# CNED SETS: COUNTABLY NEGLIGIBLE FOR EXTREMAL DISTANCES

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Abstract. The author has recently introduced the class of *CNED* sets in Euclidean space, generalizing the classical notion of *NED* sets, and shown that they are quasiconformally removable. A set $E$ is *CNED* if the conformal modulus of a curve family is not affected when one restricts to the subfamily intersecting $E$ at countably many points. We prove that several classes of sets that were known to be removable are also *CNED*, including sets of $\sigma$-finite Hausdorff $(n-1)$-measure and boundaries of domains with $n$-integrable quasihyperbolic distance. Thus, this work puts in common framework many known results on the problem of quasiconformal removability and suggests that the *CNED* condition should also be necessary for removability. We give a new necessary and sufficient criterion for closed sets to be $(C)NED$. Applying this criterion, we show that countable unions of closed $(C)NED$ sets are $(C)NED$. Therefore we enlarge significantly the known classes of quasiconformally removable sets.

Contents



## 1. Introduction

1.1. **Definitions.** Before presenting our results, we first discuss some background. We assume throughout that $n \geq 2$. For an open set $U \subset \mathbb{R}^n$ and two continua $F_1, F_2 \subset U$ the family of curves joining $F_1$ and $F_2$ inside $U$ is denoted by $\Gamma(F_1, F_2; U)$. For a set $E \subset \mathbb{R}^n$ we denote by $\mathcal{F}_0(E)$ the family of curves in $\mathbb{R}^n$ that

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do not intersect $E$, except possibly at the endpoints, and by $\mathcal{F}_\sigma(E)$ the family of curves in $\mathbb{R}^n$ that intersect $E$ at countably many points, not counting multiplicity.

A set $E \subset \mathbb{R}^n$ is *negligible for extremal distances* if for every pair of non-empty, disjoint continua $F_1, F_2 \subset \mathbb{R}^n$ we have

$$\mathrm{Mod}_n \Gamma(F_1, F_2; \mathbb{R}^n) = \mathrm{Mod}_n(\Gamma(F_1, F_2; \mathbb{R}^n) \cap \mathcal{F}_0(E)).$$

In this case, we write $E \in \mathit{NED}$; note that we suppress the dimension $n$ in this notation. If, instead, there exists a uniform constant $M \geq 1$ such that

$$\mathrm{Mod}_n \Gamma(F_1, F_2; \mathbb{R}^n) \leq M \cdot \mathrm{Mod}_n(\Gamma(F_1, F_2; \mathbb{R}^n) \cap \mathcal{F}_0(E)),$$

we say that $E$ is *weakly NED* and we write $E \in \mathit{NED}^w$. We remark that we do not require $E$ to be closed. For closed sets, the classes $\mathit{NED}$ and $\mathit{NED}^w$ agree [AS74].

The author in [Nta] introduced the class of *CNED* sets, that is, *countably negligible for extremal distances.* We say that a set $E \subset \mathbb{R}^n$ is of class *CNED* if

$$\mathrm{Mod}_n \Gamma(F_1, F_2; \mathbb{R}^n) = \mathrm{Mod}_n(\Gamma(F_1, F_2; \mathbb{R}^n) \cap \mathcal{F}_\sigma(E))$$

for every pair of non-empty, disjoint continua $F_1, F_2 \subset \mathbb{R}^n$. In this case we write $E \in \mathit{CNED}$. As above, we also define the class $\mathit{CNED}^w$ in the obvious manner. Again, $E$ need not be closed and the dimension $n$ is suppressed in this notation. For closed sets we show in Theorem 4.1 that the classes *CNED* and $\mathit{CNED}^w$ agree. The monotonicity of modulus implies that $\mathit{NED} \subset \mathit{CNED} \subset \mathit{CNED}^w$.

1.2. **Properties of negligible sets.** Closed *NED* sets have been studied extensively in the plane by Ahlfors and Beurling in [AB50], where they proved that these sets coincide with the closed sets $E \subset \mathbb{C}$ that are *removable for conformal embeddings* or else *S-removable*; that is, every conformal embedding of $\mathbb{C} \setminus E$ into $\mathbb{C}$ is the restriction of a Möbius transformation. See also Pesin's work [Pes56]. Equivalently, we may replace conformal with quasiconformal maps in this definition. See [You15] for a survey. Väisälä initiated the study of closed *NED* sets in higher dimensions [Väi62], proving that closed sets of Hausdorff $(n-1)$-measure zero are of class *NED*. The result of Ahlfors–Beurling was partially generalized in higher dimensions by Aseev–Syčev [AS74] and Vodopyanov–Goldshtein [VG77], who proved that if a closed set $E \subset \mathbb{R}^n$, $n \geq 3$, is of class *NED*, then it is removable for quasiconformal embeddings. The converse is not known in dimensions $n \geq 3$. Finally, a characterization of closed *NED* sets in $\mathbb{R}^n$ was provided by Vodopyanov–Goldshtein [VG77], who proved that closed *NED* sets coincide with sets that are removable for the Sobolev space $W^{1,n}$. We direct the reader to the introduction of [Ase09] for a survey of the known results. *NED* sets are closely related to quasiextremal distance (*QED*) exceptional sets, introduced by Gehring–Martio [GM85]. As remarked, here we will work with *NED* sets that are not necessarily closed.

The relation between *CNED* sets and quasiconformal maps was unveiled in [Nta]. We state a special case of the main theorem.

**Theorem 1.1.** *Let $E \subset \mathbb{R}^n$ be a closed CNED set. Then every homeomorphism of $\mathbb{R}^n$ that is quasiconformal on $\mathbb{R}^n \setminus E$ is quasiconformal on $\mathbb{R}^n$.*

In fact, in [Nta, Theorem 1.3] the set $E$ is not assumed to be closed, in which case the quasiconformality of $f$ in the set $\Omega \setminus E$ has to be interpreted appropriately, using the metric definition or a variant.

Closed sets $E$ satisfying the conclusion of Theorem 1.1 are called *removable for quasiconformal homeomorphisms* or else *QCH-removable.* The difference to $S$-removable sets that we discuss above is that here we study global homeomorphisms of $\mathbb{R}^n$, instead of topological embeddings of $\mathbb{R}^n \setminus E$. Moreover, note that if a planar set is $S$-removable, then it is *QCH*-removable. Although we have satisfactory characterizations of the former sets by Ahlfors–Beurling [AB50] in dimension 2, it is a notoriously difficult problem to characterize *QCH*-removable sets even in dimension 2. The current work suggests that *CNED* sets could provide a characterization.

There are many open problems related to *QCH*-removable sets, one of which is the problem of *local removability* [Bis94, Question 4], [Nta19, Question 2]: if a closed set $E$ is *QCH*-removable, is it true that every topological embedding of an open set $\Omega \subsetneq \mathbb{R}^n$ into $\mathbb{R}^n$ that is quasiconformal in $\Omega \setminus E$, is quasiconformal in $\Omega$? For *CNED* sets an affirmative answer is provided by [Nta, Theorem 1.3].

Another open problem is whether the union of two *QCH*-removable closed sets is removable [JS00]. While for disjoint sets the answer is affirmative, in general, for intersecting sets this is known only in the cases of totally disconnected sets and quasicircles [You16, Theorem 4]. We prove here that countable unions of closed *NED* and *CNED* sets are *NED* and *CNED*, respectively.

**Theorem 1.2.** *Let $E_i$, $i \in \mathbb{N}$, be a countable collection of closed subsets of $\mathbb{R}^n$.*

(i) *If $E_i$ is NED for each $i \in \mathbb{N}$, then $\displaystyle\bigcup_{i\in\mathbb{N}} E_i$ is NED.*

(ii) *If $E_i$ is CNED for each $i \in \mathbb{N}$, then $\displaystyle\bigcup_{i\in\mathbb{N}} E_i$ is CNED.*

In the case that a countable union of closed *NED* sets is *closed*, this result follows from the Baire category theorem [You15, Section 4]. The case of *non-closed* unions is significantly more complicated, since they could even be dense. The proof is given in Section 5 and relies on an intricate characterization of *NED* and *CNED* sets from Section 4. We give a vague formulation of this characterization here.

**Theorem 1.3.** *A closed set $E \subset \mathbb{R}^n$ is NED (resp. CNED) if and only if for every $n$-integrable metric $\rho\, ds$, almost every path $\gamma$ in $\mathbb{R}^n$ can be perturbed by an arbitrarily small amount of $\rho$-length to avoid the set $E$ (resp. to intersect the set $E$ at countably many points).*

See Theorem 4.1 for a precise statement. Another application of this characterization is the removability of *CNED* sets for continuous Sobolev functions. The proof is given in Section 4.4.

**Theorem 1.4.** *Let $E \subset \mathbb{R}^n$ be a closed CNED set. Then every continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$ with $f \in W^{1,n}(\mathbb{R}^n \setminus E)$ lies in $W^{1,n}(\mathbb{R}^n)$.*

1.3. **Examples of negligible sets.** So far we understand some general classes of *QCH*-removable sets. First, sets of $\sigma$-finite Hausdorff $(n-1)$-measure are removable as shown by Besicovitch [Bes31] in dimension 2 and by Gehring [Geh62] in higher dimensions. Thus, we can say that such sets are removable for *rectifiability* reasons.

Next, it is known that sets with good geometry, such as quasicircles, are removable in dimension 2. More generally, in all dimensions, boundaries of John domains, Hölder domains, and domains with $n$-integrable quasihyperbolic distance are removable [Jon95, JS00, KN05]. Roughly speaking, all of these sets have either no outward cusps or some outward cusps, but not too many on average. Thus, these sets are removable for *geometric* reasons.

Finally, *NED* sets are removable as well due to [AB50] in dimension 2 and [AS74, VG77] in higher dimensions. Thus, one could say that *NED* sets are removable because they are small in a *potential theoretic* sense. Note that all *NED* sets are necessarily totally disconnected in dimension 2.

The three classes of sets are mutually singular in a sense. Namely, there are rectifiable sets that have bad geometry and are large from a potential theoretic point of view. For example, consider a rectifiable curve with a dense set of both inward and outward cusps. Likewise, there are sets with good geometry that are not rectifiable and are large for potential theory. As an example, take a quasicircle of Hausdorff dimension larger than 1. Finally, there are sets that are small in a potential theoretic sense, but are large in terms of rectifiability and have bad geometry. For example, consider a Cantor set $E \subset \mathbb{R}$ of measure zero and Hausdorff dimension 1, and then take the set $E \times E$; this is an *NED* set by [AB50, Theorem 10] since its projections to the coordinate directions have measure zero.

A natural question is whether one can reconcile these three different worlds. In other words, is there a common reason for which all of the above classes of sets are *QCH*-removable? We provide an affirmative answer to this question.

**Theorem 1.5.** *The following sets are of class CNED in* $\mathbb{R}^n$.

(i) *Sets of class NED.*
(ii) *Sets of $\sigma$-finite Hausdorff $(n-1)$-measure.*
(iii) *Boundaries of domains with $n$-integrable quasihyperbolic distance.*

The class of sets in (iii) is defined and discussed in Section 6.1, where we also give the proof. This class includes quasicircles, boundaries of John domains, and boundaries of Hölder domains. As discussed earlier, (i) is immediate; however, the other conclusions are new. The technique used for the proof of (ii) allows us to generalize a result of Väisälä [Väi62], stating that a closed set $E \subset \mathbb{R}^n$ with Hausdorff $(n-1)$-measure zero is of class *NED*, to non-closed sets.

**Theorem 1.6.** *Let $E \subset \mathbb{R}^n$ be a set of Hausdorff $(n-1)$-measure zero. Then $E$ is NED.*

Theorem 1.5 (ii) and Theorem 1.6 are proved in Section 3 with the aid of the notion of a *family of curve perturbations*; see Theorem 3.11. Roughly speaking, such curve families contain almost every parallel translate of a curve and almost every radial segment. The main theorem of that section is Theorem 3.4, which asserts that the modulus of a curve family remains unaffected, if one restricts to the intersection of that family with a family of curve perturbations.

Combining Theorems 1.1, 1.2, and 1.5, we obtain the next removability result.

**Theorem 1.7.** *Let $E \subset \mathbb{R}^n$ be a closed set that admits a decomposition into countably many sets $E_i$, $i \in \mathbb{N}$, each of which is contained in a closed set that is either NED, or has $\sigma$-finite Hausdorff $(n-1)$-measure, or is the boundary of a domain with $n$-integrable quasihyperbolic distance. Then $E$ is CNED and QCH-removable.*

We also present some further examples. We show in Theorem 6.6 that planar sets (not necessarily closed) whose projection to each coordinate axis has measure zero are $NED^w$, generalizing a result of Ahlfors–Beurling for closed sets [AB50, Theorem

10]. Theorem 6.6 is used in the proof of Theorem 1.9 below. In Section 6.3 we present an example of a non-measurable *CNED* set, constructed by Sierpiński.

The results of this paper have already found an application in the problem of *rigidity of circle domains*. A connected open set $\Omega$ in the Riemann sphere $\widehat{\mathbb{C}}$ is a *circle domain* if each boundary component of $\Omega$ is either a circle or a point. A circle domain $\Omega$ is *conformally rigid* if every conformal map from $\Omega$ onto another circle domain is the restriction of a Möbius transformation of $\widehat{\mathbb{C}}$. He–Schramm [HS94] proved that circle domains whose boundary has $\sigma$-finite Hausdorff 1-measure are rigid. Later, Younsi and the author [NY20] proved the rigidity of circle domains with $n$-integrable quasihyperbolic distance (as in Theorem 1.5 (iii)). It is conjectured that rigidity of a circle domain is equivalent to *QCH*-removability of the boundary. The next result incorporates all previous results and provides strong evidence towards this conjecture.

**Theorem 1.8** ([Nta23])**.** *A circle domain is conformally rigid if every compact subset of its point boundary components is CNED.*

The proof features especially Theorem 1.2 and the characterization of *CNED* sets given in Theorem 4.1.

1.4. **Examples of non-negligible sets.** We remark that in Theorem 1.2 we are not requiring that the union of the closed sets $E_i$ be closed. However, both cases of the theorem fail without assuming that each individual set $E_i$ is closed.

**Theorem 1.9.** *There exist Borel NED sets $E_1, E_2 \subset \mathbb{R}^2$ such that $E_1 \cup E_2$ is a closed set that is neither NED nor CNED nor QCH-removable.*

The proof is given in Section 7. One can construct a more basic example with $E_1 \cup E_2 \notin$ *NED* as follows. Tukia [Tuk89] gives an example of a set $E_1 \subset [0,1]$ of full measure that can be mapped under a quasiconformal map of $\mathbb{C}$ onto a set of 1-measure zero in the real line. Note that sets of 1-measure zero are *NED* by Theorem 1.6 and such *NED* sets are quasiconformally invariant by Corollary 4.2. Thus, $E_1$ is *NED*. Also, $E_2 = [0,1] \setminus E_1$ is *NED* because it has measure zero. However, $E_1 \cup E_2 = [0,1]$, which is not totally disconnected, so it is not *NED*.

Compared to *NED* sets, it is significantly harder to produce sets that are not *CNED*. For the proof of Theorem 1.9 we use tools from the existing literature, and in particular from a work of Wu [Wu98], to construct *NED* sets $E_1$ and $E_2$ such that $E_1 \cup E_2$ is the product of a Cantor set in $\mathbb{R}$ with $[0,1]$. Such sets are not *QCH*-removable (see [Car51] or the Introduction of [Nta20]) and thus they are not *CNED* by Theorem 1.1; this can be proved directly in this simple situation.

As a corollary of Theorem 1.9 and [Nta, Theorem 1.3], we obtain that unions of exceptional sets for the metric definition of quasiconformality are not necessarily exceptional. Here $H_f$ denotes the metric distortion of a map $f$; see [Nta].

**Corollary 1.10.** *There exist Borel sets $E_1, E_2 \subset \mathbb{R}^2$ such that $E_1 \cup E_2$ is closed and for each $i \in \{1,2\}$, every homeomorphism $f$ of $\mathbb{R}^2$ with*

$$\sup_{x \in \mathbb{R}^2 \setminus E_i} H_f(x) < \infty$$

*is quasiconformal, but there exists a homeomorphism $g$ of $\mathbb{R}^2$ that is quasiconformal in $\mathbb{R}^2 \setminus (E_1 \cup E_2)$ and not quasiconformal in $\mathbb{R}^2$.*

Finally, we end the introduction with some remarks on non-removable sets. In dimension 2 it is known that all sets of positive area are not *QCH*-removable [KW96]. There are Jordan curves of Hausdorff dimension 1 that are non-removable [Kau84, Bis94]. Moreover, if $C\subset \mathbb{R}$ is a Cantor set, then $C\times [0,1]$ is non-removable as we discussed above. More interestingly, Wu [Wu98] proved that if $E,F\subset\mathbb{R}$ are Cantor sets and $E\notin \mathit{NED}$, then $E\times F$ is non-removable; the converse is not true since *NED* sets can have positive length [AB50]. More recently, the current author studied the problem of removability for fractals with infinitely many complementary components and proved that the Sierpiński gasket and all Sierpiński carpets are non-removable [Nta19, Nta21]. The latter result was generalized to higher dimensional carpets, known as Sierpiński spaces, by the author and Wu [NW20].

Gaskets and carpets fall into the general class of *residual sets of packings*. A packing $\mathcal{P}$ in $\mathbb{R}^n$ is a collection of bounded, connected open sets $D_i$, $i\in \mathbb{N}\cup\{0\}$, such that $D_i\subset D_0$ for every $i\in\mathbb{N}$ and $D_i\cap D_j=\emptyset$ for $i,j\in\mathbb{N}$ with $i\neq j$. The residual set of the packing $\mathcal{P}$ is the set

$$\overline{D_0}\setminus \bigcup_{i\in\mathbb{N}} D_i.$$

We observe below that in many cases such residual sets are not *CNED*.

**Proposition 1.11.** *Let $\mathcal{P}=\{D_i\}_{i\in\mathbb{N}\cup\{0\}}$ be a packing in $\mathbb{R}^n$ such that $\partial D_i\cap\partial D_j$ is countable for $i\neq j$, $i,j\in\mathbb{N}\cup\{0\}$. Then the residual set of $\mathcal{P}$ is not CNED.*

It was earlier observed that such residual sets in the plane can have Hausdorff dimension 1 but not $\sigma$-finite Hausdorff 1-measure [MN22]. Proposition 1.11 covers the Sierpiński gasket and all Sierpiński carpets.

1.5. **Open problems.** Based on the results of this work, it is natural to propose the following problem, whose resolution would answer many of the open questions related to removable sets.

**Problem 1.12.** *Do QCH-removable sets coincide with closed CNED sets?*

We also formulate a series of questions for *CNED* sets.

*Question* 1.13. If $E\subset\mathbb{R}^n$ is a closed set that is not *CNED*, does there exist a homeomorphism of $\mathbb{R}^n$ that is quasiconformal in $\mathbb{R}^n\setminus E$ and maps $E$ to a set of positive $n$-measure?

A positive answer to this question would resolve Problem 1.12 and thus it would also resolve among others the problems of local removability and of removability of unions of removable sets mentioned earlier. For closed *NED* sets in the plane the answer to the corresponding question is already known to be affirmative by Ahlfors–Beurling [AB50]: if $E\notin \mathit{NED}$, then there exists a conformal embedding $f\colon \mathbb{R}^2\setminus E\to\mathbb{R}^2$ such that the complement of $f(\mathbb{R}^2\setminus E)$ has positive area.

*Question* 1.14. If $E\subset\mathbb{R}^n$ is a closed set that is not *CNED*, does there exist a totally disconnected closed subset of $E$ that is not *CNED*?

If yes, it would suffice to answer Question 1.13 for totally disconnected sets, which could be more approachable. Note that *NED* sets in the plane are already totally disconnected, so this makes the study of these sets more accessible.

**Problem 1.15.** *Do removable sets for continuous $W^{1,n}$ functions coincide with closed CNED sets?*

This problem is motivated by Theorem 1.4. Obviously, the answer would be positive if the answer to Problem 1.12 were positive, in which case, $QCH$-removable sets would coincide with removable sets for continuous $W^{1,n}$ functions. This is another open problem discussed in [Bis94, JS00].

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## 2. Preliminaries

2.1. **Notation and definitions.** We denote the Euclidean distance between points $x, y \in \mathbb{R}^n$ by $|x-y|$. For $x \in \mathbb{R}^n$ and $0 \leq r < R$ we denote by $B(x,R)$ the open ball $\{y \in \mathbb{R}^n : |x-y| < R\}$ and by $A(x;r,R)$ the open ring $\{y \in \mathbb{R}^n : r < |x-y| < R\}$. The corresponding closed ball and ring are denoted by $\overline{B}(x,r)$ and $\overline{A}(x;r,R)$, respectively. If $B$ is an open (resp. closed) ball, then for $\lambda > 0$ we denote by $\lambda B$ the open (resp. closed) ball with the same center as $B$ and radius multiplied by $\lambda$. We also set $S^{n-1}(x,r) = \partial B(x,r)$. The open $\varepsilon$-neighborhood of a set $E \subset \mathbb{R}^n$ is denoted by $N_\varepsilon(E)$.

We use the notation $m_n$ for the $n$-dimensional Lebesgue measure in $\mathbb{R}^n$, $m_n^*$ for the outer $n$-dimensional Lebesgue measure, and $\int_{\mathbb{R}^n} \rho(x)\, dx$ or simply $\int \rho$ for the Lebesgue integral of a Lebesgue measurable extended function $\rho\colon \mathbb{R}^n \to [-\infty, \infty]$, if it exists. For such a function $\rho$, if $B$ is a measurable set with $m_n(B) \in (0,\infty)$, we define

$$\fint_B \rho = \frac{1}{m_n(B)} \int_B \rho.$$

For simplicity, extended functions will be called functions. A non-negative function is assumed to take values in $[0,\infty]$.

The cardinality of a set $E$ is denoted by $\#E$. For quantities $A$ and $B$ we write $A \lesssim B$ if there exists a constant $c > 0$ such that $A \leq cB$. If the constant $c$ depends on another quantity $H$ that we wish to emphasize, then we write instead $A \leq c(H)B$ or $A \lesssim_H B$. Moreover, we use the notation $A \simeq B$ if $A \lesssim B$ and $B \lesssim A$. As previously, we write $A \simeq_H B$ to emphasize the dependence of the implicit constants on the quantity $H$. All constants in the statements are assumed to be positive even if this is not stated explicitly and the same letter may be used in different statements to denote a different constant.

For $s \geq 0$ the *s-dimensional Hausdorff measure* $\mathscr{H}^s(E)$ of a set $E \subset \mathbb{R}^n$ is defined by

$$\mathscr{H}^s(E) = \lim_{\delta \to 0} \mathscr{H}^s_\delta(E) = \sup_{\delta > 0} \mathscr{H}^s_\delta(E),$$

where

$$\mathscr{H}^s_\delta(E) = \inf\left\{ c(s) \sum_{j=1}^{\infty} \operatorname{diam}(U_j)^s : E \subset \bigcup_j U_j,\ \operatorname{diam}(U_j) < \delta \right\}$$

for a normalizing constant $c(s) > 0$ so that the $n$-dimensional Hausdorff measure agrees with Lebesgue measure in $\mathbb{R}^n$. Note that $c(1) = 1$. The quantity $\mathscr{H}^s_\delta(E)$, $\delta \in (0,\infty]$, is called the *s-dimensional Hausdorff δ-content* of $E$. If $\delta = \infty$ we simply call this quantity the *s-dimensional Hausdorff content* of $E$. A standard fact that we will use is that

$$\mathscr{H}^s(E) = 0 \text{ if and only if } \mathscr{H}^s_\infty(E) = 0.$$

We note that if $E \subset \mathbb{R}^n$ is a connected set, then (see [BBI01, Lemma 2.6.1, p. 53])

$$\mathscr{H}^1(E) \geq \mathscr{H}^1_\infty(E) = \operatorname{diam}(E).$$

**Lemma 2.1** ([Boj88]). *Let $p \geq 1$ and $\lambda > 0$. Suppose that $\{B_i\}_{i\in\mathbb{N}}$ is a collection of balls in $\mathbb{R}^n$ and $a_i$, $i \in \mathbb{N}$, is a sequence of non-negative numbers. Then*

$$\left\| \sum_{i\in\mathbb{N}} a_i \chi_{\lambda B_i} \right\|_{L^p(\mathbb{R}^n)} \leq c(n,p,\lambda) \left\| \sum_{i\in\mathbb{N}} a_i \chi_{B_i} \right\|_{L^p(\mathbb{R}^n)}.$$

2.2. **Rectifiable paths.** A *path* or *curve* is a continuous function $\gamma\colon I \to \mathbb{R}^n$, where $I \subset \mathbb{R}$ is a compact interval. The *trace* of a path $\gamma$ is the image $\gamma(I)$ and will be denoted by $|\gamma|$. The *endpoints* of a path $\gamma\colon [a,b] \to \mathbb{R}^n$ are the points $\gamma(a), \gamma(b)$ and we set $\partial\gamma = \{\gamma(a), \gamma(b)\}$. We say that a path $\widetilde{\gamma}$ is a *weak subpath* of a path $\gamma$ if $\#\widetilde{\gamma}^{-1}(x) \leq \#\gamma^{-1}(x)$ for every $x \in \mathbb{R}^n$. In particular, this implies that $|\widetilde{\gamma}| \subset |\gamma|$. A path $\widetilde{\gamma}$ is a *(strong) subpath* of a path $\gamma\colon I \to \mathbb{R}^n$ if $\widetilde{\gamma}$ is the restriction of $\gamma$ to a closed subinterval of $I$. Note that a strong subpath is always a weak subpath, but not vice versa. A path $\gamma$ is *simple* if it is injective. Equivalently, $\#\gamma^{-1}(x) = 1$ for each $x \in |\gamma|$. It is well-known that every path has a simple weak subpath with the same endpoints [Wil70, Theorem 31.2, p. 219].

If $\gamma$ is a path and $E \subset \mathbb{R}^n$ is a set, then we say that $\gamma$ *avoids* the set $E$ if $E \cap |\gamma| = \emptyset$ and intersects $E$ at (e.g.) finitely many points if $E \cap |\gamma|$ is a finite set; note that we are not taking into account the multiplicity in the latter case.

If $\gamma_i\colon [a_i, b_i] \to \mathbb{R}^n$, $i = 1, 2$, are paths such that $\gamma_1(b_1) = \gamma_2(a_2)$, then we can define the *concatenation* of the two paths to be the path $\gamma\colon [a_1, b_2] \to \mathbb{R}^n$ such that $\gamma|_{[a_1,b_1]} = \gamma_1$ and $\gamma|_{[a_2,b_2]} = \gamma_2$. If $x, y \in \mathbb{R}^n$, then we denote the line segment from $x$ to $y$ by $[x, y]$.

The *length* of a path $\gamma$ is the total variation of the function $\gamma$ and is denoted by $\ell(\gamma)$. A path is *rectifiable* if it has finite length. Let $\gamma\colon [a,b] \to \mathbb{R}^n$ be a path and $s\colon [c,d] \to [a,b]$ be an increasing or decreasing continuous surjection. Then the path $\gamma \circ s\colon [c,d] \to \mathbb{R}^n$ is called a *reparametrization* of $\gamma$ (by the function $s$). Every rectifiable path $\gamma$ admits a unique reparametrization $\widetilde{\gamma}\colon [0, \ell(\gamma)] \to \mathbb{R}^n$ by an increasing function so that $\ell(\widetilde{\gamma}|_{[0,t]}) = t$ for all $t \in [0, \ell(\gamma)]$. The path $\widetilde{\gamma}$ is called the *arclength parametrization* of $\gamma$.

