix^i\in\Bbbk[x,y,z]_n$ with $g_i\in\Bbbk[y,z]_{n-i}$, we get

$$\begin{aligned}\{g,x\}&=\frac{\partial g}{\partial y}\{y,x\}+\frac{\partial g}{\partial z}\{z,x\}=-\frac{\partial g}{\partial y}(x^2+y^2)+\frac{\partial g}{\partial z}(2yz)\\&=-\frac{\partial g_{n-1}}{\partial y}x^{n+1}-\frac{\partial g_{n-2}}{\partial y}x^n+\left(T_1(g_{n-1})-\frac{\partial g_{n-3}}{\partial y}\right)x^{n-1}+\cdots\\&\quad+\left(T_{n-2}(g_2)-\frac{\partial g_0}{\partial y}\right)x^2+T_{n-1}(g_1)x+T_n(g_0).\end{aligned}$$

Hence by choosing $g_0,g_1,\dots,g_{n-1}$ for the coefficients of $x^0,x^1,\dots,x^{n-1}$ inductively, we can achieve **Claim 1**.

Claim 2: Modulo a Hamiltonian derivation, we can set

$$\begin{aligned}\phi(x)&=xzf(x,z)+y^3zg(x,y^2z),\\\phi(y)&=(\frac{3}{2}xy+yz)f(x,z)+(\frac{3}{2}y^4-xy^2z)g(x,y^2z)+(x^2+y^2)p,\\\phi(z)&=-(3xz+2z^2)f(x,z)-3y^3zg(x,y^2z)-2yzp\end{aligned}$$

where $f(x,z)\in\Bbbk[x,z]_{n-1}$, $g(x,y^2z)\in\Bbbk[x,y^2z]_{n-3}$ and $p\in\Bbbk[x,y,z]_{n-1}$. Furthermore, we can assume p does not contain x^{n-1}.

Proof: By **Claim 1**, up to a Hamiltonian derivation, we can write

$$\phi(x)=xzf(x,z)+y^3zg(x,y^2z)+az^{n+1}+bx^ny+cx^{n+1}$$

for some coefficients $a,b,c\in\Bbbk$. One can directly check that $\Bbbk[\Omega]\subseteq Z$. Therefore, we have

$$\phi(\Omega)=(3x^2+2xz)\phi(x)+2yz\phi(y)+(x^2+y^2)\phi(z)=0.$$

By comparing degrees (of x,y,z), we see that xz^{n+2} is not a summand of $2yz\phi(z)$ and $(x^2+y^2)\phi(y)$. Hence, the coefficient of xz^{n+2} in $\phi(\Omega)$ is $2a$. It follows that $a=0$. If $c\neq 0$, then by considering y-degree we have that $\phi(z)$ contains the summand $-3cx^{n+1}$. But then the coefficient of $x^{n+1}y^2$ is $-3c$. Thus, $c=0$. A similar argument shows that $b=0$. So we can write $\phi(x)$ as in **Claim 2**, and the expressions of $\phi(y)$ and $\phi(z)$ follow immediately. Finally, by further subtracting $\{ax^n,-\}$ from ϕ, we can replace p with $p-anx^{n-1}$. So, by choosing a suitable scalar a, we can assume p does not contain x^{n-1}.

Claim 3: We have

$$\begin{aligned}\{p,x\}=&-(2z+\frac{3}{2}x)f(x,z)-(3xz+2z^2)\frac{\partial f(x,z)}{\partial z}+zx\frac{\partial f(x,z)}{\partial x}\\&+(3y^3-2xyz)g(x,y^2z)+y^3z\frac{\partial g(x,y^2z)}{\partial x}-2xy^3z^2\frac{\partial g(x,y^2z)}{\partial(y^2z)}.\end{aligned}$$

Proof: A long and tedious calculation yields **Claim 3**.

Now we write

$$f=\sum_{i=1}^{n}a_ix^{n-i}z^{i-1}\quad\text{and}\quad g=\sum_{i=1}^{\lfloor\frac{n}{3}\rfloor}b_ix^{n-3i}(y^2z)^{i-1}$$

for some $a_i, b_i \in \Bbbk$ where we set $a_i = 0$ if $i \notin [1, n]$ and $b_i = 0$ if $i \notin [1, \lfloor \frac{n}{3} \rfloor]$. Define

$$c_i := (n-3i)a_i - (\frac{3}{2} + 3i)a_{i+1}.$$

Then **Claim 3** can be rewritten as

$$\begin{aligned}\{p, x\} = &\sum_{i=0}^{n} c_i z^i x^{n-i} + \sum_{i=1}^{\lfloor \frac{n}{3} \rfloor} (3b_i y^{2i+1} z^{i-1}) x^{n-3i} \\ &- \sum_{i=1}^{\lfloor \frac{n}{3} \rfloor} (2ib_i y^{2i-1} z^i) x^{n+1-3i} + \sum_{i=1}^{\lfloor \frac{n}{3} \rfloor} \left((n-3i) b_i y^{2i+1} z^i\right) x^{n-1-3i}.\end{aligned}$$

