THE NON-ORIENTABLE 4-GENUS OF 11 CROSSING NON-ALTERNATING KNOTS

MEGAN FAIRCHILD

ABSTRACT. The non-orientable 4-genus of a knot K in S^3 is defined to be the minimum first Betti number of a non-orientable surface F smoothly embedded in B^4 so that K bounds F. We will survey the tools used to compute the non-orientable 4-genus, and use various techniques to calculate this invariant for non-alternating 11 crossing knots. We also will view obstructions to a knot bounding a Möbius band given by the double branched cover of S^3 branched over K.

1 INTRODUCTION

Knots bounding orientable surfaces, both in S^3 and B^4, have been extensively studied, however much is still to be learned about the non-orientable surfaces in B^4 bounded by knots. Recently, the non-orientable 4-genus of torus knots has been computed for all knots $T(2,q)$ and $T(3,q)$ by Allen [1], and most knots $T(4,q)$ by Binns, Kang, Simone, Truöl, and Sabloff [2, 15]. The non-orientable 4-genus of double twist knots was calculated by Hoste, Shanahan, and Van Cott [6], and knots with 10 or fewer crossings have also been computed in detail by Ghanbarian, Jabuka, and Kelly [3, 7], with much focus on alternating knots. This paper aims to shed light on the non-alternating case and strategies to calculate the non-orientable 4-genus. We will explore various techniques in finding this invariant, as well as examining obstructions to knots bounding a Möbius band.

For this paper, a knot K is in S^3. The orientable 4-genus of a knot is the minimum genus of an orientable surface smoothly embedded in the 4-ball that is bounded by K and is denoted $g_4(K)$, and knots with $g_4(K) = 0$ are called slice knots. Following Murakami and Yasuhara in [13], the non-orientable 4-genus of a knot K, denoted $\gamma_4(K)$, is defined to be the minimum first Betti number of non-orientable surfaces F smoothly embedded in B^4 bounded by K, that is $\min\{b_1(F) | \partial F = K\}$. Note that the first Betti number is defined to be $b_1(F) = \dim H_1(F;\mathbb{Q})$. We have, by definition, for any knot K, $\gamma_4(K) \geq 1$ where equivalence applies when K bounds a Möbius band. Slice knots that bound a smooth disk embedded in B^4 have non-orientable 4-genus one, as we may attach a non-oriented band to such an embedded disk.

Theorem 1.1. *For the 185 non-alternating 11 crossing knots,*

(a) 121 knots have $\gamma_4(K) = 1$
(b) 58 knots have $\gamma_4(K) = 2$

The remaining 6 knots have $\gamma_4(K) = 1$ *or* 2.

The paper is organized as follows: Section 2 is the background on knot invariants, double branched covers, and useful bounds and obstructions for the non-orientable 4-genus. Section 3 is a survey of the techniques used to solve this problem as well as results.

Acknowledgements. The author was partially supported by NSF Grant No. DMS-1907654 and NSF RTG Grant No. DMS-2231492. I would like to thank Chuck Livingston, Pat Gilmer, and Slaven Jabuka for helpful conversations and comments. Additional thanks to Dror Bar-Natan for permitting my use of the Knot Atlas figures [9].

2 BACKGROUND

We begin by reviewing knot invariants and examining bounds for the non-orientable 4-genus as well as obstructions to a knot bounding a Möbius band. First, the crossing number of a knot is denoted $n(K)$ and is the crossing number of a diagram of a knot with the fewest crossings that could be drawn on the plane to represent the knot. The unknotting number of a knot $u(K)$ is the minimum number of crossing changes required to transform K into the unknot. Similarly, $u_s(K)$ is the minimum number of crossing changes to change K into a slice knot. The 4-dimensional clasp number, $c_4(K)$, is the minimum number of double points of transversely immersed 2-disks in the 4-ball bounded by K [13]. We then have the following triple inequality from Jabuka and Kelly [7]:

$$g_4(K) \leq c_4(K) \leq u_s(K) \leq u(K)$$

The smooth orientable 4-genus of a knot also offers an upper bound for the non-orientable 4-genus [7]:

$$\gamma_4(K) \leq 2g_4(K) + 1$$

Similar to the orientable 4-genus, we obtain an upper bound for the non-orientable 4-genus from the non-orientable 3-genus of a knot called the *crosscap number* [10], which is the minimum genus non-orientable surface a knot bounds in S^3, denoted $c(K)$, so we have $\gamma_4(K) \leq c(K)$.

Following the notation of Murakami and Yasuhara [12], we define $\Gamma_4(K) = \min\{b_1(F) | \partial F = K\}$, or similarly $\Gamma_4(K) = \min\{2g_4(K), \gamma_4(K)\}$, and thus $\Gamma_4(K) \leq \gamma_4(K)$. Murakami and Yasuhara then give us the following proposition [13]:

Proposition 2.1 (Proposition 2.3 in [13])**.** *For any knot K, the following inequalities hold.*

$$\Gamma_4(K) \leq \begin{cases} c_4(K) & \text{if } c_4(K) \text{ is even} \\ c_4(K) + 1 & \text{otherwise} \end{cases}$$

$$\gamma_4(K) \leq \begin{cases} c_4(K) & \text{if } c_4(K) \text{ is even and } c_4(K) \neq 2 \\ c_4(K) + 1 & \text{otherwise} \end{cases}$$

Corollary 2.2 (Corollary 2.4 in [13])**.** *For a knot K, if $g_4(K) = c_4(K) \geq 1$, then $\Gamma_4(K) = \gamma_4(K)$.*

The crossing number of a knot offers an upper bound, so we have [12]:

$$\Gamma(K) \leq \left\lfloor \frac{n(K)}{2} \right\rfloor \text{ and } \gamma_4(K) \leq \left\lfloor \frac{n(K)}{2} \right\rfloor$$

The signature of a knot $\sigma(K)$ is defined to be the signature of the sum of knot's Seifert matrix and its transpose, $\sigma(V + V^t)$. The Arf invariant of a knot is denoted $\mathrm{Arf}(K)$ and is a concordance invariant in $\mathbb{Z}_2$ which is calculated using the Seifert form of a knot [10]. These two invariants provide a lower bound for the non-oriented 4-genus of a knot, so we have the following proposition.

Proposition 2.3 (Proposition 2.4 in [3])**.** *Given a knot K, if $\sigma(K) + 4\mathrm{Arf(K)} \equiv 4 \pmod 8$, then $\gamma_4(K) \geq 2$.*

Double Branched Cover

Recall the definition of the non-orientable 4-genus is $\gamma_4(K) = \min\{b_1(F) | \partial F = K\}$ and note that $b_1(F) = \mathrm{dim} H_1(F, \mathbb{Q})$. Let K in S^3 bound a connected surface F in B^4 and denote the double branched cover of B^4 branched over F as $D_F(B^4)$. Gilmer and Livingston proved in [4], Lemma 1, that $b_2(D_F(B^4)) = b_1(F)$. The reasoning here is that the double branched cover of S^3 branched over K, denoted $D_K(S^3)$, is a rational homology sphere and $H_1(D_F(B^4);\mathbb{Q}) = 0$. We thus may use the linking form of $D_K(S^3)$ to provide information on the intersection form of $D_F(B^4)$.

We also have that the first homology of $D_K(S^3)$ is finite, so we have a linking form λ, and this is explored in detail by Murakami and Yasuhara in [13]

$$\lambda : H_1(D_K(S^3);\mathbb{Z}) \times H_1(D_K(S^3);\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

Given a Goeritz matrix G for K (see Section III for details), we have that G is a relation matrix for $H_1(D_K(S^3);\mathbb{Z})$ and the linking form λ is given by $\pm G^{-1}$, where the sign depends on the orientation of $D_K(S^3)$ [13]. The double branched cover is a useful tool in obstructing knots from bounding a Möbius band or a Klein bottle.

Corollary 2.4 (Corollary 3 in [4]). *Suppose that $H_1(D_K(S^3)) = \mathbb{Z}_n$ where n is the product of primes, all with odd exponent. Then if K bounds a Möbius band in B^4, there is a generator $a \in H_1(D_K(S^3))$ such that $\lambda(a,a) = \pm 1/n$*

Theorem 2.5 (Theorem 4 in [4]). *Suppose that $H_1(D_K(S^3)) = \mathbb{Z}_p \oplus \mathbb{Z}_p$ where p is prime. Then if K bounds a punctured Klein bottle in B^4, the discriminant of the linking form is $\pm 1 \in \mathbb{F}_p^*/(\mathbb{F}_p^*)^2$*

Theorem 2.6 (Theorem 11 in [4]). *Suppose that $H_1(D_K(S^3)) = \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_q$ where $q \equiv 1 \in \boldsymbol{F}_p^*/(\boldsymbol{F}_p^*)^2$. If $H_1(D_K(S^3))$ is the boundary of a 4-manifold W with second Betti number 2 which has an indefinite intersection form, then the linking form restricted to $\mathbb{Z}_p \oplus \mathbb{Z}_p \subset H_1(D_K(S^3))$ is metabolic.*

3 RESULTS AND TECHNIQUES

There are a total of 185 prime knots that are non-alternating and have 11 crossings, according to the KnotInfo Database [10]. Of those knots, there are 16 that are smoothly slice and thus have $\gamma_4(K) = 1$.

Remark 3.1. There are 16 non-alternating 11 crossing knots that are slice and thus bound a Möbius band:

$$11n_4,\ 11n_{21},\ 11n_{37},\ 11n_{39},\ 11n_{42},\ 11n_{49},\ 11n_{50},\ 11n_{67},$$

$$11n_{73},\ 11n_{74},\ 11n_{83},\ 11n_{97},\ 11n_{116},\ 11n_{132},\ 11n_{139},\ 11n_{172}$$

Proposition 3.2. *The following knots have $\gamma_4(K) = 1$:*

$$11n_1,\ 11n_3,\ 11n_5,\ 11n_6,\ 11n_7,\ 11n_8,\ 11n_9,\ 11n_{11},\ 11n_{13},\ 11n_{14},\ 11n_{15},$$

$$11n_{16},\ 11n_{18},\ 11n_{19},\ 11n_{20},\ 11n_{23},\ 11n_{24},\ 11n_{25},\ 11n_{26},\ 11n_{27},\ 11n_{31},\ 11n_{34},$$

$$11n_{36},\ 11n_{41},\ 11n_{44},\ 11n_{45},\ 11n_{46},\ 11n_{47},\ 11n_{52},\ 11n_{54},\ 11n_{57},\ 11n_{58},\ 11n_{59},$$

$$11n_{60},\ 11n_{62},\ 11n_{64},\ 11n_{65},\ 11n_{66},\ 11n_{68},\ 11n_{69},\ 11n_{70},\ 11n_{71},\ 11n_{75},\ 11n_{76},$$

$$11n_{77},\ 11n_{78},\ 11n_{79},\ 11n_{80},\ 11n_{81},\ 11n_{82},\ 11n_{86},\ 11n_{87},\ 11n_{88},\ 11n_{89},\ 11n_{91},$$

$$11n_{93},\ 11n_{94},\ 11n_{96},\ 11n_{102},\ 11n_{104},\ 11n_{105},\ 11n_{106},\ 11n_{107},\ 11n_{110},\ 11n_{111},$$

$$11n_{113},\ 11n_{117},\ 11n_{118},\ 11n_{120},\ 11n_{121},\ 11n_{122},\ 11n_{123},\ 11n_{124},\ 11n_{126},\ 11n_{127},$$

$$11n_{128},\ 11n_{129},\ 11n_{134},\ 11n_{135},\ 11n_{136},\ 11n_{142},\ 11n_{143},\ 11n_{145},\ 11n_{146},\ 11n_{147},$$

$$11n_{148},\ 11n_{150},\ 11n_{151},\ 11n_{152},\ 11n_{153},\ 11n_{154},\ 11n_{157},\ 11n_{158},\ 11n_{160},\ 11n_{162},$$

$$11n_{163},\ 11n_{164},\ 11n_{167},\ 11n_{168},\ 11n_{169},\ 11n_{170},\ 11n_{173},\ 11n_{180},\ 11n_{181},\ 11n_{183}$$

