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RESEARCH ARTICLE

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A note on limiting Calderon–Zygmund theory for transformed n-Laplace systems in divergence form

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Abstract

We consider rotated n-Laplace systems on the unit ball $B_1 \subset \mathbb{R}^n$ of the form

$$-\mathrm{div}\left(Q|\nabla u|^{n-2}\nabla u\right) = \mathrm{div}(G),$$

where $u \in W^{1,n}(B_1;\mathbb{R}^N)$, $Q \in W^{1,n}(B_1;SO(N))$, and $G \in L^{\left(\frac{n}{n-1},q\right)}(B_1;\mathbb{R}^n \otimes \mathbb{R}^N)$ for some $0 < q < \frac{n}{n-1}$. We prove that $\nabla u \in L^{(n,q(n-1))}_{loc}$ with estimates. As a corollary, we obtain that solutions to $\Delta_n u \in \mathcal{H}^1$, where $\mathcal{H}^1$ is the Hardy space, have a higher integrability, namely, $\nabla u \in L^{(n,n-1)}_{loc}$.

MSC 2020
35B65, 35J70, 35B45, 35H30, 35J70, 35J92, 46E35 (primary)

Contents

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1 | INTRODUCTION

In the theory of critical harmonic maps into manifolds $u : B^2 \subset \mathbb{R}^2 \to \mathcal{M}$, cf. [7, 8, 13], the Hardy space is an important tool, since any map $u \in W^{1,2}(B^2, \mathbb{R}^N)$ satisfying

$$\Delta u \in \mathcal{H}^1$$

is continuous, and the Hardy space naturally appears via commutators and div-curl terms [4]. The continuity statement is false for the n-Laplacian when $n \geqslant 3$: in 1995, Firoozye [5] exhibited discontinuous maps $u \in W^{1,n}(B^n, \mathbb{R}^N)$ satisfying

$$\operatorname{div}(|\nabla u|^{n-2}\nabla u) \in \mathcal{H}^1.$$

Indeed the regularity theory for n-harmonic maps into general manifolds is an interesting and difficult open question, see [15] for an overview and [10, 12] for two recent results.

Formally the result of [5] is not surprising: 'Inverting' the div (i.e., pretending it to be the half-Laplacian), we can wishfully hope for

$$|\nabla u|^{n-1} \in L^{\left(\frac{n}{n-1},1\right)},$$

where $L^{\left(\frac{n}{n-1},1\right)}$ is a Lorentz space — since Sobolev embedding implies $(-\Delta)^{-\frac{1}{2}}\mathcal{H}^1 \subset L^{\left(\frac{n}{n-1},1\right)}$ — and thus

$$\nabla u \in L^{(n,n-1)}.$$

While $\nabla u \in L^{(n,1)}$ implies u is continuous (this is what we have in the case $n = 2$), for $n \geqslant 3$, standard function space theory tells us that there are many counterexamples $u \in W^{1,n}$ satisfying $\nabla u \in L^{(n,n-1)}$ but $u \notin C^0$.

The purpose of this short note is to make this intuition more precise, somewhat giving a positive version of Firoozye's example. It also extends known limiting results for the p-Laplacian in [1, 2] (observe, however we restrict to $p = n$). Our main result is the following statement, where $B_r = B(0, r)$ is the ball of $\mathbb{R}^n$ with radius r centered at the origin.

Theorem 1.1. *Let $q \in (0, \frac{n}{n-1})$. There exists a small $\varepsilon = \varepsilon(n, N, q)$ and even smaller $\gamma = \gamma(\varepsilon)$, we have the following.*

Assume $G \in L^{\left(\frac{n}{n-1},q\right)}(B_1; \mathbb{R}^n \otimes \mathbb{R}^N)$, $A \in W^{1,n}(B_1; GL(N))$, $\|A\|_{L^\infty} + \|A^{-1}\|_{L^\infty} \leqslant 2N^2$, and $u \in W^{1,n}(B_1; \mathbb{R}^N)$ satisfy the system

$$-\operatorname{div}(|\nabla u|^{n-2}A\nabla u) = \operatorname{div} G \text{ in } B_1,$$

with the bound

$$\|\nabla A^{-1}\|_{L^n(B_1)} + \|\nabla A\|_{L^n(B_1))} \leqslant \gamma.$$

Then, for every $\theta \in (0, \frac{1}{4})$, it holds $\nabla u \in L^{(n,q(n-1))}(B_\theta)$ with the following estimate:

$$\|\nabla u\|^{n-1}_{L^{(n,q(n-1))}(B_\theta)} \leqslant c(n, N, q, \theta)\left(\|G\|_{L^{\left(\frac{n}{n-1},q\right)}(B_1)} + \|\nabla u\|^{n-1}_{L^{n-\varepsilon}(B_1)} \right).$$

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Remark 1.2. We can make a few remarks on the above statement:

(1) The quantity $2N^2$ in the bound $\|A\|_{L^\infty} + \|A^{-1}\|_{L^\infty} \leqslant 2N^2$ is arbitrary and can be replaced by any large constant depending only on n, N.
(2) We include the term $A \in GL(N)$, because the theory of n-harmonic maps allows for a change of gauge, either the Uhlenbeck–Coulomb gauge $A \in SO(N)$ [17] or the Rivière's gauge $A \in GL(N)$ [13], for the relation, see also [14].
(3) In the case $q = \frac{1}{n-1}$ of Theorem 1.1, we deduce that u is continuous.
(4) The arguments of the proof only provide a constant c in the last estimate which goes to $+\infty$ as $\theta \to \frac{1}{4}$.

In terms of the Hardy space, we obtain the following as an immediate consequence:

Corollary 1.3. *Assume* $f \in \mathcal{H}^1_{loc}(B_1; \mathbb{R}^N)$, $A \in W^{1,n}(B_1; GL(N))$, $\|A\|_{L^\infty} + \|A^{-1}\|_{L^\infty} \leqslant 2N^2$, *and* $u \in W^{1,n}(B_1; \mathbb{R}^N)$ *satisfy the system*

$$-\mathrm{div}(|\nabla u|^{n-2} A \nabla u) = f \text{ in } B_1,$$

with the bound

$$\|\nabla A^{-1}\|_{L^n(B_1)} + \|\nabla A\|_{L^n(B_1)} \leqslant \gamma.$$

Then, for every $\theta \in (0, \frac{1}{4})$, *it holds* $\nabla u \in L^{(n,n-1)}(B_\theta)$ *with the following estimate:*

$$\|\nabla u\|^{n-1}_{L^{(n,n-1)}(B_\theta)} \leqslant c(n, N, \theta)\Big(\|f\|_{\mathcal{H}^1(B_1)} + \|\nabla u\|^{n-1}_{L^{n-\varepsilon}(B_1)} \Big).$$

Proof. We solve $\Delta\phi = f$ in B_1, with $\phi = 0$ on ∂B_1. Since $f \in \mathcal{H}^1$, it holds $\nabla\phi \in L^{\left(\frac{n}{n-1},1\right)}(B_1)$. Corollary 1.3 follows from Theorem 1.1 with $G = \nabla\phi$. □

Observe that Corollary 1.3 is sharp in dimension $n = 2$. We will argue that Corollary 1.3 is also sharp in dimension $n \geqslant 3$ in some sense in Section 4. Observe also that the bound $2N^2$ in the L^∞-estimate of A is arbitrary, see Remark 1.2.

