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L^2 Stability and Weak-BV Uniqueness for Nonisentropic Euler Equations

Geng Chen[1,*] and Alexis F. Vasseur[2]

[1] *Department of Mathematics, University of Kansas, Lawrence, KS, 66049, USA.*
[2] *Department of Mathematics, University of Texas at Austin, Austin, TX 78712, USA.*

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Dedicated to Professor Gui-Qiang Chen on the occasion of his 60th birthday

Abstract. We prove the L^2 stability for weak solutions of non-isentropic Euler equations in one space dimension whose initial data are perturbed from a small BV data under the L^2 distance. Using this result, we can show the uniqueness of small BV solutions among a large family of weak solutions.

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Key words: Compressible Euler system, uniqueness, stability, relative entropy, conservation law.

1 Introduction

The 1-d compressible Euler equations, widely used for compressible inviscid flow such as gas dynamics, can be written in the Eulerian coordinates, as

$$\begin{aligned}
&\rho_t+(\rho w)_{x'}=0,\\
&(\rho w)_t+(\rho w^2+p)_{x'}=0,\\
&\left(\frac{1}{2}\rho w^2+\rho\mathcal{E}\right)_t+(\rho w^3+vp)_{x'}=0,
\end{aligned} \tag{1.1}$$

*Corresponding author. *Email addresses:* `gengchen@ku.edu` (G. Chen), `vasseur@math.utexas.edu` (A. F. Vasseur)

where three equations represent conservation of mass, momentum and energy, respectively. When we use a Lagrangian frame, co-moving with the fluid, given by $x=\int \rho dx'$, the equations become

$$\begin{aligned} &\tau_t - w_x = 0, \\ &w_t + p_x = 0, \\ &\left(\frac{1}{2}w^2 + \mathcal{E}\right)_t + (wp)_x = 0, \end{aligned} \tag{1.2}$$

whose solution is equivalent to (1.1) [36]. Here, $(t,x)\in\mathbb{R}^+\times\mathbb{R}$ are time and space, $\tau=1/\rho$ is the specific volume, p is pressure, w is fluid velocity, and $\mathcal{E}$ is the specific internal energy. For convenience, let us use the Lagrangian coordinates.

The system is closed by specifying a constitutive law. For convenience, we consider a polytropic ideal γ-law gas, with equation of state

$$\mathcal{E} = c_v\theta = \frac{p\tau}{\gamma-1}, \quad p\tau = \bar{R}\theta, \tag{1.3}$$

so that

$$p = Ke^{\frac{S}{c_v}}\tau^{-\gamma}. \tag{1.4}$$

Here S is the entropy, θ is the temperature, $\bar{R},K,c_v$ are positive constants, and $\gamma>1$ is the adiabatic gas constant. These state variables satisfy the Gibbs relation

$$\theta dS = d\mathcal{E} + pd\tau. \tag{1.5}$$

The Lagrangian sound speed is given by

$$c = \sqrt{-p_\tau} = \sqrt{K\gamma}\,\tau^{-\frac{\gamma+1}{2}}e^{\frac{S}{2c_v}}. \tag{1.6}$$

Euler equations (1.2) can be written in the form of hyperbolic conservation laws

$$u_t + \big(f(u)\big)_x = 0, \quad t>0, \quad x\in\mathbb{R} \tag{1.7}$$

with

$$(u_1,u_2,u_3) = \left(\tau, w, \frac{1}{2}w^2 + \mathcal{E}\right), \tag{1.8}$$

and by (1.3),

$$(f_1,f_2,f_3) = \left(-u_2, \frac{(\gamma-1)\big(u_3-u_2^2/2\big)}{u_1}, \frac{(\gamma-1)u_2\big(u_3-u_2^2/2\big)}{u_1}\right). \tag{1.9}$$

Our result also holds for system (1.1) in the Eulerian coordinates.

We use $\mathcal{V}_0$ to denote any bounded set on $\{u_1>0\}$ uniformly away from $\{u_1=0\}$, and we denote by $\mathcal{V}$ its interior. Then the flux function f satisfies

$$f=(f_1,f_2,f_3)\in[C(\mathcal{V}_0)]^3\cap[C^4(\mathcal{V})]^3.$$

We will consider only entropic solutions of (1.2), that is, solutions which verify additionally

$$\big(\eta(u)\big)_t+\big(q(u)\big)_x\le 0,\quad t>0,\quad x\in\mathbb{R}, \tag{1.10}$$

where in this paper, for (1.2) we use the entropy and entropy-flux pair

$$\eta=-S=c_v\left((1-\gamma)\ln\tau-\ln\mathcal{E}+\ln\frac{K}{\gamma-1}\right)$$

and $q=0$. It is easy to check that, for smooth solutions,

$$q'=\eta' f'. \tag{1.11}$$

For non-isentropic Euler equations, (1.10) tells that the entropy function S increases after passing an 1 or 3 shock, which is consistent to the Lax entropy condition. More precisely, we ask that for all $\phi\in C_0^\infty([0,\infty)\times\mathbb{R})$ verifying $\phi\ge 0$,

$$\begin{aligned}&\int_0^\infty\int_{-\infty}^\infty\left[\phi_t(t,x)\eta\big(u(t,x)\big)+\phi_x(t,x)q\big(u(t,x)\big)\right]dxdt\\&\qquad+\int_{-\infty}^\infty\phi(0,x)\eta\big(u^0(x)\big)dx\ge 0,\end{aligned} \tag{1.12}$$

where $u^0:\mathbb{R}\to\mathbb{R}$ is the prescribed initial data for the solution u.

The global existence of small BV solution for hyperbolic conservation laws (1.7) was first established by Glimm [20]. After Glimm's method, currently referred to as the Glimm scheme or random choice method, there are two other frameworks which can be used to prove the small BV existence for general hyperbolic conservation laws: the front tracking scheme (see [3, 15]) and the vanishing viscosity method [2].

The uniqueness and stability issues are trickier. Recently, for the general system with n-unknowns, Bressan and Guerra [7] and Bressan and De Lellis [5] proved the uniqueness for small BV entropy solution of (1.7), based on the earlier framework of Bressan [3, 6, 8, 9], by verifying the Tame's oscillation condition for BV solutions. Note that the unconditional uniqueness for entropy solutions in the 2×2 case was previously proved in [13] using the a-contraction theory. The

L^1 stability for small BV solutions of general systems of hyperbolic conservation laws, was established in 1990s [4, 10], or see [3]. In the authors' paper [13], for system with two unknowns, we established a general L^2 stability theory among a class of general L^2 perturbations. In this paper, we will extend the L^2 stability and weak-BV uniqueness result in [13] to non-isentropic Euler equations (1.2) satisfying (1.4) with three unknowns.

1.1 Main results

We restrict our study to the solutions verifying the so-called strong trace property.

Definition 1.1 (Strong Trace Property)**.** *Let $u\in L^\infty(\mathbb{R}^+\times\mathbb{R})$. We say that u verifies the strong trace property if for any Lipschitzian curve $t\to X(t)$, there exists two bounded functions $u_-,u_+\in L^\infty(\mathbb{R}^+)$ such that for any $T>0$*

$$\lim_{n\to\infty}\int_0^T \sup_{y\in(0,1/n)} \left|u\big(t,X(t)+y\big)-u_+(t)\right|dt$$
$$=\lim_{n\to\infty}\int_0^T \sup_{y\in(-1/n,0)} \left|u\big(t,X(t)+y\big)-u_-(t)\right|dt=0.$$

Strong traces properties were first proved for multivariable conservation laws [34], see also ([27,30]). The technique was later used to get more structural information on the solutions (see [16,33]). For systems, the question whether bounded weak solutions in $\mathcal{S}^T_{weak}$ verify the strong trace property is mostly open.

For convenience, we will use later the notation $u_+(t)=u(t,X(t)+)$, and $u_-(t)=u(t,X(t)-)$. We can then define the wildest space of solutions that we consider in the paper

$$\mathcal{S}_{weak}=\big\{u\in L^\infty(\mathbb{R}^+\times\mathbb{R}:\mathcal{V}_0)\text{ weak solution to (1.7)-(1.10), verifying Definition (1.1)}\big\}. \tag{1.13}$$

Note that this space has no smallness condition.

The aim of this paper is to show the stability of a smaller class of solutions for nonisentropic Euler equations, namely solutions with small BV norms, when perturbations are taken in the wider space $\mathcal{S}_{weak}$. More precisely, for any domain $\mathcal{O}$ such that $\overline{\mathcal{O}}\subset\mathcal{V}$, consider the following class of solutions:

$$\mathcal{S}_{BV,\varepsilon}=\big\{u\in L^\infty\big(\mathbb{R}^+,BV(\mathbb{R}:\mathcal{O})\big)\text{ solution to (1.7)-(1.10) with }\|u(t)\|_{BV(\mathbb{R})}\le\varepsilon\text{ for }t\ge 0\big\}. \tag{1.14}$$

Our main result is the following theorem.

Theorem 1.1. *Consider the entropy solution of* (1.2), (1.4) *or equivalently* (1.7)-(1.9). *For any open set $\mathcal{O}$ such that $\overline{\mathcal{O}}\subset\mathcal{V}$, there exists $\varepsilon>0$ such that the following is true.*

Let $u\in\mathcal{S}_{BV,\varepsilon}$ be a BV solution with initial value u^0. Assume that $u_n\in\mathcal{S}_{weak}$ is a sequence of wild solutions, uniformly bounded in $L^\infty(\mathbb{R}^+\times\mathbb{R})$, with initial values $u_n^0\in L^\infty(\mathbb{R})$. If u_n^0 converges to u^0 in $L^2(\mathbb{R})$, then for every $T>0, R>0, u_n$ converges to u in $L^\infty(0,T;L^2(-R,R))$. Especially, u is unique in the class $\mathcal{S}_{weak}$.

Note that any BV function verifies the strong trace property (1.1). Hence, any BV solution to (1.7)-(1.10) belongs to $\mathcal{S}_{weak}$. We can easily derive the following uniqueness result by (1.1) without the assumption on strong trace property.

Theorem 1.2. *Consider the entropy solution of* (1.2), (1.4) *or equivalently* (1.7)-(1.9). *Then, for any open set $\mathcal{O}$ such that $\overline{\mathcal{O}}\subset\mathcal{V}$, there exists $\varepsilon>0$ such that any solution in $\mathcal{S}_{BV,\varepsilon}$ with initial value u^0 is unique among the functions*

$$\big\{v\in L^\infty\big([0,T];BV(\mathbb{R})\big),\forall T>0 \text{ and solution to (1.7)-(1.10)}\big\}$$

with the same initial value.