If $\rho\colon \mathbb{R}^n \to [0,\infty]$ is a Borel function and $\gamma$ is a rectifiable path, then one can define the line integral $\int_\gamma \rho\, ds$ using the arclength parametrization of $\gamma$; see [Väi71, Chapter 1, pp. 8–9]. Namely, if $\gamma\colon [0, \ell(\gamma)] \to \mathbb{R}^n$ is parametrized by arclength, then

$$\int_\gamma \rho\, ds = \int_0^{\ell(\gamma)} \rho(\gamma(t))\, dt.$$

We gather some properties of line integrals below.

**Lemma 2.2.** *Let $\gamma$ be a rectifiable path, $\widetilde{\gamma}$ be a weak subpath of $\gamma$, and $\rho\colon \mathbb{R}^n \to [0,\infty]$ be a Borel function. The following statements are true.*

(i) $\displaystyle\int_\gamma \rho\, ds = \int_{\mathbb{R}^n} \rho(x) \#\gamma^{-1}(x)\, d\mathscr{H}^1(x).$

(ii) $\displaystyle\int_{|\gamma|} \rho\, d\mathscr{H}^1 \leq \int_\gamma \rho\, ds$ *with equality if $\gamma$ is simple.*

(iii) $\displaystyle\int_{\widetilde{\gamma}} \rho\, ds \leq \int_{\gamma} \rho\, ds.$

(iv) *If $G \subset \mathbb{R}^n$ is a Borel set such that $\mathscr{H}^1(G \cap |\gamma|) = 0$, then*

$$\int_{\gamma} \rho\, ds = \int_{\gamma} \rho \chi_{\mathbb{R}^n \setminus G}\, ds.$$

*In particular, the above statements hold for $\rho = 1$, in which case* $\displaystyle\int_{\gamma} \rho\, ds = \ell(\gamma)$.

*Proof.* Part (i) follows from [Fed69, Theorem 2.10.13, p. 177]. The inequality and equality in (ii) follow from (i). Since $\widetilde{\gamma}$ is a weak subpath of $\gamma$, we have $\#\widetilde{\gamma}^{-1}(x) \leq \#\gamma^{-1}(x)$. Thus (i) implies (iii). Part (iv) also follows from (i) upon observing that $\chi_{\mathbb{R}^n \setminus G}(x) \#\gamma^{-1}(x) = \#\gamma^{-1}(x)$ for $x \notin G \cap |\gamma|$ and thus for $\mathscr{H}^1$-a.e. $x \in \mathbb{R}^n$. $\square$

2.3. **Modulus.** Let $\Gamma$ be a family of curves in $\mathbb{R}^n$. A Borel function $\rho\colon \mathbb{R}^n \to [0, \infty]$ is *admissible* for the path family $\Gamma$ if

$$\int_{\gamma} \rho\, ds \geq 1$$

for all rectifiable paths $\gamma \in \Gamma$. For $p \geq 1$ we define the *p-modulus* of $\Gamma$ as

$$\operatorname{Mod}_p \Gamma = \inf_{\rho} \int \rho^p,$$

where the infimum is taken over all admissible functions $\rho$ for $\Gamma$. By convention, $\operatorname{Mod}_p \Gamma = \infty$ if there are no admissible functions for $\Gamma$. Note that unrectifiable paths do not affect modulus. Hence, we will assume that families of $p$-modulus zero appearing in the next considerations contain all unrectifiable paths. We will use the following standard facts about modulus:

(M1) The modulus $\operatorname{Mod}_p$ is an outer measure in the space of all curves in $\mathbb{R}^n$. In particular, it obeys the monotonicity and countable subadditivity laws.

(M2) If every path of a family $\Gamma_1$ has a subpath lying in a family $\Gamma_2$, then $\operatorname{Mod}_p \Gamma_1 \leq \operatorname{Mod}_p \Gamma_2$.

(M3) If $\Gamma_0$ is a path family with $\operatorname{Mod}_p \Gamma_0 = 0$, then the family of paths $\gamma$ that have a weak subpath in $\Gamma_0$ also has $n$-modulus zero (by Lemma 2.2 (iii)). Moreover, the family of paths that have a reparametrization contained in $\Gamma_0$ also has $p$-modulus zero.

(M4) If $\rho\colon \mathbb{R}^n \to [0, \infty]$ is a Borel function with $\rho = 0$ a.e., then for the family $\Gamma_0$ of paths $\gamma$ with $\int_{\gamma} \rho\, ds > 0$ we have $\operatorname{Mod}_n \Gamma_0 = 0$.

(M5) The modulus $\operatorname{Mod}_p$ obeys the *serial* law: if $\Gamma_i$, $i \in \mathbb{N}$, are curve families supported in disjoint Borel sets, then

$$\operatorname{Mod}_p \left( \bigcup_{i=1}^{\infty} \Gamma_i \right) \geq \sum_{i=1}^{\infty} \operatorname{Mod}_p \Gamma_i.$$

(M6) Let $E \subset \mathbb{R}^n$ be a set with $m_n(E) = 0$. Let $\Gamma_0$ be the family of paths $\gamma$ such that $\mathscr{H}^1(|\gamma| \cap E) > 0$. Then $\operatorname{Mod}_p \Gamma_0 = 0$.

(M7) For $p = n$ the modulus $\operatorname{Mod}_n$ is invariant under conformal maps.

(M8) Let $\Gamma=\Gamma(A(x;r,R))$ be the family of curves in $\mathbb{R}^n$ joining the boundary components of the ring $A(x;r,R)$. Then
$$\mathrm{Mod}_n\,\Gamma=\omega_{n-1}\left(\log\frac{R}{r}\right)^{1-n}.$$

Here $\omega_{n-1}$ is the area of the unit sphere in $\mathbb{R}^n$. See [Väi71, Chapter 1] and [HKST15, Sections 5.2–5.3] for more details about modulus and proofs of these facts.

For a path $\gamma\colon I\to\mathbb{R}^n$ and $x\in\mathbb{R}^n$ we define $\gamma+x$ to be the path $I\ni t\mapsto \gamma(t)+x$.

**Lemma 2.3.** *Let $\Gamma$ be a family of paths in $\mathbb{R}^n$ with $\mathrm{Mod}_p\,\Gamma=0$.*

(i) *For each rectifiable path $\gamma$ and for a.e. $x\in\mathbb{R}^n$ we have $\gamma+x\notin\Gamma$.*
(ii) *For each line segment $L$ parallel to a direction $v\in\mathbb{R}^n$ and for $\mathscr{H}^{n-1}$-a.e. $x\in\{v\}^\perp$ we have $L+x\notin\Gamma$.*
(iii) *For each $x\in\mathbb{R}^n$ and $w\in S^{n-1}(0,1)$ define $\gamma_w(t)=x+tw$, $t\geq 0$. Then for $0<r<R$ we have $\gamma_w|_{[r,R]}\notin\Gamma$ for $\mathscr{H}^{n-1}$-a.e. $w\in S^{n-1}(0,1)$.*

*Proof.* For each $\varepsilon>0$ there exists a function $\rho$ that is admissible for $\Gamma$ with $\int\rho^p<\varepsilon$.

For (i), let $\gamma$ be a rectifiable path in $\mathbb{R}^n$, parametrized by arclength. Let $r>0$ and $G_r$ be the set of $x\in B(0,r)$ such that $\gamma+x\in\Gamma$. We also fix $R>0$ such that $|\gamma+x|\subset B(0,R)$ whenever $x\in B(0,r)$. Then by Chebychev's inequality and Fubini's theorem we have
$$\begin{aligned} m_n^*(G_r)&\leq m_n\left(\left\{x\in B(0,r): \int_{\gamma+x}\rho\,ds\geq 1\right\}\right)\leq \int_{B(0,r)}\int_0^{\ell(\gamma)}\rho(\gamma(t)+x)\,dtdx\\ &=\int_0^{\ell(\gamma)}\int_{B(0,r)}\rho(\gamma(t)+x)\,dxdt\leq \ell(\gamma)\|\rho\|_{L^1(B(0,R))}.\end{aligned}$$
As $\varepsilon\to 0$, we have $\|\rho\|_{L^p(B(0,R))}\to 0$, so $\|\rho\|_{L^1(B(0,R))}\to 0$. This shows that $m_n(G_r)=0$ for all $r>0$. Thus, $\gamma+x\notin\Gamma$ for a.e. $x\in\mathbb{R}^n$.

For part (ii), let $G_r$ be the set of $x\in B(0,r)\cap\{v\}^\perp$ such that $L+x\in\Gamma$. Then
$$\mathscr{H}^{n-1}(G_r)\leq \mathscr{H}^{n-1}\left(\left\{x\in B(0,r)\cap\{v\}^\perp: \int_{L+x}\rho\,ds\geq 1\right\}\right)\leq \|\rho\|_{L^1(D)},$$
where $D$ is the cylinder of radius $r$ with axis $L$. We now let $\varepsilon\to 0$ as above.

Finally, for (iii), let $G$ be the set of $w\in S^{n-1}(0,1)$ such that $\gamma_w|_{[r,R]}\in\Gamma$. Then, using polar integration we have
$$\begin{aligned}\mathscr{H}^{n-1}(G)&\leq \mathscr{H}^{n-1}\left(\left\{w\in S^{n-1}(0,1): \int_{\gamma_w|_{[r,R]}}\rho\,ds\geq 1\right\}\right)\\ &\leq \int_{S^{n-1}(0,1)}\int_r^R\rho(x+tw)\,dtd\mathscr{H}^{n-1}(w)\lesssim_n r^{-n+1}\|\rho\|_{L^1(B(x,R))}.\end{aligned}$$
As before, we let $\varepsilon\to 0$ to obtain the conclusion. $\square$

2.4. **Elementary properties of *NED* and *CNED* sets.** We first recall the definitions. For an open set $U\subset\mathbb{R}^n$ and for any two closed sets $F_1,F_2\subset\mathbb{R}^n$ the family of curves joining $F_1$ and $F_2$ inside $U$ is denoted by $\Gamma(F_1,F_2;U)$. In other words, this family contains the curves $\gamma\colon[a,b]\to\overline{U}$ with $\gamma(a)\in F_1$, $\gamma(b)\in F_2$, and $\gamma((a,b))\subset U$. For a set $E\subset\mathbb{R}^n$ we denote by $\mathcal{F}_0(E)$ the family of curves in $\mathbb{R}^n$ that do not intersect $E$, except possibly at the endpoints; that is, $\mathcal{F}_0(E)$ contains the curves $\gamma\colon [a,b] \to \mathbb{R}^n$ such that $|\gamma| \setminus \gamma(\{a,b\})$ does not intersect $E$. Moreover, we define $\mathcal{F}_\sigma(E)$ to be the family of curves in $\mathbb{R}^n$ that intersect $E$ in countably many points; that is, the set $E \cap |\gamma|$ is countable (finite or infinite).

Let $E \subset \mathbb{R}^n$ be a set and $\Omega \subset \mathbb{R}^n$ be an open set. The set $E$ lies in $\mathit{NED}(\Omega)$ if for every pair of non-empty, disjoint continua $F_1, F_2 \subset \Omega$ we have

$$\operatorname{Mod}_n \Gamma(F_1, F_2; \Omega) = \operatorname{Mod}_n(\Gamma(F_1, F_2; \Omega) \cap \mathcal{F}_0(E)).$$

If we have instead

$$\operatorname{Mod}_n \Gamma(F_1, F_2; \Omega) \leq M \cdot \operatorname{Mod}_n(\Gamma(F_1, F_2; \Omega) \cap \mathcal{F}_0(E))$$

for a uniform constant $M \geq 1$, then $E$ lies in $\mathit{NED}^w(\Omega)$. A set $E \subset \mathbb{R}^n$ lies in $\mathit{CNED}(\Omega)$ and $\mathit{CNED}^w(\Omega)$ if the above equality and inequality, respectively, hold with $\mathcal{F}_\sigma(E)$ in place of $\mathcal{F}_0(E)$. In the case that $\Omega = \mathbb{R}^n$, we simply use the notation $\mathit{NED}$, $\mathit{NED}^w$, $\mathit{CNED}$, and $\mathit{CNED}^w$.

We will use the notation $*\mathit{NED}(\Omega)$ and $\mathcal{F}_*(E)$ for $\mathit{NED}(\Omega)$ or $\mathit{CNED}(\Omega)$ and for $\mathcal{F}_0(E)$ or $\mathcal{F}_\sigma(E)$, respectively. Similarly, we will use the notation $*\mathit{NED}^w(\Omega)$ for $\mathit{NED}^w(\Omega)$ or $\mathit{CNED}^w(\Omega)$. By the monotonicity of modulus, if $E \in *\mathit{NED}(\Omega)$ (resp. $*\mathit{NED}^w(\Omega)$) and $F \subset E$, then $F$ lies in $*\mathit{NED}(\Omega)$ (resp. $*\mathit{NED}^w(\Omega)$). Moreover, if $E \in *\mathit{NED}^w(\Omega)$, it is immediate that $E \cap \Omega$ must have empty interior.

A set is *non-degenerate* if it contains more than one points. For two non-degenerate sets $F_1, F_2 \subset \mathbb{R}^n$ we define the *relative distance* $\Delta(F_1, F_2)$ by

$$\Delta(F_1, F_2) = \frac{\operatorname{dist}(F_1, F_2)}{\min\{\operatorname{diam}(F_1), \operatorname{diam}(F_2)\}}.$$

**Lemma 2.4.** *Let $E \subset \mathbb{R}^n$ be a set with the following property. There exist constants $t, \phi > 0$ such that for each $x_0 \in \mathbb{R}^n$ there exists $r_0 > 0$ with the property that for every pair of non-degenerate, disjoint continua $F_1, F_2 \subset B(x_0, r_0)$ with $\Delta(F_1, F_2) \leq t$ we have*

$$\operatorname{Mod}_n(\Gamma(F_1, F_2; \mathbb{R}^n) \cap \mathcal{F}_*(E)) \geq \phi.$$

*If, in addition, $E$ is Lebesgue measurable, then $m_n(E) = 0$.*

*Proof.* Since $E$ is measurable, it differs from a Borel subset by a set of measure zero. Note also that the family $\mathcal{F}_*(E)$ increases when we pass to a subset of $E$. Thus, we assume that $E$ is Borel itself. Suppose that $m_n(E) > 0$ and that $x_0$ is a Lebesgue density point of $E$. Let $r_0$ be as in the assumption and consider $r < r_0$. Define $F_1 = \partial B(x_0, r)$ and $F_2 = \partial B(x_0, ar)$, where $a \in (0,1)$ is chosen so that

$$\Delta(F_1, F_2) = \frac{1-a}{2a} \leq t.$$

Consider the function

$$\rho(x) = \frac{1}{|x - x_0| \log(a^{-1})} \chi_{A(x_0; ar, r)}(x).$$

Then $\rho$ is admissible for $\Gamma(F_1, F_2; \mathbb{R}^n)$; see for example [Hei01, 7.14, pp. 52–53]. We set $\rho_1 = \rho \chi_{\mathbb{R}^n \setminus E}$, which is Borel measurable. If $\gamma$ is a curve in $\Gamma(F_1, F_2; \mathbb{R}^n) \cap \mathcal{F}_*(E)$, then $\gamma$ intersects $E$ at countably many points, so by Lemma 2.2 (iv) we have

$$\int_\gamma \rho_1 \, ds = \int_\gamma \rho \, ds \geq 1.$$

Hence, $\rho_1$ is admissible for $\Gamma(F_1,F_2;\mathbb{R}^n)\cap\mathcal{F}_*(E)$. We conclude that

$$\phi \leq \mathrm{Mod}_n(\Gamma(F_1,F_2;\mathbb{R}^n)\cap\mathcal{F}_*(E)) \leq \int_{\mathbb{R}^n\setminus E}\rho^n \leq \frac{m_n(B(x_0,r)\setminus E)}{(\log(a^{-1}))^n a^n r^n}.$$

Letting $r\to 0$ contradicts the assumption that $x_0$ is a density point of $E$. □

**Lemma 2.5.** *Let $\Omega\subset\mathbb{R}^n$ be an open set and $E\subset\Omega$ be a set with $E\in *NED^w(\Omega)$.*

(i) *If $\overline{E}\subset\Omega$, then $E$ satisfies the assumption of Lemma 2.4.*
(ii) *If $E$ is Lebesgue measurable, then $m_n(E)=0$.*

It was proved in [Väi62, Theorem 1] that closed *NED* sets have measure zero. However, we remark that there exists a non-measurable set, constructed by Sierpiński for a different purpose, that is of class *CNED*; see the discussion in Section 6.3.

*Proof.* Let $\overline{E}\subset\Omega$ as in (i). If $x_0\notin\overline{E}$, there exists a ball $B(x_0,r_0)\subset\mathbb{R}^n\setminus E$. For any pair of non-degenerate, disjoint continua $F_1,F_2\subset B(x_0,r_0)$ we have

$$\mathrm{Mod}_n(\Gamma(F_1,F_2;\mathbb{R}^n)\cap\mathcal{F}_*(E)) \geq \mathrm{Mod}_n\Gamma(F_1,F_2;B(x_0,r_0))$$

by the monotonifity of modulus. If $x_0\in\overline{E}$, consider a ball $B(x_0,r_0)\subset\Omega$. Since $E\in *NED^w(\Omega)$, there exists a uniform constant $M\geq 1$ such that for $F_1,F_2\subset B(x_0,r_0)$ as above, we have

$$\begin{aligned}\mathrm{Mod}_n(\Gamma(F_1,F_2;\mathbb{R}^n)\cap\mathcal{F}_*(E)) &\geq \mathrm{Mod}_n(\Gamma(F_1,F_2;\Omega)\cap\mathcal{F}_*(E))\\ &\geq M^{-1}\cdot\mathrm{Mod}_n\Gamma(F_1,F_2;\Omega)\\ &\geq M^{-1}\cdot\mathrm{Mod}_n\Gamma(F_1,F_2;B(x_0,r_0)).\end{aligned}$$

Each open ball in Euclidean space is a *Loewner space* [Hei01, Example 8.24 (a), p. 65], so the latter modulus is uniformly bounded from below, provided that $\Delta(F_1,F_2)\leq 1$. Therefore the assumption of Lemma 2.4 is satisfied.

For (ii), note that each closed set $K\subset E$ is contained in $\Omega$, lies in $*NED^w(\Omega)$, and by part (i) satisfies the assumption of Lemma 2.4. By Lemma 2.4, $m_n(K)=0$. Since $E$ is measurable, $m_n(E)=0$. □

2.5. **Comparison to classical definition of *NED* sets.** According to the classical definition, a *closed* set $E\subset\mathbb{R}^n$ is *NED* if for every pair of non-empty, disjoint continua $F_1,F_2\subset\mathbb{R}^n\setminus E$ we have

$$\mathrm{Mod}_n\Gamma(F_1,F_2;\mathbb{R}^n)=\mathrm{Mod}_n\Gamma(F_1,F_2;\mathbb{R}^n\setminus E)=\mathrm{Mod}_n(\Gamma(F_1,F_2;\mathbb{R}^n)\cap\mathcal{F}_0(E)).$$

In our definition, we required the stronger condition that the above equality holds for all disjoint continua $F_1,F_2\subset\mathbb{R}^n$ regardless of whether they intersect the set $E$. The reason for allowing such a generality in our approach is that we impose no topological assumptions on $E$, which could be even dense in $\mathbb{R}^n$; therefore it would be too restrictive and unnatural to work with continua $F_1,F_2\subset\mathbb{R}^n\setminus E$.

We show that the two definitions agree. The proof relies on some results relating $n$-modulus with *$n$-capacity.* Let $U\subset\mathbb{R}^n$ be an open set and $F_1,F_2\subset\overline{U}$ be disjoint continua. The $n$-capacity of the *condenser* $(F_1,F_2;U)$ is defined as

$$\mathrm{Cap}_n(F_1,F_2;U)=\inf_u\int_U|\nabla u|^n,$$

where the infimum is taken over all functions $u$ that are continuous in $U\cup F_1\cup F_2$, ACL in $U$ [Väi71, Section 26, p. 88], with $u=0$ in a neighborhood of $F_1$ and $u=1$ in a neighborhood of $F_2$ [Hes75, Theorem 3.3]. Equivalently, one can replace in this definition continuous ACL functions with the Dirichlet space $L^{1,n}(U)$ of locally integrable functions in $U$ with distributional derivatives of first order lying in $L^n(U)$. It has been shown by Hesse [Hes75, Theorem 5.5] that whenever $F_1, F_2$ are continua in $U$ (specifically, not intersecting $\partial U$), then

$$\mathrm{Cap}_n(F_1, F_2; U) = \mathrm{Mod}_n \Gamma(F_1, F_2; U).$$

This result was generalized to the case that $F_1, F_2 \subset \overline{U}$, provided that $U$ is a *QED domain*, by Herron–Koskela [HK90]; see also the work of Shlyk [Shl93] for a more general result. By definition, a connected open set $U \subset \mathbb{R}^n$ is a *QED* domain if there exists a constant $M \geq 1$ such that

$$\mathrm{Mod}_n \Gamma(F_1, F_2; \mathbb{R}^n) \leq M \cdot \mathrm{Mod}_n \Gamma(F_1, F_2; U)$$

for all pairs of non-empty, disjoint continua $F_1, F_2 \subset U$.

Suppose that $E$ is an *NED* set according to the classical definition. Then $U = \mathbb{R}^n \setminus E$ is a *QED* domain, whose closure is $\mathbb{R}^n$; classical *NED* sets have empty interior [Väi62]. Thus, the result of Herron–Koskela gives

$$\mathrm{Cap}_n(F_1, F_2; \mathbb{R}^n \setminus E) = \mathrm{Mod}_n \Gamma(F_1, F_2; \mathbb{R}^n \setminus E)$$

for all non-empty, disjoint continua $F_1, F_2 \subset \mathbb{R}^n$. Summarizing, in order to show the equivalence of the classical definition to the current one, it suffices to show that

$$\mathrm{Cap}_n(F_1, F_2; \mathbb{R}^n) = \mathrm{Cap}_n(F_1, F_2; \mathbb{R}^n \setminus E)$$

for all non-empty, disjoint continua $F_1, F_2 \subset \mathbb{R}^n$. Observe that this equality already holds if $F_1, F_2 \subset \mathbb{R}^n \setminus E$. Hence, $E$ is *removable for $n$-capacity*, in the sense of Vodopyanov–Goldshtein [VG77]. By [VG77, Theorem 3.1, p. 46], such sets coincide with the sets that are removable for the Dirichlet space $L^{1,n}$. That is, $m_n(E) = 0$ and if $u \in L^{1,n}(\mathbb{R}^n \setminus E)$, then $u \in L^{1,n}(\mathbb{R}^n)$ and the distributional derivatives of $u$ are the same in both spaces. Now, let $F_1, F_2$ be any non-empty, disjoint continua in $\mathbb{R}^n$; in particular, they might intersect the set $E$, as in the current definition of *NED* sets. We trivially have

$$\mathrm{Cap}_n(F_1, F_2; \mathbb{R}^n \setminus E) \leq \mathrm{Cap}_n(F_1, F_2; \mathbb{R}^n).$$

Since $E$ is removable for the Dirichlet space $L^{1,n}$, we also have the reverse inequality, completing the proof.

## 3. Families of curve perturbations

Let $\mathcal{F}$ be a path family in $\mathbb{R}^n$. We define $\partial\mathcal{F}$ to be the set of points $x \in \mathbb{R}^n$ that are endpoints of some path of $\mathcal{F}$ and $x \notin |\gamma| \setminus \partial\gamma$ for any path $\gamma \in \mathcal{F}$. In other words, $\partial\mathcal{F}$ contains endpoints of paths in $\mathcal{F}$ that are not interior points of any path of $\mathcal{F}$. For example, if $\mathcal{F}$ is the family $\Gamma(F_1, F_2; U)$, where $U$ is a ring $A(0; r, R)$ and $F_1, F_2$ are the boundary components of $U$, then $\partial\mathcal{F} = F_1 \cup F_2$. Another example is the family $\mathcal{F}_0(E)$ for some set $E \subset \mathbb{R}^n$; recall its definition from Section 2.4. Then, $\partial\mathcal{F}_0(E) \supset E$. Indeed, every point $x \in E$ can be considered as a constant path in $\mathcal{F}_0(E)$; recall that paths of $\mathcal{F}_0(E)$ can have endpoints in $E$. Moreover, if $x \in E$, then $x \notin |\gamma| \setminus \partial\gamma$ for any $\gamma \in \mathcal{F}_0(E)$; thus $x \in \partial\mathcal{F}_0(E)$.

**Definition 3.1.** We say that a path family $\mathcal{F}$ in $\mathbb{R}^n$ is a *family of curve perturbations*, or else, a *$P$-family*, if

(P1) for all non-constant rectifiable paths $\gamma$ in $\mathbb{R}^n$ we have $\gamma + x \in \mathcal{F}$ for a.e. $x \in \mathbb{R}^n$,

(P2) for every $x \in \mathbb{R}^n$ and $r > 0$, the radial segment $t \mapsto x + tw$, $0 \leq t \leq r$, lies in $\mathcal{F}$ for $\mathscr{H}^{n-1}$-a.e. $w \in S^{n-1}(0,1)$,
(P3) $\mathcal{F}$ is closed under strong subpaths and reparametrizations, and
(P4) if two paths $\gamma_1, \gamma_2 \in \mathcal{F}$ have a common endpoint that does not lie in $\partial\mathcal{F}$, then the concatenation of $\gamma_1$ with $\gamma_2$ on that endpoint lies in $\mathcal{F}$.

Property (P4) holds always for families that are closed under concatenations; for example the family $\mathcal{F}_\sigma(E)$ of curves intersecting a given set $E$ at countably many points is such. The reason for requiring that the common endpoint of $\gamma_1$ and $\gamma_2$ does not lie in $\partial\mathcal{F}$ is that we wish to accommodate curve families such as $\mathcal{F}_0(E)$, which contains paths that do not intersect $E$ *except possibly at the endpoints*. Since $\partial\mathcal{F}_0(E) \supset E$, we remark that $\mathcal{F}_0(E)$ always satisfies (P4). Finally, we note that (P3) always holds for $\mathcal{F}_0(E)$ and $\mathcal{F}_\sigma(E)$. We summarize these remarks below.

*Remark* 3.2. Properties (P3) and (P4) always hold for the families $\mathcal{F}_0(E)$ and $\mathcal{F}_\sigma(E)$, for each $E \subset \mathbb{R}^n$.

