Robust Synergistic Hybrid Feedback

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Abstract— Synergistic hybrid feedback refers to a collection of feedback laws that allow for global asymptotic stabilization of a compact set through the following switching logic: given a collection of Lyapunov functions that are indexed by a logic variable, whenever the currently selected Lyapunov function exceeds the value of another function in the collection by a given margin, then a switch to the corresponding feedback law is triggered. This kind of feedback has been under development over the past decade, and it has led to multiple solutions for global asymptotic stabilization problems on compact manifolds. The contributions of this paper include a synergistic controller design in which the logic variable is not necessarily constant between jumps, a synergistic hybrid feedback that is able to tackle the presence of parametric uncertainty, backstepping of adaptive synergistic hybrid feedbacks, and a demonstration of the proposed solutions to the problem of global obstacle avoidance.

Index Terms— Hybrid Systems, Adaptive Control, Robotics, Uncertain Systems

I. INTRODUCTION

A. Background and Motivation

In this paper, we address the problem of designing a controller for continuous-time plants of the form

$$\dot{x}_p = F_p(x_p, u_p, \theta) \tag{1}$$

where $x_p \in \mathcal{X}_p$ denotes the state of the plant, $u_p \in \mathcal{U}_p$ is the input, and θ represents the plant parameters. The main goal is to design a controller that globally asymptotically stabilizes a compact set $\mathcal{A}$ for the closed-loop system both when the parameter θ is known, but also when it is only known to belong to a given compact set Ω.

In the presence of topological obstructions, this objective is not attainable via continuous feedback and, even though it might be attainable through discontinuous feedback, the resulting closed-loop system may not be robust to arbitrarily small noise (cf. [1] and [2]). These limitations of continuous and discontinuous feedbacks constitute the motivation for the development of synergistic hybrid feedback.

In this paper, we present two novel synergistic hybrid controllers for global asymptotic stabilization of a compact set for a closed-loop system. The first controller design considers that the parameter θ is known, while the second controller design considers that θ is unknown but belongs to a known compact set Ω. Additionally, we apply the proposed controller designs to the problem of global asymptotic stabilization of a compact set for control affine systems and to a problem of global obstacle avoidance.

B. Literature Review

Synergistic hybrid feedback is a hybrid control strategy that consists of a collection of potential functions that asymptotically stabilize a given compact set by gradient descent feedback. If, for all equilibria that do not lie within the given compact set, there exists another function in the collection that has a lower value and does not share the same equilibria, then it is possible to achieve global asymptotic stabilization of the given compact set through hysteretic switching (see, e.g., [3]).

Synergistic hybrid feedback came to prominence with the work [4] on quaternion-based feedback for global asymptotic stabilization attitude tracking, thereby solving the attitude control problem (cf. [5]). The framework of synergistic hybrid feedback provides not only a solution to the problem of attitude control but, more importantly, it provides a robust solution for global asymptotic stabilization on compact manifolds. The works [6], [7] and [8] leverage the concepts at the root of synergistic hybrid feedback and use them to design controllers that are applicable to a broad class of systems. However, most of the contributions to this class of hybrid controllers are on the control of robotic systems, such as pendulum stabilization [9], vector-based rigid body stabilization [10], [11], [12], tracking for marine and aerial vehicle [13], [14], [15], and rigid body trajectory tracking [16], [17], [18]. Within the field of robotics, we single out the problem of obstacle avoidance, which is also addressed in this paper.

Obstacle avoidance is an important and longstanding problem that reflects the need to drive the state of a system from one place to another while avoiding obstacles in its way. Several solutions to this problem have been proposed over the last few decades as highlighted in [19]. In particular, it is possible to find both stochastic [20] and deterministic approaches [21] to tackle the obstacle avoidance problem.

However, it was shown in [22] that in a "sphere world," there is at least one saddle equilibrium point for each obstacle within the state space, thus precluding global asymptotic stabilization of a setpoint by continuous feedback. To address this limitation, hybrid control solutions to the problem of obstacle avoidance were proposed in [23], [24], [8], [25], [26], and [27].

Although not directly addressed in this paper, the concepts of synergistic hybrid feedback have also been used for observer design and optimization in [28] and [29], respectively.

C. Contributions

The contributions in this paper are as follows: 1) We develop a dynamic synergistic hybrid feedback controller for global asymptotic stabilization of a broad class of dynamical systems. Similarly to [18] and [15], we consider that the distinguishing feature of synergistic hybrid feedback is the switching logic, thus we depart from earlier works which were limited to controller variables that were constant during flows, and we propose a controller design that encompasses all the aforementioned works on synergistic hybrid feedback as special cases. The significance of this contribution stems from the fact that we provide conditions for global asymptotic stability at the level of the controller data so that practitioners can easily verify whether the proposed controller design is applicable to their problem of interest; 2) We provide a modification to the synergistic controller that takes into account the presence of parametric uncertainty. In this way, we extend the results of [30] and [14] to a more general setting, and we improve the jump dynamics to enable faster convergence of solutions to the hybrid closed-loop system to a desired compact set; 3) We demonstrate how a synergistic hybrid feedback controller can be used to develop an adaptive synergistic controller for the stabilization of compact sets for affine control systems under matched uncertainties; 4) We show that the proposed adaptive controller is amenable to hybrid backstepping; 5) We apply the proposed controllers to the problem of global obstacle avoidance in the presence of parametric uncertainty and illustrate the behavior of the closed-loop system through simulations.

The paper is organized as follows: in Section III we present the main assumptions on the plant dynamics. In Section IV-B provides conditions under which the closed-loop system is well-posed. In Section IV-C we provide sufficient conditions for global asymptotic stability of a compact set for the closed-loop system. In Section V, we develop the concept of robust synergistic hybrid feedback. In Section VI, we apply the synergistic approach to the development of an adaptive synergistic controller for stabilization of affine control systems subject to matched uncertainties. In Section VII, we apply the proposed controller to the problem of global obstacle avoidance. In Section VIII, we present some concluding remarks.

A preliminary version of this paper was presented at the 2019 ACC with a simpler synergistic controller design for global asymptotic stabilization of control affine systems and without the full proofs (cf. [30]). An extended version of this paper can be found at [31].

II. NOTATION & PRELIMINARIES

The Cartesian Product $\mathbb{R}^n = \mathbb{R} \times \ldots \times \mathbb{R}$ of n copies of the real line together with scalar multiplication and component-wise addition of vectors is known as n-dimensional Euclidean space. The Euclidean metric topology is the one induced by the metric $x \mapsto |x| := \sqrt{x^\top x}$. The n-dimensional Euclidean space has the topology generated by a countable basis of open balls of the form $c + \epsilon \mathbb{B} := \{x \in \mathbb{R}^n : |x - c| < \epsilon\}$, where $c \in \mathbb{R}^n$ and $\epsilon > 0$. More generally, given a set $\Omega \subset \mathbb{R}^n$, we define $\Omega + \epsilon \mathbb{B} := \bigcup_{c \in \Omega} c + \epsilon \mathbb{B}$. The operators ∂S and $\overline{S}$ denote the boundary and the closure of a set S, respectively. Given a function $f : \mathbb{R}^m \to \mathbb{R}^n$, the preimage of a set $U \subset \mathbb{R}^n$ through f is $f^{-1}(U) := \{x \in \mathbb{R}^m : f(x) \in U\}$. Similarly, the image of a set W through f is $f(W) := \{y \in \mathbb{R}^n : y = f(x) \text{ for some } x \in W\}$. Given a subset S of $X := X_1 \times X_2$, the projection of S onto X_1 is represented by $\pi_{X_1}(S) := \{x_1 \in X_1 : (x_1, x_2) \in S \text{ for some } x_2 \in X_2\}$. Similarly, the projection of S onto X_2 is denoted by $\pi_{X_2}(S) := \{x_2 \in X_2 : (x_1, x_2) \in S \text{ for some } x_1 \in X_1\}$. A set-valued map M from $S \subset \mathbb{R}^m$ to the power set of some Euclidean space $\mathbb{R}^n$ is represented by $M : S \rightrightarrows \mathbb{R}^n$. The domain of a set-valued map is given by $\operatorname{dom} M := \{x \in \mathbb{R}^n : M(x) \neq \emptyset\}$. Given a subset S of $\mathbb{R}^m$, a set-valued map $M : S \rightrightarrows \mathbb{R}^n$ is said to be outer semicontinuous (osc) relative to S if its graph, given by $\operatorname{gph} M := \{(x, y) \in S \times \mathbb{R}^n : y \in M(x)\}$, is closed relative to $S \times \mathbb{R}^n$. The set-valued map M is locally bounded at $x \in \mathbb{R}^m$ if there exists a neighborhood U_x of x such that $M(U_x) \subset \mathbb{R}^n$ is bounded. It is locally bounded relative to S if the set-valued mapping from $\mathbb{R}^m$ to $\mathbb{R}^n$ defined by $M(x)$ for $x \in S$ and $\emptyset$ for $x \notin S$ is locally bounded at each $x \in S$. It is convex-valued if $M(x)$ is convex for each $x \in S$. The tangent cone to a set $S \subset \mathbb{R}^n$ at a point $x \in \mathbb{R}^n$, denoted by $\mathsf{T}_x S$, is the set of all vectors $w \in \mathbb{R}^n$ for which there exists $x_i \in S$, $\tau_i > 0$ with $x_i \to x$, τ_i convergent to 0 from above, and $w = \lim_{i \to \infty} \frac{x_i - x}{\tau_i}$. Given a differentiable function $F : \mathbb{R}^{m \times n} \to \mathbb{R}^{p \times q}$, we define $\mathcal{D}F(X) := \frac{\partial \operatorname{vec}(F)}{\partial \operatorname{vec}(X)^\top}(X)$ for each $X \in \mathbb{R}^{m \times n}$, where $\operatorname{vec}(A) := [e_1^\top A^\top \ \ldots \ e_m^\top A^\top]^\top$ for each $A \in \mathbb{R}^{m \times n}$ and $e_i \in \mathbb{R}^m$ is a vector of zeros, except for the i-th component, which is 1. If F has multiple arguments, say $(X, Y) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{k \times \ell}$, we define $\mathcal{D}_X F(X, Y) := \frac{\partial \operatorname{vec}(F)}{\partial \operatorname{vec}(X)^\top}(X, Y)$ for each $(X, Y) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{k \times \ell}$. If $F : \mathbb{R}^n \to \mathbb{R}$, then $\nabla F(x) := \mathcal{D}F(x)^\top$ for each $x \in \mathbb{R}^n$. If $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, then $\nabla_x F(x, y) := \mathcal{D}_x F(x, y)^\top$ for each $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ and $\nabla_y F(x, y) := \mathcal{D}_y F(x, y)^\top$ for each $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$. Clarke's generalized directional derivative of a function $V : \mathbb{R}^n \to \mathbb{R}$ in the direction v, is defined as follows (c.f. [32, Eq. (1)]): $V^\circ(x; v) := \limsup_{\substack{y \to x \\ \lambda \searrow 0}} \frac{V(y + \lambda v) - V(y)}{\lambda}$.

A hybrid system $\mathcal{H}$ with state space $\mathbb{R}^n$ is defined in [33] and [34] as

$$\begin{aligned} \dot{\xi} &\in F(\xi) & \xi &\in C \\ \xi^+ &\in G(\xi) & \xi &\in D \end{aligned} \tag{2}$$

where $\xi \in \mathbb{R}^n$ is the state, $C \subset \mathbb{R}^n$ is the flow set, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the flow map, $D \subset \mathbb{R}^n$ denotes the jump set, and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ denotes the jump map. A solution ξ to $\mathcal{H}$ is parametrized by (t, j), where t denotes ordinary time and j denotes the jump time, and its domain $\operatorname{dom} \xi \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain: for each $(T, J) \in$

dom ξ, dom $\xi \cap ([0,T] \times \{0,1,\dots J\})$ can be written in the form $\cup_{j=0}^{J-1}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_J$, where $I_j := [t_j, t_{j+1}]$ and the t_j's define the jump times. A solution ξ to a hybrid system is said to be *maximal* if it cannot be extended by flowing nor jumping and *complete* if its domain is unbounded.

A set S is said to be forward pre-invariant for a hybrid system (2) if each maximal solution of (2) starting in S remains in S. It is said to be forward invariant if it is forward pre-invariant and each maximal solution from S is complete (see e.g. [34, Chapters 3 and 7]).

The hybrid basic conditions provide a set of sufficient conditions for well-posedness and they are as follows (cf. [33, Assumption 6.5]):

(A1) C and D are closed subsets of $\mathbb{R}^n$;
(A2) $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is osc and locally bounded relative to C, $C \subset \operatorname{dom} F$, and $F(x)$ is convex for every $x \in C$;
(A3) $G: \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is osc and locally bounded relative to D, and $D \subset \operatorname{dom} G$.

Given a function $V: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that is Lipschitz continuous on a neighborhood of C in (2) and $u_c: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, we say that the growth of V along flows of (2) is bounded by u_c if the following holds:

$$V^\circ(\xi; f) \leq u_c(\xi) \qquad \forall \xi \in C, \ \forall f \in F(\xi) \cap \mathsf{T}_\xi C. \qquad (3)$$

If, for some function $u_d: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$,

$$V(\xi') - V(\xi) \leq u_d(\xi) \qquad \forall \xi \in D, \ \forall \xi' \in G(\xi), \qquad (4)$$

then we say that the growth of V along jumps of (2) is bounded by u_d. If both (3) and (4) hold, then we say that the growth of V along solutions to (2) is bounded by u_c, u_d. A function $V: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is positive definite relative to a set $\mathcal{A}$ if $V(x) = 0 \iff x \in \mathcal{A}$.

A compact set $\mathcal{A}$ is said to be *stable* for (2) if for every $\epsilon > 0$ there exists $\delta > 0$ such that every solution ϕ to (2) with $|\phi(0,0)|_\mathcal{A} \leq \delta$ satisfies $|\phi(t,j)|_\mathcal{A} \leq \epsilon$ for all $(t,j) \in \operatorname{dom}\phi$; *globally pre-attractive* for (2) if every solution ϕ to (2) is bounded and, if it is complete, then also $\lim_{t+j \to +\infty} |\phi(t,j)|_\mathcal{A} = 0$; *globally pre-asymptotically stable* for (2) if it is both stable and globally pre-attractive. If every maximal solution to (2) is complete then one may drop the prefix "pre."

Given a hybrid system $\mathcal{H}$, a set $S \subset \mathbb{R}^n$ is said to be: weakly forward invariant if for every $\xi \in S$ there exists at least one complete solution contained in S; weakly backward invariant if for every $\xi \in S$, every $\tau > 0$, there exists at least one maximal solution ϕ from S such that for some $(t^\star, j^\star) \in \operatorname{dom}\phi$, $t^\star + j^\star \geq \tau$, it is the case that $\phi(t^\star, j^\star) = \xi$ and $\phi(t,j) \in S$ for all $(t,j) \in \operatorname{dom}\phi$ with $t + j \leq t^\star + j^\star$; weakly invariant if it is both weakly forward invariant and weakly backward invariant.

III. PROBLEM SETUP

To stabilize the plant given in (1), we propose the following state-based hybrid controller

$$\left. \begin{aligned} \dot{\chi}_c &\in \hat{F}_c(x_p, \chi_c, x_c, u_c) \\ \dot{x}_c &\in F_c(x_p, \chi_c, x_c) \end{aligned} \right\} \quad (x_p, \chi_c, x_c) \in C, \ u_c \in \mathcal{U}_c$$

$$\left. \begin{aligned} \chi_c^+ &= \chi_c \\ x_c^+ &\in G_c(x_p, \chi_c, x_c) \end{aligned} \right\} \quad (x_p, \chi_c, x_c) \in D$$

$$(5)$$

where $\chi_c \in \hat{\mathcal{X}}_c$ and $x_c \in \mathcal{X}_c$ represent different components of the state of the controller, $\hat{F}_c$ and F_c are the flow maps associated with χ_c and x_c, respectively, C denotes the flow set, G_c defines the update law for jumps of x_c and D is the jump set. The key differences between χ_c and x_c are the fact that χ_c does not change its value during jumps and also that the flows of χ_c depend on a virtual input variable $u_c \in \mathcal{U}_c$.