Theorem 1.1 can be seen as a weak-BV stability result, similar to the weak-strong stability result of Dafermos and DiPerna. In fact, the perturbed solution might be large data solutions in our L^2 based theory. Since the work of Dafermos [14] and DiPerna [17], it is known that on any span of time $[0,T]$ where a solution of the system is Lipschitz in x, the solution is L^2 stable (for L^2 perturbations on the initial value) among the large class of solutions which are bounded weak entropic solutions to the same system. This implies the well known weak-strong uniqueness principle: as long as a solution is Lipschitz, it is unique among any other bounded weak solution. To be more precise, let us denote the two classes of solutions

$$\begin{aligned}\mathcal{S}^T_{reg}=\big\{&u\in L^\infty([0,T]\times\mathbb{R}:\mathcal{V}) \text{ solution to (1.7)-(1.10)}\\ &\text{with } \|\partial_x u(t)\|_{L^\infty([0,T]\times\mathbb{R})}\le C \text{ for } C>0\big\},\\ \mathcal{S}^T_{weak}=\{&u\in L^\infty([0,T]\times\mathbb{R}:\mathcal{V}_0) \text{ weak solution to (1.7)-(1.10)}\}.\end{aligned}$$

Let $\mathcal{O}$ be a compact subset of $\mathcal{V}$, and u be a solution in $\mathcal{S}^T_{reg}$ with values in $\mathcal{O}$, and with initial value u^0. The result of Dafermos and DiPerna implies that if $(u_n)_{n\in\mathbb{N}}$ is a sequence of solutions in $\mathcal{S}^T_{weak}$ such that their initial values $(u_n^0)_{n\in\mathbb{N}}$ converge in $[L^2(\mathbb{R})]^3$ to u^0, then $(u_n)_{n\in\mathbb{N}}$ converges in $L^\infty(0,T;L^2(\mathbb{R}))$ to u. Especially, it implies the uniqueness of solutions in $\mathcal{S}^T_{reg}$ among the bigger class of solutions $\mathcal{S}^T_{weak}$ (weak-strong uniqueness). In [13], we extended this result, in the context of 1-d systems with two unknowns, going from the Lipschitz space $\mathcal{S}^T_{reg}$ to the BV space $\mathcal{S}_{BV,\varepsilon}$. In this paper, we establish the L^2 theory for non-isentropic Euler equation with three unknowns. Note, however, that the wild solutions of $\mathcal{S}_{weak}$ need to have the extra strong trace property compared to solutions of $\mathcal{S}^T_{weak}$.

Comparing to [13], the main difficulty for results in the current paper is how to cope with the contact discontinuity in the 2-nd characteristic family. Our idea is to use the dissipation found in [32]. To avoid a shift in the contact discontinuity, there is a very specific additional restriction on weight function $a(x,t)$ in the relative entropy norm on both sides of a contact discontinuity. So how to find an appropriate weight function $a(x,t)$ satisfying all constrains for shocks and contact discontinuities is the most difficult issue in this paper.

The result of this paper is part of a general program to study asymptotic limits to small BV solutions of conservation laws. A major obstacle to study such limits, is the lack of uniform BV control with respect to the asymptotic parameter. This is what impede the extension of inviscid limits beyond the case of artificial viscosity [2]. The main idea of the program is to bypass the search for such small BV control, replacing it by weak-BV principles. This program was recently completed in the case of the inviscid limit of the Navier-Stokes equation in the barotropic case [12]. It relies on the weak-BV principle obtained in this context in [13,21], and the uniform stability of viscous layers with respect to the viscosity obtained in [23,24].

The paper is structured as follows. In Section 2, we introduce the weighted relative entropy norm with shifts. Section 3 is dedicated to the L^2 study of single waves. The most important one concerns the study of a single shock, where Proposition 3.1 has been proved in [21]. In Section 4, we define the weighted function then prove the time decay of this function. Section 5 is dedicated to the proof of Proposition 2.2. Finally, we prove the main Theorem 1.1 in Section 6. The modified front tracking algorithm is introduced in Appendix A.

2 Weighted relative entropy and shifts

For any $g \in C^1(\mathcal{V})$, let us denote the vector valued function $g' = Dg$. Then, we denote the first, second and third eigenvalues and associated right eigenvectors of f' on $\mathcal{V}$ as $\lambda_1, r_1, \lambda_2, r_2$ and λ_3, r_3, corresponding to the 1,2 and 3 characteristic families, respectively. It is easy to verify the following conditions used in [21].

Proposition 2.1. *The entropy solution of the Euler system* (1.2), (1.4) *or equivalently* (1.7)-(1.10) *satisfies*

(a) *For any* $u\in\mathcal{V}$: $\lambda_1(u)=-c<0=\lambda_2(u)<c=\lambda_3(u)$, *where* c *is the Lagrangian sound speed in* (1.6).

(b) *For any* $u\in\mathcal{V}$, *and* $i=1,3$: $\lambda_i'(u)\cdot r_i(u)\neq 0$, *i.e. these two families are genuinely nonlinear in the sense of Lax. This means the* 1 *and* 3 *waves are shocks and rarefactions.*

(c) *For any* $u\in\mathcal{V}$, *and* $i=2$: $\lambda_i'(u)\cdot r_i(u)=0$, *i.e. the second family is linearly degenerate in the sense of Lax. This means the* 2 *wave is a contact discontinuity.*

(d) *For any* $b\in\mathcal{V}$, *and any left eigenvector* ℓ *of* $f'(b)$: *the function* $u\to\ell\cdot f(u)$ *is either convex or concave on* $\mathcal{V}$.

(e) *There exists* $L>0$ *such that for any* $u\in\mathcal{V}$ *and* $i=1,3$: $|\lambda_i(u)|\leq L$.

(f) *For* $u_L\in\mathcal{V}$, *we denote* $s\to S^1_{u_L}(s)$ *the* 1*-shock curve through* u_L *defined for* $s>0$. *We choose the parametrization such that* $s=|u_L-S^1_{u_L}(s)|$. *Hence,* $(u_L,S^1_{u_L}(s),\sigma^1_{u_L}(s))$ *is the* 1*-shock with left hand state* u_L *and strength* s. *Similarly, we define* $s\to S^3_{u_R}$ *to be the* 3*-shock curve such that* $(u_R,S^3_{u_R},\sigma^3_{u_R}(s))$ *is the* 3*-shock with right-hand state* u_R *and strength* s. σ *is the speed of shock. We assume that these curves are defined globally in* $\mathcal{V}$ *for every* $u_L\in\mathcal{V}$ *and* $u_R\in\mathcal{V}$.

(g) *(for 1-shocks) If* (u_L,u_R) *is an entropic Rankine-Hugoniot discontinuity with shock speed* σ, *then* $\sigma>\lambda_1(u_R)$. *This is the Lax entropy condition.*

(h) *(for 1-shocks) If* (u_L,u_R) *(with* $u_L\in B_\epsilon(d)$*) is an entropic Rankine-Hugoniot discontinuity with shock speed* σ *verifying* $\sigma\leq\lambda_1(u_L)$, *then* u_R *is in the image of* $S^1_{u_L}$. *That is, there exists* $s_{u_R}\in[0,s_{u_L})$ *such that* $S^1_{u_L}(s_{u_R})=u_R$ *(and hence* $\sigma=\sigma^1_{u_L}(s_{u_R})$*).*

(i) *(for 3-shocks) If* (u_L,u_R) *is an entropic Rankine-Hugoniot discontinuity with shock speed* σ, *then* $\sigma<\lambda_3(u_L)$. *This is the Lax entropy condition.*

(j) *(for 3-shocks) If* (u_L,u_R) *(with* $u_R\in B_\epsilon(d)$*) is an entropic Rankine-Hugoniot discontinuity with shock speed* σ *verifying* $\sigma\geq\lambda_3(u_R)$, *then* u_L *is in the image of* $S^3_{u_R}$. *That is, there exists* $s_{u_L}\in[0,s_{u_R})$ *such that* $S^3_{u_R}(s_{u_L})=u_L$ *(and hence* $\sigma=\sigma^3_{u_R}(s_{u_L})$*).*

(k) *For* $u_L\in\mathcal{V}$, *and for all* $s>0, d\eta(u_L|S^1_{u_L}(s))/ds>0$ *(the shock "strengthens" with s). Similarly, for* $u_R\in\mathcal{V}$, *and for all* $s>0$, $d\eta(u_R|S^3_{u_R}(s))/ds>0$. *Moreover, for each* $u_L,u_R\in\mathcal{V}$ *and* $s>0$, $d\sigma^1_{u_L}(s)/ds<0$ *and* $d\sigma^3_{u_R}(s)/ds>0$.

The proof of this proposition is classical. See, for example, [15]. The proof of our main result is based on the relative entropy method first introduced by Dafermos [14] and DiPerna [17]. From the assumption of the existence of a convex entropy η, we define an associated pseudo-distance defined for any $a,b\in\mathcal{V}_0\times\mathcal{V}$,

$$\eta(a|b)=\eta(a)-\eta(b)-\nabla\eta(b)(a-b). \tag{2.1}$$

The quantity $\eta(a|b)$ is called the relative entropy of a with respect to b, and is equivalent to $|a-b|^2$. We also define the relative entropy-flux: For $a,b\in\mathbb{R}^2$,

$$q(a;b)=q(a)-q(b)-\nabla\eta(b)\big(f(a)-f(b)\big). \tag{2.2}$$

The strength of this notion is that if u is a weak solution of (1.7)-(1.10), then u verifies also the full family of entropy inequalities for any $b\in\mathcal{V}$ constant,

$$\big(\eta(u|b)\big)_t+\big(q(u;b)\big)_x\leq 0. \tag{2.3}$$

Similar to the Kruzkov theory for scalar conservation laws, (2.3) provides a full family of entropies measuring the distance of the solution to any fixed values b in $\mathcal{V}$. The main difference is that the distance is equivalent to the square of the L^2 norm rather than the L^1 norm. Same as for the Kruzkov theory, (2.3) provides directly the stability of constant solutions (by integrating in x the inequality). Modulating the inequality with a smooth function $t,x\to b(t,x)$ provides the well-known weak-strong uniqueness result. Precisely, the relative entropy is an L^2 theory in the following sense.

Lemma 2.1. *For any fixed compact set $V\subset\mathcal{V}$, there exists $c^*,c^{**}>0$ such that for all $(u,v)\in\mathcal{V}_0\times V$,*

$$c^*|u-v|^2\leq\eta(u|v)\leq c^{**}|u-v|^2. \tag{2.4}$$

The constants c^*,c^{**} depend on bounds on the second derivative of η in V, and on the continuity of η on $\mathcal{V}_0$.

