Robust Global Hybrid Passive Complementary Filter on $\mathrm{SO}(2)$

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***Abstract*— We consider the problem of global attitude filtering on the special orthogonal group. Using hybrid systems theory, we propose a global and robust hybrid attitude filter on $\mathrm{SO}(2)$ that is inspired from the passive complementary filter. We show that the proposed filter is input-to-state stable with respect to noise in the measurements. In the absence of measurement noise, this filter renders the identity globally exponentially stable for the attitude error system. Simulations illustrate the results.**

I. INTRODUCTION

Attitude estimation is an enabler for feedback control in numerous applications, especially in aerospace and robotics. As mentioned in [1], attitude estimation is a two-step process that involves i) estimating the orientation of one frame of reference (usually fixed to a rotating rigid body) with respect to another frame of reference (usually a fixed inertial frame) and ii) filtering noise induced from noisy measurements. The attitude of a rigid body is inherently represented by an element of the special orthogonal group $\mathrm{SO}(3)$. As $\mathrm{SO}(3)$ is a non-Euclidean Lie group, the standard nonlinear filtering techniques, like the extended Kalman filter, though employed at times, do not provide desirable results [2].

As a result, there has been significant effort to design filters directly on $\mathrm{SO}(3)$. The seminal work [3] proposes three types of nonlinear complementary filters on $\mathrm{SO}(3)$. The passive complementary filter, proposed in [3], assumes attitude measurements to be on $\mathrm{SO}(3)$, whereas the explicit complementary filter uses vectorial measurements from IMU sensors to obtain an estimate of the attitude. These filters are widely used in aerospace and robotic applications [4]. The passive complementary filter is obtained by minimizing a smooth cost function on $\mathrm{SO}(3)$ [5], and it is shown to render the identity almost globally asymptotically stable for the attitude error system [3]. In fact, one cannot use a smooth feedback law to obtain global asymptotic stability of a compact set on $\mathrm{SO}(3)$, as any smooth potential function on $\mathrm{SO}(3)$ will have at least four critical points [6]. This motivates the use of hybrid systems theory to obtain global stability results on $\mathrm{SO}(3)$.

An explicit complementary filter-inspired hybrid attitude estimator is proposed in [7]. It employs a synergistic potential function-based switching mechanism, as introduced in [8], to render the identity globally asymptotically stable for the attitude error system. The work in [9] also proposes a hybrid estimator with synergistic potential function-based switching that results in global exponential stability results. As the synergistic potential function-based switching is a hysteresis-based approach, robustness of the filter with respect to measurement noise is ensured.

Due to the wide applications of the nonlinear complementary filter, this paper aims to develop a passive complementary filter-based hybrid filter that renders the identity globally exponentially stable for the attitude error system. We consider the case of planar rotations so that the attitude evolves on $\mathrm{SO}(2)$. As $\mathrm{SO}(2)$ is a special case of $\mathrm{SO}(3)$, the preceding discussion about $\mathrm{SO}(3)$ applies for $\mathrm{SO}(2)$. For simplicity, we assume that the attitude measurements lie on $\mathrm{SO}(2)$ and that the angular velocity measurements are bias-free. The work in [10] provides a comprehensive stability analysis for a passive complementary filter with these assumptions. Assuming noisy measurements, we leverage the passive complementary filter in [10] to obtain a global hybrid attitude filter on $\mathrm{SO}(2)$. The proposed hybrid filter employs a logic-based switching and renders the identity input-to-state stable (ISS) for the attitude error system. In the absence of measurement noise, the proposed filter renders the identity globally exponentially stable for the attitude error system. We also comment that the proposed hybrid filter on $\mathrm{SO}(2)$ cannot be directly extended to $\mathrm{SO}(3)$ due to topological obstructions.

The remainder of this paper is arranged as follows. Section II introduces relevant preliminaries. Section III discusses the passive complementary filter with noisy measurements. Section IV proposes a hybrid attitude filter on $\mathrm{SO}(2)$. Section V presents the simulation results, and Section VI concludes the paper. Due to space constraints, the proofs will be presented elsewhere.

II. PRELIMINARIES

A. Notation

The set of nonnegative integers, and real and nonnegative numbers is denoted by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_{\geq 0}$, respectively. The set of all n-dimensional rotation matrices is defined as $\mathrm{SO}(n) := \{R \in \mathbb{R}^{n\times n} : R^\top R = RR^\top = I,\ \det R = 1\}$. The Lie algebra of $\mathrm{SO}(n)$ is defined as $\mathfrak{so}(n) := \{X \in \mathbb{R}^{n\times n} : X + X^\top = 0\}$. The n-sphere is defined as $\mathbb{S}^n := \{v \in \mathbb{R}^{n+1} : v^\top v = 1\}$. We define the *cross* map $\cdot_\times : \mathbb{R} \to \mathfrak{so}(2)$ as $v_\times := \left[\begin{smallmatrix} 0 & -v \\ v & 0 \end{smallmatrix}\right]$ for each $v \in \mathbb{R}$. The inverse of the cross map is defined by $\mathrm{vex} : \mathfrak{so}(2) \to \mathbb{R}$,

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$v_\times \mapsto \mathrm{vex}(v_\times) \coloneqq v$. For any matrix $A \in \mathbb{R}^{n\times n}$, its skew symmetric projection is given by $\mathbb{P}_a(A) \coloneqq (A - A^\top)/2$.

The closure of a set $\mathcal{X}$ is denoted by $\overline{\mathcal{X}}$, and its interior is denoted by int $\mathcal{X}$. For any set $\mathcal{X}$ endowed with a distance metric $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_{\geq 0}$, the distance between a point $x \in \mathcal{X}$ and a nonempty set $\mathcal{A} \subset \mathcal{X}$ is defined as $|x|_\mathcal{A} \coloneqq \inf_{y\in\mathcal{A}} d(x, y)$. The distance of point $R \in \mathrm{SO}(2)$ to a nonempty set $\mathcal{A} \subset \mathrm{SO}(2)$ is defined as $|R|^2_\mathcal{A} = \inf_{X\in\mathcal{A}} \frac{1}{4}\mathrm{tr}\big(I - X^\top R\big)$.