If $\hat{F}_c$ in (5) defining the dynamics of χ_c is given, then χ_c can become part of the state of (1) and the stated objective can be attained through the design of a hybrid controller $\mathcal{H}_c := (C, F_c, D, G_c)$ with state $x_c \in \mathcal{X}_c$ and dynamics

$$\begin{aligned} \dot{x}_c &\in F_c(x, x_c) & (x, x_c) \in C \\ x_c^+ &\in G_c(x, x_c) & (x, x_c) \in D \end{aligned}$$

assigning $u := (u_p, u_c) \in \mathcal{U} := \mathcal{U}_p \times \mathcal{U}_c$ via a feedback law $(x, x_c) \mapsto \kappa(x, x_c)$, where $x := (x_p, \chi_c) \in \mathcal{X} := \mathcal{X}_p \times \hat{\mathcal{X}}_c$ is the state of the system to control with dynamics described by the following differential inclusion

$$\dot{x} \in F_\theta(x, x_c, u) := F_p(x_p, u_p, \theta) \times \hat{F}_c(x_p, \chi_c, x_c, u_c), \quad (7)$$

where θ is a constant.

This formulation enables the controller design for systems whose dynamics depend on the controller state. For example, given a plant with dynamics $\dot{x}_p = f_p(x_p) + H_p(x_p)u_p + W_p(x_p)\theta$ where f_p, H_p, W_p are functions with the appropriate dimensions, suppose that χ_c represents the reference trajectory to be tracked and that it is generated by the system $\dot{\chi}_c = f_p(\chi_c) + H_p(\chi_c)\xi_d$ for some signal ξ_d. Defining $u := u_p$ and $x_c := \xi_d$, we have that $F_\theta(x, x_c, u) = (f_p(x_p) + H_p(x_p)u + W_p(x_p)\theta, f_p(\chi_c) + H_p(\chi_c)x_c)$. More practically, x can be considered to be the part of the state of the closed-loop system that remains unchanged during jumps.

Given sets $\mathcal{X}$, $\mathcal{X}_c$, and $\mathcal{U}$, we consider a dynamical system with state $x \in \mathcal{X}$ that is governed by the dynamics (7) where $x_c \in \mathcal{X}_c$ is a controller variable, $u \in \mathcal{U}$ is the input, θ is a constant parameter that belongs to a compact set Ω and F_θ is a set-valued map with the following properties.

Assumption 1. Given sets $\mathcal{X}$, $\mathcal{X}_c$, $\mathcal{U}$, θ belonging to a compact set Ω, and F_θ as in (7) the following properties hold:

(S1) Each set $\mathcal{X}$, $\mathcal{X}_c$ and $\mathcal{U}$ is a closed nonempty subset of some Euclidean space;
(S2) The set-valued map F_θ is outer semicontinuous, locally bounded, and convex-valued.

The condition (S1) allows for the use of the analysis tools for hybrid dynamical systems that are provided in [33] which consider sets as subspaces of Euclidean spaces with the Euclidean metric topology. Condition (S2) is used to prove that the resulting closed-loop system satisfies the hybrid basic conditions.

Remark 1. Since the sets $\mathcal{X}$, $\mathcal{X}_c$ and $\mathcal{U}$ are closed relative to their Euclidean ambient spaces, then any of their closed subsets are also closed in the ambient space and locally compact Hausdorff (cf. [35, Lemma 4.29]).

In Section IV, we develop a dynamic synergistic controller with the objective of globally asymptotically stabilizing a compact set for the resulting closed-loop system under the assumption that θ is known. In Section V, we modify the

dynamic synergistic controller to allow for $\theta \in \Omega$ to be unknown, when Ω is known.

IV. DYNAMIC SYNERGISTIC HYBRID FEEDBACK

A. Controller Design

Dynamic synergistic hybrid feedback (relative to the plant in Section III) is a hybrid control strategy that renders a compact set $\mathcal{A} \subset \mathcal{X} \times \mathcal{X}_c$ globally asymptotically stable for the closed-loop system. It is composed of a feedback law

$$\kappa : \operatorname{dom} \kappa \to \mathcal{U} \tag{8}$$

and of the hybrid dynamics that are described in the sequel.

Given a function

$$V : \operatorname{dom} V \to \mathbb{R}_{\geq 0} \cup \{+\infty\}, \tag{9}$$

satisfying $\mathcal{X} \times \mathcal{X}_c \subset \operatorname{dom} V$ with $\operatorname{dom} V$ open in the Euclidean space containing $\mathcal{X} \times \mathcal{X}_c$,[1] and a set-valued map

$$D_c : \mathcal{X} \times \mathcal{X}_c \rightrightarrows \mathcal{X}_c \tag{10}$$

we define

$$\nu_V(x, x_c) := \min\{V(x, g) : g \in D_c(x, x_c)\}, \tag{11a}$$
$$\varrho_V(x, x_c) := \arg\min\{V(x, g) : g \in D_c(x, x_c)\}, \tag{11b}$$
$$\mu_V(x, x_c) := V(x, x_c) - \nu_V(x, x_c) \tag{11c}$$

for each $(x, x_c) \in \mathcal{X} \times \mathcal{X}_c$. Given a set-valued map F_c defined on $\mathcal{X} \times \mathcal{X}_c$, we define the hybrid controller dynamics as follows:

$$\dot{x}_c \in F_c(x, x_c) \qquad (x, x_c) \in C \tag{12a}$$
$$x_c^+ \in \varrho_V(x, x_c) \qquad (x, x_c) \in D \tag{12b}$$

where

$$\begin{aligned} C &:= \{(x, x_c) \in \mathcal{X} \times \mathcal{X}_c : \mu_V(x, x_c) \leq \delta(x, x_c)\}, \\ D &:= \{(x, x_c) \in \mathcal{X} \times \mathcal{X}_c : \mu_V(x, x_c) \geq \delta(x, x_c)\}, \end{aligned} \tag{13}$$

and $\delta : \mathcal{X} \times \mathcal{X}_c \to \mathbb{R}$ is a continuous function. The switching logic in (12) implements the following functionality: if the solutions to the closed-loop system reach a state (x, x_c) where $\mu_V(x, x_c)$ is greater than or equal to the predefined value of $\delta(x, x_c)$, then the variable x_c is reset to some point $g \in \varrho_V(x, x_c)$ and the feedback law changes its value from $\kappa(x, x_c)$ to $\kappa(x, g)$. Since the hybrid controller (12) is derived from κ, V, D_c, and F_c, we represent (12) using the 4-tuple (κ, V, D_c, F_c).

The hybrid closed-loop system $\mathcal{H} := (C, F_{cl}, D, G_{cl})$ resulting from the interconnection between (7) and (κ, V, D_c, F_c) is given by

$$\begin{pmatrix} \dot{x} \\ \dot{x}_c \end{pmatrix} \in F_{cl}(x, x_c) := \begin{pmatrix} F_\theta(x, x_c, \kappa(x, x_c)) \\ F_c(x, x_c) \end{pmatrix} \quad (x, x_c) \in C \tag{14a}$$

$$\begin{pmatrix} x^+ \\ x_c^+ \end{pmatrix} \in G_{cl}(x, x_c) := \begin{pmatrix} x \\ \varrho_V(x, x_c) \end{pmatrix} \quad (x, x_c) \in D. \tag{14b}$$

Remark 2. Notice that, if $\delta(x, x_c) \geq 0$ for all $(x, x_c) \in \mathcal{X} \times \mathcal{X}_c$, then it follows from the construction of the hybrid controller (κ, V, D_c, F_c) that $V(x, g) - V(x, x_c) \leq 0$ for each

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[1] The function V maps values in $\mathcal{X} \times \mathcal{X}_c$ to the one-point compactification of $\mathbb{R}_{\geq 0}$. More generally, given a topological space X that is noncompact, locally compact Hausdorff space, and an object ∞ not in X, the one point compactification of X is a topological space X^* with the topology: $\mathcal{T} = \{\text{open subsets of } X\} \cup \{U \subset X^* : X^* \backslash U \text{ is a compact subset of } X\}$.

$(x, x_c) \in D$ and each $g \in \varrho_V(x, x_c)$. In other words, if the function δ is nonnegative for all $(x, x_c) \in \mathcal{X} \times \mathcal{X}_c$, then the function V does not increase during jumps along solutions to (14).

The controller design presented in this section is informed by many preceding synergistic hybrid feedback controllers. As mentioned in Section I-C, we preserve the switching logic of the synergistic controllers in [34, Chapter 7], in the sense that controller switching is triggered when the difference between the current value of V and its lowest possible value exceeds a predefined threshold $\delta > 0$. The main difference between the controller design presented in this paper and synergistic controllers in the literature is that, here, x_c does not necessarily belong to a finite set. Instead, the flows of x_c are described more generally by a differential inclusion, and we constrain its jumps using a set-valued map D_c which is instrumental to the backstepping procedure described in Section VI-C.

In the sequel, we introduce the assumptions on the controller that allow for the global asymptotic stabilization of a compact subset of the state space. The proofs of some of the more standard results within the literature on synergistic hybrid feedback are omitted due to space constraints, but can be found in [31].

B. Basic Properties of the Closed-Loop System

In this section, we provide some conditions on (8), (9) and (10) which ensure that the closed-loop system (14) satisfies the hybrid basic conditions of [33, Assumption 6.5] and that maximal solutions to (14) are complete. To this end, we introduce the following definitions.

Definition 1. Given a compact subset $\mathcal{A}$ of $\mathcal{X} \times \mathcal{X}_c$, κ, V, D_c and F_c we say that the hybrid controller (κ, V, D_c, F_c) is a synergistic candidate relative to $\mathcal{A}$ for (7) if the following conditions hold:

(C1) The optimization problem in (11) is feasible for each $(x, x_c) \in \mathcal{X} \times \mathcal{X}_c$, in the sense that, for each $(x, x_c) \in \mathcal{X} \times \mathcal{X}_c$, there exists $g \in D_c(x, x_c)$ such that $V(x, g) < +\infty$.

(C2) F_c is outer semicontinuous, locally bounded, and convex-valued.

(C3) V is continuous, positive definite relative to $\mathcal{A}$, and $V^{-1}([0, c])$ is compact for each $c \in \mathbb{R}_{\geq 0}$;

(C4) The set-valued map D_c is outer semicontinuous, lower semicontinuous, and locally bounded;

(C5) The function κ is continuous and

$$\{(x, x_c) \in \mathcal{X} \times \mathcal{X}_c : V(x, x_c) < +\infty\} \subset \operatorname{dom} \kappa.$$

Given a synergistic candidate relative to $\mathcal{A}$, the condition (C3) guarantees that sublevel sets of V are compact and the conditions (C3) and (C4) guarantee that the synergy gap function μ_V in (11c) is continuous and that ϱ_V is outer semicontinuous, as proved in the next result.

Lemma 1. Given sets $\mathcal{X}$, $\mathcal{X}_c$, $\mathcal{U}$, and F_θ as in (7), suppose that Assumption 1 holds. Given a compact subset $\mathcal{A}$ of $\mathcal{X} \times \mathcal{X}_c$, if (κ, V, D_c, F_c) is a synergistic candidate relative to $\mathcal{A}$ for (7), then the following hold:

1) The function ν_V in (11a) is continuous;

2) The set-valued map ϱ_V in (11b) *is outer semicontinuous and* $\varrho_V(x, x_c)$ *is compact for each* $(x, x_c) \in \mathcal{X} \times \mathcal{X}_c$;

3) *The function* μ_V *in (11c) is continuous.*

Proof. It follows from (C4) that D_c is outer semicontinuous, hence $D_c(x, x_c)$ is closed for each $(x, x_c) \in \mathcal{X} \times \mathcal{X}_c$. Since D_c is also assumed to be locally bounded in (C4), we have that $D_c(x, x_c)$ is compact for each $(x, x_c) \in \mathcal{X} \times \mathcal{X}_c$. In addition, the outer semicontinuity and local boundedness of D_c imply that D_c is upper semicontinuous (cf. [33, Lemma 5.15]). Since D_c is assumed to be lower semicontinuous in (C4), we have that D_c is continuous. Since V is continuous by condition (C3), it follows from [36, Theorem 9.14] that ν_V is continuous and that ϱ_V is compact-valued and upper semicontinuous. Since $\mathcal{X}$ is locally compact Hausdorff (cf. Remark 1), it follows from [35, Proposition 4.27] that each point $(x, x_c) \in \mathcal{X} \times \mathcal{X}_c$ has a precompact neighborhood U_x. Since ϱ_V is compact-valued and upper semicontinuous, it follows from [36, Proposition 9.7] that $\varrho_V(\overline{U_x})$ is compact. It follows from the fact that $\varrho_V(U_x)$ is a subset of the compact set $\varrho_V(\overline{U_x})$ that ϱ_V is locally bounded. Since ϱ_V is compact-valued it is, in particular, closed-valued, hence it follows from [33, Lemma 5.15] that ϱ_V is outer semicontinuous. It follows from (C1) that $\nu_V(x, x_c) < +\infty$ for each $(x, x_c) \in \mathcal{X} \times \mathcal{X}_c$, hence the function μ_V is continuous because it is the composition of continuous functions. $\square$

The hybrid basic conditions in [33, Assumption 6.5] are very important to the synthesis of hybrid controllers, because they guarantee that the resulting hybrid closed-loop systems are endowed with nominal robustness to a wide range of perturbations/sensor noise and, in particular, they enable the application of invariance principles for hybrid systems (cf. [33, Chapter 8]). In the following result, we show that these conditions follow directly from the regularity of (11c) and (11) that was proved in Lemma 1.

Corollary 1. *Given sets* $\mathcal{X}$, $\mathcal{X}_c$, $\mathcal{U}$, *and* F_θ *as in* (7)*, suppose that Assumption 1 holds. Given a compact subset* $\mathcal{A}$ *of* $\mathcal{X} \times \mathcal{X}_c$, *if* (κ, V, D_c, F_c) *is a synergistic candidate relative to* $\mathcal{A}$ *for* (7)*, then the hybrid closed-loop system* (14) *satisfies (A1), (A2), and (A3).*

C. Global Asymptotic Stability of $\mathcal{A}$

In this section, we present further assumptions on the hybrid controller (κ, V, D_c, F_c) that allow for the global asymptotic stabilization of a compact set $\mathcal{A}$ for (14).

Definition 2. Given a compact subset $\mathcal{A}$ of $\mathcal{X} \times \mathcal{X}_c$, we say that a synergistic candidate relative to $\mathcal{A}$ for (7) with data (κ, V, D_c, F_c), is synergistic relative to $\mathcal{A}$ for (7) if:

(C6) The function V is Lipschitz continuous on a neighborhood of C and the growth of V along flows of (14) is bounded by u_c with

$$u_c(x, x_c) \le 0 \qquad \forall (x, x_c) \in \mathcal{X} \times \mathcal{X}_c;$$

(C7) The largest weakly invariant subset of

$$\dot{x} \in F_\theta(x, x_c, \kappa(x, x_c)) \quad \dot{x}_c \in F_c(x, x_c) \tag{15}$$

in $\overline{u_c^{-1}(0)}$, denoted by Ψ, is such that

$$\overline{\delta}_1 := \inf\{\mu_V(x, x_c) : (x, x_c) \in \Psi \backslash \mathcal{A}\} > 0. \tag{16}$$

If one considers V as a Lyapunov function candidate, then condition (C6) implies that V is nonincreasing along flows to the closed-loop system (14), implying that there exists a choice of δ which renders $\mathcal{A}$ stable for (14). We say that δ is positive if $\delta(x, x_c) > 0$ for each $(x, x_c) \in \mathcal{X} \times \mathcal{X}_c$, and that a hybrid controller (κ, V, D_c, F_c) is synergistic relative to $\mathcal{A}$ for (7) with synergy gap exceeding δ if it is synergistic relative to $\mathcal{A}$ for (7) and $\delta(x, x_c) < \mu_V(x, x_c)$ for each $(x, x_c) \in \Psi \backslash \mathcal{A}$ with Ψ defined in (C7). Condition (16) means that (κ, V, D_c, F_c) is synergistic relative to $\mathcal{A}$ for (7) for some positive δ, and, consequently, all the points in the largest weakly invariant subset of (15) in $\overline{u_c^{-1}(0)}$ that are not in $\mathcal{A}$ lie in the jump set of (14), allowing us to prove that $\mathcal{A}$ is globally asymptotically stable for the closed-loop system (14).