This elementary lemma follows directly from Taylor's theorem (see [29,35]). For the family of Euler systems, it is well known that the relative entropy provides a contraction property for rarefaction function $t,x\to b(t,x)$, even in multi-D [19]. This is because it verifies Proposition 2.1(d) (see Section 3).

However, when modulating the inequality with discontinuous functions b with shocks, the situation diverges significantly from the Kruzkov situation. This is due to the fact that the L^2 norm is not as well suited as the L^1 norm for the study of stability of shocks. The method was used by DiPerna [17] to show the uniqueness of single shocks (see also Chen and Frid [11] for the Riemann problem of the Euler equation). In [35], it was proposed to use the method to obtain stability of discontinuous solutions. The main idea was that the L^2 norm can capture very well the stability of the profile of the shock (up to a shift), even if the shift itself is more sensitive [29]. Leger [28] showed that in the scalar settings, the shock profiles (modulo shifts) have a contraction property in L^2, reminiscent to the L^1 contraction of the Kruzkov theory.

It was shown in [31] that the contraction property is usually false for systems. However, it can be recovered by weighting the relative entropy [22]. More precisely, consider a fixed shock (u_L,u_R,s). It was shown that there exists $0<a_1<a_2$

such that, for any wild solutions $u\in\mathcal{S}_{weak}$, we can construct a Lipschitz shift function $h:\mathbb{R}^+\to\mathbb{R}$ with

$$\frac{d}{dt}\left\{a_1\int_{-\infty}^{h(t)}\eta\big(u(t,x)|u_L\big)\,dx+a_2\int_{h(t)}^{\infty}\eta\big(u(t,x)|u_R\big)\,dx\right\}\le 0. \tag{2.5}$$

Note that this formula for $a_1=a_2$, and $h(t)=st$ would imply the contraction property of the shock for the relative entropy. But the result, to be valid, needs the weights a_i, and the shifts h, giving the name to the method: A-contraction with shifts.

Let us emphasize that the L^2 based a-contraction is not true without the notion of shifts. This is a major obstruction to consider solutions with several waves. Conservation laws have finite speeds of propagation. Therefore, usually, considering a finite amount of waves is equivalent to studying a single one, at least, as long as they do not interact. Because of the shifts, it is not obvious anymore in this theory. The general idea, is that one shift by singularity is needed. Those shifts depend crucially on the perturbation. It is therefore needed to prevent that this artificial shifts do not force a 1-shock to stick and holds to a 2-contact or a 3-shock, making the whole process to collapse. This problem was solved in [25], allowing the treatment of the Riemann problem. The main idea is that the shifts can be constructed based on perturbed characteristic curves associated to the wild solution. Also recall that the 2-contact of (1.2) always has zero speed, while 1 and 3 shocks have negative and positive speeds, respectively.

For BV solutions, because of the generation of infinitely many shifts, the estimate (2.5) is significantly weakened. Our main proposition is the following.

Proposition 2.2. *We consider* (1.7)-(1.10) *and let $d\in\mathcal{V}$. Then there exist $C,v,\varepsilon>0$ such that the following is true:*

For any $m>0,R,T>0,u^0\in BV(\mathbb{R})$ such that $\|u^0\|_{BV(\mathbb{R})}\le\varepsilon$ and $\|u^0-d\|_{L^\infty(\mathbb{R})}\le\varepsilon$, and any wild solution $u\in\mathcal{S}_{weak}$, there exists $\psi\colon\mathbb{R}^+\times\mathbb{R}\to\mathcal{V}$ such that for almost every $0<s<t<T$,

$$\begin{aligned}
&\|\psi(t,\cdot)\|_{BV(\mathbb{R})}\le C\|u^0\|_{BV(\mathbb{R})},\\
&\|\psi(t,\cdot)-\psi(s,\cdot)\|_{L^1}\le C|t-s|,\\
&\|\psi(t,\cdot)-u(t,\cdot)\|_{L^2((-R+vt,R-vt))}\le C\left(\|u^0-u(0,\cdot)\|_{L^2(-R,R)}+\frac{1}{m}\right),
\end{aligned}$$

the function ψ verifies the condition $A.1$ with constant C.

It would be natural to try to take for the function ψ, the unique BV solution with initial value u^0 of Theorem A.1. However, functions ψ which verify the proposition are not solutions to (1.7). Instead, the proposition shows that if the initial value $u(0,\cdot)$ is L^2 close to a set of small BV functions, then $u(t,\cdot)$ stays L^2 close, for every time $t>0$, to a slightly bigger set of small BV functions.

3 Relative entropy for the Riemann problem

We first state the refined a-contraction property of shocks for the weighted relative entropy with shifts. This result is proved in [21]. Note that the constant L is defined in Proposition 2.1(e) as the upper bound of the sound speed c.

Proposition 3.1. *Consider system* (1.7)-(1.10)*. Let $d\in\mathcal{V}$. Then there exist constants $\alpha_1,\alpha_2,\hat{\lambda}$ and $A,\varepsilon>0$ with $\alpha_1<0<\alpha_2$ and $\hat{\lambda}\geq 2L$ such that the following is true:*

Consider any shock (u_L,u_R) with $|u_L-d|+|u_R-d|\leq\varepsilon$, any $u\in\mathcal{S}_{weak}$, any $\bar{t}\in[0,\infty)$, and any $x_0\in\mathbb{R}$. Let σ be the strength of the shock $\sigma=|u_L-u_R|$. Then for any $a_1>0,a_2>0$ verifying

$$
\begin{aligned}
&1-2C\sigma\leq\frac{a_2}{a_1}\leq 1-\frac{C\sigma}{2}, \quad \textit{if } (u_L,u_R) \textit{ is a 1-shock},\\
&1+\frac{C\sigma}{2}\leq\frac{a_2}{a_1}\leq 1+2C\sigma, \quad \textit{if } (u_L,u_R) \textit{ is a 3-shock},
\end{aligned}
$$

there exists a Lipschitz shift function $h:[\bar{t},\infty)\to\mathbb{R}$ with $h(\bar{t})=x_0$ such that the following dissipation functional verifies:

$$
\begin{aligned}
&a_2\big[q\big(u\big(t,h(t)+\big);u_R\big)-\dot{h}(t)\eta\big(u\big(t,h(t)+\big)\,|u_R\big)\big]\\
&\quad -a_1\big[q\big(u\big(t,h(t)-\big);u_L\big)-\dot{h}(t)\eta\big(u\big(t,h(t)-\big)\,|u_L\big)\big]\leq 0
\end{aligned}
\tag{3.1}
$$

for almost all $t\in[\bar{t},\infty)$. Moreover, if (u_L,u_R) is a 1-shock, then for almost all $t\in[\bar{t},\infty)$,

$$
-\frac{\hat{\lambda}}{2}\leq\dot{h}(t)\leq\alpha_1<0.
$$

Similarly, if (u_L,u_R) is a 3-shock, then for almost all $t\in[\bar{t},\infty)$,

$$
0<\alpha_2\leq\dot{h}(t)\leq\frac{\hat{\lambda}}{2}.
$$

Integrating (2.3) with $b=u_L$ for $x\in(-\infty,h(t))$, integrating (2.3) with $b=u_R$ for $x\in(h(t),\infty)$, summing the results, and using (3.1) together with the strong traces property (1.1) provides the contraction property

$$
\frac{d}{dt}\int_{\mathbb{R}}a(t,x)\eta\big(u(t,x)|\psi(t,x)\big)\,dx\leq 0 \tag{3.2}
$$

with a piecewise weighted function

$$a(t,x)=\begin{cases} a_1, & x\le h(t), \\ a_2, & x>h(t) \end{cases}$$

in the case of a single shock as long as, a_2/a_1 is between $1+C(-1)^i\sigma/2$ and $1+2C(-1)^i\sigma$, when (u_L,u_R) is a i-shock. It shows that the variation of the a function has to be negative for a 1-shock, positive for a 3-shock, and can be chosen with strength of the same order as the size of the shock. The estimates on $\dot{h}$ show that we keep a finite speed of propagation, and that a shift of a 1-shock cannot overtake a 2-contact or the shift of a 3-shock if it started on its left. This is true also for its symmetric case. Recall the speed of a 2-contact is always zero, and there is no shift on contact discontinuity.

This is important because when we introduce shifts into the solution to a Riemann problem with two shocks, both shock speeds move with artificial velocities. We need to ensure that the positions of the shocks do not touch at some time after the initial time to preserve the property of classical solutions to the Riemann problem, where shocks born from a solution to a Riemann problem will never touch. This is also true when there is one shock and one contact, or two shocks and one contact in the solution of Riemann problem.

We need a similar control for approximations of rarefactions via the front tracking method. We begin to show that, by Proposition 2.1(b, d), the real rarefaction has a contraction property without the need of shift.

Lemma 3.1. *Consider the entropy solution of* (1.7)-(1.10). *Let $\bar{u}(y),v_L\le y\le v_R$, be a rarefaction wave for* (1.7). *Then for any $u\in\mathcal{S}_{weak}$ and every $t>0$ we have*

$$\begin{aligned}&\frac{d}{dt}\int_{v_Lt}^{v_Rt}\eta\left(u(t,x)|\bar{u}\left(\frac{x}{t}\right)\right)dx\\ &\le q\big(u(t,v_Lt+);\bar{u}(v_L)\big)-q\big(u(t,v_Rt-)|\bar{u}(v_R)\big)\\ &\qquad -v_L\eta\big(u(t,v_Lt+)|\bar{u}(v_L)\big)+v_R\eta\big(u(t,v_Rt-)|\bar{u}(v_R)\big).\end{aligned}$$

Proposition 3.2. *Consider the entropy solution of* (1.7)-(1.10). *There exists a constant $C>0$ such that the following is true:*

For any $\bar{u}(y),v_L\le y\le v_R$, rarefaction wave for (1.7), *denote*

$$\delta=|v_L-v_R|+\sup_{y\in[v_L,v_R]}|u_L-\bar{u}(y)|,\quad \bar{u}(v_L)=u_L,\quad \bar{u}(v_R)=u_R.$$

Then for any $u\in\mathcal{S}_{weak}$, any $v_L\le v\le v_R$, and any $t>0$ we have

$$\begin{aligned}\int_0^t &\Big\{q\big(u(t,tv+);u_R\big)-q\big(u(t,tv-);u_L\big)\\&\qquad -v\big(\eta\big(u(t,tv+)|u_R\big)-\eta\big(u(t,tv-)|u_L\big)\big)\Big\}dt\\&\le C\delta|u_L-u_R|t.\end{aligned}$$

The proof of these results are as same as those in [13]. Finally, we provide a dissipation result for a contact discontinuity, proved by [32].