Write $p = \sum_{i=0}^{n-1} p_i x^i$ with $p_i \in \Bbbk[y, z]_{n-1-i}$ and $p_{n-1} = 0$. As in **Claim 1**, we get

$$\{p, x\} = \sum_{i=-1}^{n-1} \left(T_i(p_{n-1-i}) - \frac{\partial p_{n-3-i}}{\partial y} \right) x^{n-1-i},$$

where we set $T_{-1} = 0$ and $p_{-2} = p_{-1} = p_n = 0$. Hence we have

$$\begin{aligned} T_{3i-1}(p_{n-3i}) - \frac{\partial p_{n-3i-2}}{\partial y} &= c_{3i} z^{3i} + 3b_i y^{2i+1} z^{i-1}, \\ T_{3i-2}(p_{n+1-3i}) - \frac{\partial p_{n-1-3i}}{\partial y} &= c_{3i-1} z^{3i-1} - 2ib_i y^{2i-1} z^i, \\ T_{3i}(p_{n-1-3i}) - \frac{\partial p_{n-3-3i}}{\partial y} &= c_{3i+1} z^{3i+1} + (n-3i) b_i y^{2i+1} z^i \end{aligned} \tag{E6.10.1}$$

for all $1 \le i \le \lfloor \frac{n}{3} \rfloor$ together with $-\frac{\partial p_{n-2}}{\partial y} = c_0$ and $-\frac{\partial p_{n-3}}{\partial y} = c_1 z$. In particular, if $n \equiv 2 \pmod 3$, then $T_{n-1}(p_0) = c_n z^n$.

Claim 4: We have $f = 0$.

Proof: We show that $c_0 = \cdots = c_n = 0$, which implies that $a_1 = \cdots = a_n = 0$. By the above equations, it suffices to show that $T_{n-i-1}(p_i) - \frac{\partial p_{i-2}}{\partial y} \in y\Bbbk[y, z]$ for $0 \le i \le n$. We claim that $p_i \in \Bbbk[y^2, z]$ for $0 \le i \le n$. It is clear when $i = 0$. Suppose it works for p_i for all $i \le m$. Then inductively, the above equations imply that $T_{n-m-2}(p_{m+1}) \in \operatorname{span}_{\Bbbk}\{y^{2i+1} z^j \mid \text{for all possible } i, j\}$. Note that $\ker(T_{n-m-2}) \in \Bbbk[y^2, z]$. Our claim follows by the definition of T_{n-m-2}. Since $\operatorname{img}(T_{n-i-1}) \in (y)$, we get $T_{n-i-1}(p_i) - \frac{\partial p_{i-2}}{\partial y} \in y\Bbbk[y, z]$ for $0 \le i \le n$.

Claim 5: We have $g = 0$.

Proof: By **Claim 4**, we can take $f = 0$. We will only treat the case $n \equiv 0 \pmod 3$ here and other cases will follow in a similar manner. We write $n = 3s$ and group the equations (E6.10.1) into $s+1$ parts named by (Ei) with $0 \le i \le s$. In details, (E0) is given by

$$T_{n-1}(p_0) = 3b_s y^{2s+1} z^{s-1}, \tag{E0.1}$$

$$T_{n-2}(p_1) = -2sb_s y^{2s-1} z^s, \tag{E0.2}$$

$$T_{n-3}(p_2) - \frac{\partial p_0}{\partial y} = 3b_{s-1} y^{2s-1} z^{s-1}. \tag{E0.3}$$

For $1 \le i \le s-2$, (Ei) is given by

$$T_{n-3i-1}(p_{3i}) - \frac{\partial p_{3i-2}}{\partial y} = 3b_{s-i} y^{2s-2i+1} z^{s-i-1}, \tag{Ei.1}$$

$$T_{n-3i-2}(p_{3i+1}) - \frac{\partial p_{3i-1}}{\partial y} = -2(s-i) b_{s-i} y^{2s-2i-1} z^{s-i}, \tag{Ei.2}$$

$$T_{n-3i-3}(p_{3i+2}) - \frac{\partial p_{3i}}{\partial y} = (3s - 3(s-i-1)) b_{s-i-1} y^{2s-2i-1} z^{s-i-1}. \tag{Ei.3}$$

Moreover, (E(s-1)) and (Es) are given by

$$T_2(p_{n-3}) - \frac{\partial p_{n-5}}{\partial y} = 3b_1 y^3, \tag{E(s-1).1}$$

$$T_1(p_{n-2}) - \frac{\partial p_{n-4}}{\partial y} = -2b_1 yz, \tag{E(s-1).2}$$

$$-\frac{\partial p_{n-3}}{\partial y} = 0, \tag{E(s-1).3}$$

$$-\frac{\partial p_{n-2}}{\partial y} = 0. \tag{Es}$$

As in **Claim 4**, we know $p_i \in \Bbbk[y^2, z]_{n-1-i}$. Assign the lexicographic order with $y > z$ on all monomials in $\Bbbk[y, z]$. We prove the following statement inductively for all $0 \le i \le s-2$ with $n = 3s$:

$$\begin{aligned} p_{3i} &= -\frac{3}{2} b_{s-i} (y^2)^{s-i} z^{s-i-1} + \text{lower terms in } \Bbbk[y^2, z]_{n-3i-1}, \\ p_{3i+1} &= \alpha_i (y^2)^{s-i-1} z^{s-i} + \text{lower terms in } \Bbbk[y^2, z]_{n-3i-2}, \\ p_{3i+2} &= \beta_i (y^2)^{s-i-1} z^{s-i-1} + \text{lower terms in } \Bbbk[y^2, z]_{n-3i-3} \end{aligned}$$

for some $\alpha_i, \beta_i \in \Bbbk$ and $(i+1)b_{s-i-1} = (s-i)b_{s-i}$. When $i = 0$, we use (E0.1) and (E0.2) to get

$$p_0 = -\frac{3}{2} b_s y^{2s} z^{s-1} \quad \text{and} \quad p_1 = -s b_s y^{2s-2} z^s$$

for T_{n-1} and T_{n-2} are injective. So (E0.3) implies that

$$T_{n-3}(p_2) = 3b_{s-1} y^{2s-1} z^{s-1} + \frac{\partial p_0}{\partial y} = 3(b_{s-1} - s b_s) y^{2s-1} z^{s-1}.$$

Since $3 \mid n-3$, we have $\ker(T_{n-3}) = \Bbbk y^{2s-2} z^{s-1}$ and $y^{2s-1} z^{s-1} \notin \operatorname{img}(T_{n-3})$. We get $b_{s-1} = s b_s$ and $p_2 = \beta_0 y^{2s-2} z^{s-1}$ for some $\beta_0 \in \Bbbk$. Suppose the statement holds for p_{3i}, p_{3i+1} and p_{3i+2}. Then (E(i+1).1) implies that

$$\begin{aligned} &T_{n-3i-4}(p_{3i+3}) \\ &= 3b_{s-i-1} y^{2s-2i-1} z^{s-i-2} + \frac{\partial}{\partial y} \left(\alpha_i (y^2)^{s-i-1} z^{s-i} + \text{lower terms in } \Bbbk[y^2, z]_{n-3i-2} \right) \\ &= 3b_{s-i-1} y^{2s-2i-1} z^{s-i-2} + \text{lower terms in } y(\Bbbk[y^2, z]_{n-3i-4}). \end{aligned}$$

Since T_{n-3i-4} is injective, we get

$$p_{3i+3} = -\frac{3}{2} b_{s-i-1} (y^2)^{s-i-1} z^{s-i-2} + \text{lower terms in } \Bbbk[y^2, z]_{n-3i-4}.$$

Similarly from (E(i+1).2), we get

$$p_{3i+4} = \alpha_{i+1} (y^2)^{s-i-2} z^{s-i-1} + \text{lower terms in } \Bbbk[y^2, z]_{n-3i-5}$$

for some $\alpha_{i+1} \in \Bbbk$. Finally, (E(i+1).3) implies that

$$\begin{aligned} T_{n-3i-6}(p_{3i+5}) &= (3s - 3(s-i-2))\, b_{s-i-2} y^{2s-2i-3} z^{s-i-2} \\ &\quad + \frac{\partial}{\partial y} \left(-\frac{3}{2} b_{s-i-1} (y^2)^{s-i-1} z^{s-i-2} + \text{lower terms in } \Bbbk[y^2, z]_{n-3i-4} \right) \\ &= 3\left((i+2) b_{s-i-2} - (s-i-1) b_{s-i-1} \right) y^{2s-2i-3} z^{s-i-2} \\ &\quad + \text{lower terms in } y(\Bbbk[y^2, z]_{n-3i-6}). \end{aligned}$$

Note that $\ker(T_{n-3i-6}) = \Bbbk y^{2s-2i-4} z^{s-i-2}$ and $y^{2s-2i-3} z^{s-i-2} \notin \operatorname{img}(T_{n-3i-6})$ for $3 \mid n-3i-6$. So we get $(i+2) b_{s-i-2} = (s-i-1) b_{s-i-1}$ and we can write

$$p_{3i+5} = \beta_{i+1} (y^2)^{s-i-2} z^{s-i-2} + \text{lower terms in } \Bbbk[y^2, z]_{n-3i-6}$$

for some $\beta_{i+1} \in \Bbbk$. This completes our induction argument. From the above result, we have

$$p_{n-5} = p_{3(s-2)+1} = \alpha_{s-2} y^2 z^2 + \text{lower terms in } \Bbbk[y^2, z]_4.$$

From (E(s-1).1): $T_2(p_{n-3}) - \frac{\partial p_{n-5}}{\partial y} = 3b_1 y^3$, we get $p_{n-3} = -\frac{3}{2}b_1 y^2 + \lambda z^2$ for some $\lambda \in \Bbbk$. Moreover, (E(s-1).3): $-\frac{\partial p_{n-3}}{\partial y} = 0$ implies that $b_1 = 0$. Again from the above statement, we have all $b_i = 0$ and $g = 0$.