**Lemma 3.3.** *The intersection of countably many $P$-families $\mathcal{F}_i$, $i \in \mathbb{N}$, is again a $P$-family.*

*Proof.* Let $\mathcal{F} = \bigcap_{i\in\mathbb{N}} \mathcal{F}_i$. Properties (P1), (P2), and (P3) are immediate for $\mathcal{F}$. For (P4), note that every constant path in $\mathbb{R}^n$ lies in $\mathcal{F}_i$ for each $i \in \mathbb{N}$; this follows by combining (P2) with (P3). Thus, if $x \in \partial\mathcal{F}_i$ for some $i \in \mathbb{N}$, then $x$ is the endpoint of a (constant) path in $\mathcal{F}$. Moreover, $x \notin |\gamma| \setminus \partial\gamma$ for any path $\gamma$ of $\mathcal{F}_i \supset \mathcal{F}$, so

$$\bigcup_{i\in\mathbb{N}} \partial\mathcal{F}_i \subset \partial\mathcal{F}.$$

Now, if $\gamma_1, \gamma_2 \in \mathcal{F}$ have a common endpoint that does not lie in $\partial\mathcal{F}$, then it also does not lie in $\partial\mathcal{F}_i$ for any $i \in \mathbb{N}$. Hence, by (P4), the concatenation of $\gamma_1$ with $\gamma_2$ lies in $\mathcal{F}_i$ for each $i \in \mathbb{N}$. This proves that (P4) holds for $\mathcal{F}$. $\square$

In Section 3.2 we will see important examples of such families. Specifically, if $\mathscr{H}^{n-1}(E) = 0$, then the family $\mathcal{F}_0(E)$ is a $P$-family and if $E$ has $\sigma$-finite Hausdorff $(n-1)$-measure, then $\mathcal{F}_\sigma(E)$ is a $P$-family.

3.1. **The invariance theorem.** The main result of the section states that $n$-modulus is not affected if we restrict a path family to a $P$-family.

**Theorem 3.4.** *Let $\mathcal{F}$ be a family of curve perturbations in $\mathbb{R}^n$. Then for every open set $U \subset \mathbb{R}^n$ and all pairs of non-empty, disjoint continua $F_1, F_2 \subset U$ we have*

$$\operatorname{Mod}_n \Gamma(F_1, F_2; U) = \operatorname{Mod}_n(\Gamma(F_1, F_2; U) \cap \mathcal{F})$$

We first establish several auxiliary results.

**Lemma 3.5.** *Let $\mathcal{F}$ be a family of curve perturbations in $\mathbb{R}^n$. Let $A = A(0; r, R)$, $0 < 7r \leq R$, and $F_1, F_2 \subset \mathbb{R}^n$ be disjoint continua such that every sphere $S^{n-1}(0, \rho)$, $r < \rho < R$, intersects both $F_1$ and $F_2$. Then*

$$\operatorname{Mod}_n(\Gamma(F_1, F_2; A) \cap \mathcal{F}) \geq c(n) \log\left(\frac{R}{r}\right).$$

The statement is also true for $r < R < 7r$ but we do not prove this for the sake of brevity. The statement without restricting to the family $\mathcal{F}$ is classical and can be found in [Väi71, Theorem 10.12, p. 31]. Our proof relies on the next lemma.

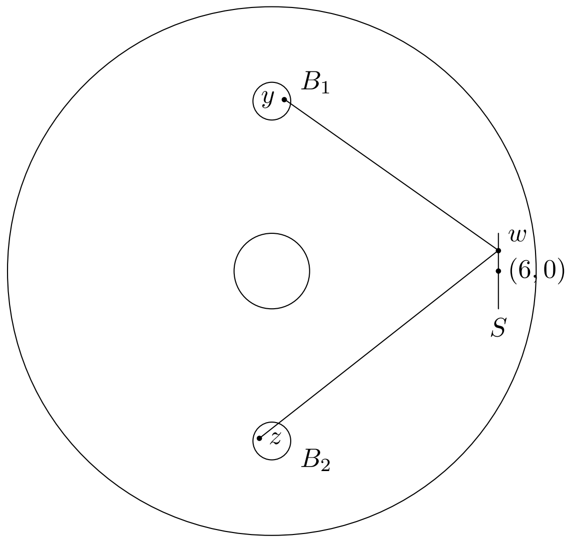


FIGURE 1

**Lemma 3.6** ([KK00, Lemma 2.1])**.** *Let $u\colon \mathbb{R}^n \to [0,\infty]$ be a Borel function and $F \subset \mathbb{R}^n$ be a continuum. Suppose that for each $y \in F$ there exists a set $D_y \subset S^{n-1}(y,1)$ with $\mathscr{H}^{n-1}(D_y) \geq a > 0$ for some $a > 0$ such that*

$$\int_{[y,w]} u\, ds \geq 1$$

*for each $w \in D_y$. Then*

$$\int_{\mathbb{R}^n} u^n \geq c(n,a)\operatorname{diam}(F).$$

*Proof of Lemma 3.5.* We will first prove the statement for $R = 7r$. We will perform several normalizations and reductions. By applying a scaling, we may assume that $r = 1$; note that $n$-modulus is unaffected by scaling and that the family $\mathcal{F}$ is mapped to a possibly different family of curve perturbations, which we denote by $\mathcal{F}$ for simplicity. There exist closed balls $B_i = \overline{B}(z_i, 1/2)$ with $z_i \in F_i \cap S^{n-1}(0, 9/2)$, and disjoint continua $F_i' \subset F_i \cap B_i$ with $\operatorname{diam}(F_i') \geq 1/2$ for $i = 1, 2$. We will find a uniform lower bound for $\operatorname{Mod}_n(\Gamma(F_1', F_2'; A) \cap \mathcal{F})$, which will give a uniform lower bound for $\operatorname{Mod}_n(\Gamma(F_1, F_2; A) \cap \mathcal{F})$ by the monotonicity of modulus. From now on, we denote $F_i'$ by $F_i$, $i = 1, 2$, for simplicity.

By applying an isometry, we assume that $B_1$ and $B_2$ are symmetric with respect to the hyperplane $\mathbb{R}^{n-1} \times \{0\}$ and their centers have non-negative first coordinate. The choice of the balls and of the normalization is such that for all points $w$ in the $(n-1)$-dimensional disk $S = B((6,0,\dots,0),1) \cap (\{6\} \times \mathbb{R}^{n-1})$ and for all $y \in B_i$, $i = 1, 2$, the segment $[y, w]$ lies in the original ring $A$; see Figure 1.

We remark that $1/2 \leq \operatorname{diam}(F_i) \leq 1$, $\operatorname{diam}(S) = 2$, and $1 \leq \operatorname{dist}(F_i, S) \leq 14 = 2R$ for $i = 1, 2$. Thus, $\operatorname{diam}(S) \simeq \operatorname{diam}(F_i) \simeq \operatorname{dist}(F_i, S) \simeq 1$. For $y \in F_1$, $w \in S$, and $z \in F_2$ consider the concatenation $\gamma(y, w, z)$ of the line segments $[y, w]$ and $[w, z]$. Note that $\gamma(y, w, z) \subset A$ and $\gamma(y, w, z) \in \Gamma(F_1, F_2; A) \cap \mathcal{F}$ for each $y \in F_1$, $z \in F_2$, and a.e. $w \in S$, by the properties of a $P$-family. Indeed, (P2) and (P3) imply that for a.e. $w \in S$ the segments $[y,w]$ and $[w,z]$ lie in $\mathcal{F}$. Moreover, the same properties imply that a.e. $w \in S$ does not lie in $\partial\mathcal{F}$. Hence, by (P4), the concatenation of the segments $[y,w]$ and $[w,z]$ lies in $\mathcal{F}$.

For each fixed $y \in F_i$ consider the map $S \ni w \mapsto \Phi_y(w) = \frac{w-y}{|w-y|}$. By the relative position of $y$ and $S$, this map is uniformly bi-Lipschitz. Thus, if $S' \subset S$ and $\mathscr{H}^{n-1}(S') \geq a$ for some $a > 0$, then $\mathscr{H}^{n-1}(\Phi_y(S')) \gtrsim_n a$. We note that the implicit constants are independent of the point $y \in F_i$.

Now, let $\rho$ be an admissible function for $\Gamma(F_1, F_2; A) \cap \mathcal{F}$. We have

$$\int_{\gamma(y,w,z)} \rho\, ds \geq 1$$

for all $y \in F_1$, $z \in F_2$ and a.e. $w \in S$. Suppose that for each $y \in F_1$ there exists $S_y \subset S$ with $\mathscr{H}^{n-1}(S_y) \geq \frac{1}{2}\mathscr{H}^{n-1}(S)$ such that we have

$$\int_{[y,w]} \rho\, ds \geq 1/2$$

for all $w \in S_y$. Then for the set $D_y = \Phi_y(S_y)$ we have $\mathscr{H}^{n-1}(D_y) \gtrsim_n 1$. Lemma 3.6 now implies that

$$\int \rho^n \gtrsim_n 1.$$

The other case is that there exists $y \in F_1$ such that there exists a subset $S'$ of $S$ with $\mathscr{H}^{n-1}(S') \geq \frac{1}{2}\mathscr{H}^{n-1}(S)$, and

$$\int_{[y,w]} \rho\, ds < 1/2$$

for each $w \in S'$. This implies that for each $z \in F_2$ and for a.e. $w \in S'$ we have

$$\int_{[z,w]} \rho\, ds \geq 1/2.$$

As before, Lemma 3.6 gives that

$$\int \rho^n \gtrsim_n 1.$$

Therefore, we have shown that

$$\mathrm{Mod}_n(\Gamma(F_1, F_2; A) \cap \mathcal{F}) \geq c(n) > 0$$

for a uniform constant $c(n)$ depending only on $n$, whenever $R = 7r$.

In the general case, let $k \in \mathbb{N}$ be the largest integer such that $R \geq 7^k r$. Consider the rings $A_i = A(0; 7^{i-1}r, 7^i r)$, $i \in \{1, \dots, k\}$. By the serial law (M5), we have

$$\mathrm{Mod}_n(\Gamma(F_1, F_2; A) \cap \mathcal{F}) \geq \sum_{i=1}^{k} \mathrm{Mod}_n(\Gamma(F_1, F_2; A_i) \cap \mathcal{F}) \gtrsim_n k \gtrsim_n \log(R/r). \quad \square$$

**Lemma 3.7.** *Let $x \in \mathbb{R}^n$, $R > 0$, and $\rho\colon \mathbb{R}^n \to [0,\infty]$ be a Borel function with $\rho \in L^n(\mathbb{R}^n)$.*

(i) *For $M > 0$, let $\Gamma_M$ be the family of paths $\gamma$ that intersect the ball $B(x,R)$ and satisfy $\ell(\gamma) > MR$. Then $\mathrm{Mod}_n \Gamma_M \to 0$ as $M \to \infty$.*

(ii) *Let $\Gamma$ be a path family with $\operatorname{Mod}_n \Gamma > a$ for some $a > 0$ such that each path $\gamma \in \Gamma$ intersects the ball $B(x, R)$. Then there exists a path $\gamma \in \Gamma$ with*

$$\int_\gamma \rho\, ds \leq c(n,a) \left( \int_{B(x,c(n,a)R)} \rho^n \right)^{1/n} \quad \text{and} \quad \ell(\gamma) \leq c(n,a)R.$$

*Proof.* Both statements are conformally invariant. Hence, using a conformal transformation, we may assume that $x = 0$ and $R = 1$. For $M > 1$, the family $\Gamma_M$ is contained in the union of the families

$$\Gamma_1 = \{\gamma : \ell(\gamma) > M \text{ and } |\gamma| \subset B(0, \sqrt{M})\},$$
$$\Gamma_2 = \{\gamma : |\gamma| \cap \partial B(0,1) \neq \emptyset \text{ and } |\gamma| \cap \partial B(0, \sqrt{M}) \neq \emptyset\}.$$

By the subadditivity of modulus, it suffices to show that $\operatorname{Mod}_n \Gamma_i$ converges to $0$ as $M \to \infty$ for $i = 1, 2$. The function $M^{-1}\chi_{B(0,\sqrt{M})}$ is admissible for $\Gamma_1$, so $\operatorname{Mod}_n \Gamma_1 \leq c(n) M^{-n/2}$. The modulus of $\Gamma_2$ is given by the explicit formula $\operatorname{Mod}_n \Gamma_2 = c(n)(\log \sqrt{M})^{1-n}$;see property (M8). This proves part (i).

Now we prove (ii). Let $M = M(n,a)$ be sufficiently large, so that $\operatorname{Mod}_n \Gamma_M < a/2$. Define $\rho_1 = \rho \chi_{B(0,M+1)}$ and let $\Gamma_1$ be the family of paths $\gamma \in \Gamma$ such that

$$\int_\gamma \rho_1\, ds > 2^{1/n} a^{-1/n} \|\rho_1\|_{L^n(\mathbb{R}^n)}.$$

Then the function

$$2^{-1/n} a^{1/n} \|\rho_1\|^{-1}_{L^n(\mathbb{R}^n)} \rho_1$$

is admissible for $\Gamma_1$, provided that $\|\rho_1\|_{L^n(\mathbb{R}^n)} \neq 0$, in which case we have $\operatorname{Mod}_n \Gamma_1 \leq a/2$. If $\|\rho_1\|_{L^n(\mathbb{R}^n)} = 0$, then $\operatorname{Mod}_n \Gamma_1 = 0$ by property (M4). Also, let $\Gamma_2$ be the family of paths $\gamma \in \Gamma$ such that $\ell(\gamma) > M$, so $\operatorname{Mod}_n \Gamma_2 \leq \operatorname{Mod}_n \Gamma_M < a/2$. By the subadditivity of modulus we have $\operatorname{Mod}_n(\Gamma_1 \cup \Gamma_2) < a < \operatorname{Mod}_n \Gamma$. It follows that $\Gamma \setminus (\Gamma_1 \cup \Gamma_2)$ has positive modulus, and in particular it is non-empty. Thus, there exists a path $\gamma \in \Gamma$ with $\ell(\gamma) \leq M$ and

$$\int_\gamma \rho_1\, ds \leq 2^{1/n} a^{-1/n} \|\rho_1\|_{L^n(\mathbb{R}^n)}.$$

Finally, note that $|\gamma| \subset B(0, M+1)$ since $|\gamma| \cap B(0,1) \neq \emptyset$ and $\ell(\gamma) \leq M$. Thus,

$$\int_\gamma \rho\, ds = \int_\gamma \rho_1\, ds,$$

which completes the proof, with $c(n,a) = \max\{M(n,a) + 1, 2^{1/n} a^{-1/n}\}$. ☐

For a continuum $F \subset \mathbb{R}^n$ and $r > 0$ we define $F^r$ to be the continuum $F + \overline{B}(0,r) = \{x + y : x \in F, y \in \overline{B}(0,r)\}$.

**Lemma 3.8.** *Let $\mathcal{F}$ be a family of curve perturbations in $\mathbb{R}^n$. Then for every open set $U \subset \mathbb{R}^n$ and all pairs of non-empty, disjoint continua $F_1, F_2 \subset U$ we have*

$$\operatorname{Mod}_n(\Gamma(F_1, F_2; U) \cap \mathcal{F}) = \lim_{r \to 0} \operatorname{Mod}_n(\Gamma(F_1^r, F_2^r; U) \cap \mathcal{F}).$$

Our proof relies on the properties of $P$-families, which is a new concept, but the main ideas originate in the proof of [Väi62, Lemma 2].

*Proof.* Note that $\Gamma(F_1,F_2;U)\cap\mathcal{F}\subset\Gamma(F_1^r,F_2^r;U)\cap\mathcal{F}$ for every $r>0$, so one inequality is immediate. Also, if $F_i$ is a point $x_0$ for some $i=1,2$, then there exists $R>0$ such that by properties (M2) and (M8) we have

$$\mathrm{Mod}_n(\Gamma(F_1^r,F_2^r;U)\cap\mathcal{F})\leq \mathrm{Mod}_n\,\Gamma(A(x_0;r,R))=c(n)\left(\log\frac{R}{r}\right)^{1-n}$$

for all sufficiently small $r>0$. Taking $r\to 0$, we obtain the desired conclusion.

We suppose that $\mathrm{diam}(F_i)>0$ for $i=1,2$. Let $\rho\in L^n(\mathbb{R}^n)$ be admissible for $\Gamma(F_1,F_2;U)\cap\mathcal{F}$. We will show that for each $q<1$ we have

$$\int_\gamma \rho\, ds\geq q$$

for all sufficiently small $r>0$ and $\gamma\in\Gamma(F_1^r,F_2^r;U)\cap\mathcal{F}$. This will imply that

$$\limsup_{r\to 0}\mathrm{Mod}_n(\Gamma(F_1^r,F_2^r;U)\cap\mathcal{F})\leq q^{-n}\int\rho^n.$$

Letting $q\to 1$ and then infimizing over $\rho$ gives the desired

$$\limsup_{r\to 0}\mathrm{Mod}_n(\Gamma(F_1^r,F_2^r;U)\cap\mathcal{F})\leq \mathrm{Mod}_n(\Gamma(F_1,F_2;U)\cap\mathcal{F}).$$

Arguing by contradiction, assume that there exists $0<q<1$ and $r_k\to 0^+$ such that for each $k\in\mathbb{N}$ there exists a path $\gamma_{r_k}\in\Gamma(F_1^{r_k},F_2^{r_k};U)\cap\mathcal{F}$ with

$$\int_{\gamma_{r_k}}\rho\, ds<q<1.$$

By passing to a subpath, we assume in addition that $|\gamma_{r_k}|$ is disjoint from $F_1\cup F_2$; here we use property (P3). We fix $R_0>0$ such that $F_i^{R_0}\subset U$, $\mathrm{diam}(F_i)>2R_0$ for $i=1,2$, and $F_1^{R_0}\cap F_2^{R_0}=\emptyset$. For each $r_k<R_0$ and $i=1,2$, there exists $x_{i,k}\in F_i$ such that $|\gamma_{r_k}|$ connects the boundary components of the ring $A_{i,k}=A(x_{i,k};r_k,R_0)$. Note that $F_i$ also connects the boundary components of the ring $A_{i,k}$, since $\mathrm{diam}(F_i)>\mathrm{diam}(A_{i,k})$. By passing to a further subpath, we assume in addition that the endpoints of $\gamma_{r_k}$ lie in the inner boundary components of $A_{i,k}$.

We fix $\varepsilon=(1-q)/2>0$. By Lemma 3.5 we have that if $r_k<R_0/8$, then

$$\mathrm{Mod}_n(\Gamma(|\gamma_{r_k}|,F_i;A_{i,k})\cap\mathcal{F})\geq c(n)\log(R_0/r_k)\gtrsim_n 1.$$

Lemma 3.7 (ii) implies that if $r_k$ is sufficiently small, depending on $\varepsilon$, then there exists a path $\gamma_i\in\Gamma(|\gamma_{r_k}|,F_i;A_{i,k})\cap\mathcal{F}$ such that

$$\int_{\gamma_i}\rho\, ds\leq c(n)\left(\int_{B(x_{i,k},c(n)r_k)}\rho^n\right)^{1/n}<\varepsilon.$$

We concatenate $\gamma_i$, $i=1,2$, with a suitable subpath of $\gamma_{r_k}$; note that the endpoint of $\gamma_i$ that lies in $|\gamma_{r_k}|$ is not in $\partial\mathcal{F}$ because it is an interior point of a path of $\mathcal{F}$. By property (P4), the concatenation lies in $\mathcal{F}$. In this way, we obtain a path $\gamma\in\Gamma(F_1,F_2;U)\cap\mathcal{F}$ such that

$$\int_\gamma\rho\, ds\leq\int_{\gamma_{r_k}}\rho\, ds+\int_{\gamma_1}\rho\, ds+\int_{\gamma_2}\rho\, ds<q+2\varepsilon=1.$$

This contradicts the admissibility of $\rho$. $\square$

*Remark* 3.9. From the proof we see that Lemma 3.8 is valid more generally for families $\mathcal{F}$ satisfying (P3), (P4), and the conclusion of Lemma 3.5 (or a variant of it, such as Lemma 6.7, which uses rectangular instead of spherical rings). Recall that $\mathcal{F}_*(E)$ always satisfies (P3) and (P4); see Remark 3.2.

Our ultimate preliminary result before the proof of Theorem 3.4 is the following theorem, which is a version of the Lebesgue differentiation theorem for line integrals.

**Theorem 3.10.** *Let $\rho\colon \mathbb{R}^n \to [-\infty,\infty]$ be a Borel function with $\rho \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ for some $p>1$. Then there exists a path family $\Gamma_0$ with $\operatorname{Mod}_p \Gamma_0 = 0$ such that for every rectifiable path $\gamma \notin \Gamma_0$ we have $\int_\gamma |\rho|\, ds < \infty$ and*

$$\lim_{r\to 0} \fint_{B(0,r)} \int_{\gamma+x} \rho\, ds = \int_\gamma \rho\, ds. \tag{3.1}$$

*Proof.* By the subadditivity of modulus, it suffices to prove the statement for paths $\gamma$ contained in a compact set. Thus, we may assume that $\rho \in L^p(\mathbb{R}^n)$. Note that $\int_\gamma |\rho|\, ds < \infty$ for all paths $\gamma$ outside a curve family of $p$-modulus zero. For continuous functions $\rho$ with compact support (3.1) is trivially true for all rectifiable paths, by uniform continuity. For the general case, for fixed $\lambda > 0$ consider the family $\Gamma_\lambda$ of rectifiable paths $\gamma$ with $\int_\gamma |\rho|\, ds < \infty$ and

$$\limsup_{r\to 0} \left| \fint_{B(0,r)} \int_{\gamma+x} \rho\, ds - \int_\gamma \rho\, ds \right| > \lambda.$$

It suffices to show that $\operatorname{Mod}_p \Gamma_\lambda = 0$ for each $\lambda>0$. Let $\phi$ be a continuous function with compact support. Then, $\Gamma_\lambda \subset \Gamma_1 \cup \Gamma_2$, where $\Gamma_1$ is the family of rectifiable paths $\gamma$ with

$$\limsup_{r\to 0} \fint_{B(0,r)} \int_{\gamma+x} |\rho-\phi|\, ds > \lambda/2$$

and $\Gamma_2$ is the family of rectifiable paths $\gamma$ with

$$\int_\gamma |\rho - \phi|\, ds > \lambda/2.$$

We note that

$$\operatorname{Mod}_p \Gamma_2 \leq 2^p \lambda^{-p} \|\rho-\phi\|^p_{L^p(\mathbb{R}^n)}.$$

Moreover, if $\gamma$ is parametrized by arclength, we have

$$\fint_{B(0,r)} \left( \int_{\gamma+x} |\rho-\phi|\, ds \right) dx = \int_0^{\ell(\gamma)} \left( \fint_{B(\gamma(t),r)} |\rho-\phi| \right) dt \leq \int_\gamma M(\rho-\phi)\, ds,$$

where $M(\cdot)$ denotes the centered Hardy–Littlewood maximal function. Hence,

$$\int_\gamma M(\rho-\phi)\, ds > \lambda/2$$

for $\gamma \in \Gamma_1$. The Hardy–Littlewood maximal $L^p$-inequality [Zie89, Theorem 2.8.2, p. 84] implies that

$$\operatorname{Mod}_p \Gamma_1 \leq 2^p\lambda^{-p} \|M(\rho-\phi)\|^p_{L^p(\mathbb{R}^n)} \leq c(n,p) 2^p \lambda^{-p} \|\rho-\phi\|^p_{L^p(\mathbb{R}^n)}.$$

Altogether,

$$\operatorname{Mod}_p \Gamma_\lambda \leq \operatorname{Mod}_p \Gamma_1 + \operatorname{Mod}_p \Gamma_2 \lesssim_{n,p,\lambda} \|\rho-\phi\|^p_{L^p(\mathbb{R}^n)}$$

Since $\phi$ was arbitrary, we conclude that $\operatorname{Mod}_p \Gamma_\lambda = 0$, as desired. $\square$

*Proof of Theorem 3.4.* By the monotonicity of modulus, it suffices to prove that

$$\operatorname{Mod}_n \Gamma(F_1, F_2; U) \leq \operatorname{Mod}_n(\Gamma(F_1, F_2; U) \cap \mathcal{F}).$$

By Lemma 3.8, it suffices to prove that

$$\operatorname{Mod}_n \Gamma(F_1, F_2; U) \leq \operatorname{Mod}_n(\Gamma(F_1^r, F_2^r; U) \cap \mathcal{F})$$

for all sufficiently small $r > 0$. We fix $r > 0$ so that $F_1^r, F_2^r \subset U$. Let $\rho\colon \mathbb{R}^n \to [0, \infty]$ be an admissible function for $\Gamma(F_1^r, F_2^r; U) \cap \mathcal{F}$ with $\rho \in L^n(\mathbb{R}^n)$. Consider the curve family $\Gamma_0$ with $\operatorname{Mod}_n \Gamma_0 = 0$, given by Theorem 3.10 and corresponding to $\rho$. Let $\gamma \in \Gamma(F_1, F_2; U) \setminus \Gamma_0$ be a rectifiable path. Since $\mathcal{F}$ is a family of curve perturbations, by (P1), for a.e. $x \in B(0, r)$ we have $\gamma + x \in \Gamma(F_1^r, F_2^r; U) \cap \mathcal{F}$. Now, the admissibility of $\rho$ for $\Gamma(F_1^r, F_2^r; U) \cap \mathcal{F}$ and Theorem 3.10 imply that $\int_\gamma \rho \, ds \geq 1$, so $\rho$ is admissible for $\Gamma(F_1, F_2; U) \setminus \Gamma_0$. Therefore,

$$\operatorname{Mod}_n \Gamma(F_1, F_2; U) = \operatorname{Mod}_n(\Gamma(F_1, F_2; U) \setminus \Gamma_0) \leq \operatorname{Mod}_n(\Gamma(F_1^r, F_2^r; U) \cap \mathcal{F}). \quad \square$$

3.2. **Examples of families of curve perturbations.** The next theorem, combined with Theorem 3.4, gives Theorem 1.5 (ii) and Theorem 1.6.

**Theorem 3.11.** *Let $E \subset \mathbb{R}^n$ be a set.*

(i) *If $\mathscr{H}^{n-1}(E) = 0$, then $\mathcal{F}_0(E)$ is a $P$-family.*

(ii) *If $E$ has $\sigma$-finite Hausdorff $(n-1)$-measure, then $\mathcal{F}_\sigma(E)$ is a $P$-family.*

The case of Hausdorff $(n-1)$-measure zero, as in (i), has already been considered by Väisälä [Väi62, Lemma 5], where it is proved that for a.e. $x \in \mathbb{R}^n$ we have $\gamma + x \in \mathcal{F}_0(E)$; that is, (P1) is satisfied. Recall also that (P3) and (P4) are always satisfied for $\mathcal{F}_0(E)$ and $\mathcal{F}_\sigma(E)$; see Remark 3.2. We first prove two preliminary lemmas that will be used in the proof of both cases (i) and (ii) of the theorem.