Lemma 2. *Given sets* $\mathcal{X}$, $\mathcal{X}_c$, $\mathcal{U}$, *and* F_θ *as in* (7)*, suppose that Assumption 1 holds. Given a compact subset* $\mathcal{A}$ *of* $\mathcal{X} \times \mathcal{X}_c$, *if* (κ, V, D_c, F_c) *is a synergistic candidate relative to* $\mathcal{A}$ *for* (7) *satisfying (C6) and* $\delta(x, x_c) \ge 0$ *for each* $(x, x_c) \in \mathcal{X} \times \mathcal{X}_c$, *then each sublevel set of* V *is forward pre-invariant for* (14)*. If, for each* $(x, x_c) \in C \backslash D$,

(VC) *there exists a neighborhood* U *of* (x, x_c) *such that* $F_{cl}(\xi) \cap \mathsf{T}_\xi C \ne \emptyset$, *for every* $\xi \in U \cap C$

then each maximal solution to (14) *is complete and, consequently, each sublevel set of* V *is forward invariant.*

Proof. It follows from the discussion in Remark 2 that the growth of V along jumps of (14) is bounded by u_d with

$$u_d(x, x_c) \le \begin{cases} -\delta(x, x_c) & \text{if } (x, x_c) \in D \\ -\infty & \text{otherwise} \end{cases} \tag{17}$$

for each $(x, x_c) \in \mathcal{X} \times \mathcal{X}_c$. Together with condition (C6) it follows that the growth of V along solutions to (14) is bounded by u_c, u_d satisfying

$$u_c(x, x_c) \le 0, \quad u_d(x, x_c) \le 0 \tag{18}$$

for each $(x, x_c) \in \mathcal{X} \times \mathcal{X}_c$. It follows that each solution ϕ to (14) with initial condition ξ satisfies $V(\phi(t, j)) \le V(\xi)$ for all $(t, j) \in \text{dom}\,\phi$, hence each sublevel set of V is forward pre-invariant for (14).

It follows from Corollary 1 that (14) satisfies the hybrid basic conditions, hence we can use [33, Proposition 6.10] to prove the completeness of each maximal solution to (14). Since $C \cup D = \mathcal{X} \times \mathcal{X}_c$, then there are no solutions to (14) starting outside the union between the jump and flow sets. It follows from (VC) that (VC) in [33, Proposition 6.10] is satisfied, hence each maximal solution to (14) either "blows up," leaves $C \cup D$ in finite time or is complete (cf. conditions *(a)*, *(b)* and *(c)* of [33, Proposition 6.10]). Since $G_{cl}(D) \subset C \cup D$, no solution can leave $C \cup D$ after a jump (hence, condition (c) in [33, Proposition 6.10] does not occur). Since each sublevel set of V is compact and forward pre-invariant, then solutions to (14) do not "blow up" (condition (b) in [33, Proposition 6.10] does not occur). It follows that each maximal solution to (14) is complete. $\square$

Lemma 3. *Given sets* $\mathcal{X}$, $\mathcal{X}_c$, $\mathcal{U}$, *and* F_θ *as in* (7)*, suppose that Assumption 1 holds. Given a compact subset* $\mathcal{A}$ *of* $\mathcal{X} \times \mathcal{X}_c$, *if* (κ, V, D_c, F_c) *is a synergistic candidate relative to* $\mathcal{A}$ *for* (7) *that satisfies (C6) and* $\delta(x, x_c) \ge 0$ *for each* $(x, x_c) \in \mathcal{X} \times \mathcal{X}_c$, *then the set* $\mathcal{A}$ *is stable for* (14)*.*

Proof. Since $\mathcal{X} \times \mathcal{X}_c \subset \text{dom}\, V$, it follows that $\mu_V(x, x_c)$ is defined for all $(x, x_c) \in \mathcal{X} \times \mathcal{X}_c$; hence, for any given continuous function $\delta : \mathcal{X} \times \mathcal{X}_c \to \mathbb{R}$, at least one of the following conditions holds: 1) $\mu_V(x, x_c) \geq \delta(x, x_c)$; 2) $\mu_V(x, x_c) \leq \delta(x, x_c)$. It follows from (13) that $\overline{C} \cup D = \mathcal{X} \times \mathcal{X}_c \subset \text{dom}\, V$. Since $\text{dom}\, V$ is also assumed to be open in the Euclidean space containing $\mathcal{X} \times \mathcal{X}_c$, it follows that $\text{dom}\, V$ contains a neighborhood of $\mathcal{A} \cap (C \cup D \cup G_{cl}(D))$. Positive definiteness of V with respect to $\mathcal{A}$ and continuity of V follows from (C3). From condition (C6) and from (17), it follows that V is locally Lipschitz on a neighborhood of $\overline{C}$ and that the bounds [34, Eqs.(3.18), (3.19)] are satisfied. Since $\mathcal{A}$ is compact and the hybrid basic conditions are satisfied (cf. Corollary 1), it follows from [34, Theorem 3.19] that $\mathcal{A}$ is stable for (7). $\qquad\square$

Theorem 1. *Given sets $\mathcal{X}$, $\mathcal{X}_c$, $\mathcal{U}$, and F_θ as in (7), suppose that Assumption 1 holds. Given a compact subset $\mathcal{A}$ of $\mathcal{X} \times \mathcal{X}_c$ and a positive function $\delta : \mathcal{X} \times \mathcal{X}_c \to \mathbb{R}$, if (κ, V, D_c, F_c) is synergistic relative to $\mathcal{A}$ for (7) with synergy gap exceeding δ, then the set $\mathcal{A}$ is globally pre-asymptotically stable for (14). If, for each $(x, x_c) \in C \backslash D$, (VC) is satisfied, then $\mathcal{A}$ is globally asymptotically stable for (14).*

Proof. Stability of $\mathcal{A}$ is proved in Lemma 3 and completeness of solutions is demonstrated in Lemma 2. The global pre-asymptotic stability of $\mathcal{A}$ for (14) follows from pre-attractivity of $\mathcal{A}$ for (14), which is demonstrated next through an application of [33, Theorem 8.2].

It follows from Lemmas 3 and 2 that the growth of V along solutions to (14) is bounded by u_c, u_d satisfying (18). In addition, since each sublevel set of V is compact as per (C3), we have that each solution to (14) is bounded. Thus, it follows from [33, Theorem 8.2] that every complete solution approaches the largest weakly invariant set

$$V^{-1}(r) \cap \left(\overline{u_c^{-1}(0)} \cup (u_d^{-1}(0) \cap G_{cl}(u_d^{-1}(0))) \right) \qquad (19)$$

for some r in the image of V. From (17) and the fact that δ is positive, it follows that $u_d^{-1}(0) = \emptyset$, hence (19) can be rewritten as

$$V^{-1}(r) \cap \overline{u_c^{-1}(0)}. \qquad (20)$$

It follows from (C7), the assumption that (κ, V, D_c, F_c) has synergy gap exceeding δ and the definition of D in (12) that the largest weakly invariant subset of (14) in (20) does not include points that are not in $\mathcal{A}$ and, consequently, $\mathcal{A}$ is globally pre-attractive for (14). Global asymptotic stability of $\mathcal{A}$ for (14) follows from global pre-asymptotic stability if each maximal solution to (14) is complete, which is guaranteed by Lemma 2 under condition (VC). $\qquad\square$

Remark 3. Theorem 1 provides conditions which guarantee that each maximal solution to (14) is complete. This means that, each maximal solution ϕ to (14) satisfies $\lim_{(t,j) \in \text{dom}\, \phi} t + j = +\infty$. However, we have not ruled out the possibility of Zeno solutions, i.e., complete solutions that are bounded in the t-direction. This possibility is ruled out by [37, Lemma 2.7] because (14) satisfies (A3) and $G_{cl}(D) \cap D = \emptyset$ when δ is positive.

Note that, if there exists an accumulation point of $\Psi \backslash \mathcal{A}$ in $\mathcal{A}$ in a topological sense, then $\overline{\delta}_1$ in (16) is equal to 0. Therefore, the topology of Ψ and $\mathcal{A}$ may preclude global

asymptotic stabilization of $\mathcal{A}$ for (14) since (C7) is not met. Conversely, if one is able to show that $\overline{\delta}_1 > 0$, then $\Psi \backslash \mathcal{A}$ does not have accumulation points in $\mathcal{A}$. With additional conditions on δ, we are able to show that there exists a neighborhood of $\mathcal{A}$ contained in C. In this case, asymptotic convergence to $\mathcal{A}$ implies that maximal solutions are *eventually continuous*, i.e., there exists some (hybrid) time after which any maximal solution approaching $\mathcal{A}$ does not have any more jumps.

Proposition 1. *Given sets $\mathcal{X}$, $\mathcal{X}_c$, $\mathcal{U}$, and F_θ as in (7), suppose that Assumption 1 holds. Given a compact subset $\mathcal{A}$ of $\mathcal{X} \times \mathcal{X}_c$, if the hybrid controller (κ, V, D_c, F_c) is synergistic relative to $\mathcal{A}$ and $\underline{\delta} := \inf\{\delta(x, x_c) : (x, x_c) \in \mathcal{X} \times \mathcal{X}_c\}$ satisfies $\underline{\delta} \in (0, \overline{\delta}_1)$, where $\overline{\delta}_1$ is given in (C7), then there exists a neighborhood of $\mathcal{A}$ that is contained in C.*

Remark 4. Note that, if the hybrid controller (κ, V, D_c, F_c) is synergistic relative to $\mathcal{A}$ for (7), then δ can be chosen as a constant $\Delta \in \mathbb{R}$ as long as $\Delta \in (0, \overline{\delta}_1)$. In this case, the conditions of Proposition 1 hold, thus the fact that δ is state-dependent does not constrain the global asymptotic stability results and it provides more flexibility to the design of the hybrid controller.

V. ROBUST SYNERGISTIC HYBRID FEEDBACK

In this section, we propose a new kind of synergistic hybrid controller that, unlike the controller of Section IV, is able to handle the case where θ is unknown, but belongs to a known compact set Ω. In this direction, we start by defining the following collection of functions $\mathcal{V} := \{V_\theta\}_{\theta \in \Omega}$ where

$$(x, x_c, \theta) \mapsto V(x, x_c, \theta) := V_\theta(x, x_c), \qquad (21)$$

is a convenient notation that allows us to consider V_θ not only as a function of (x, x_c) but also of θ. Then, we define a hybrid controller with data $(\kappa, \mathcal{V}, D_c, F_c, G_c)$ as follows:

$$\dot{x}_c \in F_c(x, x_c) \qquad\qquad (x, x_c) \in C_\Omega \qquad (22a)$$

$$x_c^+ \in G_c(x, x_c) \qquad\qquad (x, x_c) \in D_\Omega \qquad (22b)$$

where

$$C_\Omega := \left\{ (x, x_c) \in \mathcal{X} \times \mathcal{X}_c : \min_{\theta \in \Omega} \mu_{V_\theta}(x, x_c) \leq \delta(x, x_c) \right\}$$

$$D_\Omega := \left\{ (x, x_c) \in \mathcal{X} \times \mathcal{X}_c : \min_{\theta \in \Omega} \mu_{V_\theta}(x, x_c) \geq \delta(x, x_c) \right\}$$
$$(23)$$

with $\delta : \mathcal{X} \times \mathcal{X}_c \to \mathbb{R}$ continuous, and

$$\min_{\theta \in \Omega} \mu_{V_\theta}(x, x_c) = \min_{\theta \in \Omega} \{V_\theta(x, x_c) - \nu_{V_\theta}(x, x_c)\}$$

$$= \min_{\theta \in \Omega} \left\{ V(x, x_c, \theta) - \min_{g \in D_c(x, x_c)} V(x, g, \theta) \right\}$$

for each $(x, x_c) \in \mathcal{X} \times \mathcal{X}_c$, in accordance with the definitions (11c), (11a) and (21). One of the key differences in the controller (22) relative to the standard synergistic controller (12) is that flow and jump sets logic requires a minimization over the unknown parameter θ, which has the potential effect of delaying controller switching relative to the case where θ is known. Also, unlike the jump map of the hybrid controller (12), which is constructed from the data V and D_c following (11b), the jump map G_c in (22) is left undefined for the sake of generality. Finally, owing to the

fact that (22) is derived from κ, $\mathcal{V}$, D_c, F_c, and G_c, we refer to (22) using the 5-tuple $(\kappa, \mathcal{V}, D_c, F_c, G_c)$.

In the sequel, we demonstrate how this hybrid controller extends the concept of a synergistic controller in order to address the case where $\theta \in \Omega$ is not known.

Definition 3. Given a compact set Ω, a continuous function $\delta : \mathcal{X} \times \mathcal{X}_c \to \mathbb{R}$, a collection of compact subsets $\mathscr{A} := \{\mathcal{A}_\theta\}_{\theta \in \Omega}$ of $\mathcal{X} \times \mathcal{X}_c$, and a collection of continuous functions $\mathscr{V} := \{V_\theta\}_{\theta \in \Omega}$, we say that the hybrid controller $(\kappa, \mathscr{V}, D_c, F_c, G_c)$ is *synergistic relative to $\mathscr{A}$ for* (7) *with robustness margin* Ω if it satisfies the following conditions:

(R1) Each function $V_\theta : \operatorname{dom} V_\theta \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ in $\mathscr{V}$ is continuous and satisfies $\mathcal{X} \times \mathcal{X}_c \subset \operatorname{dom} V_\theta$ for each $\theta \in \Omega$;

(R2) The set-valued map G_c is outer semicontinuous and locally bounded;

(R3) For each $\theta \in \Omega$, we assume that
$$V_\theta(x, x_c) - V_\theta(x, g) \geq \min_{\theta \in \Omega} \mu_{V_\theta}(x, x_c) \qquad (24)$$
for each $(x, x_c) \in D_\Omega$ and each $g \in G_c(x, x_c)$;

(R4) For each $\theta \in \Omega$, the hybrid controller $(\kappa, V_\theta, D_c, F_c)$ is synergistic relative to $\mathcal{A}_\theta$ for (7).

Remark 5. Note that the set Ω might be uncountable, thus the collections $\mathscr{A}$ and $\mathscr{V}$ are not necessarily finite nor countable.

Remark 6. The condition (R3) guarantees that
$$V_\theta(x, x_c) - V_\theta(x, g) \geq \delta(x, x_c)$$
for each $(x, x_c) \in D_\Omega$ and each $g \in G_c(x, x_c)$, which implies that the function V_θ does not increase during jumps if $\delta(x, x_c) \geq 0$ for all $(x, x_c) \in \mathcal{X} \times \mathcal{X}_c$ (cf. Remark 2).

It is possible to construct G_c in (22) from the data $\mathscr{V}$ and D_c, but this requires additional assumptions, as shown in the following remark.

Remark 7. Suppose that Ω is compact and convex and that D_c in (10) is convex and compact for each $(x, x_c) \in \mathcal{X} \times \mathcal{X}_c$. Given $\mathscr{V} := \{V_\theta\}_{\theta \in \Omega}$, suppose that G_c defined as
$$G_c(x, x_c) := \operatorname*{arg\,max}_{g \in D_c(x,x_c)} \quad \min_{\theta \in \Omega} \{V(x, x_c, \theta) - V(x, g, \theta)\}$$
$$\forall (x, x_c) \in \mathcal{X} \times \mathcal{X}_c \quad (25)$$
is outer semicontinuous with $V(x, x_c, \theta) = V_\theta(x, x_c)$ for each $(x, x_c, \theta) \in \mathcal{X} \times \mathcal{X}_c \times \Omega$. Furthermore, suppose that, for each $(x, x_c) \in \mathcal{X} \times \mathcal{X}_c$, the function $h(g, \theta) := V_\theta(x, x_c) - V_\theta(x, g)$ is continuous, quasi-concave as a function of g, and quasi-convex as a function of θ.[2] Then, it follows from [38, Theorem 3.4] that the $\min$ and $\max$ operations in $\max_{g \in D_c(x,x_c)} \min_{\theta \in \Omega} h(g, \theta)$ commute, yielding
$$\max_{g \in D_c(x,x_c)} \min_{\theta \in \Omega} h(g, \theta) = \min_{\theta \in \Omega} \max_{g \in D_c(x,x_c)} h(g, \theta)$$
$$= \min_{\theta \in \Omega} \left\{ V_\theta(x, x_c) - \min_{g \in D_c(x,x_c)} V_\theta(x, g) \right\}. \quad (26)$$

[2]A function $(x, y) \mapsto h(x, y)$ on $X \times Y$ is quasi-concave as a function of x if the set $\{x \in X : h(x, y) \geq c\}$ is convex for each $y \in Y$ and each $c \in \mathbb{R}$. The function h is quasi-convex as a function of y if the set $\{y \in X : h(x, y) \leq c\}$ is convex for each $x \in X$ and each $c \in \mathbb{R}$.

