

Fractional Sobolev isometric immersions of planar domains

SIRAN LI, MOHAMMAD REZA PAKZAD AND ARMIN SCHIKORRA

Abstract. We discuss C^1 regularity and developability of isometric immersions of flat domains into $\mathbb{R}^3$ enjoying a local fractional Sobolev $W^{1+s, \frac{2}{s}}$ regularity for $2/3 \leq s < 1$, generalizing the known results on Sobolev and Hölder regimes. Ingredients of the proof include analysis of the weak Codazzi-Mainardi equations of the isometric immersions and study of $W^{2, \frac{2}{s}}$ planar deformations with symmetric Jacobian derivative and vanishing distributional Jacobian determinant. On the way, we also show that the distributional Jacobian determinant, conceived as an operator defined on the Jacobian matrix, behaves like the determinant of gradient matrices under products by scalar functions.

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1. Introduction

In this article we prove the C^1 regularity and developability of isometric immersions of class $W^{1+s, p}$ of two-dimensional domains Ω into $\mathbb{R}^3$ for $2/3 \leq s < 1$ and $sp \geq 2$, thereby generalizing the results of [42] for the Sobolev regime $s = 1$, $p \geq 2$ and of [13] for the Hölder regime $s > 2/3$, $p = \infty$. The proofs are obtained by adapting the ideas of a few of the results appearing in [13, 35, 36, 42] to the fractional Sobolev case.

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1.1. Background

There are several motivations for the study of isometric immersions of low regularity. A first one arises from the strong divergence in the respective behaviors of C^1 and C^2 isometric immersions of two-dimensional domains. This phenomenon, known as the flexibility and rigidity dichotomy, has other parallels, *e.g.*, for the solutions of the Euler equations in fluid dynamics. We shall direct the readers to [13] and the references therein for a survey of the literature on the historic problem of developability in differential geometry, alongside its connections to the above mentioned dichotomy in nonlinear PDEs and convex integration and to a conjecture by Misha Gromov [20, Section 3.5.5.C, Open Problems 34-36].

The second motivation stems from the calculus of variations and nonlinear elastic plate theory. Surfaces with L^2 integrable second fundamental form and curvature functionals such as the Willmore energy have a long history in geometric analysis and calculus of variations. In the context of nonlinear elasticity, the Kirchhoff model stipulates that the deformation of a piece of paper under body forces or boundary conditions minimizes the Willmore functional subject to the isometric constraint. In this context, and following the methods of Kirchheim [31], the C^1 regularity and developability of isometric immersions with L^2 integrable second fundamental form were proved by the second author in [42]. This result has had many applications in nonlinear elastic plate theory, namely in proving density of smooth isometries in the class of $W^{2,2}$ isometric immersions [24, 42], in derivation and regularity analysis of the Euler-Lagrange equations for the Kirchhoff's models on plates [25, 26], in derivation of plate and shell theories from 3d nonlinear elasticity via Γ -convergence [17, 27], in stability analysis for nonlinear plates [33], and finally in the confinement problem for unstretchable elastic sheets [12, 52].

The results of this paper give us the possibility to broaden the analysis by proposing similar models involving deformations of lower regularities, but with still some control on the curvature of the image surfaces. Indeed, as shown in Section 5, an isometric immersion u of regularity $W^{1+s,p}$ admits a second fundamental form $\Pi(u)$ of regularity $W^{s-1,p}$ if $1/2 < s < 1$ and $p \geq 2/s$. This way we can define a fractional Willmore-like curvature functional

$$\mathcal{I}(u) := \|\Pi(u)\|_{W^{s-1,p}}$$

on the class of such immersions. This variational model, which we can justifiably name *the fractional Kirchhoff plate model*, is rather phenomenological; nevertheless, mathematically, many of the above mentioned problems on the standard model can be reformulated in this new context and explored. As an example, it can be asked whether its minimizers will enjoy the same regularity as those of the standard model established in [25], or will develop new types of singularities. The results of the present article concern the class of admissible deformations of this model in the regularity regime $s \geq 2/3$ and could pave the way for proving regularity of the minimizers in the footsteps of [25].

Finally, our last motivation for the study of weakly regular isometric immersions is that it is connected to many interesting problems in nonlinear and geomet-

ric analysis: It has lead to the development of interesting methods in geometric measure theory and geometric function theory [29, 30, 35, 36], and as we shall see below, to problems on the distributional Jacobian determinant, see also [18, 36] in this regard.

1.2. Main results

Our first result is complementary to a theorem of Pakzad [42, Theorem II], which is the case $s = 1$ of our Theorem 1.1, and to the recent work for $u \in c^{1, \frac{2}{3}}(\Omega, \mathbb{R}^3) \supset C^{1, \frac{2}{3} + \varepsilon}(\Omega, \mathbb{R}^3)$ by De Lellis–Pakzad [13, Theorem 1]. Following [13, Definition 1], we say a C^1 mapping u of a two-dimensional domain Ω is developable if given any point $x \in \Omega$, u is either affine around x , or its Jacobian derivative ∇u is constant along the connected component of the intersection of a line passing through x with Ω . See also [13, Section 2] for equivalent conditions. We refer to Subsection 2.1 for definitions and notation regarding fractional Sobolev spaces.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be an open set. Consider the class of $W_{\text{loc}}^{1+s, \frac{2}{s}}$ isometric immersions:*

$$I_{\text{loc}}^{1+s, \frac{2}{s}}(\Omega, \mathbb{R}^3) := \left\{ u \in W_{\text{loc}}^{1+s, \frac{2}{s}}(\Omega, \mathbb{R}^3) : (\nabla u)^T \nabla u = \text{Id a.e. in } \Omega \right\}.$$

Then any $u \in I_{\text{loc}}^{1+s, \frac{2}{s}}(\Omega, \mathbb{R}^3)$ with $\frac{2}{3} \leq s < 1$ is $c^{1, \frac{s}{2}}$ -regular and developable.

Remark 1.2. Here $c^{1, \alpha}$ denotes all mappings whose derivatives of components lie locally in the little Hölder space $c^{0, \alpha}$, which is the closure of smooth functions in the $C^{0, \alpha}$ norm.

As a consequence we also obtain the extension of [42, Corollary 1.1] to fractional Sobolev spaces, cf. [12, 52].

Corollary 1.3. *There exists $\rho_0 > 0$ such that whenever $s \geq \frac{2}{3}$ there is no $W_{\text{loc}}^{1+s, \frac{2}{s}}$ isometric immersion of the 2-dimensional disk into a three-dimensional Euclidean ball of radius $r < \rho_0$.*

Note that $\rho_0 < \frac{1}{2}$, as the images of such immersions will always contain segments larger than the unit segment.

Remark 1.4. The same statements hold true for isometric immersions of $W^{1+s, p}$ -regularity with $s \geq 2/3$, $sp > 2$. If $s > 2/3$, $p > 3$, this fact follows from Theorem 1.1 by the embedding of $W_{\text{loc}}^{s, p}$ into $W_{\text{loc}}^{2/3, 3}$. In the case $s = 2/3$, $p > 3$ this embedding fails, but following the footsteps of [13], a proof for the developability statement can be achieved, which we leave to the reader. We have concentrated on the more challenging borderline case $sp = 2$.

Remark 1.5. Theorem 1.1 may fail for isometric immersions of $W^{1+s,p}$ -regularity if $sp < 2$. Indeed, for any $0 < s \leq 1$ and $p < 2/s$, the 1-homogeneous map $u : B^1 \rightarrow \mathbb{R}^3$ expressed in polar coordinates as

$$u(r, \theta) := \left(\frac{1}{2}r \cos(2\theta), \frac{1}{2}r \sin(2\theta), \frac{1}{2}\sqrt{3}r \right)$$

is a $W^{1+s,p}$ isometric immersion of the 2-dimensional disk into $\mathbb{R}^3$ but has a conical singularity at the origin. It clearly does not belong to C^1 and fails to be developable.

Remark 1.6. Following [41] for $s = 1, p = 2$, we expect that the isometric immersion can be shown to be C^1 up to the boundary if its $W^{1+s, \frac{2}{s}}(\Omega)$ norm is finite and $\partial\Omega$ is of class $C^{1,\alpha}$ for some $\alpha > 0$. This boundary regularity fails if $\partial\Omega$ is merely of class C^1 [41, Remark 7].

Remark 1.7. To establish the result, directly following the arguments of [13] is not enough. Indeed, observe that u is a priori not even assumed to be in C^1 . But this is not the only difficulty, as we will explain in Section 4 and Appendix A. We will hence adapt a new approach. In particular, Theorem 1.10 below is a new contribution devised to bypass the new obstacles for the case $s = 2/3$.

To set up our second and third results, we first remind following Brezis-Nguyen [7] that for any domain $\Omega \subset \mathbb{R}^n$, and f belonging to the optimal space $W_{\text{loc}}^{\frac{n-1}{n}, n}(\Omega, \mathbb{R}^n)$, the Jacobian determinant $\text{Jac}(f) := \text{Det}(\nabla f)$ is well defined as a distribution in $\mathcal{D}'(\Omega)$, see also Sickel-Youssfi [48]. We also refer to the fundamental works on the distributional Jacobian developed by Reshetnyak [43], Wentz [53], Ball [2], Tartar [50], Müller [40], Coifman-Lions-Meyer-Semmes [9], and Brezis-Nirenberg [8]. In view of the embedding theorems for the fractional Sobolev spaces, $\text{Jac}(f)$ is well defined for $f \in W_{\text{loc}}^{s,p}(\Omega, \mathbb{R}^n)$ if $\frac{n-1}{n} < s \leq 1$ and $p \geq \frac{n^2}{ns+1}$. In particular, it can be established by the methods of [7] that if $p = n/s$, $\text{Jac}(f) \in W_{\text{loc}}^{n(s-1), 1/s}(\Omega)$, – the proof is explained in [36, Lemma 1.3], cf. Lemma 8.1.

Our analysis establishes a connection between isometric immersions of fractional Sobolev regularity and deformations of plane domains f with symmetric Jacobians ∇f and vanishing distributional Jacobian determinants $\text{Jac}(f)$. In particular, the developability of isometric immersions is proved using the following similar statement for these deformations.

Theorem 1.8. *Let $\Omega \subset \mathbb{R}^2$ be an open set. Assume that $s \geq 2/3$ and $f \in W_{\text{loc}}^{s, \frac{2}{s}}(\Omega, \mathbb{R}^2)$ with its distributional Jacobian satisfying*

$$\text{curl } f = 0 \quad \text{and} \quad \text{Jac}(f) = 0 \quad \text{in } \mathcal{D}'(\Omega).$$

Then, $f \in c^{0, \frac{s}{2}}(\Omega)$ and for any point $x \in \Omega$, f is either constant around x , or it is constant along the connected component of the intersection of a line passing through x with Ω .

See statements similar to Theorem 1.8 in [31, Proposition 2.29] (for Lipschitz maps), [42, Proposition 1.1] (for $W^{1,2}$ -maps), and [35, Theorem 1.3] for Hölder continuous maps. The continuity of any f as in Theorem 1.8 was already shown in [36, Theorem 1.6].

Remark 1.9. As in Remark 1.5, Theorem 1.8 fails for $f \in W^{s,p}$ with $\frac{4}{2s+1} \leq p < \frac{2}{s}$, even for $s = 1$. We refer to the so-called “fish-like example” discussed in [16]: Letting $c = 0$, $f = \nabla u$ satisfies $\operatorname{curl}(f) = 0$ and $\operatorname{Jac}(f) = \mathcal{H}u = 0$, however f is not even continuous.

Another new contribution of this article, which will turn out to be crucial in proving Theorem 1.1 in the critical threshold $s = \frac{2}{3}$, directly regards the properties of distributional Jacobian determinants. As we shall see in Proposition 2.5 the distributional product $\lambda \nabla g$ is well defined provided $\lambda \in L^\infty \cap W^{s, \frac{n}{s}}(\Omega)$ and $g \in W^{s, \frac{n}{s}}(\Omega; \mathbb{R}^n)$ if $s > 1/2$. In view of this fact, the following seemingly natural behavior of the distributional Jacobian determinant can be proven:

Theorem 1.10. *Let $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, or $\Omega = \mathbb{R}^n$. Assume that $s \in [\frac{n}{n+1}, 1)$, $\lambda \in L^\infty \cap W^{s, \frac{n}{s}}(\Omega)$, $f, g \in W^{s, \frac{n}{s}}(\Omega; \mathbb{R}^n)$, and that*

$$\nabla f = \lambda \nabla g. \quad (1.1)$$

Then for any $\phi \in C_c^\infty(\Omega)$,

$$\operatorname{Jac}(f)[\phi] = \operatorname{Jac}(g)[\lambda^n \phi].$$

Note that since $s \geq \frac{n}{n+1}$,

$$\lambda^n \phi \in W_0^{s, \frac{n}{s}}(\Omega) \hookrightarrow W_0^{(1-s)n, \frac{1}{1-s}}(\Omega) = W_{00}^{(1-s)n, \frac{1}{1-s}}(\Omega),$$

and so the right-hand side in the above Jacobian determinant identity is well defined.

The outline of this paper is as follows. In Section 2 we begin with some preliminaries on fractional Sobolev spaces and gather some important statements to be used later in the article. In Section 3 we discuss developability of fractional Sobolev 2d deformations with symmetric Jacobian derivative and vanishing distributional Jacobian determinant. In the subsequent Sections 4 and 5, we will set out to define a notion of second fundamental form for fractional Sobolev isometries and to derive a weak version of Codazzi-Mainardi system of equations for it. In Section 6 the developability and regularity of each component are shown. We will complete the proof of Theorem 1.1 in Section 7 and present a proof of Theorem 1.10 in Section 8. In Appendix A, it is briefly shown, as a side-note, how Theorem 1.10 can be bypassed in case $s > 2/3$. In Appendix B we introduce a notion of fractional absolute continuity in order to give a simple proof of the known fact from [23] that the image of a $W^{s,p}(\mathbb{R}, \mathbb{R}^2)$ deformation is of Lebesgue measure zero provided $s > 1/2$ and $sp > 1$.

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2. Fractional Sobolev spaces, an overview and some facts

2.1. Notation

We will work with the Slobodeckij or Gagliardo fractional Sobolev space, also sometimes referred to as the Besov space. Namely, for any open set $\Omega \subset \mathbb{R}^n$, non-negative integer k , $0 < s < 1$ and $1 \leq p < \infty$, we define the fractional $W^{s,p}$ -seminorm of a mapping $f \in L^1_{\text{loc}}(\Omega, \mathbb{R}^N)$ by

$$[f]_{W^{s,p}(\Omega)} := \left(\int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} dx dy \right)^{1/p},$$

and we set for any integer $k \geq 0$ (identifying $W^{0,p}$ with L^p when $k = 0$),

$$W^{k+s,p}(\Omega) := \left\{ f \in W^{k,p}(\Omega) : [D^k f]_{W^{s,p}(\Omega)} < \infty \right\},$$

which is a Banach space with the norm

$$\|f\|_{W^{k+s,p}(\Omega)} := \|f\|_{W^{k,p}(\Omega)} + [D^k f]_{W^{s,p}(\Omega)}.$$

$W_0^{k+s,p}(\Omega)$ is defined to be the closure of $C_c^\infty(\Omega)$ in this space. Note that $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k+s,p}(\mathbb{R}^n)$ [14, Theorem 2.4]. If Ω is a bounded smooth domain, there is a bounded linear extension operator mapping $W^{s,p}(\Omega)$ to $W^{s,p}(\mathbb{R}^n)$ [14, 54]. For any such Ω , or for $\Omega = \mathbb{R}^n$, and $1 \leq p < \infty$, these spaces coincide with the Besov-Triebel-Lizorkin type spaces $B_{p,p}^s(\Omega) = F_{p,p}^s(\Omega)$ according to [44, Proposition 2.1.2 and Section 2.4]. Indeed the identification can be established as these spaces are the real (s, p) -interpolation between L^p and $W^{1,p}$ spaces, see [38, Example 1.8] and [3, Theorem 6.2.4].

When $1 < p < \infty$, the Lions-Magenes Sobolev space $W_{00}^{k+s,p}(\Omega)$ introduced in [37] is the closed subspace of $W^{k+s,p}(\mathbb{R}^n)$ defined by

$$W_{00}^{k+s,p}(\Omega) := \left\{ f \in W^{k+s,p}(\mathbb{R}^n) : \text{supp } f \subset \overline{\Omega} \right\},$$

equipped with the induced seminorm $[f]_{W_{00}^{k+s,p}(\Omega)}$ and norm $\|f\|_{W_{00}^{k+s,p}(\Omega)}$. We refer to [51, Section 4.3.2] for more references and for the following properties: $W_{00}^{k+s,p}(\Omega)$ can also be identified as the set of those elements of $W_0^{k+s,p}(\Omega)$ whose extensions by 0 outside of Ω belong to $W^{k+s,p}(\mathbb{R}^n)$. $C_c^\infty(\Omega)$ is dense in $W_{00}^{k+s,p}(\Omega)$ and we have

$$\|f\|_{W_0^{k+s,p}(\Omega)} \lesssim \|f\|_{W_{00}^{k+s,p}(\Omega)}.$$

If $sp \neq 1$ and $\partial\Omega$ is sufficiently regular the linear operator extending $f \in C_c^\infty(\Omega)$ by 0 outside of Ω to $f_0 \in W^{k+s,p}(\mathbb{R}^n)$ satisfies

$$\|f_0\|_{W^{k+s,p}(\mathbb{R}^n)} \lesssim \|f\|_{W^{k+s,p}(\Omega)},$$

which implies $W_{00}^{k+s,p}(\Omega) = W_0^{k+s,p}(\Omega)$. If $sp = 1$ this is not the case and $W_{00}^{k+s,\frac{1}{s}}(\Omega)$ is a proper dense subspace of $W_0^{k+s,\frac{1}{s}}(\Omega)$ when $\Omega \neq \mathbb{R}^n$.

If Ω is a bounded smooth domain or if $\Omega = \mathbb{R}^n$, we set for $0 < s < 1$, $1 < p < \infty$:

$$W^{-s,p'}(\Omega) := (W_{00}^{s,p}(\Omega))',$$

with $1/p + 1/p' = 1$, as a subset of distributions in $\mathcal{D}'(\Omega)$. Our definition departs from [44, Section 2.1.1 and Section 2.4.1] but by [51, Theorem 4.8.1], these two definitions coincide. Therefore the extension property is still valid for negative differentiability exponent: For a bounded smooth domain Ω , and $0 < s < 1$, any element of $W^{-s,p}$ can be extended by a bounded linear operator to an element of $W^{-s,p}(\mathbb{R}^n)$ [44, Theorem 2.4.2/2]. Moreover by [44, Proposition 2.1.4/2]

$$W^{s,p}(\Omega) = \{f \in \mathcal{D}'(\Omega) : f \in W^{s-1,p}(\Omega) \text{ and } Df \in W^{s-1,p}(\Omega, \mathbb{R}^n)\},$$

with equivalence of norms

$$\|f\|_{W^{s,p}(\Omega)} \approx \|f\|_{W^{s-1,p}(\Omega)} + \|Df\|_{W^{s-1,p}(\Omega)}. \quad (2.1)$$

For $t > -1$, the vector-valued spaces $W^{t,p}(\Omega, \mathbb{R}^N)$ are defined to be all $\mathbb{R}^N$ -valued mappings whose components lie in $W^{t,p}(\Omega)$. We will omit the target $\mathbb{R}^N$ when there is no ambiguity.