**Lemma 3.12.** *Let $E \subset \mathbb{R}^n$ and $\gamma$ be a non-constant rectifiable path. For $N \in \mathbb{N}$, let $F_N$ be the set of $x \in \mathbb{R}^n$ such that $E \cap |\gamma + x|$ contains at least $N$ points. Then*

$$m_n^*(F_1) \leq c(n) \max\{\ell(\gamma), \operatorname{diam}(E)\} \operatorname{diam}(E)^{n-1} \quad \textit{and}$$
$$m_n^*(F_N) \leq c(n) \ell(\gamma) N^{-1} \mathscr{H}^{n-1}(E).$$

*Proof.* First we show the second inequality. For $k \in \mathbb{N}$ we define $F_{N,k}$ to be the set of $x \in F_N$ such that $E \cap |\gamma + x|$ contains $N$ points with mutual distances bounded below by $1/k$. We have $F_{N,k+1} \supset F_{N,k}$, $F_N = \bigcup_{k=1}^\infty F_{N,k}$, and $m_n^*(F_N) = \lim_{k \to \infty} m_n^*(F_{N,k})$; see [Bog07, Proposition 1.5.12]. We estimate $m_n^*(F_{N,k})$. We fix a large $k \in \mathbb{N}$ so that $\frac{1}{2k} < \operatorname{diam}(|\gamma|)/4$. Consider an arbitrary cover of $E$ by sets $U_i$, $i \in I$, with $\operatorname{diam}(U_i) < \frac{1}{2k}$. For each $i \in I$ there exists a closed ball $B_i \supset U_i$ of radius $r_i = \operatorname{diam}(U_i) < \frac{1}{2k} < \operatorname{diam}(|\gamma|)/4$. Define the function

$$\rho = \frac{1}{N} \sum_{i \in I} \frac{1}{r_i} \chi_{2B_i}.$$

If $x \in F_{N,k}$, then $|\gamma + x|$ intersects at least $N$ balls $B_i$ and is not contained in any ball $2B_i$. Therefore,

$$\int_{\gamma + x} \rho \, ds \geq 1.$$

By Chebychev's inequality and Fubini's theorem, we have

$$m_n^*(F_{N,k}) \le \ell(\gamma)\|\rho\|_{L^1(\mathbb{R}^n)} \simeq_n \ell(\gamma)N^{-1}\sum_{i\in I}\operatorname{diam}(U_i)^{n-1}.$$

The cover of $E$ by sets $U_i$, $i \in I$, of diameter less than $\frac{1}{2k}$ was arbitrary, so

$$m_n^*(F_{N,k}) \lesssim_n \ell(\gamma)N^{-1}\mathscr{H}^{n-1}_{(2k)^{-1}}(E).$$

Letting $k \to \infty$ gives

$$m_n^*(F_N) \lesssim_n \ell(\gamma)N^{-1}\mathscr{H}^{n-1}(E).$$

For the first inequality, consider two cases. If $\operatorname{diam}(|\gamma|) > 4\operatorname{diam}(E)$, then we cover $E$ by a closed ball $B$ of radius $r$ with $0 \le \operatorname{diam}(E) < r < \operatorname{diam}(|\gamma|)/4$. If $x \in F_1$, then $|\gamma + x|$ intersects $B$ and is not contained in $2B$. The above argument for $N = 1$ gives $m_n^*(F_1) \lesssim_n \ell(\gamma)r^{n-1}$. Now, we let $r \to \operatorname{diam}(E)$ to obtain $m_n^*(F_1) \lesssim_n \ell(\gamma)\operatorname{diam}(E)^{n-1}$. Next, assume that $\operatorname{diam}(|\gamma|) \le 4\operatorname{diam}(E)$. In this case, if $x \in F_1$, then $|\gamma + x| \subset \overline{B}(x_0, 5\operatorname{diam}(E))$ for a fixed $x_0 \in E$. Thus, $m_n^*(F_1) \le m_n(\overline{B}(x_0, 5\operatorname{diam}(E))) = c(n)\operatorname{diam}(E)^n$. □

**Lemma 3.13.** *Let $E \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$, $r > 0$, and for $w \in S^{n-1}(0,1)$ define $\gamma_w(t) = x + tw$, $r/2 \le t \le r$. For $N \in \mathbb{N}$, let $F_N$ be the set of $w \in S^{n-1}(0,1)$ such that $E \cap |\gamma_w|$ contains at least $N$ points. Then*

$$\mathscr{H}^{n-1}(F_1) \le c(n)r^{-n+1}\min\{r, \operatorname{diam}(E)\}^{n-1} \quad \textit{and}$$
$$\mathscr{H}^{n-1}(F_N) \le c(n)r^{-n+1}N^{-1}\mathscr{H}^{n-1}(E).$$

*Proof.* For the first inequality, note that $F_1 + x$ is equal to the radial projection of $E \cap \{y \in \mathbb{R}^n : r/2 \le |x-y| \le r\}$ to the sphere $S^{n-1}(x,1)$. This projection is the composition of a uniformly Lipschitz map (projection of $\{y \in \mathbb{R}^n : r/2 \le |x-y| \le r\}$ to $S^{n-1}(x,r)$) with a scaling by $1/r$. Thus,

$$\operatorname{diam}(F_1) \lesssim r^{-1}\operatorname{diam}(E \cap \{y \in \mathbb{R}^n : r/2 \le |x-y| \le r\}) \lesssim r^{-1}\min\{r, \operatorname{diam}(E)\}.$$

Moreover, $F_1$ is contained in the intersection of a ball $B_0 = \overline{B}(x_0, \operatorname{diam}(F_1))$, where $x_0 \in F_1$, with $S^{n-1}(0,1)$. Thus,

$$\mathscr{H}^{n-1}(F_1) \le \mathscr{H}^{n-1}(B_0 \cap S^{n-1}(0,1)) \simeq_n \operatorname{diam}(F_1)^{n-1}.$$

This completes the proof of the first inequality.

For the second inequality, we proceed as in the proof of Lemma 3.12, by defining $F_{N,k}$ to be the set of $w \in S^{n-1}(0,1)$ such that $E \cap |\gamma_w|$ contains $N$ points with mutual distances bounded below by $1/k$. We define the function $\rho$ exactly as in Lemma 3.12, using an arbitrary cover of $E$ by sets $U_i$ and corresponding balls $B_i \supset U_i$ with $r_i = \operatorname{diam}(U_i) < \frac{1}{2k} < \frac{r}{8}$. Note that if $w \in F_{N,k}$, then

$$\int_{\gamma_w} \rho\, ds = \int_{r/2}^{r} \rho(x + tw)\, ds \ge 1.$$

By Chebychev's inequality and polar integration, it follows that

$$\begin{aligned}\mathscr{H}^{n-1}(F_{N,k}) &\le \int_{S^{n-1}(0,1)}\int_{r/2}^{r}\rho(x+tw)\,dt d\mathscr{H}^{n-1}(w)\\ &\lesssim_n r^{-n+1}\|\rho\|_{L^1(\mathbb{R}^n)} \simeq_n r^{-n+1}N^{-1}\sum_{i\in I}\operatorname{diam}(U_i)^{n-1}.\end{aligned}$$

We now proceed as before, infimizing over the covers of $E$ and letting $k \to \infty$. □

*Proof of Theorem 3.11.* By Remark 3.2, (P3) and (P4) are automatically satisfied for $\mathcal{F}_0(E)$ and $\mathcal{F}_\sigma(E)$. We will establish below properties (P1) and (P2).

Suppose first that $\mathscr{H}^{n-1}(E)=0$ as in (i). If $\gamma$ is a non-constant rectifiable path, by the second inequality of Lemma 3.12 (for $N=1$) we have that $E\cap|\gamma+x|=\emptyset$, i.e., $\gamma+x\in\mathcal{F}_0(E)$, for a.e. $x\in\mathbb{R}^n$. Hence, (P1) holds.

Next, if $x\in\mathbb{R}^n$, $r>0$, and $w\in S^{n-1}(0,1)$, define $\gamma_w(t)=x+tw$, $0\leq t\leq r$. By applying Lemma 3.13 countably many times (for $N=1$) to the paths $\gamma_w|_{[2^{-k}r,2^{-k+1}r]}$, we have $E\cap\gamma_w([2^{-k}r,2^{-k+1}r])=\emptyset$ for all $k\in\mathbb{N}$ and for a.e. $w\in S^{n-1}(0,1)$. Hence, $E\cap\gamma_w((0,r])=\emptyset$ for a.e. $w\in S^{n-1}(0,1)$. Recall that a path in $\mathcal{F}_0(E)$, by definition, is allowed to intersect $E$ only at the endpoints. Hence, $\gamma_w\in\mathcal{F}_0(E)$ for a.e. $w\in S^{n-1}(0,1)$. This completes the proof of (P2) and of (i).

Next, we suppose that $E$ has $\sigma$-finite Hausdorff $(n-1)$ measure, as in (ii). We write $E=\bigcup_{k=1}^\infty E_k$, where $\mathscr{H}^{n-1}(E_k)<\infty$ for each $k\in\mathbb{N}$. If we show that $\mathcal{F}_\sigma(E_k)$ is a $P$-family for each $k\in\mathbb{N}$, then $\mathcal{F}_\sigma(E)$ will also be a $P$-family, since the intersection of countably many $P$-families is a $P$-family by Lemma 3.3. Hence, for (ii) it suffices to assume that $\mathscr{H}^{n-1}(E)<\infty$.

Let $\gamma$ be a non-constant rectifiable path. We define $F$ to be the set of $x\in\mathbb{R}^n$ such that $E\cap|\gamma+x|$ is infinite and consider the set $F_N$ as in Lemma 3.12. Observe that $F=\bigcap_{N=1}^\infty F_N$. Since

$$m_n^*(F_N)\lesssim_n \ell(\gamma)N^{-1}\mathscr{H}^{n-1}(E),$$

by letting $N\to\infty$ we obtain $m_n(F)=0$. This proves (P1).

For (P2) we fix $x\in\mathbb{R}^n$, $r>0$, and for $w\in S^{n-1}(0,1)$ consider the segment $\gamma_w(t)=x+tw$, $0\leq t\leq r$, as above. For fixed $k\in\mathbb{N}$ we apply Lemma 3.13 to the paths $\gamma_w|_{[2^{-k}r,2^{-k+1}r]}$ and conclude (by letting $N\to\infty$) that the set $E\cap\gamma_w([2^{-k}r,2^{-k+1}r])$ is finite for a.e. $w\in S^{n-1}(0,1)$. Hence, for a.e. $w\in S^{n-1}(0,1)$ the set $E\cap|\gamma_w|$ is countable, i.e., $\gamma_w\in\mathcal{F}_\sigma(E)$. □

## 4. Criteria for negligibility

In this section we prove criteria for the membership of a set $E$ in *NED* or *CNED*. The first of these criteria is crucially used in the proof of Theorem 1.2 regarding the unions of *NED* and *CNED* sets. Recall that $*$*NED* denotes either *NED* or *CNED*. Also, recall from Section 2.4 the definitions of $*\mathit{NED}(\Omega)$ and $*\mathit{NED}^w(\Omega)$, and the definition of the relative distance $\Delta(F_1,F_2)$ of two sets $F_1,F_2\subset\mathbb{R}^n$. Let $\gamma\colon[a,b]\to\mathbb{R}^n$ be a non-constant path. If $[c,d]\subset(a,b)$, then the strong subpath $\gamma|_{[c,d]}$ of $\gamma$ is called *strict*.

**Theorem 4.1** (Main criterion)**.** *Let $E\subset\mathbb{R}^n$ be a set such that either $E$ is closed or $m_n(\overline{E})=0$. The following are equivalent.*

(I) *$E\in *\mathit{NED}(\Omega)$ for all open sets $\Omega\subset\mathbb{R}^n$.*

(II) *$E\in *\mathit{NED}$.*

(III) *$E\in *\mathit{NED}^w(\Omega)$ for some open set $\Omega\subset\mathbb{R}^n$ with $\Omega\supset\overline{E}$.*

(IV) *There exist constants $t,\phi>0$ such that for each $x_0\in\mathbb{R}^n$ there exists $r_0>0$ with the property that for every pair of non-degenerate, disjoint continua $F_1,F_2\subset B(x_0,r_0)$ with $\Delta(F_1,F_2)\leq t$ we have*

$$\mathrm{Mod}_n(\Gamma(F_1,F_2;\mathbb{R}^n)\cap\mathcal{F}_*(E))\geq\phi.$$

(V) *For each Borel function $\rho\colon\mathbb{R}^n\to[0,\infty]$ with $\rho\in L^n_{\mathrm{loc}}(\mathbb{R}^n)$ there exists a path family $\Gamma_0$ with $\mathrm{Mod}_n\Gamma_0=0$ such that Conclusion A below holds for* *each rectifiable path $\gamma \notin \Gamma_0$ with distinct endpoints. Moreover, $\Gamma_0$ has the property that if $\{\eta_j\}_{j\in J}$ is a finite collection of paths outside $\Gamma_0$ and $\gamma$ is a path with $|\gamma| \subset \bigcup_{j\in J} |\eta_j|$, then $\gamma \notin \Gamma_0$.*

(VI) *For each Borel function $\rho\colon \mathbb{R}^n \to [0,\infty]$ with $\rho \in L^n_{\mathrm{loc}}(\mathbb{R}^n)$ there exists a path family $\Gamma_0$ with $\operatorname{Mod}_n \Gamma_0 = 0$ such that Conclusion B below holds for each rectifiable path $\gamma \notin \Gamma_0$ with distinct endpoints.*

*Moreover, the following implications are true for all sets $E \subset \mathbb{R}^n$.*

$$\text{(V)} \Rightarrow \text{(VI)} \Rightarrow \text{(I)} \Rightarrow \text{(II)} \Rightarrow \text{(III)} \Rightarrow \text{(IV)}$$

**Conclusion A** ($\mathrm{A}(E,\rho,\gamma)$)**.** For each open neighborhood $U$ of $|\gamma|\setminus\partial\gamma$ and for each $\varepsilon > 0$ there exists a collection of paths $\{\gamma_i\}_{i\in I}$ and a simple path $\widetilde{\gamma}$ such that

(A-i) $\widetilde{\gamma} \in \mathcal{F}_*(E)$,

(A-ii) $\partial\widetilde{\gamma} = \partial\gamma$, $|\widetilde{\gamma}| \setminus \partial\gamma \subset (|\gamma| \setminus E) \cup \displaystyle\bigcup_{i\in I} |\gamma_i|$, and $\bigcup_{i\in I} |\gamma_i| \subset U$,

(A-iii) $\displaystyle\sum_{i\in I} \ell(\gamma_i) < \varepsilon$, and

(A-iv) $\displaystyle\sum_{i\in I} \int_{\gamma_i} \rho\, ds < \varepsilon$.

The paths $\gamma_i$, $i \in I$, may be taken to lie outside a given path family $\Gamma'$ with $\operatorname{Mod}_n \Gamma' = 0$. If $\overline{E} \cap \partial\gamma = \emptyset$, then $I$ may be taken to be finite. In general, the trace of each strict subpath of $\widetilde{\gamma}$ intersects finitely many traces $|\gamma_i|$, $i \in I$.

**Conclusion B** ($\mathrm{B}(E,\rho,\gamma)$)**.** For each open neighborhood $U$ of $|\gamma|$ and for each $\varepsilon > 0$ there exists a simple path $\widetilde{\gamma}$ such that

(B-i) $\widetilde{\gamma} \in \mathcal{F}_*(E)$,

(B-ii) $\partial\widetilde{\gamma} = \partial\gamma$ and $|\widetilde{\gamma}| \subset U$,

(B-iii) $\ell(\widetilde{\gamma}) \le \ell(\gamma) + \varepsilon$, and

(B-iv) $\displaystyle\int_{\widetilde{\gamma}} \rho\, ds \le \int_{\gamma} \rho\, ds + \varepsilon$.

Note that the implications (I) $\Rightarrow$ (II) $\Rightarrow$ (III) are trivial. Moreover, (III) $\Rightarrow$ (IV) follows immediately from Lemma 2.5. Conclusion B in (VI) is only a less technical statement that follows easily from Conclusion A in (V). Indeed, (B-iii) and (B-iv) follow from (A-ii), (A-iii), and (A-iv), using Lemma 2.2 (ii). Hence, we will show implications (IV) $\Rightarrow$ (V), which is the most technical one, and (VI) $\Rightarrow$ (I).

Roughly speaking, Conclusions A and B say that with small cost we can alter the path $\gamma$ to bring it inside the curve family $\mathcal{F}_*(E)$. The assumption that $E$ is closed or $m_n(\overline{E}) = 0$ will be crucially used in the proof of (IV) $\Rightarrow$ (V) (see Lemma 4.6) and it is doubtful whether this implication holds without that assumption.

An immediate corollary of Theorem 4.1 is the quasiconformal and bi-Lipschitz invariance of compact $*$*NED* sets.

**Corollary 4.2.** *Let $E \subset \mathbb{R}^n$ be a bounded set such that either $E$ is closed or $m_n(\overline{E}) = 0$. Let $\Omega \subset \mathbb{R}^n$ be an open set with $\overline{E} \subset \Omega$, and $f\colon \Omega \to \mathbb{R}^n$ be a quasiconformal embedding. If $E \in *NED$, then $f(E) \in *NED$.*

*Proof.* Under either assumption, we have $m_n(\overline{E}) = 0$ by Lemma 2.5. Observe that $f(\overline{E}) = \overline{f(E)}$ and that this is a compact subset of $f(\Omega)$ having $n$-measure zero by quasiconformality. Since $f$ distorts the $n$-modulus of each curve family in $\Omega$ by a bounded multiplicative factor, we see that $f(E) \in *NED^w(f(\Omega))$. By Theorem 4.1, we conclude that $f(E) \in *NED$. $\square$

We also prove an alternative criterion in terms of $P$-families; recall the definition of a $P$-family from Section 3. The result is also true for the case of *NED* sets but we do not prove it here to avoid some technicalities.

**Theorem 4.3** ($P$-family criterion)**.** *Let $E \subset \mathbb{R}^n$ be a set such that either $E$ is closed or $m_n(\overline{E}) = 0$. The following are equivalent.*

(I) $E \in CNED$.

(II) *For each Borel function $\rho\colon \mathbb{R}^n \to [0,\infty]$ with $\rho \in L^n(\mathbb{R}^n)$ the following statements are true.*

  (II-1) *For each rectifiable path $\gamma$, a.e. $x \in \mathbb{R}^n$, and every strong subpath $\eta$ of $\gamma + x$ with distinct endpoints, Conclusion $B(E,\rho,\eta)$ is true.*

  (II-2) *For $x \in \mathbb{R}^n$, $0 < r < R$, and $w \in S^{n-1}(0,1)$ define $\gamma_w(t) = x + tw$, $t \in [r,R]$. Then for $\mathscr{H}^{n-1}$-a.e. $w \in S^{n-1}(0,1)$, and for every strong subpath $\eta$ of $\gamma_w$, Conclusion $B(E,\rho,\eta)$ is true.*

(III) *For each Borel function $\rho\colon \mathbb{R}^n \to [0,\infty]$ with $\rho \in L^n(\mathbb{R}^n)$ there exists a $P$-family $\mathcal{F}$ such that if $\gamma \in \mathcal{F}$ is a rectifiable path, then Conclusion $B(E,\rho,\eta)$ is true for each strong subpath $\eta$ of $\gamma$ with distinct endpoints.*

Ahlfors–Beurling [AB50, Theorem 10] proved that if a closed set $E \subset \mathbb{R}^n$ is *NED* then any two points in $\mathbb{R}^n \setminus E$ can be joined by a curve in $\mathbb{R}^n \setminus E$ of short length. The analogous statement is true for closed *CNED* sets.

**Corollary 4.4.** *Let $E \subset \mathbb{R}^n$ be a closed set with $E \in CNED$. Then for every $\varepsilon > 0$ and points $x, y \in \mathbb{R}^n$ there exists a path $\gamma \in \mathcal{F}_\sigma(E)$ connecting $x$ and $y$ with $\ell(\gamma) \leq |x-y| + \varepsilon$.*

*Proof.* Let $x, y \in \mathbb{R}^n$ be distinct points, $\gamma$ be the line segment $[x,y]$, and $\varepsilon > 0$. Let $\rho \equiv 0$ and consider the $P$-family $\mathcal{F}$ given by Theorem 4.3 (III). By property (P1) of the $P$-family $\mathcal{F}$, for a.e. $z \in \mathbb{R}^n$ the path $\gamma + z$ lies in $\mathcal{F}$. Using spherical coordiantes we see that for a.e. $r > 0$ and for $\mathscr{H}^{n-1}$-a.e. $w \in S^{n-1}(0,1)$ the above is true for $z = rw$. We fix $r < \varepsilon/5$ such that this is true. Using property (P2) of a $P$-family, for $\mathscr{H}^{n-1}$-a.e. $w \in S^{n-1}(0,1)$ the radial segments $\gamma_w^x(t) = x + tw$ and $\gamma_w^y(t) = y + tw$, $0 \leq t \leq r$, lie in $\mathcal{F}$. Thus, there exists $w \in S^{n-1}(0,1)$ such that both radial segments lie in $\mathcal{F}$ and $\gamma + rw \in \mathcal{F}$. We now apply Conclusion B (B-iii) (with $\varepsilon = r$) to each of $\gamma_w^x$, $\gamma_w^y$, and $\gamma + rw$, to conclude that there exist paths $\eta_x, \eta_y, \eta \in \mathcal{F}_\sigma(E)$ with the same endpoints as $\gamma_w^x, \gamma_w^y, \gamma + rw$, respectively, such that $\ell(\eta_x) \leq 2r$, $\ell(\eta_y) \leq 2r$, and $\ell(\gamma + rw) \leq |x-y| + r$. Concatenating these paths gives a path in $\mathcal{F}_\sigma(E)$ connecting $x$ and $y$ with length bounded by $|x-y| + 5r < |x-y| + \varepsilon$. $\square$

4.1. **Auxiliary results.** We will need some auxiliary results before we prove Theorems 4.1 and 4.3. The following lemma is elementary.

**Lemma 4.5.** *Let $E \subset \mathbb{R}^n$ be a compact set with $m_n(E) = 0$, $g\colon \mathbb{R}^n \to [0,\infty]$ be a function in $L^1(\mathbb{R}^n)$, and $\lambda > 0$. For each $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < r < \delta$ and $B_i = B(x_i, r)$, $i \in \{1,\dots,N\}$, is a family of pairwise disjoint balls* *centered at $E$, then*

$$\sum_{i=1}^{N} \int_{\lambda B_i} g < \varepsilon.$$

*Proof.* Let $\lambda, \varepsilon > 0$. Since $E$ is compact with $m_n(E) = 0$ and $g \in L^1(\mathbb{R}^n)$, there exists $\delta > 0$ such that

$$\int_{N_{\lambda\delta}(E)} g < C(n,\lambda)^{-1}\varepsilon \tag{4.1}$$

for a constant $C(n,\lambda) > 0$ to be determined. Let $0 < r < \delta$ and consider a family of finitely many disjoint balls $B_i = B(x_i, r)$, $i \in \{1, \dots, N\}$, centered at $E$. Suppose that $\lambda > 1$. If $x \in \lambda B_i$, then $B_i \subset \lambda B_i \subset B(x, 2\lambda r)$. Since the balls $B_i$, $i \in \{1, \dots, N\}$, are disjoint, by volume comparison we see that

$$\sum_{i=1}^{N} \chi_{\lambda B_i} \leq 2^n \lambda^n \chi_{\bigcup_{i=1}^N \lambda B_i} \leq 2^n \lambda^n \chi_{N_{\lambda r}(E)}.$$

The same inequality is trivially true when $0 < \lambda \leq 1$ with constant 1 in place of $2^n\lambda^n$. We now set $C(n,\lambda) = \max\{1, 2^n\lambda^n\}$ and by (4.1) we obtain

$$\sum_{i=1}^{N} \int_{\lambda B_i} g = \int g \sum_{i=1}^{N} \chi_{\lambda B_i} \leq C(n,\lambda) \int_{N_{\lambda r}(E)} g < \varepsilon. \qquad \square$$

**Lemma 4.6.** *Let $E \subset \mathbb{R}^n$ be a closed set with $m_n(E) = 0$. Then for each non-negative function $\rho \in L^n_{\mathrm{loc}}(\mathbb{R}^n)$ and for each $\lambda > 0$*

(i) *there exists a path family $\Gamma_0$ with $\mathrm{Mod}_n \Gamma_0 = 0$ and*
(ii) *for each $m \in \mathbb{N}$ there exists a family of balls $\{B_{i,m} = B(x_{i,m}, r_{i,m})\}_{i \in I_m}$ covering $E$ with $r_{i,m} < 1/m$, $i \in I_m$,*

*such that for every non-constant curve $\gamma \notin \Gamma_0$ we have*

$$\begin{aligned} &\lim_{m\to\infty} \sum_{i: B_{i,m} \cap |\gamma| \neq \emptyset} r_{i,m} \left( \fint_{\lambda B_{i,m}} \rho^n \right)^{1/n} = 0 \quad \textit{and} \\ &\lim_{m\to\infty} \sum_{i: B_{i,m} \cap |\gamma| \neq \emptyset} r_{i,m} = 0. \end{aligned} \tag{4.2}$$

*Proof.* First, we reduce to the case that $E$ is compact. Suppose that the statement is true for compact sets. For $k \in \mathbb{N}$, let $A_k = \{x \in \mathbb{R}^n : k-1 \leq |x| \leq k\}$. Then for each $k \in \mathbb{N}$, there exists a curve family $\Gamma_k$ with $\mathrm{Mod}_n \Gamma_k = 0$ and for each $m \in \mathbb{N}$ there exists a family of balls $\{B_{i,m}\}_{i \in I_{m,k}}$ covering $E \cap A_k$, with radii less than $1/m$, so that (4.2) is true for non-constant paths $\gamma \notin \Gamma_k$. We discard the balls not intersecting $E \cap A_k$, if any. Let $I_m = \bigcup_{k\in\mathbb{N}} I_{m,k}$ and $\Gamma_0 = \bigcup_{k \in \mathbb{N}} \Gamma_k$. Note that $\mathrm{Mod}_n \Gamma_0 = 0$ by the subadditivity of modulus. We claim that (4.2) is true for the balls $\{B_{i,m}\}_{i\in I_m}$.

If $\gamma$ is a non-constant path outside $\Gamma_0$, then $\gamma$ is contained in the union of finitely many sets $A_k$, $k \in \mathbb{N}$. Moreover, there exists $k_0 \in \mathbb{N}$ such that $B_{i,m} \cap |\gamma| = \emptyset$ for all $i \in I_{m,k}$, $m \in \mathbb{N}$, and $k > k_0$. Thus, in (4.2), the sums over the indices $i \in I_{m,k}$ such that $B_{i,m} \cap |\gamma| \neq \emptyset$ are zero for all $k > k_0$. For $k \leq k_0$, the limits of the corresponding sums vanish, since $\gamma \notin \Gamma_k$. Since limits commute with finite sums, we obtain (4.2) for the family of balls $\{B_{i,m}\}_{i\in I_m}$.

Suppose now that $E$ is compact. Let $g \in L^n(\mathbb{R}^n)$ be a non-negative function, to be specified later. By the $5B$-covering lemma ([Hei01, Theorem 1.2]) for each $r>0$ we can find a cover of $E$ by finitely many balls of radius $r$ centered at $E$ so that the balls with the same centers and radius $r/5$ are disjoint. Combining this with Lemma 4.5 (for $5\lambda$ in place of $\lambda$), for each $m \in \mathbb{N}$ we may find a cover of $E$ by balls $B_{i,m} = B(x_{i,m}, r_m)$, $i \in I_m$, centered at $E$, such that $r_m < 1/m$, the balls $\frac{1}{5}B_{i,m}$ are disjoint, and

$$\sum_{i\in I_m} \int_{\lambda B_{i,m}} g^n < \frac{1}{2^m}.$$

For $m \in \mathbb{N}$, we define the Borel function

$$h_m = \sum_{i\in I_m} \left( \fint_{\lambda B_{i,m}} g^n \right)^{1/n} \chi_{2B_{i,m}}.$$

By Lemma 2.1, we have

$$\begin{aligned}\sum_{m\in\mathbb{N}} \|h_m\|_{L^n(\mathbb{R}^n)} &\lesssim_n \sum_{m\in\mathbb{N}} \left\| \sum_{i\in I_m} \left( \fint_{\lambda B_{i,m}} g^n \right)^{1/n} \chi_{\frac{1}{5}B_{i,m}} \right\|_{L^n(\mathbb{R}^n)} \\ &\lesssim_{n,\lambda} \sum_{m\in\mathbb{N}} \left( \sum_{i\in I_m} \int_{\lambda B_{i,m}} g^n \right)^{1/n} \lesssim_{n,\lambda} \sum_{m\in\mathbb{N}} \frac{1}{2^{m/n}} < \infty.\end{aligned}$$

By a variant of Fuglede's lemma [Väi71, Theorem 28.1], there exists a curve family $\Gamma_0$ with $\operatorname{Mod}_n \Gamma_0 = 0$ such that for each path $\gamma \notin \Gamma_0$ we have

$$\lim_{m\to\infty} \int_\gamma h_m \, ds = 0.$$

Implicitly we assume that $\Gamma_0$ contains all curves that are not rectifiable.