B. Nonautonomous Hybrid Systems on SO(2)

In this paper, we design an estimation algorithm for a system on SO(2) that involves a logic variable. This algorithm can be represented by considering a hybrid system on SO(2) with a logic variable $q \in Q \subset \mathbb{N}$, given as [11]

$$\mathcal{H} : \begin{cases} \dot{x} &= f(x, v, w) \quad (x, v, w) \in C \\ x^+ &= g(x, v, w) \quad (x, v, w) \in D \\ \zeta &= h(x, v, w) \end{cases} \tag{1}$$

where $x \coloneqq (R, q) \in \mathcal{X} \coloneqq \mathrm{SO}(2) \times Q$ is the state of the system, $v \in \mathcal{V}$ is the input, $w \in \mathcal{W}$ is the disturbance acting on the system, and ζ is the output. We assume that $\mathcal{W}$ has a group structure with the identity element $e \in \mathcal{W}$. The set $C \subset \mathcal{X}\times\mathcal{V}\times\mathcal{W}$ is the flow set on which flows are permitted, and $D \subset \mathcal{X} \times \mathcal{V} \times \mathcal{W}$ is the jump set on which jumps are permitted. The function $f : C \to \mathrm{TSO}(2) \times \{0\}$ denotes the flow map and $g : D \to \mathcal{X}$ denotes the jump map. During flows, $\mathcal{H}$ enforces q to be constant, i.e., $\dot{q} = 0$.

A solution (x, v, w) to $\mathcal{H}$ is parametrized by $(t, j) \in \mathbb{R}_{\geq 0}\times \mathbb{N}$, where t denotes the ordinary time that has passed, and j denotes the number of times the solution has jumped. The domain of the solution, denoted by $\mathrm{dom}\,(x, v, w)$, is a hybrid time domain [12, Definition 2.3]. The input-disturbance pair (v, w) can be interpreted as a hybrid input to $\mathcal{H}$ (see [11, Definition 2.27]). The notion of a solution to $\mathcal{H}$ is adapted from [11, Definition 2.29]. A solution is maximal if it cannot be extended, and it is complete if its domain is unbounded.

Now, we define the following stability notions, in the presence of a disturbance w, that are tailored for (1).

Definition 1 (Input-to-state stability notions)**.** The system (1) is said to be *locally input-to-state stable* (LISS) with respect to a nonempty, compact set $\widetilde{\mathcal{A}} \coloneqq \mathcal{A} \times \{\underline{q}\} \subset \mathrm{SO}(2) \times Q$ if there exists a class-$\mathcal{KL}$ function β, a class-$\mathcal{K}$ function γ, and scalars $\kappa \in (0, \max_{R\in\mathrm{SO}(2)} |R|_\mathcal{A})$ and $k_w > 0$ such that each maximal solution $(t, j) \mapsto (x(t, j), v(t, j), w(t, j))$ to (1) with $|x(0, 0)|_{\widetilde{\mathcal{A}}} \leq \kappa$ and $|w_\#|_e \leq k_w$ satisfies

$$|x(t, j)|_{\widetilde{\mathcal{A}}} \leq \beta(|x(0, 0)|_{\widetilde{\mathcal{A}}}, t + j) + \gamma(|w_\#|_e) \tag{2}$$

for all $(t, j) \in \mathrm{dom}\,(x, v, w)$, where $|w_\#|_e \coloneqq \sup_{(t,j)\in\mathrm{dom}\, w} |w(t, j)|_e$, and $\mathrm{dom}\,(x, v, w) = \mathrm{dom}\, x = \mathrm{dom}\, v = \mathrm{dom}\, w$. If (2) holds for each $x(0, 0) \in \mathcal{X}$, then (1) is *input-to-state stable* (ISS) with respect to $\widetilde{\mathcal{A}}$.

Definition 2 (Robust pre-forward invariance of a set)**.** A nonempty set $U \subset \mathrm{SO}(2) \times Q$ is said to be robustly pre-forward invariant for (1) if every solution $(t, j) \mapsto (x(t, j), v(t, j), w(t, j))$ to (1) with $x(0, 0) \in U$ satisfies $x(t, j) \in U$ for all $(t, j) \in \mathrm{dom}\,(x, v, w)$.

Now, we provide conditions for $\mathcal{H}$ to be well-posed. In particular, $\mathcal{H}$ is well-posed if it satisfies the following assumption [12].

Assumption 1 (Hybrid basic conditions)**.** For the hybrid system $\mathcal{H}$ in (1),

(A1) C and D are closed subsets of $\mathcal{X} \times \mathcal{V} \times \mathcal{W}$,
(A2) $f : C \to \mathrm{TSO}(2) \times \{0\}$ is continuous,
(A3) $g : D \to \mathrm{SO}(2) \times Q$ is continuous.