It is also useful to define for $0 < s < 1$ and $1 < p < \infty$ the homogenous norm

$$\begin{aligned} \|f\|_{\dot{W}^{-s,p'}(\Omega)} &:= \sup \left\{ f[\phi] : \phi \in C_c^\infty(\Omega) \text{ and } [\phi]_{W_{00}^{s,p}(\Omega)} \leq 1 \right\} \\ &\geq \|f\|_{W^{-s,p'}(\Omega)}, \end{aligned} \quad (2.2)$$

where here and throughout the article $f[\phi]$ denotes the action of the distribution f on ϕ . We denote the corresponding space of finite-norm distributions by $\dot{W}^{-s,p'}(\Omega)$, and note that $C_c^\infty(\Omega)$ is dense in $\dot{W}^{-s,p'}(\Omega)$. It follows from (2.1) through a standard scaling argument that

$$[f]_{W^{s,p}(\mathbb{R}^n)} \lesssim \|Df\|_{\dot{W}^{s-1,p}(\mathbb{R}^n)}. \quad (2.3)$$

We conclude our presentation of fractional Sobolev spaces by a final useful observation. For $n \geq 2$ let the differential and integral operators $\Delta_{\mathbb{R}^n}$, $\Delta_{\mathbb{R}^n}^{-1}$ and the Riesz transform $\mathcal{R}$ be respectively defined by the Fourier symbols $|\xi|^2$, $|\xi|^{-2}$ and $i\xi/|\xi|$. It is known that $\Delta_{\mathbb{R}^n}^{-1}$ is a well-defined operator and coincides (modulo a

conventional sign) with the Newtonian potential on $L^2(\mathbb{R}^n) \supset C_c^\infty(\mathbb{R}^n)$. By a classical theorem [19, Corollary 5.2.8] the Riesz transform is a bounded operator from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n, \mathbb{R}^n)$ when $1 < p < \infty$. It is a linear operator commuting with differentiation, hence, via the interpolation property [38, Theorem 1.6] and a scaling argument, and in view of the fact that $\mathcal{R} \cdot \mathcal{R}f = -f$ we obtain that

$$[\mathcal{R}f]_{W^{s,p}(\mathbb{R}^n, \mathbb{R}^n)} \approx [f]_{W^{s,p}(\mathbb{R}^n)}$$

for any $0 < s < 1$ and $1 < p < \infty$. An argument by duality yields the similar estimate

$$\|\mathcal{R}f\|_{\dot{W}^{-s,p'}(\mathbb{R}^n, \mathbb{R}^n)} \approx \|f\|_{\dot{W}^{-s,p'}(\mathbb{R}^n)}.$$

Combining this fact with (2.3), we obtain

$$\begin{aligned} [D\Delta_{\mathbb{R}^n}^{-1}f]_{W^{s,p}(\mathbb{R}^n)} &\lesssim \|D(D\Delta_{\mathbb{R}^n}^{-1})f\|_{\dot{W}^{s-1,p}(\mathbb{R}^n)} \\ &= \|\mathcal{R} \otimes \mathcal{R}f\|_{\dot{W}^{s-1,p}(\mathbb{R}^n)} \approx \|f\|_{\dot{W}^{s-1,p}(\mathbb{R}^n)}. \end{aligned} \quad (2.4)$$

2.2. Mollification and commutator estimates

For a given smooth bounded domain $\Omega \subset \mathbb{R}^n$ we fix an extension operator and for any $f \in W^{s,p}(\Omega)$, we still denote its extension by $f \in W^{s,p}(\mathbb{R}^n)$. Throughout the paper, we fix a standard mollifier $\varphi \in C_c^\infty(B^1)$, $\int_{B^1} \varphi = 1$. For any mapping $f \in W^{s,p}(\Omega)$ with Ω as above, we let f_ε be the mollifications of the extension $f_\varepsilon := f * \varphi_\varepsilon$, where $\varphi_\varepsilon(x) := \frac{1}{\varepsilon^n} \varphi(\frac{x}{\varepsilon})$. The following estimates, which are reminiscent of [10, 11, 13] will be used in our analysis:

Lemma 2.1. *Let $0 < s < 1$, $f, g \in W^{s,p}(\Omega)$, where either $\Omega \subset \mathbb{R}^n$ is smooth and bounded or $\Omega = \mathbb{R}^n$. Then:*

- (i) $\|f_\varepsilon - f\|_{L^p} \leq o(\varepsilon^s)$;
- (ii) $\forall k \geq 1 \|\nabla^k f_\varepsilon\|_{L^p} \leq o(\varepsilon^{s-k})$;
- (iii) If $p \geq 2$, $\forall k \geq 0 \|\nabla^k(f_\varepsilon g_\varepsilon - (fg)_\varepsilon)\|_{L^{p/2}} \leq o(\varepsilon^{2s-k})$,

where the bound function $o(\cdot)$ depends on p, φ and the extension constant of Ω .

Proof. (i) By the extension property of smooth bounded domains it is sufficient to prove the estimates for $\Omega = \mathbb{R}^n$. Let for $x, y \in \mathbb{R}^n$

$$\delta_x f(y) := f(y - x) - f(y).$$

We have by Hölder's inequality

$$\begin{aligned} &\|f_\varepsilon - f\|_{L^p}^p \\ &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \delta_x f(y) \varphi_\varepsilon(x) dx \right|^p dy = \int_{\mathbb{R}^n} \left| \int_{\{|x| \leq \varepsilon\}} \delta_x f(y) \varphi_\varepsilon(x) dx \right|^p dy \\ &\leq \int_{\mathbb{R}^n} \left(\int_{\{|x| \leq \varepsilon\}} (|x|^{-(s+\frac{n}{p})} |\delta_x f(y)|)^p dx \right) \left(\int_{\{|x| \leq \varepsilon\}} (|x|^{(s+\frac{n}{p})} |\varphi_\varepsilon(x)|)^{p'} \right)^{\frac{p}{p'}} dy \\ &\leq C \varepsilon^{sp} \int_{\mathbb{R}^n} \int_{\{|x| \leq \varepsilon\}} |x|^{-(sp+n)} |f(y-x) - f(y)|^p dx dy \leq \varepsilon^{sp} o(1), \end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, and the last inequality is a consequence of the dominated convergence and Fubini theorems, in view of the fact that the integrand belongs to $L^1(\mathbb{R}^n \times \mathbb{R}^n)$.

(ii) Similarly as for (i) we write:

$$\begin{aligned}
 & \left\| \nabla^k f_\varepsilon \right\|_{L^p}^p \\
 &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(y-x) \nabla^k (\varphi_\varepsilon)(x) dx \right|^p dy = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \delta_x f(y) \varepsilon^{-k} \left(\nabla^k \varphi \right)_\varepsilon(x) dx \right|^p dy \\
 &\leq \int_{\mathbb{R}^n} \left(\int_{\{|x| \leq \varepsilon\}} \left(|x|^{-(s+\frac{n}{p})} |\delta_x f(y)| \right)^p dx \right) \\
 &\quad \times \left(\int_{\{|x| \leq \varepsilon\}} \left(\varepsilon^{-k} |x|^{(s+\frac{n}{p})} \left| \left(\nabla^k \varphi \right)_\varepsilon(x) \right| \right)^{p'} \right)^{\frac{p}{p'}} dy \\
 &\leq C \varepsilon^{(s-k)p} \int_{\mathbb{R}^n} \int_{\{|x| \leq \varepsilon\}} |x|^{-(sp+n)} |f(y-x) - f(y)|^p dx dy \leq \varepsilon^{(s-k)p} o(1),
 \end{aligned}$$

which is the desired estimate.

(iii) First we observe that for all $k \geq 0$

$$\begin{aligned}
 & \left| \int_{\mathbb{R}^n} \delta_x f(y) \delta_x g(y) \nabla^k (\varphi_\varepsilon)(x) dx \right| \\
 &= \left| \int_{\mathbb{R}^n} \frac{\delta_x f(y)}{|x|^{s+n/p}} \frac{\delta_x g(y)}{|x|^{s+n/p}} |x|^{2(s+n/p)} \varepsilon^{-k} \left(\nabla^k \varphi \right)_\varepsilon(x) dx \right| \\
 &\leq \left\| \frac{\delta_x f(y)}{|x|^{s+n/p}} \frac{\delta_x g(y)}{|x|^{s+n/p}} \right\|_{L^{\frac{p}{2}}(\{|x| \leq \varepsilon\})} \left\| |x|^{2(s+n/p)} \varepsilon^{-k} \left(\nabla^k \varphi \right)_\varepsilon(x) \right\|_{L^{\frac{p}{p-2}}(\{|x| \leq \varepsilon\})} \\
 &\leq C \varepsilon^{2s-k} \left\| \frac{\delta_x f(y)}{|x|^{s+n/p}} \right\|_{L^p(\{|x| \leq \varepsilon\})} \left\| \frac{\delta_x g(y)}{|x|^{s+n/p}} \right\|_{L^p(\{|x| \leq \varepsilon\})}. \tag{2.5}
 \end{aligned}$$

For $k = 0$ we write for all $y \in \mathbb{R}^n$:

$$(f_\varepsilon g_\varepsilon - (fg)_\varepsilon)(y) = (f_\varepsilon - f)(g_\varepsilon - g)(y) - \int_{\mathbb{R}^n} \delta_x f(y) \delta_x g(y) \varphi_\varepsilon(x) dx.$$

The $L^{p/2}$ norms of the first term is estimated by $o(\varepsilon^{2s})$, using part (i), (ii) and Hölder's inequality. Now, integrating the $\frac{p}{2}$ th power of the second term over the parameter y , and using (2.5) will yield the $o(1)$ factor and complete the proof.

If $k \geq 1$, it is sufficient to note that for all $y \in \mathbb{R}^n$:

$$\begin{aligned} \nabla^k(f_\varepsilon g_\varepsilon - (fg)_\varepsilon)(y) &= \sum_{j=0}^k \nabla^j f_\varepsilon \otimes \nabla^{k-j} g_\varepsilon(y) - \nabla^k(fg)_\varepsilon(y) \\ &= \sum_{j=1}^{k-1} \nabla^j f_\varepsilon \otimes \nabla^{k-j} g_\varepsilon(y) + (f_\varepsilon - f) \nabla^k g_\varepsilon(y) \\ &\quad + (g_\varepsilon - g) \nabla^k f_\varepsilon(y) \\ &\quad - \int_{\mathbb{R}^n} \delta_x f(y) \delta_x g(y) \nabla^k(\varphi_\varepsilon)(x) dx. \end{aligned}$$

The $L^{p/2}$ norms of the terms in the first summation are estimated by $o(\varepsilon^{2s-k})$, using part (ii) and Hölder's inequality. The second and third terms are estimated using (i). Finally, integrating its $\frac{p}{2}$ th power of the last term and once more applying (2.5) leads to an $o(\varepsilon^{2s-k})$ control as desired. $\square$

Remark 2.2. The estimates in Lemma 2.1 are not optimal and seem to characterize the spaces $b_{p,\infty}^s$ [44, Definition 2.1.3/1], which are larger than $W^{s,p}$. We conjecture that results of the paper can still be achieved for the borderline space $b_{3,\infty}^{2/3}$ through the same approach.

Corollary 2.3. *Let $s \in (0, 1)$ and $p \geq 2$. If $f, g \in W^{s,p}(\Omega)$, where either Ω is smooth and bounded or $\Omega = \mathbb{R}^n$, then*

$$\lim_{\varepsilon \rightarrow 0} \|f_\varepsilon g_\varepsilon - (fg)_\varepsilon\|_{W^{2s, \frac{p}{2}}} = 0.$$

Proof. The idea is to use the interpolation inequality [38, Corollary 1.1.7]

$$\|h\|_{W^{\theta, \frac{p}{2}}} \lesssim \|h\|_{L^{\frac{p}{2}}}^{1-\theta} \|h\|_{W^{1, \frac{p}{2}}}^{\theta}$$

for all $h \in W^{1, \frac{p}{2}}$ and $0 \leq \theta \leq 1$. For $0 < s \leq \frac{1}{2}$, we apply Lemma 2.1(iii) for $k = 0$ and $k = 1$ to $h := f_\varepsilon g_\varepsilon - (fg)_\varepsilon$ with $\theta = 2s$ to obtain:

$$\|f_\varepsilon g_\varepsilon - (fg)_\varepsilon\|_{W^{2s, \frac{p}{2}}} \leq o\left(\varepsilon^{(1-2s)2s+2s(2s-1)}\right) = o(1).$$

Similarly, if $\frac{1}{2} < s < 1$, we let $h := \nabla(f_\varepsilon g_\varepsilon - (fg)_\varepsilon)$ and $\theta = 2s - 1$ and we apply again Lemma 2.1(iii) for $k = 0, 1, 2$, and the interpolation estimate, which together yield:

$$\|f_\varepsilon g_\varepsilon - (fg)_\varepsilon\|_{L^{\frac{p}{2}}} \leq o(\varepsilon^{2s})$$

and

$$\|\nabla(f_\varepsilon g_\varepsilon - (fg)_\varepsilon)\|_{W^{2s-1, \frac{p}{2}}} \leq o\left(\varepsilon^{(1-(2s-1))(2s-1)+(2s-1)(2s-2)}\right) = o(1). \quad \square$$

We will also need the following elementary estimate, which in fact states the known embedding of $W^{s, \frac{n}{s}}(\mathbb{R}^n)$ into VMO [8, Section I.2, Example 2]:

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f \in W^{s, \frac{n}{s}}(\Omega)$. Then for all $x \in \Omega$, and $\varepsilon < \text{dist}(x, \partial\Omega)$,*

$$\lim_{\varepsilon \rightarrow 0} \oint_{B_\varepsilon(x)} |f - f_\varepsilon(x)|^{\frac{n}{s}} = 0.$$

Proof. It is sufficient to show that

$$\|f - f_\varepsilon(x)\|_{L^{\frac{n}{s}}(B_\varepsilon(x))} \leq o(\varepsilon^s),$$

which follows from the a variant of fractional Poincaré inequality which is valid for all $s \in (0, 1)$ and $1 \leq p < \infty$:

$$\|f - f_\varepsilon(x)\|_{L^p(B_\varepsilon(x))} \leq C\varepsilon^s [f]_{W^{s,p}(B_\varepsilon(x))},$$

and can be proved similarly as in [15, Proposition 2.1], where we have replaced the average of f on the ball by $f_\varepsilon(x)$.

Here we provide another proof. For a fixed $x \in \Omega$ we have by Lemma 2.1(i) and $p = \frac{n}{s}$:

$$\begin{aligned} \|f - f_\varepsilon(x)\|_{L^{\frac{n}{s}}(B_\varepsilon(x))} &\leq \|f - f_\varepsilon\|_{L^{\frac{n}{s}}(B_\varepsilon(x))} + \|f_\varepsilon - f_\varepsilon(x)\|_{L^{\frac{n}{s}}(B_\varepsilon(x))} \\ &\leq o(\varepsilon^s) + \|f_\varepsilon - f_\varepsilon(x)\|_{L^{\frac{n}{s}}(B_\varepsilon(x))}. \end{aligned}$$

It remains to bound the second term, for which can apply the standard Poincaré inequality for any $f \in L^1(\Omega)$ with the proper scaling on the ball $B_\varepsilon(x)$

$$\|f_\varepsilon - f_\varepsilon(x)\|_{L^{\frac{n}{s}}(B_\varepsilon(x))} \leq C\varepsilon \|\nabla f_\varepsilon\|_{L^{\frac{n}{s}}(B_\varepsilon(x))}, \quad (2.6)$$

to obtain, this time via Lemma 2.1(ii) the desired estimate. Note that we have the right to use $f_\varepsilon(x)$ as the normalization constant since $\frac{n}{s} > n$ and $W^{1, \frac{n}{s}}$ embeds in $C^{0,1-s}$. $\square$

2.3. Distributional products in fractional Sobolev spaces

In Section 5 we will define a notion of second fundamental form for fractional Sobolev isometries through the first part of the following result. We will present a proof following the methodology of [34], which then is adapted to subsequently show the complementary second part, which, in particular, will be used in proving Theorem 1.10 in Section 8.

Proposition 2.5. *Let $n \geq 2$, $1/2 < s < 1$, $f \in W^{s, \frac{n}{s}}(\mathbb{R}^n)$.*

- (i) *Let $\mu \in W^{s, \frac{n}{s}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then, for any $\alpha \in \{1, \dots, n\}$, the product $\mu \partial_\alpha f$ is well defined as a distribution on $\mathbb{R}^n$, and*

$$\|\mu \partial_\alpha f\|_{\dot{W}^{s-1, \frac{n}{s}}(\mathbb{R}^n)} \lesssim [f]_{W^{s, \frac{n}{s}}(\mathbb{R}^n)} \left([\mu]_{W^{s, \frac{n}{s}}(\mathbb{R}^n)} + \|\mu\|_{L^\infty(\mathbb{R}^n)} \right);$$

(ii) Let $\mu_k \in W^{s, \frac{n}{s}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with

$$\sup_k \left(\|\mu_k\|_{L^{\frac{n}{s}}(\mathbb{R}^n)} + \|\mu_k\|_{L^\infty(\mathbb{R}^n)} \right) < \infty.$$

Assume moreover that $[\mu_k]_{W^{s, \frac{n}{s}}(\mathbb{R}^n)} \xrightarrow{k \rightarrow \infty} 0$. Then, for any $\alpha \in \{1, \dots, n\}$,

$$\|\mu_k \partial_\alpha f\|_{\dot{W}^{s-1, \frac{n}{s}}(\mathbb{R}^n)} \xrightarrow{k \rightarrow \infty} 0.$$

Proof. We will first show (i). Remember that the harmonic extension of $f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ to $\mathbb{R}_+^{n+1}$ is defined by the Poisson extension operator [19, Example 2.1.13]

$$f^h(t, x) := C_n \int_{\mathbb{R}^n} \frac{t}{(|x - z|^2 + t^2)^{\frac{n+1}{2}}} f(z) dz \quad (2.7)$$

and the operator can be extended to $W^{s, \frac{n}{s}}(\mathbb{R}^n)$ [28, 34]. Let $\phi \in C_c^\infty(\mathbb{R}^n)$ and let μ^h , f^h , and ϕ^h be the harmonic extensions of μ , f , and ϕ , respectively, on to $\mathbb{R}_+^{n+1}$.

The one-dimensional integration by parts [34] allows us to define

$$\mu \partial_\alpha f[\phi] := - \int_{\mathbb{R}_+^{n+1}} \partial_{n+1} \left(\mu^h \partial_\alpha f^h \phi^h \right). \quad (2.8)$$

By (2.2), we are going to estimate

$$\|\mu \partial_\alpha f\|_{\dot{W}^{s-1, \frac{n}{s}}(\mathbb{R}^n)} = \sup \left\{ \mu \partial_\alpha f[\phi] : \phi \in C_c^\infty(\mathbb{R}^n) \text{ and } [\phi]_{W^{1-s, \frac{n}{n-s}}(\mathbb{R}^n)} \leq 1 \right\}.$$

So let us fix one $\phi \in C_c^\infty(\mathbb{R}^n)$ with $[\phi]_{W^{1-s, \frac{n}{n-s}}(\mathbb{R}^n)} \leq 1$. We bound

$$|\mu \partial_\alpha f[\phi]| \lesssim \int_{\mathbb{R}_+^{n+1}} |D\mu^h| |Df^h| |\phi^h| + |\mu^h| |Df^h| |D\phi^h|,$$

as we can always tackle the ∂_α term (which is in $\mathbb{R}^n$ -direction) via integration by parts. Here and hereafter, D is the $\mathbb{R}^{n+1}$ -dimensional gradient.

We first claim that

$$\int_{\mathbb{R}_+^{n+1}} |D\mu^h| |Df^h| |\phi^h| \lesssim [\mu]_{W^{s, \frac{n}{s}}(\mathbb{R}^n)} [f]_{W^{s, \frac{n}{s}}(\mathbb{R}^n)} [\phi]_{W^{1-s, \frac{n}{n-s}}(\mathbb{R}^n)}. \quad (2.9)$$

We have

$$\begin{aligned}
& \int_{\mathbb{R}_+^{n+1}} |D\mu^h| |Df^h| |\phi^h| \\
& \leq \int_{\mathbb{R}^n} |\mathcal{M}\phi(x)| \int_0^\infty |D\mu^h(x,t)| |Df^h(x,t)| dt dx \\
& \leq \int_{\mathbb{R}^n} |\mathcal{M}\phi(x)| \left(\int_0^\infty |D\mu^h(x,t)|^2 dt \right)^{\frac{1}{2}} \left(\int_0^\infty |Df^h(x,t)|^2 dt \right)^{\frac{1}{2}} dx \\
& \lesssim \|\mathcal{M}\phi\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} \left(\int_0^\infty |D\mu^h(x,t)|^2 dt \right)^{\frac{2n}{2}} dx \right)^{\frac{1}{2n}} \\
& \quad \times \left(\int_{\mathbb{R}^n} \left(\int_0^\infty |Df^h(t,x)|^2 dt \right)^{\frac{2n}{2}} dx \right)^{\frac{1}{2n}}.
\end{aligned}$$

Here we have used, for the Hardy-Littlewood maximal function $\mathcal{M}$, the fact that

$$|\phi^h(x,t)| \lesssim \mathcal{M}\phi(x).$$

Also recall the characterization of the homogeneous Triebel-Lizorkin spaces (listed, e.g., in [28, 34]):

$$\|f\|_{\dot{F}_{p,q}^\alpha} \approx \left(\int_{\mathbb{R}^n} \left(\int_0^\infty |t^{1-\frac{1}{q}-\alpha} Df^h|^q dt \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}.$$

So, in light of the maximal theorem, we have shown that

$$\int_{\mathbb{R}_+^{n+1}} |D\mu^h| |Df^h| |\phi^h| \lesssim \|\phi\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \|\mu\|_{\dot{F}_{2n,2}^{\frac{1}{2}}(\mathbb{R}^n)} \|f\|_{\dot{F}_{2n,2}^{\frac{1}{2}}(\mathbb{R}^n)}.$$

Thus, we can immediately conclude (2.9) from the embeddings [44, Proposition 2.2.1 and Theorem 2.2.3(ii)] and scaling arguments:

$$\begin{aligned}
\|\phi\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} & \leq [\phi]_{W^{1-s, \frac{n}{n-s}}(\mathbb{R}^n)} \leq 1, \\
\|\mu\|_{\dot{F}_{2n,2}^{\frac{1}{2}}(\mathbb{R}^n)} & \lesssim \|\mu\|_{\dot{F}_{\frac{n}{s}, \frac{n}{s}}^s(\mathbb{R}^n)} = [\mu]_{W^{s, \frac{n}{s}}(\mathbb{R}^n)}, \\
\|f\|_{\dot{F}_{2n,2}^{\frac{1}{2}}(\mathbb{R}^n)} & \lesssim [f]_{W^{s, \frac{n}{s}}(\mathbb{R}^n)},
\end{aligned}$$

as long as $s > 1/2$.