Finally, we can show that ϕ is Hamiltonian. By all the above claims, up to a proper Hamiltonian derivation, we can take a Poisson derivation ϕ of degree n as

$$\phi(x) = 0, \quad \phi(y) = (x^2 + y^2)p, \quad \phi(y) = -2yzp$$

for some $p \in \Bbbk[x, y, z]_{n-1}$. From $\{\phi(x), y\} + \{x, \phi(y)\} = \phi(x^2 + y^2)$, we get $\{p, x\} = 0$ or $(x^2 + y^2)p_y = 2yzp_z$. We show that $p = p(x, \Omega)$ by induction on the degree of f. It is clear that we can write $p_y = 2yzq$ and $p_z = (x^2 + y^2)q$ for some $q \in \Bbbk[x, y, z]$ of degree $\deg(p) - 3$. Then $p_{zy} = p_{yz}$ implies that $(x^2 + y^2)q_y = 2yzq_z$. So our induction hypothesis implies that $q = q(x, \Omega)$. Take any polynomial $h(x, \Omega)$ such that $\frac{\partial h(x,\Omega)}{\partial \Omega} = q$. An easy calculation shows that $h_y = p_y$ and $h_z = p_z$. So $p - h \in \Bbbk[x]$. This proves our claim. Now take any $Q(x, \Omega)$ such that $\frac{\partial Q(x,\Omega)}{\partial x} = p(x, \Omega)$. Then one checks that $\phi = \{Q, -\}$ and ϕ is Hamiltonian. □

Remark 6.11. If A is a non-unimodular quadratic Poisson polynomial algebra $\Bbbk[x_1, \cdots, x_n]$, then by Corollary 0.3, A^δ is unimodular for some graded Poisson derivation δ of A (in fact $\delta = \frac{1}{\sum_i \deg x_i}\mathbf{m}$). By (E4.3.1), $rgt(A) = rgt(A^\delta)$. If one can calculate rgt for all unimodular quadratic Poisson structures on $\Bbbk[x_1, \cdots, x_n]$, then the above formula provides a way of computing $rgt(A)$ when A is not unimodular.

Note that all 13 classes of non-unimodular quadratic Poisson structures on $\Bbbk[x_1, x_2, x_3]$ were listed explicitly in [8] (also see [7,22]). For each class, the modular derivation $\mathbf{m}$ is easy to compute. Therefore rgt can be calculated by the method mentioned in the above paragraph.

7. Rigidity, H-ozoneness, and PH^1-minimality

In this section we will study some connections between rigidity of graded twisting, ozone derivations and the first Poisson cohomology.

Let A be a general Poisson algebra with Poisson center Z. Let $Pd(A)$ be the Lie algebra of all Poisson derivations of A and let $Hd(A)$ be the Lie ideal of $Pd(A)$ of all Hamiltonian derivations. Recall from (E1.5.4) that the first Poisson cohomology of A is defined to be

$$PH^1(A) := Pd(A)/Hd(A). \tag{E7.0.1}$$

If A is $\mathbb{Z}$-graded, then so is $PH^1(A)$. Part (1) of the following definition is Definition 6.8.

Definition 7.1. Let A be a Poisson algebra.

(1) A Poisson derivation ϕ of A is called *ozone* if $\phi(z) = 0$ for all $z \in Z$.
(2) Let $Od(A)$ denote the Lie algebra of all ozone Poisson derivations of A.
(3) We say A is H-*ozone* if $Od(A) = Hd(A)$, namely, if every ozone Poisson derivation is Hamiltonian.

It is clear that $Od(A)$ is a Lie ideal of $Pd(A)$ and

$$Hd(A) \subseteq Od(A) \subseteq Pd(A).$$

In general, not every ozone Poisson derivation is Hamiltonian.

For the rest of this section, we only consider locally finite connected $\mathbb{N}$-graded Poisson algebras A with $A_i \neq 0$ for some $i > 0$. Later we will only consider $A = \Bbbk[x, y, z]$ where $\deg(x) = \deg(y) = \deg(z) = 1$. In this case, the Euler derivation defined in (E0.0.2) is a nonzero Poisson derivation.

Lemma 7.2. *Let A be a connected graded Poisson algebra with center Z. Suppose Z is a domain. Then $ZE \cap Od(A) = 0$ if $Z \neq \Bbbk$. As a consequence, $ZE \cap Hd(A) = 0$ and the canonical map $ZE \to PH^1(A)$ sending zE to the coset $zE + Hd(A)$ is a graded injective Z-module map.*

Proof. Let f be any homogeneous element in Z. It is easy to check that fE is a Poisson derivation. So ZE is an abelian Lie subalgebra of $Pd(A)$. Moreover, one can check that $Pd(A)$ is a Z-module.

Next we assume that $Z \neq \Bbbk$. Let ϕ be in $ZE \cap Od(A)$ and we can write it as $\phi = fE$ for some $f \in Z$. Let $z \in Z$ be a nonzero element of positive degree. Then $\phi(z) = 0$ as $\phi \in Od(A)$. Since $\phi = fE$, we obtain that $0 = f(\deg z)z$. This implies that $f = 0$ or $\phi = 0$. Hence $ZE \cap Od(A) = 0$.