Proposition 3.3. *The following knots have* $\gamma_4(K) = 2$*:*

$$11n_2,\ 11n_{10},\ 11n_{12},\ 11n_{22},\ 11n_{28},\ 11n_{29},\ 11n_{30},\ 11n_{32},\ 11n_{33},\ 11n_{35},$$
$$11n_{38},\ 11n_{43},\ 11n_{48},\ 11n_{51},\ 11n_{53},\ 11n_{55},\ 11n_{56},\ 11n_{61},\ 11n_{63},\ 11n_{72},$$
$$11n_{84},\ 11n_{85},\ 11n_{90},\ 11n_{92},\ 11n_{95},\ 11n_{98},\ 11n_{99},\ 11n_{100},\ 11n_{101},\ 11n_{103},$$
$$11n_{108},\ 11n_{109},\ 11n_{112},\ 11n_{114},\ 11n_{115},\ 11n_{119},\ 11n_{125},\ 11n_{130},\ 11n_{131},\ 11n_{133},$$
$$11n_{137},\ 11n_{138},\ 11n_{140},\ 11n_{141},\ 11n_{144},\ 11n_{149},\ 11n_{155},\ 11n_{156},\ 11n_{161},\ 11n_{165}$$
$$11n_{171},\ 11n_{174},\ 11n_{175},\ 11n_{176},\ 11n_{179},\ 11n_{182},\ 11n_{184},\ 11n_{185},$$

Constraints on Invariants

The knot invariant information for this paper was extracted from Knot Info [10].

Lemma 3.4. *If* K *is a knot satisfying* $\sigma(K)+4\mathrm{Arf(K)} \equiv 4 \pmod 8$*, and* $c_4(K) = 1$*, then* $\gamma_4(K) = 2$*.*

The result is clear from Proposition 2.1 and Corollary 2.3. We now examine knots that have $g_4(K) = u(K) = 1$ (or $g_4(K) = u_s(K) = 1$) and $\sigma(K) + 4\mathrm{Arf(K)} \equiv 4 \pmod 8$ to see the following knots have $\gamma_4(K) = 2$:

$$11n_{12},\ 11n_{28},\ 11n_{48},\ 11n_{53},\ 11n_{55},\ 11n_{85},\ 11n_{100}$$
$$11n_{114},\ 11n_{115},\ 11n_{119},\ 11n_{130},\ 11n_{156},\ 11n_{179},\ 11n_{182}$$

All knots listed above satisfy $\sigma(K) + 4\mathrm{Arf(K)} \equiv 4 \pmod 8$. Since they satisfy $g_4(K) = 1 = u(K)$ (or $g_4(K) = u_s(K) = 1$) by the hypothesis, we have $c_4(K) = 1$, and thus by Lemma 3.4 we may conclude $\gamma_4(K) = 2$.

Lemma 3.5. *Given* K *is a knot satisfying* $\sigma(K) + 4\mathrm{Arf(K)} \equiv 4 \pmod 8$*, and* $c_4(K) = 2$*, then* $\gamma_4(K) = 2$*.*

By Corollary 2.2, we have $\Gamma_4(K) = \gamma_4(K)$, and thus applying Proposition 2.1 we achieve $\gamma_4(K) \leq 2$. Therefore, $\gamma_4(K) = 2$. We now observe that the following knots have $\gamma_4(K) = 2$:

$$11n_2,\ 11n_{35},\ 11n_{95},\ 11n_{103},\ 11n_{108},\ 11n_{109},\ 11n_{144},\ 11n_{149},\ 11n_{174},\ 11n_{175},\ 11n_{185}$$

The knots listed above all satisfy $\sigma(K) + 4\mathrm{Arf(K)} \equiv 4 \pmod 8$ and thus $\gamma_4(K) \geq 2$. Additionally, these knots all satisfy $g_4(K) = u(K) = 2$, and thus $c_4(K) = 2$.

Non-Oriented Band Moves

The primary method used in calculations was via non-oriented band moves. We begin with an oriented knot K and an oriented band, $[0,1] \times [0,1]$. Following the conventions of Jabuka and Kelly [7], we attach the band to K in the sense that the orientation of the band agrees with the orientation of K on $[0,1] \times \{0\}$ but disagrees on $[0,1] \times \{1\}$, or vise versa. One then does surgery along the band. The result of non-orientable band surgery will always be a knot, while the result after *orientable* band surgery is a link. Non-orientable band surgery is explored by Moore and Vazquez in [11] and is called *non-coherent band surgery.*

The notation for a knot K that has been transformed into a knot K' by a non-oriented band move is $K \xrightarrow{h} K'$ where h is either 0, 1, or -1, determined by the number of half twists given to h with respect to the blackboard framing. These three band moves can be seen in the Figure 3.1. From left to right, we have $\xrightarrow{0}$ is the band move without a twist, $\xrightarrow{-1}$ is the band move with a left-handed half twist, and $\xrightarrow{1}$ is the band move with a right handed half twist.

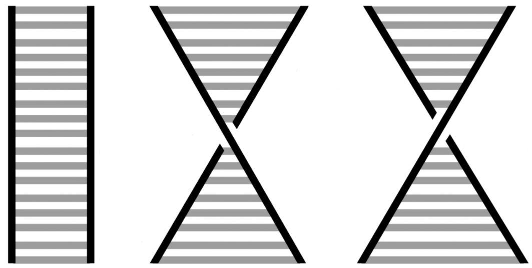


FIGURE 3.1. Band Moves with $h = 0, -1, 1$ from left to right

Proposition 3.6 (Proposition 2.4 in [7]). *If the knots K and K' are related by a non-oriented band move, then*

$$\gamma_4(K) \leq \gamma_4(K') + 1$$

If a knot K is related to a slice knot K' by a non-oriented band move, then $\gamma_4(K) = 1$.

Proof of Theorem 1.1 part (a). Every knot listed in Proposition 3.2 is either a slice knot or one non-oriented band move away from a slice knot. See Figure 4.3 - Figure 4.11 for details.

Lemma 3.7. *The following knots have $\gamma_4(K) = 2$:*

$$11n_{10},\ 11n_{12},\ 11n_{30},\ 11n_{32},\ 11n_{43},\ 11n_{48},\ 11n_{51},\ 11n_{55},\ 11n_{61},\ 11n_{72}$$
$$11n_{85},\ 11n_{90},\ 11n_{98},\ 11n_{103},\ 11n_{130},\ 11n_{133}$$

We now recall Proposition 2.3 and note the knots listed in the above lemma all satisfy $\sigma(K) + 4\mathrm{Arf}(\mathrm{K}) \equiv 4 \pmod 8$. So we know the above knots have $\gamma_4(K) \geq 2$. The above listed knots all are one non-oriented band move away from a knot K' so that $\gamma_4(K') = 1$ (see Figure 4.12 - Figure 4.15), thus we conclude $\gamma_4(K) = 2$.

Linking Form Calculation

We look for a knot K so that $\sigma(K) + 4\mathrm{Arf}(\mathrm{K}) \equiv 0, \pm 2 \pmod 8$, and thus K does not meet the obstruction from Proposition 2.3. We calculate the linking form of $H_1(D_K(S^3))$ to see if K meets the obstruction from Corollary 2.4. The first thing we do is calculate the Goeritz matrix for K. We will do an example here, but an interested reader is referred to Gordan and Litherland [5].

To construct the Goeritz matrix, we first make a checkerboard coloring of a knot.

Each white region is labeled R_i and the unbounded region is R_0. We then assign a value to each crossing C, $\eta(C) = \pm 1$, via the figure below, and following the conventions from Gordan and Litherland [5].

Next, we construct a matrix G' with the algorithm:

$$g'(i,j) = \begin{cases} -\sum \eta(C) \text{ where the sum ranges over all crossings } C \text{ incident to } R_i \text{ and } R_j, i \neq j \\ -\sum_{k \neq i} g'(i,k) = g'(i,i) \text{ if } i = j \end{cases}$$

Then, the Goeritz matrix G is obtained from G' by deleting the 0^{th} row and column. The determinant of G is an invariant of the knot, and G is a linking matrix for $H_1(D_K(S^3))$ [5, 13].

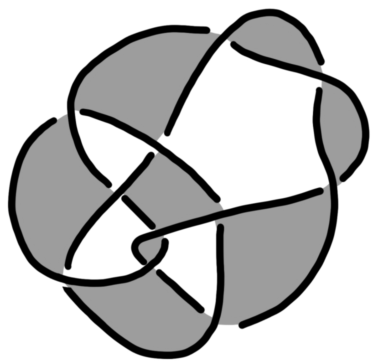


FIGURE 3.2. Checkerboard coloring for $11n_{155}$

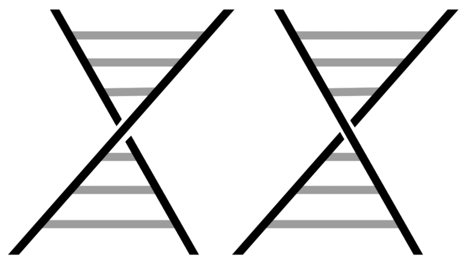


FIGURE 3.3. left: $\eta(C) = 1$, right: $\eta(C) = -1$

Now, we may calculate the linking form. As previously mentioned, $\pm G^{-1}$ represents the linking form λ where $\lambda : H_1(D_K(S^3);\mathbb{Z}) \times H_1(D_K(S^3);\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$. To continue the example, we have G and G^{-1} for the knot $11n_{155}$ as:

$$G = \begin{bmatrix} 3 & -1 & 0 & -1 \\ -1 & 5 & -1 & 0 \\ 0 & -1 & 0 & 2 \\ -1 & 0 & 2 & 0 \end{bmatrix} \qquad G^{-1} = \begin{bmatrix} \frac{20}{51} & \frac{2}{17} & \frac{10}{51} & \frac{1}{17} \\ \frac{2}{17} & \frac{4}{17} & \frac{1}{17} & \frac{2}{17} \\ \frac{10}{51} & \frac{1}{17} & \frac{5}{51} & \frac{9}{17} \\ \frac{1}{17} & \frac{2}{17} & \frac{9}{17} & \frac{1}{17} \end{bmatrix}$$

Now we have the linking form $\lambda(g,g) = \pm 20/51$. Suppose $11n_{155}$ bounds a Möbius band. We wish to find an $n \in \mathbb{Z}$ so that $\lambda(ng,ng) = \pm 1/51$. This means $\pm 20/51 = \lambda(ng,ng) = n^2\lambda(g,g) = \pm 20n^2/51 = \pm 1/51$, so $20n^2 \equiv \pm 1 \pmod{51}$. A quick calculation shows this is not possible, and thus $11n_{155}$ does not bound a Möbius band.