Next we estimate

$$\begin{aligned}
 & \int_{\mathbb{R}_+^{n+1}} |\mu^h| |Df^h| |D\phi^h| \\
 & \leq \left(\int_{\mathbb{R}^n} \left(\int_0^\infty |t^{1-\frac{n-s}{n}-(1-s)} D\phi^h|^{\frac{n}{n-s}} dt \right) dx \right)^{\frac{n-s}{n}} \\
 & \quad \times \left(\int_{\mathbb{R}^n} \left(\int_0^\infty |\mu^h t^{1-\frac{s}{n}-s} Df^h|^{\frac{n}{s}} dt \right) dx \right)^{\frac{s}{n}} \\
 & \leq \|\mu^h\|_{L^\infty(\mathbb{R}_+^{n+1})} \underbrace{[\phi]_{W^{1-s, \frac{n}{n-s}}(\mathbb{R}^n)}}_{\leq 1} [f]_{W^{s, \frac{n}{s}}(\mathbb{R}^n)}.
 \end{aligned} \tag{2.10}$$

Now it is sufficient to observe that by the maximum principle

$$\|\mu^h\|_{L^\infty(\mathbb{R}_+^{n+1})} \leq \|\mu\|_{L^\infty(\mathbb{R}^n)}$$

to conclude together with (2.8) and (2.9) with

$$\|\mu \partial_\alpha f\|_{\dot{W}^{s-1, \frac{n}{s}}(\mathbb{R}^n)} \lesssim [\mu]_{W^{s, \frac{n}{s}}(\mathbb{R}^n)} [f]_{W^{s, \frac{n}{s}}(\mathbb{R}^n)} + \|\mu\|_{L^\infty(\mathbb{R}^n)} [f]_{W^{s, \frac{n}{s}}(\mathbb{R}^n)},$$

which finishes the proof of (i).

(ii) does not directly follow from (i). We first analyse the asymptotic behavior of μ_k . Note that since $W^{s, \frac{n}{s}}(\mathbb{R}^n)$ is reflexive, μ_k is weakly sequentially compact in $W^{s, \frac{n}{s}}$. We shall see that $\mu_k \rightharpoonup 0$ weakly in $W^{s, \frac{n}{s}}(\mathbb{R}^n)$. Indeed, take any weakly convergent subsequence, relabelled μ_k , $\mu_k \rightharpoonup \mu \in W^{s, \frac{n}{s}}(\mathbb{R}^n)$. Let B_R be the open ball of radius $R > 0$ centered at origin in $\mathbb{R}^n$. For any $R > 0$, $\mu_k|_{B_R}$ is a bounded sequence in $W^{s, \frac{n}{s}}(B_R)$ and hence by [14, Theorem 7.1], it is precompact in $L^{n/s}(B_R)$. Since the limit of convergent subsequences cannot be anything other than $\mu|_{B_R}$, we conclude that for each $R > 0$, $\mu_k \rightarrow \mu$ strongly in $L^{n/s}(B_R)$, and so for some subsequence, μ_{k_j} converges almost everywhere in B_R to μ . This implies that

$$\lim_{j \rightarrow \infty} \frac{\mu_{k_j}(x) - \mu_{k_j}(y)}{|x - y|^{2s}} = \frac{\mu(x) - \mu(y)}{|x - y|^{2s}}$$

for almost every $(x, y) \in B_R \times B_R$. On the other hand,

$$[\mu_{k_j}]_{W^{s, \frac{n}{s}}(B_R)} \leq [\mu_{k_j}]_{W^{s, \frac{n}{s}}(\mathbb{R}^n)} \rightarrow 0$$

by the main assumption, which implies, again up to a subsequence of μ_{k_j} , that the same limit vanishes for almost every $(x, y) \in B_R \times B_R$. As a consequence $\mu|_{B_R}$ must be constant for all $R > 0$, and since $\mu \in W^{s, \frac{n}{s}}(\mathbb{R}^n)$, we obtain that $\mu \equiv 0$ is the unique weak accumulation point of the original sequence μ_k . We finally conclude that for all $R > 0$, $\|\mu_k\|_{L^{n/s}(B_R)} \rightarrow 0$.

In order to prove (ii), we note that it is sufficient to assume $f \in C_c^\infty(\mathbb{R}^n)$. Indeed, let $f_j \in C^\infty(\mathbb{R}^n)$ be such that $[f_j - f]_{W^{s-1, \frac{n}{s}}(\mathbb{R}^n)} \rightarrow 0$. If

$$\lim_{k \rightarrow \infty} \|\mu_k \partial_\alpha f_j\|_{W^{s-1, \frac{n}{s}}(\mathbb{R}^n)} = 0, \quad (2.11)$$

as proved below, then

$$\|\mu_k \partial_\alpha f\|_{W^{s-1, \frac{n}{s}}(\mathbb{R}^n)} \leq \|\mu_k \partial_\alpha (f - f_j)\|_{W^{s-1, \frac{n}{s}}(\mathbb{R}^n)} + \|\mu_k \partial_\alpha f_j\|_{W^{s-1, \frac{n}{s}}(\mathbb{R}^n)}$$

converges to 0 too since because of (i) and the uniform boundedness of μ_k the first term on the right-hand side is arbitrarily small for large j .

Now we prove (2.11). Let $f \in C_c^\infty(\mathbb{R}^n)$ and assume that $\text{supp } f$ lies in the open ball B_ρ in $\mathbb{R}^n$. Fix a smooth cut-off function $\eta \in C_c^\infty(B_{\rho+1})$ such that $\eta \equiv 1$ on B_ρ . We observe that for all k and for all $\phi \in C_c^\infty(\mathbb{R}^n)$

$$\begin{aligned} (\eta \mu_k) \partial_\alpha f[\phi] &= \int_{\mathbb{R}^n} (\partial_\alpha f) \eta \mu_k \phi = \int_{B_\rho} (\partial_\alpha f) \eta \mu_k \phi = \int_{B_\rho} \partial_\alpha f \mu \phi \\ &= \int_{\mathbb{R}^n} \partial_\alpha f \mu \phi = \mu \partial_\alpha f[\phi]. \end{aligned}$$

This implies $\mu \partial_\alpha f = (\eta \mu_k) \partial_\alpha f$ and it is sufficient now to prove that

$$\lim_{k \rightarrow \infty} \|(\eta \mu_k) \partial_\alpha f\|_{W^{s-1, \frac{n}{s}}(\mathbb{R}^n)} = 0. \quad (2.12)$$

In order to do so, we have to analyse the sequence $\eta \mu^k$ and its harmonic extension $(\eta \mu_k)^h$ to $\mathbb{R}_+^{n+1}$. We have

$$\|\eta \mu_k\|_{L^{\frac{n}{s}}(\mathbb{R}^n)} \leq \|\eta\|_{L^\infty(\mathbb{R}^n)} \|\mu_k\|_{L^{\frac{n}{s}}(B_{\rho+1})} \xrightarrow{k \rightarrow \infty} 0,$$

and

$$\begin{aligned} [\eta \mu_k]_{W^{s, \frac{n}{s}}(\mathbb{R}^n)} &\leq \|\eta\|_{L^\infty(\mathbb{R}^n)} [\mu_k]_{W^{s, \frac{n}{s}}(\mathbb{R}^n)} \\ &\quad + 2 \left(\int_{B_{\rho+1}} |\mu_k(y)|^{\frac{n}{s}} \int_{\mathbb{R}^n} \frac{|\eta(x) - \eta(y)|^{\frac{n}{s}}}{|x - y|^{2n}} dx dy \right)^{\frac{s}{n}} \\ &\lesssim [\mu_k]_{W^{s, \frac{n}{s}}(\mathbb{R}^n)} + \|\mu_k\|_{L^{\frac{n}{s}}(B_{\rho+1})} \xrightarrow{k \rightarrow \infty} 0. \end{aligned}$$

Now, following the first inequality in (2.10), applied to f and to the sequence $\eta \mu_k$, together with (2.8) and (2.9), we obtain:

$$\begin{aligned} \|\eta \mu_k \partial_\alpha f\|_{W^{s-1, \frac{n}{s}}(\mathbb{R}^n)} &\lesssim [\eta \mu_k]_{W^{s, \frac{n}{s}}(\mathbb{R}^n)} [f]_{W^{s, \frac{n}{s}}(\mathbb{R}^n)} \\ &\quad + \left\| (\eta \mu_k)^h t^{1-\frac{s}{n}-s} Df^h \right\|_{L^{\frac{n}{s}}(\mathbb{R}_+^{n+1})}. \end{aligned}$$

Since $[\eta\mu_k]_{W^{s, \frac{n}{s}}(\mathbb{R}^n)} \xrightarrow{k \rightarrow \infty} 0$, we conclude the proof of the theorem once we show

$$\lim_{k \rightarrow \infty} \left\| (\eta\mu_k)^h t^{1-\frac{s}{n}-s} Df^h \right\|_{L^{\frac{n}{s}}(\mathbb{R}_+^{n+1})} = 0. \quad (2.13)$$

For this, we observe that

$$\left\| t^{1-\frac{s}{n}-s} Df^h \right\|_{L^{\frac{n}{s}}(\mathbb{R}_+^{n+1})} \lesssim [f]_{W^{s, \frac{n}{s}}(\mathbb{R}^n)} < \infty$$

and by the maximum principle

$$\sup_k \left\| (\eta\mu_k)^h \right\|_{L^\infty(\mathbb{R}_+^{n+1})} \leq \sup_k \|\eta\mu_k\|_{L^\infty(\mathbb{R}^n)} < \infty.$$

Let

$$G_k := (\eta\mu_k)^h t^{1-\frac{s}{n}-s} \left| Df^h \right|.$$

Then we have

$$\sup_k |G_k(x, t)| \lesssim t^{1-\frac{s}{n}-s} \left| Df^h(x, t) \right| \quad \forall x, t \in \mathbb{R}_+^{n+1}.$$

On the other hand, we have from the convergence $\eta\mu_k \rightarrow 0$ in $L^{\frac{n}{s}}(\mathbb{R}^n)$ that every subsequence of $\eta\mu_k$ has a subsequence $\eta\mu_{k_j} \xrightarrow{j \rightarrow \infty} 0$ almost everywhere in $\mathbb{R}^n$. Since $\eta\mu_k$ are compactly supported in $B_{\rho+1}$, they belong to $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and hence the Poisson integral formula (2.7) is valid. Now, the uniform boundedness of $\eta\mu_k$ in $L^\infty(\mathbb{R}^n)$ and dominated convergence applied to (2.7) imply that $(\eta\mu_{k_j})^h$, and hence G_{k_j} , converge to 0 almost everywhere in $\mathbb{R}_+^{n+1}$. By dominated convergence we then find

$$\lim_{j \rightarrow \infty} \|G_{k_j}\|_{L^{\frac{n}{s}}(\mathbb{R}_+^{n+1})} = 0.$$

A standard argument now implies (2.13) and we conclude the proof as (2.12) is shown. $\square$

The following corollary is a local version of Proposition 2.5:

Corollary 2.6. *Let $n \geq 2$ and $1/2 < s < 1$. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain and $f \in W^{s, \frac{n}{s}}(\Omega)$.*

- (i) *Let $\mu \in W^{s, \frac{n}{s}}(\Omega) \cap L^\infty(\Omega)$. Then, for any $\alpha \in \{1, \dots, n\}$, the product $\mu \partial_\alpha f$ is well defined as a distribution on Ω and*

$$\|\mu \partial_\alpha f\|_{\dot{W}^{s-1, \frac{n}{s}}(\Omega)} \lesssim [f]_{W^{s, \frac{n}{s}}(\Omega)} ([\mu]_{W^{s, \frac{n}{s}}(\Omega)} + \|\mu\|_{L^\infty(\Omega)}).$$

Moreover, for any $\mu \in C^\infty(\overline{\Omega})$ and $\phi \in W_{00}^{1-s, \frac{n}{n-s}}(\Omega)$ we have

$$\mu \partial_\alpha f[\phi] = \partial_\alpha f[\mu\phi]; \quad (2.14)$$

(ii) Let $\mu_k \in W^{s, \frac{n}{s}}(\Omega) \cap L^\infty(\Omega)$ be such that

$$\sup_k \|\mu_k\|_{L^\infty(\Omega)} < \infty \quad \text{and} \quad \lim_{k \rightarrow \infty} \|\mu_k\|_{W^{s, \frac{n}{s}}(\Omega)} = 0. \quad (2.15)$$

Then, for any $\alpha \in \{1, \dots, n\}$,

$$\|\mu_k \partial_\alpha f\|_{\dot{W}^{s-1, \frac{n}{s}}(\Omega)} \xrightarrow{k \rightarrow \infty} 0.$$

Remark 2.7. Note that the mere boundedness of $\|\mu_k\|_{L^{\frac{n}{s}}(\Omega)}$ is no more sufficient for the local version of Proposition 2.5-(ii) to be true. $\mu_k \equiv 1$ is a trivial counterexample.

Proof. Given $f \in W^{s, \frac{n}{s}}(\Omega)$ and $\mu \in W^{s, \frac{n}{s}}(\Omega) \cap L^\infty(\Omega)$, we extend them to $\tilde{f}, \tilde{\mu}$ using a bounded linear operator to the whole $\mathbb{R}^n$ and we consider the mollified sequence $\tilde{f}_\varepsilon$ and $\tilde{\mu}_\varepsilon$. By Proposition 2.5 we have for any $\phi \in C_c^\infty(\Omega)$, extended by 0 outside Ω to $\tilde{\phi}$ over $\mathbb{R}^n$,

$$\int_{\Omega} \tilde{\mu}_\varepsilon \partial_\alpha \tilde{f}_\varepsilon \phi = \int_{\mathbb{R}^n} \tilde{\mu}_\varepsilon \partial_\alpha \tilde{f}_\varepsilon \tilde{\phi} \longrightarrow \int_{\mathbb{R}^n} \tilde{\mu} \partial_\alpha \tilde{f} \tilde{\phi} \text{ as } \varepsilon \rightarrow 0.$$

We define for $\phi \in C_c^\infty(\Omega)$

$$\mu \partial_\alpha f[\phi] := \tilde{\mu} \partial_\alpha \tilde{f}[\tilde{\phi}], \quad (2.16)$$

which, in view of the fact that

$$[\tilde{\phi}]_{W^{s,p}(\mathbb{R}^n)} \lesssim [\phi]_{W_{00}^{s,p}(\Omega)},$$

satisfies the desired estimate in (i). Approximating f and ϕ in their respective spaces by smooth sequences $\tilde{f}_k \in C^\infty(\bar{\Omega})$ and $\tilde{\phi}_k \in C_c^\infty(\Omega)$ and passing to the limit using the newly established estimates on Ω yields (2.14).

As for (ii), Proposition 2.5-(ii) is applicable to the extensions $\tilde{\mu}_k$ because of the assumptions in (2.15) since in that case $\|\tilde{\mu}_k\|_{L^\infty(\mathbb{R}^n)}$ are uniformly bounded and we have

$$\|\tilde{\mu}_k\|_{W^{s, \frac{n}{s}}(\mathbb{R}^n)} \lesssim \|\mu_k\|_{W^{s, \frac{n}{s}}(\Omega)} \xrightarrow{k \rightarrow \infty} 0.$$

This implies (ii) as formulated.

Note that a diagonal argument and part (ii) also prove the independence of the definition from the choice of extensions. $\square$

Corollary 2.8. Let $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain or $\Omega = \mathbb{R}^n$. Assume that $1/2 < s < 1$, $g \in \dot{W}^{s-1, \frac{n}{s}}(\Omega)$ and $\mu \in W^{s, \frac{n}{s}}(\Omega) \cap L^\infty(\Omega)$. Then the product μg is well defined as a distribution on Ω and

$$\|\mu g\|_{\dot{W}^{s-1, \frac{n}{s}}(\Omega)} \lesssim \|g\|_{\dot{W}^{s-1, \frac{n}{s}}(\Omega)} \left([\mu]_{W^{s, \frac{n}{s}}(\Omega)} + \|\mu\|_{L^\infty(\Omega)} \right).$$

Moreover, if $\mu_k \in W^{s, \frac{n}{s}}(\Omega) \cap L^\infty(\Omega)$ with

$$\begin{cases} \sup_k \|\mu_k\|_{L^\infty(\Omega)} < \infty \quad \text{and} \\ \begin{cases} \lim_{k \rightarrow \infty} \|\mu_k\|_{W^{s, \frac{n}{s}}(\Omega)} = 0 & \text{if } \Omega \neq \mathbb{R}^n \\ \sup_k \|\mu_k\|_{L^{\frac{n}{s}}(\mathbb{R}^n)} < \infty \text{ and } \lim_{k \rightarrow \infty} [\mu_k]_{W^{s, \frac{n}{s}}(\mathbb{R}^n)} = 0 & \text{otherwise,} \end{cases} \end{cases}$$

then

$$\|\mu_k g\|_{\dot{W}^{s-1, \frac{n}{s}}(\Omega)} \xrightarrow{k \rightarrow \infty} 0.$$

Remark 2.9. When $\Omega = \mathbb{R}^n$, $1/2 < s < 1$ and g belongs to the larger space $W^{s-1, \frac{n}{s}}(\mathbb{R}^n)$, the product μg can be defined as an element of $W^{s-1, \frac{n}{s}}(\mathbb{R}^n)$ and its continuity can be shown using [44, Theorem 4.6.2/2], where the Triebel-Lizorkin theory of spaces and the notion of paraproducts are used. Another proof can be given through duality based on [6, Lemma 6]. Indeed, for $1/2 < s < 1$, let $1 < t = n/s < \infty$, $0 < \theta = (1-s)/s < 1$, $1 < p = n/(n-s) < \infty$, and $1 < r = n/(n-1) < \infty$, and note that

$$\frac{1}{r} + \frac{\theta}{t} = \frac{1}{p}.$$

Hence, [6, Lemma 6] implies that for all $\phi \in W^{1-s, \frac{n}{n-s}}(\mathbb{R}^n)$

$$\begin{aligned} & \|\mu \phi\|_{W^{1-s, \frac{n}{n-s}}(\mathbb{R}^n)} \\ & \lesssim \|\mu\|_{L^\infty(\mathbb{R}^n)} \|\phi\|_{W^{1-s, \frac{n}{n-s}}(\mathbb{R}^n)} + \|\mu\|_{W^{s, \frac{n}{s}}(\mathbb{R}^n)}^\theta \|\mu\|_{L^\infty(\mathbb{R}^n)}^{1-\theta} \|\phi\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \\ & \lesssim \|\mu\|_{L^{\frac{2s-1}{s}}(\mathbb{R}^n)}^{\frac{2s-1}{s}} \left(\|\mu\|_{L^\infty(\mathbb{R}^n)}^{\frac{1-s}{s}} + \|\mu\|_{W^{s, \frac{n}{s}}(\mathbb{R}^n)}^{\frac{1-s}{s}} \right) \|\phi\|_{W^{1-s, \frac{n}{n-s}}(\mathbb{R}^n)}. \end{aligned}$$

Now it is sufficient to define, for $g \in W^{s-1, \frac{n}{s}}(\mathbb{R}^n)$,

$$\mu g[\phi] := g[\mu \phi],$$

and we obtain the estimate

$$\|\mu g\|_{W^{s-1, \frac{n}{s}}(\mathbb{R}^n)} \lesssim \|\mu\|_{L^{\frac{2s-1}{s}}(\mathbb{R}^n)}^{\frac{2s-1}{s}} \left(\|\mu\|_{L^\infty(\mathbb{R}^n)}^{\frac{1-s}{s}} + \|\mu\|_{W^{s, \frac{n}{s}}(\mathbb{R}^n)}^{\frac{1-s}{s}} \right) \|g\|_{W^{s-1, \frac{n}{s}}(\mathbb{R}^n)}$$

by duality.