III. Passive Complementary Filter

The rotational kinematics of a rigid body on SO(2) are given by

$$\dot{R} = R\Omega_\times \tag{3}$$

where $t \mapsto \Omega(t) \in \mathbb{R}$ represents the angular velocity in the body fixed frame. One of the standard tools for estimating the rigid body attitude that satisfies the kinematics in (3) is the nonlinear complementary filter [3]. In this filter, the measurements of the angular velocity Ω and the rotation matrix R are given as Ω^y and R^y, respectively. Then, the goal of the filter is to generate a filtered estimate $\hat{R}$ of the true state R using the measurements. In other words, the filter error defined as $\widetilde{R} \coloneqq \hat{R}^\top R$ should asymptotically approach I. We employ the following measurement model:

$$R^y = R\mathcal{N}_R, \qquad \Omega^y = \Omega + \eta_\omega, \tag{4}$$

where $t \mapsto \mathcal{N}_R \in \mathrm{SO}(2)$ and $t \mapsto \eta_\omega(t) \in \mathbb{R}$ are signals of time representing the bounded measurement noise in attitude and gyroscope measurements, respectively, with the bounds given by

$$\theta_R \coloneqq \sup_{t\in\mathrm{dom}\,\mathcal{N}_R} |\mathcal{N}_R(t)|_I, \quad \overline{\eta}_\omega \coloneqq \sup_{t\in\mathrm{dom}\,\eta_\omega} |\eta_\omega(t)|. \tag{5}$$

We say that the noise signal $t \mapsto (\mathcal{N}_R(t), \eta_\omega(t))$ is *admissible* if $(\mathcal{N}_R(t), \eta_\omega(t)) \in \mathcal{W}$ for all t in its domain, where

$$\mathcal{W} \coloneqq \{X \in \mathrm{SO}(2) : |X|_I \leq \theta_R\} \times [-\overline{\eta}_\omega, \overline{\eta}_\omega]. \tag{6}$$

We use this measurement model, along with the Passive Complementary Filter proposed in [3, Theorem 4.2], and obtain the following result.

Theorem 1. *Given the rotational kinematics in* (3) *with a bounded and locally absolutely continuous input signal* $t \mapsto \Omega(t)$*, a constant* $k_p > 0$*, and measurements* R^y *and* Ω^y *satisfying* (4), (5), *and* $\theta_R \leq \sqrt{(5 + 2\sqrt{5})/10}$*, the filter*

$$\dot{\hat{R}} = \hat{R}\Big(\Omega^y + k_p\omega(\widetilde{R}^y)\Big)_\times, \quad \hat{R}(0) \in \mathrm{SO}(2), \tag{7}$$

where $\omega(\widetilde{R}^y) \coloneqq \mathrm{vex}\Big(\mathbb{P}_a(\widetilde{R}^y)\Big)$ *and* $\widetilde{R}^y \coloneqq \hat{R}^\top R^y$ *is such that*

(a) the kinematics of the filter error $\widetilde{R}$ *are given as*

$$\begin{aligned} \dot{\widetilde{R}} &= f(\widetilde{R}, \Omega, w_c) \\ &\coloneqq [\widetilde{R}, \Omega_\times] - \eta_{\omega\times}\widetilde{R} - k_p\omega(\widetilde{R}\mathcal{N}_R)_\times\widetilde{R}, \end{aligned} \tag{8}$$

where $w_c := (\mathcal{N}_R, \eta_\omega) \in \mathcal{W}$*, and* $[\widetilde{R}, \Omega_\times] := \widetilde{R}\Omega_\times - \Omega_\times \widetilde{R}$*,*

(b) the filter error (8) *is LISS with respect to* $\{I\} \subset \mathrm{SO}(2)$*; in particular, for each* $\kappa \in (0,1)$ *and each* $\varepsilon \in (0,2)$*, if* $|\hat{R}(0)^\top R(0)|_I \leq \kappa$ *then, for each* $t \mapsto (\mathcal{N}_R(t), \eta_\omega(t)) \in \mathcal{W}$ *with bounds as in* (5) *such that*

$$\mathfrak{h}(\overline{\eta}_\omega) + \mathfrak{g}(\theta_R) \in \left[0, \varepsilon k_p \kappa^2(1-\kappa^2)\right], \tag{9}$$

where $\mathfrak{h}(s) := s/4$ *and* $\mathfrak{g}(s) := \frac{k_p}{2}\left(2s^2 + \sqrt{s^2(1-s^2)}\right)$ *for all* $s \geq 0$*, every solution* $t \mapsto (\widetilde{R}(t), v_c(t), w_c(t))$ *to* (8) *satisfies*

$$|\widetilde{R}(t)|_I^2 \leq e^{-2k_p(1-\kappa^2)t}|\widetilde{R}(0)|_I^2 + \frac{\mathfrak{h}(\overline{\eta}_\omega) + \mathfrak{g}(\theta_R)}{2k_p(1-\kappa^2)} \tag{10}$$

for all $t \in \mathrm{dom}\,(\widetilde{R}, v_c, w_c)$.

Note that in the absence of measurement noise, we have $\theta_R = \overline{\eta}_\omega = 0$. In this setting, it follows from Theorem 1 that the set $\{I\} \cup \mathbb{U}_{\mathrm{SO}(2)}$ is pre-forward invariant for (8), where

$$\mathbb{U}_{\mathrm{SO}(2)} := \{X \in \mathrm{SO}(2) : |X|_I = 1\}.$$

The set $\mathbb{U}_{\mathrm{SO}(2)}$ denotes the set of $180°$ rotations from the identity orientation, and has Lebesgue measure zero in $\mathrm{SO}(2)$. Therefore, if the measurements are noise-free, it follows from Theorem 1 that I is almost globally exponentially stable for (8), and $\mathbb{U}_{\mathrm{SO}(2)}$ is unstable for (8). As a result, in the forthcoming sections, we refer to $\mathbb{U}_{\mathrm{SO}(2)}$ as the unstable and pre-forward invariant set for (8). Furthermore, for each solution to (8) starting from $\mathbb{U}_{\mathrm{SO}(2)}$, appropriately chosen arbitrarily small measurement noise can prevent the filter error from converging to the identity.