Results

Theorem 3.8 (Theorem 2 in [4])**.** *Let K in S^3 be a knot. The linking form $(H_1(D_K(S^3),\lambda)$ splits as a direct sum $(G_1,\lambda_1) \oplus (G_2,\lambda_2)$ where (G_2,λ_2) is metabolic and (G_1,λ_1) has a presentation of rank $\lambda_1(F)$.*

Lemma 3.9. *Let K in S^3 be a knot and suppose that $H_1(D_K(S^3)) = \mathbb{Z}_{p^2q}$ where p is prime and q is a product of primes, all with odd exponent. Then if K bounds a Möbius band in B^4, there is a generator $a \in H_1(D_K(S^3))$ such that either $\lambda(a,a) = \pm 1/p^2q$ or $\lambda(a,a) = \pm 1/q$.*

Proof. As we see in Theorem 3.8, $(H_1(D_K(S^3)),\lambda)$ splits as a direct sum $(G_1,\lambda_1) \oplus (G_2,\lambda_2)$ where (G_2,λ_2) is metabolic and λ_1 is presented by the linking matrix of $D_K(S^3)$, which has a presentation of rank one. As q is square-free, we have that $\mathbb{Z}_q$ is completely contained (as a subgroup) in G_1. Then either $\mathbb{Z}_{p^2}$ is completely contained in G_2, which implies it is metabolic, or $\mathbb{Z}_{p^2}$ is contained in G_1.

If $\mathbb{Z}_{p^2}$ is completely contained in G_2, then there exists a subgroup H of $\mathbb{Z}_{p^2}$ so that $|H|^2 = p^2$ and $\lambda(g, g') = 0$ for any $g, g' \in H$, since λ_2 is metabolic. Then, as λ_1 must have a presentation of rank one, we have that the presentation matrix must be of the form $(\pm|G_1|) = (\pm q)$. Therefore, the linking form λ_1 on G_1 is given by $\pm 1/q$.

If $\mathbb{Z}_{p^2}$ is completely contained in G_1, a similar argument shows λ_1 is given by $\pm 1/q$ □

The following knots:

$$11n_{22},\ 11n_{29},\ 11n_{33},\ 11n_{56},\ 11n_{84},\ 11n_{92},\ 11n_{101},\ 11n_{112},\ 11n_{125},\ 11n_{131},\ 11n_{138},\ 11n_{155},\ 11n_{176},\ 11n_{184}$$

have the respective linking forms:

$$\frac{42}{55},\ \frac{14}{51},\ \frac{22}{51},\ \frac{12}{35},\ \frac{18}{35},\ \frac{2}{15},\ \frac{19}{39},\ \frac{53}{55},\ \frac{61}{63},\ \frac{39}{67},\ \frac{13}{15},\ \frac{20}{51},\ \frac{11}{63},\ \frac{2}{87}$$

All of the 14 linking forms listed above satisfy the obstruction from Corollary 2.4 and Lemma 3.9. Additionally, all of these knots have an non-orientable band move to a knot K' where $\gamma_4(K') = 1$ (Figures 4.12 -4.15). Thus, each of these knots has non-orientable 4-genus equal to 2.

Knot Floer Homology

Ozsváth, Stipsicz, and Szabó explored non-orientable knot floer homology and how the Upsilon invariant provides lower bounds for the non-orientable 4-genus [14]. Given K is a knot, denote $\Upsilon_K(1)$ as $\upsilon(K)$ (lower case Upsilon), and then we have:

$$\left|\upsilon(K) - \frac{\sigma(K)}{2}\right| \leq \gamma_4(K)$$

However, if K is not an L-space knot, this invariant is rather difficult to compute. Additionally, we have from [14] that for an alternating (or quasi-alternating) knot K,

$$\upsilon(K) = \frac{\sigma(K)}{2}$$

For the 185 non-alternating 11-crossing knots, only 3 are not quasi-alternating. Of those 3, two are slice and one is not. This is thus not a useful lower bound for the knots being considered in this paper. However, this is a useful invariant for torus knots, demonstrated in detail by Binns, Kang, Simone, and Truöl in [2]. Additionally, Allen explored a geography problem where the Upsilon invariant was wonderfully utilized in [1].

4 SPECIAL CASES

Lemma 4.1. *The knot $11n_{38}$ does not bound a Möbius band.*

The knot $11n_{38}$ has $H_1(D_K(S^3)) = \mathbb{Z}_3$ and thus the linking form is represented by the 1×1 matrix $[1/3]$. This is clear, as the non-zero elements of $\mathbb{Z}_3$ are 1 and -1. Then, if K bounds a Möbius band F in B^4, we have $b(F) = b(D_F(B^4)) = 1$ and $D_F(B^4)$ is negative definite [4]. From Theorem 3 in [5], we have that the intersection form on $H_2(D_F(B^4)))$ is represented by the linking matrix on $H_1(D_K(S^3))$, which can be viewed from the entries in the Goeritz matrix. The Goeritz matrix G is a 4×4 matrix that is indefinite, and when diagonalized, $G = SJS^{-1}$, the matrix J is also indefinite. We may suppose that there exists a presentation matrix that represents the linking form, and by checking the diagonal entries on $-G^{-1}$, we have that $1/3$ represents the form. This implies the manifold is positive definite, which is a contradiction. Thus, $11n_{38}$ does not bound a Möbius band. We then have that there is a non-orientable band move from $11n_{38}$ to the trefoil knot, which has $\gamma_4(3_1) = 1$, therefore we may conclude that $\gamma_4(11n_{38}) = 2$. The figure below was obtained from Knot Atlas [9].

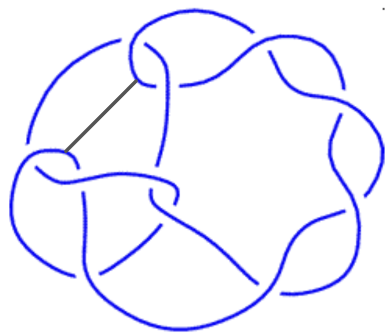


FIGURE 4.1. A non-oriented band move from $11n_{38} \xrightarrow{0} 3_1$

We thus have a combination of Lemmas 3.4, 3.5, and 4.1, Proposition 3.6, and Theorem 3.8 showing Proposition 3.3 is true, thus proving part (b) of Theorem 1.1.

Lemma 4.2. *The knots* $11n_{17}$, $11n_{40}$, $11n_{159}$, $11n_{166}$, $11n_{177}$ *and* $11n_{178}$ *all have* $\gamma_4(K) = 1$ *or 2.*

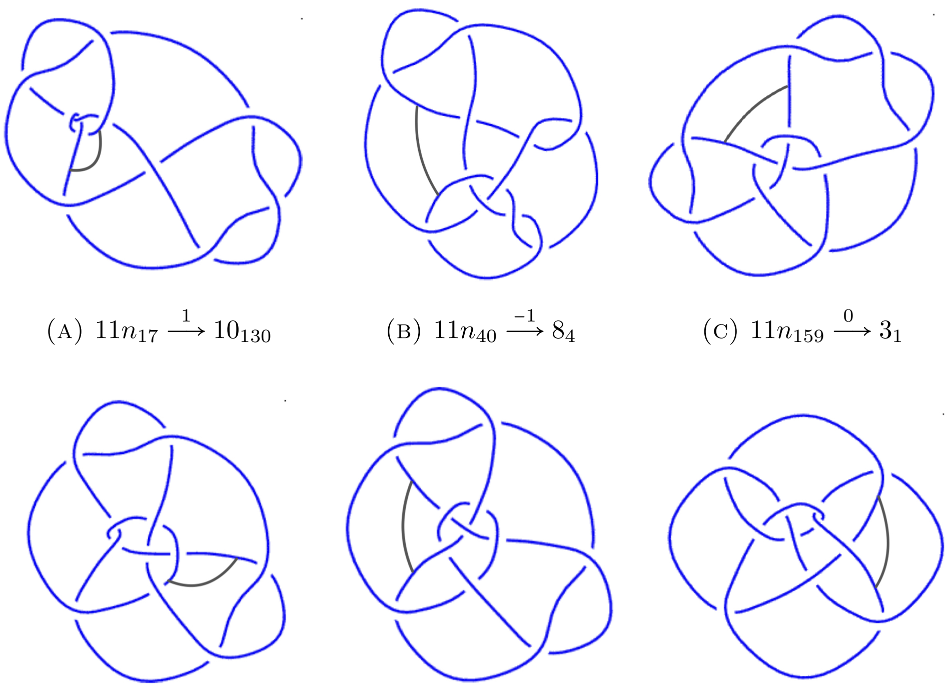
We have the following table:

Knot	linking form	definiteness of $D_F(B^4)$	4-genus
$11n_{17}$	$1/47$	positive	1
$11n_{40}$	$-1/79$	negative	1
$11n_{159}$	$1/71$	positive	1
$11n_{166}$	$1/59$	positive	1
$11n_{177}$	$1/83$	positive	1
$11n_{178}$	$-1/95$	negative	1

Proof. Denote K as a knot listed in Lemma 4.2. We first examine the knot signature and Arf invariant to see $\sigma(K) + 4\mathrm{Arf(K)} \equiv \pm 2 \pmod 8$. Thus, no obstruction arises from Proposition 2.3, so we may only conclude $\gamma_4(K) \geq 1$. We then move on to examining the linking form of K. Note that the determinant of K, $d = \det(K)$, is either a prime number or a product of exactly 2 prime numbers. As $d = |H_1(D_K(S^3))|$, we cannot have a splitting of $H_1(D_K(S^3))$ into $G_1 \oplus G_2$ where

G_2 is metabolic, since d is square free. We thus see that the linking form λ for each knot is of the form $\pm 1/d$. We also compare the linking form of the knot to the definitness of $D_F(B^4)$. The sign of the 4-manifold $D_F(B^4)$ corresponds to the sign of the quadratic form [4], thus the linking form, and we see that our signs are corresponding for the linking form and definiteness of $D_F(B^4)$. Additionally, each knot is one band move away from a knot K' so that $\gamma_4(K') = 1$, see Figure 4.2, and thus $\gamma_4(K) \leq 2$. We thus cannot find an obstruction to these knots bounding a Möbius band, but also cannot find the desired band move to a slice knot. Therefore, $\gamma_4(K) \leq 2$ for the knots in Lemma 4.2. □

This concludes the proof for Theorem 1.1.



(A) $11n_{17} \xrightarrow{1} 10_{130}$ (B) $11n_{40} \xrightarrow{-1} 8_4$ (C) $11n_{159} \xrightarrow{0} 3_1$

(D) $11n_{166} \xrightarrow{1} 10_{142}$ (E) $11n_{177} \xrightarrow{0} 3_1$ (F) $11n_{178} \xrightarrow{-1} 9_{32}$

Figure 4.2. Non-oriented band moves from the knots $11n_{17}, 11n_{40}, 11n_{159}, 11n_{166}, 11n_{177},$ and $11n_{178}$ to knots with non-orientable genus 1.