Outline: The starting point of our arguments is based on recent estimates by the authors [10], which in turn are strongly motivated by Kuusi and Mingione's seminal [9], combined with covering arguments to estimate level sets. Then we adapt ideas of [11] to obtain our result. In Section 2, we recall the definition of Lorentz spaces and the necessary preliminary results. In Section 3, we prove Theorem 1.1. In Section 4, we discuss the regularity of Firoozye's example.

2 | PRELIMINARY ESTIMATES

In this section, we define some notations and recall the necessary preliminary estimates on Lorentz spaces, p-harmonic maps, and maximal functions.

In the rest of the paper, we will denote $B(x, r) \subset \mathbb{R}^n$ the ball of radius $r > 0$ and center $x \in \mathbb{R}^n$. If $x = 0$, we will denote $B_r = B(0, r)$. If $\lambda > 0$ and $B = B(x, r)$ is a ball, we will denote $\lambda B := B(x, \lambda r)$.

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We now recall the definitions and relevant properties of Lorentz spaces. For further reading, see, for instance, [6, section 1.4]. Let $\Omega \subset \mathbb{R}^n$ be an open set. Given a function $f : \Omega \to \mathbb{R}$, we define its decreasing rearrangement $f^* : [0, |\Omega|) \to [0, \infty)$ by

$$\forall t > 0, \quad f^*(t) := \inf \{\lambda \geqslant 0 : |\{x \in \Omega : |f(x)| > \lambda\}| \leqslant t\}.$$

Given $p \in (0, \infty)$ and $q \in (0, +\infty]$, a function $f : \Omega \to \mathbb{R}$ belongs to the Lorentz space $L^{(p,q)}(\Omega)$ if the following quantity is finite:

$$\|f\|_{L^{(p,q)}(\Omega)} := \begin{cases} \left(\int_0^\infty \left(f^*(s) s^{\frac{1}{p}}\right)^q \frac{ds}{s}\right)^{\frac{1}{q}} & \text{if } q < \infty, \\ \sup_{s>0} s^{\frac{1}{p}} f^*(s) & \text{if } q = \infty. \end{cases}$$

Lorentz spaces are refinements of the Lebesgue spaces in the following sense. Given $p \geqslant 1$ and $0 < q < r \leqslant +\infty$ and $|\Omega| < \infty$, it holds $L^{(p,p)}(\Omega) = L^p(\Omega)$ and $L^{(p,q)}(\Omega) \subset L^{(p,r)}(\Omega)$. Given $r > 0$, $p > 1$, and $q \in (0, \infty]$, it holds

$$\|f^r\|_{L^{(p,q)}(\Omega)} = \|f\|^r_{L^{(pr,qr)}(\Omega)}.$$

From the regularity of p-harmonic maps (vectorial, but with unconstrained target), cf. [16, Theorem 3.2], we have the following:

Theorem 2.1. *Let $p \in (1, n]$ and $\theta_0 \in (0, 1)$. There exists $c = c(n, N, p, \theta_0) > 0$ such that the following holds. Consider a ball $B(x, r) \subset \mathbb{R}^n$ and $v \in W^{1,p}(B(x, r); \mathbb{R}^N)$ such that $\Delta_p v = 0$ on $B(x, r)$. Then it holds*

$$\|\nabla v\|_{L^\infty(B(x,\theta_0 r))} \leq c \left(⨍_{B(x,r)} |\nabla v|^p \right)^{\frac{1}{p}}.$$

Remark 2.2. The above estimate has been proved for $\theta_0 = \frac{1}{4}$ in [16, Theorem 3.2]. By a covering argument, we can choose $\theta_0 \in (0, 1)$ arbitrarily, up to increasing the constant c.

The following is the initial estimate we need for our purposes, which was proved in [10, Corollary 5.2.].

Lemma 2.3. *Let $\sigma \in (0, 1)$. There exists $\varepsilon_1 = \varepsilon_1(n, N, \sigma) > 0$ such that the following holds.*
For any $\varepsilon \in (0, \varepsilon_1)$, there exists $\gamma_1 = \gamma_1(n, N, \varepsilon, \sigma) > 0$ with the following properties.
There exists $C_0 = C_0(n, N, \sigma, \varepsilon) > 0$ such that the following hold.
Assume $u \in W^{1,n}(B(x, r); \mathbb{R}^N)$ satisfies

$$\operatorname{div}(A|\nabla u|^{n-2} \nabla u) = \operatorname{div} G \quad \text{in } B(x, r),$$

where $A \in W^{1,n}(B(x, r); GL(N))$, $\|A\|_{L^\infty} + \|A^{-1}\|_{L^\infty} \leqslant 2N^2$, and

$$\|\nabla A\|_{L^n(B(x,r))} + \|\nabla A^{-1}\|_{L^n(B(x,r))} \leqslant \gamma_1. \tag{2.1}$$

There exists a radius $\rho \in \left[\frac{1}{2}r, \frac{3}{4}r\right]$ *such that if* $v \in W^{1,n}(B(x,\rho);\mathbb{R}^N)$ *satisfies*

$$\begin{cases} \Delta_n v = 0 \text{ in } B(x,\rho), \\ v = u \quad \text{on } \partial B(x,\rho), \end{cases}$$

then it holds

$$\left(\fint_{B(x,\rho)} |\nabla u - \nabla v|^{n-\varepsilon}\right)^{\frac{1}{n-\varepsilon}} \leq C_0(\sigma,\varepsilon)\left(\fint_{B(x,r)} |G|^{\frac{n-\varepsilon}{n-1}}\right)^{\frac{1}{n-\varepsilon}} + \sigma\left(\fint_{B(x,r)} |\nabla u|^{n-\varepsilon}\right)^{\frac{1}{n-\varepsilon}}. \tag{2.2}$$

Remark 2.4. In the statement of [10, Corollary 5.2], the integral quantity involving G in the right-hand side of (2.2) is $\left(\fint_{B(x,r)} |G|^{\frac{n}{n-1}}\right)^{\frac{1}{n}}$. However, the above estimate is obtained by following step by step the proof of [10, Lemma 4.2] in the case $f = 0$. The only change is the estimate of the term I which has to be replaced, with the notations of the proof of [10, Lemma 4.2], with the Hölder inequality $I \leqslant C\|G\|_{L^{\frac{n-\varepsilon}{n-1}}} \|\nabla a\|_{L^{\frac{n-\varepsilon}{1-\varepsilon}}}$.