It follows from (26) and (11c) that $\max_{g \in D_c(x,x_c)} \min_{\theta \in \Omega} h(g, \theta) = \min_{\theta \in \Omega} \mu_{V_\theta}(x, x_c)$. We conclude that, for each $\theta \in \Omega$, the following holds $V_\theta(x, x_c) - V_\theta(x, g) \geq \min_{\theta \in \Omega} \mu_{V_\theta}(x, x_c)$, for each $(x, x_c) \in \mathcal{X} \times \mathcal{X}_c$ and each g belonging to (25), hence condition (24) is verified.

The assumption that the hybrid controller $(\kappa, \mathscr{V}, D_c, F_c, G_c)$ is synergistic relative to $\mathscr{A}$ for (7) with robustness margin Ω ensures that the hybrid closed-loop system
$$\begin{pmatrix} \dot{x} \\ \dot{x}_c \end{pmatrix} \in F_{cl}(x, x_c) := \begin{pmatrix} F_\theta(x, x_c, \kappa(x, x_c)) \\ F_c(x, x_c) \end{pmatrix} \quad (x, x_c) \in C_\Omega$$
$$(27a)$$
$$\begin{pmatrix} x^+ \\ x_c^+ \end{pmatrix} \in G_\Omega(x, x_c) := \begin{pmatrix} x \\ G_c(x, x_c) \end{pmatrix} \quad (x, x_c) \in D_\Omega$$
$$(27b)$$
satisfies the hybrid basic conditions as proved next.

Lemma 4. *Given sets $\mathcal{X}$, $\mathcal{X}_c$, $\mathcal{U}$, and F_θ as in* (7), *suppose that Assumption 1 holds. Given a compact set Ω and a collection of compact subsets $\mathscr{A} := \{\mathcal{A}_\theta\}_{\theta \in \Omega}$ of $\mathcal{X} \times \mathcal{X}_c$ if $(\kappa, \mathscr{V}, D_c, F_c, G_c)$ is synergistic relative to $\mathscr{A}$ for* (7) *with robustness margin Ω, then the hybrid closed-loop system* (27) *satisfies (A1), (A2), and (A3).*

Proof. The continuity of μ_{V_θ} (for a fixed $\theta \in \Omega$) is established in Lemma 1. It follows from the continuity of $(x, x_c, \theta) \mapsto V(x, x_c, \theta) = V_\theta(x, x_c)$ that is assumed in (R1), compactness of Ω and from [36, Theorem 9.14] that the function
$$(x, x_c) \mapsto \min_{\theta \in \Omega} \mu_{V_\theta}(x, x_c) \qquad (28)$$
is continuous on $\mathcal{X} \times \mathcal{X}_c$. It follows from the continuity of (28) and of δ that C_Ω and D_Ω are closed, because they are the preimage of the closed sets $(-\infty, 0]$ and $[0, +\infty]$, respectively. It follows from Assumption 1, (C2), and (C5) that the flow map F_Ω is outer semicontinuous, locally bounded and convex-valued. It follows from (R2) that G_Ω is outer semicontinuous, and locally bounded relative to D_Ω. $\qquad \square$

In the sequel, we demonstrate that, for each $\theta \in \Omega$, the set $\mathcal{A}_\theta \in \mathscr{A}$ is globally asymptotically stable under appropriate assumptions on δ. The next result asserts forward pre-invariance of sublevel sets of $V_\theta \in \mathscr{V}$ for the closed-loop system (27) when δ is a continuous and nonnegative function.

Lemma 5. *Given sets $\mathcal{X}$, $\mathcal{X}_c$, $\mathcal{U}$, and F_θ as in* (7), *suppose that Assumption 1 holds. Given a compact set Ω and a collection of compact subsets $\mathscr{A} := \{\mathcal{A}_\theta\}_{\theta \in \Omega}$ of $\mathcal{X} \times \mathcal{X}_c$, if $(\kappa, \mathscr{V}, D_c, F_c, G_c)$ is synergistic relative to $\mathscr{A}$ for* (7) *with robustness margin Ω and if $\delta(x, x_c) \geq 0$ for each $(x, x_c) \in \mathcal{X} \times \mathcal{X}_c$, then, for each $\theta \in \Omega$, each sublevel set of V_θ is forward pre-invariant for* (27). *If, for each $(x, x_c) \in C_\Omega \setminus D_\Omega$,*

(VC') *there exists a neighborhood U of (x, x_c) such that $F_{cl}(\xi) \cap \mathsf{T}_\xi C_\Omega \neq \emptyset$, for every $\xi \in U \cap C_\Omega$*

then each maximal solution to (27) *is complete and, consequently, each sublevel set of V_θ is forward invariant for* (27).

Proof. As explained in Remark 6, it follows from (R3) and (23) that $V_\theta(x, x_c) - V_\theta(x, g) \geq \delta(x, x_c)$ for each

$g \in G_c(x, x_c)$ and each $(x, x_c) \in D_\Omega$. Hence, the growth of V_θ during jumps of (27) is bounded by

$$u_{d,\theta}(x, x_c) := \begin{cases} -\delta(x, x_c) & \text{if } (x, x_c) \in D_\Omega \\ -\infty & \text{otherwise} \end{cases} \quad (29)$$

for each $(x, x_c) \in \mathcal{X} \times \mathcal{X}_c$. Since $(\kappa, \mathcal{V}, D_c, F_c, G_c)$ is synergistic relative to $\mathscr{A}$ for (7) with robustness margin Ω, it follows that $(\kappa, V_\theta, D_c, F_c)$ is synergistic relative to $\mathcal{A}_\theta$ for (7) and, due to this assumption, the remainder of the proof follows closely that of Lemma 2. From condition (C6) it follows that the growth of V_θ along solutions to (27) is bounded by $u_{c,\theta}, u_{d,\theta}$, with $u_{c,\theta}(x, x_c) \leq 0$ and $u_{d,\theta}(x, x_c) \leq 0$ for each $(x, x_c) \in \mathcal{X} \times \mathcal{X}_c$. This implies that sublevel sets of V_θ are forward pre-invariant for (27). The completeness of solutions under (VC') follows closely the proof in Lemma 2, thus it is omitted here. $\qquad \square$

Let Ψ_θ denote the largest weakly invariant subset of

$$(\dot{x}, \dot{x}_c) \in F_{cl}(x, x_c) \qquad (x, x_c) \in \overline{u_{c,\theta}^{-1}(0)}$$

where $u_{c,\theta}$ is the upper bound on the growth of V_θ during flows of (27) as defined in (C6). Given a function $\delta : \mathcal{X} \times \mathcal{X}_c \to \mathbb{R}$ and a hybrid controller $(\kappa, \mathcal{V}, D_c, F_c, G_c)$ that is synergistic relative to $\mathscr{A}$ for (7) with robustness margin Ω, we say that it has synergy gap exceeding δ if, for each $\theta \in \Omega$ and each $(x, x_c) \in \Psi_\theta \backslash \mathcal{A}_\theta$, $\delta(x, x_c) < \mu_{V_\theta}(x, x_c)$.

Theorem 2. *Given sets $\mathcal{X}, \mathcal{X}_c, \mathcal{U},$ and F_θ as in (7), suppose that Assumption 1 holds. Given a compact set Ω, a positive function $\delta : \mathcal{X} \times \mathcal{X}_c \to \mathbb{R}$, and a collection of compact subsets $\mathscr{A} := \{\mathcal{A}_\theta\}_{\theta \in \Omega}$ of $\mathcal{X} \times \mathcal{X}_c$, if $(\kappa, \mathcal{V}, D_c, F_c, G_c)$ is synergistic relative to $\mathscr{A}$ for (7) with robustness margin Ω and synergy gap exceeding δ, then, for each $\theta \in \Omega$, the set $\mathcal{A}_\theta$ is globally pre-asymptotically stable for (27). If, for each $(x, x_c) \in C_\Omega \backslash D_\Omega$, (VC') is satisfied, then $\mathcal{A}_\theta$ is globally asymptotically stable for (27).*

Proof. For each $\theta \in \Omega$, it follows from (C3) that each sublevel set of V_θ is compact and, since it is also forward pre-invariant as shown in Lemma 5, we have that each solution to (27) is bounded. In addition, it follows from the proof of Lemma 5 that the growth of V_θ along jumps of (27) is bounded by (29) and, since $\delta(x, x_c) > 0$ by assumption, it follows from [33, Theorem 8.2] that each complete solution to (27) approaches the largest weakly invariant subset of $V_\theta^{-1}(r) \cap \overline{u_{c,\theta}^{-1}(0)}$ for some r in the image of V_θ, which is to say that each complete solution to (27) approaches $\Psi_\theta \cap C_\Omega$. Since each point $(x, x_c) \in \Psi_\theta \backslash \mathcal{A}_\theta$ belongs to $D_\Omega \backslash C_\Omega$ by Condition (C7), it follows that each complete solution to (27) converges to $\mathcal{A}_\theta$, which concludes the proof of global pre-attractivity of $\mathcal{A}_\theta$ for (27). The proof of stability of $\mathcal{A}_\theta$ for (14) follows closely the proof of Lemma 3. We conclude that $\mathcal{A}_\theta$ is globally pre-asymptotically stable for (27). Global asymptotic stability of $\mathcal{A}_\theta$ for (27) under condition (VC') follows directly from global pre-asymptotic stability and completeness of solutions, as shown in Lemma 5. $\qquad \square$

In the next section, we apply the proposed controller to the design of adaptive synergistic feedback control laws for a class of affine systems with matched uncertainties.

VI. Adaptive Backstepping of Synergistic Hybrid Feedback for Affine Control Systems

A. Nominal Synergistic Hybrid Feedback

In this section, we apply the controller design of Section V to the problem of global asymptotic stabilization of a compact set $\mathcal{A} \subset \mathcal{X} \times \mathcal{X}_c$ for a control affine system subject to parametric uncertainty, where $\mathcal{X}$ and $\mathcal{X}_c$ denote the spaces of the state and controller variables, respectively. In this direction, let F_θ in (7) be given by

$$F_\theta(x, x_c, u) := f(x, x_c) + H(x, x_c)u + W(x, x_c)\theta \quad (30)$$

for each $(x, x_c, u) \in \mathcal{X} \times \mathcal{X}_c \times \mathcal{U}$, where u denotes an input variable subject to the constraint $u \in \mathcal{U}$, and

$$\theta \in \Omega := \{\theta \in \mathbb{R}^\ell : |\theta| \leq \theta_0\} \quad (31)$$

represents the parametric uncertainty of the model whose norm is assumed to be bounded by a known parameter $\theta_0 \in \mathbb{R}_{\geq 0}$. The controller design in this section is applicable under the assumption of matched uncertainties stated next.

Assumption 2. There exists a continuously differentiable function $\widehat{W}$ such that $W(x, x_c) = H(x, x_c)\widehat{W}(x, x_c)$ for each $(x, x_c) \in \mathcal{X} \times \mathcal{X}_c$.

In addition, we assume that we are given a synergistic hybrid controller for the nominal (unperturbed) system as defined next.

Definition 4. Given a compact set $\mathcal{A} \subset \mathcal{X} \times \mathcal{X}_c$ and a continuous function $\delta : \mathcal{X} \times \mathcal{X}_c \to \mathbb{R}$, the hybrid controller $(\kappa_0, V_0, D_c, F_c)$ is said to be *nominally* synergistic relative to $\mathcal{A}$ for (30) with synergy gap exceeding δ if it is synergistic relative to $\mathcal{A}$ for

$$\dot{x} = F_0(x, x_c, u) := f(x, x_c) + H(x, x_c)u \quad (32)$$

with synergy gap exceeding δ, and V_0 is continuously differentiable on $\{(x, x_c) \in \mathcal{X} \times \mathcal{X}_c : V_0(x, x_c) < +\infty.\}$.

The dynamical system (32) is obtained from (30) by considering that there are no perturbations, i.e., $\theta = 0$. It follows from Theorem 1 that $\mathcal{A}$ is globally asymptotically stable for the closed-loop system $\mathcal{H}$ in (14) resulting from the interconnection of (32) and a nominally synergistic controller relative to $\mathcal{A}$ for (30) when $\theta = 0$. In the following sections, we employ the results of Sections IV and V in order to develop a synergistic controller that is able to cope with parametric uncertainty from a given nominally synergistic controller.

In the next section, we present variations of the nominal synergistic controller to deal with parametric uncertainty.

B. Adaptive Synergistic Hybrid Feedback

In this section, we modify the nominal synergistic controller given in Section VI-A to globally asymptotically stabilize

$$\mathcal{A}_{1,\theta} := \mathcal{A} \times \{\theta\} \quad (33)$$

for the closed-loop system when θ in (30) is nonzero.[3] In this direction, let $\hat{\theta} \in \mathbb{R}^\ell$ denote an estimate of the parameter

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[3] As the controller design exploits ideas in the literature of adaptive control, we refer the reader to [39] and [40] for an overview of adaptive controller design and backstepping under the influence of model uncertainty.

θ that is generated via

$$\dot{\hat{\theta}} = \Gamma_1 \operatorname{Proj}(W(x, x_c)^\top \nabla_x V_0(x, x_c), \hat{\theta}), \qquad (34)$$

where $\Gamma_1 \in \mathbb{R}^{\ell \times \ell}$ is a positive definite matrix and Proj : $\mathbb{R}^\ell \times \mathbb{R}^\ell \to \mathbb{R}^\ell$ is given by

$$\operatorname{Proj}(\eta, \hat{\theta}) := \begin{cases} \eta & \text{if } \operatorname{p}(\hat{\theta}) \leq 0 \text{ or } \nabla \operatorname{p}(\hat{\theta})^\top \eta \leq 0 \\ \left(I_\ell - \dfrac{\operatorname{p}(\hat{\theta}) \nabla \operatorname{p}(\hat{\theta}) \nabla \operatorname{p}(\hat{\theta})^\top}{\nabla \operatorname{p}(\hat{\theta})^\top \nabla \operatorname{p}(\hat{\theta})} \right) \eta & \text{otherwise} \end{cases}$$
$$(35)$$

for each $(\eta, \hat{\theta}) \in \mathbb{R}^\ell \times \mathbb{R}^\ell$,

$$\operatorname{p}(\hat{\theta}) := \frac{\hat{\theta}^\top \hat{\theta} - \theta_0^2}{\epsilon^2 + 2\epsilon\theta_0} \qquad (36)$$

for each $\hat{\theta} \in \mathbb{R}^\ell$, with $\epsilon > 0$ and $\theta_0 > 0$ given in (31), and W as in (30). The function Proj in (35) has the following properties (cf. [41]):

(P1) Proj is Lipschitz continuous;

(P2) Each solution $t \mapsto \hat{\theta}(t)$ to $\dot{\hat{\theta}} = \Gamma_1 \operatorname{Proj}(\eta(t), \hat{\theta})$, from $\hat{\theta} \in \Omega + \epsilon\overline{\mathbb{B}}$ with input $t \mapsto \eta(t)$ satisfies $\operatorname{rge} \hat{\theta} \subset \Omega + \epsilon\overline{\mathbb{B}}$;