Concordance

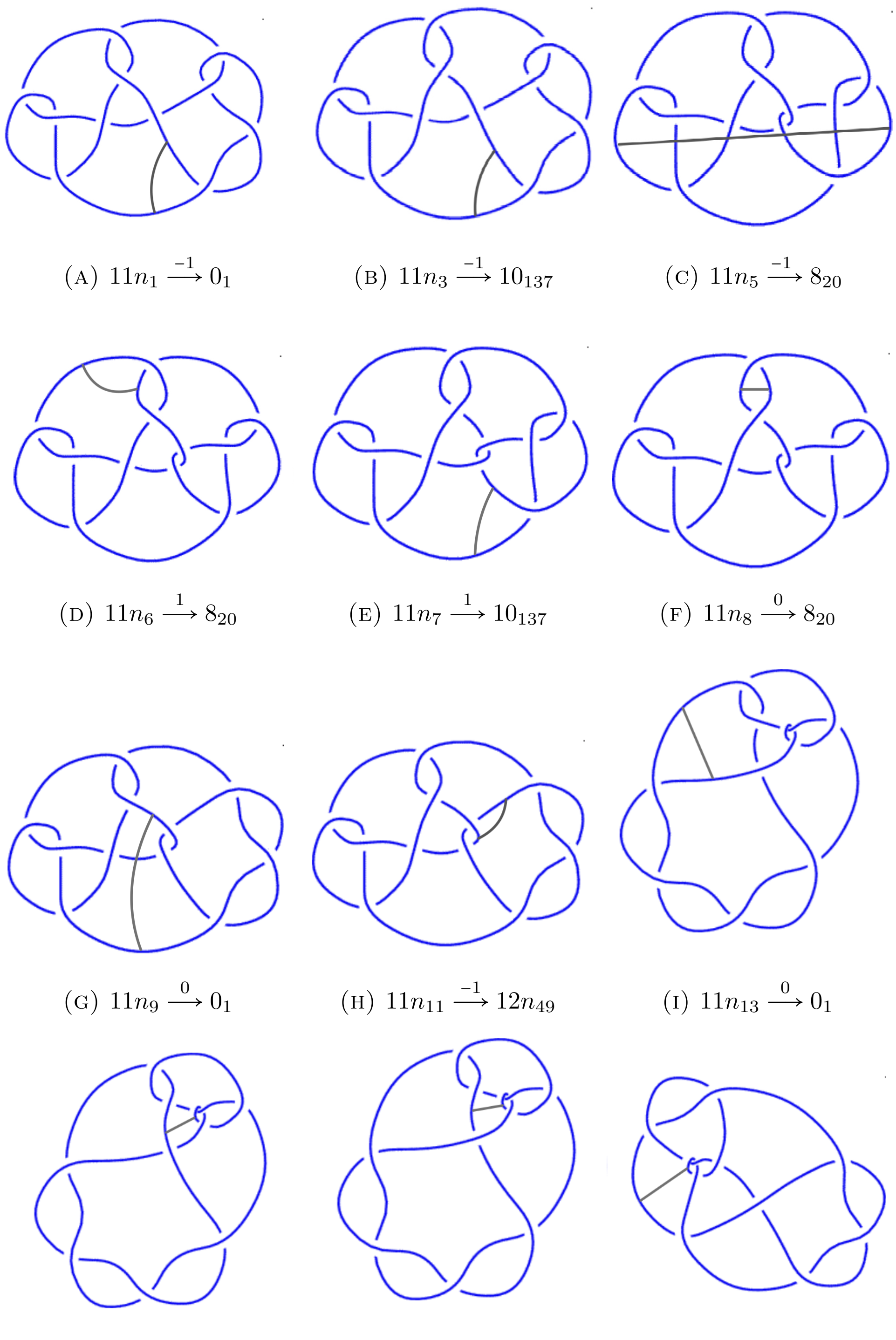
Given K and J two concordant knots, it is well known that $g_4(K) = g_4(J)$ and easy to see that $\gamma_4(K) = \gamma_4(J)$. Thus one may wonder if studying concordance of knots may help us solve this non-orientable 4-genus problem. For the six remaining knots, their concordance genus is known [10], however the knots to which they are concordant is still unknown.

Question 4.3. Is $11n_{40}$ concordant to 10_{57}?

10_{57} is a wonderful candidate for concordance to $11n_{40}$, just by a simple analysis of their invariants [10]. If the answer to Question 4.3 is yes, then the knot $11n_{40}$ has $\gamma_4(11n_{40}) = 1$.

Conjecture 4.4. The knots $11n_{17}$, $11n_{159}$, $11n_{166}$, $11n_{177}$, and $11n_{178}$ are not concordant to any knot with 11 or fewer crossings. Moreover, $11n_{17}$, $11n_{159}$, and $11n_{166}$ are not concordant to any knot with 12 or fewer crossings.

It should be noted that Kearny has found the concordance genus of 11-crossing knots in [8], as well as specific concordances from 11-crossing knots to knots of lower crossings.



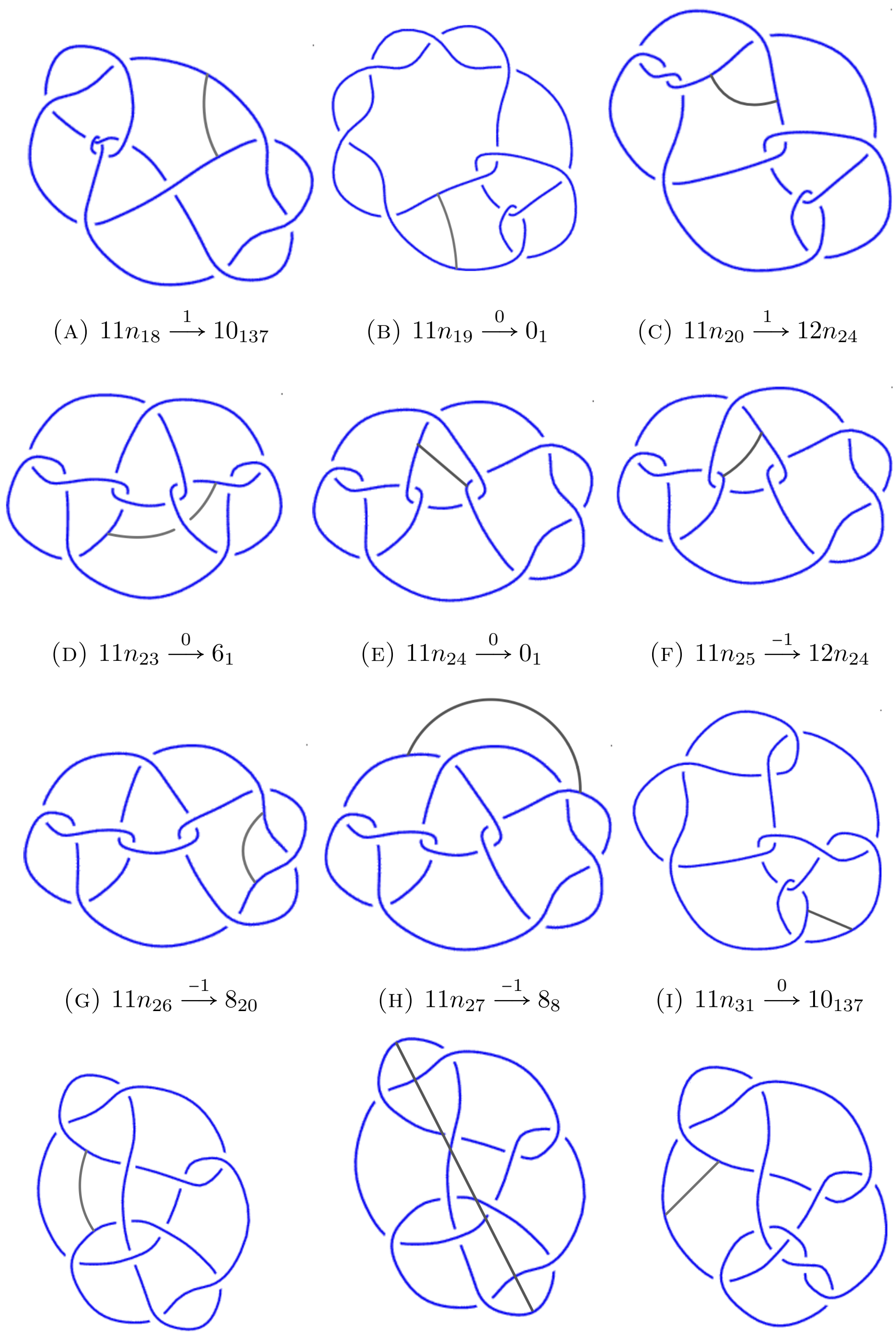
(A) $11n_1 \xrightarrow{-1} 0_1$ (B) $11n_3 \xrightarrow{-1} 10_{137}$ (C) $11n_5 \xrightarrow{-1} 8_{20}$

(D) $11n_6 \xrightarrow{1} 8_{20}$ (E) $11n_7 \xrightarrow{1} 10_{137}$ (F) $11n_8 \xrightarrow{0} 8_{20}$

(G) $11n_9 \xrightarrow{0} 0_1$ (H) $11n_{11} \xrightarrow{-1} 12n_{49}$ (I) $11n_{13} \xrightarrow{0} 0_1$

(J) $11n_{14} \xrightarrow{0} 6_1$ (K) $11n_{15} \xrightarrow{0} 0_1$ (L) $11n_{16} \xrightarrow{0} 0_1$

FIGURE 4.3. Non-oriented band moves from the knots $11n_1, 11n_3, 11n_5, 11n_6, 11n_7, 11n_8, 11n_9, 11n_{11}, 11n_{13}, 11n_{14}, 11n_{15}$, and $11n_{16}$ to smoothly slice knots.



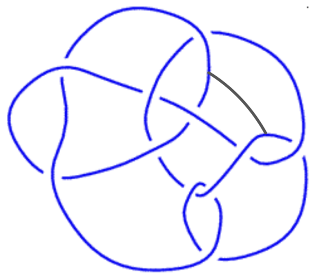
(A) $11n_{18} \xrightarrow{1} 10_{137}$ (B) $11n_{19} \xrightarrow{0} 0_1$ (C) $11n_{20} \xrightarrow{1} 12n_{24}$

(D) $11n_{23} \xrightarrow{0} 6_1$ (E) $11n_{24} \xrightarrow{0} 0_1$ (F) $11n_{25} \xrightarrow{-1} 12n_{24}$

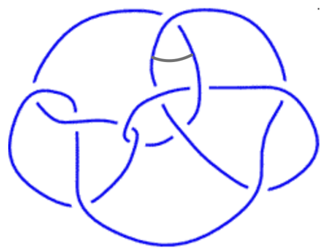
(G) $11n_{26} \xrightarrow{-1} 8_{20}$ (H) $11n_{27} \xrightarrow{-1} 8_8$ (I) $11n_{31} \xrightarrow{0} 10_{137}$

(J) $11n_{34} \xrightarrow{0} 0_1$ (K) $11n_{36} \xrightarrow{0} 10_{129}$ (L) $11n_{41} \xrightarrow{0} 8_{20}$

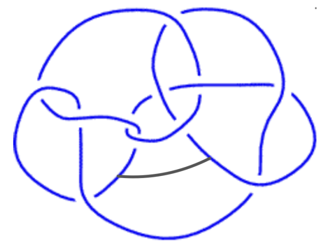
FIGURE 4.4. Non-oriented band moves from the knots $11n_{18}, 11n_{19}, 11n_{20}$, $11n_{23}, 11n_{24}, 11n_{25}, 11n_{26}, 11n_{27}, 11n_{31}, 11n_{34}, 11n_{36}$, and $11n_{41}$ to smoothly slice knots.