Lemma 3.2. *Consider the entropy solution of* (1.7)-(1.10)*. The contact discontinuity* (u_L,u_R) *at* $x=\beta$ *is uniformly stable in the sense that, for any* $u\in\mathcal{S}_{weak}$ *and every* $t>0$ *we have*

$$H(t):=\frac{d}{dt}\left\{\int_{h_1(t)}^{\beta}\theta_L\eta(u|u_L)dx+\int_{\beta}^{h_2(t)}\theta_R\eta(u|u_R)dx\right\}\le F^+-F^-,$$

where

$$\begin{aligned}F^+&=\theta_L\big[q\big(u\big(R,h_1(R)+\big);u_L\big)-\dot{h}_1(R)\eta\big(u\big(R,h_1(R)+\big)|u_L\big)\big],\\F^-&=\theta_R\big[q\big(u\big(R,h_2(R)-\big);u_R\big)-\dot{h}_2(R)\eta\big(u\big(R,h_2(R)-\big)|u_R\big)\big],\end{aligned}$$

and $h_1(t)$*,* $h_2(t)$ *are Lipschitz curves.*

Proof. First, recall that $\eta=-S,q=0$ and the Gibbs relation (1.5). Also by (1.7) and (1.2), $f=(-w,p,wp)$. By (1.10), we have $-\partial_t S(u|u_R)+\partial_x q(u;u_R)\le 0$, where $q(a;b)=dS(b)\cdot(f(a)-f(b))$. We deduce that

$$H(t)\le\theta_L dS_L\cdot\big(f_- -f(u_L)\big)-\theta_R dS_R\cdot\big(f_+-f(u_R)\big)+F^+-F^-,$$

where $f_\pm$ denote the right/left traces of $f(u)$ along $x=\beta$. The conservation laws tells us, not only that these traces are well defined as bounded measurable functions, but also that they coincide. As we know on a contact discontinuity $w_R=w_L$, $p_R=p_L$ and $f(u_R)=f(u_L)$, hence $\theta_L dS_L=\theta_R dS_R$. Finally, we get

$$H(t)\le F^+-F^-.$$

The proof is complete. □

In this lemma, the weights on two sides of a contact discontinuity must have an exact ratio θ_R/θ_L. Later, we will define the weight function $a(t,x)$ very carefully in order to keep this ratio. There will be no shift on the contact discontinuity and rarefaction wave.

4 The weight function $a(t,x)$

In the proof of Proposition 2.2, the function ψ will be defined through a modification of the front tracking algorithm, very similar to the one used in [13] for 2×2 system. For completeness, we include a brief description of the modified front tracking algorithm in the appendix.

We always denote the strength of a wave as $|\sigma|$. In particular for 2-wave, we define

$$\sigma=\frac{1}{CJ}\Delta\theta, \tag{4.1}$$

where the constant C is defined in Proposition 3.1 and

$$J=\frac{1}{2}\inf_{\mathcal{O}}\theta>0, \tag{4.2}$$

where the temperature θ defined in (1.3) always has a positive lower bound in $\mathcal{V}$. For 1-wave, 3-wave, and non-physical shock, the choice of wave strength does not give us any convenience. For simplicity, we just choose $\sigma=|\Delta u|$ for non-physical shock and $|\sigma|=|\Delta u_1|=|\Delta\tau|$ for 1-wave and 3-wave, respectively, where $\Delta f=f_+-f_-$ measures the difference of f on two sides of waves. Here, in the first and third family, σ takes positive sign on a shock and negative value on a rarefaction front. Note this choice of wave strength is only for convenience. One can choose other wave strengths.

Now we consider any modified front tracking approximate solutions ψ_ν defined in the appendix. Recall, $L(t)$ defined in (A.11) is the total variation of ψ_ν, and $Q(t)$ defined in (A.12) is the interaction potential. Clearly, $L(t)$ and $Q(t)$ stay constant along time intervals between consecutive collisions of fronts and changes only across points of wave interaction.

For any pairwise interaction considered in the accurate and simplified solver in the modified front tracking scheme, one has the following estimates (see [3]).

Proposition 4.1. *First consider the accurate solver. Call σ',σ'' the strengths of two interacting wave-fronts, and let $\sigma_1,\sigma_2,\sigma_3$ be the strengths of the outgoing waves of the first, second and third families, respectively.*

- *If wave with strength σ' is an i-wave and wave with strength σ'' is a j-wave with $i\neq j$, then*

$$|\sigma_i-\sigma'|+|\sigma_j-\sigma''|+\sum_{k\neq i,j}|\sigma_k|\leq C_0|\sigma'\sigma''|. \tag{4.3}$$

- *If both waves with strengths σ' and σ'' belong to the i-th family, then*

$$|\sigma_i-(\sigma'+\sigma'')|+\sum_{i\neq j}|\sigma_j|\leq C_0|\sigma'\sigma''|(|\sigma'|+|\sigma''|). \tag{4.4}$$

Then for simplified solver,

- *If both two waves with strengths σ' and σ'' are not pseudoshock, while there is an outgoing pseudoshock, then the strength of the pseudoshock is bounded by $C_0|\sigma'\sigma''|$.*
- *If one incoming wave with strength σ' is a pseudoshock, while another wave with strength σ'' is not a pseudoshock, then the wave strength difference of incoming and outgoing pseudoshocks is bounded by $C_0|\sigma'\sigma''|$.*

Here we can always choose ε small enough, especially smaller than the ε of Proposition 3.1, and such that

$$C_0\varepsilon\leq 1. \tag{4.5}$$

At any time $t\geq 0$, the weight function $a(t,\cdot)$ is defined as follows. As shown in Fig. 1, $a(t,\cdot)$ consists of n constant states $a^{(1)}$ to $a^{(n)}$ from left to right. Assume x_0 is a point of interaction, where there may exist more than one outgoing wave. $a^{(i)}$ and $a^{(i+1)}$ are the left and right values on two sides of x_0.

Define

$$a^{(1)}(t)=1+C\big(L(t)+\kappa Q(t)\big), \tag{4.6}$$

where C is the positive constant defined in Proposition 3.1. We first show that when ε is small enough,

$$0<1+2CU<\frac{1}{J}\inf_{\mathcal{O}}\theta, \tag{4.7}$$

where U is a uniform constant upper bound of

$$L(t)+\kappa Q(t).$$

In fact, by (A.13), we know that $U\leq L(0)+\kappa Q(0)$. Here, by the definition of J in (4.2), we know $\inf_{\mathcal{O}}\theta/J>2$, so when ε is sufficiently small, U is sufficiently small then (4.7) holds. Then inductively, we define

$$a^{(i+1)}(t)=a^{(i)}(t)+C\left(-\big(\sigma_1^{(i)}\big)_+ +\big(\sigma_3^{(i)}\big)_+ +\frac{J\hat{a}}{\hat{\theta}}\sigma_2^{(i)}\right), \tag{4.8}$$

where $\hat{\theta}$ is the value of θ between outgoing 1 and 2 waves, as shown in Fig. 1,

$$\hat{a}=a^{(i)}(t)-C\big(\sigma_1^{(i)}\big)_+.$$

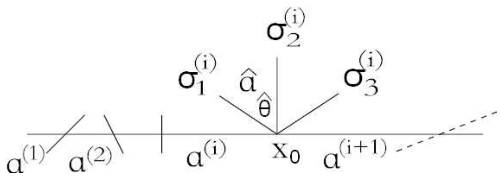

Figure 1: How to define the piecewise constant function $a(t,\cdot)$ at a time $t\geq 0$?

Here the strengths $\sigma_1^{(i)},\sigma_2^{(i)},\sigma_3^{(i)}$ of outgoing waves at the point of interaction might all be nonzero, see the interaction point x_0 in Fig. 1. While when there is no wave interaction, at most one of $\sigma_1^{(i)},\sigma_2^{(i)},\sigma_3^{(i)}$ is nonzero. Here rarefaction and pseudoshock do not impact the value of function a.

For convenience, we introduce the following measure $\mu(t,\cdot)$ as a sum of Dirac measures in x:

$$\begin{aligned}\mu(t,x)&=-\sum_{i:1-\text{shock}}\sigma^{(i)}\delta_{\{x_i(t)\}}+\sum_{i:3-\text{shock}}\sigma^{(i)}\delta_{\{x_i(t)\}}+\sum_{i:2-\text{wave}}\frac{J\hat{a}}{\hat{\theta}}\sigma^{(i)}\delta_{\{x_i(t)\}}\\&=-\sum_{i:1-\text{wave}}(\sigma^{(i)})_+\delta_{\{x_i(t)\}}+\sum_{i:3-\text{wave}}(\sigma^{(i)})_+\delta_{\{x_i(t)\}}+\sum_{i:2-\text{wave}}\frac{J\hat{a}}{\hat{\theta}}\sigma^{(i)}\delta_{\{x_i(t)\}},\end{aligned}$$

where $\hat{a}$ and $\hat{\theta}$ denote the a and θ values to the left of the contact discontinuity, respectively, using Fig. 1.

In a summary,

$$a(t)=1+C\left(L(t)+\kappa Q(t)+\int_{-\infty}^{x}\mu(t,x)dx\right),$$

where the constant C is defined in Proposition 3.1. Note that the function a is piecewise constant, with discontinuities only along shock curves or contact discontinuities. In particular it is constant across rarefaction curves and pseudoshock curves. We show that the function a has the following properties.

Proposition 4.2. *There exists $C_1>0$ such that for every $\varepsilon>0$ small enough,*

$$1\leq a(t,x)\leq 1+C_1\varepsilon. \tag{4.9}$$

For every time without wave interaction, and for every x such that a 1-shock σ_i is located at $x=x_i(t)$

$$1-2C|\sigma_i|\leq\frac{a(t,x_i(t)+)}{a(t,x_i(t)-)}\leq 1-\frac{1}{2}C|\sigma_i|. \tag{4.10}$$

For every time without wave interaction, and for every x such that a 3-shock σ_i is located at $x=x_i(t)$

$$1+\frac{1}{2}C|\sigma_i|\le\frac{a\big(t,x_i(t)+\big)}{a\big(t,x_i(t)-\big)}\le 1+2C|\sigma_i|, \tag{4.11}$$

where C is the positive constant defined in Proposition 3.1.