Since $Hd(A) \subseteq Od(A)$, $ZE \cap Hd(A) = 0$. So the map

$$ZE \to Pd(A)/Hd(A) =: PH^1(A)$$

is injective.

Finally if $Z = \Bbbk$, then $ZE = \Bbbk E$. It is trivial to see that $ZE \cap Hd(A) = 0$ for $E \notin Hd(A)$. □

By the above lemma, the minimal possibility of $PH^1(A)$ is ZE. This motivates the following definition.

Definition 7.3. Let A be a nontrivial connected graded Poisson algebra with Poisson center Z. Suppose Z is a domain.

(1) We say A is PH^1-minimal if $PH^1(A) \cong ZE$.
(2) We say A has an *Euler-ozone decomposition* if

$$Pd(A) = ZE \rtimes Od(A).$$

(3) We say A has an *Euler-Hamiltonian decomposition* if

$$Pd(A) = ZE \rtimes Hd(A).$$

By Remark 5.3, if A is the Poisson polynomial algebra being the semiclassical limit of the 4-dimensional (resp. 3-dimensional) Sklyanin algebra, then it is PH^1-minimal. Note that the dimension of $Gpd(A)$ is the constant term of $h_{Pd(A)}(t)$. So A is rigid of graded twisting if and only if the constant term of $h_{Pd(A)}(t)$ is 1 [Remark 5.2]. Therefore we have

$$A \text{ is } PH^1\text{-minimal} \quad \Rightarrow \quad rgt(A) = 0. \tag{E7.3.1}$$

Proposition 7.4. *Let A be a connected graded Poisson algebra. Then the following are equivalent.*

(i) *A is PH^1-minimal.*
(ii) *$h_{PH^1(A)}(t) = h_Z(t)$ provided Z is a domain.*
(iii) *$h_{Pd(A)}(t) = h_A(t)$.*

Proof. (i) $\Leftrightarrow$ (ii) The equivalence follows from the definition.

(ii) $\Leftrightarrow$ (iii) It is clear that the map $A \to Hd(A)$ sending $a \to H_a$ is surjective. The kernel is the center Z. So $h_{Hd(A)}(t) = h_A(t) - h_Z(t)$. By (E7.0.1), $h_{PH^1(A)}(t) = h_{Pd(A)}(t) - h_{Hd(A)}(t)$. Therefore

$$\begin{aligned} h_{Pd(A)}(t) - h_A(t) &= h_{PH^1(A)}(t) + h_{Hd(A)}(t) - h_A(t) \\ &= h_{PH^1(A)}(t) + h_A(t) - h_Z(t) - h_A(t) \\ &= h_{PH^1(A)}(t) - h_Z(t). \end{aligned}$$

The assertion follows. □

Let A be the Poisson algebra in Example 6.6(Case 3). This Poisson algebra can be considered as the semiclassical limit of the 3-dimensional Sklyanin algebra. The potential Ω has an isolated singularity. By [29, Proposition 4.5] (and [35, Theorem 5.1]), $PH^1(A) = ZE$. Consequently, A is PH^1-minimal.

Lemma 7.5. *Let A be a connected graded Poisson algebra. Assume that Z is a non-trivial domain. If A is PH^1-minimal, then A is H-ozone and has an Euler-Hamiltonian decomposition.*

Proof. Since A is PH^1-minimal, $ZE \cong PH^1(A) = Pd(A)/Hd(A)$. This implies that $Pd(A) = ZE \rtimes Hd(A)$. So A has an Euler-Hamiltonian decomposition.

For a graded Poisson derivation ϕ of degree d, by the Euler-Hamiltonian decomposition, we have

$$\phi = fE + H_a.$$

Let z be a nonzero central element of positive degree. Then $\phi(z) = |z|fz + H_a(z) = |z|fz$. If ϕ is ozone, $0 = \phi(z) = |z|fz$ which implies that $f = 0$ and $\phi = H_a$ as desired. □

Lemma 7.6. *Let A be a connected graded Poisson domain. Suppose A is H-ozone.*

(1) *Every Poisson normal element in A is Poisson central.*
(2) *Suppose A is a Poisson polynomial ring. Then A is unimodular.*

Proof. (1) Let x be a nonzero Poisson normal element. Then it is the sum of homogeneous Poisson normal elements. So we can assume that x is homogeneous. Let ϕ be the log-Hamilton derivation $x^{-1}H_x$. Since $H_x(z)=0$ for all z in the center Z, ϕ is ozone. By the hypothesis, ϕ is Hamiltonian, namely, $\phi = H_y$ for some element y. Since ϕ has degree 0, $\deg y = 0$ (or $y \in \Bbbk$) and consequently, $\phi = 0$. This implies that x is central.