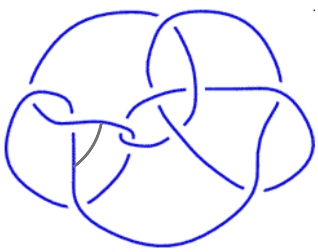
(A) $11n_{44} \xrightarrow{1} 6_1$



(B) $11n_{45} \xrightarrow{1} 10_{129}$

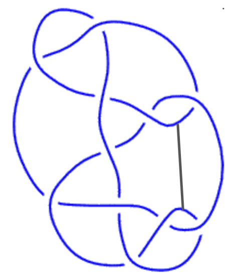


(C) $11n_{46} \xrightarrow{0} 6_1$

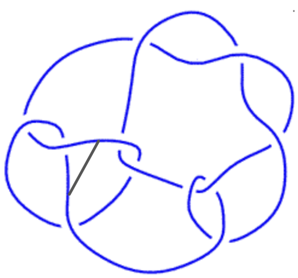


(D) $11n_{47} \xrightarrow{0} 8_{20}$

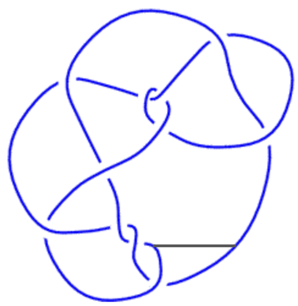
(E) $11n_{52} \xrightarrow{1} 12n_{170}$



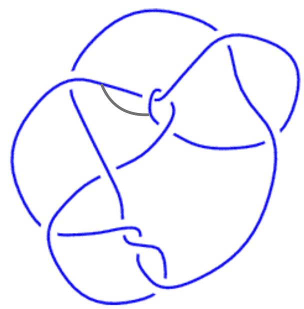
(F) $11n_{54} \xrightarrow{0} 6_1$



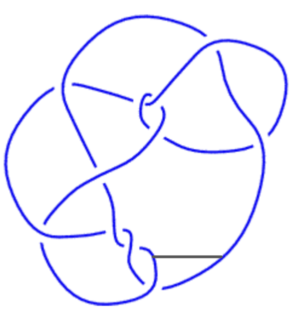
(G) $11n_{57} \xrightarrow{0} 0_1$



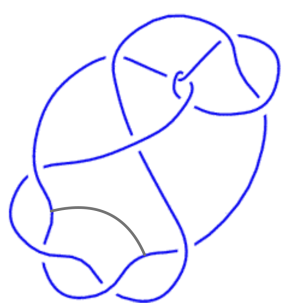
(H) $11n_{58} \xrightarrow{-1} 8_{20}$



(I) $11n_{59} \xrightarrow{0} 8_9$



(J) $11n_{60} \xrightarrow{-1} 8_{20}$

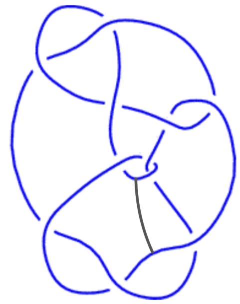


(K) $11n_{62} \xrightarrow{1} 0_1$

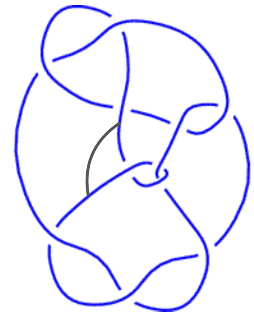
(L) $11n_{64} \xrightarrow{0} 0_1$

FIGURE 4.5. Non-oriented band moves from the knots $11n_{44}, 11n_{45}, 11n_{46}, 11n_{47}, 11n_{52}, 11n_{54}, 11n_{57}, 11n_{58}, 11n_{59}, 11n_{60}, 11n_{62}$, and $11n_{64}$ to smoothly slice knots.

(A) $11n_{65} \xrightarrow{0} 9_{46}$

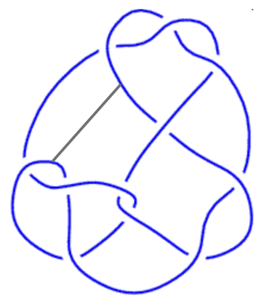


(B) $11n_{66} \xrightarrow{0} 8_{8}$

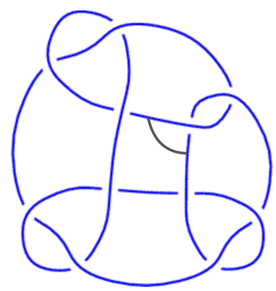


(C) $11n_{68} \xrightarrow{-1} 10_{129}$

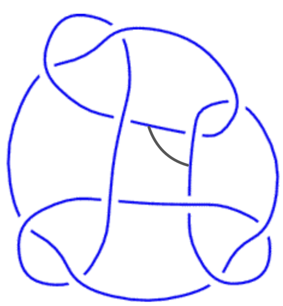
(D) $11n_{69} \xrightarrow{0} 8_{20}$



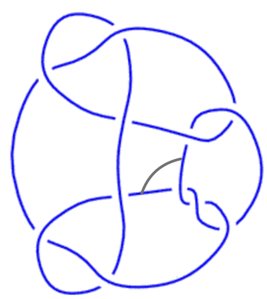
(E) $11n_{70} \xrightarrow{0} 8_{20}$



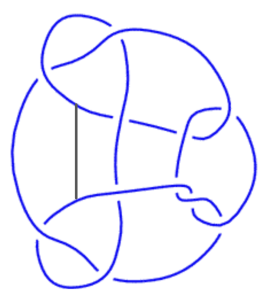
(F) $11n_{71} \xrightarrow{-1} 12n_{556}$



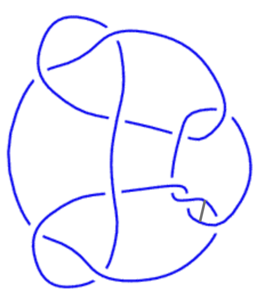
(G) $11n_{75} \xrightarrow{1} 12n_{553}$



(H) $11n_{76} \xrightarrow{0} 8_{20}$

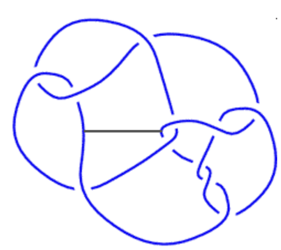


(I) $11n_{77} \xrightarrow{-1} 8_{20}$



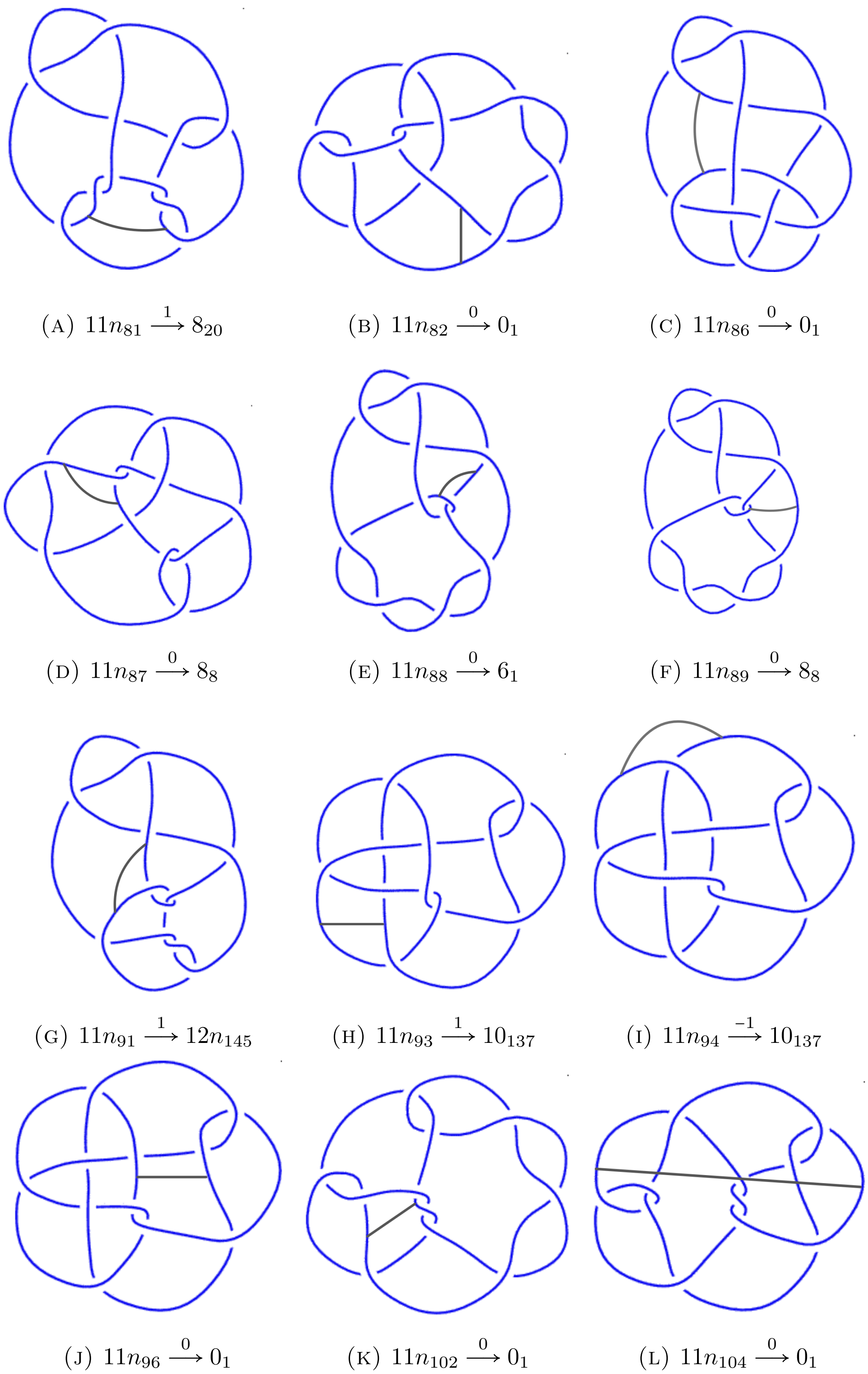
(J) $11n_{78} \xrightarrow{0} 8_{20}$

(K) $11n_{79} \xrightarrow{0} 0_{1}$



(L) $11n_{80} \xrightarrow{-1} 0_{1}$

Figure 4.6. Non-oriented band moves from the knots $11n_{65}, 11n_{66}, 11n_{68}, 11n_{69}, 11n_{70}, 11n_{71}, 11n_{75}, 11n_{76}, 11n_{77}, 11n_{78}, 11n_{79}$, and $11n_{80}$ to smoothly slice knots.