For every time without wave interaction, and for every x_0 such that a 2-contact σ_i is located at $x=x_0$

$$\frac{a(t,x_0+)}{a(t,x_0-)}=\frac{\theta(t,x_0+)}{\theta(t,x_0-)}, \tag{4.12}$$

on two sides of the contact.

For every time t with a wave interaction, and almost every x

$$a(t+,x)\le a(t-,x). \tag{4.13}$$

Proof. **Step 1.** We notice that, if we can assume that $J\hat{a}/\hat{\theta}\le 1$ in (4.8), then by (4.7)

$$1\le a^{(i)}(t)\le 1+C\big(2L(t)+\kappa Q(t)\big)\le 1+2CU\le\frac{1}{J}\inf_{\mathcal{O}}\theta$$

So using $\hat{a}\le a^{(i)}(t)$, and by induction, we know that

$$1\le a^{(i)}(t)\le 1+CU<\frac{\inf_{\mathcal{O}}\theta}{J}$$

for any i, then

$$1\le a\le 1+CU.$$

By (A.13), $U<4\varepsilon$ when ε is sufficiently small, so we can find a $C_1=4C$ such that (4.9) is satisfied for any small ε. Note, we always have

$$\frac{J\hat{a}}{\hat{\theta}}\le 1. \tag{4.14}$$

So for ε small enough, $1/2<1/a(t,x_i(t)-)<2$. Now

$$\begin{aligned}\frac{a\big(t,x_i(t)+\big)}{a\big(t,x_i(t)-\big)}-1&=\frac{1}{a\big(t,x_i(t)-\big)}\Big(a\big(t,x_i(t)+\big)-a\big(t,x_i(t)-\big)\Big)\\&=\frac{1}{a\big(t,x_i(t)-\big)}C|\sigma_i|\alpha\end{aligned}$$

with $\alpha = 1$ if the shock σ_i is a 3-shock, and $\alpha = -1$ if it is a 1-shock. This shows (4.10) and (4.11).

For any 2-wave, $CJ\sigma = \Delta\theta$, using (4.1). So, by (4.8), it is easy to get (4.12).

Step 2. Finally, we come to prove the time decay of a. Consider a time t with a wave interaction. From the definition of the a function,

$$\begin{aligned}&\sup_{\mathbb{R}} \big(a(t+,x) - a(t-,x)\big) \\ \le\, & C\left(\int_{\mathbb{R}} |\mu(t+) - \mu(t-)|\,dx + \big(\Delta L(t) + \kappa \Delta Q(t)\big)\right).\end{aligned} \tag{4.15}$$

Still assume that waves interact at $x = x_0$. The interacting wave fronts are $\sigma'\sigma''$ leading to outgoing physical waves $\sigma_1, \sigma_2, \sigma_3$ in three families, and a possible non-physical wave denoted by σ_u. We study $\mu(t+) - \mu(t-)$ by considering separately all the possible kind of interactions.

(i). We first consider interactions including pseudoshocks. For convenience, we just call them simplified interactions. Recall, σ_i for the i-th outgoing wave in the simplified solver is equal to the sum of σ for all incoming waves in the i-th family. We then consider simplified interactions including a 2-wave. Since all 2-waves have zero speed, any two 2-waves will never interaction with each other. So we only have to consider two cases given in Fig. 2.

First, we consider the left picture of Fig. 2. To compare $\mu(t+)$ and $\mu(t-)$, we notice that

$$\frac{\hat{a}}{\hat{\theta}}\sigma_2 - \frac{a_-}{\theta_-}\sigma' = O(\sigma'\sigma''),$$

where we use $\sigma_2 = \sigma'$ in the simplified solver and

$$\left|\frac{\hat{a}}{\hat{\theta}} - \frac{a_-}{\theta_-}\right| = O(\sigma'')$$

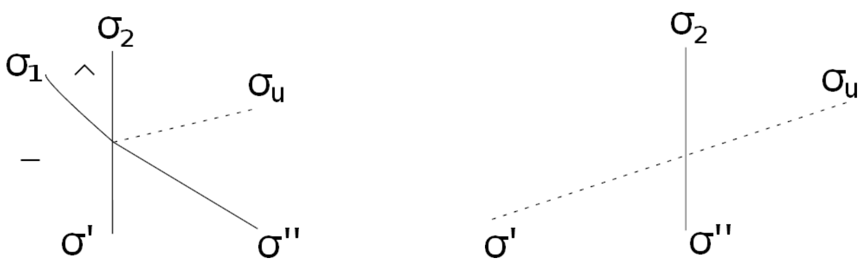

Figure 2: Two pairwise interaction including pseudoshock (dash lines).

by Proposition 4.1. So

$$
\begin{aligned}
&|\mu(t+)-\mu(t-)| \leq O(\sigma'\sigma''),\\
&|\Delta L| = |\sigma_u| \leq O(\sigma'\sigma''),
\end{aligned}
$$

which can be bounded by $-\Delta Q$ when κ is sufficiently large. So

$$a(t+,x)-a(t-,x)\leq 0.$$

For the right picture of Fig. 2, we can show $a(t+,x)-a(t-,x)\leq 0$ similarly, so do all simplified interactions including a contact discontinuity. For simplified interactions without contact discontinuity, we always have

$$\int_{-\infty}^{x}\big(\mu(t+)-\mu(t-)\big)\leq 0,$$

since the total strength of 1-shock or 3-shock does not increase after interaction. So $a(t+,x)-a(t-,x)\leq 0$ in this case.

It remains to consider the cases involving the accurate solver. They correspond to the three cases in Fig. 3. The second and third cases have symmetric cases of 2-wave and 3-wave, and 3-wave and 3-wave interactions, respectively, which can be treated symmetrically.

We consider separately the three cases corresponding to Proposition 4.1.

(ii). If σ'' is a 1-wave and σ' is a 3-wave. Using the definition of μ to justify the first equality below, the fact that $y\to(y)_+$ is Lipschitz with constant 1 and (4.14) for the second inequality, (4.3) for the third inequality, and for ε small enough to get the last inequality, we have

$$
\begin{aligned}
|\mu(t+)-\mu(t-)| &= \delta_{\{x_0\}}\left|(\sigma_3)_+-(\sigma_1)_+-\big((\sigma')_+-(\sigma'')_+\big)+\frac{\hat{a}}{\hat{\theta}}\sigma_2\right|\\
&\leq \delta_{\{x_0\}}\big(|(\sigma_3)_+-(\sigma')_+|+|(\sigma_1)_+-(\sigma'')_+|+|\sigma_2|\big)\\
&\leq \delta_{\{x_0\}}\big(|\sigma_3-\sigma'|+|\sigma_1-\sigma''|+|\sigma_2|\big)\\
&\leq \delta_{\{x_0\}}C_0|\sigma'\sigma''|\\
&\leq -\big(\Delta L(t)+\kappa\Delta Q(t)\big)\,\delta_{\{x_0\}},
\end{aligned}
$$

when we choose κ sufficiently large. By using (4.15) and the fact that $\Delta L(t)+\kappa\Delta Q(t)<0$, gives

$$\sup_{\mathbb{R}}\big(a(t+,x)-a(t-,x)\big)\leq 0.$$

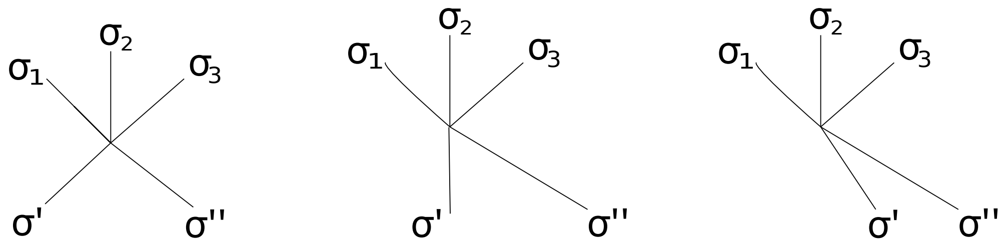

Figure 3: Pairwise interactions using accurate solver. Each line segment represents a shock or a rarefaction jump or a contact disconuity.

(iii). If σ'' is a 1-wave and σ' is a 2-wave. Similar as the case in the left picture of Fig. 2, we can show

$$\frac{\hat{a}}{\hat{\theta}}\sigma_2-\frac{a_-}{\theta_-}\sigma'=O(\sigma'\sigma''),$$

where we also use Proposition 4.1. Then following the similar steps as in case (ii), we can prove that

$$\sup_{\mathbb{R}}\big(a(t+,x)-a(t-,x)\big)\le 0,$$

when we choose κ sufficiently large.

(iv). Both σ' and σ'' are 1-waves. In this case,

$$\begin{aligned}|\mu(t+)-\mu(t-)|&=\delta_{\{x_0\}}\left|(\sigma_3)_+-(\sigma_1)_+-\big(-(\sigma')_+-(\sigma'')_+\big)+\frac{\hat{a}}{\hat{\theta}}\sigma_2\right|\\&\le\delta_{\{x_0\}}\big(|(\sigma_3)_+|+|(\sigma_1)_+-(\sigma')_+-(\sigma'')_+|+|\sigma_2|\big)\\&\le\delta_{\{x_0\}}\big(|(\sigma_3)|+|(\sigma_1)_+-(\sigma')_+-(\sigma'')_+|+|\sigma_2|\big).\end{aligned}$$

We need to separate cases depending on the nature of the incoming waves.

(iv)-1. If σ' and σ'' are rarefactions, there is no interaction.

(iv)-2. If σ' and σ'' are shocks, then σ_1 is also a shock, and

$$|(\sigma_1)_+-(\sigma')_+-(\sigma'')_+|=|\sigma_1-\sigma'-\sigma''|.$$

Since the outgoing 1 and 3 waves are both reflected waves which are in the second order,

$$|\mu(t+)-\mu(t-)|\le\delta_{\{x_0\}}C_0|\sigma'\sigma''|\le-\big(\Delta L(t)+\kappa\Delta Q(t)\big)\delta_{\{x_0\}}.$$

(iv)-3. Finally, if one of σ' and σ'' is a shock, let say σ', and the other a rarefaction, let say σ''. Then

$$\begin{aligned}\left|(\sigma_1)_+-(\sigma')_+-(\sigma'')_+\right|=\left|(\sigma_1)_+-\sigma'\right|&\le\left|\sigma''-(\sigma_1)_-\right|+\left|\sigma_1-\sigma'-\sigma''\right|\\&\le-\big(\Delta L(t)+\kappa\Delta Q(t)\big),\end{aligned}$$

since $|\sigma''-(\sigma_1)_-|\le-\Delta L(t)+C_0\varepsilon|\sigma'||\sigma''|$ with sufficiently large κ.