We will work on dyadic cubes and balls. To that extent, we give some definitions. Given $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ and $\ell > 0$, we consider the cube centered in a and having side-length ℓ:

$$Q_\ell(a) := \left[a_1 - \frac{\ell}{2}, a_1 + \frac{\ell}{2}\right] \times \cdots \times \left[a_n - \frac{\ell}{2}, a_n + \frac{\ell}{2}\right] \subset \mathbb{R}^n.$$

Given $r > 0$, we will denote $rQ_\ell(a) := Q_{r\ell}(a)$. Given a ball $B \subset \mathbb{R}^n$, the *inner cube* of B is the cube $Q \subset B$ having the same center and maximal side-length. Dyadic subcubes of a cube $Q = Q_\ell(a)$ are defined by induction as follows. We denote $\mathcal{C}_0 := \{Q\}$ and $\mathcal{C}_1$ the family of subcubes of Q obtained by dividing Q in 2^n cubes having disjoint interior and such that each of them have side-length $\frac{\ell}{2}$. Given an integer $k \geqslant 1$, assume that $\mathcal{C}_k$ have been defined. We consider the family $\mathcal{C}_{k+1}$ of subcubes of Q obtained by dividing each cube $\tilde{Q} \in \mathcal{C}_k$ in 2^n subcubes having disjoint interior and such that the side-length of each of these subcubes is equal to half of the side-length of $\tilde{Q}$. A subcube $\tilde{Q} \subset Q$ is called *dyadic* if $\tilde{Q} \in \bigcup_{k\geqslant 1} \mathcal{C}_k$. Given $k \geqslant 1$ and $Q_1 \in \mathcal{C}_k$, there exists a unique $Q_0 \in \mathcal{C}_{k-1}$, called the *predecessor* of Q_1, such that $Q_1 \subset Q_0$. A predecessor is defined only for strict subcubes of Q and is always a dyadic subcube of Q.

We will work with a covering argument, for this we need the the following result from [3, Lemma 1.2].

Lemma 2.5. *Let* $Q_0 \subset \mathbb{R}^n$ *be a cube. Assume* $X \subset Y \subset Q_0$ *are measurable sets such that the following properties hold:*

1. *there exists* $\delta > 0$ *such that* $|X| < \delta|Q_0|$,
2. *if* $Q \subsetneq Q_0$ *is a dyadic subcube, then the inequality* $|X \cap Q| > \delta|Q|$ *implies that the predecessor* $\tilde{Q}$ *of* Q *is contained in* Y.

Then it holds $|X| < \delta|Y|$.

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We also have the following result for the uncentered, restricted maximal function operator, see [11, Theorem 7]. Given $f \in L^1(B_1)$, we set

$$M_{B_1} f(x) := \sup \left\{ \fint_{B(y,r)} |f| \, : \, x \in B(y,r) \subset B_1 \right\}.$$

Lemma 2.6. *Let $t > 1$ and $q \in (0, \infty]$. There exists a constant $c = c(n,t,q) > 0$ such that the following holds. Consider a ball $B_1 \subset \mathbb{R}^n$ and $g \in L^{(t,q)}(B_1)$. Then,*

$$\|M_{B_1} g\|_{L^{(t,q)}(B_1)} \leqslant c \|g\|_{L^{(t,q)}(B_1)}.$$

3 | LEVEL-SET ESTIMATES: PROOF OF THEOREM 1.1

In this section, we adapt the techniques of [11] to obtain a proof of Theorem 1.1. We always assume that u, A, and G are solutions as in Theorem 1.1.

We fix the parameter $\theta_0 \in (0,1)$ in Theorem 2.1 for the whole section. We will prove Theorem 1.1 for $\theta = \frac{\theta_0}{4}$, see (3.10). The final result will follow from the fact that θ_0 is arbitrary.

Step 1: Level-set decay

Lemma 3.1. *Let $\theta = \frac{\theta_0}{4}$. There exist a universal constant $\Gamma = \Gamma(n,N) > 1$ such that the following holds. We denote Q_θ as the inner cube of B_θ.*

For every $T > 1$, there exists $\varepsilon_2 = \varepsilon_2(n,N,T) \in (0,1)$ such that the following holds.

For any $\varepsilon \in (0, \varepsilon_2)$, there exists $\eta = \eta(n,N,\varepsilon,T) \in (0,1)$ and $\gamma_2 = \gamma_2(n,N,\varepsilon,T) \in (0,1)$ such that, if

$$\|\nabla A\|_{L^n(B_1)} + \|\nabla A^{-1}\|_{L^n(B_1)} \leqslant \gamma_2,$$

then the following holds. For any dyadic subcube Q of Q_θ and any $\lambda > \lambda_0$, where

$$\lambda_0 := \frac{2^n}{|B_1|\theta_0^n} \|\nabla u\|^{n-\varepsilon}_{L^{n-\varepsilon}(B_1)}, \tag{3.1}$$

the following holds.

Assume that the predecessor $\tilde{Q}$ of Q is contained in Q_θ and that the following inequality holds:

$$\left| Q \cap \left\{ x \in Q_\theta \, : \, M_{B_1}[|\nabla u|^{n-\varepsilon}](x) > \Gamma T \lambda, \; M_{B_1}\left[|G|^{\frac{n-\varepsilon}{n-1}} \right](x) \leqslant \eta \lambda \right\} \right| > T^{-\frac{2n}{n-\varepsilon}} |Q|. \tag{3.2}$$

Then $\tilde{Q}$ satisfies

$$\tilde{Q} \subset \left\{ x \in Q_\theta \, : \, M_{B_1}[|\nabla u|^{n-\varepsilon}](x) > \lambda \right\}. \tag{3.3}$$

Proof. We will define θ later in (3.10), for the moment, we consider only that $\theta \leqslant \frac{1}{4}$, in order to have $4Q \subset B_1$. By contradiction, we assume that (3.2) is valid but (3.3) fails.

Since (3.3) is wrong but $\tilde{Q} \subset B_\theta$, there exists $x_0 \in \tilde{Q}$ such that

$$M_{B_1}[|\nabla u|^{n-\varepsilon}](x_0) \leqslant \lambda. \tag{3.4}$$

Let B the unique ball having $3Q$ as inner cube, $B \subset 4Q \subset B_1$. Then we have

$$\fint_B |\nabla u|^{n-\varepsilon} \leq \lambda.$$

From (3.2), there exists $x_1 \in Q$ such that $M_{B_1}\left[|G|^{\frac{n-\varepsilon}{n-1}}\right](x_1) \leqslant \eta\lambda$. Since $x_1 \in B \subset B_1$, we also have

$$\fint_B |G|^{\frac{n-\varepsilon}{n-1}} \leq \eta\lambda.$$

From Lemma 2.3, there exists a radius $\rho \in (\frac{1}{2}, \frac{3}{4})$ such that the n-harmonic extension $v \in W^{1,n}(\rho B; \mathbb{R}^N)$ of u satisfies

$$\begin{aligned} \fint_{\rho B} |\nabla u - \nabla v|^{n-\varepsilon} &\leq C_0(\sigma, \varepsilon) \fint_B |G|^{\frac{n-\varepsilon}{n-1}} + \sigma \fint_B |\nabla u|^{n-\varepsilon} \\ &\leq C_0(\sigma, \varepsilon)\eta\lambda + \sigma\lambda. \end{aligned} \tag{3.5}$$

Furthermore, it holds by Theorem 2.1

$$\left(\fint_{\frac{\theta_0}{2}B} |\nabla v|^{2n}\right)^{\frac{1}{2n}} \leq c(n) \left(\fint_{\frac{1}{2}B} |\nabla v|^{n-\varepsilon}\right)^{\frac{1}{n-\varepsilon}}. \tag{3.6}$$