(2) Let $\mathbf{m}$ be the modular derivation of A. It follows from the definition that it is ozone. Since $\deg \mathbf{m} = 0$, by the hypothesis, $\mathbf{m} = H_y$ for some element y of degree 0. Hence $y \in \Bbbk$ and consequently, $\mathbf{m} = 0$. The assertion follows. $\square$

By far we have proved the following diagram

$$\begin{array}{ccc} A \text{ is } PH^1\text{-minimal} & \xrightarrow{\text{Lemma 7.5}} & A \text{ is } H\text{-ozone} \\ {\scriptstyle (E7.3.1)}\downarrow & & \downarrow{\scriptstyle \text{Lemma 7.6(2)}} \\ rgt(A)=0 & \xrightarrow[\text{Lemma 4.4(3)}]{} & A \text{ is unimodular.} \end{array}$$

Next we show that some of the conditions are equivalent under extra hypotheses.

Lemma 7.7. *Suppose A is a connected graded Poisson algebra such that $Z = \Bbbk[z]$ where z is homogeneous with $\deg z > 0$.*

(1) *If $Pd(A)_{\leq -1} = 0$, then A has an Euler-ozone decomposition.*
(2) *Suppose A is H-ozone. Then $rgt(A) = 0$.*

Proof. (1) Let ϕ be a Poisson derivation of A of degree i. By the hypothesis, $i \geq 0$.

Case 1: $\deg z \mid i$. Since $\phi(z)$ is central, $\phi(z) = az^n$ for some $a \in \Bbbk$ and $n \geq 0$. Then $\phi' := \phi - \frac{a}{|z|}z^{n-1}E$ satisfies $\phi'(z) = 0$. So ϕ' is ozone. Therefore $\phi = \frac{a}{|z|}z^{n-1}E + \phi'$.

Case 2: $\deg z \nmid i$. Since $\phi(z)$ is central, it must be 0. Therefore ϕ is ozone.

Combining these two cases, every Poisson derivation is the sum of fE for some $f \in Z$ and an ozone derivation.

(2) By Lemma 4.4(4), it suffices to show that $\dim_{\Bbbk} Gpd(A) = 1$, or equivalently, $Gpd(A) = \Bbbk E$. Let $\phi \in Gpd(A)$ and $\phi(z) = az$ for some $a \in \Bbbk$. Let δ be $\phi - \frac{a}{|z|}E$. Then $\delta \in Gpd(A)$ is ozone. By the hypothesis, δ is Hamiltonian, say $\delta = H_f$ for some homogeneous element $f \in A$. Since $\deg \delta = 0$, $\deg f = 0$. Since A is connected graded, $H_f = 0$ and consequently, $\delta = 0$. Thus $\phi = \frac{a}{|z|}E$ and $Gpd(A) = \Bbbk E$. $\square$

Now we are ready to prove Theorem 0.6.

Proof of Theorem 0.6. (1) $\Rightarrow$ (2): Since $rgt(A) = 0$, every graded Poisson derivation δ is of the form cE. Then $E \wedge \delta = 0$. By (E2.3.1), $\langle a, b\rangle = \{a, b\}$. So $A = A^{\delta}$. The assertion follows.

(2) $\Rightarrow$ (1): By Corollary 0.3, there is a Poisson derivation δ such that A^{δ} is unimodular. Since $A^{\delta} \cong A$ for all δ, A is unimodular. Suppose to the contrary that A is not rigid. Then there is a Poisson derivation δ not in ZE. Thus, by Theorem 0.2, the modular derivation of A^{δ} is

$$\mathbf{n} = 0 + \left(\sum_{i=1}^{3} \deg x_i\right)\delta - \operatorname{div}(\delta)E$$

which cannot be zero as $\operatorname{div}(\delta) \in \Bbbk$ [Lemma 1.2(3)]. Therefore A^{δ} is not isomorphic to A, yielding a contradiction.

(5) $\Leftrightarrow$ (6): Under the hypothesis of (5), A is PH^1-minimal. One implication follows by Lemma 7.5 and the other is clear.

(6) $\Rightarrow$ (7): See the proof of Lemma 7.5.

(7) $\Rightarrow$ (1): This is Lemma 7.7(2).

(3) $\Leftrightarrow$ (5): This is Proposition 7.4.

(1) $\Leftrightarrow$ (8): This is Corollary 6.7.

(8) $\Rightarrow$ (6,7): If $\Omega = \frac{1}{3}(x^3 + y^3 + z^3) + \lambda xyz$ with $\lambda^3 \neq -1$, then by the comments before Lemma 7.5, A is PH^1-minimal. Hence A is H-ozone since (5) $\Leftrightarrow$ (6). If Ω is $x^3 + y^2z$ or $x^3 + x^2z + y^2z$, it follows by Lemmas 6.9 and 6.10 that A is H-ozone. In all three cases in Example 6.6, one can check easily that $Pd(A)_{\leq -1} = 0$. By Lemma 7.7(1), A has an Euler-ozone decomposition. Since A is H-ozone, A has an Euler-Hamiltonian decomposition.

(5) $\Rightarrow$ (1): The assertion follows from Remark 5.2.