When σ' is a rarefaction and σ'' a shock, we can show the desired result similarly. Gathering all the cases, we obtain

$$\sup_{\mathbb{R}}\big(a(t+,x)-a(t-,x)\big)\le 0.$$

The proof is complete. □

5 Proof of Proposition 2.2

This section is devoted to the proof of Proposition 2.2.

The relation in (4.13) verifies that the ratio of weights on two sides of any contact discontinuity is θ_R/θ_L, which satisfies the requirement in Lemma 3.2. With the help of Lemma 3.2 for contact discontinuities, one can now prove Proposition 2.2 in a very similar way as for the system of two variables in [13], after proving properties of weight function $a(t,x)$ in Proposition 4.1.

In our front-tracking procedure, we stop and restart the clock every time there is a collision between waves (when the waves initiated from distinct Riemann problems). Weak solutions u to (1.7) naturally lie in $C^0(\mathbb{R}^+;W^{-1,\infty}(\mathbb{R}))$. Note that the formulation of the entropy inequality (1.12) holds with a boundary term for $t=0$, and this classically implies that u is continuous in time at $t=0$ with values in $L^1_{\text{loc}}(\mathbb{R})$. Because $L^1_{\text{loc}}(\mathbb{R})$ is a strong topology, it implies that $\eta(u)$ is also continuous at $t=0$ in the same topology in x. However, because $\eta(u)$ verifies only inequality (1.10), $\eta(u)$ does not share this regularity in time for $t>0$. Therefore, $\eta(u)$ is well defined only almost everywhere in time. However, this technicality of stopping and restarting the clock at any time t is not a real issue, and its resolution can be formalized with the use of approximate limits as follows. For a reference on approximate limits, see [18, pp. 55–57].

Lemma 5.1 ([26, Lemma 2.5], Stop and Restart the Clock).
Let $u\in L^\infty(\mathbb{R}^+\times\mathbb{R})$ be a weak solution to (1.7) *with initial data u^0. Further, assume that u is entropic for the entropy η, i.e. verifies* (1.10) *in the sense of distribution. Assume*

also that u verifies the strong trace property (1.1). *Then for all $\tilde{u}\in\mathcal{V}_0$, and for all $c,d\in\mathbb{R}$ with $c<d$, the following approximate right- and left-hand limits:*

$$\operatorname{ap}\lim_{t\to t_0^{\pm}}\int_c^d \eta\big(u(t,x)|\tilde{u}\big)\,dx \tag{5.1}$$

exist for all $t_0\in(0,\infty)$ and verify

$$\operatorname{ap}\lim_{t\to t_0^{-}}\int_c^d \eta\big(u(t,x)|\tilde{u}\big)\,dx\geq \operatorname{ap}\lim_{t\to t_0^{+}}\int_c^d \eta\big(u(t,x)|\tilde{u}\big)\,dx. \tag{5.2}$$

Furthermore, the approximate right-hand limit exists at $t_0=0$ and verifies

$$\int_c^d \eta\big(u^0(x)|\tilde{u}\big)\,dx\geq \operatorname{ap}\lim_{t\to t_0^{+}}\int_c^d \eta\big(u(t,x)|\tilde{u}\big)\,dx. \tag{5.3}$$

The proof of Lemma 5.1 follows exactly the proof of [26, Lemma 2.5]. For this reason, we do not include a proof here.

We gather in the following lemma useful simple properties of the relative quantities.

Lemma 5.2. *For any $\mathcal{O}$ open subset of $\mathcal{V}$ with $\overline{\mathcal{O}}\subset\mathcal{V}$, there exists a constant $C>0$ such that*

$$\begin{aligned}
&|q(a;b)|\leq C\eta(a|b), && \forall (a,b)\in\mathcal{V}_0\times\overline{\mathcal{O}},\\
&|q(a;b_1)-q(a;b_2)|\leq C|b_1-b_2|, && \forall (b_1,b_2)\in\overline{\mathcal{O}}^2,\quad a\in\mathcal{V},\\
&|\eta(a|b_1)-\eta(a|b_2)|\leq C|b_1-b_2|, && \forall (b_1,b_2)\in\overline{\mathcal{O}}^2,\quad a\in\mathcal{V}.
\end{aligned}$$

We now prove Proposition 2.2. First we fix the value $\tilde{u}$ to be bigger than both $\hat{\lambda}$ and the constant C of Lemma 5.2. Take $0<\varepsilon<1/2$ small enough such that Theorem A.1, Proposition 3.1, and Proposition A.1 hold true. For any initial value u^0, and wild solution $u\in\mathcal{S}_{weak}$, we consider the family of solutions ψ_ν of the modified front tracking method. We want now to choose a particular one. Fix $T,R>0$, and $p\in\mathbb{N}$. First we insure that the initial value verifies

$$\|u^0-\psi_\nu(0,\cdot)\|_{L^2(-R,R)}\leq\frac{1}{p}.$$

This fixes N_ν. Then we fix $\delta_\nu = 1/(pT)$. Thanks to Lemma A.2, we can choose ε_ν such that

$$\sup_{r\in[0,T]}\sum_{i\in\mathcal{P}(r)}|\sigma_i| \le \frac{1}{pT}.$$

We denote by ψ the associated solution to the modified front tracking method ψ_ν. Especially, it verifies

$$\|u(0,\cdot)-\psi(0,\cdot)\|_{L^2(-R,R)} \le \|u^0-u(0,\cdot)\|_{L^2(-R,R)}+\frac{1}{p}, \tag{5.4}$$

$$T\delta_\nu \sup_{t\in[0,T]} L(t) \le \frac{1}{p}, \tag{5.5}$$

$$T\sup_{r\in[0,T]}\sum_{i\in\mathcal{P}(r)}|\sigma_i| \le \frac{1}{p}. \tag{5.6}$$

Proposition A.1 provides three of the four properties of Proposition 2.2. It remains only to show the control in L^2 of $\psi(t,\cdot)-u(t,\cdot)$. Recall that as in Appendix A, for every time $r>0$, we denote by $\mathcal{P}(r)$ the set of i corresponding to non-physical waves.

Consider two successive interaction times $t_j<t_{j+1}$ of the front tracking solution ψ. Let the curves of discontinuity between the two times $t_j<t_{j+1}$ be $h_1,\ldots,h_N$ for $N\in\mathbb{N}$ such that

$$h_1(t)<\cdots<h_N(t) \tag{5.7}$$

for all $t\in(t_j,t_{j+1})$. We only work on the cone of information, so we define for all times t,

$$h_0(t) = -R+vt, \tag{5.8}$$

$$h_{N+1} = R-vt. \tag{5.9}$$

Note that there are no interactions between wave fronts in ψ and the cone of information (coming from h_0 and h_{N+1}). For any $t\in[t_j,t_{j+1}]$, note that on

$$\mathcal{Q}=\{(r,x): t_j<r<t, h_i(r)<x<h_{i+1}(r)\},$$

the function $\psi(r,x)=b$ is constant. Moreover, by construction, the weight function $a(r,x)$ is also constant on this set. Therefore, integrating (2.3) on $\mathcal{Q}$, and using the strong trace property of Definition 1.1, we find

$$\operatorname{ap}\lim_{s\to t^-}\int_{h_i(t)}^{h_{i+1}(t)} a(t-,x)\eta\big(u(s,x)|\psi(t,x)\big)\,dx$$

$$\leq \operatorname{ap}\lim_{s\to t_j{}^+} \int_{h_i(t_j)}^{h_{i+1}(t_j)} a(t_j+,x)\eta\big(u(s,x)|\psi(x,t_j)\big)\,dx + \int_{t_j}^{t} \big(F_i^+(r)-F_{i+1}^-(r)\big)\,dr,$$

where

$$\begin{aligned} F_i^+(r) = a\big(r,h_i(r)+\big)\big[&q\big(u\big(r,h_i(r)+\big);\psi\big(r,h_i(r)+\big)\big) \\ &-\dot{h}_i(r)\eta\big(u\big(r,h_i(r)+\big)|\psi\big(r,h_i(r)+\big)\big)\big], \\ F_i^-(r) = a\big(r,h_i(r)-\big)\big[&q\big(u\big(r,h_i(r)-\big);\psi\big(r,h_i(r)-\big)\big) \\ &-\dot{h}_i(r)\eta\big(u\big(r,h_i(r)-\big)|\psi\big(r,h_i(r)-\big)\big)\big]. \end{aligned}$$

We sum in i, and combine the terms corresponding to i into one sum, and the terms corresponding to $i+1$ into another sum, to find

$$\begin{aligned} &\operatorname{ap}\lim_{s\to t^-} \int_{-R+vt}^{R-vt} a(t-,x)\eta\big(u(s,x)|\psi(t,x)\big)\,dx \\ \leq &\operatorname{ap}\lim_{s\to t_j{}^+} \int_{-R+vt_j}^{R-vt_j} a(t_j+,x)\eta\big(u(s,x)|\psi(t_j,x)\big)\,dx + \sum_{i=1}^{N}\int_{t_j}^{t}\big(F_i^+(r)-F_i^-(r)\big)\,dr, \end{aligned}$$

where we have used that $F_0^+ \leq 0$ and $F_{N+1}^- \geq 0$ thanks to the first statement of Lemma 5.2, the definition of v, and the fact that $\dot{h}_0 = -v = -\dot{h}_{N+1}$.

We decompose the sum into three sums, one corresponding to the 1 and 3 shock fronts, one for the rarefaction fronts, and one for the pseudoshocks. Here by Lemma 3.2 and (4.12), we know that contact discontinuities will not contribute to the sum of F_i^+ and F_i^-. Then the following estimates for shocks and pseudoshocks are very similar as in [13]. We include the proof to make this paper self-contained.