Note that if $\gamma$ is a non-constant rectifiable curve, $B_{i,m} \cap |\gamma| \neq \emptyset$, and $m$ is sufficiently large so that $\operatorname{diam}(|\gamma|) > 4m^{-1} > 4r_m$, then $|\gamma|$ is not contained in $2B_{i,m}$, so

$$\int_\gamma \chi_{2B_{i,m}} \, ds \geq r_m.$$

Thus,

$$\int_\gamma h_m \, ds \geq \sum_{i: B_{i,m}\cap|\gamma|\neq\emptyset} r_m \left( \fint_{\lambda B_{i,m}} g^n \right)^{1/n}.$$

We conclude that if $\gamma$ is a non-constant curve outside $\Gamma_0$, then

$$\lim_{m\to\infty} \sum_{i: B_{i,m}\cap|\gamma|\neq\emptyset} r_m \left( \fint_{\lambda B_{i,m}} g^n \right)^{1/n} = 0. \tag{4.3}$$

We finally set $g = (\rho+1)\chi_{B(0,R)}$ in the above manipulations, where $\rho$ is the given function from the statement and $B(0,R)$ is a large ball containing the compact set $E$. Note that for all large $m \in \mathbb{N}$ we have $\lambda B_{i,m} \subset B(0,R)$ for all $i \in I_m$. Then for every non-constant curve $\gamma \notin \Gamma_0$ we obtain (4.2) from (4.3). $\square$

**Lemma 4.7.** *Let $E \subset \mathbb{R}^n$ be a set, $\rho\colon \mathbb{R}^n \to [0,\infty]$ be a Borel function, and $\gamma$ be a rectifiable path with distinct endpoints such that $|\gamma| \cap \overline{E}$ is totally disconnected. If Conclusion $A(E,\rho,\eta)$ is true for each strict subpath $\eta$ of $\gamma$ with distinct endpoints that do not lie in the set $\overline{E}$, then Conclusion $A(E,\rho,\gamma)$ is also true. The above statement remains true for Conclusion B in place of Conclusion A.*

*Proof.* We only present the argument for Conclusion A since the argument for Conclusion B is similar and less technical. Let $U$ be an open neighborhood of $|\gamma| \setminus \partial\gamma$ and $\varepsilon > 0$. It suffices to prove that a strong subpath of $\gamma$ with the same endpoints satisfies Conclusion A. Consider a parametrization $\gamma\colon [a,b] \to \mathbb{R}^n$. Then there exists an open subinterval $J$ of $[a,b]$ such that $\gamma(J) \subset U$ and $\gamma_{\overline{J}}$ has the same endpoints as $\gamma$. Without loss of generality, we assume that $J=(0,1)$ and we denote the path $\gamma|_{[0,1]}$ by $\gamma$.

Suppose first that $\overline{E} \cap \partial\gamma = \emptyset$. Since $\overline{E}$ is closed, there exist paths $\gamma|_{[0,t_1]}$, $\gamma|_{[t_2,1]}$ that do not intersect $\overline{E}$, and a strict subpath $\eta = \gamma|_{[t_1,t_2]}$ with $\overline{E} \cap \partial\eta = \emptyset$. By assumption, there exists a simple path $\widetilde{\eta}$ with the same endpoints as $\eta$ and *finitely many* paths $\eta_i$, $i \in I$, inside $U$ as in Conclusion A$(E,\rho,\eta)$. Concatenating $\widetilde{\eta}$ with $\gamma|_{[0,t_1]}$ and $\gamma|_{[t_2,1]}$, and then passing to a simple weak subpath, gives the desired path $\widetilde{\gamma}$ that verifies Conclusion A$(E,\rho,\gamma)$.

Next, suppose that $\overline{E} \cap \partial\gamma \neq \emptyset$. Consider a sequence $a_j \in (0,1)$, $j \in \mathbb{Z}$, such that $a_{j-1} < a_j$ for each $j \in \mathbb{Z}$ and $\bigcup_{j\in\mathbb{Z}} [a_{j-1},a_j] = (0,1)$. Let $\gamma^j = \gamma|_{[a_{j-1},a_j]}$, which is a strict subpath of $\gamma$. Since $\gamma((0,1))$ is disjoint from $\{\gamma(0),\gamma(1)\}$, the points $a_j$ can be chosen so that $\gamma^j$ has distinct endpoints for $j \in \mathbb{Z}$. Since $|\gamma| \cap \overline{E}$ is totally disconnected, we may further choose the points $a_j$ so that $\gamma(a_j) \notin \overline{E}$ for each $j \in \mathbb{Z}$.

By assumption, the strict subpath $\gamma^j$ of $\gamma$ satisfies Conclusion A$(E,\rho,\gamma^j)$ for each $j \in \mathbb{Z}$. Thus, we obtain paths $\widetilde{\gamma}^j$ and $\gamma_i^j$, $i \in \{1,\dots,N_j\}$, as in the conclusion, for $\varepsilon 2^{-|j|}$ in place of $\varepsilon$, and such that $\bigcup_{i=1}^{N_j} |\gamma_i^j| \subset U$. In particular, by (A-i) we have $\widetilde{\gamma}^j \in \mathcal{F}_*(E)$. If $\mathcal{F}_*(E) = \mathcal{F}_0(E)$, since the endpoints of $\widetilde{\gamma}^j$ do not lie in $E$, we have $|\widetilde{\gamma}^j| \cap E = \emptyset$. By discarding some paths $\gamma_i^j$, we assume that $|\gamma_i^j|$ intersects $|\widetilde{\gamma}^j|$ for all $i \in \{1,\dots,N_j\}$.

We consider parametrizations $\widetilde{\gamma}^j : [a_{j-1},a_j] \to U$ with $\widetilde{\gamma}^j|_{\{a_{j-1},a_j\}} = \gamma^j|_{\{a_{j-1},a_j\}}$, $j \in \mathbb{Z}$. Then we create a curve $\widetilde{\gamma}\colon [0,1] \to \overline{U}$ such that $\widetilde{\gamma}((0,1)) \subset U$, by concatenating these paths. Namely, we define $\widetilde{\gamma}(0) = \gamma(0)$, $\widetilde{\gamma}(1) = \gamma(1)$, and $\widetilde{\gamma}|_{[a_{j-1},a_j]} = \widetilde{\gamma}^j$ for $j \in \mathbb{Z}$. Note that $\ell(\widetilde{\gamma}^j) \le \ell(\gamma^j) + \varepsilon 2^{-|j|}$, which follows from Conclusion A. We conclude that $\operatorname{diam}(|\widetilde{\gamma}^j|) \to 0$, as $|j| \to \infty$, so $\widetilde{\gamma}$ is continuous. By passing to a weak subpath that has the same endpoints, we assume that $\widetilde{\gamma}$ is simple.

Property (A-i) is immediate for $\widetilde{\gamma}$. Property (A-ii) also holds if $\{\gamma_i\}_{i\in I}$ is the family $\{\gamma_i^j\}_{j\in\mathbb{Z}, i\in\{1,\dots,N_j\}}$. Indeed, all these paths are contained in $U$, and by the properties of the paths $\widetilde{\gamma}^j$ we have

$$|\widetilde{\gamma}| \setminus \partial\widetilde{\gamma} \subset \bigcup_{j\in\mathbb{Z}} |\widetilde{\gamma}^j| \subset \bigcup_{j\in\mathbb{Z}} \left( (|\gamma^j| \setminus E) \cup \bigcup_{i=1}^{N_j} |\gamma_i^j| \right) \subset (|\gamma| \setminus E) \cup \bigcup_{j\in\mathbb{Z}} \bigcup_{i=1}^{N_j} |\gamma_i^j|.$$

Finally, we have

$$\sum_{j\in\mathbb{Z}} \sum_{i=1}^{N_j} \ell(\gamma_i^j) \le \sum_{j\in\mathbb{Z}} \varepsilon 2^{-|j|} = 3\varepsilon \quad \text{and} \quad \sum_{j\in\mathbb{Z}} \sum_{i=1}^{N_j} \int_{\gamma_i^j} \rho\, ds \le 3\varepsilon.$$

Thus (A-iii) and (A-iv) hold as well.

Now, we verify the last part of Conclusion A. The paths $\gamma_i^j$ may be taken to lie outside a given curve family $\Gamma'$ with $\operatorname{Mod}_n \Gamma' = 0$, since they were obtained via Conclusion A. If $\eta$ is a strict subpath of $\widetilde{\gamma}$, then $|\eta|$ intersects $\bigcup_{i=1}^{N_j} |\gamma_i^j|$ for finitely many $j \in \mathbb{Z}$. Indeed, as $|j| \to \infty$, the paths $\{\gamma_i^j\}_{i\in\{1,\dots,N_j\}}$ have arbitrarily small lengths by the above, and their traces intersect $|\widetilde{\gamma}^j|$, which is contained in arbitrarily small neighborhoods of the endpoints of $\widetilde{\gamma}$. Since $|\eta|$ has positive distance from the endpoints of $\widetilde{\gamma}$, we conclude that $|\gamma_i^j|$ cannot intersect $|\eta|$ if $|j|$ is sufficiently large. This completes the proof. $\square$

4.2. **Proof of the main criterion.** As we have discussed, it suffices to show implication (VI) $\Rightarrow$ (I) for all sets $E$, and implication (IV) $\Rightarrow$ (V), which is the most technical one, for sets $E$ that are closed or whose closure has measure zero.

*Proof of Theorem 4.1:* (VI) $\Rightarrow$ (I). Let $\Omega \subset \mathbb{R}^n$ be an open set and $F_1, F_2 \subset \Omega$ be a pair of non-empty, disjoint continua. By the monotonicity of modulus, it suffices to show that

$$\operatorname{Mod}_n \Gamma(F_1, F_2; \Omega) \leq \operatorname{Mod}_n(\Gamma(F_1, F_2; \Omega) \cap \mathcal{F}_*(E)).$$

Let $\rho \in L^n(\mathbb{R}^n)$ be an admissible function for $\Gamma(F_1, F_2; \Omega) \cap \mathcal{F}_*(E)$. We consider an exceptional curve family $\Gamma_0$ with $\operatorname{Mod}_n \Gamma_0 = 0$ as in (VI). Let $\gamma \in \Gamma(F_1, F_2; \Omega) \setminus \Gamma_0$, so Conclusion B$(E, \rho, \gamma)$ is true. Thus, for any $\varepsilon > 0$ there exists a rectifiable path $\widetilde{\gamma} \in \Gamma(F_1, F_2; \Omega) \cap \mathcal{F}_*(E)$ with

$$1 \leq \int_{\widetilde{\gamma}} \rho \, ds \leq \int_{\gamma} \rho \, ds + \varepsilon.$$

Letting $\varepsilon \to 0$ shows that $\rho$ is admissible for $\Gamma(F_1, F_2; \Omega) \setminus \Gamma_0$. Since $\operatorname{Mod}_n \Gamma_0 = 0$, the proof is completed. $\square$

*Proof of Theorem 4.1:* (IV) $\Rightarrow$ (V). By the assumption (IV), which coincides with the assumption of Lemma 2.4, we have $m_n(E) = 0$. Therefore, under either of the initial assumptions of Theorem 4.1, we have $m_n(\overline{E}) = 0$. Let $\rho \colon \mathbb{R}^n \to [0, \infty]$ be a Borel function in $L^n_{\mathrm{loc}}(\mathbb{R}^n)$. By Lemma 4.6, for each $m \in \mathbb{N}$ there exists a family of balls $\{B_{i,m}\}_{i \in I_m}$ covering the set $\overline{E}$, with radii converging uniformly in $I_m$ to 0 as $m \to \infty$, and such that (4.2) is true for all paths outside an exceptional family with $n$-modulus zero and for a value of $\lambda > 0$ to be specified. Let $\Gamma_0$ be the exceptional family of paths $\gamma$ that either do not satisfy (4.2), or $\mathscr{H}^1(|\gamma| \cap \overline{E}) > 0$. By Lemma 4.6 and property (M6), $\operatorname{Mod}_n \Gamma_0 = 0$. Note that the path family $\Gamma_0$ has the required property for (V), that if $\{\eta_j\}_{j \in J}$ is a finite collection of paths outside $\Gamma_0$ and $\gamma$ is a path with $|\gamma| \subset \bigcup_{j \in J} |\eta_j|$, then $\gamma \notin \Gamma_0$.

We will show that Conclusion A$(E, \rho, \gamma)$ holds for all paths $\gamma \notin \Gamma_0$ with distinct endpoints. For $\gamma \notin \Gamma_0$, the set $|\gamma| \cap \overline{E}$ is totally disconnected. In view of Lemma 4.7, it suffices to show that Conclusion A$(E, \rho, \eta)$ is true for each subpath $\eta$ of $\gamma$ with distinct endpoints not lying in the closed set $\overline{E}$. By the defining properties of the curve family $\Gamma_0$, all subpaths of $\gamma$ also lie outside $\Gamma_0$. Therefore, it suffices to show that Conclusion A$(E, \rho, \gamma)$ is true for each path $\gamma \notin \Gamma_0$ with distinct endpoints *not lying in* $\overline{E}$.

Let $\gamma \notin \Gamma_0$ be a path with distinct endpoints not lying in $\overline{E}$. As a final reduction, we consider a simple weak subpath $\eta$ of $\gamma$ that has the same endpoints. Note that $\eta \notin \Gamma_0$ by the defining properties of $\Gamma_0$ and that is suffices to show Conclusion A$(E,\rho,\eta)$, which implies Conclusion A$(E,\rho,\gamma)$. Hence, we may impose the additional restriction that $\gamma$ is *simple*.

We fix a simple path $\gamma \notin \Gamma_0$ with distinct endpoints not lying in $\overline{E}$, $\varepsilon>0$, and an open neighborhood $U$ of $|\gamma|\setminus\partial\gamma$. We fix an injective parametrization $\gamma\colon[0,1]\to\mathbb{R}^n$ and note that $\gamma(0),\gamma(1)\notin\overline{E}$ and $\gamma((0,1))\subset U$.

By the compactness of $|\gamma|\cap\overline{E}$ and the assumption (IV), there exists a finite cover of $|\gamma|\cap\overline{E}$ by open balls $V_1,\dots,V_L$ such that for every $i\in\{1,\dots,L\}$ and for any non-degenerate, disjoint continua $F_1,F_2\subset 2V_i$,

$$\text{if } \Delta(F_1,F_2)\le t \text{ then } \operatorname{Mod}_n(\Gamma(F_1,F_2;\mathbb{R}^n)\cap\mathcal{F}_*(E))\ge\phi. \tag{4.4}$$

Observe that if a set $D$ intersects $|\gamma|\cap\overline{E}$ and has sufficiently small diameter, namely $\operatorname{diam}(D)\le\min\{2^{-1}\operatorname{diam}(V_i): i\in\{1,\dots,L\}\}$, then $D\subset 2V_i$ for some $i\in\{1,\dots,L\}$. Hence, (4.4) holds for any non-degenerate, disjoint continua $F_1,F_2\subset D$. We also fix $a>1$, depending on $t$, so that if $A=A(x;r,ar)$ is any ring and $F_1,F_2\subset\overline{A}$ are disjoint continua connecting the boundary components of $A$, then

$$\Delta(F_1,F_2)\le\frac{2}{a-1}\le t.$$

If $m\in\mathbb{N}$ is fixed and sufficiently large, by (4.2) we have

$$\sum_{i:B_{i,m}\cap|\gamma|\neq\emptyset} r_{i,m}\left(\fint_{\lambda B_{i,m}}\rho^n\right)^{1/n}<\varepsilon \quad\text{and}\quad \sum_{i:B_{i,m}\cap|\gamma|\neq\emptyset} r_{i,m}<\varepsilon. \tag{4.5}$$

By the compactness of $|\gamma|\cap\overline{E}$, there exists a finite subcollection $D_1,\dots,D_N$ of $\{B_{i,m}\}_{i\in I_m}$ covering the set $|\gamma|\cap\overline{E}$. We also assume that $D_i$ intersects $|\gamma|\cap\overline{E}$ for each $i\in\{1,\dots,N\}$ and we denote the radius of $D_i$ by $r_i$. Since the endpoints of $\gamma$ do not lie in $\overline{E}$, we have $|\gamma|\cap\overline{E}\subset U$. Thus,

$$\delta=\operatorname{dist}(|\gamma|\cap\overline{E},\partial\gamma\cup\partial U)>0.$$

If $m$ is sufficiently large so that $2(a+\lambda)r_i<\delta$ for each $i\in\{1,\dots,N\}$, we have

$$\bigcup_{i=1}^N(a+\lambda)D_i\subset U \quad\text{and}\quad \bigcup_{i=1}^N aD_i\cap\partial\gamma=\emptyset. \tag{4.6}$$

Finally, we choose an even larger $m$, so that (4.4) holds for any non-degenerate, disjoint continua $F_1,F_2\subset(a+1)D_i$, $i\in\{1,\dots,N\}$.

We set $\widetilde{\gamma}_0=\gamma$. We will define inductively simple paths $\widetilde{\gamma}_k$, $k\in\{0,\dots,N\}$, with the same endpoints as $\gamma$. Once $\widetilde{\gamma}_{k-1}$ has been defined for some $k\in\{1,\dots,N\}$ and has the same endpoints as $\gamma$, we define $\widetilde{\gamma}_k$ as follows. If $D_k\cap|\widetilde{\gamma}_{k-1}|=\emptyset$, then we set $\gamma_k=\emptyset$ (i.e., the empty path) and $\widetilde{\gamma}_k=\widetilde{\gamma}_{k-1}$. Suppose $D_k\cap|\widetilde{\gamma}_{k-1}|\neq\emptyset$. Consider an injective parametrization $\widetilde{\gamma}_{k-1}\colon[0,1]\to\mathbb{R}^n$. By (4.6) the endpoints of $\widetilde{\gamma}_{k-1}$ do not lie in $aD_k$, thus there exist two moments $0<s_1<s_2<1$ such that $\widetilde{\gamma}_{k-1}(s_1),\widetilde{\gamma}_{k-1}(s_2)\in\partial D_k$ and $\widetilde{\gamma}_{k-1}([0,1]\setminus(s_1,s_2))\cap D_k=\emptyset$. Moreover, there exist moments $s_1'<s_1$ and $s_2'>s_2$ such that $\widetilde{\gamma}_{k-1}(s_1'),\widetilde{\gamma}_{k-1}(s_2')\in\partial(aD_k)$, $G_1=\widetilde{\gamma}_{k-1}([s_1',s_1])\subset\overline{aD_k}\setminus D_k$ and $G_2=\widetilde{\gamma}_{k-1}([s_2,s_2'])\subset\overline{aD_k}\setminus D_k$; see Figure 2. Note that $G_1$ and $G_2$ are disjoint since the path $\widetilde{\gamma}_{k-1}$ is simple, and that they connect the boundary components of the ring $aD_k\setminus\overline{D_k}$. Since $\Delta(G_1,G_2)\le t$ (by

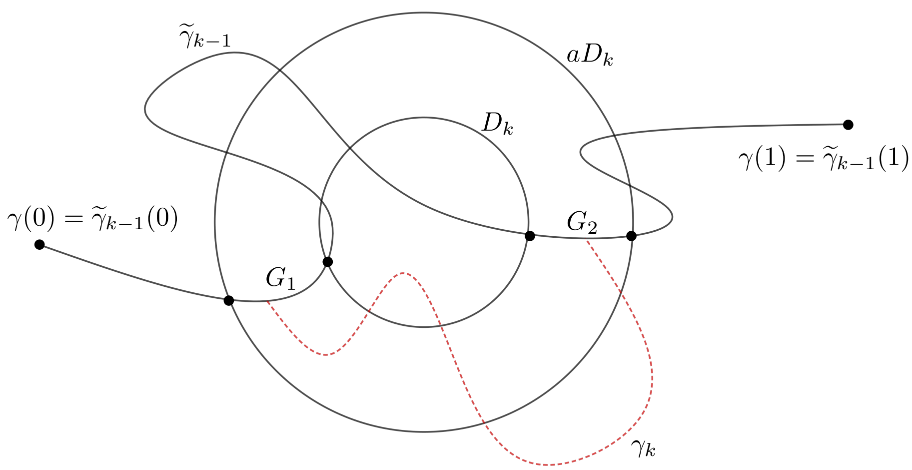


FIGURE 2. The construction of $\widetilde{\gamma}_k$ from $\widetilde{\gamma}_{k-1}$.

the choice of $a$) and $G_1, G_2 \subset (a+1)D_k$, by (4.4) we have

$$\mathrm{Mod}_n(\Gamma(G_1, G_2; \mathbb{R}^n) \cap \mathcal{F}_*(E)) \geq \phi.$$

Note that if $\Gamma'$ is a given path family with $\mathrm{Mod}_n \Gamma' = 0$ as in the end of Conclusion A, then we also have

$$\mathrm{Mod}_n(\Gamma(G_1, G_2; \mathbb{R}^n) \cap \mathcal{F}_*(E) \setminus \Gamma') \geq \phi.$$

Each path of $\Gamma(G_1, G_2; \mathbb{R}^n) \cap \mathcal{F}_*(E) \setminus \Gamma'$ intersects the ball $(a+1)D_k$. By Lemma 3.7 there exists a path $\gamma_k \in \Gamma(G_1, G_2; \mathbb{R}^n) \cap \mathcal{F}_*(E) \setminus \Gamma'$ such that

$$\int_{\gamma_k} \rho\, ds \lesssim_{n,\phi,a} r_k \left( \fint_{c(n,\phi,a)D_k} \rho^n \right)^{1/n} \quad \text{and} \quad \ell(\gamma_k) \leq c(n,\phi,a) r_k. \tag{4.7}$$

We now set $\lambda = c(n, \phi, a)$ and note that $|\gamma_k| \subset (a+\lambda)D_k \subset U$ by (4.6). Also, the endpoints of $\gamma_k$ lie in $|\widetilde{\gamma}_{k-1}|$. We concatenate $\gamma_k$ with suitable subpaths of $\widetilde{\gamma}_{k-1}$ that do not intersect $D_k$ to obtain a path $\widetilde{\gamma}_k$ that has the same endpoints as $\gamma$. If necessary, we replace $\widetilde{\gamma}_k$ with a simple weak subpath that has the same endpoints. By construction we have

$$|\widetilde{\gamma}_k| \subset (|\widetilde{\gamma}_{k-1}| \setminus D_k) \cup (|\gamma_k| \setminus \partial\gamma_k).$$

Inductively, we see that

$$|\widetilde{\gamma}_k| \subset \left( |\gamma| \setminus \bigcup_{i=1}^{k} D_i \right) \cup \bigcup_{i=1}^{k} (|\gamma_i| \setminus \partial\gamma_i). \tag{4.8}$$

For $k = N$ we obtain a simple path $\widetilde{\gamma} = \widetilde{\gamma}_N$ with the same endpoints as $\gamma$. Since $|\gamma| \cap \overline{E} \subset \bigcup_{i=1}^N D_i$, by (4.8) we have

$$|\widetilde{\gamma}| \subset (|\gamma| \setminus \overline{E}) \cup \bigcup_{i=1}^{N} (|\gamma_i| \setminus \partial\gamma_i) \quad \text{and} \quad |\widetilde{\gamma}| \cap E \subset \bigcup_{i=1}^{N} (|\gamma_i| \setminus \partial\gamma_i) \cap E. \tag{4.9}$$

If $\mathcal{F}_*(E) = \mathcal{F}_0(E)$, then $(|\gamma_i| \setminus \partial\gamma_i) \cap E = \emptyset$ since $\gamma_i \in \mathcal{F}_0(E)$ for each $i \in \{1, \dots, N\}$. Thus, $|\widetilde{\gamma}| \cap E = \emptyset$ and $\widetilde{\gamma} \in \mathcal{F}_0(E)$. If $\mathcal{F}_*(E) = \mathcal{F}_\sigma(E)$, then $|\widetilde{\gamma}_i| \cap E$ is countable for each $i \in \{1, \dots, N\}$, so $|\widetilde{\gamma}| \cap E$ is countable and $\widetilde{\gamma} \in \mathcal{F}_\sigma(E)$. Thus, (A-i) is satisfied. From (4.9) we obtain immediately

$$|\widetilde{\gamma}| \subset (|\gamma| \setminus E) \cup \bigcup_{i=1}^{N} |\gamma_i|.$$

By construction, we have $\bigcup_{i=1}^N |\gamma_i| \subset \bigcup_{i=1}^N (a+\lambda) D_i \subset U$. Thus, we have established property (A-ii). Finally, by (4.5) and (4.7) we have

$$\sum_{i=1}^{N} \ell(\gamma_i) \lesssim \sum_{i=1}^{N} r_i \lesssim \varepsilon \;\; \text{and} \;\; \sum_{i=1}^{N} \int_{\gamma_i} \rho\, ds \lesssim \sum_{i=1}^{N} r_i \left( \fint_{\lambda D_i} \rho^n \right)^{1/n} \lesssim \varepsilon.$$

These inequalities address (A-iii) and (A-iv). The last part of Conclusion A, asserting that the collection $\{\gamma_i\}_i$ is finite, whenever $\overline{E} \cap \partial\gamma = \emptyset$, is also true. □

### 4.3. **Proof of the $P$-family criterion.**

*Proof of Theorem 4.3:* (I) $\Rightarrow$ (II). Let $\rho$ be a non-negative Borel function with $\rho \in L^n(\mathbb{R}^n)$. By Theorem 4.1 (VI), there exists a path family $\Gamma_0$ with $\operatorname{Mod}_n \Gamma_0 = 0$ such that Conclusion B$(E, \rho, \gamma)$ is true for all paths $\gamma \notin \Gamma_0$ with distinct endpoints. Moreover, by enlarging $\Gamma_0$ while still requiring that $\operatorname{Mod}_n \Gamma_0 = 0$, we may assume that if $\gamma \notin \Gamma_0$, then all subpaths of $\gamma$ also have this property; see (M3). By Lemma 2.3, for each rectifiable path $\gamma$ and for a.e. $x \in \mathbb{R}^n$ we have $\gamma + x \notin \Gamma_0$; in particular, the same holds for all strong subpaths of $\gamma + x$, as required in part (II-1). Moreover, if $\gamma_w$ is as in (II-2), by the same lemma, for $\mathscr{H}^{n-1}$-a.e. $w \in S^{n-1}(0,1)$ the path $\gamma_w$ and all of its strong subpaths lie outside $\Gamma_0$. This completes the proof. □

*Proof of Theorem 4.3:* (III) $\Rightarrow$ (I). Let $\rho \in L^n(\mathbb{R}^n)$ be an admissible function for $\Gamma(F_1, F_2; \mathbb{R}^n) \cap \mathcal{F}_\sigma(E)$, where $F_1, F_2$ are disjoint continua. Consider the $P$-family $\mathcal{F}$ with the given properties; note that $\mathcal{F}$ depends on $\rho$. Let $\gamma \in \Gamma(F_1, F_2; \mathbb{R}^n) \cap \mathcal{F}$ be a rectifiable path. By the properties of $\mathcal{F}$, each strong subpath of $\gamma$ with distinct endpoints, and in particular $\gamma$, satisfies Conclusion B. Hence, for each $\varepsilon > 0$ there exists a rectifiable path $\widetilde{\gamma} \in \Gamma(F_1, F_2; \mathbb{R}^n) \cap \mathcal{F}_\sigma(E)$ such that

$$1 \leq \int_{\widetilde{\gamma}} \rho\, ds \leq \int_{\gamma} \rho\, ds + \varepsilon.$$

As $\varepsilon \to 0$, this shows that $\rho$ is admissible for $\Gamma(F_1, F_2; \mathbb{R}^n) \cap \mathcal{F}$. Hence,

$$\operatorname{Mod}_n(\Gamma(F_1, F_2; \mathbb{R}^n) \cap \mathcal{F}) \leq \int \rho^n.$$

Since $\mathcal{F}$ is a $P$-family, by Theorem 3.4, we have

$$\operatorname{Mod}_n \Gamma(F_1, F_2; \mathbb{R}^n) = \operatorname{Mod}_n(\Gamma(F_1, F_2; \mathbb{R}^n) \cap \mathcal{F}) \leq \int \rho^n.$$

Infimizing over $\rho$, gives $\operatorname{Mod}_n \Gamma(F_1,F_2;\mathbb{R}^n) \leq \operatorname{Mod}_n(\Gamma(F_1,F_2;\mathbb{R}^n)\cap \mathcal{F}_\sigma(E))$, so $E$ is *CNED*. $\square$

For the proof of the last implication (II) $\Rightarrow$ (III), it is crucial that $\mathcal{F}_\sigma(E)$ is closed under concatenations, a fact that is not true for $\mathcal{F}_0(E)$. Although the result is true for $\mathcal{F}_0(E)$, the proof is more involved and we only present the argument for $\mathcal{F}_\sigma(E)$.