(4) $\Rightarrow$ (1): The assertion follows from Remark 5.2

(8) $\Rightarrow$ (4): In all three cases, Z is $\Bbbk[\Omega]$ (this is a well-known fact and a special case of it is [26, Lemma 1], also see Lemmas 7.8 and 7.9 later), which has Hilbert series $\frac{1}{1-t^3}$. The assertion follows from (5) since (5) is equivalent to (8).

(4) $\Leftrightarrow$ (9): It follows from (E1.5.5) that $h_{PH^3(A)}(t) - h_{PH^2(A)}(t) = h_{PH^0(A)}(t) - h_{PH^1(A)}(t) + t^{-3}$. We know that $PH^0(A) = Z$. So the assertion follows from the fact that $h_{PH^1(A)}(t) = h_{PH^0(A)}(t) = \frac{1}{1-t^3}$ if and only if $h_{PH^3(A)}(t) - h_{PH^2(A)}(t) = t^{-3}$. $\square$

Before we prove Corollary 0.7, we will need to calculate the Hilbert series of $PH_0(A)$ for A in the first two cases of Example 6.6. By definition and [28, p.2357], the 0th Poisson homology of the Poisson polynomial algebra $\Bbbk[x, y, z]$ is

$$PH_0(A) \cong \frac{A}{\{A, A\}} = \frac{A}{\big(H_x(A) + H_y(A) + H_z(A)\big)}.$$

Case 1: $\Omega = x^3 + y^2 z$. We use the G-grading introduced in the proof of Lemma 6.9, namely, $\deg_G x = 0$, $\deg_G y = 1$ and $\deg_G z = -2$. Let

$$\begin{aligned}
a_{i,j,0,l} &= x^i y^j z^0 \Omega^l, \quad i, j, l \geq 0,\\
b_{i,0,k,l} &= x^i y^0 z^k \Omega^l, \quad i, l \geq 0, k \geq 1,\\
c_{i,1,k,l} &= x^i y z^k \Omega^l, \quad i, l \geq 0, k \geq 1,\\
&\mathbb{A} := \{a_{i,j,0,l} \mid i, j, l \geq 0\},\\
&\mathbb{B} := \{b_{i,0,k,l} \mid i, l \geq 0, k \geq 1\},\\
&\mathbb{C} := \{c_{i,1,k,l} \mid i, l \geq 0, k \geq 1\}.
\end{aligned}$$

If X is a subset of elements in A, we use $\Bbbk X$ to denote the $\Bbbk$-linear span of X.

Lemma 7.8. *Retain the above notations.*

(1) $\mathbb{A} \cup \mathbb{B} \cup \mathbb{C}$ *is a* $\Bbbk$*-linear basis of* A.

(2)

$$\begin{aligned}
H_x(a_{i,j,0,l}) &= j a_{i,j+1,0,l},\\
H_x(b_{i,0,k,l}) &= (-2k) c_{i,1,k,l},\\
H_x(c_{i,1,k,l}) &= \begin{cases} (1-2k)(b_{i,0,k-1,l+1} - b_{i+3,0,k-1,l}) & k > 1,\\ (1-2k)(a_{i,0,0,l+1} - a_{i+3,0,0,l}) & k = 1. \end{cases}
\end{aligned}$$

(3)

$$\begin{aligned}
H_y(a_{i,j,0,l}) &= (-i) a_{i-1,j+2,0,l},\\
H_y(b_{i,0,k,l}) &= \begin{cases} (-i) b_{i-1,0,k-1,l+1} + (i+3k) b_{i+2,0,k-1,l} & k > 1,\\ (-i) a_{i-1,0,0,l+1} + (i+3) a_{i+2,0,0,l} & k = 1, \end{cases}\\
H_y(c_{i,1,k,l}) &= \begin{cases} (-i) c_{i-1,1,k-1,l+1} + (i+3k) c_{i+2,1,k-1,l} & k > 1,\\ (-i) a_{i-1,1,0,l+1} + (i+3) a_{i+2,1,0,l} & k = 1. \end{cases}
\end{aligned}$$

(4)

$$\begin{aligned}
H_z(a_{i,j,0,l}) &= \begin{cases} 2i a_{i-1,j-1,0,l+1} + (-2i-3j) a_{i+2,j-1,0,l} & j > 0,\\ 2i c_{i-1,1,1,l} & j = 0, \end{cases}\\
H_z(b_{i,0,k,l}) &= 2i c_{i-1,1,k+1,l},\\
H_z(c_{i,1,k,l}) &= 2i b_{i-1,0,k,l+1} + (-3-2i) b_{i+2,0,k,l}.
\end{aligned}$$

(5) $A/(H_x(A) + H_y(A) + H_z(A))$ *has a* $\Bbbk$*-linear basis*

$$\{a_{0,0,0,0}, a_{1,0,0,0}\} \cup \{a_{0,1,0,l}\}_{l \geq 0} \cup \{a_{1,1,0,l}\}_{l \geq 0} \cup \{b_{0,0,k,0}\}_{k \geq 1} \cup \{b_{1,0,k,0}\}_{k \geq 1}.$$

(6) *The Hilbert series of* $PH_0(A)$ *is*

$$\frac{(1+t)^3}{1-t^3}.$$

(7) *The Hilbert series of* $PH^0(A) = h_Z(t)$ *is*

$$\frac{1}{1-t^3}.$$

Proof. This follows from a tedious and direct computation. □

The proof of the above lemma is routine and long, but very elementary, only using easy linear algebra arguments. To save space the details are omitted here. Note that Lemma 7.8(7) is a well-known fact.