Combining (3.4) and (3.5), we deduce that

$$\begin{aligned} \left(\fint_{\frac{1}{2}B} |\nabla v|^{n-\varepsilon}\right)^{\frac{1}{n-\varepsilon}} &\leq c \left(\fint_{\rho B} |\nabla (v-u)|^{n-\varepsilon}\right)^{\frac{1}{n-\varepsilon}} + c\left(\fint_B |\nabla u|^{n-\varepsilon}\right)^{\frac{1}{n-\varepsilon}} \\ &\leq ((c(n) + C_0(\sigma, \varepsilon)\eta)\,\lambda)^{\frac{1}{n-\varepsilon}}. \end{aligned}$$

From (3.6), we obtain

$$\fint_{\frac{\theta_0}{2}B} |\nabla v|^{2n} \leq c(n)((1 + C_0(\sigma, \varepsilon)\eta)\lambda)^{\frac{2n}{n-\varepsilon}}. \tag{3.7}$$

Now, we have all the ingredients to estimate the quantity

$$\left|\left\{x \in Q : M_{B_1}[|\nabla u|^{n-\varepsilon}](x) > \Gamma T \lambda\right\}\right|.$$

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First, we compare with the more restricted maximal function

$$M_{B_{\theta_0/2}}[|\nabla u|^{n-\varepsilon}](x) := \sup\left\{ \fint_{B(y,r)} |\nabla u|^{n-\varepsilon} : x \in B(y,r) \subset B_{\theta_0/2} \right\}.$$

We obtain the following relation:

$$M_{B_1}[|\nabla u|^{n-\varepsilon}](x) = \max\left(M_{B_{\theta_0/2}}[|\nabla u|^{n-\varepsilon}](x); \sup\left\{ \fint_{B(y,r)} |\nabla u|^{n-\varepsilon} : x \in B(y,r) \not\subset B_{\theta_0/2} \right\}\right). \tag{3.8}$$

Given $x \in B_{\theta_0/4}$, we estimate the second term using that a ball $B(y,r) \not\subset B_{\theta_0/2}$ containing x must have a radius $r \geqslant \frac{\theta_0}{2}$: It holds

$$\sup\left\{ \fint_{B(x,r)} |\nabla u|^{n-\varepsilon} : B(x,r) \not\subset B_{\theta_0/2} \right\} \leq \frac{2^n}{|B_1|\theta_0^n} \|\nabla u\|_{L^{n-\varepsilon}(B_1)}^{n-\varepsilon} =: \lambda_0. \tag{3.9}$$

We define

$$\theta := \frac{\theta_0}{4} < \frac{1}{4}. \tag{3.10}$$

If $\lambda > \lambda_0$, we deduce from (3.8) and (3.9):

$$\begin{aligned} \left|\left\{ x \in Q : M_{B_1}[|\nabla u|^{n-\varepsilon}](x) > \Gamma T\lambda \right\}\right| &= \left|\left\{ x \in Q : M_{B_{\theta_0/2}}[|\nabla u|^{n-\varepsilon}](x) > \Gamma T\lambda \right\}\right| \\ &\leqslant \left|\left\{ x \in Q : M_{B_{\theta_0/2}}[|\nabla v|^{n-\varepsilon}](x) > \frac{1}{2^n}\Gamma T\lambda \right\}\right| \\ &\quad + \left|\left\{ x \in Q : M_{B_{\theta_0/2}}[|\nabla (u-v)|^{n-\varepsilon}](x) > \frac{1}{2^n}\Gamma T\lambda \right\}\right|. \end{aligned}$$

With standard estimates on maximal functions, we find

$$\left|\left\{ x \in Q : M_{B_1}[|\nabla u|^{n-\varepsilon}](x) > \Gamma T\lambda \right\}\right| \leqslant \frac{C(n)}{(\Gamma T\lambda)^{\frac{2n}{n-\varepsilon}}} \int_Q |\nabla v|^{2n} + \frac{C(n)}{\Gamma T\lambda} \int_Q |\nabla (u-v)|^{n-\varepsilon}.$$

Combining this with (3.5) and (3.7), we arrive at

$$\begin{aligned} \left|\left\{ x \in Q : M_{B_1}[|\nabla u|^{n-\varepsilon}](x) > \Gamma T\lambda \right\}\right| &\leqslant \frac{C(n)|Q|}{(\Gamma T\lambda)^{\frac{2n}{n-\varepsilon}}}((1 + C_0(\sigma,\varepsilon)\eta)\lambda)^{\frac{2n}{n-\varepsilon}} \\ &\quad + \frac{C(n)|Q|}{\Gamma T\lambda}(C_0(\sigma,\varepsilon)\eta + c(n)\sigma)\lambda \\ &\leqslant \frac{C(n)|Q|}{(\Gamma T)^{\frac{2n}{n-\varepsilon}}}(1 + C_0(\sigma,\varepsilon)\eta)^{\frac{2n}{n-\varepsilon}} + \frac{C(n)|Q|}{\Gamma T}(C_0(\sigma,\varepsilon)\eta + c(n)\sigma). \end{aligned} \tag{3.11}$$

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We first choose $\sigma = \sigma(n, T)$ small enough in order to obtain

$$c(n)\sigma \leqslant \frac{1}{2T^{\frac{2n}{n-1}-1}} \leqslant \frac{1}{2T^{\frac{2n}{n-\varepsilon}-1}}.$$

This choice fixes $\varepsilon_2 = \varepsilon_1(n, T)$ thanks to Lemma 2.3. Then, for any $\varepsilon \in (0, \varepsilon_2)$, we obtain a constant $\gamma_2 = \gamma_1(n, N, \varepsilon, \sigma)$ and we choose $\eta = \eta(n, \varepsilon, T)$ small enough, so that we obtain

$$C_0(\sigma, \varepsilon)\eta + c(n)\sigma \leqslant \frac{1}{T^{\frac{2n}{n-1}-1}} \leqslant \frac{1}{T^{\frac{2n}{n-\varepsilon}-1}},$$
$$(1 + C_0(\sigma, \varepsilon)\eta)^{\frac{2n}{n-\varepsilon}} \leqslant 2.$$

Coming back to (3.11) with these choices, we obtain

$$\left|\left\{x \in Q : M_{B_1}[|\nabla u|^{n-\varepsilon}](x) > \Gamma T\lambda\right\}\right| \leqslant \frac{C(n)|Q|}{T^{\frac{2n}{n-\varepsilon}}}\left(\frac{1}{\Gamma^{\frac{2n}{n-\varepsilon}}} + \frac{1}{\Gamma}\right) \leqslant \frac{C(n)|Q|}{T^{\frac{2n}{n-\varepsilon}}}\left(\frac{1}{\Gamma^2} + \frac{1}{\Gamma}\right). \tag{3.12}$$

We now choose $\Gamma = \Gamma(n)$ large enough to have

$$C(n)\left(\frac{1}{\Gamma^2} + \frac{1}{\Gamma}\right) < \frac{1}{2}. \tag{3.13}$$

Hence, we obtain

$$\left|\left\{x \in Q : M_{B_1}[|\nabla u|^{n-\varepsilon}](x) > \Gamma T\lambda\right\}\right| \leqslant \frac{|Q|}{2T^{\frac{2n}{n-\varepsilon}}}.$$

This is a contradiction to (3.2), and we can conclude. □

Step 2: Application of Lemma 2.5

Lemma 3.2. *Let $\theta = \frac{\theta_0}{4}$. There exists a universal constant $\Gamma = \Gamma(n, N) > 1$ such that the following holds. Let Q_θ be the inner cube of B_θ.*

For every $T > 1$, there exists $\varepsilon_2 = \varepsilon_2(n, N, T) \in (0, 1)$ such that the following holds.