Proof. If $\Omega = \mathbb{R}^n$, in view of (2.4), it suffices to apply Proposition 2.5 to components of $f := D\Delta_{\mathbb{R}^n}^{-1}g$, if necessary by approximating g in $\dot{W}^{s-1, \frac{n}{s}}(\mathbb{R}^n)$ by a sequence of $C_c^\infty(\mathbb{R}^n)$ functions. If Ω is a bounded smooth domain, we fix an

extension operator $g \rightarrow \tilde{g}$ from $W^{s-1, \frac{n}{s}}(\Omega)$ into $W^{s-1, p}(\mathbb{R}^n)$, and a function $\eta \in C_c^\infty(\mathbb{R}^n)$ such that $\eta \equiv 1$ on $\overline{\Omega}$. We have

$$\|\eta \tilde{g}\|_{\dot{W}^{s-1, \frac{n}{s}}(\mathbb{R}^n)} \lesssim \|\tilde{g}\|_{W^{s-1, \frac{n}{s}}(\mathbb{R}^n)} \lesssim \|g\|_{W^{s-1, \frac{n}{s}}(\Omega)} \leq \|g\|_{\dot{W}^{s-1, \frac{n}{s}}(\Omega)}.$$

Hence $\eta \tilde{g} \in \dot{W}^{s-1, \frac{n}{s}}(\mathbb{R}^n)$ is a bounded extension of g to the whole $\mathbb{R}^n$ and for any extension $\tilde{\mu} \in W^{s, \frac{n}{s}} \cap L^\infty(\mathbb{R}^n)$ of μ , the product $\tilde{\mu}(\eta \tilde{g})$ is well defined. We set $\mu g[\phi] := (\tilde{\mu}(\eta \tilde{g}))[\phi]$ for all $\phi \in C_c^\infty(\Omega)$. We can now argue as in the proof of Corollary 2.6 in order to establish the properties of the distributional product μg and its independence from the choice of the extension operators or η . $\square$

3. A Proof of Theorem 1.8

Our reasoning for proving Proposition 3.2 is a combination of the arguments used in the proofs of [42, Proposition 1.1] and [35, Theorem 1.3]. First, analogous to [35, Proposition 7.1], we show that assuming the proper fractional Sobolev regularity, the degree formula is valid for f :

Lemma 3.1. *Assume $\Omega \subset \mathbb{R}^2$ is an open smooth bounded set, or $\Omega = \mathbb{R}^2$, and fix $s \geq 2/3$ and $f \in W^{s, 2/s} \cap C^0(\Omega, \mathbb{R}^2)$. For any $\tilde{\Omega} \Subset \Omega$ and any $g \in C_c^\infty(\mathbb{R}^2 \setminus f(\partial\tilde{\Omega}))$, one has*

$$\int_{\mathbb{R}^2} g(y) \deg(f, \tilde{\Omega}; y) dy = \text{Jac}(f)[g \circ f].$$

In particular, if $\text{Jac}(f) > 0$, then $\deg(f, \tilde{\Omega}; y)$ is non-negative whenever it is well defined and moreover

$$\forall y \in f(\tilde{\Omega}) \setminus f(\partial\tilde{\Omega}) \quad \deg(f, \tilde{\Omega}; y) \geq 1, \quad (3.1)$$

since the degree must be positive for such y .

By definition $\text{Jac}(f) > 0$ if, for all non-negative $\phi \in C_c^\infty(\Omega)$, $\text{Jac}(f)[\phi] > 0$, unless $\phi \equiv 0$.

Proof. Consider the mollified functions $f_\varepsilon := f * \varphi_\varepsilon \in C^\infty(\Omega, \mathbb{R}^2)$, as defined in Subsection 2.2. Since f_ε converges locally uniformly to f , similar as in [35, Proposition 7.1] we have

$$\begin{aligned} \deg(f, \tilde{\Omega}; y) &= \deg(f_\varepsilon, \tilde{\Omega}; y) \quad \text{for all } y \in \text{supp } g; \\ \int_{\tilde{\Omega}} (g \circ f_\varepsilon(z)) \det \nabla f_\varepsilon(z) dz &= \int_{\mathbb{R}^2} g(y) \deg(f_\varepsilon, \tilde{\Omega}; y) dy, \end{aligned}$$

for small enough ε . So it suffices to show that

$$\int_{\tilde{\Omega}} (g \circ f_\varepsilon(z)) \det \nabla f_\varepsilon(z) dz \longrightarrow \text{Jac}(f)[g \circ f] \quad \text{as } \varepsilon \rightarrow 0. \quad (3.2)$$

But the left-hand side of (3.2) equals $\text{Jac}(f_\varepsilon)[g \circ f_\varepsilon]$, which converges to $\text{Jac}(f)[g \circ f]$ by Lemma 8.1. This proves (3.2), and hence the assertion follows. $\square$

Next we show that if further $\text{Jac}(f) \equiv 0$ and $\text{curl } f = 0$, then the image $f(\Omega)$ is of zero measure. In view of [32, Corollary 1.1.2] (since f is curl-free, f has locally a gradient structure) and [13, Proposition 2.1], it follows that f is either locally constant around a point or constant in segments joining the boundary of Ω on both sides. The local Hölder regularity $C^{0,s/2}$ is a straightforward consequence of the Fubini theorem for fractional Sobolev spaces [44, 2.3.4/2] and the Sobolev embedding theorem in one dimensions [14, Theorem 8.2] after the application of the local bilipschitz change of variable introduced in the proof of [13, Lemma 2.11]. The little Hölder regularity follows in view of density of smooth mappings in $W^{s,\frac{2}{s}}(\mathbb{R})$ for $s > 0$. This will conclude the proof of Theorem 1.8.

Proposition 3.2. *Let Ω , s , and f be as in the assumptions of Theorem 1.8. Then $f(\Omega)$ has zero Lebesgue measure. In particular it has empty interior.*

Proof. Without loss of generality and by considering compactly contained subsets of Ω we can assume that Ω is bounded and smooth. Following Kirchheim [31] and as in the arguments of Pakzad [42, Lemma 2.1] and Li-Schikorra [36, Theorem 1.6], consider the auxiliary maps

$$f^{(\delta)}(x, y) := f(x, y) + \delta(-y, x)^\top. \quad (3.3)$$

Let $\widetilde{\Omega} \Subset \Omega$ be an open set. Since $f^{(\delta)} \rightarrow f$ uniformly as $\delta \searrow 0$, there exists a number δ_κ small enough such that

$$\|f - f^{(\delta_\kappa)}\|_{C^0(\widetilde{\Omega})} \leq \kappa.$$

One may choose δ_κ to be decreasing in κ . As a consequence, $f(\widetilde{\Omega})$ lies in the κ -neighbourhood of $f^{(\delta_\kappa)}(\widetilde{\Omega})$. Thus

$$\mathcal{L}^2\left(f(\widetilde{\Omega}) \Delta f^{(\delta_\kappa)}(\widetilde{\Omega})\right) \leq C\kappa^2$$

for some constant C depending only on $\widetilde{\Omega}$. Therefore, by sending $\kappa \rightarrow 0$, we may infer that

$$\lim_{\delta \searrow 0} \mathcal{L}^2\left(f^{(\delta)}(\widetilde{\Omega})\right) = \mathcal{L}^2\left(f(\widetilde{\Omega})\right). \quad (3.4)$$

On the other hand, once again by setting $f_\varepsilon := f * \varphi_\varepsilon$ and $f_\varepsilon^{(\delta)}(x, y) := f_\varepsilon(x, y) + \delta(-y, x)^\top$, we note that $f^{(\delta)}$ is the $W^{s,2/s}$ -limit of $f_\varepsilon^{(\delta)}$ and hence for all $\phi \in C_c^\infty(\Omega)$:

$$\text{Jac}(f^{(\delta)})[\phi] = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \det(\nabla f_\varepsilon^{(\delta)}) \phi = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \det(\nabla f_\varepsilon) \phi + \int_{\Omega} \delta^2 \phi = \int_{\Omega} \delta^2 \phi,$$

where we used the facts that $\text{curl } f_\varepsilon = 0$ and $\text{Jac}(f) = 0$. We deduce that $\text{Jac}(f^{(\delta)}) \equiv \delta^2 > 0$. Note that by [36, Theorem 1.1] $f^{(\delta)}$ is continuous.

We take a nondecreasing sequence of non-negative $g_k \in C_c^\infty(\mathbb{R}^2 \setminus f^{(\delta)}(\partial\widetilde{\Omega}))$ converging pointwise to $\chi_{\mathbb{R}^2 \setminus f^{(\delta)}(\partial\widetilde{\Omega})}$. Applying Lemma 3.1 and the monotone convergence theorem we have

$$\begin{aligned} & \int_{\mathbb{R}^2 \setminus f^{(\delta)}(\partial\widetilde{\Omega})} \deg(f^{(\delta)}, \widetilde{\Omega}; y) \, dy \\ &= \lim_{k \rightarrow \infty} \int_{\mathbb{R}^2} g_k(y) \deg(f^{(\delta)}, \widetilde{\Omega}; y) \, dy = \lim_{k \rightarrow \infty} \int_{\widetilde{\Omega}} (g_k \circ f^{(\delta)}) \delta^2 \quad (3.5) \\ &= \delta^2 \mathcal{L}^2(\widetilde{\Omega} \setminus (f^{(\delta)})^{-1}(f^{(\delta)}(\partial\widetilde{\Omega}))) \leq \delta^2 \mathcal{L}^2(\widetilde{\Omega}). \end{aligned}$$

For any $x \in \Omega$, we let $B_x \Subset \Omega$ be a disk centered at x in a manner so that $f^{(\delta)} \in W^{s, 2/s}(\partial B_x)$. This is possible by the Fubini theorem for fractional Sobolev spaces, which is a well-known fact, see [49] in view of [1, Lemma 7.68]. A similar proof recently appeared in [36, Lemma 2.2]; for other proofs see [44, Theorem 2.3.4/2] or [45, Lemma 2.6]. Now, Theorem B.1 yields $\mathcal{L}^2(f^{(\delta)}(\partial B_x)) = 0$. Therefore, applying (3.1) and (3.5) to $\widetilde{\Omega} = B_x$ we have:

$$\begin{aligned} \mathcal{L}^2(f^{(\delta)}(B_x)) &= \int_{\mathbb{R}^2} \chi_{f^{(\delta)}(B_x)} = \int_{\mathbb{R}^2 \setminus f^{(\delta)}(\partial B_x)} \chi_{f^{(\delta)}(B_x)} \\ &\leq \int_{\mathbb{R}^2 \setminus f^{(\delta)}(\partial B_x)} \deg(f^{(\delta)}, B_x; y) \, dy \leq \delta^2 \mathcal{L}^2(B_x). \end{aligned}$$

It follows by (3.4) that for all $x \in \Omega$, $\mathcal{L}^2(f(B_x)) = 0$. The conclusion follows. $\square$

4. Mollifying $W^{1+s, 2/s}$ isometric immersions

Given an isometric immersion $u \in I^{1+s, \frac{2}{s}}(\Omega, \mathbb{R}^3)$ on a bounded smooth domain $\Omega \subset \mathbb{R}^2$, with $s > 1/2$, we will study the geometry of a sequence of mollified mappings $u_\varepsilon := u * \varphi_\varepsilon$. One difficulty is that the mapping u_ε is not isometric anymore, and a priori might fail to be an immersion. We will also need to define the Gauss map $\vec{n}^\varepsilon$ by the formula

$$\vec{n}^\varepsilon := \frac{\partial_1 u_\varepsilon \wedge \partial_2 u_\varepsilon}{|\partial_1 u_\varepsilon \wedge \partial_2 u_\varepsilon|}.$$

But $\vec{n}^\varepsilon$ is well defined only if $|\partial_1 u_\varepsilon \wedge \partial_2 u_\varepsilon|(x) \neq 0$ for a.e. $x \in \widetilde{\Omega}$. Actually, for $\vec{n}^\varepsilon$ to be smooth we need a uniform lower bound on $|\partial_1 u_\varepsilon \wedge \partial_2 u_\varepsilon|$; in other words we need that u_ε is an immersion at least for small enough $\varepsilon > 0$, a fact that is true but by no means trivial. This is the subject of the following lemma, which also discusses the behavior of the pull-back metric induced by u_ε , i.e.:

$$\mathfrak{g}^\varepsilon := (\nabla u_\varepsilon)^T \nabla u_\varepsilon = u_\varepsilon^* \mathfrak{c}.$$

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain, $0 < s < 1$ and $u \in I^{1+s, \frac{2}{s}}(\Omega, \mathbb{R}^3)$. Let $\tilde{\Omega} \Subset \Omega$. Then there exists $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$,*

$$\forall x \in \tilde{\Omega} \quad |\partial_1 u_\varepsilon \wedge \partial_2 u_\varepsilon|(x) = \sqrt{\det g^\varepsilon} > \frac{1}{2} \quad (4.1)$$

and as a consequence g^ε is a Riemannian metric, $u_\varepsilon : \tilde{\Omega} \rightarrow \mathbb{R}^3$ is a smooth immersion on $\tilde{\Omega}$ and the unit normal $\vec{n}^\varepsilon$ and the second fundamental form $\Pi_{ij}^\varepsilon := \partial_{ij} u_\varepsilon \cdot \vec{n}^\varepsilon$ are well defined. Moreover, the following statements hold true:

- (i) $\lim_{\varepsilon \rightarrow 0} \|g^\varepsilon - \mathfrak{e}\|_{C^0(\tilde{\Omega})} = 0$;
- (ii) $\lim_{\varepsilon \rightarrow 0} \|(g^\varepsilon)^{-1} - \mathfrak{e}\|_{C^0(\tilde{\Omega})} = 0$;
- (iii) $\|g^\varepsilon - \mathfrak{e}\|_{L^{\frac{2}{s}}(\tilde{\Omega})} \leq o(\varepsilon^s)$ and $\|\nabla g^\varepsilon\|_{L^{\frac{2}{s}}(\tilde{\Omega})} + \|\nabla(g^\varepsilon)^{-1}\|_{L^{\frac{2}{s}}(\tilde{\Omega})} \leq o(\varepsilon^{s-1})$;
- (iv) $\|g^\varepsilon - \mathfrak{e}\|_{L^{\frac{1}{s}}(\tilde{\Omega})} \leq o(\varepsilon^{2s})$ and for $k \geq 1$, $\|\nabla^k g^\varepsilon\|_{L^{\frac{1}{s}}(\tilde{\Omega})} \leq o(\varepsilon^{2s-k})$;
- (iv) $\lim_{\varepsilon \rightarrow 0} \|g^\varepsilon - \mathfrak{e}\|_{W^{2s, \frac{1}{s}}(\tilde{\Omega})} = 0$.

Remark 4.2. $W^{s, \frac{2}{s}}$ barely fails to embed in L^∞ in two dimensions and the C^0 convergence of the metrics g^ε , which is a key feature of the statement, is not trivial.

Proof. Consider the smooth manifold

$$O(2, 3) := \{A \in \mathbb{R}^{3 \times 2} : A^T A = \text{Id}\},$$

and note that if $u \in I^{1+s, \frac{2}{s}}(\Omega)$, then $\nabla u \in O(2, 3)$ a.e. in Ω . We claim that the Jacobian derivatives ∇u_ε of the mollified sequence u_ε are uniformly close to $O(2, 3)$ on $\tilde{\Omega}$. Note that $W^{s, \frac{2}{s}}(\tilde{\Omega})$ does not embed in $L^\infty(\tilde{\Omega})$ and so ∇u_ε are not necessarily uniformly close to ∇u .

Lacking an L^∞ estimate, the main idea is to use the approach of Schoen and Uhlenbeck [46] and to apply the standard BMO estimate

$$\|\nabla u_\varepsilon - \nabla u\|_{BMO} \leq \|\nabla u_\varepsilon - \nabla u\|_{W^{s, \frac{2}{s}}},$$

on small balls around a point $x \in \Omega$. See also [8, Section I.1] for a discussion of this topic and its applications in a larger context and [5] regarding its application in approximating fractional Sobolev mappings into manifolds.

Indeed, applying Lemma 2.4 we have for all $x \in \tilde{\Omega}$ and $\varepsilon < \text{dist}(\tilde{\Omega}, \partial\Omega)$

$$|\text{dist}(\nabla u_\varepsilon(x), O(2, 3))|^{\frac{n}{s}} \leq \int_{B_\varepsilon(x)} |\nabla u(y) - \nabla u_\varepsilon(x)|^{\frac{n}{s}} dy \leq o(1),$$

where

$$\text{dist}(\nabla u_\varepsilon(x), O(2, 3)) := \inf_{A \in O(2, 3)} |\nabla u_\varepsilon(x) - A|,$$

and $|A|$ denotes the Hilbert-Schmidt norm of a matrix A . Let $A(x) \in O(2, 3)$ be the matrix for which the infimum is attained. Therefore

$$\begin{aligned} \|\mathfrak{g}^\varepsilon - \mathfrak{e}\|_{C^0(\widetilde{\Omega})} &= \sup_{x \in \widetilde{\Omega}} \left| (\nabla u_\varepsilon(x))^T \nabla u_\varepsilon(x) - \text{Id} \right| \\ &= \sup_{x \in \widetilde{\Omega}} \left| (\nabla u_\varepsilon)^T \nabla u_\varepsilon - A^T A \right| (x) \leq o(1), \end{aligned}$$

which proves (i). In particular, since $|\partial_1 u_\varepsilon \wedge \partial_2 u_\varepsilon| = \sqrt{\det \mathfrak{g}^\varepsilon}$ we also obtain

$$\lim_{\varepsilon \rightarrow 0} \|\partial_1 u_\varepsilon \wedge \partial_2 u_\varepsilon - 1\|_{C^0(\widetilde{\Omega})} = \lim_{\varepsilon \rightarrow 0} \left\| \sqrt{\det \mathfrak{g}^\varepsilon} - 1 \right\|_{C^0(\widetilde{\Omega})} = 0.$$

This establishes (4.1). Statement (ii) follows by straightforward calculations using the above uniform estimates. Since ∇u_ε stays uniformly bounded in $L^\infty(\widetilde{\Omega})$, applying Lemma 2.1(ii) to the sequence $\nabla^2 u_\varepsilon$ yields (iii). Finally (iv) and (v) follow respectively from the commutator estimate Lemma 2.1(iii) and Corollary 2.3 since

$$((\nabla u)^T \nabla u) * \varphi_\varepsilon = \mathfrak{e} * \varphi_\varepsilon = \mathfrak{e} \text{ in } \widetilde{\Omega}$$

for all $\varepsilon < \text{dist}(\widetilde{\Omega}, \partial\Omega)$. □

We can therefore define the second fundamental form of $\mathfrak{g}^\varepsilon$ on $\widetilde{\Omega}$ by

$$\Pi_{ij}^\varepsilon := \partial_{ij} u_\varepsilon \cdot \vec{n}^\varepsilon. \quad (4.2)$$

Also, remember that for any Riemannian metric $\mathfrak{g} \in \mathbb{R}_{\text{sym}+}^{2 \times 2}$, its Christoffel symbols are defined by

$$\Gamma_{ij}^l(\mathfrak{g}) := \frac{1}{2} \mathfrak{g}^{lm} (\partial_i \mathfrak{g}_{mj} + \partial_j \mathfrak{g}_{im} - \partial_m \mathfrak{g}_{ij}),$$

with the Einstein summation convention, where $\mathfrak{g}^{lm}$ are the components of $\mathfrak{g}^{-1}$. We define therefore the tensor Γ^ε by:

$$\Gamma^\varepsilon := \left[\Gamma_{ij}^{l,\varepsilon} \right]_{i,j,l \in \{1,2\}}, \quad \Gamma_{ij}^{l,\varepsilon} := \Gamma_{ij}^l(\mathfrak{g}^\varepsilon),$$

with the usual convention $|\Gamma^\varepsilon| := (\sum_{i,j,l=1}^2 |\Gamma_{ij}^{l,\varepsilon}|^2)^{\frac{1}{2}}$.