IV. Hybrid Passive Complementary Filter on $\mathrm{SO}(2)$

In this section, using the structure in (1), we design the hybrid filter by employing a logic variable $q \in Q$, where Q will be defined later in this section. Our design approach for the hybrid filter is as follows: when $\widetilde{R}^y$ is *close enough* to I, the filter is defined by (7). We call this filter the *local filter* and denote it by $\hat{\mathcal{H}}_0$. When $\widetilde{R}^y$ is *far enough* from I, a passive complementary filter, called the *global filter* and denoted $\hat{\mathcal{H}}_1$, is designed such that $\widetilde{R}$ converges to $R^\top R^{*\top} R \in \mathrm{SO}(2)$ (which is *close* to identity) instead of actually converging to I, where $R^* \in \mathrm{SO}(2)$ is defined such that $|R^*|_I$ is close to zero (i.e., $0 < |R^*|_I < c_1$ for some $c_1 \in (0,1)$). Since $|R^\top R^{*\top} R|_I = |R^*|_I$, $|\widetilde{R}|_I$ converges to $|R^*|_I$. This property ensures that the estimate generated by the filter does not remain in $\mathbb{U}_{\mathrm{SO}(2)}$ owing to item a in Theorem 1. As a consequence, the global filter brings $\widetilde{R}$ into a *small enough* neighborhood of I so that the local filter can be used.

Now, we concretely define the notions of *close enough* and *far enough*. Fix constants $c_0, c_1 \in \mathbb{R}$, and $R^* \in \mathrm{SO}(2)$ such that

$$0 < |R^*|_I < c_1 < c_0 < 1, \quad \text{and} \quad c_1^2 < 1 - |R^*|_I^2. \tag{11}$$

These constants solidify the notion of $\widetilde{R}^y$ being *close enough* and *far enough* from $I \in \mathrm{SO}(2)$ as follows:

- the measured filter error $\widetilde{R}^y$ is said to be close enough to I if $\widetilde{R}^y \in C_0$, with C_0 defined as
$$C_0 := \{X \in \mathrm{SO}(2) : |X|_I \leq c_0\}, \tag{12}$$
- the measured filter error $\widetilde{R}^y$ is said to be far enough from I if $\widetilde{R}^y \in C_1$, where C_1 is defined as
$$C_1 := \{X \in \mathrm{SO}(2) : |X|_I \geq c_1\}. \tag{13}$$

A. The Global Filter on $\mathrm{SO}(2)$

To formalize the *global filter* $\hat{\mathcal{H}}_1$, define

$$R_1 := R^* R, \qquad \widetilde{R}_1 := \hat{R}^\top R_1 = \hat{R}^\top R^* R. \tag{14}$$

Therefore, R_1 represents a rotation of R by R^*, and $\widetilde{R}_1$ represents the error rotation between $\hat{R}$ and R_1. Thus, if $\hat{R}$ converges to R_1, the corresponding filter error $\widetilde{R}_1$ converges to I and $\widetilde{R}$ converges to $R^\top R^{*\top} R$. Next, it is easy to see from (3) that R_1 satisfies the following kinematics:

$$\dot{R}_1 = R_1 \Omega_\times. \tag{15}$$

Therefore, the Passive Complementary Filter (7) can now be used to estimate R_1. In particular, with $\overline{k}_p > 0$, we define the following global filter, for the case when $\widetilde{R}^y$ is *far enough* from I:

$$\dot{\hat{R}} = \hat{R}\Big(\Omega^y + \overline{k}_p \omega(\widetilde{R}_1^y)\Big)_\times, \quad \hat{R}(0) \in \mathrm{SO}(2), \tag{16}$$

where $\widetilde{R}_1^y = \hat{R}^\top R_1^y$. Using Theorem 1, it follows that $\{I\} \subset \mathrm{SO}(2)$ is LISS for

$$\dot{\widetilde{R}}_1 = f(\widetilde{R}_1, \Omega, w_c), \tag{17}$$

where w_c and the function f are as defined in (8) and below it. Hence, for *small enough* noise and for all initial conditions such that $\widetilde{R}_1(0) \notin \mathbb{U}_{\mathrm{SO}(2)}$, it follows from item b of Theorem 1 that the filter in (16) ensures convergence of $\hat{R}$ to a small enough neighborhood of R_1. Note that the set $\mathbb{U}_{\mathrm{SO}(2)} = \{\widetilde{R}_1 \in \mathrm{SO}(2) : |\widetilde{R}_1|_I = 1\}$ for the global filter can be written in terms of $\widetilde{R}$ using the following equivalence.

Proposition 1. *Given* $R^* \in \mathrm{SO}(2)$ *such that* $|R^*|_I \in (0,1)$*,* $|\widetilde{R}_1|_I = 1 \iff |\widetilde{R}|_I^2 = 1 - |R^*|_I^2$*.*

B. Operation and Logic of the Hybrid Filter on $\mathrm{SO}(2)$

The proposed hybrid passive complementary filter on $\mathrm{SO}(2)$ is a hysteresis-based hybrid filter that guarantees global estimation and, due to implementing hysteresis-based switching, ensures that there is no chattering (see [13]). Next, we present its construction.

We employ a logic variable $q \in Q := \{0,1\}$ to implement hysteresis-based switching. The idea is to use the local filter $\hat{\mathcal{H}}_0$ when $q = 0$ and the measured filter error $\widetilde{R}^y$ is close enough to I, and use the global filter $\hat{\mathcal{H}}_1$ when $q = 1$ and the measured filter error is far enough from I.

The logic of the proposed hybrid filter is as follows:

Case 1 If $q = 0$ and $\widetilde{R}^y$ is close enough to I (i.e., $\widetilde{R}^y \in C_0$), the proposed hybrid filter employs the local filter $\hat{\mathcal{H}}_0$, defined in (7). Due to Theorem 1, exponential convergence of $\widetilde{R}$ to a small neighborhood of I follows, where the size of the neighborhood is determined by the size of the noise, through the last term in (10).

Case 2 If $q = 1$ and $\widetilde{R}^y$ is far enough from I (i.e., $\widetilde{R}^y \in C_1$), the proposed hybrid filter employs the global filter $\hat{\mathcal{H}}_1$, defined in (16). The evolution of this filter will cause $\widetilde{R}^y$ to get close enough to I in finite time, resulting in **Case 4**.