For every $\varepsilon \in (0, \varepsilon_2)$, we define

$$\lambda_1 := \max\left(\frac{2^n\Gamma}{|B_1|\theta_0^n}, \frac{1}{|B_\theta|}\right)T^{\frac{2n}{n-\varepsilon}}\|\nabla u\|^{n-\varepsilon}_{L^{n-\varepsilon}(B_1)}.$$

There exists $\eta = \eta(n, N, \varepsilon, T) \in (0, 1)$ and $\gamma_2 = \gamma_2(n, N, \varepsilon, T) \in (0, 1)$ such that, if

$$\|\nabla A\|_{L^n(B_1)} + \|\nabla A^{-1}\|_{L^n(B_1)} \leqslant \gamma_2,$$

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then for every $k \in \mathbb{N}$, it holds

$$\left|\left\{x \in Q_\theta : M_{B_1}[|\nabla u|^{n-\varepsilon}](x) > (\Gamma T)^{k+1}\lambda_1\right\}\right| \leqslant \frac{1}{T^{\frac{2n}{n-\varepsilon}}}\left|\left\{x \in Q_\theta : M_{B_1}[|\nabla u|^{n-\varepsilon}](x) > (\Gamma T)^{k}\lambda_1\right\}\right| + \left|\left\{x \in Q_\theta : M\left[|G|^{\frac{n-\varepsilon}{n-1}}\right](x) > \eta(\Gamma T)^{k}\lambda_1\right\}\right|. \tag{3.14}$$

Proof. We consider the cases where $\lambda = (\Gamma T)^k\lambda_1$ in Lemma 3.1 for any $k \in \mathbb{N}^*$ and for some λ_1 to be chosen later. The goal is to apply Lemma 2.5 with $Q_0 = Q_\theta$, $\delta = T^{-\frac{2n}{n-\varepsilon}}$ and the sets

$$X = \left|\left\{x \in Q_\theta : M_{B_1}[|\nabla u|^{n-\varepsilon}](x) > (\Gamma T)^{k+1}\lambda_1, M_{B_1}\left[|G|^{\frac{n-\varepsilon}{n-1}}\right](x) \leqslant \eta(\Gamma T)^{k}\lambda_1\right\}\right|, \tag{3.15}$$

$$Y = \left|\left\{x \in Q_\theta : M_{B_1}[|\nabla u|^{n-\varepsilon}](x) > (\Gamma T)^{k}\lambda_1\right\}\right|. \tag{3.16}$$

From Lemma 3.1, (2) in Lemma 2.5 is satisfied from $\lambda_1 = \lambda_0$. However, we need to increase λ_1 in order to obtain (1).

To do so, we first consider λ_1 of the form $\alpha T^{\frac{2n}{n-\varepsilon}}\lambda_0$, for some universal constant $\alpha = \alpha(n, N, \theta_0) \geqslant \Gamma$, in order to satisfy (1) in Lemma 2.5. From the definition of λ_0 in (3.1), we have from standard estimates of maximal functions

$$\left|\left\{x \in Q_\theta : M_{B_1}[|\nabla u|^{n-\varepsilon}](x) > \alpha T^{\frac{2n}{n-\varepsilon}}\lambda_0\right\}\right| \leqslant \frac{1}{\alpha T^{\frac{2n}{n-\varepsilon}}\lambda_0}\int_{B_1}|\nabla u|^{n-\varepsilon} \leqslant \frac{|B_1|\theta_0^n}{2^n\alpha T^{\frac{2n}{n-\varepsilon}}}.$$

We define $\alpha = \alpha(n, N, \theta_0)$ by the relation

$$\alpha = \max\left(\Gamma, \frac{|B_1|\theta_0^n}{2^n|Q_\theta|}\right).$$

That is, we have

$$\left|\left\{x \in Q_\theta : M_{B_1}[|\nabla u|^{n-\varepsilon}](x) > \alpha T^{\frac{2n}{n-\varepsilon}}\lambda_0\right\}\right| \leqslant \frac{|Q_\theta|}{T^{\frac{2n}{n-\varepsilon}}}.$$

We define $\lambda_1 \geqslant \lambda_0$ by the relation

$$\lambda_1 := \alpha T^{\frac{2n}{n-\varepsilon}}\lambda_0 = \max\left(\frac{1}{|Q_\theta|}, \frac{2^n\Gamma}{|B_1|\theta_0^n}\right)T^{\frac{2n}{n-\varepsilon}}\int_{B_1}|\nabla u|^{n-\varepsilon}. \tag{3.17}$$

We now check (1) of Lemma 2.5 for the set X defined in (3.15) and λ_1 defined in (3.17). Fix an integer $k \geqslant 1$. Since $\Gamma T \geqslant 1$, we have

$$\left|\left\{x \in Q_\theta : M_{B_1}[|\nabla u|^{n-\varepsilon}](x) > (\Gamma T)^{k}\lambda_1\right\}\right| \leqslant \left|\left\{x \in Q_\theta : M_{B_1}[|\nabla u|^{n-\varepsilon}](x) > \lambda_1\right\}\right| \leqslant \frac{|Q_\theta|}{T^{\frac{2n}{n-\varepsilon}}}.$$

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Thus, we can apply Lemma 2.5 with the choices $Q_0 = Q_\theta$, $\delta = T^{-\frac{2n}{n-\varepsilon}}$ and X, Y defined in (3.15)–(3.16). We obtain

$$\left|\left\{x \in Q_\theta : M_{B_1}[|\nabla u|^{n-\varepsilon}](x) > (\Gamma T)^{k+1}\lambda_1, M_{B_1}\left[|G|^{\frac{n-\varepsilon}{n-1}}\right](x) \leqslant \eta(\Gamma T)^k\lambda_1\right\}\right|$$
$$\leqslant \frac{1}{T^{\frac{2n}{n-\varepsilon}}}\left|\left\{x \in Q_\theta : M_{B_1}[|\nabla u|^{n-\varepsilon}](x) > (\Gamma T)^k\lambda_1\right\}\right|.$$

Therefore, it holds

$$\begin{aligned}&\left|\left\{x \in Q_\theta : M_{B_1}[|\nabla u|^{n-\varepsilon}](x) > (\Gamma T)^{k+1}\lambda_1\right\}\right|\\ \leqslant&\left|\left\{x \in Q_\theta : M_{B_1}[|\nabla u|^{n-\varepsilon}](x) > (\Gamma T)^{k+1}\lambda_1, M_{B_1}\left[|G|^{\frac{n-\varepsilon}{n-1}}\right](x) \leqslant \eta(\Gamma T)^k\lambda_1\right\}\right|\\ &+\left|\left\{x \in Q_\theta : M\left[|G|^{\frac{n-\varepsilon}{n-1}}\right](x) > \eta(\Gamma T)^k\lambda_1\right\}\right|\\ \leqslant&\frac{1}{T^{\frac{2n}{n-\varepsilon}}}\left|\left\{x \in Q_\theta : M_{B_1}[|\nabla u|^{n-\varepsilon}](x) > (\Gamma T)^k\lambda_1\right\}\right|\\ &+\left|\left\{x \in Q_\theta : M\left[|G|^{\frac{n-\varepsilon}{n-1}}\right](x) > \eta(\Gamma T)^k\lambda_1\right\}\right|.\end{aligned}$$