Corollary 4.3. *Let Ω , $\widetilde{\Omega}$, s and u be as in Lemma 4.1. Then:*

- (i) $\|\Pi^\varepsilon\|_{L^{\frac{2}{s}}(\widetilde{\Omega})} \leq o(\varepsilon^{s-1})$ and $\|\Pi^\varepsilon\|_{\dot{W}^{s-1, \frac{2}{s}}(\widetilde{\Omega})} \leq C$;
- (ii) $\|\Gamma^\varepsilon\|_{L^{\frac{1}{s}}(\widetilde{\Omega})} \leq o(\varepsilon^{2s-1})$ and $\|\nabla \Gamma^\varepsilon\|_{L^{\frac{1}{s}}(\widetilde{\Omega})} \leq o(\varepsilon^{2s-2})$;
- (iii) If $s \geq \frac{1}{2}$, then $\lim_{\varepsilon \rightarrow 0} \|\Gamma^\varepsilon\|_{W^{2s-1, \frac{1}{s}}(\widetilde{\Omega})} = 0$.

Proof. (i) follows from Lemma 2.1(ii) applied to $\nabla^2 u_\varepsilon$ and from Corollary 2.6 with the estimate

$$\|\Pi^\varepsilon\|_{\dot{W}^{s-1, \frac{2}{s}}(\tilde{\Omega})} \leq \left(\|\vec{n}_\varepsilon\|_{L^\infty(\tilde{\Omega})} + [\vec{n}_\varepsilon]_{W^{s, \frac{2}{s}}(\tilde{\Omega})} \right) [\nabla u_\varepsilon]_{W^{s, \frac{2}{s}}(\tilde{\Omega})} \leq C,$$

where the uniform bounds on $\vec{n}_\varepsilon$ are obvious from (4.1) and the similar bounds on ∇u_ε . Applying Lemma 4.1(iii)-(iv) we obtain (ii) on $\tilde{\Omega}$:

$$\|\Gamma^\varepsilon\|_{L^{\frac{1}{s}}} \leq \|(\mathfrak{g}^\varepsilon)^{-1}\|_{L^\infty} \|\nabla \mathfrak{g}\|_{L^{\frac{1}{s}}} \leq o(\varepsilon^{2s-1}),$$

and

$$\|\nabla \Gamma^\varepsilon\|_{L^{\frac{1}{s}}} \leq \|\nabla(\mathfrak{g}^\varepsilon)^{-1}\|_{L^{\frac{2}{s}}} \|\nabla \mathfrak{g}^\varepsilon\|_{L^{\frac{2}{s}}} + \|(\mathfrak{g}^\varepsilon)^{-1}\|_{L^\infty} \|\nabla^2 \mathfrak{g}^\varepsilon\|_{L^{\frac{1}{s}}} \leq o(\varepsilon^{2s-2}).$$

Interpolating these two estimates as in Corollary 2.3 yields (iii). $\square$

Our next statements regard the asymptotic behavior of $\det \Pi^\varepsilon$, which enjoys a better than expected convergence due to its almost Jacobian determinant structure, and of $\operatorname{curl} \Pi^\varepsilon$:

Proposition 4.4. *Let $\Omega, \tilde{\Omega}, u$ be as in Lemma 4.1 with $s \geq 1/2$. Then for all $\phi \in C_c^\infty(\tilde{\Omega})$*

$$\left| \int_{\tilde{\Omega}} (\det \Pi^\varepsilon) \phi \right| \leq o(\varepsilon^{2s-1}) \|\nabla \phi\|_{L^{\frac{1}{1-s}}(\tilde{\Omega})} + o(1) \|\phi\|_{L^\infty(\tilde{\Omega})}.$$

Proof. By [22, Equations (2.1.2)] and the Gauss equation [22, Equations (2.1.7)] we have on $\tilde{\Omega}$:

$$\begin{aligned} \det \Pi^\varepsilon &= R_{2121}(\mathfrak{g}^\varepsilon) = \mathfrak{g}_{1m}^\varepsilon (\partial_1 \Gamma_{22}^{m,\varepsilon} - \partial_2 \Gamma_{21}^{m,\varepsilon} + \Gamma_{1s}^{m,\varepsilon} \Gamma_{22}^{s,\varepsilon} - \Gamma_{2s}^{m,\varepsilon} \Gamma_{21}^{s,\varepsilon}) \\ &= \partial_1 (\mathfrak{g}_{1m}^\varepsilon \Gamma_{22}^{m,\varepsilon}) - \partial_2 (\mathfrak{g}_{1m}^\varepsilon \Gamma_{21}^{m,\varepsilon}) + O(|\Gamma^\varepsilon|^2) \\ &= 2\partial_{12} \mathfrak{g}_{12}^\varepsilon - \partial_{11} \mathfrak{g}_{22}^\varepsilon - \partial_{22} \mathfrak{g}_{11}^\varepsilon + O(|\Gamma^\varepsilon|^2) \\ &= -\operatorname{curl}^T \operatorname{curl} \mathfrak{g}^\varepsilon + O(|\Gamma^\varepsilon|^2). \end{aligned}$$

Hence, using the embedding of $W^{2s-1, \frac{1}{s}}(\mathbb{R}^2)$ into $L^2(\mathbb{R}^2)$ and Corollary 4.3(iii) for $s \geq \frac{1}{2}$ we obtain:

$$\begin{aligned} \left| \int_{\tilde{\Omega}} (\det \Pi^\varepsilon) \phi \right| &\lesssim \int_{\tilde{\Omega}} |(\operatorname{curl}^T \operatorname{curl} \mathfrak{g}^\varepsilon) \phi| + \|\Gamma^\varepsilon\|_{L^2(\tilde{\Omega})}^2 \|\phi\|_{L^\infty(\tilde{\Omega})} \\ &\lesssim \left| \int_{\tilde{\Omega}} (\operatorname{curl} \mathfrak{g}^\varepsilon) \cdot \nabla^\perp \phi \right| + \|\Gamma^\varepsilon\|_{W^{2s-1, \frac{1}{s}}(\tilde{\Omega})}^2 \|\phi\|_{L^\infty(\tilde{\Omega})} \\ &\lesssim \|\nabla \mathfrak{g}^\varepsilon\|_{L^{\frac{1}{s}}(\tilde{\Omega})} \|\nabla \phi\|_{L^{\frac{1}{1-s}}(\tilde{\Omega})} + o(1) \|\phi\|_{L^\infty(\tilde{\Omega})}, \end{aligned}$$

which concludes the proof in view of Lemma 4.1(iii). $\square$

Proposition 4.5. *Let $\Omega, \tilde{\Omega}, u$ be as in Lemma 4.1 with $\frac{2}{3} \leq s < 1$. Then*

$$\|\partial_2 \Pi_{i1}^\varepsilon - \partial_1 \Pi_{i2}^\varepsilon\|_{L^1(\tilde{\Omega})} \leq o(1) \quad \text{for } i \in \{1, 2\}. \quad (4.3)$$

Remark 4.6. An L^1 estimate for $\text{curl } \Pi^\varepsilon$ is not enough for a Hodge decomposition for Π^ε , hence a better than L^1 estimate is crucial for completing the same proof as in [13] for our main theorem. We will hence adapt a new approach as explained in the following section.

Proof. The Codazzi-Mainardi equations [22, Equation (2.1.6)] for the immersion u_ε read

$$\partial_2 \Pi_{i1}^\varepsilon - \partial_1 \Pi_{i2}^\varepsilon = \Pi_{i1}^\varepsilon \Gamma_{i2}^{l,\varepsilon} - \Pi_{i2}^\varepsilon \Gamma_{i1}^{l,\varepsilon}.$$

Now since $s \geq \frac{2}{3}$, $\frac{1}{2} < s' := \frac{2-s}{2} \leq s$, the embedding

$$W^{s, \frac{2}{s}}(\mathbb{R}^2) \hookrightarrow W^{s', \frac{2}{s'}}(\mathbb{R}^2)$$

implies that $u \in I^{1+s', \frac{2}{s'}}(\Omega, \mathbb{R}^3)$. Applying Corollary 4.3(i) for s and Corollary 4.3(ii) for s' yields, for any $\phi \in C_c^\infty(\Omega)$,

$$\|\partial_2 \Pi_{i1}^\varepsilon - \partial_1 \Pi_{i2}^\varepsilon\|_{L^1(\tilde{\Omega})} \lesssim \|\Pi^\varepsilon\|_{L^{\frac{2}{s}}(\tilde{\Omega})} \|\Gamma^\varepsilon\|_{L^{\frac{1}{s'}}(\tilde{\Omega})} \leq o(\varepsilon^{s-1}) o(\varepsilon^{2s'-1}) = o(1). \quad \square$$

5. Second fundamental form for $W^{1+s, 2/s}$ isometric immersions for $\frac{1}{2} < s < 1$

Given an isometric immersion $u \in I_{\text{loc}}^{1+s, \frac{2}{s}}(\Omega, \mathbb{R}^3)$ and a bounded smooth domain $\tilde{\Omega} \Subset \Omega$, with $s > 1/2$, we shall define a weak notion of the second fundamental form Π as a distribution in $\dot{W}^{s-1, \frac{2}{s}}(\tilde{\Omega}, \mathfrak{gl}(2))$. In order to apply the results of Section 4, note that for a $\delta > 0$ small enough $\tilde{\Omega} \Subset \tilde{\Omega}_\delta \Subset \tilde{\Omega}_{2\delta} \Subset \Omega$ and $u \in I^{1+s, \frac{2}{s}}(\tilde{\Omega}_{2\delta}, \mathbb{R}^3)$, where $\tilde{\Omega}_\delta := \{x \in \mathbb{R}^2 : \text{dist}(x, \tilde{\Omega}) < \delta\}$.

The second fundamental form of a given immersion $u : \tilde{\Omega} \rightarrow \mathbb{R}^3$ in the chart defined by u itself is expressed by the product

$$\Pi_{ij} := \partial_{ij} u \cdot \vec{n}, \quad (5.1)$$

where for all $x \in \tilde{\Omega}$, $\vec{n}(x)$ is the unit normal to the immersed surface $u(\tilde{\Omega})$ at $u(x)$. Under our regularity assumptions, and since by the isometry condition for a.e. $x \in \tilde{\Omega}$ we have $|\partial_i u(x)| = 1$, we have $\nabla u \in L^\infty(\tilde{\Omega}, \mathbb{R}^{3 \times 2})$, and so the unit normal

$$\vec{n} := \partial_1 u \wedge \partial_2 u \quad \text{belongs to} \quad W^{s, \frac{2}{s}} \cap L^\infty(\tilde{\Omega}). \quad (5.2)$$

On the other hand, $\partial_{ij} u \in W_{\text{loc}}^{s-1, \frac{2}{s}}(\tilde{\Omega}, \mathbb{R}^3)$, and could be a mere distribution. However, the existence of the distributional product Π_{ij} under these regularity assumptions is justified by Proposition 2.5. To summarize we state the following definition:

Definition 5.1. Let $\Omega \subset \mathbb{R}^2$ be an open set and let $u \in I_{\text{loc}}^{1+s, \frac{2}{s}}(\Omega, \mathbb{R}^3)$ with $\frac{1}{2} < s < 1$. Then, through Proposition 2.5, we define its (weak) second fundamental form

$$\Pi = \Pi(u) := [\Pi_{ij}]_{i,j \in \{1,2\}} \in W_{\text{loc}}^{s-1, \frac{2}{s}}(\Omega, \mathfrak{gl}(2)),$$

by (5.1), namely,

$$\Pi_{ij}[\phi] := \sum_{k=1}^3 \left(\bar{n}^k \partial_{ij} u^k \right) [\phi]$$

for all $\phi \in W_0^{1-s, \frac{2}{2-s}}(\Omega)$ with $\text{supp } \phi \Subset \Omega$.

Proposition 5.2. Let $\Omega \subset \mathbb{R}^2$ be an open set, $\frac{2}{3} \leq s < 1$ and $u \in I_{\text{loc}}^{1+s, \frac{2}{s}}(\Omega, \mathbb{R}^3)$. For all bounded smooth domain $\tilde{\Omega} \Subset \Omega$ the following holds.

- (i) $\lim_{\varepsilon \rightarrow 0} \|\bar{n}^\varepsilon - \bar{n}\|_{W^{s, \frac{2}{s}}(\tilde{\Omega})} = 0;$
- (ii) $\lim_{\varepsilon \rightarrow 0} \|\Pi^\varepsilon - \Pi\|_{\dot{W}^{s-1, \frac{2}{s}}(\tilde{\Omega})} = 0.$

Proof. Note that for any subsequence, we can always find a subsequence of ∇u_ε converging point-wise to ∇u and that ∇u_ε are uniformly L^∞ -bounded in ε . Hence, a straightforward norm calculation and dominated convergence theorem implies (i). To show (ii), we write

$$\begin{aligned} \Pi_{ij}^\varepsilon - \Pi_{ij} &= \sum_{k=1}^3 \bar{n}^{\varepsilon, k} \partial_{ij} u_\varepsilon^k - \bar{n}^k \partial_{ij} u^k = \sum_{k=1}^3 \bar{n}^{\varepsilon, k} \left(\partial_{ij} u_\varepsilon^k - \partial_{ij} u^k \right) \\ &\quad + \sum_{k=1}^3 \left(\bar{n}^{\varepsilon, k} - \bar{n}^k \right) \partial_{ij} u^k, \end{aligned}$$

where $\bar{n}^\varepsilon = (\bar{n}^{\varepsilon, 1}, \bar{n}^{\varepsilon, 2}, \bar{n}^{\varepsilon, 3})$. Now in view of (i), the convergence of each summing term in the $\dot{W}^{s-1, \frac{2}{s}}(\tilde{\Omega})$ norm follows in order from the first and second parts of Corollary 2.6. $\square$

An immediate conclusion of Proposition 4.5 is the following statement regarding the second fundamental form of u when $s \geq \frac{2}{3}$:

Lemma 5.3. Let $\frac{2}{3} \leq s < 1$ and $u \in I_{\text{loc}}^{1+s, \frac{2}{s}}(\Omega, \mathbb{R}^3)$. Assume that $\tilde{\Omega} \Subset \Omega$ is a simply connected bounded smooth domain and let Π be as in Definition 5.1. Then there exists $f \in W^{s, \frac{2}{s}}(\tilde{\Omega}, \mathbb{R}^2)$ such that $\Pi = \nabla f$ in the sense of distributions.

Proof. An immediate consequence of Proposition 4.5 is that Π satisfies the Codazzi equations in the sense of distributions, i.e., $\text{curl } \Pi = 0$:

$$\partial_2 \Pi_{11} - \partial_1 \Pi_{12} = 0 \quad \text{and} \quad \partial_2 \Pi_{21} - \partial_1 \Pi_{22} = 0 \quad \text{in } \mathcal{D}'(\tilde{\Omega}_\delta). \quad (5.3)$$

Let us consider a direct regularisation of the second fundamental form Π . With Π defined as in Definition 5.1, we set

$$\Pi_\varepsilon := \Pi * \varphi_\varepsilon \in C^\infty(\widetilde{\Omega}_\delta; \mathfrak{gl}(2)). \quad (5.4)$$

Here $\Pi_\varepsilon \rightarrow \Pi$ in $W^{s-1, \frac{2}{s}}(\widetilde{\Omega}_\delta)$ as $\varepsilon \rightarrow 0$. The order of convolution and differentiation can be interchanged, so Π_ε satisfies (5.3) in $\mathcal{D}'(\widetilde{\Omega})$ for $\varepsilon < \delta$. Therefore, since $\widetilde{\Omega}$ is simply-connected, there exists $f^\varepsilon \in C^\infty(\Omega, \mathbb{R}^2)$ such that $\Pi_\varepsilon = \nabla f^\varepsilon$. By standard elliptic regularity theory we may choose f^ε to be convergent to some f in $W^{s, \frac{2}{s}}$. Since Π_ε converges only in a very weak norm, and we must be careful that the traces of the solutions are well defined on the boundary, hereby we justify these estimates.

In order to find the sequence f^ε , we first solve for

$$\begin{cases} \Delta \widetilde{\Pi}_\varepsilon = \Pi_\varepsilon & \text{in } \widetilde{\Omega} \\ \operatorname{curl} \widetilde{\Pi}_\varepsilon = \partial_\nu \widetilde{\Pi} \cdot \tau - \partial_\tau \widetilde{\Pi} \cdot \nu = 0 & \text{on } \partial \widetilde{\Omega} \\ \widetilde{\Pi} \cdot \nu = 0 & \text{on } \partial \widetilde{\Omega} \end{cases}$$

where ν and $\tau := \nu^\perp$ are respectively the outward normal and tangential fields to $\partial \widetilde{\Omega}$. Note that the above system is a basic elliptic system discussed at length in the literature of elliptic systems for differential forms, see, e.g., [47, Lemma 1.6.5]. However, from another point of view, if we flatten the boundary the Dirichlet and Neumann boundary conditions decouple and so there is no problem in directly applying the theory of elliptic equations. By [44, Theorem 3.4.3/3(i)], $\widetilde{\Pi}_\varepsilon$ satisfies the estimate

$$\|\widetilde{\Pi}_\varepsilon\|_{W^{1+s, \frac{2}{s}}(\widetilde{\Omega})} \lesssim \|\Pi_\varepsilon\|_{W^{s-1, \frac{2}{s}}(\widetilde{\Omega})} \leq C.$$

Taking the curl of the equation, we note that $\operatorname{curl} \widetilde{\Pi}_\varepsilon$ is harmonic and vanishes on the boundary, hence $\operatorname{curl} \widetilde{\Pi}_\varepsilon \equiv 0$ in $\widetilde{\Omega}$. Now we use the identity

$$\nabla \operatorname{div} \widetilde{\Pi}_\varepsilon - \Delta \widetilde{\Pi}_\varepsilon = -\nabla^\perp \operatorname{curl} \widetilde{\Pi}_\varepsilon,$$

to deduce that $f^\varepsilon := \operatorname{div} \widetilde{\Pi}_\varepsilon$ satisfies $\nabla f^\varepsilon = \Pi_\varepsilon$ with the estimate

$$\|f^\varepsilon\|_{W^{s, \frac{2}{s}}(\widetilde{\Omega})} \lesssim \|\widetilde{\Pi}_\varepsilon\|_{W^{1+s, \frac{2}{s}}(\widetilde{\Omega})} \lesssim 1.$$

Therefore f^ε converges in the sense of distributions to some $f \in W^{s, \frac{2}{s}}(\widetilde{\Omega})$ satisfying $\nabla f = \Pi$. $\square$

6. Developability of components and $c^{1, \frac{s}{2}}$ -regularity

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain and assume that $u \in I_{\operatorname{loc}}^{1+s, \frac{2}{s}}(\Omega, \mathbb{R}^3)$ with $\frac{2}{3} \leq s < 1$. Then, for each $m \in \{1, 2, 3\}$, the component u^m satisfies*

$$\operatorname{Jac}(\nabla u^m) \equiv 0 \text{ in } \mathcal{D}'(\Omega)$$

and as a consequence it is $c^{1, \frac{s}{2}}$ -regular and developable by Theorem 1.8.