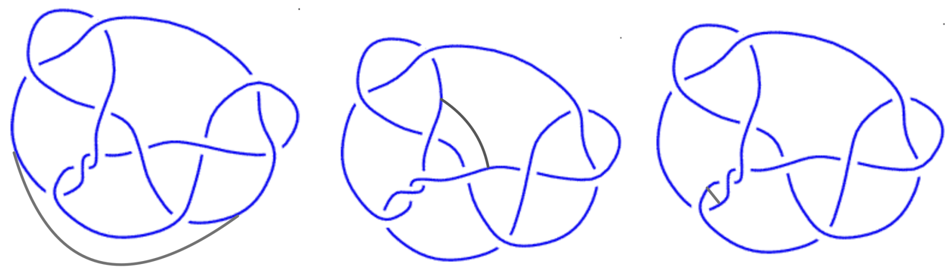
(A) $11n_{81} \xrightarrow{1} 8_{20}$ (B) $11n_{82} \xrightarrow{0} 0_1$ (C) $11n_{86} \xrightarrow{0} 0_1$

(D) $11n_{87} \xrightarrow{0} 8_8$ (E) $11n_{88} \xrightarrow{0} 6_1$ (F) $11n_{89} \xrightarrow{0} 8_8$

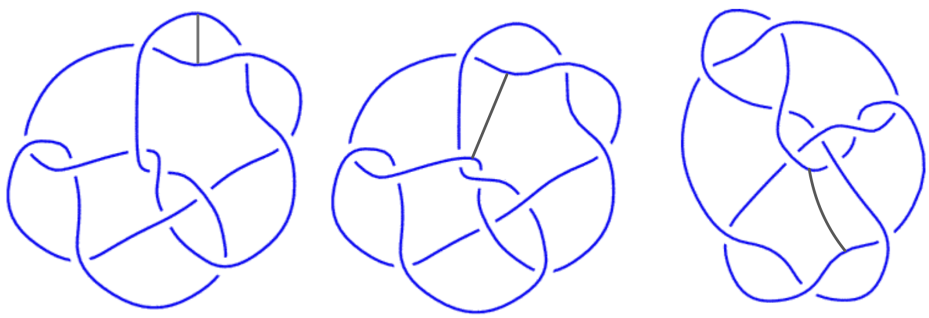
(G) $11n_{91} \xrightarrow{1} 12n_{145}$ (H) $11n_{93} \xrightarrow{1} 10_{137}$ (I) $11n_{94} \xrightarrow{-1} 10_{137}$

(J) $11n_{96} \xrightarrow{0} 0_1$ (K) $11n_{102} \xrightarrow{0} 0_1$ (L) $11n_{104} \xrightarrow{0} 0_1$

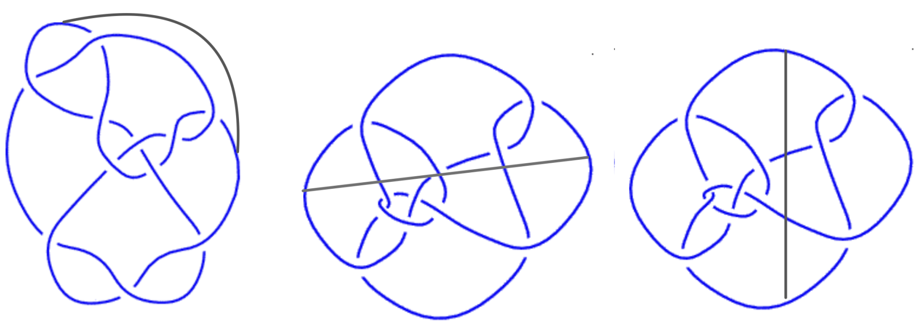
FIGURE 4.7. Non-oriented band moves from the knots $11n_{81}, 11n_{82}, 11n_{86}, 11n_{87}, 11n_{88}, 11n_{89}, 11n_{91}, 11n_{93}, 11n_{94}, 11n_{96}, 11n_{102}$, and $11n_{104}$ to smoothly slice knots.



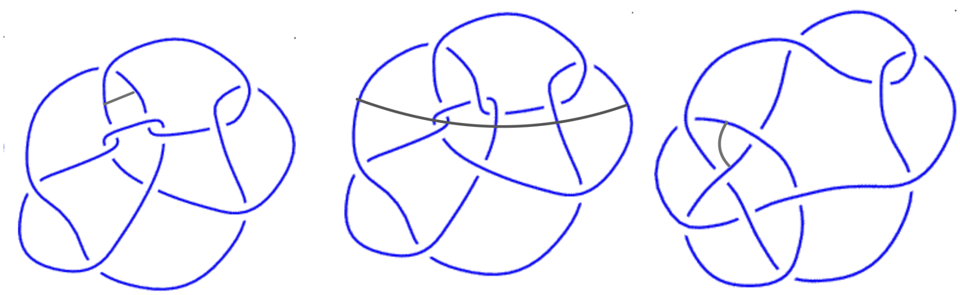
(A) $11n_{105} \xrightarrow{1} 9_{27}$ (B) $11n_{106} \xrightarrow{0} 0_1$ (C) $11n_{107} \xrightarrow{0} 0_1$



(D) $11n_{110} \xrightarrow{0} 8_8$ (E) $11n_{111} \xrightarrow{1} 0_1$ (F) $11n_{113} \xrightarrow{-1} 9_{47}$

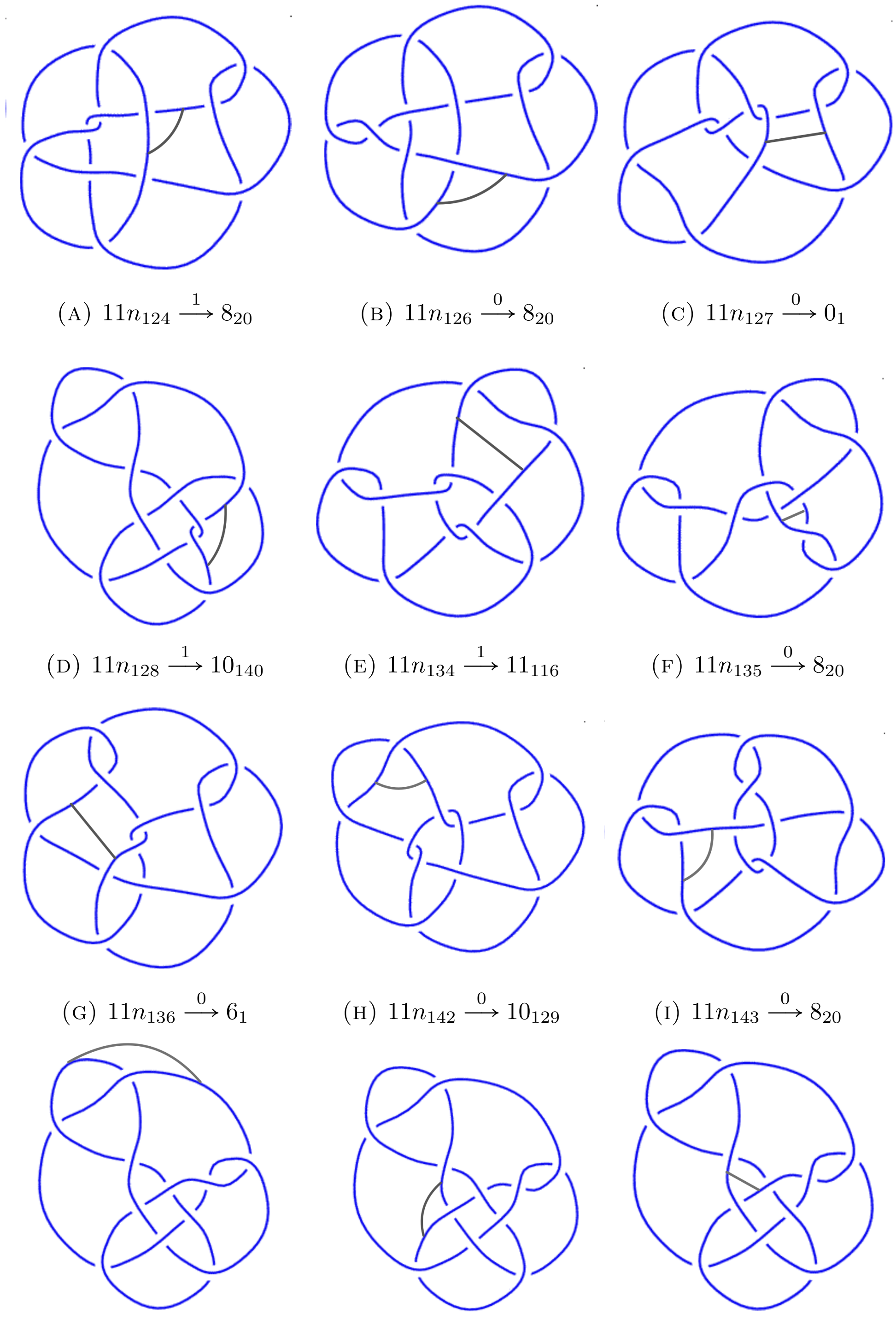


(G) $11n_{117} \xrightarrow{1} 12n_{414}$ (H) $11n_{118} \xrightarrow{0} 0_1$ (I) $11n_{120} \xrightarrow{1} 12n_{312}$



(J) $11n_{121} \xrightarrow{0} 0_1$ (K) $11n_{122} \xrightarrow{0} 0_1$ (L) $11n_{123} \xrightarrow{0} 9_{27}$

FIGURE 4.8. Non-oriented band moves from the knots $11n_{105}, 11n_{106}, 11n_{107}, 11n_{110}, 11n_{111}, 11n_{113}, 11n_{117}, 11n_{118}, 11n_{120}, 11n_{121}, 11n_{122},$ and $11n_{123}$ to smoothly slice knots.



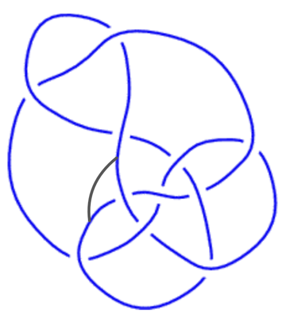
(A) $11n_{124} \xrightarrow{1} 8_{20}$ (B) $11n_{126} \xrightarrow{0} 8_{20}$ (C) $11n_{127} \xrightarrow{0} 0_1$

(D) $11n_{128} \xrightarrow{1} 10_{140}$ (E) $11n_{134} \xrightarrow{1} 11_{116}$ (F) $11n_{135} \xrightarrow{0} 8_{20}$

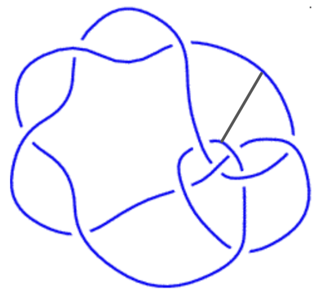
(G) $11n_{136} \xrightarrow{0} 6_1$ (H) $11n_{142} \xrightarrow{0} 10_{129}$ (I) $11n_{143} \xrightarrow{0} 8_{20}$

(J) $11n_{145} \xrightarrow{1} 6_1$ (K) $11n_{146} \xrightarrow{-1} 10_{137}$ (L) $11n_{147} \xrightarrow{0} 6_1$

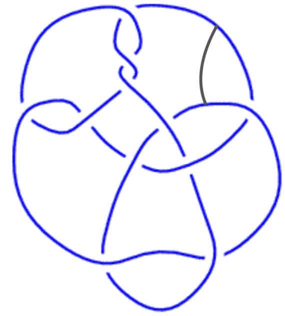
FIGURE 4.9. Non-oriented band moves from the knots $11n_{124}, 11n_{126}, 11n_{127}, 11n_{128}, 11n_{134}, 11n_{135}, 11n_{136}, 11n_{142}, 11n_{143}, 11n_{145}, 11n_{146},$ and $11n_{147}$ to smoothly slice knots.