*Proof of Theorem 4.3:* (II) $\Rightarrow$ (III). First, we show that $m_n(E)=0$ in the case that $E$ is closed. Fix an open ball $B$ and consider the Borel function $\rho=(1-\chi_E)\chi_B$, which lies in $L^n(\mathbb{R}^n)$. By (II-1), Conclusion $\mathrm{B}(E,\rho,\gamma)$ is true for a.e. line segment $\gamma$ parallel to a given coordinate direction. Let $\gamma$ be such a line segment that is contained in the ball $B$. For each $\varepsilon>0$ there exists a path $\widetilde{\gamma}\in \mathcal{F}_\sigma(E)$ contained in $B$, with the same endpoints as $\gamma$, and

$$\int_{\widetilde{\gamma}} \rho\, ds \leq \int_\gamma \rho\, ds +\varepsilon.$$

By Lemma 2.2 (iv) we have $\int_{\widetilde{\gamma}} \chi_E\, ds=0$ since $|\gamma|\cap E$ is countable. Thus,

$$\ell(\gamma)\leq \ell(\widetilde{\gamma}) =\int_{\widetilde{\gamma}} 1\, ds =\int_{\widetilde{\gamma}} \rho\, ds \leq \int_\gamma \rho\, ds+\varepsilon =\ell(\gamma)-\int_\gamma \chi_E\, ds+\varepsilon.$$

By letting $\varepsilon\to 0$, we obtain $\int_\gamma \chi_E\, ds=0$, so $\mathscr{H}^1(|\gamma|\cap E)=0$ by Lemma 2.2 (ii). This is true for a.e. line segment $\gamma$ in $B$ parallel to a coordinate direction, so $m_n(E\cap B)=0$ by Fubini's theorem. The ball $B$ was arbitrary, so $m_n(E)=0$.

Let $\rho$ be a non-negative Borel function in $L^n(\mathbb{R}^n)$ and let $\mathcal{F}$ be the family of rectifiable paths $\gamma$ such that Conclusion $\mathrm{B}(E,\rho,\eta)$ holds for each strong subpath $\eta$ of $\gamma$ with distinct endpoints. We show that $\mathcal{F}$ is a $P$-family.

First, we see that (P1) is satisfied. That is, if $\gamma\colon [a,b]\to\mathbb{R}^n$ is a non-constant rectifiable path, then $\gamma+x\in\mathcal{F}$ for a.e. $x\in\mathbb{R}^n$. This is true by the assumption (II-1). For (P2), we fix $x\in\mathbb{R}^n$ and $R>0$, and we have to show that for a.e. $w\in S^{n-1}(0,1)$ the radial segment $\gamma_w(t)=x+tw$, $t\in[0,R]$, lies in $\mathcal{F}$. By the assumption (II-2), for each $0<r<R$ and for a.e. $w\in S^{n-1}(0,1)$ the radial segment $\gamma_w|_{[r,R]}$ lies in $\mathcal{F}$. For $r_k=2^{-k}R$, $k\in\mathbb{N}$, we see that $\gamma_w|_{[r_k,R]}\in\mathcal{F}$ for a.e. $w\in S^{n-1}(0,1)$ and for all $k\in\mathbb{N}$. We fix $w\in S^{n-1}(0,1)$ such that $\mathscr{H}^1(|\gamma_w|\cap \overline{E})=0$ and $\gamma_w|_{[r_k,R]}\in\mathcal{F}$ for all $k\in\mathbb{N}$. Since $m_n(\overline{E})=0$, these statements hold for a.e. $w\in S^{n-1}(0,1)$. Our goal is to show that Conclusion $\mathrm{B}(E,\rho,\eta)$ is true for every strong subpath $\eta$ of $\gamma_w$; this will imply that $\gamma_w\in\mathcal{F}$, as desired. Every *strict* subpath of $\eta$ is a strong subpath of $\gamma_w|_{[r_k,R]}$ for some $k\in\mathbb{N}$. Since $\gamma_w|_{[r_k,R]}\in\mathcal{F}$, every strict subpath of $\eta$ satisfies Conclusion B. Since $|\eta|\cap\overline{E}$ is totally disconnected, we conclude from Lemma 4.7 that $\eta$ satisfies Conclusion B.

From the definition of $\mathcal{F}$ it is clear that (P3) is always satisfied. We finally have to prove (P4). Note that $\mathcal{F}_*(E)=\mathcal{F}_\sigma(E)$ in (B-i). Let $\gamma_1,\gamma_2$ be two paths in $\mathcal{F}$ that have a common endpoint and let $\gamma$ be their concatenation. Consider a strong subpath $\eta$ of $\gamma$ that has distinct endpoints. Then $\eta$ is either a strong subpath of $\gamma_1$ or $\gamma_2$, or $\eta$ is the concatenation of strong subpaths $\eta_1$ of $\gamma_1$ and $\eta_2$ of $\gamma_2$. In the latter case, which is the nontrivial one, since $\eta_1$ and $\eta_2$ satisfy Conclusion B, and in particular (B-i), there exists paths $\widetilde{\eta}_1,\widetilde{\eta}_2\in\mathcal{F}_\sigma(E)$ as in Conclusion B with the same endpoints as $\eta_1,\eta_2$, by (B-ii). Concatenating $\widetilde{\eta}_1$ with $\widetilde{\eta}_2$ gives a path $\widetilde{\eta}\in\mathcal{F}_\sigma(E)$, which shows that $\eta$ also satisfies Conclusion B. Thus, $\gamma\in\mathcal{F}$, as desired. $\square$

4.4. **Application to Sobolev removability.** We prove Theorem 1.4 as an application of Theorem 4.1. Specifically, we show that closed *CNED* sets $E \subset \mathbb{R}^n$ are removable for continuous $W^{1,n}$ functions; that is, every continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$ with $f \in W^{1,n}(\mathbb{R}^n \setminus E)$ lies in $W^{1,n}(\mathbb{R}^n)$.

*Proof of Theorem 1.4.* Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be continuous with $f \in W^{1,n}(\mathbb{R}^n \setminus E)$. Then the classical gradient $\nabla f$ exists almost everywhere in $\mathbb{R}^n \setminus E$. Since $m_n(E) = 0$, there exists a Borel representative of $|\nabla f|$ on $\mathbb{R}^n$. By Fuglede's theorem [Väi71, Theorem 28.2, p. 95], there exists a curve family $\Gamma_1$ with $\operatorname{Mod}_n \Gamma_1 = 0$ such that for each $\gamma \notin \Gamma_1$ we have $\int_\gamma |\nabla f|\, ds < \infty$ and the function $f$ is absolutely continuous on every subpath $\gamma|_{[a,b]}$ of $\gamma$ with $\gamma((a,b)) \subset \mathbb{R}^n \setminus E$. Moreover,

$$|f(\gamma(b)) - f(\gamma(a))| \leq \int_{\gamma|_{(a,b)}} |\nabla f|\, ds.$$

Let $\gamma\colon [a,b] \to \mathbb{R}^n$ be a path outside $\Gamma_1$. The set $(a,b) \setminus \gamma^{-1}(E)$ is a countable union of disjoint open intervals $(a_i,b_i)$, $i \in \mathbb{N}$. Using the continuity of $f$ and the above inequality, we have

$$\begin{aligned} m_1(f(|\gamma| \setminus E)) &\leq \sum_{i\in\mathbb{N}} m_1(f(\gamma([a_i,b_i]))) \leq \sum_{i\in\mathbb{N}} (\max_{[a_i,b_i]} f\circ\gamma - \min_{[a_i,b_i]} f\circ\gamma) \\ &\leq \sum_{i\in\mathbb{N}} \int_{\gamma|_{(a_i,b_i)}} |\nabla f|\, ds \leq \int_\gamma |\nabla f|\, ds. \end{aligned} \tag{4.10}$$

In particular, if $\gamma \in \mathcal{F}_\sigma(E) \setminus \Gamma_1$, then

$$m_1(f(|\gamma|)) = m_1(f(|\gamma| \setminus E)) \leq \int_\gamma |\nabla f|\, ds. \tag{4.11}$$

Since $m_n(E) = 0$, in order to show that $f \in W^{1,n}(\mathbb{R}^n)$, it suffices to show that there exists a path family $\Gamma_0$ with $\operatorname{Mod}_n \Gamma_0 = 0$, such that $f$ is absolutely continuous along each path $\gamma \notin \Gamma_0$. Let $\Gamma_2$ be the path family given by Theorem 4.1 (V), with $\operatorname{Mod}_n \Gamma_2 = 0$ and let $\Gamma_0 = \Gamma_1 \cup \Gamma_2$. We fix a path $\gamma \notin \Gamma_0$ and a subpath $\beta = \gamma|_{[a,b]}$. Note that $\beta \notin \Gamma_2$ by the properties of $\Gamma_2$ in Theorem 4.1 (V). Hence, Conclusion $\mathrm{A}(E, |\nabla f|, \beta)$ is true. For each $\varepsilon > 0$ there exists a path $\widetilde{\beta} \in \mathcal{F}_\sigma(E)$ with the same endpoints as $\beta$ and a collection of paths $\beta_i \notin \Gamma_1$, $i \in I$, such that

$$|\widetilde{\beta}| \setminus \partial\widetilde{\beta} \subset (|\beta| \setminus E) \cup \bigcup_{i\in I} |\beta_i| \quad \text{and} \quad \sum_{i\in I} \int_{\beta_i} |\nabla f|\, ds < \varepsilon.$$

Thus, using (4.10) and (4.11), we have

$$\begin{aligned} |f(\beta(b)) - f(\beta(a))| &\leq m_1(f(|\widetilde{\beta}|)) = m_1(f(|\widetilde{\beta}| \setminus \partial\beta)) \\ &\leq m_1(f(|\beta| \setminus E)) + \sum_{i\in I} m_1(f(|\beta_i|)) \\ &\leq \int_\beta |\nabla f|\, ds + \sum_{i\in I} \int_{\beta_i} |\nabla f|\, ds \\ &\leq \int_\beta |\nabla f|\, ds + \varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$ shows that $f$ is absolutely continuous along $\gamma$. □

## 5. Unions of negligible sets

In this section we prove Theorem 1.2, which asserts that the union of countably many closed *NED* (resp. *CNED*) sets is *NED* (resp. *CNED*). The proof is based on Theorem 4.1 from the preceding section. We first prove an auxiliary lemma.

**Lemma 5.1.** *Let $G_i$, $i\in\mathbb{N}$, be a sequence of continua in $\mathbb{R}^n$ with $G_i\subset G_{i+1}$, $i\in\mathbb{N}$, and $\sup_{i\in\mathbb{N}}\mathscr{H}^1(G_i)<\infty$. If $G=\bigcup_{i\in\mathbb{N}}G_i$, then $\mathscr{H}^1(\overline{G}\setminus G)=0$. In particular if $\gamma\colon[0,1]\to\overline{G}$ is a rectifiable path and $\rho\colon\mathbb{R}^n\to[0,\infty]$ is a Borel function, then*

$$\int_\gamma \rho\, ds=\int_\gamma \rho\chi_G\, ds.$$

See Proposition 4A and Corollary 4I in [Fre92] for a more general result.

*Proof.* Without loss of generality, $\operatorname{diam}(G_1)>0$. Let $d=\operatorname{diam}(G_1)$ and we fix $k\in\mathbb{N}$. Since $G_k$ is closed, for each $x\in\overline{G}\setminus G\subset\overline{G}\setminus G_k$ there exists $r_x>0$ such that $B(x,r_x)\cap G_k=\emptyset$. By the $5B$-covering lemma ([Hei01, Theorem 1.2]), there exists family of balls $B_i=B(x_i,r_i)$ with $x_i\in\overline{G}\setminus G$ and $r_i<d$, $i\in\mathbb{N}$, such that $\overline{G}\setminus G\subset\bigcup_{i\in\mathbb{N}}B_i$, the balls $\frac{1}{5}B_i$ are disjoint, and $\frac{1}{5}B_i\cap G_k=\emptyset$ for each $i\in\mathbb{N}$.

Fix $i\in\mathbb{N}$. Since $x_i\in\overline{G}$, there exists a point $y\in\frac{1}{10}B_i\cap G$. The sequence $\{G_j\}_{j\in\mathbb{N}}$ is increasing, so there exists $j>k$ such that $y\in\frac{1}{10}B_i\cap G_j$. Since $\operatorname{diam}(G_j)\geq d>\operatorname{diam}(\frac{1}{5}B_i)$ and $G_j$ is a continuum, there exists a connected set in $\frac{1}{5}B_i\cap G_j$ that connects $\partial B(x_i,r_i/10)$ to $\partial B(x_i,r_i/5)$. In combination with the fact that $\frac{1}{5}B_i\cap G_k=\emptyset$, we obtain

$$\mathscr{H}^1\left((G\setminus G_k)\cap\frac{1}{5}B_i\right)\geq\mathscr{H}^1\left(G_j\cap\frac{1}{5}B_i\right)\geq r_i/10.$$

We now have

$$\mathscr{H}^1_\infty(\overline{G}\setminus G)\leq\sum_{i\in\mathbb{N}}2r_i\leq 20\sum_{i\in\mathbb{N}}\mathscr{H}^1\left((G\setminus G_k)\cap\frac{1}{5}B_i\right)\leq 20\mathscr{H}^1(G\setminus G_k).$$

Since $\mathscr{H}^1(G)=\sup_{i\in\mathbb{N}}\mathscr{H}^1(G_i)<\infty$ and $G\setminus G_k$ decreases to $\emptyset$ as $k\to\infty$, we have $\mathscr{H}^1(\overline{G}\setminus G)=\mathscr{H}^1_\infty(\overline{G}\setminus G)=0$. This completes the proof of the first statement. The last statement follows from Lemma 2.2 (iv). $\square$

*Proof of Theorem 1.2.* We split the proof into several parts.

**Initial setup.** Define $E=\bigcup_{i\in\mathbb{N}}E_i$ and let $\rho\in L^n_{\mathrm{loc}}(\mathbb{R}^n)$ be a non-negative Borel function. For each $i\in\mathbb{N}$ there exists an exceptional family $\Gamma_i$ with $\operatorname{Mod}_n\Gamma_i=0$, given by Theorem 4.1 (V). Also by property (M6) the family $\Gamma'$ of paths $\gamma$ with $\mathscr{H}^1(|\gamma|\cap E)>0$ has $n$-modulus zero. Define $\Gamma_0=\Gamma'\cup\bigcup_{i\in\mathbb{N}}\Gamma_i$. By the properties of $\Gamma_i$, the path family $\Gamma_0$ has the property that if $\{\eta_j\}_{j\in J}$ is a finite collection of paths outside $\Gamma_0$ and $\gamma$ is a path with $|\gamma|\subset\bigcup_{j\in J}|\eta_j|$, then $\gamma\notin\Gamma_0$. Moreover $|\gamma|\cap E$ is totally disconnected for all paths $\gamma\notin\Gamma_0$.

Our goal is to show that Conclusion B$(E,\rho,\gamma)$ is true for each $\gamma\notin\Gamma_0$. By the last part of Theorem 4.1 and (VI), this suffices for $E$ to be $*$*NED*. We fix a path $\gamma\notin\Gamma_0$ with distinct endpoints, $\varepsilon>0$, and a neighborhood $U$ of $|\gamma|$. It suffices to show Conclusion B for a weak subpath of $\gamma$ with the same endpoints; also all weak subpaths of $\gamma$ lie outside $\Gamma_0$ by the properties of $\Gamma_0$. Thus, by replacing $\gamma$ with a simple weak subpath, we assume that $\gamma$ is simple and $\gamma\notin\Gamma_0$.

We introduce some notation. Define $\mathcal{W}_\sigma = \bigcup_{m=1}^\infty \mathbb{N}^m$ and $\mathcal{W}_0 = \mathbb{N}$. We use the notation $\mathcal{W}_*$ for $\mathcal{W}_\sigma$ or $\mathcal{W}_0$, depending on whether we are working with *CNED* or *NED* sets, respectively. Each element $w \in \mathbb{N}^m \subset \mathcal{W}_\sigma$ is called a *word* of *length* $l(w) = m$. For $w \in \mathcal{W}_0$ we also define $l(w) = w$. The *empty word* $\emptyset$ has length 0.

**Induction assumption.** Set $\widetilde{\gamma}_0 = \alpha_\emptyset = \gamma$ and note that $\alpha_\emptyset \notin \Gamma_0$. Let $U_\emptyset$ be a neighborhood of $|\alpha_\emptyset| \setminus \partial\alpha_\emptyset$ with $\operatorname{diam}(U_\emptyset) \leq 2\operatorname{diam}(|\alpha_\emptyset|)$ and $\overline{U_\emptyset} \subset U$. Suppose that for $m \geq 0$ we have defined simple paths $\widetilde{\gamma}_k$, $k \in \{0, \dots, m\}$, and collections of paths $\{\gamma_i\}_{i\in I_w}$ for $l(w) \leq m-1$ such that

(A-1) $\widetilde{\gamma}_k \in \mathcal{F}_*(\bigcup_{i=1}^k E_i)$ for $k \in \{0, \dots, m\}$,
(A-2) $\partial\widetilde{\gamma}_k = \partial\gamma$ and $|\widetilde{\gamma}_k| \subset |\gamma| \cup \bigcup_{l(w)\leq k-1} \bigcup_{i\in I_w} |\gamma_i|$ for $k \in \{0, \dots, m\}$,
(A-3) $\sum_{l(w)\leq m-1} \sum_{i\in I_w} \ell(\gamma_i) < \varepsilon$, and
(A-4) $\sum_{l(w)\leq m-1} \sum_{i\in I_w} \int_{\gamma_i} \rho\, ds < \varepsilon$.

Moreover, suppose we have defined the following objects:

(A-5) Simple paths $\{\alpha_w\}_{l(w)\leq m}$ such that $\{\alpha_w\}_{l(w)=k}$ is a collection of parametrizations of the closures of the components of $|\widetilde{\gamma}_k| \setminus (E_1 \cup \dots \cup E_k)$, for each $k \in \{0, \dots, m\}$. In addition, for $l(w) \leq m$, each strict subpath of $\alpha_w$ lies outside $\Gamma_0$. We remark that in the case of $\mathcal{W}_* = \mathcal{W}_0$, there is only one component of $|\widetilde{\gamma}_k| \setminus (E_1 \cup \dots \cup E_k)$ and $\alpha_k = \widetilde{\gamma}_k$.
(A-6) Open sets $\{U_w\}_{l(w)\leq m}$ such that $U_w$ is a neighborhood of $|\alpha_w| \setminus \partial\alpha_w$ with $\operatorname{diam}(U_w) \leq 2\operatorname{diam}(|\alpha_w|)$ for $l(w) \leq m$ and the collection $\{U_w\}_{l(w)=k}$ is disjointed for each $k \in \{0, \dots, m\}$. Moreover, if $m \geq 1$ and $l(w) \leq m-1$, then $U_{(w,j)} \subset U_w$ for $j \in \mathbb{N}$ in the case $\mathcal{W}_* = \mathcal{W}_\sigma$ and $U_{w+1} \subset U_w$ in the case $\mathcal{W}_* = \mathcal{W}_0$. Finally, $\overline{U_w}$ does not intersect $E_1 \cup \dots \cup E_{l(w)}$, except possibly at the endpoints of $\alpha_w$, for $l(w) \leq m$.

Finally, we require the compatibility property

(A-7) $|\widetilde{\gamma}_m| \setminus \bigcup_{l(w)=k} U_w = |\widetilde{\gamma}_k| \setminus \bigcup_{l(w)=k} U_w$ for $k \in \{0, \dots, m\}$.

**Inductive step.** We now define $\widetilde{\gamma}_{m+1}$ as follows. Fix $w \in \mathcal{W}_*$ with $l(w) = m$. By (A-5) each strict subpath $\eta$ of $\alpha_w$ avoids the path family $\Gamma_0$; thus Conclusion $\mathrm{A}(E_{m+1}, \rho, \eta)$ is true and $\mathscr{H}^1(|\eta| \cap E_{m+1}) = 0$. The latter implies that $|\alpha_w| \cap E_{m+1}$ is totally disconnected. By Lemma 4.7, Conclusion $\mathrm{A}(E_{m+1}, \rho, \alpha_w)$ is true. Thus, for each $\delta_w > 0$ there exists a simple path $\widetilde{\alpha}_w$ and paths $\gamma_i \notin \Gamma_0$, $i \in I_w$, such that

(A'-1) $\widetilde{\alpha}_w \in \mathcal{F}_*(E_{m+1})$,
(A'-2) $\partial\widetilde{\alpha}_w = \partial\alpha_w$ and $|\widetilde{\alpha}_w| \subset |\alpha_w| \cup \bigcup_{i\in I_w} |\gamma_i|$ and $\bigcup_{i\in I_w} |\gamma_i| \subset U_w$,
(A'-3) $\sum_{i\in I_w} \ell(\gamma_i) < \delta_w$, and
(A'-4) $\sum_{i\in I_w} \int_{\gamma_i} \rho\, ds < \delta_w$.

If we choose a sufficiently small $\delta_w$, we can ensure that (A-3) and (A-4) are true for the index $m+1$ in place of $m$. By (A'-2), the path $\widetilde{\alpha}_w$ is obtained by modifying $\alpha_w$ within $U_w$. By (A-6), $\overline{U_w}$ does not intersect $E_1 \cup \dots \cup E_m$, except possibly at the endpoints of $\alpha_w$. Therefore, $\widetilde{\alpha}_w \in \mathcal{F}_*(E_1 \cup \dots \cup E_m)$. Combining this with (A'-1), we obtain $\widetilde{\alpha}_w \in \mathcal{F}_*(E_1 \cup \dots \cup E_{m+1})$.

Using (A-5), we define $\widetilde{\gamma}_{m+1}$ by replacing each $\alpha_w$ in $\widetilde{\gamma}_m$, where $l(w) = m$, with $\widetilde{\alpha}_w$; see Figure 3. In the case of $\mathcal{W}_\sigma$, we need to ensure that this procedure gives a path. Indeed, by (A'-2) and (A'-3) we have $\ell(\widetilde{\alpha}_w) \leq \ell(\alpha_w) + \delta_w$, so if $\delta_w$ is sufficiently small, then we obtain a path $\widetilde{\gamma}_{m+1}$. By (A-5), the endpoints of $\widetilde{\gamma}_m$ do

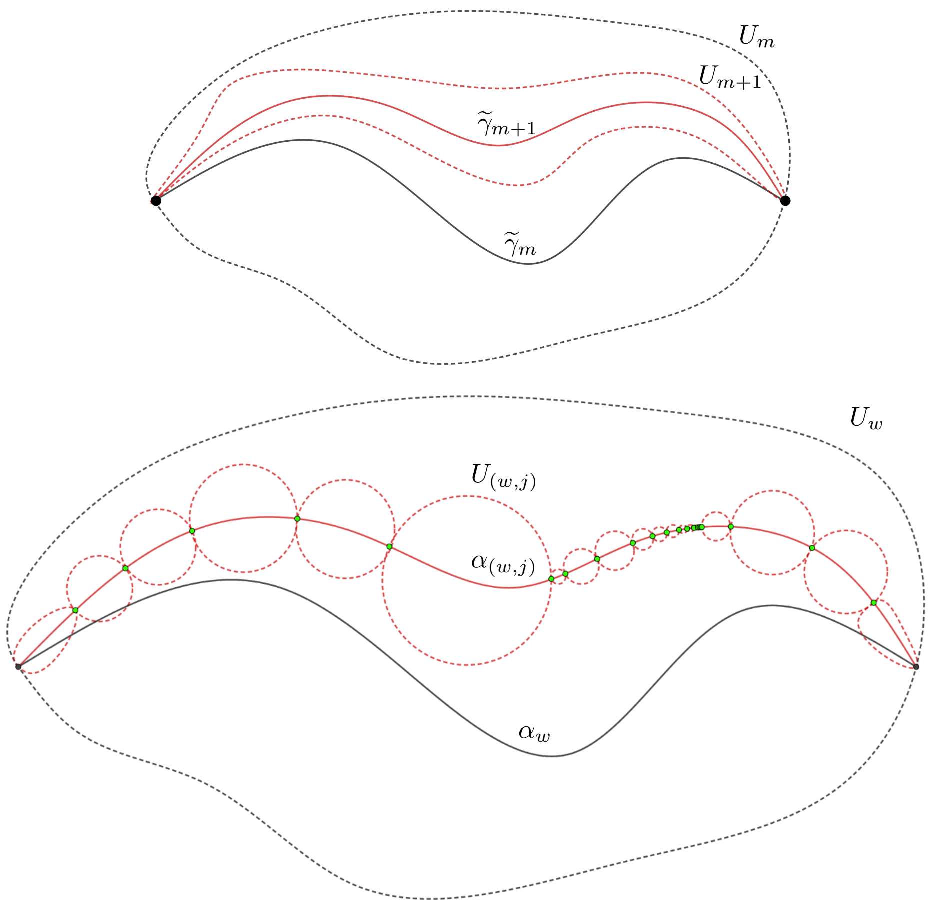


FIGURE 3. Construction of $\widetilde{\gamma}_{m+1}$ from $\widetilde{\gamma}_m$. Top figure: the case of $\mathcal{W}_0$. Bottom figure: the case of $\mathcal{W}_\sigma$. The red curve is $\widetilde{\alpha}_w$ and the green points denote the set $(|\widetilde{\gamma}_{m+1}| \cap E_{m+1}) \cap U_w$, which is countable. In fact, a large part of $\widetilde{\alpha}_w$ should be shared with $\alpha_w$ by (A'-2) but we do not indicate this to simplify the figure.

not lie in $|\alpha_w| \setminus \partial\alpha_w$ for any $w$ with $l(w) = m$, so they are not modified. Thus, $\widetilde{\gamma}_{m+1}$ has the same endpoints as $\gamma$. Moreover, by (A-6), the regions $\{U_w\}_{l(w)=m}$, where the paths $\widetilde{\alpha}_w$ differ from $\alpha_w$ are pairwise disjoint. Hence $\widetilde{\gamma}_{m+1}$ is a simple path, since $\widetilde{\gamma}_m$ is simple.