Case 3 If $q = 0$ and $\widetilde{R}^y$ is far enough from I (i.e., $\widetilde{R}^y \in D_0$), the value of q is reset to one, which causes the proposed filter to use the global filter according to **Case 2**.

Case 4 If $q = 1$ and $\widetilde{R}^y$ is close enough to I (i.e., $\widetilde{R}^y \in D_1$), the value of q is reset to zero, which results in **Case 1** and ensures exponential convergence of $\widetilde{R}$ to a small neighborhood of I.

Note that due to the characterization of *close enough* and *far enough*, the regions of operation for **Case 1** and **Case 2** are the sets C_0 and C_1, respectively. To ensure convergence of $\widetilde{R}$ to a small neighborhood of I, we design the region of operation of the filter to be disjoint from the unstable and pre-forward invariant set for its error dynamics. For **Case 2**, Proposition 1, along with our choice of c_1 and R^*, ensures that the region of operation of the global filter and its unstable and pre-forward invariant $\mathbb{U}_{\mathrm{SO}(2)}$ set are disjoint.

The regions of operation for **Case 3** and **Case 4** are defined to contain all the points that are not in the regions of operation for **Case 1** and **Case 2**, respectively. In particular, we define the following sets:

$$D_0 := \overline{\mathrm{SO}(2) \setminus C_0} \quad \text{and} \quad D_1 := \overline{\mathrm{SO}(2) \setminus C_1}. \tag{18}$$

The set D_0 is the region of operation for **Case 3** and the set D_1 is the region of operation for **Case 4**. Note that, for each $q \in Q$, D_q contains in its interior the unstable and pre-forward invariant set of the filter $\hat{\mathcal{H}}_q$.

When $q = 0$, the sets C_0 and D_0 capture the notion of *close enough* and *far enough* from I, respectively. Similarly, for $q = 1$, the sets D_1 and C_1 capture the notion of *close enough* and *far enough* from I, respectively. Figure 1 illustrates the construction of the flow sets and the jump sets.

C. The Hybrid Filter

In this section, we formalize the hybrid filter outlined in the previous section. Following **Case 1**-**Case 4** and the logic therein, we model the hybrid filter as a hybrid system of the form (1). The filter, denoted by $\hat{\mathcal{H}}_{\mathrm{PCF}}$, has state $x := (\hat{R}, q) \in \mathcal{X} := \mathrm{SO}(2) \times Q$, output $\zeta := \hat{R}$, and data $(\hat{C}, \hat{F}, \hat{D}, \hat{G}, \zeta)$ defined below. Following (7) and (16), the input of $\hat{\mathcal{H}}_{\mathrm{PCF}}$ is given by the measurements $u := (R^y, \Omega^y) \in \mathcal{U} := \mathrm{SO}(2)\times\mathbb{R}$. This input, which naturally is a continuous-time signal, is defined on the hybrid time domain of a solution x, namely, $\mathrm{dom}\, u = \mathrm{dom}\, x$, so that u flows when x flows, and jumps to the same value when x jumps. More formally, given a signal

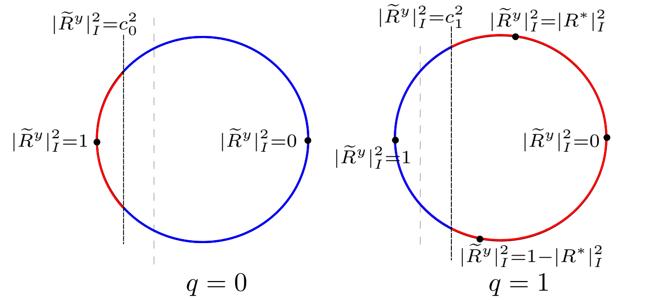


Fig. 1: Since SO(2) is isomorphic to $\mathbb{S}^1$, the unit circle on the left represents the values of $\widetilde{R}^y \in \mathrm{SO}(2)$ for $q = 0$, whereas the one on the right represents $\widetilde{R}^y$ for $q = 1$. Points on the circle that lie on the same vertical dashed line have the same value of $|\widetilde{R}^y|_I$. The blue and the red regions represent the flow set C_q and the jump set D_q, respectively, for each $q \in Q$.

$t \mapsto (R^y(t), \Omega^y(t))$ and a solution x, we define the hybrid input $(t, j) \mapsto u(t, j) = (R^y(t, j), \Omega^y(t, j))$ as

$$u(t, j) := (R^y(t), \Omega^y(t)) \quad \forall (t, j) \in \mathrm{dom}\, x. \tag{19}$$

Next, to define the hybrid filter $\hat{\mathcal{H}}_{\mathrm{PCF}}$, we follow the logic from **Case 1**-**Case 4** and define the flow set $\hat{C}$ as the union of the region of operation for **Case 1** and **Case 2**. The flow map $\hat{F}$ is defined such that q remains constant during flows, and $\hat{R}$ flows according to the filter $\hat{\mathcal{H}}_q$. The jump set $\hat{D}$ is defined to be the set of all points resulting in **Case 3** and **Case 4**. Furthermore, following the logic in **Case 3** and **Case 4**, the jump map is defined such that q toggles from zero to one and vice versa, respectively.