Thanks to Propositions 3.1 and 4.2, for any i corresponding to a shock front

$$F_i^+(r)-F_i^-(r)\leq 0 \quad \text{for almost every} \quad t_j<r<t.$$

Denote $\mathcal{R}$ the set of i corresponding to approximated rarefaction fronts. Then for any $i\in\mathcal{R}$ by construction, $a(h_i(r)+,r)=a(h_i(r)-,r)$. And from Proposition 3.2, and (5.5)

$$\sum_{i\in\mathcal{R}}\int_{t_j}^{t}\big(F_i^+(r)-F_i^-(r)\big)\,dr \leq C\delta_\nu(t-t_j)\sum_{i\in\mathcal{R}}|\sigma_i| \leq C\delta_\nu(t-t_j)L(t) \leq \frac{C}{pT}(t-t_j).$$

Consider now the case when $i\in\mathcal{P}(r)$. Recall that pseudoshocks travel with supersonic (greater-than-characteristic) speed $\hat{\lambda}$. Thus, we must have that for almost every time r: $u(r,h_i(r)+)=u(r,h_i(r)-)$. This is because if $u(r,h_i(r)+)\neq u(r,h_i(r)-)$, then the shock $\big(u(r,h_i(r)+),u(r,h_i(r)-),\hat{\lambda}\big)$ would be traveling with speed greater than any of the eigenvalues of Df, a contradiction. By construction of the a function, we know that a does not have a jump across pseudoshocks, so we have also $a(r,h_i(r)+)=a(r,h_i(r)-)$. Therefore, thanks to the second and third estimates of Lemma 5.2,

$$F_i^+(r)-F_i^-(r)\leq C\big|\psi\big(r,h_i(r)+\big)-\psi\big(r,h_i(r)-\big)\big|=C|\sigma_i|.$$

Then, from (5.6) we receive

$$\sum_{i\in\mathcal{P}(r)}\int_{t_j}^t\big(F_i^+(r)-F_i^-(r)\big)\,dr\leq C(t-t_j)\sum_{i\in\mathcal{P}(t)}|\sigma_i|\leq\frac{C(t-t_j)}{pT}.$$

Gathering all the families of waves, we find

$$\begin{aligned}&\operatorname{ap}\lim_{s\to t^-}\int_{-R+vt}^{R-vt}a(t-,x)\eta\big(u(s,x)|\psi(t,x)\big)\,dx\\
\leq&\operatorname{ap}\lim_{s\to t_j{}^+}\int_{-R+vt_j}^{R-vt_j}a(t_j+,x)\eta\big(u(s,x)|\psi(t_j,x)\big)\,dx+\frac{C(t-t_j)}{pT}.\end{aligned}$$

Consider now any $0<t<T$ and denote $0<t_1<\cdots<t_J$ the times of wave interactions before $t, t_0=0$, and $t_{J+1}=t$. Using the convexity of η, Lemma 5.1, and (4.13) we find

$$\begin{aligned}&\int_{-R+vt}^{R-vt}a(t,x)\eta\big(u(t,x)|\psi(t,x)\big)\,dx-\int_{-R}^{R}a(0,x)\eta\big(u(0,x)|\psi(0,x)\big)\,dx\\
\leq&\operatorname{ap}\lim_{s\to t^+}\int_{-R+vt}^{R-vt}a(t,x)\eta\big(u(s,x)|\psi(t,x)\big)\,dx-\int_{-R}^{R}a(0,x)\eta\big(u(0,x)|\psi(0,x)\big)\,dx\\
\leq&\sum_{j=1}^{J+1}\Bigg(\operatorname{ap}\lim_{s\to t_j{}^+}\int_{-R+vt_j}^{R-vt_j}a(t_j-,x)\eta\big(u(s,x)|\psi(t,x)\big)\,dx\\
&\qquad-\operatorname{ap}\lim_{s\to t_{j-1}^+}\int_{-R+vt_{j-1}}^{R-vt_{j-1}}a(t_{j-1}-,x)\eta\big(u(s,x)|\psi(t,x)\big)\,dx\Bigg)\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{j=1}^{J+1}\Bigg(\text{ap}\lim_{s\to t_j^-}\int_{-R+vt_j}^{R-vt_j} a(t_j-,x)\eta\big(u(s,x)|\psi(t,x)\big)\,dx \\
&\qquad -\text{ap}\lim_{s\to t_{j-1}^+}\int_{-R+vt_{j-1}}^{R-vt_{j-1}} a(t_{j-1}-,x)\eta\big(u(s,x)|\psi(t,x)\big)\,dx\Bigg) \\
&\le \sum_{j=1}^{J+1}\Bigg(\text{ap}\lim_{s\to t_j^-}\int_{-R+vt_j}^{R-vt_j} a(t_j-,x)\eta\big(u(s,x)|\psi(t,x)\big)\,dx \\
&\qquad -\text{ap}\lim_{s\to t_{j-1}^+}\int_{-R+vt_{j-1}}^{R-vt_{j-1}} a(t_{j-1}+,x)\eta\big(u(s,x)|\psi(t,x)\big)\,dx\Bigg) \\
&\le \sum_{j=1}^{J+1} C\frac{(t_j-t_{j-1})}{Tp} \le C\frac{t}{Tp}\le \frac{C}{p}.
\end{aligned}
$$

Using that $|a-1|<1/2$ and (5.4), we get that for every $0<t<T$,

$$
\int_{-R+vt}^{R-vt} \eta\big(u(t,x)|\psi(t,x)\big)\,dx \le 2\|u^0-u(0,\cdot)\|^2_{L^2(-R,R)}+\frac{C}{p}.
$$

Choosing p big enough such that $C/p<1/m$ gives the result.

6 Proof of Theorem 1.1

For each $d\in\overline{\mathcal{O}}$, consider $\varepsilon_d>0$ such that both Proposition 2.2 and Theorem A.1 are valid. The union (over d) of the balls $B_{\varepsilon_d/2}(d)$ cover the compact $\overline{\mathcal{O}}$, so there exists a finite subcover. Denote $\varepsilon>0$ the smallest of the $\varepsilon_{d_i}/2$ for this finite subcover.

By passing to a subsequence if necessary, we assume that $\|u_m^0-u^0\|_{L^2}\le 1/m$. From Proposition 2.2 we have a sequence of functions ψ_m (for all $m\in\mathbb{N}$), uniformly bounded in $L^\infty(\mathbb{R}^+,BV(\mathbb{R}))$. Moreover, ψ_m verify condition A.1 and (A.19) uniformly, and they verify for all time $t>0$

$$
\|\psi_m(t,\cdot)-u_m(t,\cdot)\|_{L^2(\mathbb{R})}\le\frac{2}{m}. \tag{6.1}
$$

From Lemma A.3, there exists $\psi\in L^\infty(\mathbb{R}^+\times\mathbb{R})$ verifying the bounded variation condition (A.2) such that for every $T>0,R>0,\psi_m$ converges in $C^0(0,T:L^2(-R,R))$

to ψ. Together with (6.1), u_m converges in $L^\infty(0,T:L^2(-R,R))$ to ψ. Since the convergence is strong and u_m verifies (1.7) and (1.10), the limit ψ is also solution to (1.7) and (1.10), with initial value u^0. From Theorem A.1 or directly using the recent uniqueness result in [5,7], it is the unique solution verifying Definition A.2.

Applying the result to the constant sequence $\overline{u}_n = u$, the fixed BV function with initial value u^0 from the hypotheses of the theorem, shows that u is also this unique solution. Therefore, $\psi = u$. This ends the proof of Theorem 1.1.

Appendix A. Modified front tracking algorithm

In the proof of Proposition 2.2, the function ψ will be defined through a modification of the front tracking algorithm, very similar to the one used in [13] for 2×2 system. For completeness, we include here a brief description of the modified front tracking algorithm. This appendix is largely from [13].

The bounded variation condition in [9] will be also introduced. And our modified front tracking solution verifies this condition.

For the construction of the ψ we are about to give, the modification to the front tracking algorithm (as presented in [1]) consists in changing the velocity of the shocks. The shocks move with an artificial velocity dictated by the shift functions of Proposition 3.1, instead of moving with the Rankine-Hugoniot speed.

Recall that given a Riemann problem with two constant states u_- and u_+ sufficiently close, a solution with at most three constant states, connected by either shocks or rarefaction fans, can always be found. More precisely, there exist C^2 curves $\sigma \mapsto T_i(\sigma)(u_-)$, $i=1,2,3$, parametrized by arclength such that

$$u_+ = T_3(\sigma_3)\circ T_2(\sigma_2)\circ T_1(\sigma_1)(u_-) \tag{A.1}$$

for some σ_1 and σ_2. We define $u_0 := u_-$ and

$$u_1 := T_1(\sigma_1)(u_0), \tag{A.2}$$

$$u_2 := T_2(\sigma_2)\circ T_1(\sigma_1)(u_0), \tag{A.3}$$

$$u_3 := T_3(\sigma_3)\circ T_2(\sigma_2)\circ T_1(\sigma_1)(u_0). \tag{A.4}$$

We use the convention that, when σ_i is positive (negative) the states u_{i-1} and u_i are separated by an i-shock (i-rarefaction) wave. Further, the strength of the i-wave is defined as $|\sigma_i|$.

For given initial data u^0, let ψ_ν^0 be a sequence of piecewise-constant functions approximating u^0 in L^2 on $(-R,R)$. (We will choose ν later such as to give us the required $\psi = \psi_\nu$.) Let N_ν be the number of discontinuities in the function ψ_ν and choose a parameter δ_ν controlling the maximum strength of the (approximate) rarefaction fronts.

We now introduce the two Riemann solvers. One will be used when the product of the strengths of the colliding waves is large, the other will be used when the product of the strengths is small or one of the incoming waves is non-physical (also known as a pseudoshock).

A.1 The Riemann solvers

The Riemann solvers will use non-physical waves (also known as pseudoshocks). These are waves connecting two states (let us call them u_- and u_+), and traveling with a fixed velocity $\hat{\lambda} > 0$ defined in Proposition 3.1. Therefore, it is greater than all characteristic speeds on $\mathcal{V}$ and greater than the speed of the shifts (which have a uniform bound on their speeds). We define this non-physical wave to have strength $|\sigma| := |u_- - u_+|$ and we say it belongs to the third wave family. Remark that since all non-physical waves travel with the same speed $\hat{\lambda}$, they cannot interact with each other.

Assume that at a positive time $\bar{t}$, there is an interaction at the point $\bar{x}$ between two waves of families i_α, i_β and strengths $\sigma'_\alpha, \sigma'_\beta$, respectively, with $1 \le i_\alpha, i_\beta \le 3$. Let σ'_α denote the left incoming wave. Let u_-, u_+ be the Riemann problem generated by the interaction, and let $\sigma_1, \sigma_2, \sigma_3$ and u_0, u_1, u_2, u_3 be defined as earlier. Finally, we can now define the accurate and simplified Riemann solvers.