Proof. The argument follows closely that of [13, Theorem 3]. Let us fix $m \in \{1, 2, 3\}$ and set

$$g := \nabla u^m \in W_{\text{loc}}^{s, \frac{2}{s}}(\Omega, \mathbb{R}^2).$$

Let $\tilde{\Omega} \Subset \Omega$. For $\delta > 0$ small enough we have $u \in I^{1+s, \frac{2}{s}}(\tilde{\Omega}_\delta, \mathbb{R}^3)$. For $\varepsilon < \delta$ we let u_ε be the mollified sequence of immersions with the properties discussed in Section 4. Note that by [22, Equation (2.1.3)] we have

$$\partial_{ij} u_\varepsilon^m = \Gamma_{ij}^{k, \varepsilon} \partial_k u_\varepsilon^m + \Pi_{ij}^\varepsilon \vec{n}^{\varepsilon, m}. \quad (6.1)$$

Obviously $g_\varepsilon = \nabla u_\varepsilon^m$ and hence, for all $\phi \in C_c^\infty(\tilde{\Omega})$,

$$\begin{aligned} \int_{\tilde{\Omega}} \text{Jac}(g_\varepsilon) \phi &= \int_{\tilde{\Omega}} \det(\nabla^2 u_\varepsilon^m) \phi = \int_{\tilde{\Omega}} \det(\Gamma^\varepsilon \cdot \nabla u_\varepsilon^m + \Pi^\varepsilon \vec{n}^{\varepsilon, m}) \phi \\ &= \int_{\tilde{\Omega}} \det(\Pi^\varepsilon) (\vec{n}^{\varepsilon, m})^2 \phi + \int_{\tilde{\Omega}} \det(\Gamma^\varepsilon \cdot \nabla u_\varepsilon^m) \phi \\ &\quad + \int_{\tilde{\Omega}} \vec{n}^{\varepsilon, m} \Pi^\varepsilon : \text{cof}(\Gamma^\varepsilon \cdot \nabla u_\varepsilon^m) \phi \\ &= I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon. \end{aligned}$$

We claim that as $\varepsilon \rightarrow 0$ the limit of each term I_j^ε is 0, which will complete the proof as $\text{Jac}(g)$ is the distributional limit of $\text{Jac}(g_\varepsilon)$ [36, Lemma 1.3]. By Proposition 4.4

$$\begin{aligned} |I_1^\varepsilon| &\leq o(\varepsilon^{2s-1}) \left(\|(\nabla \vec{n}^{\varepsilon, m}) \phi\|_{L^{\frac{1}{1-s}}(\tilde{\Omega})} + \|\nabla \phi\|_{L^{\frac{1}{1-s}}(\tilde{\Omega})} \right) + o(1) \|(\vec{n}^{\varepsilon, m})^2 \phi\|_{L^\infty(\tilde{\Omega})} \\ &\leq o(\varepsilon^{2s-1}) \|\nabla \vec{n}^{\varepsilon, m}\|_{L^{\frac{1}{1-s}}(\tilde{\Omega})} \|\phi\|_{L^\infty(\tilde{\Omega})} \\ &\quad + o(\varepsilon^{2s-1}) \|\nabla \phi\|_{L^{\frac{1}{1-s}}(\tilde{\Omega})} + o(1) \|\phi\|_{L^\infty(\tilde{\Omega})}. \end{aligned}$$

However note for $s \geq \frac{2}{3}$ the embedding

$$\|\nabla u_\varepsilon\|_{W^{2(1-s), \frac{1}{1-s}}(\tilde{\Omega})} \lesssim \|\nabla u_\varepsilon\|_{W^{s, \frac{2}{s}}(\tilde{\Omega})} \lesssim 1.$$

Therefore applying Lemma 2.1(ii) to ∇u_ε and in view of (4.1) we obtain

$$\|\nabla \vec{n}^{\varepsilon, m}\|_{L^{\frac{1}{1-s}}(\tilde{\Omega})} \leq o(\varepsilon^{2(1-s)-1}) \leq o(\varepsilon^{1-2s}).$$

We conclude for I_1^ε that

$$|I_1^\varepsilon| \leq o(1) \|\phi\|_{L^\infty(\tilde{\Omega})} + o(\varepsilon^{2s-1}) \|\nabla \phi\|_{L^{\frac{1}{1-s}}(\tilde{\Omega})} \rightarrow 0.$$

Now, regarding I_2^ε observe that ∇u_ε is uniformly bounded in L^∞ and as previously observed we can obtain by the embedding of $W^{2s-1, \frac{1}{s}}$ into L^2 , through Corollary 4.3(iii):

$$|I_2^\varepsilon| \lesssim \|\Gamma^\varepsilon\|_{L^2(\tilde{\Omega})}^2 \|\phi\|_{L^\infty(\tilde{\Omega})} = o(1).$$

Finally, to finish the proof of our claim, we estimate once again similar as in Proposition 4.5

$$|I_3^\varepsilon| \lesssim \int_{\tilde{\Omega}} |\Gamma^\varepsilon| |\Pi^\varepsilon| |\phi| \leq \|\Pi^\varepsilon\|_{L^{\frac{2}{s}}(\tilde{\Omega})} \|\Gamma^\varepsilon\|_{L^{\frac{1}{s'}}(\tilde{\Omega})} \|\phi\|_{L^\infty(\tilde{\Omega})} \leq o(1), \quad \text{for } s \geq \frac{2}{3}. \quad \square$$

7. Developability

We already know by Theorem 6.1 that each component of u is independently developable and has the required regularity. What remains to be shown is that the constancy segments and regions of the developability are the same for the three components.

Let $\tilde{\Omega}$ be any smooth bounded domain supported in Ω and let f be defined as in Lemma 5.3. We first claim that any such f is developable.

Proposition 7.1. *Let $\Omega, \tilde{\Omega}, s, u, f$ be as in Lemma 5.3. Then $\text{Jac}(f) = 0$ in $\mathcal{D}'(\tilde{\Omega})$. In particular, since $\nabla f = \Pi$ is symmetric, the conclusions of Theorem 1.8 hold true for f .*

Proof. We will once again use equation (6.1), but this time we will directly pass to the limit as $\varepsilon \rightarrow 0$. Applying Corollary 2.8 in view of Proposition 5.2, we note that

$$\Pi_{ij}^\varepsilon \vec{n}^{\varepsilon, m} \longrightarrow \Pi_{ij} \vec{n}^m \text{ in } \mathcal{D}'(\tilde{\Omega}) \text{ as } \varepsilon \rightarrow 0.$$

Also, Corollary 4.3(iii) implies that the first term in the right-hand side of (6.1) converges to 0 in $\mathcal{D}'(\tilde{\Omega})$. Since $\partial_{ij} u_\varepsilon$ converges to $\partial_{ij} u$, we conclude with the following identity for any pair $(i, j) \in \{1, 2\}^2$:

$$\partial_{ij} u^m = \Pi_{ij} \vec{n}^m.$$

Letting $g_m := \nabla u^m$, this identity reads

$$\nabla g_m = \vec{n}^m \Pi = \vec{n}^m \nabla f. \quad (7.1)$$

Note that $f, g_m \in W^{s, \frac{2}{s}}(\tilde{\Omega})$ and $\vec{n} \in W^{s, \frac{2}{s}} \cap L^\infty(\tilde{\Omega})$. Hence Theorem 1.10 yields that for any $\phi \in C_c^\infty(\tilde{\Omega})$

$$\text{Jac}(g_m)[\phi] = \text{Jac}(f) [(\vec{n}^m)^2 \phi].$$

On the other hand by Theorem 6.1 we have $\text{Jac}(g_m) = 0$, therefore for all $\phi \in C_c^\infty(\tilde{\Omega})$

$$\begin{aligned} \text{Jac}(f)[\phi] &= \text{Jac}(f) \left[\sum_{m=1}^3 (\vec{n}^m)^2 \phi \right] \\ &= \sum_{m=1}^3 \text{Jac}(f) [(\vec{n}^m)^2 \phi] = \sum_{m=1}^3 \text{Jac}(g_m)[\phi] = 0. \quad \square \end{aligned}$$

We complete the proof of Theorem 1.1. We have shown that f is continuous, and for any $x \in \widetilde{\Omega}$, it is either constant around x , or it is constant along the connected component of the intersection of a line passing through x with $\widetilde{\Omega}$. By [13, Corollary 2.10 and Lemma 2.11], for any $x \in \widetilde{\Omega}$, there exists a disk $B_x \ni x$ in $\widetilde{\Omega}$ and Lipschitz unit vector field $\vec{\eta}$ on B_x such that for all $\psi \in C_c^\infty(B_x)$

$$(\nabla f) \cdot [\psi \vec{\eta}] = \int_{B_x} f \operatorname{div}(\psi \vec{\eta}) = 0.$$

Note that the vector field $\vec{\eta}$ determines the constancy directions for f . We claim that for each m and for all $\psi \in C_c^\infty(B_x)$

$$\int_{B_x} \operatorname{div}(\psi \vec{\eta}) \nabla u^m = 0. \quad (7.2)$$

We remark that proving this claim and applying [13, Lemma 2.11 and Proposition 2.1] yields the desired simultaneous constancy of ∇u^m along the segments defined by $\vec{\eta}$ and completes the proof of our main theorem.

To prove (7.2), first note that by Proposition 5.2-(i) and Corollary 2.6 we obtain

$$\lim_{\varepsilon \rightarrow 0} \|\vec{n}^{\varepsilon, m} \nabla f - \vec{n}^m \nabla f\|_{\dot{W}^{s-1, \frac{2}{s}}(\widetilde{\Omega})} = 0,$$

which implies through (2.14)

$$(\vec{n}^m \nabla f) \cdot [\psi \vec{\eta}] = \lim_{\varepsilon \rightarrow 0} (\vec{n}^{\varepsilon, m} \nabla f) \cdot [\psi \vec{\eta}] = \lim_{\varepsilon \rightarrow 0} \nabla f \cdot [(\vec{n}^{\varepsilon, m}) \psi \vec{\eta}] = 0.$$

Combined with (7.1) we obtain that for $m = 1, 2, 3$

$$\int_{B_x} \operatorname{div}(\psi \vec{\eta}) g_m = (\nabla g_m) \cdot [\psi \vec{\eta}] = (\vec{n}^m \nabla f) \cdot [\psi \vec{\eta}] = 0,$$

which establishes (7.2) as claimed.

8. The distributional Jacobian determinant behaves like a determinant

In this section we will prove Theorem 1.10. We first gather some known preliminary results regarding the statement of the theorem.

8.1. Preliminaries

The following useful lemmas are well-known facts. They can be derived via a tedious argument based on Littlewood-Paley theory and paraproducts [48] which extends an earlier work on the limiting case $s = 1$ by [9]. Much more elegant proofs can be achieved following [7] based on the harmonic extension, see also [4], and we refer to [34] for generalizations.

Lemma 8.1 (Distributional Jacobian). *Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain or $\Omega = \mathbb{R}^n$. Assume that $\frac{n-1}{n} < s < 1$, $f \in W^{s, \frac{n}{s}}(\Omega, \mathbb{R}^n)$, $\psi \in W_0^{(1-s)n, \frac{1}{1-s}}(\Omega)$. Then*

$$\text{Jac}(f)[\psi] := \lim_{k \rightarrow \infty} \text{Jac}(f_k)[\psi_k]$$

is well defined as a distribution in $W^{n(s-1), \frac{1}{s}}$, where $f_k \in C^\infty(\overline{\Omega})$ and $\psi_k \in C_c^\infty(\Omega)$ are any two sequences of functions converging to f and ψ in their respective norms.

See, e.g., [36, Lemma 1.3] for a proof.

Lemma 8.2 ([34, Theorem 3.2]). *Let $n \geq 1$, $\lambda, g \in W^{\frac{n}{n+1}, n+1}(\mathbb{R}^n)$ and $\phi \in C_c^\infty(\mathbb{R}^n, \bigwedge^{n-2}(\mathbb{R}^n))$. Then*

$$\left| \int_{\mathbb{R}^n} \lambda dg \wedge d\phi \right| \lesssim [\lambda]_{W^{\frac{n}{n+1}, n+1}(\mathbb{R}^n)} [g]_{W^{\frac{n}{n+1}, n+1}(\mathbb{R}^n)} [\phi]_{W^{\frac{2}{n+1}, \frac{n+1}{n-1}}(\mathbb{R}^n)}.$$

In particular, by the Stokes theorem for differential forms, and by choosing suitable test forms ϕ we have the following estimates for the components:

$$\begin{aligned} \|d(\lambda dg)\|_{\dot{W}^{-\frac{2}{n+1}, \frac{n+1}{2}}(\mathbb{R}^n)} &= \|d(\lambda dg)\|_{\left(\dot{W}^{\frac{2}{n+1}, \frac{n+1}{n-1}}(\mathbb{R}^n)\right)'} \\ &\lesssim [\lambda]_{W^{\frac{n}{n+1}, n+1}(\mathbb{R}^n)} [g]_{W^{\frac{n}{n+1}, n+1}(\mathbb{R}^n)}. \end{aligned}$$

8.2. A determinant estimate

Proposition 8.3. *For any $k \in \{0, \dots, n\}$ and $\phi \in C_c^\infty(\mathbb{R}^n)$, scalar functions $a_j \in W^{\frac{n}{n+1}, n+1}(\mathbb{R}^n)$ and 1-forms $\beta_j \in W^{\frac{n-1}{n+1}, \frac{n+1}{2}}(\mathbb{R}^n, \bigwedge^1(\mathbb{R}^n))$,*

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} (da_1 \wedge \dots \wedge da_k \wedge \beta_{k+1} \wedge \dots \wedge \beta_n) \phi \right| \\ &\lesssim \left(\|\phi\|_{L^\infty} + [\phi]_{W^{\frac{n}{n+1}, n+1}} \right) \prod_{j=1}^k [a_j]_{W^{\frac{n}{n+1}, n+1}} \prod_{j=k+1}^n [\beta_j]_{W^{\frac{n-1}{n+1}, \frac{n+1}{2}}}. \end{aligned}$$

Remark 8.4. The proposition is indeed a determinant estimate:

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} \det(\nabla a_1, \dots, \nabla a_k, B_{k+1}, \dots, B_n) \phi \right| \\ &\lesssim \left(\|\phi\|_{L^\infty} + [\phi]_{W^{\frac{n}{n+1}, n+1}} \right) \prod_{j=1}^k [a_j]_{W^{\frac{n}{n+1}, n+1}} \prod_{j=k+1}^n [B_j]_{W^{\frac{n-1}{n+1}, \frac{n+1}{2}}} \end{aligned}$$

for scalar functions and vector fields of appropriate regularity.

Proof. This can be proven by the tedious arguments in [48] using Littlewood-Paley decomposition and paraproducts. Instead we follow an argument inspired by [7], with the adaptations from [34] (see also [28]). Let a^h, β^h, ϕ^h be the harmonic extensions of the corresponding forms or vectors to $\mathbb{R}_+^{n+1}$.

$$\begin{aligned} & \int_{\mathbb{R}^n} (da_1 \wedge \dots \wedge da_k \wedge \beta_{k+1} \wedge \dots \wedge \beta_n) \phi \\ &= \int_{\mathbb{R}_+^{n+1}} d \left((da_1^h \wedge \dots \wedge da_k^h \wedge \beta_{k+1}^h \wedge \dots \wedge \beta_n^h) \phi^h \right). \end{aligned}$$

Since $dd = 0$ we find

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} (da_1 \wedge \dots \wedge da_k \wedge \beta_{k+1} \wedge \dots \wedge \beta_n) \phi \right| \\ & \lesssim \sum_{\ell=k+1}^n \int_{\mathbb{R}_+^{n+1}} \left| Da_1^h \right| \cdot \dots \cdot \left| Da_k^h \right| \left| \beta_{k+1}^h \right| \dots \left| D\beta_\ell^h \right| \dots \left| \beta_n^h \right| \left| \phi^h \right| \\ & \quad + \int_{\mathbb{R}_+^{n+1}} \left| Da_1^h \right| \cdot \dots \cdot \left| Da_k^h \right| \left| \beta_{k+1}^h \right| \dots \left| \beta_n^h \right| \left| D\phi^h \right|. \end{aligned} \quad (8.1)$$

Recall that for the Hardy-Littlewood maximal function $\mathcal{M}$

$$\left| f^h(x, t) \right| \lesssim \mathcal{M}f(x), \quad (8.2)$$

and for $s \in (0, 1)$,

$$[f]_{W^{s,p}} \approx \left(\int_{\mathbb{R}^n} \left(\int_0^\infty |t^{1-\frac{1}{p}-s} Df^h|^p dt \right) dx \right)^{\frac{1}{p}}.$$

See, e.g., [28, 34]. Therefore from the Hölder inequality and Sobolev embeddings we obtain, for the first terms in (8.1),

$$\begin{aligned} & \int_{\mathbb{R}_+^{n+1}} \left| Da_1^h \right| \cdot \dots \cdot \left| Da_k^h \right| \left| \beta_{k+1}^h \right| \dots \left| D\beta_\ell^h \right| \dots \left| \beta_n^h \right| \left| \phi^h \right| \\ & \lesssim \|\mathcal{M}\phi\|_{L^\infty} \left(\prod_{l=1}^k \|Da_l^h\|_{L^{n+1}(\mathbb{R}_+^{n+1})} \right) \\ & \quad \times \left\| \beta_{k+1}^h \right\|_{L^{n+1}(\mathbb{R}_+^{n+1})} \dots \left\| D\beta_\ell^h \right\|_{L^{\frac{n+1}{2}}(\mathbb{R}_+^{n+1})} \dots \left\| \beta_n^h \right\|_{L^{n+1}(\mathbb{R}_+^{n+1})} \\ & \lesssim \|\mathcal{M}\phi\|_{L^\infty} \left(\prod_{l=1}^k [a_l^h]_{W^{1,n+1}} \right) [\beta_{k+1}^h]_{W^{1,\frac{n+1}{2}}} \dots [\beta_\ell^h]_{W^{1,\frac{n+1}{2}}} \dots [\beta_n^h]_{W^{1,\frac{n+1}{2}}} \\ & \lesssim \|\mathcal{M}\phi\|_{L^\infty} \left(\prod_{l=1}^k [a_l]_{W^{\frac{n}{n+1},n+1}} \right) \prod_{l=k+1}^n [\beta_l]_{W^{\frac{n-1}{n+1},\frac{n+1}{2}}} \end{aligned}$$

which is bounded as required in view of (8.2). The last term in (8.1) is estimated in the same manner through a Hölder estimate:

$$\begin{aligned}
 & \int_{\mathbb{R}_+^{n+1}} \left| Da_1^h \right| \cdots \left| Da_k^h \right| \left| \beta_{k+1}^h \right| \cdots \left| \beta_n^h \right| \left| D\phi^h \right| \\
 & \lesssim \left\| D\phi^h \right\|_{L^{n+1}(\mathbb{R}_+^{n+1})} \prod_{l=1}^k \left\| Da_l^h \right\|_{L^{n+1}(\mathbb{R}_+^{n+1})} \prod_{l=k+1}^n \left\| \beta_l^h \right\|_{L^{n+1}(\mathbb{R}_+^{n+1})} \\
 & \lesssim [\phi]_{W^{\frac{n}{n+1}, n+1}} \prod_{l=1}^k [a_l]_{W^{\frac{n}{n+1}, n+1}} \prod_{l=k+1}^n [\beta_l]_{W^{\frac{n-1}{n+1}, \frac{n+1}{2}}}. \quad \square
 \end{aligned}$$

8.3. Hodge decomposition

Proposition 8.5. Assume that $\lambda \in W^{\frac{n}{n+1}, n+1} \cap L^\infty(\mathbb{R}^n)$ and $g \in W^{\frac{n}{n+1}, n+1}(\mathbb{R}^n; \mathbb{R}^n)$. Then we can decompose

$$\lambda dg = da + \beta,$$

with

$$\begin{aligned}
 [a]_{W^{\frac{n}{n+1}, n+1}(\mathbb{R}^n)} & \lesssim \left(\|\lambda\|_{L^\infty} + [\lambda]_{W^{\frac{n}{n+1}, n+1}(\mathbb{R}^n)} \right) [g]_{W^{\frac{n}{n+1}, n+1}(\mathbb{R}^n)}, \\
 [\beta]_{W^{\frac{n-1}{n+1}, \frac{n+1}{2}}(\mathbb{R}^n)} & \lesssim [\lambda]_{W^{\frac{n}{n+1}, n+1}(\mathbb{R}^n)} [g]_{W^{\frac{n}{n+1}, n+1}(\mathbb{R}^n)}.
 \end{aligned}$$

Proof. On $\mathbb{R}^n$ we let $\omega := \Delta_{\mathbb{R}^n}^{-1}(\lambda dg)$. Hence

$$\Delta_{\mathbb{R}^n} \omega \equiv (dd^* + d^*d)\omega = \lambda dg.$$

Set $a := d^* \omega$ and $\beta := d^* d \omega$. Observe that

$$\Delta_{\mathbb{R}^n} d \omega = d \Delta_{\mathbb{R}^n} \omega = d(\lambda dg);$$

that is,

$$\beta = d^* d \omega = d^* \Delta_{\mathbb{R}^n}^{-1}(d(\lambda dg)).$$

Therefore in view of a component-wise application of (2.4) and Lemma 8.2 we have

$$\begin{aligned}
 [\beta]_{W^{\frac{n-1}{n+1}, \frac{n+1}{2}}(\mathbb{R}^n)} & \lesssim \|d(\lambda dg)\|_{\dot{W}^{\frac{n-1}{n+1}-1, \frac{n+1}{2}}(\mathbb{R}^n)} \\
 & = \|d(\lambda dg)\|_{\dot{W}^{-\frac{2}{n+1}, \frac{n+1}{2}}(\mathbb{R}^n)} \\
 & \lesssim [\lambda]_{W^{\frac{n}{n+1}, n+1}(\mathbb{R}^n)} [g]_{W^{\frac{n}{n+1}, n+1}(\mathbb{R}^n)}.
 \end{aligned}$$