As a result, the data $(\hat{C}, \hat{F}, \hat{D}, \hat{G}, \zeta)$ of the hybrid filter $\hat{\mathcal{H}}_{\mathrm{PCF}}$ is given as

$$\begin{aligned}
&\hat{C} := \left\{(x, u) \in \mathcal{X} \times \mathcal{U} : (\hat{R}^\top R^y, q) \in \textstyle\bigcup_{q\in Q}(C_q \times \{q\})\right\} \\
&\hat{F}(x, u) := \left(\hat{R}\hat{\kappa}(x, u),\ 0\right) \qquad \forall (x, u) \in \hat{C} \\
&\hat{D} := \left\{(x, u) \in \mathcal{X} \times \mathcal{U} : (\hat{R}^\top R^y, q) \in \textstyle\bigcup_{q\in Q}(D_q \times \{q\})\right\} \\
&\hat{G}(x, u) := (\hat{R},\ 1 - q) \qquad\qquad \forall (x, u) \in \hat{D} \\
&\zeta := \hat{R}
\end{aligned}$$

where $\hat{\kappa}(x, u) := q\hat{\kappa}_1(x, u)+(1-q)\hat{\kappa}_0(x, u)$ with $\hat{\kappa}_0(x, u) := (\Omega^y + k_p\omega(\widetilde{R}^y))_\times$ and $\hat{\kappa}_1(x, u) := (\Omega^y + \overline{k}_p\omega(\widetilde{R}^y_1))_\times$, and recall that $\widetilde{R}^y_1 = \hat{R}^\top R^* R^y$. Note that the map $\hat{G}$ is well-defined because the sets D_0 and D_1 are disjoint.

D. Hybrid Filter Error Dynamics

Recall that the state of the hybrid filter $\hat{\mathcal{H}}_{\mathrm{PCF}}$ is $x = (\hat{R}, q)$. The desired reference trajectory for $\hat{\mathcal{H}}_{\mathrm{PCF}}$ is $(R, 0)$. This results in the filter error being defined as $\widetilde{R} := \hat{R}^\top R$. To obtain the hybrid filter error dynamics, we compute the error dynamics at points in the flow set $\hat{C}$ of the filter. Since the flow map results in $\dot{\hat{R}} = \hat{F}(x, u) = \hat{R}\hat{\kappa}(x, u)$ for all $(x, u) \in \hat{C}$, the dynamics for $\widetilde{R} = \hat{R}^\top R$ are found to be

$$\dot{\widetilde{R}} = \dot{\hat{R}}^\top R + \hat{R}^\top \dot{R} = -\hat{\kappa}(x, u)\widetilde{R} + \widetilde{R}\Omega_\times. \tag{20}$$

Similarly, we compute $\widetilde{R}^+ = \hat{R}^{+\top} R = \hat{R}^\top R = \widetilde{R}$. Since the hybrid filter error $\widetilde{R}$ flows (resp., jumps) when $\hat{\mathcal{H}}_{\text{PCF}}$ flows (resp., jumps), the corresponding flow set (resp., jump set) is comprised of points $\widetilde{R}$ such that, given input R^y, the state-input pair (x, u) lies in $\hat{C}$ (resp., $\hat{D}$). We now characterize these flow and jump sets in terms of $\widetilde{R}$ as follows.

Lemma 1. *Consider constants c_0 and c_1 satisfying $0 < c_1 < c_0 < 1$. Given $\widetilde{R} \in \mathrm{SO}(2)$ and a signal $t \mapsto \mathcal{N}_R(t)$ such that θ_R is defined in (5) and satisfies $\theta_R \leq (2+\sqrt{2})/4$, the following holds:*

1) $\widetilde{R}^y \in C_0 \implies$
$$\widetilde{R} \in C_{0,w} := \{X \in \mathrm{SO}(2) : |X|_I^2 \leq c_0^2 + \mathfrak{F}(\theta_R)\},$$
2) $\widetilde{R}^y \in D_0 \implies$
$$\widetilde{R} \in D_{0,w} := \{X \in \mathrm{SO}(2) : |X|_I^2 \geq c_0^2 - \mathfrak{F}(\theta_R)\},$$
3) $\widetilde{R}^y \in C_1 \implies$
$$\widetilde{R} \in C_{1,w} := \{X \in \mathrm{SO}(2) : |X|_I^2 \geq c_1^2 - \mathfrak{F}(\theta_R)\},$$
4) $\widetilde{R}^y \in D_1 \implies$
$$\widetilde{R} \in D_{1,w} := \{X \in \mathrm{SO}(2) : |X|_I^2 \leq c_1^2 + \mathfrak{F}(\theta_R)\},$$

where $\mathfrak{F}(s) := s^2 + s\sqrt{1-s^2}$ for all $s \in [0,1]$.

For each $q \in Q$, the sets $C_{q,w}$ and $D_{q,w}$ represent the flow set and the jump set for the hybrid filter error dynamics, respectively. The subscript w denotes the dependence of these sets on the bound on attitude measurement noise.

Note that the admissible measurement noise bound θ_R needs to be small enough such that $D_{0,w} \cap D_{1,w} = \varnothing$, which prevents persistent jumping of a solution to the hybrid filter error dynamics. Furthermore, with (11) satisfied, the admissible measurement noise should allow R^* to lie in the interior of $D_{1,w}$ so that a jump can be triggered in finite time to switch from the global filter to the local filter. For each $q \in Q$, the noise must also be small enough to prevent intersection of the set $C_{q,w}$ with the unstable and pre-forward invariant sets of the corresponding filter. These conditions on the noise are captured by the following bound on $\mathfrak{F}(\theta_R)$:

$$\mathfrak{F}(\theta_R) \leq \rho_{\max} := \min\left\{\frac{c_0^2 - c_1^2}{2}, c_1^2 - |R^*|_I^2, 1 - c_0^2, \right. \\ \left. c_1^2 + |R^*|_I^2 - 1, c_1^2 - \frac{4|R^*|_I^2}{1+3|R^*|_I^2}\right\}. \tag{21}$$

The last term in $\rho_{\max}$ ensures that the filter error decays exponentially. With $\mathcal{W}$ in (6), define, for each $\rho \in [0, \rho_{\max})$, the set of admissible values for the measurement noise signal $t \mapsto (\mathcal{N}_R(t), \eta_\omega(t))$ as

$$\mathcal{W}_\rho := \mathcal{W} \cap \big(\{X \in \mathrm{SO}(2) : \mathfrak{F}(|X|_I) \leq \rho\} \times [-\overline{\eta}_\omega, \overline{\eta}_\omega]\big).$$

Note that the set $\mathcal{W}_\rho \subset \mathrm{SO}(2) \times \mathbb{R}$ is compact, which ensures that the measurement noise is bounded.