(P3) Given $\theta \in \Omega$, $(\theta - \hat{\theta})^\top \operatorname{Proj}(\eta, \hat{\theta}) \geq (\theta - \hat{\theta})^\top \eta$ for each $(\eta, \hat{\theta}) \in \mathbb{R}^\ell \times \mathbb{R}^\ell$;

with $\epsilon > 0$ as in (36). Given a hybrid controller $(\kappa_0, V_0, D_c, F_c)$ that is nominally synergistic relative to $\mathcal{A}$ for (30) with synergy gap exceeding δ, and the controller variable $x_{c,1} := (x_c, \hat{\theta}) \in \mathcal{X}_{c,1} := \mathcal{X}_c \times (\Omega + \epsilon\overline{\mathbb{B}})$, we define

$$\kappa_1(x, x_{c,1}) := \kappa_0(x, x_c) - \widehat{W}(x, x_c)\hat{\theta} \qquad (37a)$$

$$V_{1,\theta}(x, x_{c,1}) := V_0(x, x_c) + \frac{1}{2}(\theta - \hat{\theta})^\top \Gamma_1^{-1}(\theta - \hat{\theta}) \quad (37b)$$

$$D_{c,1}(x, x_{c,1}) := D_c(x, x_c) \times (\Omega + \epsilon\overline{\mathbb{B}}) \qquad (37c)$$

$$F_{c,1}(x, x_{c,1}) = \begin{bmatrix} F_c(x, x_c) \\ \Gamma_1 \operatorname{Proj}(W(x, x_c)^\top \nabla_x V_0(x, x_c), \hat{\theta}) \end{bmatrix}$$
$$(37d)$$

for each $(x, x_{c,1}) \in \mathcal{X} \times \mathcal{X}_{c,1}$, where $\widehat{W}$ comes from Assumption 2. The hybrid closed-loop system resulting from the interconnection between (30) and the hybrid controller $(\kappa_1, V_{1,\theta}, D_{c,1}, F_{c,1})$, is given by

$$(\dot{x}, \dot{x}_{c,1}) \in F_{cl,1}(x, x_{c,1}) \qquad (x, x_{c,1}) \in C_1 \qquad (38a)$$

$$(x^+, x_{c,1}^+) \in G_{cl,1}(x, x_{c,1}) \qquad (x, x_{c,1}) \in D_1 \qquad (38b)$$

where

$$C_1 := \{(x, x_{c,1}) \in \mathcal{X} \times \mathcal{X}_{c,1} : \mu_{V_{1,\theta}}(x, x_{c,1}) \leq \delta(x, x_c)\}$$

$$D_1 := \{(x, x_{c,1}) \in \mathcal{X} \times \mathcal{X}_{c,1} : \mu_{V_{1,\theta}}(x, x_{c,1}) \geq \delta(x, x_c)\}$$

and

$$F_{cl,1}(x, x_{c,1}) := \begin{bmatrix} F_\theta(x, x_c, \kappa_1(x, x_{c,1})) \\ F_{c,1}(x, x_{c,1}) \end{bmatrix}$$
$$\forall (x, x_{c,1}) \in C_1 \qquad (39a)$$

$$G_{cl,1}(x, x_{c,1}) := \begin{bmatrix} x \\ \varrho_{V_{1,\theta}}(x, x_{c,1}) \end{bmatrix} \qquad \forall (x, x_{c,1}) \in D_1$$
$$(39b)$$

where, for each $(x, x_{c,1}) \in \mathcal{X} \times \mathcal{X}_{c,1}$,

$$\nu_{V_{1,\theta}}(x, x_{c,1}) = \nu_{V_0}(x, x_c) \qquad (40a)$$

$$\varrho_{V_{1,\theta}}(x, x_{c,1}) = \varrho_{V_0}(x, x_c) \times \{\theta\} \qquad (40b)$$

$$\mu_{V_{1,\theta}}(x, x_{c,1}) = \mu_{V_0}(x, x_c) + \frac{1}{2}(\theta - \hat{\theta})^\top \Gamma_1^{-1}(\theta - \hat{\theta}) \quad (40c)$$

are the result of (11a), (11b) and (11c), respectively.

Remark 8. The functions (40) are not realizable for the hybrid controller $(\kappa_1, V_{1,\theta}, D_{c,1}, F_{c,1})$, because $\mu_{V_{1,\theta}}$ and $\varrho_{V_{1,\theta}}$ in (40) depend on the unknown constant θ. This dependence will be removed when we show that there exists $G_{c,1} : \mathcal{X} \times \mathcal{X}_{c,1} \rightrightarrows \mathcal{X}_{c,1}$ such that the hybrid controller $(\kappa_1, \mathcal{V}_1, D_{c,1}, F_{c,1}, G_{c,1})$ with $\mathcal{V} := \{V_{1,\theta}\}_{\theta \in \Omega}$ is synergistic relative to $\mathcal{A}_1 := \{\mathcal{A}_{1,\theta}\}_{\theta \in \Omega}$ for (30) with robustness margin Ω.

To design (22), we start by showing that the hybrid controller $(\kappa_1, V_{1,\theta}, D_{c,1}, F_{c,1})$ is synergistic relative to $\mathcal{A}_{1,\theta}$ for (30).

Proposition 2. *Suppose that the sets $\mathcal{X}$, $\mathcal{X}_c$, $\mathcal{U}$, and the set-valued map F_θ in (30) satisfy Assumption 1, and that Assumption 2 holds. Given $\theta \in \Omega$, a compact set $\mathcal{A} \subset \mathcal{X} \times \mathcal{X}_c$, and a hybrid controller $(\kappa_0, V_0, D_c, F_c)$ that is nominally synergistic relative to $\mathcal{A}$ for (30), the controller $(\kappa_1, V_{1,\theta}, D_{c,1}, F_{c,1})$ given in (37) is a synergistic candidate relative to $\mathcal{A}_{1,\theta}$ for (30).*

Proof. The optimization problems in (11) are feasible for each $x \in \mathcal{X}$, because they are feasible for V_0, hence (C1) is satisfied.

Since $V_{1,\theta}$ corresponds to the sum of V_0 with $(\theta - \hat{\theta})^\top \Gamma_1^{-1}(\theta - \hat{\theta})/2$ and both terms are continuous, it follows that $V_{1,\theta}$ is continuous. Since V_0 is positive definite with respect to $\mathcal{A}$ and $\hat{\theta} \mapsto (\theta - \hat{\theta})^\top \Gamma_1^{-1}(\theta - \hat{\theta})$ is positive definite relative to θ, it follows that $V_{1,\theta}$ is positive definite relative to $\mathcal{A}_{1,\theta}$. It follows from the assumption that $V_0^{-1}([0, c])$ is compact for each $c \geq 0$ and radial unboundedness of $\hat{\theta} \mapsto (\theta - \hat{\theta})^\top \Gamma_1^{-1}(\theta - \hat{\theta})$ relative to θ that $V_{1,\theta}^{-1}([0, c])$ is compact for each $c \geq 0$, thus proving that $V_{1,\theta}$ satisfies (C3).

From (37c), we have that $D_{c,1}(x, x_{c,1})$ is the Cartesian product between $D_c(x, x_c)$ and $\Omega + \epsilon\overline{\mathbb{B}}$ for each $(x, x_c) \in \mathcal{X} \times \mathcal{X}_c$. Since D_c satisfies (C4) by assumption, we have that $D_{c,1}$ also satisfies (C4). Since κ_0 satisfies (C5), then κ_1 also satisfies (C5). $\qquad\square$

Proposition 3. *Suppose that the sets $\mathcal{X}$, $\mathcal{X}_c$, $\mathcal{U}$, and the set-valued map F_θ in (30) satisfy Assumption 1, and that Assumption 2 holds. Given $\theta \in \Omega$, a compact set $\mathcal{A} \subset \mathcal{X} \times \mathcal{X}_c$, and a hybrid controller $(\kappa_0, V_0, D_c, F_c)$ that is nominally synergistic relative to $\mathcal{A}$ for (30), the controller $(\kappa_1, V_{1,\theta}, D_{c,1}, F_{c,1})$ given in (37) satisfies (C6).*

Proof. It follows from (37b), (39a) and (P3) that, for each $(x, x_{c,1}) \in \mathcal{X} \times \mathcal{X}_{c,1}$ and each $f_{cl,1} \in F_{cl,1}(x, x_{c,1})$

$$\nabla V_{1,\theta}(x, x_{c,1})^\top f_{cl,1} \leq \nabla V_0(x, x_c)^\top \begin{bmatrix} F_\theta(x, x_c, \kappa_1(x, x_{c,1})) \\ f_c \end{bmatrix}$$
$$- (\theta - \hat{\theta})^\top W(x, x_c)^\top \nabla_x V_0(x, x_c)$$

where $f_c \in F_c(x, x_c)$ is the component of $f_{cl,1}$ that determines the dynamics of x_c, i.e., $\dot{x}_c = f_c$. Replacing (30) and (37a) in (VI-B), it follows from Assumption 2 that

$$\nabla V_{1,\theta}(x, x_{c,1})^\top f_{cl,1} \leq \nabla V_0(x, x_c)^\top \begin{bmatrix} F_0(x, x_c, \kappa_0(x, x_c)) \\ f_c \end{bmatrix}$$

for each $(x, x_{c,1}) \in \mathcal{X} \times \mathcal{X}_{c,1}$. From the assumption that the hybrid controller (12) with data $(\kappa_0, V_0, D_c, F_c)$ is synergistic relative to $\mathcal{A}$ for (32), we have that $\nabla V_{1,\theta}(x, x_{c,1})^\top f_{cl,1} \leq 0$

for each $(x, x_{c,1}) \in \mathcal{X} \times \mathcal{X}_{c,1}$ and each $f_{cl,1} \in F_{cl,1}(x, x_{c,1})$, which proves (C6). $\qquad\square$

Since the hybrid controller $(\kappa_1, V_{1,\theta}, D_{c,1}, F_{c,1})$ satisfies (C6), we have that $V_{1,\theta}$ is nonincreasing along solutions to the closed-loop system (38), but satisfying (C7) requires further assumptions on the data, as shown next.

Proposition 4. *Suppose that the sets $\mathcal{X}$, $\mathcal{X}_c$, $\mathcal{U}$, and the set-valued map F_θ in (30) satisfy Assumption 1, and that Assumption 2 holds. Given $\theta \in \Omega$, a compact set $\mathcal{A} \subset \mathcal{X} \times \mathcal{X}_c$, and a hybrid controller $(\kappa_0, V_0, D_c, F_c)$ that is nominally synergistic relative to $\mathcal{A}$ for (30) with synergy gap exceeding δ, let Ψ denote the largest weakly invariant subset of*

$$(\dot{x}, \dot{x}_c) \in F_{cl,0}(x, x_c) = \begin{pmatrix} F_0(x, x_c, \kappa_0(x, x_c)) \\ F_c(x, x_c) \end{pmatrix}$$

on $(x, x_c) \in \mathcal{E} := \{(x, x_c) \in \mathcal{X} \times \mathcal{X}_c : \nabla V_0(x, x_c)^\top f_{cl,0} = 0$ for some $f_{cl,0} \in F_{cl,0}(x, x_c)\}$ and let $\Psi_{1,\theta}$ denote the largest weakly invariant subset of

$$(\dot{x}, \dot{x}_{c,1}) \in F_{cl,1}(x, x_{c,1}) \qquad (x, x_{c,1}) \in \mathcal{E}_1$$

with $\mathcal{E}_1 := \{(x, x_{c,1}) \in \mathcal{X} \times \mathcal{X}_{c,1} : \nabla V_{1,\theta}(x, x_{c,1})^\top f_{cl,1} = 0$ for some $f_{cl,1} \in F_{cl,1}(x, x_{c,1})\}$. If the projection of $\Psi_{1,\theta} \backslash \mathcal{A}_{1,\theta}$ onto $\mathcal{X} \times \mathcal{X}_c$ is a subset of $\Psi \backslash \mathcal{A}$, i.e.,[4]

$$\pi_{\mathcal{X} \times \mathcal{X}_c}(\Psi_{1,\theta} \backslash \mathcal{A}_{1,\theta}) \subset \Psi \backslash \mathcal{A}, \qquad (41)$$

then the hybrid controller $(\kappa_1, V_{1,\theta}, D_{c,1}, F_{c,1})$ in (37) is synergistic relative to $\mathcal{A}_{1,\theta}$ for (30) with synergy gap exceeding δ.

Proof. It follows from the definition of $\mu_{V_{1,\theta}}$ in (40c) that $\mu_{V_{1,\theta}}(x, x_{c,1})$ is the sum of $\mu_{V_0}(x, x_c)$ with a quadratic nonnegative term, hence

$$\mu_{V_{1,\theta}}(x, x_{c,1}) \geq \mu_{V_0}(x, x_c) \qquad (42)$$

for each $(x, x_{c,1}) \in \Psi_{1,\theta} \backslash \mathcal{A}_{1,\theta}$ and, consequently, we have that

$$\begin{aligned}
\overline{\delta}_2 &:= \inf\{\mu_{V_{1,\theta}}(x, x_c, \theta) : (x, x_{c,1}) \in \Psi_{1,\theta} \backslash \mathcal{A}_{1,\theta}\} \\
&\geq \inf\{\mu_{V_0}(x, x_c) : (x, x_{c,1}) \in \Psi_{1,\theta} \backslash \mathcal{A}_{1,\theta}\}.
\end{aligned} \qquad (43)$$

The fact that $(x, x_{c,1}) \in \Psi_{1,\theta} \backslash \mathcal{A}_{1,\theta}$ implies $(x, x_c) \in \pi_{\mathcal{X} \times \mathcal{X}_c}(\Psi_{1,\theta} \backslash \mathcal{A}_{1,\theta})$ together with (43) allow us to derive the following inequality: $\overline{\delta}_2 \geq \inf\{\mu_{V_0}(x, x_c) : (x, x_c) \in \pi_{\mathcal{X} \times \mathcal{X}_c}(\Psi_{1,\theta} \backslash \mathcal{A}_{1,\theta})\}$. It follows from (41) that $\overline{\delta}_2 \geq \inf\{\mu_{V_0}(x, x_c) : (x, x_c) \in \Psi \backslash \mathcal{A}\}$ which is greater than zero by the assumption that the controller $(\kappa_0, V_0, D_c, F_c)$ is synergistic relative to $\mathcal{A}$ for (32) with synergy gap exceeding δ. In addition, we have that $\mu_{V_{1,\theta}}(x, x_{c,1}) \geq \mu_{V_0}(x, x_c) > \delta(x, x_c)$ for each $(x, x_{c,1}) \in \Psi_{1,\theta} \backslash \mathcal{A}_{1,\theta}$, which proves that the hybrid controller $(\kappa_1, V_{1,\theta}, D_{c,1}, F_{c,1})$ in (37) is synergistic relative to $\mathcal{A}_{1,\theta}$ for (30) with synergy gap exceeding δ. $\qquad\square$

Remark 9. It should be noted that condition (41) is akin to a persistence of excitation condition, as it guarantees that the inclusion of the estimator dynamics in (34) does not add new undesired equilibria to the closed-loop system.

In the next result, we complete the construction of the robust synergistic controller (22) from the data of a nominally

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[4] Given a subset S of $X := X_1 \times X_2$, the projection of S onto X_1 is represented by $\pi_{X_1}(S) := \{x_1 \in X_1 : (x_1, x_2) \in S$ for some $x_2 \in X_2\}$. Similarly, the projection of S onto X_2 is denoted by $\pi_{X_2}(S) := \{x_2 \in X_2 : (x_1, x_2) \in S$ for some $x_1 \in X_1\}$.

synergistic controller $(\kappa_0, V_0, D_c, F_c)$, by designing a set-valued map $G_{c,1} : \mathcal{X} \times \mathcal{X}_{c,1} \rightrightarrows \mathcal{X}$ that is outer semicontinuous, locally bounded and satisfies (24).