(A) **Accurate solver**. If $\sigma_i < 0$, we let

$$p_i := \left\lceil \frac{\sigma_i}{\delta_\nu} \right\rceil, \tag{A.5}$$

where $\lceil s \rceil$ denotes the smallest integer number greater than s. For $l = 1, \ldots, p_i$ we define

$$u_{i,l} := T_i\left(\frac{l\sigma_i}{p_i}\right)(u_{i-1}), \quad x_{i,l}(t) := \bar{x} + (t - \bar{t})\lambda_i(u_{i,l}). \tag{A.6}$$

On the other hand, if $\sigma_i > 0$, we define $p_i := 1$ and

$$u_{i,l} := u_i, \quad x_{i,l}(t) := h_i(t). \tag{A.7}$$

Here, h_i is the shift function coming from Proposition 3.1. Within the context of Proposition 3.1, we take $u_L = u_{i-1}$ and $u_R = u_i$. Then, we define the approximate solution to the Riemann problem as follows:

$$v_a(t,x) := \begin{cases} u_-, & \text{if} \quad x < x_{1,1}(t), \\ u_+, & \text{if} \quad x > x_{3,p_3}(t), \\ u_i, & \text{if} \quad x_{i,p_i}(t) < x < x_{i+1,1}(t), \\ u_{i,l}, & \text{if} \quad x_{i,l}(t) < x < x_{i,l+1}(t), \quad l = 1,\dots,p_i - 1. \end{cases} \tag{A.8}$$

Note that thanks to the two last properties of Proposition 3.1, we have

$$x_{i,p_i}(t) < x_{i+1,1}(t), \quad \forall t > 0,$$

so the function is well defined.

(B) **Simplified solver**. For each $i = 1,2,3$ let σ_i'' be the sum of the strengths of the strengths of all incoming i-waves. Define

$$u' := T_3(\sigma_3'') \circ T_2(\sigma_2'') \circ T_1(\sigma_1'')(u_-). \tag{A.9}$$

Let $v_a(t,x)$ be the approximate solution of the Riemann problem (u_-, u') given by (A.8). Remark that in general $u' \neq u_+$ and thus we are introducing a non-physical front between these states. Hence, we define the simplified solution as follows:

$$v_s(t,x) := \begin{cases} v_a(t,x), & \text{if} \quad x - \bar{x} < \hat{\lambda}(t - \bar{t}), \\ u_+, & \text{if} \quad x - \bar{x} > \hat{\lambda}(t - \bar{t}). \end{cases} \tag{A.10}$$

A.2 Construction of approximate solutions

Given ν we construct the approximate solution $\psi_\nu(t,x)$ as follows. At time $t = 0$ all of the Riemann problems in ψ_ν^0 are solved accurately as in (A) (the accurate solver). By slightly perturbing the speed of a wave if necessary, we can ensure that at each time we have at most one collision, which will involve only two wavefronts. Suppose that at some time $t > 0$ there is a collision between two waves from the i_α-th and i_β-th families. Denote the strengths of the two waves by σ_α and σ_β, respectively. The Riemann problem generated by this interaction is solved as follows. Let ϵ_ν be a fixed small parameter which will be chosen later.

- If $|\sigma_\alpha \sigma_\beta| > \epsilon_\nu$ and the two waves are physical, then we use the accurate solver (A).
- If $|\sigma_\alpha \sigma_\beta| < \epsilon_\nu$ and the two waves are physical, or one wave is non-physical, then we use the simplified solver (B).

By the following lemma, for any ϵ_ν this algorithm will yield an approximate solution defined for all times $t>0$.

Lemma A.1 ([1, Lemma 2.1]). *The number of wavefronts in $\psi_\nu(t,x)$ is finite. Hence, the approximate solutions ψ_ν are defined for all $t>0$.*

This lemma is stated and proved in [1, Lemma 2.1] for piecewise constant front tracking solutions where shocks move according to Rankine-Hugoniot. We do not repeat the proof here, because using shifts in the front tracking algorithm (instead of Rankine-Hugoniot speeds) does not impact the proof. The proof is identical.

We introduce the total variation of ψ_ν as

$$L(t)=\sum|\sigma_i|=\mathrm{TV}(\psi_\nu)(t), \tag{A.11}$$

namely the sum of the strengths of all jump discontinuities that cross the t-time line, including all physical and non-physical fronts. Clearly, $L(t)$ stays constant along time intervals between consecutive collisions of fronts and changes only across points of wave interaction.

A j-wave and an i-wave, with the former crossing the t-time line to the left of the latter, are called approaching when either $i<j$, or $i=j$ and at least one of these waves is a shock, or two waves are approaching and one of them is a non-physical front. We recall then the definition of the potential for wave interactions

$$Q(t)=\sum_{i,j:\text{approaching waves}}|\sigma_i||\sigma_j|, \tag{A.12}$$

where the summation runs over all pairs of approaching waves, with strengths $|\sigma_i|$ and $|\sigma_j|$, which cross the t-line. Let us summarize some well known fact of the front tracking method which are still valid in our situation.

Proposition A.1. *There exists $\kappa>0$ such that for any ε small enough, the functional $L(t)+\kappa Q(t)$ is decreasing in time. Moreover, for any time t where waves with strength $|\sigma_i|$ and $|\sigma_j|$ interact the jump of Q at this time verifies*

$$\Delta L(t)+\kappa\Delta Q(t)\le-\frac{\kappa}{2}|\sigma_i||\sigma_j|. \tag{A.13}$$

Especially, there exists a constant $C>0$ such that for every $\nu>0,T>0$,

$$\|\psi_\nu\|_{L^\infty(0,T,BV(\mathbb{R}))}\le 2\varepsilon,$$

$$\|\psi_\nu(t,\cdot)-\psi_\nu(s,\cdot)\|_{L^1}\le C|t-s|,\quad 0<s<t<T,$$

the function ψ_ν verifies the Condition A.1 with constant C.

The proof of this proposition is classical. One can find the proof in [3] or [13]. For every time $r>0$, we denote by $\mathcal{P}(r)$ the set of i corresponding to non-physical waves. The following lemma is unchanged from [1, Lemma 3.1].

Lemma A.2 ([1, Lemma 3.1]). *If*

$$\lim_{\nu\to\infty}\epsilon_\nu\left(N_\nu+\frac{1}{\delta_\nu}\right)^k=0 \tag{A.14}$$

for every positive integer k, then the total strength of non-physical waves in ψ_ν goes to zero uniformly in t as $\nu\to\infty$

$$\sup_{r\in[0,T]}\sum_{i\in\mathcal{P}(r)}|\sigma_i| \;\to\; 0, \quad when \quad \nu \;\to\; 0.$$

Finally, following [3], we introduce the notion of space-like curve.

Definition A.1 (Space-Like Curves). *Let $\hat{\lambda}$ be the constant in Proposition* 3.1. *Then we define a space-like curve to be a curve of the form $\{t=\gamma(x):x\in(a,b)\}$, with*

$$|\gamma(x_2)-\gamma(x_1)|<\frac{x_2-x_1}{\hat{\lambda}}, \quad \forall a<x_1<x_2<b. \tag{A.15}$$

Still following [3], we now introduce bounded variation condition.

Definition A.2 (Bounded Variation Condition). *We say that a function $u\in L^\infty(\mathbb{R}^+\times\mathbb{R})$ verifies the bounded variation condition if there exists $\delta>0$ such that, for every bounded space-like curve $\{t=\gamma(x):x\in[a',b']\}$ with*

$$|\gamma(x_1)-\gamma(x_2)|\le\delta|x_1-x_2|, \quad \forall x_1,x_2\in[a',b'], \tag{A.16}$$

the function $x\mapsto u(\gamma(x),x):=u_\gamma(x)$ is well defined and has bounded variation.

Note that taking constant functions γ shows that these functions u are BV in x. Let us now state a uniqueness result of [3,9], rephrased in our context.

Theorem A.1 ([3,9]). *For any $d\in\mathcal{V}$, there exists $\varepsilon>0$ such that for any u^0 initial value with $\|u^0\|_{BV(\mathbb{R})}\le\varepsilon$ and $\|u^0-d\|_{L^\infty(\mathbb{R})}\le\varepsilon$, there exists only one solution u of* (1.7)-(1.10) *with initial value u^0 and verifying the bounded variation condition of Definition* A.2.

Note that 1.2 replaces the bounded variation condition in Definition A.2, by only $u\in L^\infty(\mathbb{R}^+;BV(\mathbb{R}))$, and Theorem 1.1 by $u\in\mathcal{S}_{weak}$.

Furthermore, we define domination.

Definition A.3 (Domination)**.** *Given two space-like curves* $\gamma: (a,b) \to \mathbb{R}$ *and* $\gamma': (a',b') \to \mathbb{R}$*, we say that* γ *dominates* γ' *if* $a \le a' < b' \le b$ *and, moreover,*

$$\gamma(x) \le \gamma'(x) \le \min\left\{\gamma(a) + \frac{x-a}{\hat{\lambda}}, \gamma(b) + \frac{b-x}{\hat{\lambda}}\right\}, \quad \forall x \in (a',b'). \tag{A.17}$$

This property implies that γ' is entirely contained in a domain of determinacy for the curve γ. We introduce now the following condition.

Condition A.1. Let $C > 0$. Let a function $\psi \in L^\infty(\mathbb{R}^+; BV(\mathbb{R}))$ be piecewise constant. We say that it verifies the Condition A.1 with constant C, if it verifies the following.

Let γ and γ' be any two space-like curves with γ dominating γ' (Definition A.3). Then,

$$\text{Tot.Var.}\{\psi;\gamma'\} \le C\,\text{Tot.Var.}\{\psi;\gamma\}. \tag{A.18}$$

We can show that any modified front tracking solution ψ verifies the Condition A.1 using the decay of Glimm potential in Proposition A.1. Hence, one can prove that the limit of solutions to the modified front tracking algorithm inherits the bounded variation condition. In fact, to find a precise proof, one refers the reader to [3, Lemma 7.3] for classical piecewise constant approximate solutions constructed by the front tracking algorithm (without shifts), or find the proof in [13].

We can also prove the following lemma for future use. Also see the proof in [13].

Lemma A.3. *Let* $\{\psi_n\}_{n\in\mathbb{N}}$ *be a family of piecewise constant functions uniformly bounded in* $L^\infty(\mathbb{R}^+, BV(\mathbb{R}))$*. Assume that there exists* $C > 0$ *such that for every* $n \in \mathbb{N}$, ψ_n *verifies condition* A.1 *for this constant* C*, and*

$$\|\psi_n(t,\cdot) - \psi_n(s,\cdot)\|_{L^1} \le C|t-s|, \quad 0 < s < t < T. \tag{A.19}$$

Then, there exists $\psi \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ *verifying the bounded variation condition* A.2 *such that, up to a subsequence,* ψ_n *converges to* ψ *when* $n \to \infty$ *in* $C^0(0,T; L^2(-R,R))$ *for every* $T > 0, R > 0$*, and almost everywhere in* $\mathbb{R}^+ \times \mathbb{R}$.

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