The filter error dynamics is represented by the hybrid system $\widetilde{\mathcal{H}}_{\text{PCF}}^w = (\widetilde{C}_w, \widetilde{F}_w, \widetilde{D}_w, \widetilde{G}_w)$ with state $\widetilde{x} := (\widetilde{R}, q) \in \mathcal{X}$, hybrid input signal $(t,j) \mapsto \widetilde{v}(t,j) := (R(t), \Omega(t)) \in \mathcal{V} := \mathrm{SO}(2) \times \mathbb{R}$ that is obtained from the continuous-time input signal $t \mapsto (R(t), \Omega(t))$ using (19), and hybrid disturbance signal $(t,j) \mapsto \widetilde{w}(t,j) := (\mathcal{N}_R(t), \eta_\omega(t)) \in \mathcal{W}_\rho$, which is obtained from the continuous-time disturbance signal $t \mapsto (\mathcal{N}_R(t), \eta_\omega(t)) \in \mathcal{W}_\rho$. Following (20) and Lemma 1, the data of $\widetilde{\mathcal{H}}_{\text{PCF}}^w$ is given as follows:

$$\begin{aligned}
\widetilde{C}_w &:= \Big(\textstyle\bigcup_{q\in Q}(C_{q,w} \times \{q\})\Big) \times \mathcal{V} \times \mathcal{W}_\rho \\
\widetilde{F}_w(\widetilde{x}, \widetilde{v}, \widetilde{w}) &:= \Big(-\kappa_w(\widetilde{x}, \widetilde{v}, \widetilde{w})\widetilde{R} + \widetilde{R}\Omega_\times, 0\Big) \\
&\qquad\qquad \forall (\widetilde{x}, \widetilde{v}, \widetilde{w}) \in \widetilde{C}_w \\
\widetilde{D}_w &:= \Big(\textstyle\bigcup_{q\in Q}(D_{q,w} \times \{q\})\Big) \times \mathcal{V} \times \mathcal{W}_\rho \\
\widetilde{G}_w(\widetilde{x}, \widetilde{v}, \widetilde{w}) &:= (\widetilde{R}, 1-q) \qquad \forall (\widetilde{x}, \widetilde{v}, \widetilde{w}) \in \widetilde{D}_w
\end{aligned}$$

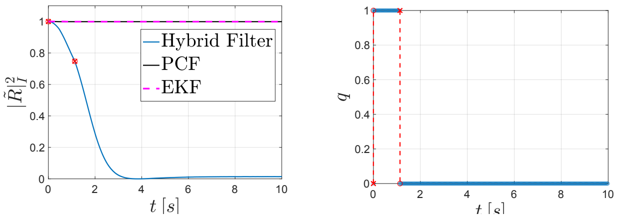
where $\kappa_w(\widetilde{x}, \widetilde{v}, \widetilde{w}) := \hat{\kappa}(x,u)$, with x (resp., u) denoting the state (resp., input) of $\hat{\mathcal{H}}_{\text{PCF}}$. These are represented using $\widetilde{x}, \widetilde{v}$, and $\widetilde{w}$, since $x = (R\widetilde{R}^\top, q)$ and $u = (R\mathcal{N}_R, \Omega + \eta_\omega)$.

Theorem 2. *Suppose that the input signal $t \mapsto \Omega(t) \in \mathbb{R}$ is bounded and locally absolutely continuous in $\operatorname{dom}\Omega$, and the noise signal $t \mapsto w_c(t) := (\mathcal{N}_R(t), \eta_\omega(t))$ is Lebesgue measurable and locally essentially bounded in $\operatorname{dom} w_c$. Then, for each constants c_0, c_1, and $R^* \in \mathrm{SO}(2)$ satisfying (11) and $4 < c_1^2(3 + \frac{1}{|R^*|_I^2})$, each positive constants k_p and $\overline{k}_p$, and each $\rho \in [0, \rho_{\max})$, the following holds:*

1) *$\widetilde{\mathcal{H}}_{\text{PCF}}^w$ and $\hat{\mathcal{H}}_{\text{PCF}}$ satisfy the hybrid basic conditions,*
2) *If $t \mapsto w_c(t)$ satisfies (5), $w_c(t) \in \mathcal{W}_\rho$ for all $t \in \operatorname{dom} w_c$, and $\operatorname{dom}\Omega = \operatorname{dom} w_c = \mathbb{R}_{\geq 0}$, then every maximal solution to $\widetilde{\mathcal{H}}_{\text{PCF}}^w$ from $\widetilde{C}_w \cup \widetilde{D}_w$ is complete and exhibits no more than two jumps,*
3) *$\widetilde{\mathcal{H}}_{\text{PCF}}^w$ is ISS with respect to the set $\widetilde{\mathcal{A}} := \{I\} \times \{0\} \subset \mathrm{SO}(2) \times Q$; in particular, for each solution $(t,j) \mapsto (\widetilde{x}(t,j), \widetilde{v}(t,j), \widetilde{w}(t,j))$ to $\widetilde{\mathcal{H}}_{\text{PCF}}^w$ and each $\varepsilon \in (0,2)$, there exist constants $K, \Lambda, \delta, \gamma_{\min} > 0$ such that, if $t \mapsto w_c(t)$ satisfies (5) and $(t,j) \mapsto \widetilde{w}(t,j) \in \mathcal{W}_\rho$ satisfies*
$$\mathfrak{h}(\overline{\eta}_\omega) + \mathfrak{g}(\theta_R) \in [0, \varepsilon\gamma_{\min}],$$
then the following holds:
$$|\widetilde{x}(t,j)|_{\widetilde{\mathcal{A}}}^2 \leq Ke^{-\Lambda t}|\widetilde{x}(0,0)|_{\widetilde{\mathcal{A}}}^2 + \delta(\mathfrak{h}(\overline{\eta}_\omega) + \mathfrak{g}(\theta_R)) \tag{22}$$
for all $(t,j) \in \operatorname{dom}(\widetilde{x}, \widetilde{v}, \widetilde{w})$.