Case 2: $\Omega = x^3 + x^2z + y^2z$. We need to prove a lemma similar to Lemma 7.8. We use the same notations as in Case 1 except that Ω is $x^3 + x^2z + y^2z$ instead of $x^3 + y^2z$.

Lemma 7.9. *Let A be as in Example 6.6(Case 2) with potential* $\Omega = x^3 + x^2z + y^2z$.

(1) $\mathbb{A} \cup \mathbb{B} \cup \mathbb{C}$ *is a* $\Bbbk$*-linear basis of A.*

(2)

$$H_x(a_{i,j,0,l}) = ja_{i,j+1,0,l} + ja_{i+2,j-1,0,l},$$
$$H_x(b_{i,0,k,l}) = (-2k)c_{i,1,k,l},$$
$$H_x(c_{i,1,k,l}) = \begin{cases} (1-2k)(b_{i,0,k-1,l+1} - b_{i+3,0,k-1,l}) + 2kb_{i+2,0,k,l} & k>1, \\ -(a_{i,0,0,l+1} - a_{i+3,0,0,l}) + 2b_{i+2,0,1,l} & k=1. \end{cases}$$

(3)

$$H_y(a_{i,j,0,l}) = (-i)a_{i-1,j+2,0,l} + (-i)a_{i+1,j,0,l},$$
$$H_y(b_{i,0,k,l}) = \begin{cases} (-i)b_{i-1,0,k-1,l+1} + (i+3k)b_{i+2,0,k-1,l} + 2kb_{i+1,0,k,l} & k>1, \\ (-i)a_{i-1,0,0,l+1} + (i+3)a_{i+2,0,0,l} + 2b_{i+1,0,1,l} & k=1, \end{cases}$$
$$H_y(c_{i,1,k,l}) = \begin{cases} (-i)c_{i-1,1,k-1,l+1} + (i+3k)c_{i+2,1,k-1,l} + 2kc_{i+1,1,k,l} & k>1, \\ (-i)a_{i-1,1,0,l+1} + (i+3)a_{i+2,1,0,l} + 2c_{i+1,1,1,l} & k=1. \end{cases}$$

(4)

$$H_z(a_{i,j,0,l}) = \begin{cases} 2ic_{i-1,1,1,l} & j=0, \\ 2ia_{i-1,0,0,l+1} + (-2i-3)a_{i+2,0,0,l} + (-2i-2)b_{i+1,0,1,l} & j=1, \\ 2ia_{i-1,1,0,l+1} + (-2i-6)a_{i+2,1,0,l} + (-2i-4)c_{i+1,1,1,l} & j=2, \\ 2ia_{i-1,j-1,0,l+1} + (-2i-3j)a_{i+2,j-1,0,l} \\ \qquad +(-2i-2j)x^{i+1}y^{j-1}z\Omega^l & j \ge 3, \end{cases}$$
$$H_z(b_{i,0,k,l}) = 2ic_{i-1,1,k+1,l},$$
$$H_z(c_{i,1,k,l}) = 2ib_{i-1,0,k,l+1} + (-3-2i)b_{i+2,0,k,l} + (-2-2i)b_{i+1,0,k+1,l}.$$

(5) $A/(H_x(A) + H_y(A) + H_z(A))$ *has a* $\Bbbk$*-linear basis*

$$\{a_{0,0,0,0}, a_{1,0,0,0}\} \cup \{a_{3i,1,0,0}\}_{i \ge 0} \cup \{a_{1+3i,1,0,0}\}_{i \ge 0} \cup \{b_{0,0,k,0}\}_{k \ge 1} \cup \{b_{1,0,k,0}\}_{k \ge 1}.$$

(6) *The Hilbert series of* $PH_0(A)$ *is*

$$\frac{(1+t)^3}{1-t^3}.$$

(7) *The Hilbert series of* $PH^0(A) = h_Z(t)$ *is*

$$\frac{1}{1-t^3}.$$

Proof. This follows from a tedious and direct computation. □

Similar to Lemma 7.8, the proof of Lemma 7.9 is routine and long (even longer than the proof of Lemma 7.8), but still very elementary. To save space the details are omitted here. Note that Lemma 7.9(7) is a well-known fact.

Finally we prove Corollary 0.7.

Proof of Corollary 0.7. (1) This is clear since $Z = \Bbbk[\Omega]$ by Lemmas 7.8 and 7.9.

(2) It follows from Theorem 0.6((8)⇒(4)).

(4) By the Poincaré duality [23, Theorem 3.5], $h_{PH^3(A)}(t) = t^{-3}h_{PH_0(A)}$. Then the assertion follows from Lemmas 7.8 and 7.9.

(3) It follows from Parts (1,2,4) and (E1.5.5). □

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Data availability

No data was used for the research described in the article.

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