□

Step 3: Lorentz spaces estimates

Proof of Theorem 1.1. Following the notations of [11], we define for any $H \geqslant 0$,

$$\mu_1(H) := \left|\left\{x \in Q_\theta : M_{B_1}[|\nabla u|^{n-\varepsilon}](x) > H\right\}\right|,$$
$$\mu_2(H) := \left|\left\{x \in Q_\theta : M_{B_1}\left[|G|^{\frac{n-\varepsilon}{n-1}}\right](x) > H\right\}\right|.$$

For any integer $k \geqslant 1$, we write the estimate (3.14) in terms of μ_1 and μ_2

$$\mu_1((\Gamma T)^{k+1}\lambda_1) \leqslant \mu_2(\eta(\Gamma T)^k\lambda_1) + T^{-\frac{2n}{n-\varepsilon}}\mu_1((\Gamma T)^k\lambda_1). \tag{3.18}$$

We define the sequences

$$a_k := \mu_1((\Gamma T)^k\lambda_1),$$
$$b_k := \mu_2((\Gamma T)^k\eta\lambda_1).$$

If $G \in L^{(\frac{n}{n-1},q)}$, then $|G|^{\frac{n-\varepsilon}{n-1}} \in L^{(\frac{n}{n-\varepsilon},q\frac{n-1}{n-\varepsilon})}$. By Lemma 2.6, it holds $M_{B_1}[|G|^{\frac{n-\varepsilon}{n-1}}] \in L^{(\frac{n}{n-\varepsilon},q\frac{n-1}{n-\varepsilon})}$. We write this in terms of μ_2

$$\left\|M_{B_1}[|G|^{\frac{n-\varepsilon}{n-1}}]\right\|^{q\frac{n-1}{n-\varepsilon}}_{L^{\left(\frac{n}{n-\varepsilon},q\frac{n-1}{n-\varepsilon}\right)}(Q_\theta)} = q\frac{n-1}{n-\varepsilon}\int_0^\infty \left(H^{\frac{n}{n-\varepsilon}}\mu_2(H)\right)^{q\frac{n-1}{n}}\frac{dH}{H} < \infty.$$

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We derive an estimate on the sequence b_k from the above inequality. We start with the following decomposition of the integral

$$\int_{\eta\lambda_1}^{\infty}\left(H^{\frac{n}{n-\varepsilon}}\mu_2(H)\right)^{q\frac{n-1}{n}}\frac{dH}{H}=\sum_{k=0}^{\infty}\int_{(\Gamma T)^k\eta\lambda_1}^{(\Gamma T)^{k+1}\eta\lambda_1}\left(H^{\frac{n}{n-\varepsilon}}\mu_2(H)\right)^{q\frac{n-1}{n}}\frac{dH}{H}.$$

Since μ_2 is a nonincreasing function, we obtain

$$\begin{aligned}\int_{\eta\lambda_1}^{\infty}\left(H^{\frac{n}{n-\varepsilon}}\mu_2(H)\right)^{q\frac{n-1}{n}}\frac{dH}{H}&\geqslant\sum_{k=0}^{\infty}b_{k+1}^{q\frac{n-1}{n}}\int_{(\Gamma T)^k\eta\lambda_1}^{(\Gamma T)^{k+1}\eta\lambda_1}\left(H^{\frac{n}{n-\varepsilon}}\right)^{q\frac{n-1}{n}}\frac{dH}{H}\\&\geqslant\sum_{k=0}^{\infty}b_{k+1}^{q\frac{n-1}{n}}\int_{(\Gamma T)^k\eta\lambda_1}^{(\Gamma T)^{k+1}\eta\lambda_1}H^{q\frac{n-1}{n-\varepsilon}-1}dH.\end{aligned}$$

If $q<\frac{n}{n-1}$, then up to reducing ε, we also have $q<\frac{n-\varepsilon}{n-1}$. Therefore, $q\frac{n-1}{n-\varepsilon}-1<0$ and we obtain the desired estimate on the summability of b_k

$$\begin{aligned}\int_{\eta\lambda_1}^{\infty}\left(H^{\frac{n}{n-\varepsilon}}\mu_2(H)\right)^{q\frac{n-1}{n}}\frac{dH}{H}&\geqslant\sum_{k=0}^{\infty}b_{k+1}^{q\frac{n-1}{n}}\left((\Gamma T)^{k+1}\eta\lambda_1\right)^{q\frac{n-1}{n-\varepsilon}-1}\int_{(\Gamma T)^k\eta\lambda_1}^{(\Gamma T)^{k+1}\eta\lambda_1}dH\\&\geqslant(\Gamma T-1)\sum_{k=0}^{\infty}b_{k+1}^{q\frac{n-1}{n}}\left((\Gamma T)^{k+1}\eta\lambda_1\right)^{q\frac{n-1}{n-\varepsilon}}.\end{aligned}$$

Since $T>1$ and $\Gamma>1$, we have the following inequality:

$$\sum_{k=0}^{\infty}b_{k+1}^{q\frac{n-1}{n}}\left((\Gamma T)^{k+1}\eta\lambda_1\right)^{q\frac{n-1}{n-\varepsilon}}\leqslant c(n)\left\|M_{B_1}[|G|^{\frac{n-\varepsilon}{n-1}}]\right\|_{L^{\left(\frac{n}{n-\varepsilon},q\frac{n-1}{n-\varepsilon}\right)}(Q_\theta)}^{q\frac{n-1}{n-\varepsilon}}.\tag{3.19}$$

We now write (3.18) in terms of the sequences $(a_k)_k$ and $(b_k)_k$. We obtain for any $k\geqslant 0$,

$$a_{k+2}\leqslant b_{k+1}+\frac{a_{k+1}}{T^{\frac{2n}{n-\varepsilon}}}\leqslant b_{k+1}+\frac{a_{k+1}}{T^2}.$$

We raise to the power $q\frac{n-1}{n}$ and multiply by $\left((\Gamma T)^{k+2}\eta\lambda_1\right)^{q\frac{n-1}{n-\varepsilon}}$ to obtain

$$\begin{aligned}\left((\Gamma T)^{k+2}\eta\lambda_1\right)^{q\frac{n-1}{n-\varepsilon}}a_{k+2}^{q\frac{n-1}{n}}&\leqslant\left((\Gamma T)^{k+2}\eta\lambda_1\right)^{q\frac{n-1}{n-\varepsilon}}2^{q\frac{n-1}{n}}\left(b_{k+1}^{q\frac{n-1}{n}}+T^{-2q\frac{n-1}{n}}a_{k+1}^{q\frac{n-1}{n}}\right)\\&\leqslant 2^{q\frac{n-1}{n}}(\Gamma T)^{q\frac{n-1}{n-\varepsilon}}\left(b_{k+1}^{q\frac{n-1}{n}}\left((\Gamma T)^{k+1}\eta\lambda_1\right)^{q\frac{n-1}{n-\varepsilon}}\right)\\&\quad+2^{q\frac{n-1}{n}}(\Gamma T)^{q\frac{n-1}{n-\varepsilon}}T^{-2q\frac{n-1}{n}}\left(a_{k+1}^{q\frac{n-1}{n}}\left((\Gamma T)^{k+1}\eta\lambda_1\right)^{q\frac{n-1}{n-\varepsilon}}\right).\end{aligned}\tag{3.20}$$