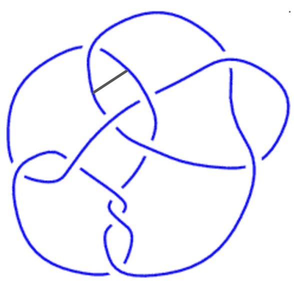
(A) $11n_{148} \xrightarrow{1} 10_{137}$



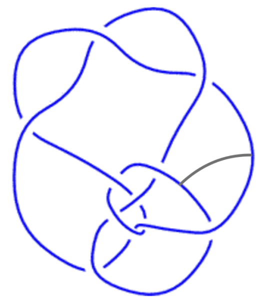
(B) $11n_{150} \xrightarrow{0} 8_8$



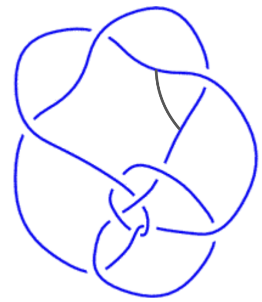
(C) $11n_{151} \xrightarrow{-1} 10_{153}$



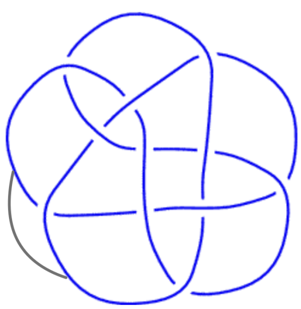
(D) $11n_{152} \xrightarrow{0} 10_{153}$



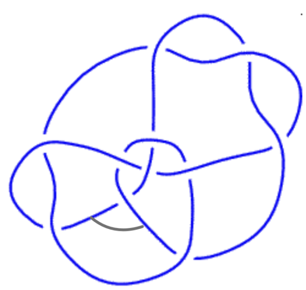
(E) $11n_{153} \xrightarrow{1} 10_{129}$



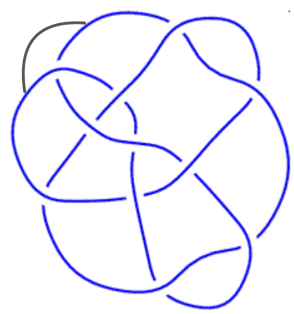
(F) $11n_{154} \xrightarrow{-1} 12n_{504}$



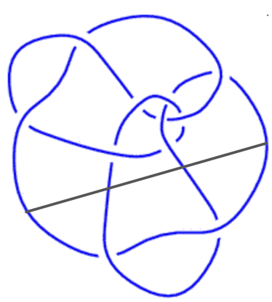
(G) $11n_{157} \xrightarrow{-1} 9_{27}$



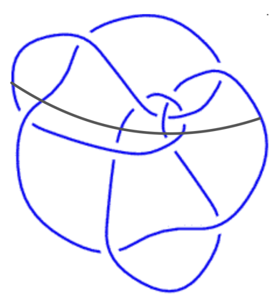
(H) $11n_{158} \xrightarrow{0} 0_1$



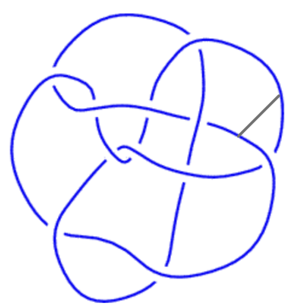
(I) $11n_{160} \xrightarrow{1} 12n_{802}$



(J) $11n_{162} \xrightarrow{-1} 10_{140}$

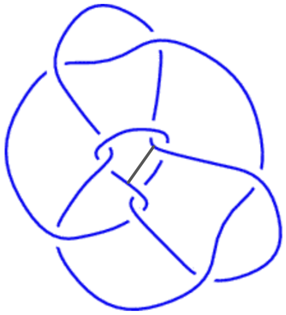


(K) $11n_{163} \xrightarrow{0} 8_8$

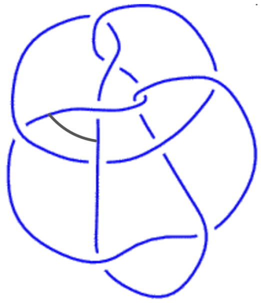


(L) $11n_{164} \xrightarrow{0} 8_{20}$

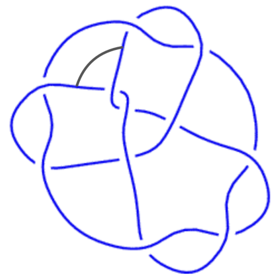
Figure 4.10. Non-oriented band moves from the knots $11n_{148}, 11n_{150}, 11n_{151}, 11n_{152}, 11n_{153}, 11n_{154}, 11n_{157}, 11n_{158}, 11n_{160}, 11n_{162}, 11n_{163},$ and $11n_{164}$ to smoothly slice knots.



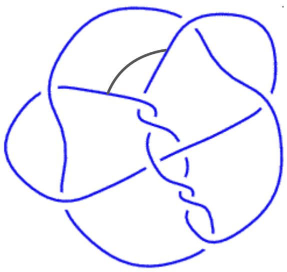
(A) $11n_{167} \xrightarrow{0} 6_1$



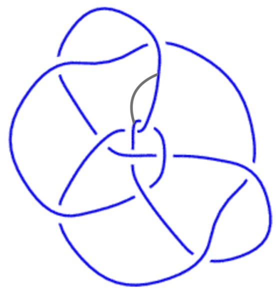
(B) $11n_{168} \xrightarrow{-1} 10_{137}$



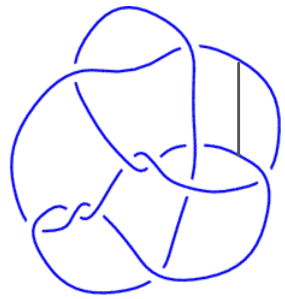
(C) $11n_{169} \xrightarrow{-1} 12n_{817}$



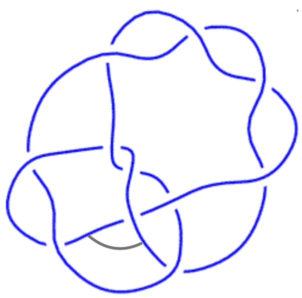
(D) $11n_{170} \xrightarrow{1} 12n_{876}$



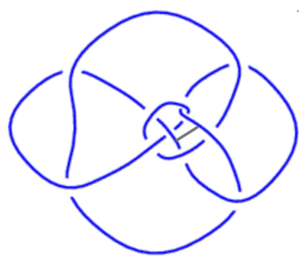
(E) $11n_{173} \xrightarrow{1} 9_{46}$



(F) $11n_{180} \xrightarrow{0} 6_1$

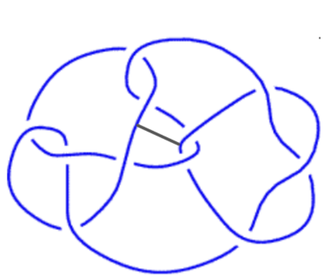


(G) $11n_{181} \xrightarrow{0} 6_1$

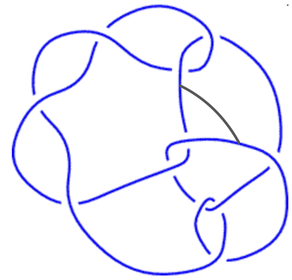


(H) $11n_{183} \xrightarrow{0} 0_1$

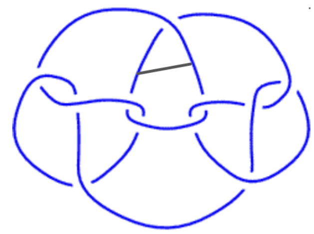
FIGURE 4.11. Non-oriented band moves from the knots $11n_{167}, 11n_{168}, 11n_{169}, 11n_{170}, 11n_{173}, 11n_{180}, 11n_{181},$ and $11n_{183}$ to smoothly slice knots.



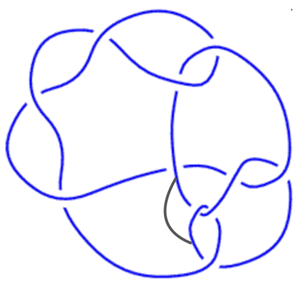
(A) $11n_{10} \xrightarrow{0} 7_6$



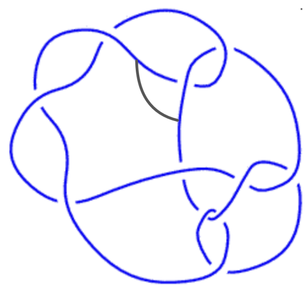
(B) $11n_{12} \xrightarrow{0} 6_2$



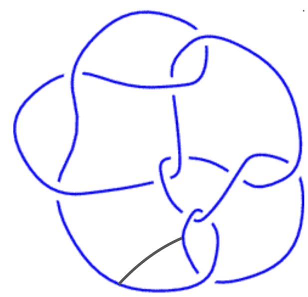
(C) $11n_{22} \xrightarrow{-1} 5_2$



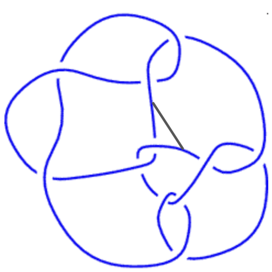
(D) $11n_{29} \xrightarrow{0} 8_6$



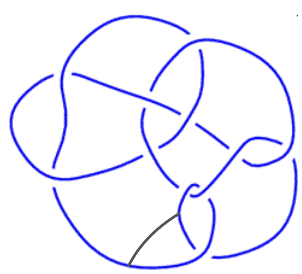
(E) $11n_{30} \xrightarrow{-1} 10_{126}$



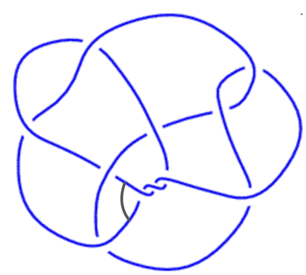
(F) $11n_{32} \xrightarrow{0} 9_{25}$



(G) $11n_{33} \xrightarrow{1} 10_{134}$

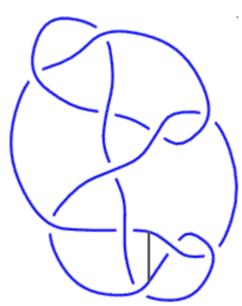


(H) $11n_{43} \xrightarrow{0} 9_{32}$



(I) $11n_{48} \xrightarrow{0} 7_2$

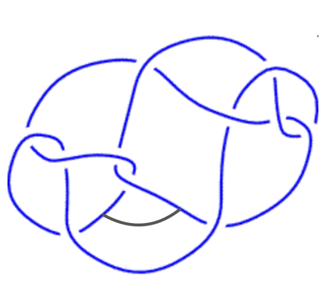
(J) $11n_{51} \xrightarrow{1} 9_8$



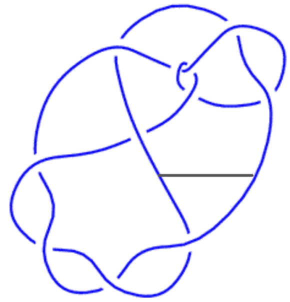
(K) $11n_{55} \xrightarrow{0} 9_{45}$

(L) $11n_{56} \xrightarrow{0} 9_{43}$

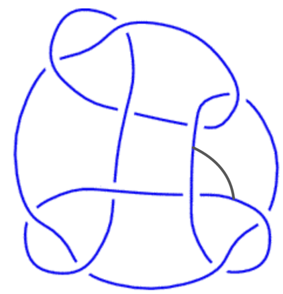
FIGURE 4.12. Non-oriented band moves from the knots $11n_{10}, 11n_{12}, 11n_{22}, 11n_{29}, 11n_{30}, 11n_{32}, 11n_{33}, 11n_{43}, 11n_{48}, 11n_{51}, 11n_{55}$, and $11n_{56}$ to knots with non-orientable genus 1.