We now verify (A-1) and (A-2). By construction and (A-5), we have

$$|\widetilde{\gamma}_{m+1}| \setminus \bigcup_{l(w)=m} (|\widetilde{\alpha}_w| \setminus \partial\widetilde{\alpha}_w) = |\widetilde{\gamma}_m| \setminus \bigcup_{l(w)=m} (|\alpha_w| \setminus \partial\alpha_w) = \partial\gamma \cup \left( |\widetilde{\gamma}_m| \cap \bigcup_{i=1}^m E_i \right).$$

In the case of $\mathcal{W}_\sigma$, since $\widetilde{\gamma}_m \in \mathcal{F}_\sigma(E_1 \cup \cdots \cup E_m)$ and $\widetilde{\alpha}_w \in \mathcal{F}_\sigma(E_1 \cup \cdots \cup E_{m+1})$ for $l(w) = m$, we conclude that $\widetilde{\gamma}_{m+1} \in \mathcal{F}_\sigma(E_1 \cup \cdots \cup E_{m+1})$, as required in (A-1). In the case $\mathcal{W}_* = \mathcal{W}_0$ we simply have $\widetilde{\gamma}_m = \alpha_m$ (see (A-5)) and $\widetilde{\gamma}_{m+1} = \widetilde{\alpha}_m$, so

$\widetilde{\gamma}_{m+1} \in \mathcal{F}_0(E_1 \cup \cdots \cup E_{m+1})$, as desired. By construction and (A'-2) we have

$$|\widetilde{\gamma}_{m+1}| \subset |\widetilde{\gamma}_m| \cup \bigcup_{l(w)=m} \bigcup_{i \in I_w} |\gamma_i|.$$

Thus, by the induction assumption we obtain (A-2) for the index $m+1$.

Since $\widetilde{\gamma}_{m+1}$ is obtained by modifying $\widetilde{\gamma}_m$ in the open sets $U_w$, $l(w) = m$, we have

$$|\widetilde{\gamma}_{m+1}| \setminus \bigcup_{l(w)=m} U_w = |\widetilde{\gamma}_m| \setminus \bigcup_{l(w)=m} U_w.$$

By (A-6), for $k \leq m$ we have $\bigcup_{l(w)=k} U_w \supset \bigcup_{l(w)=m} U_w$, so

$$|\widetilde{\gamma}_{m+1}| \setminus \bigcup_{l(w)=k} U_w = |\widetilde{\gamma}_m| \setminus \bigcup_{l(w)=k} U_w = |\widetilde{\gamma}_k| \setminus \bigcup_{l(w)=k} U_w,$$

where the last equality follows from the induction assumption (A-7). This proves the equality in (A-7) for the index $m+1$.

Next, we verify (A-5). Note that the paths $\{\widetilde{\alpha}_w\}_{l(w)=m}$, parametrize the closures of the components of $|\widetilde{\gamma}_{m+1}| \setminus (E_1 \cup \cdots \cup E_m)$; this follows from the construction and (A-5). For $l(w) = m$, let $\{\alpha_{(w,j)}\}_{j\in\mathbb{N}}$ be a collection of simple paths parametrizing the closures of the components of $|\widetilde{\alpha}_w| \setminus E_{m+1}$; see Figure 3. Then $\{\alpha_w\}_{l(w)=m+1}$ gives a collection that parametrizes the closures of the components of $|\widetilde{\gamma}_{m+1}| \setminus (E_1 \cup \cdots \cup E_{m+1})$. This verifies the first part of (A-5). Moreover, if $\mathcal{W}_* = \mathcal{W}_\sigma$, for $l(w) = m$ and $j \in \mathbb{N}$, each strict subpath $\widetilde{\eta}$ of $\alpha_{(w,j)}$ is a strict subpath of $\widetilde{\alpha}_w$. If $\mathcal{W}_* = \mathcal{W}_0$, each strict subpath $\widetilde{\eta}$ of $\alpha_{w+1}$ is a strict subpath of $\widetilde{\alpha}_w$. In either case, by (A'-2), there exists a strict subpath $\eta$ of $\alpha_w$ such that $|\widetilde{\eta}| \subset |\eta| \cup \bigcup_{i\in I_w} |\gamma_i|$. By the induction assumption (A-5), $\eta \notin \Gamma_0$. By the last part of Conclusion A (see the statement in Theorem 4.1), $|\widetilde{\eta}|$ intersects finitely many of the traces $|\gamma_i|$, $i \in I_w$. Recall that $\gamma_i \notin \Gamma_0$, $i \in I_w$. The properties of the family $\Gamma_0$ imply that $\widetilde{\eta} \notin \Gamma_0$. This completes the proof of the second part of (A-5).

We now discuss (A-6). If $\mathcal{W}_* = \mathcal{W}_0$, we define $U_{m+1}$ to be a neighborhood of $\overline{|\alpha_{m+1}|} \setminus \alpha_{m+1}$ such that $\operatorname{diam}(U_{m+1}) \leq 2\operatorname{diam}(|\alpha_{m+1}|)$, $U_{m+1} \subset U_m$, and $\overline{U_{m+1}}$ does not intersect $E_1 \cup \cdots \cup E_{m+1}$, except possibly at the endpoints of $\alpha_{m+1}$; this uses that $E_1 \cup \cdots \cup E_{m+1}$ is closed and does not intersect $|\alpha_{m+1}|$, except possibly at the endpoints. If $\mathcal{W}_* = \mathcal{W}_\sigma$, for $l(w) = m$ we define $U_{(w,1)}$ to be a neighborhood of $|\alpha_{(w,1)}| \setminus \partial\alpha_{(w,1)}$ such that $\operatorname{diam}(U_{(w,1)}) \leq 2\operatorname{diam}(|\alpha_{(w,1)}|)$, $U_{(w,1)} \subset U_w$, and $\overline{U_{(w,1)}}$ does not intersect $E_1 \cup \cdots \cup E_{m+1}$, except possibly at the endpoints of $\alpha_{(w,1)}$. Moreover, since $\alpha_w$ is simple, we may require that $\overline{U_{(w,1)}}$ is disjoint from $|\alpha_{(w,j)}| \setminus \partial\alpha_{(w,j)}$ for $j > 1$. Next, we define $U_{(w,2)}$ to be a neighborhood of $|\alpha_{(w,2)}| \setminus \partial\alpha_{(w,2)}$ with the same properties as $U_{(w,1)}$ and with $U_{(w,1)} \cap U_{(w,2)} = \emptyset$. Inductively, we define $U_{(w,j)}$ for all $j \in \mathbb{N}$ with the desired properties; see Figure 3. We have completed the proof of the inductive step.

**Completion of the proof.** Now we will show that Conclusion B$(E, \rho, \gamma)$ is true. For $m \in \mathbb{N}$ define $G_m = \bigcup_{k=0}^m |\widetilde{\gamma}_k|$. This is a continuum, since $|\widetilde{\gamma}_k|$ contains the endpoints of $\gamma$ for each $k \in \mathbb{N}$ by (A-2). We have $G_m \subset G_{m+1}$ and

$$G_m \subset |\gamma| \cup \bigcup_{l(w)\leq m-1} \bigcup_{i\in I_w} |\gamma_i|$$

for $m \in \mathbb{N}$. By Lemma 2.2 (ii) and property (A-3) we have

$$\mathscr{H}^1(G_m) \leq \ell(\gamma) + \sum_{l(w)\leq m-1} \sum_{i\in I_w} \ell(\gamma_i) \leq \ell(\gamma)+\varepsilon.$$

Thus $\sup_{m\in\mathbb{N}} \mathscr{H}^1(G_m) < \infty$, as required in Lemma 5.1. We set $G = \bigcup_{m\in\mathbb{N}} G_m$.

Since $\widetilde{\gamma}_m$ is a simple path for each $m \in \mathbb{N}$, by Lemma 2.2 (ii) we have

$$\ell(\widetilde{\gamma}_m) = \mathscr{H}^1(|\widetilde{\gamma}_m|) \leq \mathscr{H}^1(G_m) \leq \ell(\gamma)+\varepsilon.$$

By the Arzelà–Ascoli theorem, there exists a subsequence of $\widetilde{\gamma}_m$, parametrized by arclength, that converges uniformly to a path $\widetilde{\gamma}$ with the same endpoints as $\gamma$ and

$$\ell(\widetilde{\gamma}) \leq \liminf_{m\to\infty} \ell(\widetilde{\gamma}_m) \leq \ell(\gamma)+\varepsilon.$$

Hence, (B-iii) holds. Since $|\widetilde{\gamma}_m| \subset \overline{U_\emptyset}$ for each $m \in \mathbb{N}$ by (A-6), we have $|\widetilde{\gamma}| \subset \overline{U_\emptyset} \subset U$, as required in (B-ii). We assume that $\widetilde{\gamma}$ is simple by considering a weak subpath if necessary. Since $|\widetilde{\gamma}| \subset \overline{G}$, by Lemma 5.1 and Lemma 2.2 (ii) we have

$$\begin{aligned}\int_{\widetilde{\gamma}} \rho\, ds &= \int_{\widetilde{\gamma}} \rho \chi_G\, ds = \int_{|\widetilde{\gamma}|\cap G} \rho\, d\mathscr{H}^1 \leq \int_G \rho\, d\mathscr{H}^1\\ &\leq \int_\gamma \rho\, ds + \sum_{w\in\mathcal{W}_*}\sum_{i\in I_w} \int_{\gamma_i} \rho\, ds \leq \int_\gamma \rho\, ds + \varepsilon,\end{aligned}$$

where the last inequality follows from (A-4). This shows (B-iv).

Finally, we argue for (B-i). By (A-7) for $m \geq k \geq 0$ we have

$$|\widetilde{\gamma}_m| \subset |\widetilde{\gamma}_k| \cup \overline{\bigcup_{l(w)=k} U_w}. \tag{5.1}$$

By (A-6), $\operatorname{diam}(U_w) \leq 2\operatorname{diam}(|\alpha_w|)$. If there are infinitely many non-empty sets $U_w$ with $l(w)=k$, only finitely many curves $\alpha_w$ can have diameter larger than a given number; indeed by (A-5) these are subpaths of $\widetilde{\gamma}_k$ that have pairwise disjoint traces, except possibly at the endpoints. It follows that $U_w$ is contained in a small neighborhood of $|\widetilde{\gamma}_k|$ for all but finitely many $w$ with $l(w)=k$. Hence, we see that

$$\overline{\bigcup_{l(w)=k} U_w} \subset |\widetilde{\gamma}_k| \cup \bigcup_{l(w)=k} \overline{U_w}.$$

By letting $m \to \infty$ in (5.1), we conclude that

$$|\widetilde{\gamma}| \subset |\widetilde{\gamma}_k| \cup \overline{\bigcup_{l(w)=k} U_w} \subset |\widetilde{\gamma}_k| \cup \bigcup_{l(w)=k} \overline{U_w}.$$

In the case of $\mathcal{W}_\sigma$, the set in the right-hand side intersects $E_1\cup\cdots\cup E_k$ at countably many points by (A-1) and (A-6). Thus, $\widetilde{\gamma} \in \mathcal{F}_\sigma(E_1\cup\cdots\cup E_k)$ for each $k\in\mathbb{N}$, so $\widetilde{\gamma}\in\mathcal{F}_\sigma(E)$. In the case of $\mathcal{W}_0$, we have $|\widetilde{\gamma}| \subset |\widetilde{\gamma}_k| \cup \overline{U_k} \subset \overline{U_k}$ and the set $\overline{U_k}$ does not intersect $E_1\cup\cdots\cup E_k$, except possibly at the endpoints of $\alpha_k = \widetilde{\gamma}_k$. Thus, $|\widetilde{\gamma}|$ does not intersect $E_1\cup\cdots\cup E_k$, except possibly at the endpoints, for each $k \in \mathbb{N}$. We conclude that $\widetilde{\gamma} \in \mathcal{F}_0(E)$. We have completed the verification of Conclusion $\mathrm{B}(E,\rho,\gamma)$, and thus, the proof of Theorem 1.2. $\square$

## 6. Examples of negligible sets

Recall that all sets of $\sigma$-finite (resp. zero) Hausdorff $(n-1)$ measure are *CNED* (resp. *NED*), by Theorem 3.11. In this section we will discuss further examples.

6.1. **A quasihyperbolic condition for *CNED* sets.** Let $\Omega \subsetneq \mathbb{R}^n$ be a domain, i.e., a connected open set. For a point $x \in \Omega$, define $\delta_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$. We define the *quasihyperbolic distance* of two points $x_1, x_2 \in \Omega$ by

$$k_\Omega(x_1, x_2) = \inf_\gamma \int_\gamma \frac{1}{\delta_\Omega}\, ds,$$

where the infimum is taken over all rectifiable paths $\gamma$ in $\Omega$ that connect $x_1$ and $x_2$.

**Theorem 6.1.** *Let $\Omega \subset \mathbb{R}^n$ be a domain such that $k_\Omega(\cdot, x_0) \in L^n(\Omega)$ for some $x_0 \in \Omega$. Then $\partial\Omega \in CNED$. In particular, boundaries of John and Hölder domains are CNED.*

See [SS90] for the definitions of the latter two classes of domains. The condition $k_\Omega(\cdot, x_0) \in L^n(\Omega)$ appeared in the work of Jones–Smirnov [JS00], who showed its sufficiency for $\partial\Omega$ to be *QCH*-removable. The same condition has also been used in recent work of the current author [Nta20] to establish the removability of certain fractals with infinitely many complementary components for Sobolev spaces; in addition, it has appeared in work of the current author and Younsi [NY20] in establishing the rigidity of circle domains under this condition. We will use some auxiliary results from [NY20], which have been proved there in dimension 2, but the proofs apply to all dimensions.

*Remark* 6.2. Domains satisfying the condition of the theorem are bounded [NY20, Lemma 2.6] and thus have finite $n$-measure. Using this, one can show that an equivalent condition is $k_\Omega \in L^n(\Omega \times \Omega)$, so the base point $x_0$ is not of importance.

We will prepare the necessary background before proving the theorem. For a domain $\Omega \subsetneq \mathbb{R}^n$ we consider the *Whitney cube decomposition* $\mathcal{W}(\Omega)$, which is a collection of closed dyadic cubes $Q \subset \Omega$, called *Whitney cubes*, such that

(1) the cubes of $\mathcal{W}(\Omega)$ have disjoint interiors and $\bigcup_{Q \in \mathcal{W}(\Omega)} Q = \Omega$,
(2) $\operatorname{diam}(Q) \le \operatorname{dist}(Q, \partial\Omega) \le 4 \operatorname{diam}(Q)$ for all $Q \in \mathcal{W}(\Omega)$, and
(3) if $Q_1 \cap Q_2 \neq \emptyset$, then $1/4 \le \operatorname{diam}(Q_1)/\operatorname{diam}(Q_2) \le 4$, for all $Q_1, Q_2 \in \mathcal{W}(\Omega)$.

See [Ste70, Theorem 1 and Prop. 1, pp. 167–169] for the existence of the decomposition. We denote by $\ell(Q)$ the side length of a cube $Q$; this is not to be confused with the length $\ell(\gamma)$ of a path $\gamma$. Two Whitney cubes $Q_1, Q_2 \in \mathcal{W}(\Omega)$ with $\ell(Q_1) \ge \ell(Q_2)$ are *adjacent* if a face of $Q_2$ is contained in a face of $Q_1$.

**Lemma 6.3.** *Let $\Omega \subset \mathbb{R}^n$ be a domain and $Q_1, Q_2 \in \mathcal{W}(\Omega)$ be adjacent cubes. Let $F_1 \subset Q_1$ and $F_2 \subset Q_2$ be continua with $\operatorname{diam}(F_i) \ge a\ell(Q_i)$ for some $a > 0$, $i = 1, 2$, and let $\rho \colon \mathbb{R}^n \to [0, \infty]$ be a Borel function. Then there exists a rectifiable path $\gamma \in \Gamma(F_1, F_2; \operatorname{int}(Q_1 \cup Q_2))$ such that*

$$\int_\gamma \rho\, ds \le c(n,a) \left( \|\rho\|_{L^n(Q_1)} + \|\rho\|_{L^n(Q_2)} \right) \quad \textit{and} \quad \ell(\gamma) \le c(n,a)(\ell(Q_1) + \ell(Q_2)).$$

*Proof.* By Lemma 3.7, it suffices to show that $\operatorname{Mod}_n \Gamma(F_1, F_2; \operatorname{int}(Q_1 \cup Q_2))$ is uniformly bounded from below, depending only on $n$ and $a$. This can be shown by mapping $Q_1 \cup Q_2$ with a uniformly bi-Lipschitz map onto a ball. Euclidean balls are Loewner spaces; see [Hei01, Chapter 8] for the definition and properties. Hence, the desired lower bound is satisfied. $\square$

**Lemma 6.4.** *Let $\Omega \subset \mathbb{R}^n$ be a domain, $\gamma\colon [0,1] \to \overline{\Omega}$ be a path such that $\gamma((0,1)) \subset \Omega$ and $\gamma(0), \gamma(1) \in \partial\Omega$, and $\rho\colon \mathbb{R}^n \to [0,\infty]$ be a Borel function. Then there exists a path $\gamma'\colon [0,1] \to \Omega$ with $\gamma'(0) = \gamma(0)$, $\gamma'(1) = \gamma(1)$, $\gamma'((0,1)) \subset \bigcup_{\substack{Q\in\mathcal{W}(\Omega)\\ |\gamma|\cap Q\neq\emptyset}} Q$,*

$$\int_{\gamma'} \rho\, ds \leq c(n) \sum_{\substack{Q\in\mathcal{W}(Q)\\ |\gamma'|\cap Q\neq\emptyset}} \|\rho\|_{L^n(Q)} \quad \text{and} \quad \ell(\gamma') \leq c(n) \sum_{\substack{Q\in\mathcal{W}(Q)\\ |\gamma'|\cap Q\neq\emptyset}} \ell(Q).$$

*Proof.* There exists a sequence $Q_i$, $i \in \mathbb{Z}$, of distinct Whitney cubes such that $Q_i$ is adjacent to $Q_{i+1}$ for each $i \in \mathbb{Z}$, $|\gamma| \cap Q_i \neq \emptyset$ for each $i \in \mathbb{Z}$, and $Q_i \to \gamma(0)$ as $i \to -\infty$ and $Q_i \to \gamma(1)$ as $i \to \infty$; see [NY20, p. 143] for an argument.

Consider the adjacent cubes $Q_0$ and $Q_1$. Let $F_1$ be the face of $Q_0$ that is opposite to the common face of $Q_0, Q_1$ and $F_2$ be the corresponding face of $Q_1$. We apply Lemma 6.3 to obtain a path $\gamma_0$ in $Q_0 \cup Q_1$ connecting $F_1$ with $F_2$ and satisfying the conclusions of the lemma with $a = 1/2$. We now consider $Q_1$ and $Q_2$. Let $F_1$ be a subcontinuum of $|\gamma_0|$ connecting opposite sides of $Q_1$ and let $F_2$ be the face of $Q_2$ opposite to the common face between $Q_1$ and $Q_2$. Applying Lemma 6.3 with $a = 1/2$, we obtain a path $\gamma_1$ connecting $F_1$ with $F_2$ in $Q_1 \cup Q_2$. Inductively, for each $i \in \mathbb{Z}$ we obtain a path $\gamma_i$ in $Q_i \cup Q_{i+1}$ as in the conclusions of Lemma 6.3 such that $|\gamma_i| \cap |\gamma_{i+1}| \neq \emptyset$. Since $\operatorname{diam}(Q_i) \to 0$ as $|i| \to \infty$, we can concatenate the paths $\gamma_i$ to obtain a path $\gamma'$ with the desired properties. $\square$

Suppose that there exists a base point $x_0 \in \Omega$ with $k_\Omega(\cdot, x_0) \in L^n(\Omega)$. It is shown in [JS00, pp. 273–274] that there exists a tree-like family $\mathcal{G}$ of curves starting at $x_0$, connecting centers of adjacent Whitney cubes, and landing at $\partial\Omega$, that behave essentially like quasihyperbolic geodesics and so that each point of $\partial\Omega$ is the landing point of a curve of $\mathcal{G}$. For each cube $Q \in \mathcal{W}(\Omega)$ we define the *shadow* $SH(Q)$ of $Q$ to be the set of points $x \in \partial\Omega$ such that there exists a curve of $\mathcal{G}$ starting at $x_0$, passing through $Q$ and landing at $x$. We define

$$s(Q) = \operatorname{diam}(SH(Q)).$$

The set $SH(Q)$ is a compact subset of $\partial\Omega$ for each $Q \in \mathcal{W}(\Omega)$; see [NY20, Lemma 2.7]. Moreover, it is shown in [JS00, p. 275] that

$$\sum_{Q\in\mathcal{W}(\Omega)} s(Q)^n \lesssim_n \int_\Omega k(x,x_0)^n\, dx. \tag{6.1}$$

**Lemma 6.5** ([NY20, Lemma 2.10])**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain such that $k_\Omega(\cdot, x_0) \in L^n(\Omega)$ for some $x_0 \in \Omega$. For each simple path $\gamma\colon [0,1] \to \mathbb{R}^n$ and $\varepsilon > 0$ there exists a finite collection of paths $\{\gamma_i\colon [0,1] \to \overline{\Omega}\}_{i\in I}$ such that*

(i) *$\partial\gamma_i \subset \partial\Omega$ and either $\gamma_i$ is constant or $\gamma_i((0,1)) \subset \Omega$ for each $i \in I$,*

(ii) *there exists a path $\widetilde{\gamma}$ with $\partial\widetilde{\gamma} = \partial\gamma$ and $|\widetilde{\gamma}| \subset (|\gamma| \setminus \partial\Omega) \cup \bigcup_{i\in I} |\gamma_i|$,*

(iii) *if $Q \in \mathcal{W}(\Omega)$ is a Whitney cube with $|\gamma_i| \cap Q \neq \emptyset$ for some $i \in I$, then*

$$|\gamma| \cap SH(Q) \neq \emptyset \quad \text{and} \quad \ell(Q) \leq \varepsilon,$$

(iv) *$|\gamma_i|$ and $|\gamma_j|$ intersect disjoint sets of Whitney cubes $Q \in \mathcal{W}(\Omega)$ for $i \neq j$.*

*Moreover, if one replaces $\gamma_i$, for each $i \in I$, with a path $\gamma_i'\colon [0,1] \to \overline{\Omega}$ such that $\gamma_i'(0) = \gamma_i(0)$, $\gamma_i'(1) = \gamma_i(1)$, and either $\gamma_i'$ is constant or $\gamma_i'((0,1)) \subset \bigcup_{\substack{Q\in\mathcal{W}(\Omega)\\ |\gamma_i|\cap Q\neq\emptyset}} Q$, then conclusions* (i)–(iv) *also hold for the collection $\{\gamma_i'\}_{i\in I}$.*

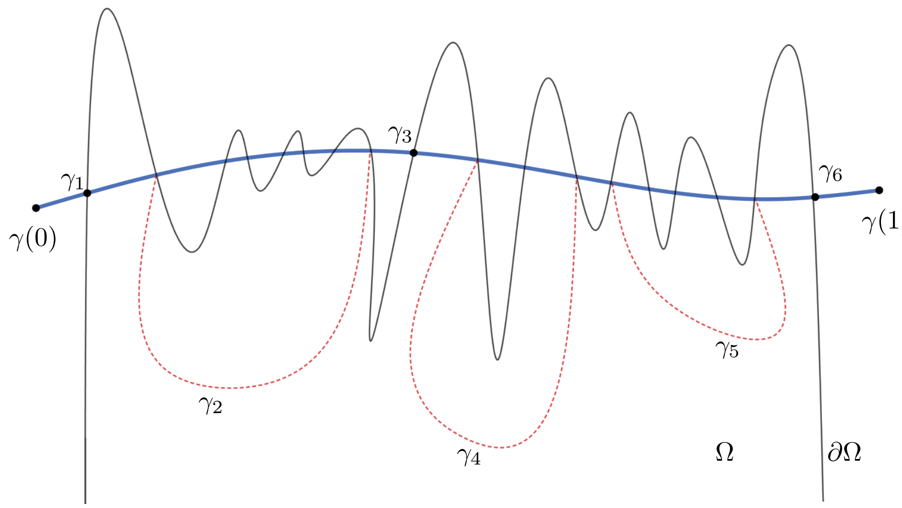


FIGURE 4. The path $\gamma$ (blue) and the paths $\gamma_i$ given by Lemma 6.5. There are three constant paths $\gamma_i$. The path $\widetilde{\gamma}$ arises by replacing the arcs of $\gamma$ between the endpoints of $\gamma_i$ with $\gamma_i$.

See Figure 4 for an illustration. The formulation of this lemma in [NY20, Lemma 2.10] is slightly different; however, the proof of (i), (ii), and (iii) is identical. The paths $\gamma_i$ are concatenations of subpaths of paths of $\mathcal{G}$ and are replacing finitely many arcs of $\gamma$ that cover the set $|\gamma| \cap \partial\Omega$. Hence, using the path $\widetilde{\gamma}$ arising from these replacements one essentially avoids the set $|\gamma| \cap \partial\Omega$, except at the endpoints of $\gamma_i$. We provide a sketch of the proof of (iv). One needs to concatenate appropriately the paths $\gamma_i$ that satisfy (i)–(iii). If two paths $\gamma_i$, $\gamma_j$, $i \neq j$, meet a common Whitney cube $Q$, then one can concatenate these paths with a line segment inside $Q$. Then one considers a subpath $\gamma_{ij}$ of the concatenation so that (i) and (ii) are satisfied with $\gamma_{ij}$ in place of $\gamma_i$ and $\gamma_j$. After finitely many concatenations, one can obtain the family $\{\widetilde{\gamma}_i\}_{i\in \widetilde{I}}$ that satisfies all conditions (i)–(iv). The last part of the lemma provides some extra freedom in the choice of the paths $\gamma_i$; conclusion (ii) for the collection $\{\gamma_i'\}_{i\in I}$ follows from the construction and the other conclusions are immediate since $\partial\gamma_i' = \partial\gamma_i$ and the trace of the path $\gamma_i'$ intersects no more Whitney cubes than $\gamma_i$ does.