Up to reducing ε again, it holds:

$$\begin{aligned}q\frac{n-1}{n-\varepsilon}-2q\frac{n-1}{n}&=q(n-1)\left(\frac{1}{n-\varepsilon}-\frac{2}{n}\right)\\&\leqslant q(n-1)\left(\frac{3}{2n}-\frac{2}{n}\right)\\&\leqslant q(n-1)\frac{1}{2n}<0.\end{aligned}$$

Therefore, using that $\Gamma = \Gamma(n)$ (see (3.13)), we can choose $T = T(n,q) > 1$ such that

$$2^{q\frac{n-1}{n}}\Gamma^{q\frac{n-1}{n-\varepsilon}}T^{q\frac{n-1}{n-\varepsilon}-2q\frac{n-1}{n}} \leqslant 2^{q\frac{n-1}{n}}\Gamma^{q}T^{q(n-1)\frac{-1}{2n}} \leqslant \frac{1}{4}. \tag{3.21}$$

Coming back to (3.20), we obtain

$$\left((\Gamma T)^{k+2}\eta\lambda_1\right)^{q\frac{n-1}{n-\varepsilon}} a_{k+2}^{q\frac{n-1}{n}} \leqslant c(n,q)\left(b_{k+1}^{q\frac{n-1}{n}}\left((\Gamma T)^{k+1}\eta\lambda_1\right)^{q\frac{n-1}{n-\varepsilon}}\right) + \frac{1}{4}\left(a_{k+1}^{q\frac{n-1}{n}}\left((\Gamma T)^{k+1}\eta\lambda_1\right)^{q\frac{n-1}{n-\varepsilon}}\right).$$

Given any integer $K \geqslant 1$, we sum the above inequalities for k between 0 and K to obtain

$$\sum_{k=0}^{K}\left((\Gamma T)^{k+2}\eta\lambda_1\right)^{q\frac{n-1}{n-\varepsilon}} a_{k+2}^{q\frac{n-1}{n}} \leqslant c(n,q)\sum_{k=0}^{K}\left(b_{k+1}^{q\frac{n-1}{n}}\left((\Gamma T)^{k+1}\eta\lambda_1\right)^{q\frac{n-1}{n-\varepsilon}}\right) + \frac{1}{4}\sum_{k=0}^{K}\left(a_{k+1}^{q\frac{n-1}{n}}\left((\Gamma T)^{k+1}\eta\lambda_1\right)^{q\frac{n-1}{n-\varepsilon}}\right).$$

By reindexing the sums over the a_k, it holds

$$\sum_{k=2}^{K+2}\left((\Gamma T)^{k}\eta\lambda_1\right)^{q\frac{n-1}{n-\varepsilon}} a_{k}^{q\frac{n-1}{n}} \leqslant c(n,q)\sum_{k=0}^{K}\left(b_{k+1}^{q\frac{n-1}{n}}\left((\Gamma T)^{k+1}\eta\lambda_1\right)^{q\frac{n-1}{n-\varepsilon}}\right) + \frac{1}{4}\sum_{k=1}^{K+1}\left(a_{k}^{q\frac{n-1}{n}}\left((\Gamma T)^{k}\eta\lambda_1\right)^{q\frac{n-1}{n-\varepsilon}}\right).$$

Therefore, the last sum of the right-hand side can be reabsorbed by the left-hand side, only the term in $k = 1$ remains on the right-hand side

$$\frac{3}{4}\sum_{k=2}^{K+2}\left((\Gamma T)^{k}\eta\lambda_1\right)^{q\frac{n-1}{n-\varepsilon}} a_{k}^{q\frac{n-1}{n}} \leqslant c(n,q)\sum_{k=0}^{K}\left(b_{k+1}^{q\frac{n-1}{n}}\left((\Gamma T)^{k+1}\eta\lambda_1\right)^{q\frac{n-1}{n-\varepsilon}}\right) + \frac{1}{4}\left(a_{1}^{q\frac{n-1}{n}}(\Gamma T\eta\lambda_1)^{q\frac{n-1}{n-\varepsilon}}\right).$$

We now consider the limit $K \to \infty$

$$\frac{3}{4}\sum_{k=2}^{\infty}\left((\Gamma T)^{k}\eta\lambda_1\right)^{q\frac{n-1}{n-\varepsilon}} a_{k}^{q\frac{n-1}{n}} \leqslant c(n,q)\sum_{k=0}^{\infty}\left(b_{k+1}^{q\frac{n-1}{n}}\left((\Gamma T)^{k+1}\eta\lambda_1\right)^{q\frac{n-1}{n-\varepsilon}}\right) + \frac{1}{4}\left(a_{1}^{q\frac{n-1}{n}}(\Gamma T\eta\lambda_1)^{q\frac{n-1}{n-\varepsilon}}\right).$$

Using $a_1 \leqslant |B_1|$, the expression of λ_1 in (3.17), and the summability of $(b_k)_k$ in (3.19), we deduce that

$$\frac{3}{4}\sum_{k=2}^{\infty}\left((\Gamma T)^{k}\eta\lambda_1\right)^{q\frac{n-1}{n-\varepsilon}} a_{k}^{q\frac{n-1}{n}} \leqslant c(n,q,\theta_0)\left(\left\|M_{B_1}[|G|^{\frac{n-\varepsilon}{n-1}}]\right\|^{q\frac{n-1}{n-\varepsilon}}_{L^{\left(\frac{n}{n-\varepsilon},q\frac{n-1}{n-\varepsilon}\right)}(B_1)} + \|\nabla u\|^{q(n-1)}_{L^{n-\varepsilon}(B_1)}\right).$$

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From Lemma 2.6, we deduce that

$$\sum_{k=2}^{\infty}\left((\Gamma T)^k\eta\lambda_1\right)^{q\frac{n-1}{n-\varepsilon}} a_k^{q\frac{n-1}{n}} \leqslant c(n,q,\theta_0)\left(\|G\|^q_{L^{\left(\frac{n}{n-1},q\right)}(B_1)} + \|\nabla u\|^{q(n-1)}_{L^{n-\varepsilon}(B_1)}\right).$$