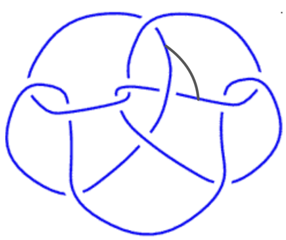
(A) $11n_{61} \xrightarrow{0} 6_2$



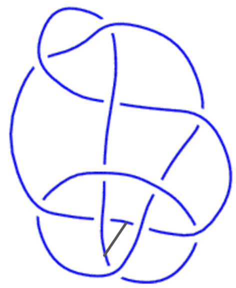
(B) $11n_{63} \xrightarrow{1} 10_{131}$



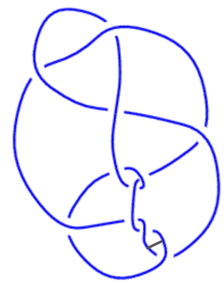
(C) $11n_{72} \xrightarrow{0} 9_{28}$



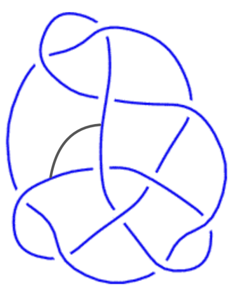
(D) $11n_{84} \xrightarrow{-1} 9_{44}$



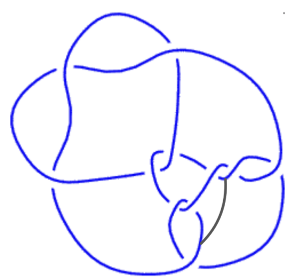
(E) $11n_{85} \xrightarrow{0} 5_2$



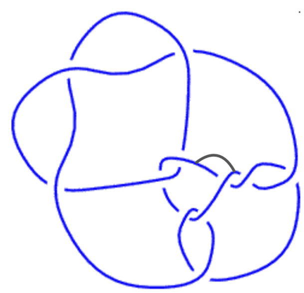
(F) $11n_{90} \xrightarrow{1} 10_147$



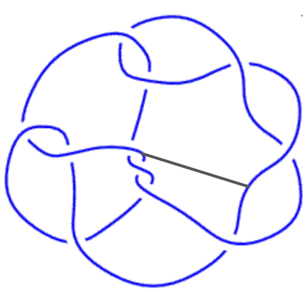
(G) $11n_{92} \xrightarrow{1} 9_8$



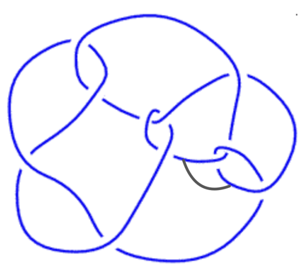
(H) $11n_{98} \xrightarrow{0} 8_6$



(I) $11n_{99} \xrightarrow{1} 10_{148}$



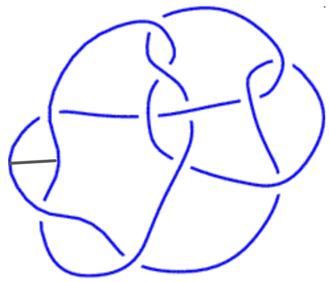
(J) $11n_{101} \xrightarrow{0} 6_2$



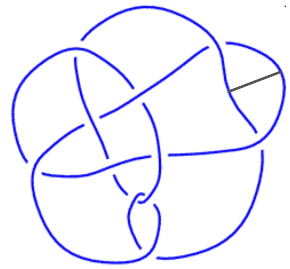
(K) $11n_{103} \xrightarrow{0} 9_{45}$

(L) $11n_{112} \xrightarrow{0} 8_6$

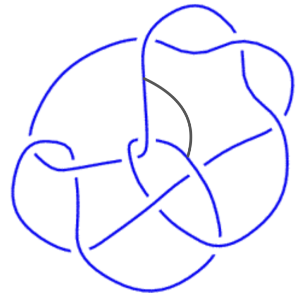
FIGURE 4.13. Non-oriented band moves from the knots $11n_{61}, 11n_{63}, 11n_{72}, 11n_{84}, 11n_{85}, 11n_{90}, 11n_{92}, 11n_{98}, 11n_{99}, 11n_{101}, 11n_{103}$, and $11n_{112}$ to knots with non- orientable genus 1.



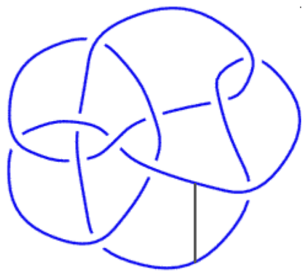
(A) $11n_{125} \xrightarrow{0} 8_{14}$



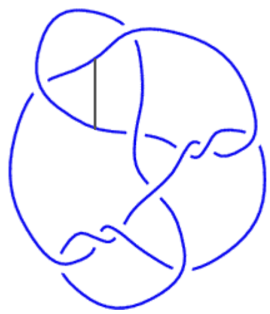
(B) $11n_{130} \xrightarrow{0} 8_{7}$



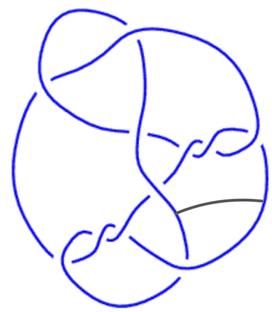
(C) $11n_{131} \xrightarrow{0} 8_{14}$



(D) $11n_{133} \xrightarrow{0} 10_{165}$

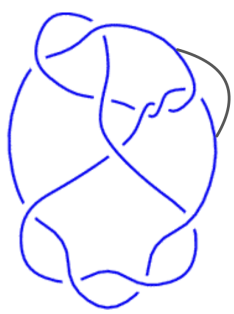


(E) $11n_{137} \xrightarrow{0} 10_{131}$

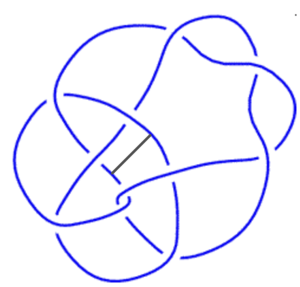


(F) $11n_{138} \xrightarrow{1} 10_{139}$

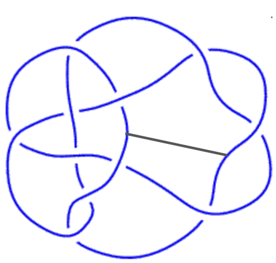
(G) $11n_{140} \xrightarrow{-1} 10_{144}$



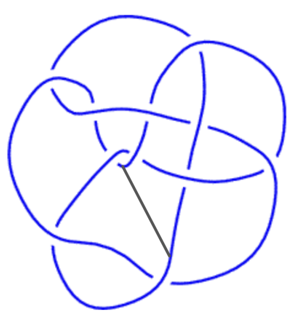
(H) $11n_{141} \xrightarrow{-1} 10_{126}$



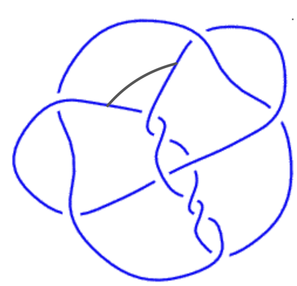
(I) $11n_{155} \xrightarrow{0} 3_{1}$



(J) $11n_{161} \xrightarrow{-1} 6_{2}$

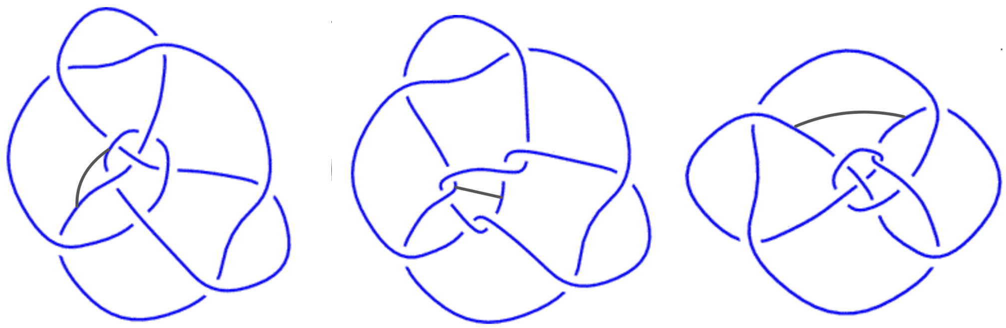


(K) $11n_{165} \xrightarrow{0} 11n_{46}$



(L) $11n_{171} \xrightarrow{1} 10_{144}$

FIGURE 4.14. Non-oriented band moves from the knots $11n_{125}, 11n_{130}, 11n_{131}, 11n_{133}, 11n_{137}, 11n_{138}, 11n_{140}, 11n_{141}, 11n_{155}, 11n_{161}, 11n_{165},$ and $11n_{171}$ to knots with non- orientable genus 1.



(A) $11n_{176} \xrightarrow{-1} 8_{10}$ (B) $11n_{179} \xrightarrow{0} 8_{14}$ (C) $11n_{184} \xrightarrow{0} 6_2$

FIGURE 4.15. Non-oriented band moves from the knots $11n_{176}, 11n_{179},$ and $11n_{184}$ to knots with non-orientable genus 1.

References

[1] Samantha Allen. Nonorientable surfaces bounded by knots: a geography problem. *New York J. Math.*, 29:1038–1059, 2023.

[2] Fraser Binns, Sungkyung Kang, Jonathan Simone, and Paula Truöl. On the nonorientable four-ball genus of torus knots. *arXiv preprint arXiv:2109.09187*, 2021.

[3] Nakisa Ghanbarian. The non-orientable 4-genus for knots with 10 crossings. *J. Knot Theory Ramifications*, 31(5):Paper No. 2250034, 46, 2022.

[4] Patrick M. Gilmer and Charles Livingston. The nonorientable 4-genus of knots. *J. Lond. Math. Soc. (2)*, 84(3):559–577, 2011.

[5] C. McA. Gordon and R. A. Litherland. On the signature of a link. *Invent. Math.*, 47(1):53–69, 1978.

[6] Jim Hoste, Patrick D. Shanahan, and Cornelia A. Van Cott. On the nonorientable 4-genus of double twist knots. *J. Knot Theory Ramifications*, 32(5):Paper No. 2350036, 30, 2023.

[7] Stanislav Jabuka and Tynan Kelly. The nonorientable 4-genus for knots with 8 or 9 crossings. *Algebr. Geom. Topol.*, 18(3):1823–1856, 2018.

[8] M. Kate Kearney. The concordance genus of 11-crossing knots. *J. Knot Theory Ramifications*, 22(13):1350077, 17, 2013.

[9] The Knot Atlas. URL: http://katlas.org/wiki/The_Hoste-Thistlethwaite_Table_of_11_Crossing_Knots, February 2024.

[10] Charles Livingston and Allison H. Moore. Knotinfo: Table of knot invariants. URL: http://knotinfo.math.indiana.edu, January 2024.

[11] Allison H. Moore and Mariel Vazquez. A note on band surgery and the signature of a knot. *Bull. Lond. Math. Soc.*, 52(6):1191–1208, 2020.

[12] Hitoshi Murakami and Akira Yasuhara. Crosscap number of a knot. *Pacific J. Math.*, 171(1):261–273, 1995.

[13] Hitoshi Murakami and Akira Yasuhara. Four-genus and four-dimensional clasp number of a knot. *Proc. Amer. Math. Soc.*, 128(12):3693–3699, 2000.

[14] Peter S. Ozsváth, András I. Stipsicz, and Zoltán Szabó. Unoriented knot Floer homology and the unoriented four-ball genus. *Int. Math. Res. Not. IMRN*, (17):5137–5181, 2017.

[15] Joshua M. Sabloff. On a refinement of the non-orientable 4-genus of torus knots. *Proc. Amer. Math. Soc. Ser. B*, 10:242–251, 2023.

Department of Mathematics, Louisiana State University

Email address: mfarr17@lsu.edu