Moreover,

$$\Delta_{\mathbb{R}^n} a = d^* \Delta_{\mathbb{R}^n} \omega = d^*(\lambda dg),$$

so again

$$a = \Delta_{\mathbb{R}^n}^{-1} d^*(\lambda dg).$$

Using (2.4) as before and Proposition 2.5, we obtain as claimed

$$\begin{aligned} [a]_{W^{\frac{n}{n+1}, n+1}(\mathbb{R}^n)} &\lesssim \|\lambda dg\|_{\dot{W}^{\frac{n}{n+1}-1, n+1}(\mathbb{R}^n)} \\ &\lesssim \left(\|\lambda\|_{L^\infty} + [\lambda]_{W^{\frac{n}{n+1}, n+1}(\mathbb{R}^n)} \right) [g]_{W^{\frac{n}{n+1}, n+1}(\mathbb{R}^n)}. \quad \square \end{aligned}$$

Proposition 8.6. Assume that $\lambda \in W^{\frac{n}{n+1}, n+1} \cap L^\infty(\mathbb{R}^n)$ and

$$f \in W^{\frac{n}{n+1}, n+1}(\mathbb{R}^n; \mathbb{R}^n).$$

Then we can decompose

$$\lambda_\varepsilon df_\varepsilon - (\lambda df)_\varepsilon = da^\varepsilon + \beta^\varepsilon$$

with

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} [a^\varepsilon]_{W^{\frac{n}{n+1}, n+1}(\mathbb{R}^n)} &= 0, \\ \lim_{\varepsilon \rightarrow 0} [\beta^\varepsilon]_{W^{\frac{n-1}{n+1}, \frac{n+1}{2}}(\mathbb{R}^n)} &= 0. \end{aligned}$$

Proof. Our arguments are similar to those for Proposition 8.5. First we consider

$$\begin{aligned} &\|d(\lambda_\varepsilon df_\varepsilon - (\lambda df)_\varepsilon)\|_{\dot{W}^{-\frac{2}{n+1}, \frac{n+1}{2}}(\mathbb{R}^n)} \\ &\leq \|d(\lambda_\varepsilon df_\varepsilon - \lambda_\varepsilon df)\|_{\dot{W}^{-\frac{2}{n+1}, \frac{n+1}{2}}(\mathbb{R}^n)} + \|d(\lambda_\varepsilon df - \lambda df)\|_{\dot{W}^{-\frac{2}{n+1}, \frac{n+1}{2}}(\mathbb{R}^n)} \\ &\quad + \|d(\lambda df - (\lambda df)_\varepsilon)\|_{\dot{W}^{-\frac{2}{n+1}, \frac{n+1}{2}}(\mathbb{R}^n)} \\ &=: \text{I}_\varepsilon + \text{II}_\varepsilon + \text{III}_\varepsilon. \end{aligned}$$

In view of Lemma 8.2, we find that

$$\begin{aligned} \text{I}_\varepsilon + \text{II}_\varepsilon &= \|d(\lambda_\varepsilon d(f_\varepsilon - f))\|_{\dot{W}^{-\frac{2}{n+1}, \frac{n+1}{2}}(\mathbb{R}^n)} \\ &\quad + \|d((\lambda_\varepsilon - \lambda)df)\|_{\dot{W}^{-\frac{2}{n+1}, \frac{n+1}{2}}(\mathbb{R}^n)} \\ &\lesssim [\lambda_\varepsilon]_{W^{\frac{n}{n+1}, n+1}(\mathbb{R}^n)} [f_\varepsilon - f]_{W^{\frac{n}{n+1}, n+1}(\mathbb{R}^n)} \\ &\quad + [\lambda_\varepsilon - \lambda]_{W^{\frac{n}{n+1}, n+1}(\mathbb{R}^n)} [f]_{W^{\frac{n}{n+1}, n+1}(\mathbb{R}^n)} \xrightarrow{\varepsilon \rightarrow 0} 0. \end{aligned}$$

In addition, we use Lemma 8.2 once again to deduce that

$$d(\lambda df) \in \dot{W}^{-\frac{2}{n+1}, \frac{n+1}{2}}(\mathbb{R}^n).$$

Thus the convolution converges:

$$\text{III}_\varepsilon = \left\| d(\lambda df) - (d(\lambda df))_\varepsilon \right\|_{\dot{W}^{-\frac{2}{n+1}, \frac{n+1}{2}}(\mathbb{R}^n)} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Putting together the convergence results for I_ε , II_ε , and III_ε , we arrive at

$$\lim_{\varepsilon \rightarrow 0} \left\| d(\lambda_\varepsilon df_\varepsilon - (\lambda df)_\varepsilon) \right\|_{\dot{W}^{-\frac{2}{n+1}, \frac{n+1}{2}}(\mathbb{R}^n)} = 0. \quad (8.3)$$

Now we proceed as in Proposition 8.5. We first solve on $\mathbb{R}^n$:

$$\Delta_{\mathbb{R}^n} \omega^\varepsilon \equiv (dd^* + d^*d)\omega^\varepsilon = \lambda_\varepsilon df_\varepsilon - (\lambda df)_\varepsilon,$$

and then set $a^\varepsilon := d^* \omega^\varepsilon$ and $\beta^\varepsilon := d^* d \omega^\varepsilon$. Observe that

$$\Delta_{\mathbb{R}^n} d \omega^\varepsilon = d \Delta_{\mathbb{R}^n} \omega^\varepsilon = d(\lambda_\varepsilon df_\varepsilon - (\lambda df)_\varepsilon).$$

That is,

$$\beta^\varepsilon = d^* d \omega^\varepsilon = d^* \Delta_{\mathbb{R}^n}^{-1} (d(\lambda_\varepsilon df_\varepsilon - (\lambda df)_\varepsilon)).$$

So, with (2.4) and (8.3) we find that

$$\begin{aligned} [\beta^\varepsilon]_{\dot{W}^{\frac{n-1}{n+1}, \frac{n+1}{2}}(\mathbb{R}^n)} &\lesssim \|d(\lambda_\varepsilon df_\varepsilon - (\lambda df)_\varepsilon)\|_{\dot{W}^{\frac{n-1}{n+1}-1, \frac{n+1}{2}}(\mathbb{R}^n)} \\ &= \|d(\lambda_\varepsilon df_\varepsilon - (\lambda df)_\varepsilon)\|_{\dot{W}^{-\frac{2}{n+1}, \frac{n+1}{2}}(\mathbb{R}^n)} \xrightarrow{\varepsilon \rightarrow 0} 0. \end{aligned}$$

Moreover, we have

$$\Delta_{\mathbb{R}^n} a^\varepsilon = d^* \Delta_{\mathbb{R}^n} \omega^\varepsilon = d^* (\lambda_\varepsilon df_\varepsilon - (\lambda df)_\varepsilon),$$

so

$$a^\varepsilon = \Delta_{\mathbb{R}^n}^{-1} d^* (\lambda_\varepsilon df_\varepsilon - (\lambda df)_\varepsilon).$$

Once again (2.4) yields

$$\begin{aligned} [a^\varepsilon]_{\dot{W}^{\frac{n}{n+1}, n+1}(\mathbb{R}^n)} &\lesssim \|\lambda_\varepsilon df_\varepsilon - (\lambda df)_\varepsilon\|_{\dot{W}^{\frac{n}{n+1}-1, n+1}(\mathbb{R}^n)} \\ &\lesssim \|\lambda_\varepsilon df_\varepsilon - \lambda df\|_{\dot{W}^{\frac{n}{n+1}-1, n+1}(\mathbb{R}^n)} \\ &\quad + \|(\lambda df)_\varepsilon - \lambda df\|_{\dot{W}^{\frac{n}{n+1}-1, n+1}(\mathbb{R}^n)}. \end{aligned}$$

We will use Proposition 2.5 repeatedly throughout the rest of the proof. Observe that $\lambda df \in \dot{W}^{\frac{n}{n+1}-1, n+1}(\mathbb{R}^n)$, so

$$\|(\lambda df)_\varepsilon - \lambda df\|_{\dot{W}^{\frac{n}{n+1}-1, n+1}(\mathbb{R}^n)} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

On the other hand,

$$\begin{aligned} \|\lambda_\varepsilon df_\varepsilon - \lambda df\|_{\dot{W}^{\frac{n}{n+1}-1, n+1}(\mathbb{R}^n)} &\leq \|(\lambda_\varepsilon - \lambda)df\|_{\dot{W}^{\frac{n}{n+1}-1, n+1}(\mathbb{R}^n)} \\ &\quad + \|\lambda_\varepsilon d(f_\varepsilon - f)\|_{\dot{W}^{\frac{n}{n+1}-1, n+1}(\mathbb{R}^n)}. \end{aligned}$$

The former term tends to zero as $\varepsilon \rightarrow 0$. For the latter term, we have

$$\begin{aligned} &\|\lambda_\varepsilon d(f_\varepsilon - f)\|_{\dot{W}^{\frac{n}{n+1}-1, n+1}(\mathbb{R}^n)} \\ &\lesssim \left(\|\lambda_\varepsilon\|_{L^\infty} + [\lambda_\varepsilon]_{W^{\frac{n}{n+1}, n+1}(\mathbb{R}^n)} \right) [f_\varepsilon - f]_{W^{\frac{n}{n+1}, n+1}(\mathbb{R}^n)}, \end{aligned}$$

which again tends to zero. $\square$

8.4. Proof of Theorem 1.10

Proof. Fix $\phi \in C_c^\infty(\Omega)$. We want to show that

$$\text{Jac}(f)[\phi] - \text{Jac}(g)[\lambda^n \phi] = 0.$$

We first boundedly extend g, λ on the whole $\mathbb{R}^n$, keeping the same names for convenience. We define $F := \lambda \nabla g$ as a distribution in $\mathbb{R}^n$, which is well defined by Proposition 2.5. Note that for all $\eta \in C_c^\infty(\Omega)$, extending η by 0 outside Ω to $\tilde{\eta}$, we obtain by (1.1) in view of (2.16):

$$F[\tilde{\eta}] = \nabla f[\eta].$$

Fix an open set $\tilde{\Omega} \Subset \Omega$ containing $\text{supp } \phi$. For ε small enough, $F_\varepsilon := F * \varphi_\varepsilon$ coincides with ∇f_ε on $\tilde{\Omega}$ and hence applying Lemma 8.1 we have

$$\text{Jac}(f)[\phi] = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \det(\nabla f_\varepsilon) \phi = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} \det(F_\varepsilon) \phi,$$

where ϕ is extended by 0 outside Ω to $\mathbb{R}^n$. Also, mollifying g and λ and once again applying Lemma 8.1 we obtain

$$\text{Jac}(g)[\lambda^n \phi] = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \det(\nabla g_\varepsilon) \lambda_\varepsilon^n \phi = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} \det(\nabla g_\varepsilon) \lambda_\varepsilon^n \phi,$$

since $\lambda_\varepsilon^n \phi \rightarrow \lambda^n \phi$ in $W_{00}^{(1-s)n, \frac{1}{1-s}}(\Omega)$. Therefore we have

$$\begin{aligned} &\text{Jac}(f)[\phi] - \text{Jac}(g)[\lambda^n \phi] \\ &= \lim_{\varepsilon \rightarrow \infty} \int_{\mathbb{R}^n} (\det(F_\varepsilon) - \det(\lambda_\varepsilon \nabla g_\varepsilon)) \phi \\ &= \lim_{\varepsilon \rightarrow \infty} \int_{\mathbb{R}^n} \left(\det((\lambda \nabla g)_\varepsilon) - \det(\lambda_\varepsilon \nabla g_\varepsilon) \right) \phi \\ &= \sum_{j=1}^n \int_{\mathbb{R}^n} \left(\lambda_\varepsilon dg_\varepsilon^1 \wedge \cdots \wedge \lambda_\varepsilon dg_\varepsilon^{j-1} \wedge \left[(\lambda dg^j)_\varepsilon - \lambda_\varepsilon dg_\varepsilon^j \right] \wedge (\lambda dg^{j+1})_\varepsilon \right. \\ &\quad \left. \wedge \cdots \wedge (\lambda dg^n)_\varepsilon \right) \phi. \end{aligned}$$

In view of the Sobolev embedding

$$W^{s, \frac{s}{n}}(\mathbb{R}^n) \hookrightarrow W^{\frac{n}{n+1}, n+1}(\mathbb{R}^n),$$

for $s \geq \frac{n}{n+1}$, and the fact that the distributional identity in the bigger space implies the one in the smaller space, we can assume that $s = \frac{n}{n+1}$. For each entry of the form $(\lambda dg^i)_\varepsilon$ and $\lambda_\varepsilon dg^i_\varepsilon$, we shall apply Hodge decomposition as in Proposition 8.5. To the difference term $(\lambda dg^j)_\varepsilon - \lambda_\varepsilon dg^j_\varepsilon$ we apply Hodge decomposition as in Proposition 8.6. We then obtain terms of the form:

$$\int_{\mathbb{R}^n} (da_1^\varepsilon \wedge \dots \wedge da_k^\varepsilon \wedge \beta_{k+1}^\varepsilon \wedge \dots \wedge \beta_n^\varepsilon) \phi,$$

where each a_j^ε and β_j^ε is bounded in its corresponding seminorm. Note that, fixing ε , the estimates in Proposition 8.3 are still valid for the above integral since by construction we can approximate each a_j^ε (respectively β_j^ε) in its seminorm by a sequence of scalar functions in $W^{\frac{n}{n+1}, n+1}(\mathbb{R}^n)$ (respectively 1-forms in

$$W^{\frac{n-1}{n+1}, \frac{n+1}{2}}\left(\mathbb{R}^n, \bigwedge^1(\mathbb{R}^n)\right).$$

Therefore, to conclude, we use Proposition 8.3: one of the terms a_j^ε or one of β_j^ε converges to zero (since it comes from the difference term), in the corresponding norm, thanks to Proposition 8.6, while the other terms are bounded by Proposition 8.5. So we obtain the claim by taking $\varepsilon \rightarrow 0$. $\square$

Appendix A. A proof of Proposition 7.1 for $s > 2/3$

As a tangential note, in this section we will sketch how a slightly weaker statement than Proposition 7.1 can be obtained without using Theorem 1.10. Hence, this provides another proof of Theorem 1.1, but only for $s > 2/3$. Hereby, we would like to highlight the importance of Theorem 1.10 in completing our proof for the critical case $s = 2/3$.

We begin with the following observation. As a corollary of the gained regularity $u \in c^{0, \frac{s}{2}}$ in Theorem 6.1, we can improve some of the estimates of previous sections and prove:

Proposition A.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain, $\frac{2}{3} \leq s < 1$ and $u \in I^{1+s, \frac{2}{s}}(\Omega, \mathbb{R}^3)$. Let $\theta \in [0, 1]$. For all $\tilde{\Omega} \Subset \Omega$*

- (i) $\|\Pi^\varepsilon\|_{L^{\frac{2}{s\theta}}(\tilde{\Omega})} \leq o(\varepsilon^{\frac{s}{2}(1+\theta)-1});$
- (ii) $\|\Gamma^\varepsilon\|_{L^{\frac{1}{s\theta}}(\tilde{\Omega})} \leq o(\varepsilon^{s(1+\theta)-1}).$

Proof. The estimates are obtained by interpolating the estimates in Corollary 4.3 with a new set of estimates obtained through $c^{0, \frac{s}{2}}$ regularity in the same manner; see [13, Equations (4.4) and (4.8)]. We leave the details to the reader. $\square$

An immediate corollary is the following better than L^1 -estimate for $\operatorname{curl} \Pi^\varepsilon$. As we previously explained in Remark 4.6, this is the missing link for following the steps of [13] in proving our main theorem. We can now obtain this estimate only for the super-critical values of $s > 2/3$.

Corollary A.2. *If $s > \frac{2}{3}$, there exists $r > 1$ such that*

$$\lim_{\varepsilon \rightarrow 0} \|\operatorname{curl} \Pi^\varepsilon\|_{L^r(\tilde{\Omega})} = 0.$$

Proof. Letting

$$\frac{1}{r} = \frac{s\theta}{2} + s\theta = \frac{3s}{2}\theta,$$

we have

$$\|\operatorname{curl} \Pi^\varepsilon\|_{L^r(\tilde{\Omega})} \leq \|\Pi\|_{L^{\frac{2}{s\theta}}} \|\Gamma^\varepsilon\|_{L^{\frac{1}{s\theta}}} \leq o\left(\varepsilon^{\frac{3s}{2}(1+\theta)-2}\right).$$

To complete the proof we need to show that there is $\theta \in (0, 1)$ such that

$$r > 1 \quad \text{and} \quad \frac{3s}{2}(1+\theta) - 2 \geq 0.$$

These are respectively equivalent to

$$\theta < \frac{2}{3s} \quad \text{and} \quad \theta \geq \frac{4}{3s} - 1.$$

But if $\frac{2}{3} < s < 1$ we have

$$0 < \frac{1}{3} < \frac{4}{3s} - 1 < \frac{2}{3s} < 1,$$

and so we can choose any $\theta \in [\frac{4}{3s} - 1, \frac{2}{3s})$. □

Once the L^r vanishing estimate for $\operatorname{curl} \Pi^\varepsilon$ is obtained, and having the usual elliptic estimates at hand, one can proceed as in [13, Proposition 4.5] to show that $\operatorname{Jac}(f) \equiv 0$ as required by Proposition 7.1. This completes the proof of Theorem 1.1 but only for $s > 2/3$ as in Section 7. Once again, we will leave the details to the interested reader.

Appendix B. Fractional absolute continuity

In proving Theorem 1.8, we used the following result, which follows by an embedding theorem from a known result for Bessel-potential spaces [23, Theorem 1.1].

Theorem B.1. *Let $u \in W^{s,p}(\mathbb{R}, \mathbb{R}^m)$ with $s \in (0, 1)$, $p \in (1, \infty)$ such that $sp > 1$ and let I be a finite interval. Then the Hausdorff dimension $\mathcal{H}\text{-dim}$ of $u^*(I)$ is at most $\frac{1}{s}$ for any interval $I \subset \mathbb{R}$. Here u^* denotes the continuous representative of u .*

Indeed, following [1, Theorem 7.63 (g)], we note that for any $p > 1$ and $\varepsilon > 0$,

$$W^{s,p}(\mathbb{R}^n) \hookrightarrow L^{s-\varepsilon,p}(\mathbb{R}^n).$$

Choosing $\varepsilon > 0$ such that $p(s - \varepsilon) - 1 > 0$, and applying [23, Theorem 1.1], we obtain Theorem B.1. (Note the notational disparity with [23], which uses $W^{s,p}$ for the Bessel-potential space $H_p^s = L^{s,p}$.)

Remark B.2. The typical space-filling curves provide counterexamples to Theorem B.1 if $sp < 1$. E.g., the Peano-curve $f : I \rightarrow \mathbb{R}^2$ that fills a square is of class $C^{1/2}$, and thus belongs to $W^{s,2}$ for any $s < \frac{1}{2}$ — however $\mathcal{H}_\infty^2(f(I)) \neq 0$.

The case $sp = 1$ is quite curious. It is known that for $u \in W^{1,1}(I, \mathbb{R}^N)$, if u^* denotes its continuous representative then $\mathcal{H}^1(u^*(I)) < \infty$. This is also based on the absolute continuity of the integral, however, in the fractional case $s < 1$ the condition $sp = 1$ does not guarantee continuity in one dimension. Indeed, it is unclear to us if there is always a representative u^* for $u \in W^{s,\frac{1}{s}}(\mathbb{R}, \mathbb{R}^N)$ such that $\mathcal{H}^{\frac{1}{s}}(u^*(I)) < \infty$.

We would like to note that Theorem B.1 also follows from a notion reminiscent of absolute continuity for fractional Sobolev maps. It is well known that Theorem B.1 holds for $s = 1$ and $p > 1$, which is a consequence of absolute continuity of $W^{1,1}$ -maps. Also it is known from the area formula and the Luzin property [21, Lemma 21] that the continuous representative of a map $u \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^m)$ for $m \geq n \geq 2$ and $p > n$ has image with $\mathcal{H}^p(u(\mathbb{R}^n)) = 0$. In this sense, Theorem B.1 is a natural extension to maps with one-dimensional domain in fractional Sobolev spaces. In this appendix we will further discuss this approach. The authors do not know of any instance in the literature where the following observations are made.