Proposition 5. *Suppose that the sets $\mathcal{X}$, $\mathcal{X}_c$, $\mathcal{U}$, and the set-valued map F_θ in (30) satisfy Assumption 1, and that Assumption 2 holds. Given Ω in (31), a compact set $\mathcal{A} \subset \mathcal{X} \times \mathcal{X}_c$, and a hybrid controller $(\kappa_0, V_0, D_c, F_c)$ that is nominally synergistic relative to $\mathcal{A}$ for (30) with synergy gap exceeding δ, $\mathscr{A}_1 := \{\mathcal{A}_{1,\theta}\}_{\theta \in \Omega}$ with $\mathcal{A}_{1,\theta}$ in (33), $\mathscr{V}_1 := \{V_{1,\theta}\}_{\theta \in \Omega}$ with $V_{1,\theta}$ in (37b), then the hybrid controller $(\kappa_1, \mathscr{V}_1, D_{c,1}, F_{c,1}, G_{c,1})$ where*

$$G_{c,1}(x, x_{c,1}) := \varrho_{V_0}(x, x_c) \times \hat{G}(\hat\theta) \qquad (44)$$

for each $(x, x_{c,1}) \in \mathcal{X} \times \mathcal{X}_{c,1}$, and

$$\begin{aligned}
\hat{G}(\hat\theta) := \arg\max_{g \in \Omega + \epsilon\overline{\mathbb{B}}} \min_{\theta \in \Omega} \ &(\theta - \hat\theta)^\top \Gamma_1^{-1}(\theta - \hat\theta) \\
&- (\theta - g)^\top \Gamma_1^{-1}(\theta - g)
\end{aligned}$$

for each $\hat\theta \in \Omega + \epsilon\overline{\mathbb{B}}$, is synergistic relative to $\mathscr{A}_1$ for (30) with robustness margin Ω and synergy gap exceeding δ.

Proof. In Proposition 4 we demonstrate that the hybrid controller $(\kappa_1, V_{1,\theta}, D_{c,1}, F_{c,1})$ is synergistic relative to $\mathcal{A}_{1,\theta}$ as required by Definition 3. It remains to be shown that the hybrid controller $(\kappa_1, \mathscr{V}_1, D_{c,1}, F_{c,1}, G_{c,1})$ satisfies assumptions (R1), (R2) and (R3). To prove (R1), one must show that $\mathcal{X} \times \mathcal{X}_{c,1} \subset \operatorname{dom} V_{1,\theta}$. From the definition of $V_{1,\theta}$ in (37b), we have that $\operatorname{dom} V_{1,\theta} = \operatorname{dom} V_0 \times (\Omega + \epsilon\overline{\mathbb{B}})$. It follows from the assumption that the hybrid controller $(\kappa_0, V_0, D_c, F_c)$ is nominally synergistic relative to $\mathcal{A}$ for (30) that $\mathcal{X} \times \mathcal{X}_c \subset \operatorname{dom} V_0$, hence $\mathcal{X} \times \mathcal{X}_{c,1} \subset \operatorname{dom} V_{1,\theta}$. The function $(x, x_{c,1}, \theta) \mapsto V_1(x, x_{c,1}, \theta) := V_{1,\theta}(x, x_{c,1})$ is continuous because it results from the composition of continuous functions, hence (R1) holds.

To prove (R2) and (R3), one must show that $G_{c,1}$ is outer semicontinuous, locally bounded and that it satisfies (24). Since $G_{c,1}(x, x_{c,1})$ is the Cartesian product of $\varrho_{V_0}(x, x_c)$ and $\hat{G}(\hat\theta)$ for each $(x, x_{c,1}) \in \mathcal{X} \times \mathcal{X}_{c,1}$ and ϱ_{V_0} is outer semicontinuous and locally bounded as proved in Lemma 1, to demonstrate that (R2) is satisfied it only remains to be shown that $\hat{G}$ is outer semicontinuous and locally bounded. Let $h(g, \theta) := (\theta - \hat\theta)^\top \Gamma_1^{-1}(\theta - \hat\theta) - (\theta - g)^\top \Gamma_1^{-1}(\theta - g)$ for each $(g, \theta) \in (\Omega + \epsilon\overline{\mathbb{B}}) \times \Omega$. Since h results from the composition of continuous functions it is also continuous. It follows from the compactness of Ω and from [36, Theorem 9.14] that

$$\underline{h}(g) := \min\{h(g, \theta) : \theta \in \Omega\} \qquad \forall g \in \Omega + \epsilon\overline{\mathbb{B}} \qquad (45)$$

is continuous. Since $D_{c,1}$ is continuous and compact-valued, it follows from the continuity of (45) and [36, Theorem 9.14] that $\hat{G}$ is compact-valued and upper semicontinuous. The remainder of the proof of outer semicontinuity and local boundedness of $\hat{G}$ follows closely that of Lemma 1, thus it will be omitted.

The fact that $G_{c,1}$ satisfies (24) follows from the observations in Remark 7 by noticing that Ω and $\Omega + \epsilon\overline{\mathbb{B}}$ are convex and compact spaces and the function h, which can be rewritten as $h(g, \theta) = 2\theta^\top \Gamma_1^{-1}(g - \hat\theta) - g^\top \Gamma_1^{-1} g + \hat\theta^\top \Gamma_1^{-1}\hat\theta$ for each $(g, \theta) \in (\Omega + \epsilon\overline{\mathbb{B}}) \times \Omega$, is quasi-concave as a function of g and quasi-convex as a function of θ. $\qquad\square$

The hybrid closed-loop system resulting from the interconnection between $(\kappa_1, \mathscr{V}_1, D_{c,1}, F_{c,1}, G_{c,1})$ and (30) is given by:

$$(\dot{x}, \dot{x}_{c,1}) \in F_{cl,1}(x, x_{c,1}) \qquad (x, x_{c,1}) \in C_{\Omega,1} \quad (46a)$$

$$(x^+, x_{c,1}^+) \in G_{\Omega,1}(x, x_{c,1}) \qquad (x, x_{c,1}) \in D_{\Omega,1} \quad (46b)$$

where

$$C_{\Omega,1} := \left\{ (x, x_{c,1}) \in \mathcal{X} \times \mathcal{X}_{c,1} : \min_{\theta \in \Omega} \mu_{V_{1,\theta}}(x, x_{c,1}) \leq \delta(x, x_c) \right\}$$

$$D_{\Omega,1} := \left\{ (x, x_{c,1}) \in \mathcal{X} \times \mathcal{X}_{c,1} : \min_{\theta \in \Omega} \mu_{V_{1,\theta}}(x, x_{c,1}) \geq \delta(x, x_c) \right\}$$

and

$$G_{\Omega,1}(x, x_{c,1}) := \begin{bmatrix} x \\ G_{c,1}(x, x_{c,1}) \end{bmatrix} \forall (x, x_{c,1}) \in D_{\Omega,1}, \quad (47)$$

Global asymptotic stability of $\mathcal{A}_{1,\theta}$ for (46) follows from the application of Theorem 2 and it is summarized in the next corollary.

Corollary 2. *Suppose that the sets $\mathcal{X}$, $\mathcal{X}_c$, $\mathcal{U}$, and the set-valued map F_θ in (30) satisfy Assumption 1, and that Assumption 2 holds. Given Ω in (31), a positive function $\delta : \mathcal{X} \times \mathcal{X}_c \mapsto \mathbb{R}$, a compact set $\mathcal{A} \subset \mathcal{X} \times \mathcal{X}_c$, and a hybrid controller $(\kappa_0, V_0, D_c, F_c)$ that is nominally synergistic relative to $\mathcal{A}$ for (30) with synergy gap exceeding δ, for each $\theta \in \Omega$, the set $\mathcal{A}_{1,\theta}$ is globally pre-asymptotically stable for (46). If, for each $(x, x_{c,1}) \in C_{\Omega,1} \backslash D_{\Omega,1}$, (VC') is satisfied, then $\mathcal{A}_{1,\theta}$ is globally asymptotically stable for (46).*

Proof. It follows from (42) that $\min\{\mu_{V_{1,\theta}}(x, x_{c,1}) : \theta \in \Omega\} \geq \mu_{V_0}(x, x_c)$ for each $(x, x_{c,1}) \in \Psi_{1,\theta} \backslash \mathcal{A}_{1,\theta}$. Since $\mu_{V_0}(x, x_c) > \delta(x, x_c)$ for each $(x, x_{c,1}) \in \Psi_{1,\theta} \backslash \mathcal{A}_{1,\theta}$ as shown in the proof of Proposition 4, and δ is positive, the conditions of Theorem 2 apply and we are able to conclude that $\mathcal{A}_{1,\theta}$ is globally asymptotically stable for (46). $\square$

C. Backstepping

Given a nominally synergistic controller $(\kappa_0, V_0, D_c, F_c)$, we extend the dynamics of the controller in Section VI-B to include the input u as a controller state:[5]

$$\dot{x}_{c,2} \in F_{c,2}(x, x_{c,2})$$
$$:= \left\{ \begin{bmatrix} f_c \\ \Gamma_1 \operatorname{Proj}(v(x, x_{c,2}), \hat{\theta}) \\ f_u(x, x_{c,2}) + \mathcal{D}_{x_c}(\kappa_1(x, x_{c,1})) f_c \end{bmatrix} : f_c \in F_c(x, x_c) \right\}$$
(48)

with $x_{c,2} := (x_{c,1}, u) \in \mathcal{X}_{c,2} := \mathcal{X}_c \times (\Omega + \epsilon\overline{\mathbb{B}}) \times \mathbb{R}^m$, $\Gamma_2 \in \mathbb{R}^{m \times m}$ positive definite, $k_u > 0$,

$$v(x, x_{c,2}) := W(x, x_c)^\top \nabla_x V_0(x, x_c)$$
$$- W(x, x_c)^\top \mathcal{D}_x(\kappa_1(x, x_{c,1}))^\top \Gamma_2^{-1}(u - \kappa_1(x, x_{c,1})) \quad (49)$$

for each $(x, x_{c,2}) \in \mathcal{X} \times \mathcal{X}_{c,2}$, and

$$f_u(x, x_{c,2}) := -\widehat{W}(x, x_c)\Gamma_1 \operatorname{Proj}(v(x, x_{c,2}), \hat{\theta})$$
$$- k_u(u - \kappa_1(x, x_{c,1})) - \Gamma_2 H(x, x_c)^\top \nabla_x V_0(x, x_c)$$
$$+ \mathcal{D}_x(\kappa_1(x, x_{c,1})) F(x, x_c, u, \hat{\theta})$$
(50)

[5]Alternatively, one may consider u as a plant state rather than a controller state, in which case u would remain constant during jumps. We have included u as a controller variable because it is an approach less often found in the literature.

which is defined for each $(x, x_{c,2}) \in \mathcal{X} \times \mathcal{X}_{c,2}$ assuming that κ_0 is continuously differentiable and that $F(x, x_c, u, \hat{\theta}) = F_{\hat{\theta}}(x, x_c, u)$ denotes the dynamics (30) with θ is equal to the estimated value $\hat{\theta}$.

Given the compact set Ω of possible (unknown) values of θ in (31), a compact set $\mathcal{A} \subset \mathcal{X} \times \mathcal{X}_c$, and a nominal synergistic controller $(\kappa_0, V_0, D_c, F_c)$ relative to $\mathcal{A}$ for (30) with synergy gap exceeding δ, the main goal of this section is to design a controller of the form (22) that is synergistic relative to $\mathscr{A}_2 := \{\mathcal{A}_{2,\theta}\}_{\theta \in \Omega}$ for (30) with robustness margin Ω and synergy gap exceeding δ, where

$$\mathcal{A}_{2,\theta} := \{ (x, x_{c,2}) \in \mathcal{X} \times \mathcal{X}_{c,2} : (x, x_{c,1}) \in \mathcal{A}_{1,\theta}, \\ u = \kappa_1(x, x_{c,1}) \}. \quad (51)$$

In this direction, we define the Lyapunov function

$$V_{2,\theta}(x, x_{c,2}) := V_{1,\theta}(x, x_{c,1}) \\ + \frac{1}{2}(u - \kappa_1(x, x_{c,1}))^\top \Gamma_2^{-1}(u - \kappa_1(x, x_{c,1})) \quad (52)$$

for each $(x, x_{c,2}) \in \mathcal{X} \times \mathcal{X}_{c,2}$ and the set-valued map

$$D_{c,2}(x, x_{c,2}) := \{ (g_{c,1}, g_u) \in \mathcal{X}_{c,2} : g_{c,1} \in D_{c,1}(x, x_{c,1}), \\ g_u = \kappa_1(x, g_{c,1}) \} \quad (53)$$

for each $(x, x_{c,2}) \in \mathcal{X} \times \mathcal{X}_{c,2}$. The choice $u = \kappa_1(x, g_{c,1})$ in (53) may seem peculiar, but it turns out that this value minimizes (52) with respect to u, hence it is suitable for the jump logic.

From the interconnection between (30) and the hybrid controller $(\kappa_2, V_{2,\theta}, D_{c,2}, F_{c,2})$ with $\kappa_2(x, x_{c,2}) = u$ for each $(x, x_{c,2}) \in \mathcal{X} \times \mathcal{X}_{c,2}$, we obtain the hybrid closed-loop system

$$(\dot{x}, \dot{x}_{c,2}) \in F_{cl,2}(x, x_{c,2}) \quad (x, x_{c,2}) \in C_2 \\ := \{ (x, x_{c,2}) \in \mathcal{X} \times \mathcal{X}_{c,2} : \mu_{V_{2,\theta}}(x, x_{c,2}) \leq \delta(x, x_c) \}$$
(54a)

$$(x^+, x_{c,2}^+) \in G_{cl,2}(x, x_{c,2}) \quad (x, x_{c,2}) \in D_2 \\ := \{ (x, x_{c,2}) \in \mathcal{X} \times \mathcal{X}_{c,2} : \mu_{V_{2,\theta}}(x, x_{c,2}) \geq \delta(x, x_c) \}$$
(54b)

where

$$F_{cl,2}(x, x_{c,2}) := \begin{bmatrix} F_\theta(x, x_c, u) \\ F_{c,2}(x, x_{c,2}) \end{bmatrix} \quad \forall (x, x_{c,2}) \in C_2 \quad (55a)$$

$$G_{cl,2}(x, x_{c,2}) := \begin{bmatrix} x \\ \varrho_{V_{2,\theta}}(x, x_{c,2}) \end{bmatrix} \quad \forall (x, x_{c,2}) \in D_2. \quad (55b)$$

Note that, from the definitions (11b) and (11c), we have the following identities for the hybrid controller $(\kappa_2, V_{2,\theta}, D_{c,2}, F_{c,2})$:

$$\varrho_{V_{2,\theta}}(x, x_{c,2}) = \{ (g_{c,1}, g_u) \in \mathcal{X}_{c,2} : g_{c,1} \in \varrho_{V_{1,\theta}}(x, x_{c,1}), \\ g_u = \kappa_1(x, g_{c,1}) \},$$

$$\mu_{V_{2,\theta}}(x, x_{c,2}) = \mu_{V_{1,\theta}}(x, x_{c,1}) + \frac{1}{2}\left| \Gamma_2^{-\frac{1}{2}}(u - \kappa_1(x, x_{c,1})) \right|^2$$

for each $(x, x_{c,2}) \in \mathcal{X} \times \mathcal{X}_{c,2}$,[6]; hence, similarly to (38), the closed-loop system (54) is impossible to implement due to dependence on θ in C_2, D_2, and $G_{cl,2}$, but, similarly to the controller of Section VI-B, this dependence will be removed with the design of a hybrid controller that is synergistic

[6]Since $\Gamma_2 \in \mathbb{R}^{m \times m}$ is assumed to be positive definite, $\Gamma_2^{-\frac{1}{2}}$ exists and is unique (cf. [42, Section 8.5]).

relative to $\mathscr{A}_2 := \{\mathcal{A}_{2,\theta}\}_{\theta \in \Omega}$ for (30) with robustness margin Ω (cf. Remark 8).

We are able to prove the following result using arguments similar to those of Proposition 4.