Arguing similar to (3.19), we obtain from the above estimate an estimate in Lorentz spaces of $M_{B_1}[|\nabla u|^{n-\varepsilon}]$

$$\begin{aligned}
&\left\|M_{B_1}[|\nabla u|^{n-\varepsilon}]\right\|^{q\frac{n-1}{n-\varepsilon}}_{L^{\left(\frac{n}{n-\varepsilon},q\frac{n-1}{n-\varepsilon}\right)}(Q_\theta)}\\
&= q\frac{n-1}{n-\varepsilon}\int_0^\infty \left(H^{\frac{n}{n-\varepsilon}}\mu_1(H)\right)^{q\frac{n-1}{n}}\frac{dH}{H}\\
&= q\frac{n-1}{n-\varepsilon}\left(\int_0^{(\Gamma T)^2\lambda_1}\left(H^{\frac{n}{n-\varepsilon}}\mu_1(H)\right)^{q\frac{n-1}{n}}\frac{dH}{H} + \sum_{k=2}^{\infty}\int_{(\Gamma T)^k\lambda_1}^{(\Gamma T)^{k+1}\lambda_1}\left(H^{\frac{n}{n-\varepsilon}}\mu_1(H)\right)^{q\frac{n-1}{n}}\frac{dH}{H}\right)\\
&\leqslant q\left(|B_1|^{q\frac{n-1}{n}}\left((\Gamma T)^2\lambda_1\right)^{q\frac{n-1}{n-\varepsilon}} + (\Gamma T-1)\sum_{k=2}^{\infty}a_k^{q\frac{n-1}{n}}\left((\Gamma T)^k\lambda_1\right)^{q\frac{n-1}{n-\varepsilon}}\right)\\
&\leqslant c(n,q,\theta_0)\left(\|G\|^q_{L^{\left(\frac{n}{n-1},q\right)}(B_1)} + \|\nabla u\|^{q(n-1)}_{L^{n-\varepsilon}(B_1)}\right).
\end{aligned}$$

We conclude the proof of Theorem 1.1,

$$\|\nabla u\|^{q(n-1)}_{L^{(n,q(n-1))}(Q_\theta)} \leqslant c(n,q,\theta_0)\left(\|G\|^q_{L^{\left(\frac{n}{n-1},q\right)}(B_1)} + \|\nabla u\|^{q(n-1)}_{L^{n-\varepsilon}(B_1)}\right). \qquad \square$$

4 | OPTIMALITY OF COROLLARY 1.3

In this section, we study the regularity of the examples obtained in [5]. Firoozye proved that for any $\alpha \in \left(0, \frac{n-2}{n-1}\right)$, the function $u_\alpha(x) = \log(1/|x|)^\alpha$ is a solution to $\Delta_n u \in \mathcal{H}^1_{loc}$ on a ball $B_{1/2} \subset \mathbb{R}^n$.

Lemma 4.1. *For every $\frac{1}{1-\alpha} < q$, it holds $\nabla u_\alpha \in L^{(n,q)}$.*

Remark 4.2. Since $\alpha < \frac{n-2}{n-1}$, it holds $\frac{1}{1-\alpha} < n-1$. In particular, it holds $\nabla u_\alpha \in L^{(n,n-1)}(B_{1/2})$ for any α. Furthermore, we have

$$\frac{1}{1-\alpha} \xrightarrow[\alpha\to\frac{n-2}{n-1}]{} n-1.$$

Thus, $L^{(n,n-1)}$ is the maximal integrability which is common to every ∇u_α.

Proof. The norm of the gradient of u_α is given by

$$\forall x \in B_{1/2},\quad |\nabla u_\alpha(x)| = \frac{\alpha}{|x|}\log\left(\frac{1}{|x|}\right)^{\alpha-1}.$$

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The nondecreasing rearrangement of $|\nabla u_\alpha|$ is given by

$$\forall t \in (0, |B_{1/2}|), \quad f(t) = \frac{\alpha}{|B_1|^{-\frac{1}{n}} t^{\frac{1}{n}}} \log\left(\frac{1}{|B_1|^{-\frac{1}{n}} t^{\frac{1}{n}}}\right)^{\alpha-1}.$$

The map $\nabla u_\alpha \in L^{(n,q)}$ if and only if

$$\int_0^{|B_{1/2}|} \left(t^{\frac{1}{n}} f(t)\right)^q \frac{dt}{t} < \infty.$$

This is equivalent to

$$\int_0^{1/2} \frac{dt}{t|\log(t)|^{q(1-\alpha)}} < \infty.$$

This is true if and only if $q(1-\alpha) > 1$. Hence, it holds $\nabla u_\alpha \in L^{(n,q)}(B_{1/2})$ for every $q > \frac{1}{1-\alpha}$. □

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REFERENCES

1. A. Alberico, A. Cianchi, and C. Sbordone, *On the modulus of continuity of solutions to the n-Laplace equation*, J. Elliptic Parabol. Equ. **1** (2015), 1–12.
2. A. Alberico, A. Cianchi, and C. Sbordone, *Continuity properties of solutions to the p-Laplace system*, Adv. Calc. Var. **10** (2017), no. 1, 1–24.
3. L. A. Caffarelli and I. Peral, *On $W^{1,p}$ estimates for elliptic equations in divergence form*, Comm. Pure Appl. Math. **51** (1998), no. 1, 1–21.
4. R. Coifman, P.-L. Lions, Y. Meyer, and S. Semmes, *Compensated compactness and Hardy spaces*, J. Math. Pures Appl. (9) **72** (1993), no. 3, 247–286.
5. N. B. Firoozye, *n-Laplacian in $\mathcal{H}^1_{loc}$ does not lead to regularity*, Proc. Amer. Math. Soc. **123** (1995), no. 11, 3357–3360.
6. L. Grafakos, *Classical fourier analysis*, 3rd ed., Graduate Texts in Mathematics, vol. 249, Springer, New York, 2014.
7. F. Hélein, *Régularité des applications faiblement harmoniques entre une surface et une variété riemannienne*, C. R. Acad. Sci. Paris Sér. I Math. **312** (1991), no. 8, 591–596.

14692120, 2024, 11, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/blms.13147, Wiley Online Library on [18/05/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

8. F. Hélein, *Harmonic maps, conservation laws and moving frames*, 2nd ed., Cambridge Tracts in Mathematics, vol. 150, Cambridge University Press, Cambridge, 2002. Translated from the 1996 French original, With a foreword by James Eells.
9. T. Kuusi and G. Mingione, *Vectorial nonlinear potential theory*, J. Eur. Math. Soc. (JEMS) **20** (2018), no. 4, 929–1004.
10. D. Martino and A. Schikorra, *Regularizing properties of n-laplace systems with antisymmetric potentials in Lorentz spaces*, Math. Ann. **389** (2024), no. 4, 3301–3348.
11. G. Mingione, *Gradient estimates below the duality exponent*, Math. Ann. **346** (2010), no. 3, 571–627.
12. M. Miśkiewicz, B. Petraszczuk, and P. Strzelecki, *Regularity for solutions of H-systems and n-harmonic maps with $n/2$ square integrable derivatives*, Nonlinear Anal. **232** (2023), Paper No. 113289, 22.
13. T. Rivière, *Conservation laws for conformally invariant variational problems*, Invent. Math. **168** (2007), no. 1, 1–22.
14. A. Schikorra, *A remark on gauge transformations and the moving frame method*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **27** (2010), no. 2, 503–515.
15. A. Schikorra and P. Strzelecki, *Invitation to H-systems in higher dimensions: known results, new facts, and related open problems*, EMS Surv. Math. Sci. **4** (2017), no. 1, 21–42.
16. K. Uhlenbeck, *Regularity for a class of non-linear elliptic systems*, Acta Math. **138** (1977), no. 3-4, 219–240.
17. K. Uhlenbeck, *Connections with L^p bounds on curvature*, Comm. Math. Phys. **83** (1982), no. 1, 31–42.

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