One of the basic Sobolev-space results is that the continuous representative f^* of a function $f \in W^{1,1}$ is absolutely continuous, that is for any $\varepsilon > 0$ there exists $\delta > 0$ such that whenever we have a pairwise disjoint collection of intervals $(I_i)_{i=1}^\infty$ with

$$\sum_i |I_i| < \delta$$

then

$$\sum_i |f^*(x) - f^*(y)| < \varepsilon.$$

This follows easily from the fundamental theorem of calculus (which holds for the continuous representative f^*)

$$f^*(a) - f^*(b) = \int_a^b f'(z) dz$$

and the absolute continuity of the integral, which says that if $g \in L^1(\Omega)$ then for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|g\|_{L^1(U)} < \varepsilon \quad \forall U \subset \Omega \text{ measurable} : |U| < \delta.$$

By a covering argument, it is also easy to show that an absolutely continuous function $f : I \subset \mathbb{R} \rightarrow \mathbb{R}^N$ must have a 1-dimensional finite Hausdorff content $\mathcal{H}_\infty^1(f(I)) < \infty$, where

$$\mathcal{H}_\infty^p(A) := \inf \left\{ \sum_i (r_i)^p : \text{there is a cover of } A \subset \bigcup_i B(r_i) \right. \\ \left. \text{with balls } B(r_i) \text{ of radius } r_i > 0 \right\}.$$

The underlying reason for Theorem B.1 is that there is a fractional generalization of a sort of absolute continuity to fractional Sobolev spaces $W^{s,p}(\mathbb{R})$ as long as $sp > 1$. Observe that for $s < 1$ there are discontinuous functions in $W^{s,p}$ with $sp = 1$.

Definition B.3 ((t,p)-absolute continuity). Let $t \geq 0$ and $p \in (0, \infty)$. A continuous function $f : \mathbb{R} \rightarrow \mathbb{R}^N$ is called (t, p) -absolutely continuous if the following holds. For any $\epsilon > 0$ there exists a $\delta > 0$ such that whenever we have a sequence of disjoint intervals $(I_i)_{i=1}^\infty$ with

$$\sum_i |I_i| < \delta$$

then

$$\sum_i \sup_{x \neq y \in I_i} \frac{|f(x) - f(y)|^p}{|x - y|^t} < \epsilon.$$

For $p = 1, t = 0$ this is the usual absolute continuity.

The following lemmas are elementary.

Lemma B.4. If $\frac{1+\tilde{t}}{1+t} \leq \frac{\tilde{p}}{p} \leq 1$, then (t, p) -absolute continuity implies $(\tilde{t}, \tilde{p})$ -absolute continuity.

Proof. Let $\lambda := \tilde{p}/p \leq 1$. For any collection of disjoint intervals I_i we have

$$\begin{aligned} & \sum_i \sup_{x \neq y \in I_i} \frac{|f(x) - f(y)|^{\tilde{p}}}{|x - y|^{\tilde{t}}} \\ &= \sum_i \sup_{x \neq y \in I_i} \left(\frac{|f(x) - f(y)|^p}{|x - y|^t} \right)^\lambda |x - y|^{\lambda t - \tilde{t}} \\ &\leq \sum_i \left(\sup_{x \neq y \in I_i} \frac{|f(x) - f(y)|^p}{|x - y|^t} \right)^\lambda |I_i|^{\lambda t - \tilde{t}} \\ &\leq \left(\sum_i \sup_{x \neq y \in I_i} \frac{|f(x) - f(y)|^p}{|x - y|^t} \right)^\lambda \left(\sum_i |I_i|^{\frac{\lambda t - \tilde{t}}{1-\lambda}} \right)^{1-\lambda}, \end{aligned}$$

where we used the Hölder inequality $\|\cdot\|_{l^1} \leq \|\cdot\|_{l^{\frac{1}{\lambda}}} \|\cdot\|_{l^{\frac{1}{1-\lambda}}}$. Since f is (t, p) -absolutely continuous, given $\epsilon > 0$, we choose $\delta_1 > 0$ such that

$$\sum_i \sup_{x \neq y \in I_i} \frac{|f(x) - f(y)|^p}{|x - y|^t} < \epsilon.$$

Note that the assumption implies

$$\frac{\lambda t - \tilde{t}}{1 - \lambda} \geq 1.$$

If $\sum_i |I_i| < \delta := \min\{\delta_1, \epsilon\}$, we hence obtain, by combining the above estimates,

$$\sum_i \sup_{x \neq y \in I_i} \frac{|f(x) - f(y)|^{\tilde{p}}}{|x - y|^{\tilde{t}}} < \epsilon^\lambda \left(\sum_i |I_i| \right)^{\frac{\lambda t - \tilde{t}}{1 - \lambda} (1 - \lambda)} < \epsilon^\lambda \delta^{\lambda t - \tilde{t}} \leq \epsilon. \quad \square$$

Lemma B.5 (Hausdorff content of (t, p) -absolutely continuous maps). *Let $f : I \rightarrow \mathbb{R}^N$ be (t, p) -absolutely continuous. If $t > 0$, then*

$$\mathcal{H}_\infty^p(f(I)) = 0.$$

If $t = 0$ we still have

$$\mathcal{H}_\infty^p(f(I)) < \infty.$$

Proof. In the definition of (t, p) -absolute continuity let $\epsilon = 1$ and obtain some $\delta > 0$. Let $\tilde{I}$ be any subinterval of I with $\text{diam}(\tilde{I}) < \frac{\delta}{2}$. For any $\sigma > 0$ we find N (a finite number that depends on σ) intervals $(I_i)_{i=1}^N$ and N intervals $(J_i)_{i=1}^N$ which are pairwise disjoint, with $|I_i|, |J_i| < \sigma$ and $\bigcup_i I_i \cup J_i = \tilde{I}$. Each $f(I_i)$ (respectively $f(J_i)$) is then contained in a ball of radius $2\sigma^{\frac{t}{p}} \sup_{x, y \in I_i} \frac{|f(x) - f(y)|}{|x - y|^{\frac{t}{p}}}$ (centered at $f(x_i)$ for some $x_i \in I_i$ (respectively $x \in J_i$)). By (t, p) -absolute continuity we then have

$$\mathcal{H}_\infty^p(f(\tilde{I})) \lesssim \sum_i \sigma^t \left(\sup_{x, y \in I_i} \frac{|f(x) - f(y)|^p}{|x - y|^t} + \sup_{x, y \in J_i} \frac{|f(x) - f(y)|^p}{|x - y|^t} \right) \lesssim \sigma^t.$$

Since this holds for any subinterval $\tilde{I}$ of diameter $\frac{\delta}{2}$, we cover I by $\approx \frac{1}{\delta}$ many such intervals and obtain

$$\mathcal{H}_\infty^p(f(I)) \lesssim \frac{1}{\delta} \sigma^t < \infty.$$

If $t > 0$ we can take σ arbitrarily small to obtain $\mathcal{H}_\infty^p(f(I)) = 0$. $\square$

In view of the above two lemmas, Theorem B.1 will follow from one last statement.

Lemma B.6. *Let $s \in (0, 1)$, $p \in (1, \infty)$ with $sp > 1$. Then the continuous representative u^* of any map $u \in W^{s,p}(\mathbb{R})$ is $(sp - 1, p)$ -absolutely continuous.*

Remark B.7. For $s = 1$ and $p = 1$ the result is still true (and it is the classical absolute continuity result for $W^{1,1}$ -maps in 1 dimension).

There cannot be such a result for when $sp = 1$, $s < 1$, since $W^{s, \frac{1}{s}}$ does not embed into the space of continuous functions. *E.g.*, take $s = \frac{1}{2}$ and $p = 2$: denote by $B^2 \subset \mathbb{R}^2$ the unit ball in $\mathbb{R}^2$ and by $B_+^2 := B^2 \cap \mathbb{R}_+^2$ the upper halfball; then $\log \log 2\sqrt{(x_1)^2 + (x_2)^2}$ belongs to $W^{1,2}(B^2)$, thus to $W^{1,2}(B_+^2)$. By the trace theorem for $I = [-1/2, 1/2]$, we find that $\log \log 2|x_1| \in W^{\frac{1}{2},2}(I)$, however this is clearly not a continuous function (let alone absolutely continuous in any sense).

Proof of Lemma B.6. Since $sp > 1$, $W^{s,p}(I)$ embeds in $C^{0,s-1/p}(I)$ for any interval (see, *e.g.*, [14, Section 8]). Indeed, for a universal constant $C > 0$ and all $a, b \in I$ we have

$$|u^*(b) - u^*(a)| \leq C[u]_{W^{s,p}(I)}|a - b|^{s-1/p},$$

which gives, for $a \neq b$,

$$\frac{|u^*(b) - u^*(a)|^p}{|a - b|^{sp-1}} \leq C[u]_{W^{s,p}(I)}^p.$$

We therefore obtain for the mutually disjoint I_i :

$$\sum_i \sup_{x \neq y \in I_i} \frac{|u^*(x) - u^*(y)|^p}{|x - y|^{sp-1}} \leq C \sum_i [u]_{W^{s,p}(I_i)}^p \leq [u]_{W^{s,p}(A)}^p,$$

where $A = \bigcup I_i$. Now, by the absolute continuity of the integral $[u]_{W^{s,p}(\mathbb{R})}^p < \infty$, for any $\epsilon > 0$, there is δ small enough such that $|A| = \sum_i |I_i| < \delta$ implies $[u]_{W^{s,p}(A)}^p < \epsilon$. $\square$

Proof of Theorem B.1. By Lemma B.6, f is $(sp-1, p)$ -absolutely continuous. Let $\tilde{p} = 1/s$ and $\tilde{s} = s$. Then

$$\frac{1 + \tilde{s}\tilde{p} - 1}{1 + sp - 1} = \frac{\tilde{p}}{p} = \frac{1}{sp} < 1.$$

The conditions of Lemma B.4 are satisfied and hence f is also $(\tilde{s}\tilde{p}-1, \tilde{p})$ -absolutely continuous. Lemma B.5 implies that the Hausdorff dimension of $u^*(I)$ is at most $\tilde{p} = 1/s$, as required. $\square$

Remark B.8. We could have also used the Sobolev embedding $W_{\text{loc}}^{s,p}(\mathbb{R}, \mathbb{R}^m) \hookrightarrow W_{\text{loc}}^{\tilde{s},\tilde{p}}(\mathbb{R}, \mathbb{R}^m)$ for any $\tilde{s} < s$ and $\tilde{p} \leq p$ [44, Proposition 2.1.2 and Theorem 2.4.4/1], but note that this is not necessarily true for $\tilde{s} = s$ [39], and some small adjustment would have become necessary.

References

- [1] R. A. ADAMS, “Sobolev Spaces”, Pure and Applied Mathematics, Vol. 65, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975.
- [2] J. M. BALL, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Rational Mech. Anal. **63** (1976/77), 337–403.
- [3] J. BERGH and J. LÖFSTRÖM, “Interpolation Spaces. An Introduction”, Grundlehren der Mathematischen Wissenschaften, Vol. 223, Springer-Verlag, Berlin-New York, 1976.
- [4] J. BOURGAIN, H. BREZIS and P. MIRONESCU, *Lifting, degree, and distributional Jacobian revisited*, Comm. Pure Appl. Math. **58** (2005), 529–551.
- [5] P. BOUSQUET, A. C. PONCE and J. VAN SCHAFTINGEN, *Density of smooth maps for fractional Sobolev spaces $W^{s,p}$ into ℓ simply connected manifolds when $s \geq 1$* , Confluentes Math. **5** (2013), 3–22.
- [6] H. BREZIS and P. MIRONESCU, *Gagliardo-Nirenberg, composition and products in fractional Sobolev spaces*, J. Evol. Equ. **1** (2001), 387–404. Dedicated to the memory of Tosio Kato.
- [7] H. BREZIS and H. NGUYEN, *The Jacobian determinant revisited*, Invent. Math. **185** (2011), 17–54.
- [8] H. BREZIS and L. NIRENBERG, *Degree theory and BMO. I. Compact manifolds without boundaries*, Selecta Math. (N.S.) **1** (1995), 197–263.
- [9] R. COIFMAN, P.-L. LIONS, Y. MEYER and S. SEMMES, *Compensated compactness and Hardy spaces*, J. Math. Pures Appl. (9) **72** (1993), 247–286.
- [10] P. CONSTANTIN, W. E. and E. S. TITI, *Onsager’s conjecture on the energy conservation for solutions of Euler’s equation*, Comm. Math. Phys. **165** (1994), 207–209.
- [11] S. CONTI, C. DE LELLIS and L. SZÉKELYHIDI, JR., *h-principle and rigidity for $C^{1,\alpha}$ isometric embeddings*, In: “Nonlinear Partial Differential Equations”, H. Holden and K. H. Karlsen (eds.), Abel Symposia, Vol. 7, Springer, Heidelberg, 2012, 83–116.
- [12] S. CONTI and F. MAGGI, *Confining thin elastic sheets and folding paper*, Arch. Ration. Mech. Anal. **187** (2008), 1–48.
- [13] C. DE LELLIS and M. R. PAKZAD, *The geometry of $C^{1,\alpha}$ flat isometric immersions*, to appear in Proc. Roy. Soc. Edinburgh Sect. A.
- [14] E. DI NEZZA, G. PALATUCCI and E. VALDINOCI, *Hitchhiker’s guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), 521–573.
- [15] I. DRELICHMAN and R. G. DURÁN, *Improved Poincaré inequalities in fractional Sobolev spaces*, Ann. Acad. Sci. Fenn. Math. **43** (2018), 885–903.
- [16] I. FONSECA and J. MALÝ, *From Jacobian to Hessian: distributional form and relaxation*, Riv. Mat. Univ. Parma (7) **4*** (2005), 45–74.
- [17] G. FRIESECKE, R. D. JAMES and S. MÜLLER, *A hierarchy of plate models derived from nonlinear elasticity by gamma-convergence*, Arch. Ration. Mech. Anal. **180** (2006), 183–236.
- [18] P. GLADBACH and H. OLBERMANN, *Coarea formulae and chain rules for the Jacobian determinant in fractional Sobolev spaces*, J. Funct. Anal. **278** (2020), Paper No. 108312, 21 pp.
- [19] L. GRAFAKOS, “Classical Fourier Analysis”, Graduate Texts in Mathematics, Vol. 249, Springer, New York, third edition, 2014.
- [20] M. GROMOV, *Geometric, algebraic, and analytic descendants of Nash isometric embedding theorems*, Bull. Amer. Math. Soc. (N.S.) **54** (2017), 173–245.
- [21] P. HAJLASZ, *Sobolev mappings, co-area formula and related topics*, In: “Proceedings on Analysis and Geometry (Russian)” (Novosibirsk Akademgorodok, 1999), K. Vodop’yanov (ed.), Izdat. Ross. Akad. Nauk Sib. Otd. Inst. Mat., Novosibirsk, 2000, 227–254.
- [22] Q. HAN and J.-X. HONG, “Isometric Embedding of Riemannian Manifolds in Euclidean Spaces”, Mathematical Surveys and Monographs, Vol. 130, American Mathematical Society, Providence, RI, 2006.

- [23] S. HENCL and P. HONZÍK, *Dimension distortion of images of sets under Sobolev mappings*, Ann. Acad. Sci. Fenn. Math. **40** (2015), 427–442.
- [24] P. HORNUNG, *Approximation of flat $W^{2,2}$ isometric immersions by smooth ones*, Arch. Ration. Mech. Anal. **199** (2011), 1015–1067.
- [25] P. HORNUNG, *Euler-Lagrange equation and regularity for flat minimizers of the Willmore functional*, Comm. Pure Appl. Math. **64** (2011), 367–441.
- [26] P. HORNUNG, *Continuation of infinitesimal bendings on developable surfaces and equilibrium equations for nonlinear bending theory of plates*, Comm. Partial Differential Equations **38** (2013), 1368–1408.
- [27] P. HORNUNG, M. LEWICKA and M. R. PAKZAD, *Infinitesimal isometries on developable surfaces and asymptotic theories for thin developable shells*, J. Elasticity **111** (2013), 1–19.
- [28] J. INGMANN, “Estimates for Commutators of Fractional Differential Operators via Harmonic Extension”, Master Thesis, 2020.
- [29] R. L. JERRARD, *Some rigidity results related to Monge-Ampère functions*, Canad. J. Math. **62** (2010), 320–354.
- [30] R. L. JERRARD and M. R. PAKZAD, *Sobolev spaces of isometric immersions of arbitrary dimension and co-dimension*, Ann. Mat. Pura Appl. (4) **196** (2017), 687–716.
- [31] B. KIRCHHEIM, “Geometry and Rigidity of Microstructures”, Habilitation Thesis, 2001.
- [32] M. V. KOROBKOV, *Properties of C^1 -smooth functions with a one-dimensional gradient range*, Sibirsk. Mat. Zh. **50** (2009), 1105–1122.
- [33] M. LECUMBERRY and S. MÜLLER, *Stability of slender bodies under compression and validity of the von Kármán theory*, Arch. Ration. Mech. Anal. **193** (2009), 255–310.
- [34] E. LENZMANN and A. SCHIKORRA, *Sharp commutator estimates via harmonic extensions*, Nonlinear Anal. **193** (2020), Paper No. 111375, 37 pp.
- [35] M. LEWICKA and M. R. PAKZAD, *Convex integration for the Monge-Ampère equation in two dimensions*, Anal. PDE **10** (2017), 695–727.
- [36] S. LI and A. SCHIKORRA, *$W^{s, \frac{n}{s}}$ -maps with positive distributional Jacobians*, Potential Anal. **55** (2021), 403–417.
- [37] J.-L. LIONS and E. MAGENES, *Problemi ai limiti non omogenei. I*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) **14** (1960), 269–308.
- [38] A. LUNARDI, “Interpolation Theory”, Edizioni della Normale, Pisa, 2018, third edition.
- [39] P. MIRONESCU and W. SICKEL, *A Sobolev non embedding*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **26** (2015), 291–298.
- [40] S. MÜLLER, *Higher integrability of determinants and weak convergence in L^1* , J. Reine Angew. Math. **412** (1990), 20–34.
- [41] S. MÜLLER and M. R. PAKZAD, *Regularity properties of isometric immersions*, Math. Z. **251** (2005), 313–331.
- [42] M. R. PAKZAD, *On the Sobolev space of isometric immersions*, J. Differential Geom. **66** (2004), 47–69.
- [43] J. G. REŠETNJAK, *The weak convergence of completely additive vector-valued set functions*, Sibirsk. Mat. Zh. **9** (1968), 1386–1394.
- [44] T. RUNST and W. SICKEL, “Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations”, De Gruyter Series in Nonlinear Analysis and Applications, Vol. 3, Walter de Gruyter & Co., Berlin, 1996.
- [45] A. SCHIKORRA and J. M. SCOTT, *Weak limits of fractional sobolev homeomorphisms are almost injective*, Studia Math. **269** (2023), 241–260.
- [46] R. SCHOEN and K. UHLENBECK, *Boundary regularity and the Dirichlet problem for harmonic maps*, J. Differential Geom. **18** (1983), 253–268.
- [47] G. SCHWARZ, “Hodge Decomposition—a Method for Solving Boundary Value Problems”, Lecture Notes in Mathematics, Vol. 1607, Springer-Verlag, Berlin, 1995.
- [48] W. SICKEL and A. YOUSSEFI, *The characterisation of the regularity of the Jacobian determinant in the framework of potential spaces*, J. London Math. Soc. (2) **59** (1999), 287–310.

- [49] R. S. STRICHARTZ, *Fubini-type theorems*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) **22** (1968), 399–408.
- [50] L. TARTAR, *Compensated compactness and applications to partial differential equations*, In: “Nonlinear Analysis and Mechanics: Heriot-Watt Symposium, Vol. IV”, R. J. Knops (ed.), Research Notes in Mathematics, Vol. 39, Pitman, Boston, Mass.-London, 1979, 136–212.
- [51] H. TRIEBEL, “Interpolation Theory, Function Spaces, Differential Operators”, North-Holland Mathematical Library, Vol. 18, North-Holland Publishing Co., Amsterdam-New York, 1978.
- [52] S. C. VENKATARAMANI, T. A. WITTEN, E. M. KRAMER and R. P. GEROGH, *Limitations on the smooth confinement of an unstretchable manifold*, J. Math. Phys. **41** (2000), 5107–5128.
- [53] H. C. WENTE, *An existence theorem for surfaces of constant mean curvature*, J. Math. Anal. Appl. **26** (1969), 318–344.
- [54] Y. ZHOU, *Fractional Sobolev extension and imbedding*, Trans. Amer. Math. Soc. **367** (2015), 959–979.

School of Mathematical Sciences
IMA-Shanghai
Shanghai Jiao Tong University
No. 6 Science Buildings
800 Dongchuan Road
Minhang District, Shanghai, China (200240)
siran.li@sjtu.edu.cn

Université de Toulon
CS 60584
83041 Toulon Cedex 9, France
reza.pakzad@univ-tln.fr

Department of Mathematics
University of Pittsburgh
301 Thackeray Hall
Pittsburgh, PA 15260, USA
armin@pitt.edu