Proposition 6. *Suppose that the sets $\mathcal{X}$, $\mathcal{X}_c$, $\mathcal{U}$, and the set-valued map F_θ in (30) satisfy Assumption 1, and that Assumption 2 holds. Given $\theta \in \Omega$, a compact set $\mathcal{A} \subset \mathcal{X} \times \mathcal{X}_c$, and a hybrid controller $(\kappa_0, V_0, D_c, F_c)$ that is nominally synergistic relative to $\mathcal{A}$ for (30) with synergy gap exceeding δ, if (41) is satisfied then the hybrid controller $(\kappa_2, V_{2,\theta}, D_{c,2}, F_{c,2})$ is synergistic relative to $\mathcal{A}_{2,\theta}$ for (30) with synergy gap exceeding δ.*

Proof. Similarly to the proof of Proposition 4, it is possible to show that properties (C1), (C3) and (C5) follow directly from the fact that $\mathcal{A}_{2,\theta}$ is compact and from the assumption that $(\kappa_0, V_0, D_c, F_c)$ is synergistic relative to $\mathcal{A}$ for (32). It follows from the continuity of $D_{c,1}$ and κ_1 that $D_{c,2}$ is continuous. That $D_{c,2}$ is compact-valued follows from compactness of $D_{c,1}$ and continuity of κ_1, hence (C4) is satisfied. From the assumption that $(\kappa_0, V_0, D_c, F_c)$ is synergistic relative to $\mathcal{A}$ for (32), we show in [31] that

$$\nabla V_{2,\theta}(x, x_{c,2})^\top f_{cl,2} \leq \nabla V_0(x, x_c)^\top F_0(x, x_c, \kappa_0(x, x_c))$$
$$- k_u(u - \kappa_1(x, x_{c,1}))^\top \Gamma_2^{-1}(u - \kappa_1(x, x_{c,1})) \leq 0 \quad (56)$$

for each $(x, x_{c,2}) \in \mathcal{X} \times \mathcal{X}_{c,2}$ satisfying $V_{2,\theta}(x, x_{c,2}) < +\infty$ and each $f_{cl,2} \in F_{cl,2}(x, x_{c,2})$, hence property (C6) is satisfied. Let $\Psi_{2,\theta}$ denote the largest weakly invariant subset of

$$(\dot{x}, \dot{x}_{c,2}) \in F_{cl,2}(x, x_{c,2}) \qquad (x, x_c) \in \mathcal{E}_2 \quad (57)$$

with $\mathcal{E}_2 := \{(x, x_{c,2}) \in \mathcal{X} \times \mathcal{X}_{c,2} : \nabla V_{2,\theta}(x, x_{c,2})^\top f_{cl,2} = 0$ for some $f_{cl,2} \in F_{cl,2}(x, x_{c,2})\}$. To verify that $(\kappa_2, V_{2,\theta}, D_{c,2}, F_{c,2})$ is synergistic relative to $\mathcal{A}_{2,\theta}$ for (48), we need to check that $\bar{\delta}_2 := \inf\{\mu_{V_{2,\theta}}(x, x_{c,2}) : (x, x_{c,2}) \in \Psi_{2,\theta} \backslash \mathcal{A}_{2,\theta}\} > 0$. It follows from (56) that $\Psi_{2,\theta} \subset \{(x, x_{c,2}) \in \mathcal{X} \times \mathcal{X}_{c,2} : (x, x_{c,1}) \in \Psi_{1,\theta}, u = \kappa_1(x, x_{c,1})\}$ where $\Psi_{1,\theta}$ is defined in Proposition 4. It follows from (51) that

$$\bar{\delta}_2 \geq \inf\{\mu_{V_{2,\theta}}(x, x_{c,2}) : (x, x_{c,1}) \in \Psi_{1,\theta} \backslash \mathcal{A}_{1,\theta},$$
$$u = \kappa_1(x, x_{c,1})\} \quad (58)$$
$$= \inf\{\mu_{V_{1,\theta}}(x, x_{c,1}) : (x, x_{c,1}) \in \Psi_{1,\theta} \backslash \mathcal{A}_{1,\theta}\}$$

which we have shown in Proposition 4 to satisfy $\bar{\delta}_2 > 0$, under assumption (41). In addition, $\mu_{V_{2,\theta}}(x, x_{c,2}) = \mu_{V_{1,\theta}}(x, x_{c,1}) \geq \mu_{V_0}(x, x_c) > \delta(x, x_c)$ for each $(x, x_{c,2}) \in \Psi_{2,\theta} \backslash \mathcal{A}_{2,\theta}$, hence the hybrid controller $(\kappa_2, V_{2,\theta}, D_{c,2}, F_{c,2})$ is synergistic relative to $\mathcal{A}_{2,\theta}$ for (30) with synergy gap exceeding δ. $\square$

To finalize the design of a robust synergistic controller, we provide the construction of the jump map $G_{c,2}$ in the next proposition.

Proposition 7. *Suppose that the sets $\mathcal{X}$, $\mathcal{X}_c$, $\mathcal{U}$, and the set-valued map F_θ in (30) satisfy Assumption 1, and that Assumption 2 holds. Given Ω in (31), a compact set $\mathcal{A} \subset \mathcal{X} \times \mathcal{X}_c$, and a hybrid controller $(\kappa_0, V_0, D_c, F_c)$ that is nominally synergistic relative to $\mathcal{A}$ for (30) with synergy gap exceeding δ, $\mathscr{A}_2 := \{\mathcal{A}_{2,\theta}\}_{\theta \in \Omega}$ with $\mathcal{A}_{2,\theta}$ in (51), $\mathscr{V}_2 := \{V_{2,\theta}\}_{\theta \in \Omega}$ with*

$V_{2,\theta}$ *in (52), the hybrid controller $(\kappa_2, \mathscr{V}_2, D_{c,2}, F_{c,2}, G_{c,2})$ where*

$$G_{c,2}(x, x_{c,2}) := \{(g_{c,1}, g_u) \in D_{c,2}(x, x_{c,2}) :$$
$$g_{c,1} \in G_{c,1}(x, x_{c,1})\} \quad (59)$$

for each $(x, x_{c,2}) \in \mathcal{X} \times \mathcal{X}_{c,2}$ is synergistic relative to $\mathscr{A}_2$ for (30) with robustness margin Ω and synergy gap exceeding δ.

Proof. In Proposition 6 we demonstrate that the hybrid controller $(\kappa_2, V_{2,\theta}, D_{c,2}, F_{c,2})$ is synergistic relative to $\mathcal{A}_{2,\theta}$ with synergy gap exceeding δ as required by Definition 3. The proof that (R1) is satisfied follows closely the proof of Proposition 5, hence it is omitted here. The outer semicontinuity and local boundedness of $G_{c,2}$ follows from outer semicontinuity and local boundedness of $G_{c,1}$ in addition to the continuity of κ_1, thus (R2) is verified. For each $(x, x_{c,2}) \in \mathcal{X} \times \mathcal{X}_{c,2}$ and for each $g_{c,2} \in \mathcal{X}_2$, we have that

$$V_{2,\theta}(x, x_{c,2}) - V_{2,\theta}(x, g_{c,2})$$
$$\geq \min_{\theta \in \Omega} V_{2,\theta}(x, x_{c,2}) - V_{2,\theta}(x, g_{c,2}). \quad (60)$$

From (59), it follows that $g_{c,2} := (g_{c,1}, g_u)$ with $g_{c,1}$ belonging to (44) and $g_u = \kappa_1(x, g_{c,1})$. Replacing (52) in (60) and plugging in the aforementioned values of $g_{c,1}$ and g_u, we have that

$$V_{2,\theta}(x, x_{c,2}) - V_{2,\theta}(x, g_{c,2})$$
$$\geq \max_{g_{c,1} \in D_{c,1}(x, x_{c,1})} \min_{\theta \in \Omega} V_{1,\theta}(x, x_{c,1}) - V_{1,\theta}(x, g_{c,1})$$
$$+ \frac{1}{2}(u - \kappa_1(x, x_{c,1}))^\top \Gamma_2^{-1}(u - \kappa_1(x, x_{c,1}))$$
$$= \max_{g_{c,2} \in D_{c,2}(x, x_{c,2})} \min_{\theta \in \Omega} V_{2,\theta}(x, x_{c,2}) - V_{2,\theta}(x, g_{c,2})$$
$$\quad (61)$$

for each $(x, x_{c,2}) \in \mathcal{X} \times \mathcal{X}_{c,2}$ and each $g_{c,2} := (g_{c,1}, g_u) \in G_{c,2}(x, x_{c,2})$. Since the $\max$ and $\min$ operators in (61) commute as shown in the proof of Proposition 5, it follows that

$$V_{2,\theta}(x, x_{c,2}) - V_{2,\theta}(x, g_{c,2}) \geq \min_{\theta \in \Omega} \mu_{V_{2,\theta}}(x, x_{c,2})$$

for each $(x, x_{c,2}) \in \mathcal{X} \times \mathcal{X}_{c,2}$ and each $g_{c,2} := (g_{c,1}, g_u) \in G_{c,2}(x, x_{c,2})$, thus verifying (R3). $\square$

The hybrid closed-loop system resulting from the interconnection between $(\kappa_2, \mathscr{V}_2, D_{c,2}, F_{c,2}, G_{c,2})$ and (30) is given by:

$$(\dot{x}, \dot{x}_{c,2}) \in F_{cl,2}(x, x_{c,2}) \qquad (x, x_{c,2}) \in C_{\Omega,2} \quad (62a)$$
$$(x^+, x_{c,2}^+) \in G_{\Omega,2}(x, x_{c,2}) \qquad (x, x_{c,2}) \in D_{\Omega,2} \quad (62b)$$

where

$$C_{\Omega,2} := \left\{(x, x_{c,2}) \in \mathcal{X} \times \mathcal{X}_{c,2} : \min_{\theta \in \Omega} \mu_{V_{2,\theta}}(x, x_{c,2}) \leq \delta(x, x_c)\right\}$$

$$D_{\Omega,2} := \left\{(x, x_{c,2}) \in \mathcal{X} \times \mathcal{X}_{c,2} : \min_{\theta \in \Omega} \mu_{V_{2,\theta}}(x, x_{c,2}) \geq \delta(x, x_c)\right\}$$

and

$$G_{\Omega,2}(x, x_{c,2}) := \begin{bmatrix} x \\ G_{c,2}(x, x_{c,2}) \end{bmatrix} \forall (x, x_{c,2}) \in D_{\Omega,2}.$$

The global asymptotic stability of $\mathcal{A}_{2,\theta}$ for (62) follows from Theorem 2 and it is stated in the next corollary for the sake of completeness. The proof is omitted because it is identical to the proof of Corollary 2

Corollary 3. *Suppose that the sets $\mathcal{X}$, $\mathcal{X}_c$, $\mathcal{U}$, and the set-valued map F_θ in (30) satisfy Assumption 1, and that Assumption 2 holds. Given Ω in (31), a positive function $\delta : \mathcal{X} \times \mathcal{X}_c \mapsto \mathbb{R}$, a compact set $\mathcal{A} \subset \mathcal{X} \times \mathcal{X}_c$, and a hybrid controller $(\kappa_0, V_0, D_c, F_c)$ that is nominally synergistic relative to $\mathcal{A}$ for (30) with synergy gap exceeding δ, for each $\theta \in \Omega$, the set $\mathcal{A}_{2,\theta}$ is globally pre-asymptotically stable for (62). If, for each $(x, x_{c,2}) \in C_{\Omega,2} \backslash D_{\Omega,2}$, (VC') is satisfied, then $\mathcal{A}_{2,\theta}$ is globally asymptotically stable for (62).*

In the next section, we apply the controllers proposed in Sections VI-B and VI-C to global asymptotic stabilization of a setpoint for a two-dimensional system in the presence of an obstacle.

VII. Synergistic Hybrid Feedback for Robust Global Obstacle Avoidance

To demonstrate the applicability of the synergistic adaptive controller of Section VI, we consider the problem of globally asymptotically stabilizing the origin for a vehicle moving on a plane with an obstacle $\mathcal{N} := z_0 + r\overline{\mathbb{B}}$ with $z_0 \in \mathbb{R}^2$ and $r > 0$ such that the origin is not contained in $\mathcal{N}$. We consider that the evolution in time of the position $z \in \mathbb{R}^2 \backslash \mathcal{N}$ of the vehicle is described by

$$\dot{z} = u + \theta \tag{63}$$

where $u \in \mathcal{U} := \mathbb{R}^2$ is the input and $\theta \in \mathbb{R}^2$ is an unknown constant. We have shown in [8, Section IV] that $\psi(z) := \begin{bmatrix} \frac{z-z_0}{|z-z_0|} & \log(|z - z_0| - r) \end{bmatrix}^\top$ is a diffeomorphism between $\mathbb{R}^2 \backslash \mathcal{N}$ and $\mathsf{S}^1 \times \mathbb{R}$, hence global asymptotic stabilization of the origin for (63) is equivalent to the global asymptotic stabilization of $\psi(0)$ for

$$\dot{x} = \mathcal{D}\psi(\psi^{-1}(x))u + \mathcal{D}\psi(\psi^{-1}(x))\theta. \tag{64}$$

with $x \in \mathcal{X} := \mathsf{S}^1 \times \mathbb{R}$. Before moving to the controller design, we show that Assumption 1 is verified for the particular problem at hand.

Proposition 8. *Let $\mathcal{X}_c$ denote a closed subset of some Euclidean space. Then, the set-valued map*

$$F_\theta(x, x_c, u) := \mathcal{D}\psi(\psi^{-1}(x))u + \mathcal{D}\psi(\psi^{-1}(x))\theta \tag{65}$$

defined for each $(x, x_c, u) \in \mathcal{X} \times \mathcal{X}_c \times \mathcal{U}$ satisfies Assumption 1 and

- *($\star$) The intersection between $F_\theta(x, x_c, u)$ and the tangent space to $\mathcal{X}$ at (x, x_c, u) is nonempty for each $(x, x_c, u) \in \mathcal{X} \times \mathcal{X}_c \times \mathcal{U}$.*

Proof. To check that the condition (S1) holds, note that the sets $\mathcal{X}$, $\mathcal{X}_c$ and $\mathcal{U}$ are closed subsets of Euclidean spaces. It follows from the fact that ψ is a diffeomorphism between $\mathbb{R}^2 \backslash \mathcal{N}$ and $\mathsf{S}^1 \times \mathbb{R}$ that $\mathcal{D}\psi(\psi^{-1}(x))$ is an isomorphism between the tangent space to $\mathbb{R}^2 \backslash \mathcal{N}$ at $\psi^{-1}(x)$ and the tangent space to $\mathsf{S}^1 \times \mathbb{R}$ at x for each $x \in \mathcal{X}$ (cf. [35, Proposition 3.6]), thus ($\star$) is verified. Since ψ is a diffeomorphism it also follows that $x \mapsto \mathcal{D}\psi(\psi^{-1}(x))$ is continuous, thus F_θ is also continuous and single-valued, hence it verifies (S2). $\square$

The following section presents a hybrid feedback controller of the form $(\kappa_0, V_0, D_c, F_c)$ that is nominally synergistic relative to

$$\mathcal{A} := \{(x, q) \in \mathcal{X} \times \mathcal{X}_c : x = \psi(0)\}. \tag{66}$$

for (65) as required by the controller design that is presented in Section VI.

A. Nominally Synergistic Hybrid Feedback Controller for Obstacle Avoidance

Inspired by the controller design in [8, Section IV], let $\phi_q(x) := \begin{bmatrix} \frac{q_2 x_1 - q_1 x_2}{1 - q_1 x_1 - q_2 x_2} & x_3 \end{bmatrix}^\top$ for each $x := (x_1, x_2, x_3) \in U_q := \{x \in \mathcal{X} : q_1 x_1 + q_2 x_2 \neq 1\}$ with $q := (q_1, q_2) \in \mathsf{S}^1$. The second component of $\phi_q(x)$ is the identity map, while the first component corresponds to the stereographic projection of $(x_1, x_2) \in \mathsf{S}^1$ from $q \in \mathsf{S}^1$.

Let controller variable x_c be equal to q, i.e., $x_c = q \in \mathcal{X}_c$, with $\mathcal{X}_c := \{q \in \mathsf{S}^1 : q^\top x_{12}^0 \leq \gamma\}$, where $\gamma \in (-1, 1)$ is a controller parameter and $x_{12}^0 \in \mathsf{S}^1$ is the shorthand notation for the first two components of $x^0 := \psi(0)$. Then, we define

$$V_0(x, q) := \begin{cases} \frac{1}{2} |\phi_q(x) - \phi_q(\psi(0))|^2 & \text{if } x \in U_q \\ +\infty & \text{otherwise} \end{cases} \tag{67}$$

for each $(x, q) \in \mathcal{X} \times \mathcal{X}_c$. The definition of $\mathcal{X}_c$ above is instrumental as it implies that $x_{12}^0 \in U_q$ for each $q \in \mathcal{X}_c$.