*Proof of Theorem 6.1.* We will verify Theorem 4.3 (II) to show that $\partial\Omega$ is *CNED*. Let $\rho\colon \mathbb{R}^n \to [0,\infty]$ be a Borel function with $\rho \in L^n(\mathbb{R}^n)$. We check condition (II-1). The proof of (II-2) is very similar and we omit it. Let $\gamma$ be a non-constant rectifiable path, and set $g = \rho + \chi_\Omega$. Since $\Omega$ is bounded (see Remark 6.2), we have $g \in L^n(\mathbb{R}^n)$. By Lemma 3.12 we have

$$\begin{aligned}\int_{\mathbb{R}^n} \sum_{\substack{\ell(Q)\leq \varepsilon\\ |\gamma+x|\cap SH(Q)\neq\emptyset}} \|g\|_{L^n(Q)}\, dx &= \sum_{\ell(Q)\leq\varepsilon} \|g\|_{L^n(Q)} \cdot m_n(\{x : |\gamma+x|\cap SH(Q)\neq\emptyset\})\\ &\lesssim_n \sum_{\ell(Q)\leq\varepsilon} \|g\|_{L^n(Q)} \cdot \max\{\ell(\gamma), s(Q)\} s(Q)^{n-1}.\end{aligned}$$

Since $s(\cdot)\in\ell^n(\mathcal{W}(\Omega))$ by (6.1), we have $s(Q)\leq \ell(\gamma)$ if $\ell(Q)\leq\varepsilon$ and $\varepsilon$ is sufficiently small. Using this fact and Hölder's inequality, we bound the above sum by

$$\ell(\gamma)\bigg(\sum_{\ell(Q)\leq\varepsilon}\int_Q g^n\bigg)^{1/n}\cdot\bigg(\sum_{\ell(Q)\leq\varepsilon} s(Q)^n\bigg)^{1-1/n}.$$

As $\varepsilon\to 0$, this converges to 0. We conclude that as $\varepsilon\to 0$ along a sequence, for a.e. $x\in\mathbb{R}^n$ we have

$$\sum_{\substack{\ell(Q)\leq\varepsilon\\ |\gamma+x|\cap SH(Q)\neq\emptyset}}\|\rho\|_{L^n(Q)}=o(1)\quad\text{and}\quad\sum_{\substack{\ell(Q)\leq\varepsilon\\ |\gamma+x|\cap SH(Q)\neq\emptyset}}\ell(Q)=o(1). \tag{6.2}$$

Let $x\in\mathbb{R}^n$ such that (6.2) holds. Let $\eta$ be a strong subpath of $\gamma+x$ with distinct endpoints. We claim that Conclusion $\mathrm{B}(\partial\Omega,\rho,\eta)$ is true. It suffices to prove this for a simple weak subpath of $\eta$ with the same endpoints that we still denote by $\eta$.

For $\varepsilon>0$ we apply Lemma 6.5 to the path $\eta$ and obtain a finite collection of paths $\{\eta_i\}_{i\in I}$. To each non-constant path $\eta_i$ we apply Lemma 6.4 and we obtain a path $\eta_i'$ that has the same endpoints as $\eta_i$ and if $|\eta_i'|\cap Q\neq\emptyset$ for some $Q\in\mathcal{W}(\Omega)$, then $|\eta_i|\cap Q\neq\emptyset$. If $\eta_i$ is constant, we set $\eta_i'=\eta_i$. The last part of Lemma 6.5 allows us to replace each $\eta_i$ with $\eta_i'$ while retaining properties (i)–(iv). By Lemma 6.5 (i) and (ii), there exists a simple path $\widetilde{\eta}$ such that

(1) $|\widetilde{\eta}|\cap\partial\Omega$ is a finite set (only the endpoints of $\eta_i'$ can lie in $\partial\Omega$), and
(2) $\partial\eta=\partial\widetilde{\eta}$ and $|\widetilde{\eta}|\subset(|\eta|\setminus\partial\Omega)\cup\bigcup_{i\in I}|\eta_i'|$.

We discard the paths $\eta_i'$ whose trace does not intersect $|\widetilde{\eta}|$. Furthermore, since the paths $\eta_i'$ satisfy the conclusions of Lemma 6.4, in combination with Lemma 6.5 (iii) and (iv), we have

(3) $\displaystyle\sum_{i\in I}\ell(\eta_i')\leq c(n)\sum_{i\in I}\sum_{\substack{\ell(Q)\leq\varepsilon\\ |\eta_i'|\cap Q\neq\emptyset}}\ell(Q)\leq c(n)\sum_{\substack{\ell(Q)\leq\varepsilon\\ |\eta|\cap SH(Q)\neq\emptyset}}\ell(Q),$ and

(4) $\displaystyle\sum_{i\in I}\int_{\eta_i'}\rho\,ds\leq c(n)\sum_{i\in I}\sum_{\substack{\ell(Q)\leq\varepsilon\\ |\eta_i'|\cap Q\neq\emptyset}}\|\rho\|_{L^n(Q)}\leq c(n)\sum_{\substack{\ell(Q)\leq\varepsilon\\ |\eta|\cap SH(Q)\neq\emptyset}}\|\rho\|_{L^n(Q)}.$

Note that (1) implies (B-i), and (2), (3), (4), together with (6.2), imply (B-iii) and (B-iv). Finally, given an open neighborhood $U$ of $|\eta|$, if $\varepsilon$ is sufficiently small, the sum of the lengths of $\eta_i'$ is small, and thus $\bigcup_{i\in I}|\eta_i'|\subset U$; this proves (B-ii). $\square$

6.2. **Projections to axes and *NED* sets.** We present a result for sets whose projections to the coordinate axes have measure zero. This result will be crucial for the proof of Theorem 1.9.

**Theorem 6.6.** *Let $E\subset\mathbb{R}^2$ be a set whose projection to each coordinate direction has 1-measure zero. Then $E\in NED^w$. If, in addition, $m_2(\overline{E})=0$, then $E\in NED$.*

This was proved for closed sets by Ahlfors–Beurling [AB50, Theorem 10]. For sets that are not closed the proof is substantially more complicated. Note that $\overline{E}$ is not *NED* in general even if it has measure zero. For example take $E=\mathbb{Q}\times\{0\}$, which is *NED* by Theorem 1.6. However, its closure is $\overline{E}=\mathbb{R}\times\{0\}$, which has measure zero, but it is not *NED* since it is not totally disconnected.

We prove Theorem 6.6 only in dimension 2, because of the following lemma, whose statement is very similar to Lemma 3.5. Except for that lemma, none of the arguments in the proof of Theorem 6.6 depend on the dimension.

**Lemma 6.7.** *Let $E \subset \mathbb{R}^2$ be a set whose projection to each coordinate direction has 1-measure zero. For $t > 0$, denote by $Q_t$ the open square centered at the origin with side length $t$ and sides parallel to the coordinate axes. Let $0 < r < R$ and suppose that $F_1, F_2 \subset \mathbb{R}^2$ are disjoint continua such that $\partial Q_t$ intersects both $F_1$ and $F_2$ for every $r < t < R$. Then*

$$\mathrm{Mod}_n(\Gamma(F_1, F_2; Q_R \setminus \overline{Q_r}) \cap \mathcal{F}_0(E)) \geq \frac{1}{4} \log\left(\frac{R}{r}\right).$$

*Proof.* We have $\partial Q_t \cap E = \emptyset$ for a.e. $t \in (r, R)$. Let $\rho$ be an admissible function for $\Gamma(F_1, F_2; Q_R \setminus \overline{Q_r}) \cap \mathcal{F}_0(E)$. Then

$$1 \leq \int_{\partial Q_t} \rho \, ds \leq \left(\int_{\partial Q_t} \rho^2 \, ds\right)^{1/2} \sqrt{4t}$$

for a.e. $t \in (r, R)$. By integration and Fubini's theorem, we have

$$\frac{1}{4} \log\left(\frac{R}{r}\right) = \int_r^R \frac{1}{4t} \, dt \leq \int_{Q_R \setminus \overline{Q_r}} \rho^2.$$

Infimizing over $\rho$ gives the conclusion. □

The proof of the following lemma is exactly the same as the proof of Lemma 3.8, where one uses Lemma 6.7 in place of Lemma 3.5; see Remark 3.9.

**Lemma 6.8.** *Let $E \subset \mathbb{R}^2$ be a set whose projection to each coordinate direction has 1-measure zero. Then for every open set $U \subset \mathbb{R}^2$ and for every pair of non-empty, disjoint continua $F_1, F_2 \subset U$ we have*

$$\mathrm{Mod}_2(\Gamma(F_1, F_2; U) \cap \mathcal{F}_0(E)) = \lim_{r \to 0} \mathrm{Mod}_2(\Gamma(F_1^r, F_2^r; U) \cap \mathcal{F}_0(E)).$$

*Proof of Theorem 6.6.* Let $F_1, F_2 \subset \mathbb{R}^2$ be non-empty, disjoint continua. We fix a small $r > 0$ so that $F_1^r \cap F_2^r = \emptyset$ and let $\rho \colon \mathbb{R}^2 \to [0, \infty]$ be a Borel function with $\rho \in L^2(\mathbb{R}^2)$ that is admissible for $\Gamma(F_1^r, F_2^r; \mathbb{R}^2) \cap \mathcal{F}_0(E)$.

Consider a sequence of open sets $\{V_m\}_{m \in \mathbb{N}}$, such that $E \subset V_{m+1} \subset V_m \subset N_{1/m}(E)$ and such that the projection of $V_m$ to each coordinate axis has measure less than $1/m$ for each $m \in \mathbb{N}$. Observe that $\bigcap_{m=1}^\infty V_m$ has 2-measure zero. For each $m \in \mathbb{N}$ define the closed set $X_m = (\mathbb{R}^2 \setminus V_m) \cup F_1^r \cup F_2^r$. Note that a.e. line parallel to a coordinate direction does not intersect the set $\bigcap_{m=1}^\infty V_m$. Hence, a.e. line parallel to a coordinate direction lies in $\bigcup_{m=1}^\infty X_m$. Moreover, if $\gamma$ is a rectifiable path in $X_m$ joining $F_1^r$ to $F_2^r$, then $\gamma$ has a subpath in $\mathbb{R}^2 \setminus V_m \subset \mathbb{R}^2 \setminus E$ joining $F_1^r$ to $F_2^r$. By the admissibility of $\rho$, we obtain

$$\int_\gamma \rho \, ds \geq 1.$$

For each $m \in \mathbb{N}$, we define on $X_m$ the function

$$g_m(x) = \min\left\{\inf_{\gamma_x} \int_{\gamma_x} \rho \, ds, 1\right\}$$

where the infimum is taken over all rectifiable paths $\gamma_x$ in $X_m$ that connect $F_1^r$ to $x$. By [JJR+07, Corollary 1.10], the function $g_m$ is measurable; the fact that

$X_m$ is closed, thus complete, is important here. One can alternatively argue using [HKST15, Lemma 7.2.13, p. 187]. Moreover, we have $0 \leq g_m \leq 1$, $g_m = 0$ on $F_1^r$, and $g_m = 1$ on $F_2^r$. Next, we show that $\rho$ is an *upper gradient* of $g_m$. That is,

$$|g_m(y) - g_m(x)| \leq \int_\gamma \rho \, ds$$

for every rectifiable path $\gamma\colon [0,1] \to X_m$ with $\gamma(0) = x$ and $\gamma(1) = y$. Since the roles of $x$ and $y$ are symmetric, we will only show that

$$g_m(y) - g_m(x) \leq \int_\gamma \rho \, ds.$$

If $g_m(y) = 1$, then this inequality is immediate, since then $g_m(y) - g_m(x) \leq 0$. Suppose $g_m(y) = \inf_{\gamma_y} \int_{\gamma_y} \rho \, ds$. We fix a curve $\gamma_x$ joining $F_1^r$ to $x$. Define a curve $\gamma_y$ by concatenating $\gamma_x$ with $\gamma$. Then

$$g_m(y) \leq \int_{\gamma_y} \rho \, ds = \int_\gamma \rho \, ds + \int_{\gamma_x} \rho \, ds.$$

Infimizing over $\gamma_x$ gives the desired inequality.

The sequence of sets $\{X_k\}_{k\in\mathbb{N}}$ is increasing. Thus, if $k \geq m$, then $g_k$ is defined by infimizing over a larger collection of paths compared to the definition of $g_m$. It follows that $0 \leq g_k \leq g_m$ in $X_m$. Therefore, for each $m \in \mathbb{N}$, $g_k$ converges pointwise as $k \to \infty$ to a measurable function $g$ in $X_m$. Moreover, by the pointwise convergence, for each $m \in \mathbb{N}$ the function $\rho$ is an upper gradient of $g$ in $X_m$, $0 \leq g \leq 1$ in $\bigcup_{m=1}^\infty X_m$, $g = 0$ in a neighborhood of $F_1$, and $g = 1$ in a neighborhood of $F_2$. On $\mathbb{R}^2 \setminus \bigcup_{m=1}^\infty X_m \subset \bigcap_{m=1}^\infty V_m$ we define $g = 0$. Thus, we have extended $g$ to a measurable function in $\mathbb{R}^2$.

We claim that $g$ is absolutely continuous in a.e. line segment parallel to a coordinate direction. Let $L$ be a line segment parallel to $e_1 = (1,0)$. By construction, for $\mathscr{H}^1$-a.e. $z \in \{e_1\}^\perp$ the line segment $L + z$ lies in $X_m$ for some $m \in \mathbb{N}$. Moreover, since $\rho \in L^2(\mathbb{R}^2)$, we have $\int_{L+z} \rho \, ds < \infty$ for a.e. $z \in \{e_1\}^\perp$. By the upper gradient inequality we conclude that $g$ is absolutely continuous in $L + z$ for a.e. $z \in \{e_1\}^\perp$ and $|g_x| \leq \rho$ almost everywhere on $L + z$. This implies that $|g_x| \leq \rho$ a.e. Similarly, $|g_y| \leq \rho$ a.e. Thus, $g$ lies in the classical Sobolev space $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$ and $|\nabla g| \leq \sqrt{2}\rho$ a.e. in $\mathbb{R}^2$; see [Zie89, Theorem 2.1.4, p. 44].

If $\mathbb{R}^2 \setminus \overline{E} \neq \emptyset$, then each $x \in \mathbb{R}^2 \setminus \overline{E}$ has a bounded open neighborhood $Y$ that is disjoint from $V_m$ for sufficiently large $m$, since $V_m \subset N_{1/m}(\overline{E})$. Thus, $Y \subset X_m$ and $g \in W^{1,2}(Y)$. Since $\rho$ is an upper gradient of $g$ in $Y$, by [Haj03, Corollary 7.15] we conclude that $|\nabla g| \leq \rho$ a.e. in $Y$. Therefore, $|\nabla g| \leq \rho$ a.e. in $\mathbb{R}^2 \setminus \overline{E}$. Summarizing, at a.e. point of $\mathbb{R}^2$ we have

$$|\nabla g| \leq \rho \chi_{\mathbb{R}^2 \setminus \overline{E}} + \sqrt{2}\rho\chi_{\overline{E}}.$$

Since $g = 0$ in a neighborhood of $F_1$ and $g = 1$ in a neighborhood of $F_2$, for each $\varepsilon > 0$ there exists (by mollification) a smooth function $g_\varepsilon$ on $\mathbb{R}^2$ with the same properties and with $\|\nabla g_\varepsilon\|_{L^2(\mathbb{R}^2)}^2 < \|\nabla g\|_{L^2(\mathbb{R}^2)}^2 + \varepsilon$; see [Zie89, Lemma 2.1.3, p. 43]. It is immediate that $|\nabla g_\varepsilon|$ is admissible for $\Gamma(F_1, F_2; \mathbb{R}^2)$. Thus,

$$\operatorname{Mod}_2 \Gamma(F_1, F_2; \mathbb{R}^2) \leq \|\nabla g_\varepsilon\|_{L^2(\mathbb{R}^2)}^2 \leq \|\rho\|_{L^2(\mathbb{R}^2 \setminus \overline{E})}^2 + 2\|\rho\|_{L^2(\overline{E})}^2 + \varepsilon.$$

First we let $\varepsilon \to 0$, and then we infimize over $\rho$ to obtain

$$\operatorname{Mod}_2 \Gamma(F_1, F_2; \mathbb{R}^2) \leq c_0 \operatorname{Mod}_2(\Gamma(F_1^r, F_2^r; \mathbb{R}^2) \cap \mathcal{F}_0(E)),$$

where $c_0 = 1$ if $m_2(\overline{E}) = 0$ and $c_0 = 2$ otherwise. Now, we let $r \to 0$ and by Lemma 6.8 we obtain

$$\operatorname{Mod}_2 \Gamma(F_1, F_2; \mathbb{R}^2) \leq c_0 \operatorname{Mod}_2(\Gamma(F_1, F_2; \mathbb{R}^2) \cap \mathcal{F}_0(E)). \qquad \square$$

6.3. **A non-measurable *CNED* set.** Sierpiński [Sie20], using the axiom of choice but not the continuum hypothesis, constructed a striking example of a non-measurable set $E \subset \mathbb{R}^2$ such that every line intersects $E$ in at most two points. This example served at that time as a counterexample to the converse of Fubini's theorem: if the slices of a planar set are measurable, then is the whole set measurable? We show here that Sierpiński's set is *CNED*. Thus, the assumption of measurability in Lemma 2.5 is necessary in order to derive that *CNED* sets have measure zero.

**Proposition 6.9.** *There exists a non-measurable set $E \subset \mathbb{R}^2$ that is CNED.*

*Proof.* Let $E$ be the non-measurable set of Sierpiński. Let $F_1, F_2 \subset \mathbb{R}^2$ be non-empty, disjoint continua. We will show that

$$\operatorname{Mod}_2 \Gamma(F_1, F_2; \mathbb{R}^2) \leq \operatorname{Mod}_2(\Gamma(F_1, F_2; \mathbb{R}^2) \cap \mathcal{F}_\sigma(E)).$$

According to a remarkable result of Aseev [Ase09, Theorem 2.1], we have

$$\operatorname{Mod}_2 \Gamma(F_1, F_2; \mathbb{R}^2) = \operatorname{Mod}_2(\Gamma(F_1, F_2; \mathbb{R}^2) \cap \mathcal{F}),$$

where $\mathcal{F}$ is the family of *piecewise linear curves with respect to* $\mathbb{R}^2 \setminus (F_1 \cup F_2)$; that is, $\gamma \in \mathcal{F}$ if each point of $|\gamma| \setminus (F_1 \cup F_2)$ has a neighborhood $V$ such that $|\gamma| \cap V$ consists of finitely many straight line segments. Hence, it suffices to show that

$$\Gamma(F_1, F_2; \mathbb{R}^2) \cap \mathcal{F} \subset \Gamma(F_1, F_2; \mathbb{R}^2) \cap \mathcal{F}_\sigma(E).$$

Let $\gamma \in \Gamma(F_1, F_2; \mathbb{R}^2) \cap \mathcal{F}$. By the properties of the set $E$, and since $\gamma$ is piecewise linear, the set $|\gamma| \cap E$ is countable. Hence, $\gamma \in \mathcal{F}_\sigma(E)$, as desired. $\square$

## 7. Examples of non-negligible sets

*Proof of Proposition 1.11.* Let $E$ be the residual set of a packing as in the statement. Let $\Gamma_0$ denote the family of non-constant curves that intersect the set

$$S = \bigcup_{\substack{i,j \in \mathbb{N} \\ i \neq j}} (\partial D_i \cap \partial D_j).$$

Since $S$ is countable, we have $\operatorname{Mod}_n \Gamma_0 = 0$. Consider continua $F_1, F_2$, contained in $D_1, D_2$, respectively. The claim that $E \notin \mathit{CNED}$ follows once we establish that

$$\Gamma(F_1, F_2; D_0) \cap \mathcal{F}_\sigma(E) \subset \Gamma_0. \tag{7.1}$$

Let $\gamma \in \Gamma(F_1, F_2; D_0) \setminus \Gamma_0$. By considering a weak subpath, we assume that $\gamma$ is a simple path with the same properties. We consider an injective parametrization $\gamma \colon [0,1] \to D_0$. Let $A$ be the set of $t \in [0,1]$ such that there exists $\delta > 0$ and $i \in \mathbb{N}$ with the property that $\gamma((t-\delta, t) \cup (t, t+\delta)) \subset D_i$ and $\gamma(t) \in \partial D_i$; that is, $\gamma$ "bounces" on $\partial D_i$ at time $t$. Then $A$ is countable and relatively open in $\gamma^{-1}(E)$.

We claim that the compact set $B = \gamma^{-1}(E) \setminus A$ is non-empty and perfect, in which case, it is uncountable. Therefore, $|\gamma| \cap E$ is uncountable and $\gamma \notin \mathcal{F}_\sigma(E)$, which completes the proof of (7.1). To see that $B$ is non-empty, let $t = \sup\{a \in$

$[0,1] : \gamma(a) \in D_1\}$. Then $\gamma(t) \in \partial D_1$, so $t \in \gamma^{-1}(E)$, $\gamma((t-\delta,t)) \cap D_1 \neq \emptyset$ for every $\delta > 0$, and $\gamma((t,1]) \cap D_1 = \emptyset$. Thus, $t \notin A$ and $t \in B$, showing that $B \neq \emptyset$. We now show perfectness. Let $t \in B$ and let $I \subset [0,1]$ be an open interval containing $t$.

*Case 1:* Suppose that $\gamma(I) \subset E$. Since $A$ is countable, there exists $s \in I \setminus A$, $s \neq t$, with $\gamma(s) \in E$. Hence, $(I \setminus \{t\}) \cap B \neq \emptyset$.

*Case 2:* Suppose that $\gamma(I) \cap D_{i_0} \neq \emptyset$ for some $i_0 \in \mathbb{N}$ and $\gamma(t) \notin \partial D_{i_0}$. Without loss of generality $\gamma(s) \in D_{i_0}$ for some $s \in I$ with $s < t$. Let $s_1 = \sup\{a \in (s,t) : \gamma(a) \in D_{i_0}\}$. Then $\gamma(s_1) \in \partial D_{i_0}$, $s_1 \neq t$, $\gamma((s_1-\delta, s_1)) \cap D_{i_0} \neq \emptyset$ for every $\delta > 0$, and $\gamma((s_1,t)) \cap D_{i_0} = \emptyset$. Thus, $s_1 \notin A$ and we have $s_1 \in I \cap B$, so $(I \setminus \{t\}) \cap B \neq \emptyset$.

*Case 3:* Suppose $\gamma(t) \in \partial D_{i_0}$ for some $i_0 \in \mathbb{N}$. Since $t \notin A$, there exists an open subinterval of $I$, say $J = (s,t)$, such that $\gamma(J)$ intersects the complement of $D_{i_0}$. If $\gamma(J) \subset E$, then by Case 1 we have $(J \setminus \{t\}) \cap B \neq \emptyset$. Suppose that $\gamma(J) \cap D_{i_1} \neq \emptyset$ for some $i_1 \neq i_0$. Then $\gamma(t) \notin \partial D_{i_1}$, since $\gamma \notin \Gamma_0$ and $\gamma$ avoids the set $\partial D_{i_1} \cap \partial D_{i_0}$. We are now reduced to Case 2 with $i_1$ in place of $i_0$ and $J$ in place of $I$. $\square$

*Remark* 7.1. One can relax the assumption of the proposition to requiring that $\partial D_i \cap \partial D_j$, $i \neq j$, has Sobolev $n$-capacity zero (see [HKST15, Section 7.2]). Then the family of curves passing through $\partial D_i \cap \partial D_j$ has $n$-modulus zero.

Next, we establish a preliminary elementary result before proving Theorem 1.9.

**Lemma 7.2.** *Let $U \subset \mathbb{R}$ be an open set and $E \subset \mathbb{R}$ be a compact set with $m_1(E) > 0$. Then there exists a sequence of similarities $\tau_i \colon \mathbb{R} \to \mathbb{R}$, $i \in \mathbb{N}$, and a set $N \subset \mathbb{R}$ with $m_1(N) = 0$ such that*

$$U = N \cup \bigcup_{i\in\mathbb{N}} \tau_i(E).$$

*Proof.* Suppose that $E \subset (a,b)$ and set $\lambda = m_1(E)/(b-a)$. Moreover, suppose that $U$ is bounded. Let $U_0 = U$ and let $I_{0,i}$, $i \in \mathbb{N}$, be the connected components of $U_0$, which are bounded open intervals. For each $i \in \mathbb{N}$, define $\tau_{0,i}$ to be the similarity that maps $(a,b)$ onto $I_{0,i}$. Then $m_1(\tau_{0,i}(E))/m_1(I_{0,i}) = \lambda$ and

$$m_1\left(\bigcup_{i\in\mathbb{N}} \tau_{0,i}(E)\right) = \lambda m_1(U_0).$$

We now define $U_1 = U_0 \setminus \bigcup_{i\in\mathbb{N}} \tau_{0,i}(E)$, which is open, and note that $m_1(U_1) = (1-\lambda)m_1(U_0)$. We proceed in the same way to obtain similarities $\tau_{1,i}$, $i \in \mathbb{N}$, that map $(a,b)$ to the connected components of $U_1$. In the $k$-th step, we obtain the set

$$U_k = U_0 \setminus \bigcup_{j=0}^{k-1} \bigcup_{i\in\mathbb{N}} \tau_{j,i}(E)$$

with $m_n(U_k) = (1-\lambda)^k m_1(U_0)$. Thus, $N = U_0 \setminus \bigcup_{j=0}^{\infty} \bigcup_{i\in\mathbb{N}} \tau_{j,i}(E)$ is a null set, as desired. If $U$ is unbounded, we can simply write it as a countable union of bounded open sets and apply the previous result to each of them. $\square$

*Proof of Theorem 1.9.* According to a construction of Wu [Wu98, Example 2], there exist Cantor sets $G, F \subset \mathbb{R}$ such that $m_1(G) = 0$, $m_1(F) > 0$, and $G \times F$ is removable for the Sobolev space $W^{1,2}$. Thus, $G \times F$ is of class *NED* $\subset$ *CNED*; this follows from [VG77].

By Lemma 7.2, there exist countably many scaled and translated copies $F_i \subset (0,1)$, $i \in \mathbb{N}$, of $F$, such that the set $E_1 = [0,1] \setminus \bigcup_{i\in\mathbb{N}} F_i$ has 1-measure zero. We let $E_2 = [0,1] \setminus E_1 = \bigcup_{i\in\mathbb{N}} F_i$. We have

$$G \times [0,1] = G \times (E_1 \cup E_2) = (G \times E_1) \cup (G \times E_2).$$

The set $G \times [0,1]$ is not *QCH*-removable (recall the discussion in the Introduction), so it is not *CNED* by Theorem 1.1; this can also be proved directly.

On the other hand, for each $i \in \mathbb{N}$ the set $G \times F_i$ is the quasiconformal image of $G \times F$, which is *NED*. Compact *NED* sets are invariant under quasiconformal maps by Corollary 4.2 (this also follows from [AB50, Theorem 4]). Thus, $G \times F_i$ is *NED* for each $i \in \mathbb{N}$. By Theorem 1.2 we conclude that

$$G \times E_2 = \bigcup_{i\in\mathbb{N}} G \times F_i \in NED\,.$$

Finally, note that the projections of the set $G \times E_1$ to the coordinate axes have measure zero. Moreover, $\overline{G \times E_1} \subset G \times [0,1]$, and the latter has 2-measure zero. By Theorem 6.6 we conclude that $G \times E_1 \in NED$. ☐

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