Remark 1. When the constant K in item 3 of Theorem 2 is greater than one, since $|\widetilde{R}|_I^2 \leq 1$ already holds for each $\widetilde{R} \in \mathrm{SO}(2)$, the exponential bound in Theorem 2 holds trivially over an initial window of hybrid time. The bound (22) assures that there exist $T > 0$ such that, for each solution $(\widetilde{x}, \widetilde{v}, \widetilde{w})$ to $\widetilde{\mathcal{H}}_{\text{PCF}}$, the first term therein is smaller than one for all $(t,j) \in \operatorname{dom}(\widetilde{x}, \widetilde{v}, \widetilde{w})$ satisfying $t \geq T$.

Remark 2. The proposed hybrid filter is a *global* filter because Proposition 1 ensures that the unstable and pre-forward invariant sets of the complementary filter (7) and of the global filter (16) do not intersect. Consequently, appropriate switching between these filters globally steers the filter error to a neighborhood of I. The direct extension of this approach to $\mathrm{SO}(3)$, however, turns out to be unsuccessful as the unstable and pre-forward invariant sets of the passive complementary filter on $\mathrm{SO}(3)$ and the global filter always intersect, irrespective of the choice of $R^* \in \mathrm{SO}(3)$. Therefore, there exists an initial attitude estimate from where convergence of $\widetilde{R}$ to I does not occur. Due to this issue that arises from the topology of the set $\mathbb{U}_0$, alternative global filters must be used



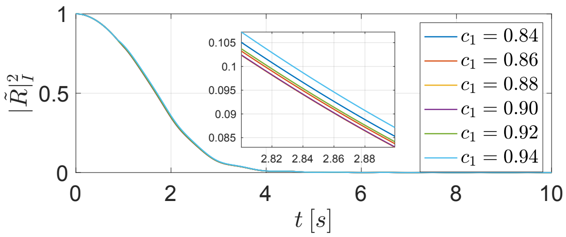


Fig. 3: Rate of convergence comparison.

instead of simply rotating the passive complementary filter, as done in this paper.

V. Simulation Results

Consider a continuous-time angular velocity signal as $\Omega(t) = \sin(t)$, with the initial condition for the rotational kinematics as $R(0,0) = -I \in \mathrm{SO}(2)$. The filter parameters are chosen as $c_0 = 0.966$, $c_1 = 0.866$, $k_p = 1$, $\overline{k}_p = 1$, and $R^* \in \mathrm{SO}(2)$ such that $|R^*|_I^2 = 0.5$. The filter is initialized as $\hat{R}(0,0) = I$, $q(0,0) = 0$. Note that the filter initialization results in $\widetilde{R}(0,0) = -I \in \mathbb{U}_{\mathrm{SO}(2)}$. With the above initialization, we simulate[1] the hybrid filter, the passive complementary filter (PCF) in (7), and the Extended Kalman Filter (EKF). Let $\tilde{\theta}$ denote the angle of rotation of the filter error $\widetilde{R}$. We set the noise $\eta_\omega = \sin(\tilde{\theta})$ if $|\widetilde{R}|_I^2 > 0.999$ and $\eta_\omega = -\mathrm{sign}(\sin(\tilde{\theta}))$ otherwise. The rotation angle of $\mathcal{N}_R$ is sampled from a uniform distribution over $[-\frac{\pi}{18}, \frac{\pi}{18}]$.

The initialization of the filter results in $(\widetilde{R}(0,0), q(0,0)) \in D_0 \times \{0\}$. The choice of noise prevents PCF and EKF from making $\widetilde{R}$ converge to I, as seen in Figure 2a, where the plots for PCF and EKF both stay at $|\widetilde{R}|_I^2 = 1$. For the hybrid filter, however, we expect that $\widetilde{R}$ converges to a small neighborhood of I, and the solution jumps twice, which is as observed from Figures 2a and 2b, where the jumps occur at $t = 0$ sec and $t \approx 1.6$ sec.

Furthermore, the simulations in Figure 3 highlight the effect of the choice of the parameter c_1 for fixed values of c_0 and R^*. The conditions in Theorem 2 define an *admissible region* for the values of c_1. As c_1 approaches the boundary of this admissible region, the rate of convergence of $|\widetilde{R}|_I^2$ decreases. This follows from Theorem 1 as the rate of convergence of $|\widetilde{R}|_I^2$ is smaller when $\widetilde{R}$ is closer to either I or $\mathbb{U}_{\mathrm{SO}(2)}$. A similar trend is observed when parameters c_0 and R^* are varied, keeping the other parameters fixed.

[1]The simulation files can be found at https://github.com/HybridSystemsLab/HybridComplementaryFilter

The runtime for one iteration of $\hat{\mathcal{H}}_{\mathrm{PCF}}$ and the PCF, for randomly chosen values of states and parameters but a fixed time step $h = 0.0001$ sec, averaged over 10^4 iterations is 3.044×10^{-6} sec and 2.295×10^{-6} sec, respectively. Though still low, the higher computation time for $\hat{\mathcal{H}}_{PCF}$ is a consequence of performing more computations, which include assessing if the state-input pair belongs to the flow and jump set and computation of the jump map.

VI. Conclusion

Inspired by the passive complementary filter, we propose a global hybrid filter on $\mathrm{SO}(2)$. Assuming that the measurements of the attitude and the angular velocity are corrupted by a bounded noise signal, the proposed filter renders the identity input-to-state stable for the hybrid filter error system. In the absence of measurement noise, the filter renders the identity globally exponentially stable for the hybrid filter error system. Simulations show that the passive complementary filter and the EKF fail to ensure global convergence, whereas the hybrid filter successful does so.

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