Defining $D_c(x, q) = \mathcal{X}_c$ for each $(x, q) \in \mathcal{X} \times \mathcal{X}_c$ and

$$\rho(x, q) := -\frac{(x_1, x_2) + q}{|(x_1, x_2) + q|}$$

for each $(x, q) \in \mathcal{X} \times \mathcal{X}_c$ satisfying $x_{12} \neq -q$, the solution to the optimization problem (11) is given by:

$$\varrho_{V_0}(x, q) = \begin{cases} \{q \in \mathsf{S}^1 : q^\top x_{12}^0 = 0\} & \text{if } x = -q \\ \rho(x, q) & \text{if } \rho(x, q)^\top x_{12}^0 \leq \gamma \\ \gamma x_{12}^0 + \sqrt{1 - \gamma^2}\Pi(x_{12}^0)\rho(x, q) & \text{otherwise} \end{cases} \tag{68}$$

for each $(x, q) \in \mathcal{X} \times \mathcal{X}_c$, where $\Pi(q) = I_2 - qq^\top$ for each $q \in \mathsf{S}^1$. To increase the rate of descent of (67) during flows, we define $F_c(x, q) := -\Gamma(q)\Pi(q)\nabla_q V_0(x, q)$ for each $(x, q) \in \mathcal{X} \times \mathcal{X}_c$, where $\Gamma(q)$ is a nonnegative continuous function on S^1 satisfying $\Gamma(q) = 0$ for each $q \in \mathsf{S}^1$ such that $q^\top x_{12}^0 \geq \gamma$. Defining the feedback law

$$\kappa_0(x, q) = -\left(\mathcal{D}\psi(\psi^{-1}(x))\right)^\top \mathcal{D}\phi_q(x)^\top(\phi_q(x) - \phi_q(\psi(0))) \tag{69}$$

for each $(x, q) \in \mathrm{dom}\,\kappa_0 = \{(x, q) \in \mathcal{X} \times \mathcal{X}_c : x \in U_q\}$, we prove that $(\kappa_0, V_0, D_c, F_c)$ is synergistic relative to $\mathcal{A}$ for (65) in the next proposition.

Proposition 9. *Given $\mathcal{A}$ in (66) and a continuous function $\delta : \mathcal{X} \times \mathcal{X}_c \to \mathbb{R}$, the hybrid controller $(\kappa_0, V_0, D_c, F_c)$ is nominally synergistic relative to $\mathcal{A}$ in (66) for (64) with synergy gap exceeding δ.*

Proof. It follows from (68) that (11) is feasible, hence (C1) is satisfied. The function F_c is continuous, thus condition (C2) is verified. Since each chart $\phi_q : U_q \to \mathbb{R}^2$ is a diffeomorphism, $\phi_q(x) = \phi_q(\psi(0))$ if and only if $x = \psi(0)$, hence V in (67) is positive definite relative to (66). Moreover, V_0 is continuous and $V_0^{-1}([0, c])$ is compact for each $c \in \mathbb{R}_{\geq 0}$, thus (C3) is verified. Since D_c is constant and equal to the compact set $\mathcal{X}_c$ for each $(x, x_c) \in \mathcal{X} \times \mathcal{X}_c$, we have that D_c is outer semicontinuous, lower semicontinuous and locally bounded, hence (C4) is verified. Condition (C5) is verified for (69) because $\mathrm{dom}\,\kappa = \{(x, x_c) \in \mathcal{X} \times \mathcal{X}_c : V_0(x, x_c) < +\infty\}$. This proves that $(\kappa_0, V_0, D_c, F_c)$ is a synergistic candidate relative to $\mathcal{A}$ for (64) when $\theta = 0$. From (11c) and (37b), we have that $\mu_{V_0}(x, q) = +\infty >$

$\delta(x, q)$ for each $(x, q) \notin U_q$ for any function δ, hence it follows that each $(x, q) \in \mathcal{X} \times \mathcal{X}_c$ satisfying $(x, q) \in U_q$ does not belong to C. Since U_q is open relative to $\mathcal{X} := \mathsf{S}^1 \times \mathbb{R}$ for each $q \in \mathcal{X}_c$, $\{(x, q) \in \mathcal{X} \times \mathcal{X}_c : x \in \mathcal{X} \backslash U_q\}$ and C are disjoint closed sets, and there exists a neighborhood of C where V_0 is Lipschitz continuous. The generalized derivative of V_0 at (x, q) is the direction $F_{cl,0}(x, q)$ is given by

$$V_0°(x, q; F_{cl,0}(x, q)) = -\Gamma(q) \left| \Pi(q) \nabla_q V_0(x, q) \right|^2$$
$$- \left| \mathcal{D}\psi(\psi^{-1}(x))^\top \mathcal{D}\phi_q(x)^\top (\phi_q(x) - \phi_q(\psi(0))) \right|^2, \quad (70)$$

for each $(x, q) \in C$, where $F_{cl,0}$ is the flow map for the closed-loop system resulting from the interconnection between $(\kappa_0, V_0, D_c, F_c)$ and (64) with $\theta = 0$ (cf. (14)). It follows from (70) that the growth of V_0 along the flows of the closed-loop system is upper bounded by 0, hence (C6) is verified. It follows from the fact that ψ and $\{\phi_q\}_{q \in \mathcal{X}_c}$ are diffeomorphisms that condition (C7) is satisfied with $\Psi = \mathcal{A}$ (cf. [8]), thus completing the proof. $\qquad \square$

An additional property of the hybrid controller $(\kappa_0, V_0, D_c, F_c)$ is that is has synergy gap exceeding δ for any continuous function δ satisfying $\delta(x, x_c) < +\infty$ for each $(x, x_c) \in \mathcal{X} \times \mathcal{X}_c$, because $\Psi \backslash \mathcal{A} = \emptyset$. Therefore, any choice of a positive δ yields global asymptotic stability of $\mathcal{A}$ for the hybrid closed-loop system resulting from the interconnection between $(\kappa_0, V_0, D_c, C_c)$ and (64) (with $\theta = 0$).

B. Adaptive Synergistic Feedback for Robust Global Obstacle Avoidance

More importantly, the synergistic controller $(\kappa_0, V_0, D_c, F_c)$ can be modified according to the procedures of Section VI to yield a controller that is synergistic relative to $\mathscr{A}_1 = \{\mathcal{A}_{1,\theta}\}_{\theta \in \Omega}$ with robustness margin Ω and synergy gap exceeding δ, because Assumptions 2 and (41) hold for the application at hand.

Proposition 10. *Given* $\Omega \subset \mathbb{R}^2$ *in* (31)*, $\mathcal{A}$ in* (66)*, a continuous function* $\delta : \mathcal{X} \times \mathcal{X}_c \to \mathbb{R}$*, the hybrid controllers* $(\kappa_0, V_0, D_c, F_c)$ *and* $(\kappa_1, V_{1,\theta}, D_{c,1}, F_{c,1})$ *derived from* $(\kappa_0, V_0, D_c, F_c)$ *using* (37)*, we have that the hybrid controller* $(\kappa_1, \mathscr{V}_1, D_{c,1}, F_{c,1}, G_{c,1})$ *with* $\mathscr{V}_1 := \{V_{1,\theta}\}_{\theta \in \Omega}$ *and* $G_{c,1}$ *in* (44) *is synergistic relative to* $\mathscr{A}_1 := \{\mathcal{A} \times \{\theta\}\}_{\theta \in \Omega}$ *for* (64) *with robustness margin* Ω *and synergy gap exceeding* δ*.*

Proof. Assumption 2 is verified with $W(x, x_c) = H(x, x_c) = \mathcal{D}\psi(\psi^{-1}(x))$ and $\widehat{W}(x, x_c) = I_2$. To verify (41), note that

$$V_1°(x, x_{c,1}; F_{cl,1}(x, x_{c,1})) \leq -\Gamma(q) \left| \Pi(q) \nabla_q V_0(x, q) \right|^2$$
$$- \left| \mathcal{D}\psi(\psi^{-1}(x))^\top \mathcal{D}\phi_q(x)^\top (\phi_q(x) - \phi_q(\psi(0))) \right|^2, \quad (71)$$

with $x_{c,1} = (q, \hat{\theta}) \in \mathcal{X}_{c,1} := \mathcal{X}_c \times (\Omega + \epsilon\overline{\mathbb{B}})$. Similarly to the proof of Proposition 9, the largest weakly invariant set $\Psi_{1,\theta}$ in $\mathcal{E}_1 = \{(x, x_{c,1}) \in \mathcal{X} \times \mathcal{X}_{c,1} : \nabla V_{1,\theta}(x, x_{c,1})^\top f_{cl,1} = 0$ for some $f_{cl,1} \in F_{cl,1}(x, x_{c,1})\}$ is such that $x = \psi(0)$. Then, it follows from (64) that

$$\mathcal{D}\psi(\psi^{-1}(x))(\theta - \hat{\theta}) = 0. \quad (72)$$

Since ψ is a diffeomorphism, we have that $\mathcal{D}\psi(\psi^{-1}(x))$ is an isomorphism between the tangent spaces to $\mathbb{R}^2 \backslash \mathcal{N}$ at

$\psi^{-1}(x)$ and to $\mathsf{S}^1 \times \mathbb{R}$ at x, thus (72) is verified if and only if $\theta = \hat{\theta}$. We conclude that $\Psi_{1,\theta} = \mathcal{A}_{1,\theta}$ and, consequently, $\pi_{\mathcal{X} \times \mathcal{X}_c}(\Psi_{1,\theta} \backslash \mathcal{A}_{1,\theta}) = \emptyset \subset \Psi \backslash \mathcal{A}$ with Ψ as in the proof of Proposition 9, thus (41) is verified. $\qquad \square$

The fact that the hybrid controller $(\kappa_1, \mathscr{V}_1, D_{c,1}, F_{c,1}, G_{c,1})$ derived from the data of the nominally synergistic controller $(\kappa_0, V_0, D_c, F_c)$ is synergistic relative to $\mathscr{A}_1$ for (64) with robustness margin Ω and synergy gap exceeding δ implies that, for each $\theta \in \Omega$, the set $\mathcal{A}_{1,\theta} := \mathcal{A} \times \{\theta\}$ is globally asymptotically stable for the hybrid closed-loop system (46) resulting from the interconnection between $(\kappa_1, \mathscr{V}_1, D_{c,1}, F_{c,1}, G_{c,1})$ and (64). Similarly, we have that, for each $\theta \in \Omega$, the set $\mathcal{A}_{2,\theta}$ in (51) is globally asymptotically stable for the hybrid closed-loop system (62) resulting from the interconnection between $(\kappa_2, \mathscr{V}_2, D_{c,2}, F_{c,2}, G_{c,2})$ and (64). In the next section, we illustrate these results by presenting some numerical results that depict the behaviour of the closed-loop systems.

C. Simulation Results

In this section, we present simulation results of the closed-loop system resulting from the interconnection between (64) and the hybrid controllers that are presented in Section VI considering that there is an obstacle $\mathcal{N} := z_0 + r\overline{\mathbb{B}}$ with $z_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}^\top$ and $r = 0.5$. Furthermore, we consider that $\theta = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^\top$ and that the controller parameters are $k_u = 1$, $\Gamma_1/2 = \Gamma_2 = I_2$, $\epsilon = 1$, $\theta_0 = 1$, and $\delta(x, q) = 1$ for each $(x, q) \in \mathcal{X} \times \mathcal{X}_c$. For this particular choice of Γ_1, we have that $\hat{G}(\hat{\theta}) = \min\{1, \theta_0/|\hat{\theta}|\}\hat{\theta}$ for each $\hat{\theta} \in \Omega + \epsilon\overline{\mathbb{B}}$, which is outer semicontinuous and locally bounded. In addition, we set $\gamma = 0.5$ and $\Gamma(q) = \max\{0, \gamma - q^\top x_{12}^0\}$ for each $q \in \mathsf{S}^1$.

Figure 1 represents the trajectory of the vehicle starting from rest at $z(0) = \begin{bmatrix} 2 & 0 \end{bmatrix}^\top$ for each of the controllers presented in Section VI. It can be verified both through Figure 1 as well as Figure 2 that the trajectories before and after backstepping are comparable, since the evolution of the distance of the vehicle to the desired setpoint is fairly similar in both cases. The bottom half of Figure 2 depicts the evolution of the estimation error, which has a smaller settling time for the closed-loop system with the controller of Section VI-C than the controller of Section VI-B for this particular simulation.

Due to the fact that the controller weighs the distance between the vehicle and the obstacle evenly across all possible values of $(x_1, x_2) \in \mathsf{S}^1$, and because the desired setpoint is located at the same distance to obstacle as the initial condition, the vehicle describes roughly a circular trajectory around the obstacle. It is possible to improve this behaviour by changing either the synergistic potential function V_0 or the diffeomorphism ψ so as to weigh the distance to the obstacle differently across different values of $(x_1, x_2) \in \mathsf{S}^1$.

To find out more about the simulation and its implementation, you may explore the source code at `https://github.com/pcasau/synergistic`.

VIII. Conclusion

Synergistic hybrid feedback has taken many forms over the years, depending on the particular dynamical system
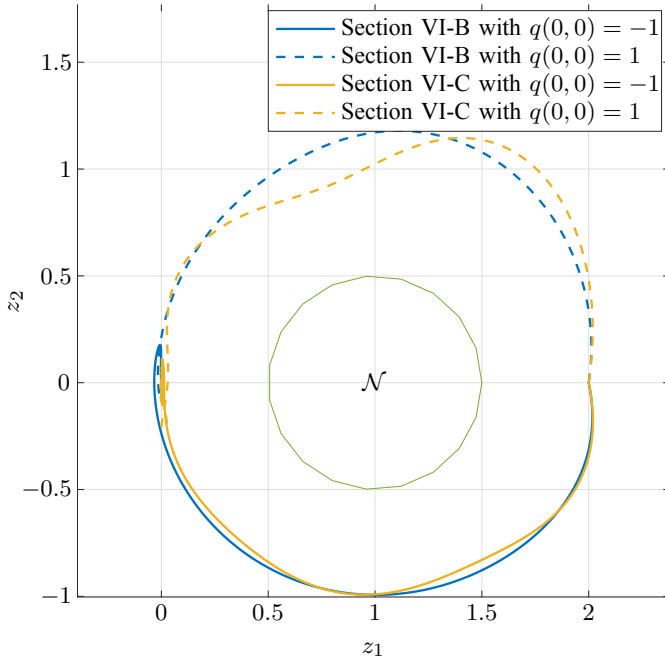
Fig. 1. Trajectories $t \mapsto z(t)$ of (63) for the closed-loop system with parameters given in Section VII-C.
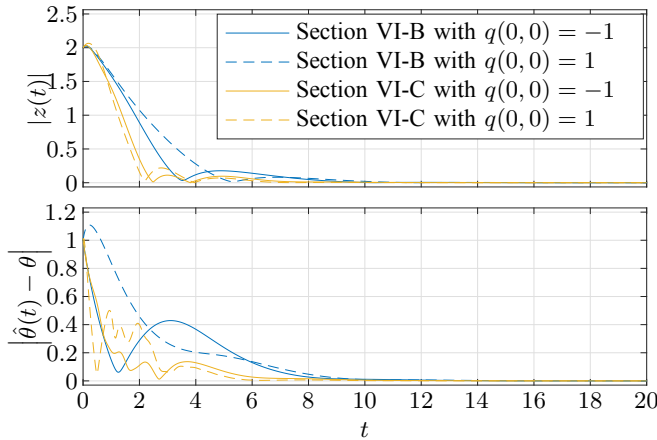

Fig. 2. Evolution in time of the distance to the origin (top) and the parametric estimation error (bottom) for the closed-loop system with parameters given in Section VII-C.

being studied. The unifying framework for synergistic hybrid feedback that we presented in this paper captures the most salient features of existing synergistic hybrid feedbacks in order to help others distinguish between the particular and the general in different instances of synergistic hybrid feedback across the literature. In addition, we provided a controller design that starts from an existing synergistic controller and modified it in order to yield an adaptive controller that is able to compensate for the presence of bounded matched uncertainties in affine control systems. Furthermore, we demonstrated that the proposed controller is amenable to backstepping and can be applied to the problem of global obstacle avoidance.

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