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ORIGINAL PAPER

A Hypergraph Analog of Dirac's Theorem for Long Cycles in 2-Connected Graphs

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Abstract

Dirac proved that each n-vertex 2-connected graph with minimum degree at least k contains a cycle of length at least $\min\{2k, n\}$. We consider a hypergraph version of this result. A *Berge cycle* in a hypergraph is an alternating sequence of distinct vertices and edges $v_1, e_2, v_2, \ldots, e_c, v_1$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ for all i (with indices taken modulo c). We prove that for $n \geq k \geq r + 2 \geq 5$, every 2-connected r-uniform n-vertex hypergraph with minimum degree at least $\binom{k-1}{r-1} + 1$ has a Berge cycle of length at least $\min\{2k, n\}$. The bound is exact for all $k \geq r + 2 \geq 5$.

Keywords Berge cycles · Extremal hypergraph theory · Minimum degree

Mathematics Subject Classification 05D05 · 05C65 · 05C38 · 05C35

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1 Introduction and Results

1.1 Terminology and Known Results for Graphs

A hypergraph H is a family of subsets of a ground set. We refer to these subsets as the *edges* of H and the elements of the ground set as the *vertices* of H. We use $E(H)$ and $V(H)$ to denote the set of edges and the set of vertices of H respectively. We say H is *r-uniform* (r-graph, for short) if every edge of H contains exactly r vertices. A *graph* is a 2-graph. For a hypergraph H and $A \subseteq V(H)$, by $H[A]$ we denote the subhypergraph of H induced by A.

The *degree* $d_H(v)$ of a vertex v in a hypergraph H is the number of edges containing v. When there is no ambiguity, we may drop the subscript H and simply use $d(v)$. The *minimum degree*, $\delta(H)$, is the minimum over degrees of all vertices of H.

A *hamiltonian cycle* in a graph is a cycle which visits every vertex. Sufficient conditions for existence of hamiltonian cycles in graphs have been well-studied. In particular, a famous result of Dirac from 1952 is:

Theorem 1 (Dirac [4]). *Let $n \geq 3$. If G is an n-vertex graph with minimum degree $\delta(G) \geq n/2$, then G has a hamiltonian cycle.*

Dirac also proved that every graph G with minimum degree $k \geq 2$ contains a cycle of length at least $k + 1$, and that this bound can be significantly strengthened when G is 2-connected.

Theorem 2 (Dirac [4]). *Let $n \geq k \geq 2$. If G is an n-vertex, 2-connected graph with minimum degree $\delta(G) \geq k$, then G has a cycle of length at least* $\min\{2k, n\}$.

This theorem is sharp by the following examples. First, for $k \geq 3$, let $V(G_1) = X_1 \cup X_2 \cup \ldots \cup X_t$ where $|X_i| = k$ and $G_1[X_i] = K_k$ for all $1 \leq i \leq t$, and there are vertices u, v such that $X_i \cap X_j = \{u, v\}$ for all $i \neq j$. Since $k \geq 3$, $\delta(G_1) = k - 1$, and each cycle in G_1 intersects at most 2 sets $X_i \setminus \{u, v\}$, thus having length at most $k + k - 2 = 2k - 2$. Another example is the graph G_2 obtained by joining every vertex of the clique K_{k-1} to every vertex of an independent set with $n - (k - 1)$ vertices. Again, $\delta(G_2) = k - 1$, and each cycle in G_2 has length at most $2(k - 1) = 2k - 2$. Moreover, G_2 is $(k - 1)$-connected. So for k large, one cannot improve the bound in Theorem 2 by requiring higher connectivity.

A refinement of Theorem 2 for bipartite graphs was obtained by Voss and Zuluaga [20], which was further refined by Jackson [14] as follows.

Theorem 3 (Jackson [14]). *Let G be a 2-connected bipartite graph with bipartition (A, B), where $|A| \geq |B|$. If each vertex of A has degree at least a and each vertex of B has degree at least b, then G has a cycle of length at least* $2\min\{|B|, a + b - 1, 2a - 2\}$. *Moreover, if $a = b$ and $|A| = |B|$, then G has a cycle of length at least* $2\min\{|B|, 2a - 1\}$.

A sharpness example for Theorem 3 is a graph $G_3 = G_3(a, b, a', b')$ for $a' \geq b' \geq a + b - 1$ obtained from disjoint complete bipartite graphs $K_{a'-b,a}$ and $K_{b,b'-a}$ by joining each vertex in the a part of $K_{a'-b,a}$ to each vertex in the b part of $K_{b,b'-a}$.

1.2 Terminology and Known Results for Uniform Hypergraphs

We consider the notion of *Berge cycles*.

Definition 1.1 A **Berge cycle of** length c in a hypergraph is an alternating list of c distinct vertices and c distinct edges $C = v_1, e_1, v_2, \ldots, e_{c-1}, v_c, e_c, v_1$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ for all $1 \leq i \leq c$ (we always take indices of cycles of length c modulo c). We call vertices $v_1, \ldots, v_c$ **the defining vertices** of C and write $V(C) = \{v_1, \ldots, v_c\}$, $E(C) = \{e_1, \ldots, e_c\}$.

Notation for Berge paths is similar. In addition, a **partial Berge path** is an alternating sequence of distinct edges and vertices beginning with an edge and ending with a vertex $e_0, v_1, e_1, v_2, \ldots, e_k, v_{k+1}$ such that $v_1 \in e_0$ and for all $1 \leq i \leq k$, $\{v_i, v_{i+1}\} \subseteq e_i$.

A series of approximations and analogs of Theorem 1 for Berge cycles in a number of classes of r-uniform hypergraphs (r*-graphs*, for short) were obtained by Bermond, Germa, Heydemann and Sotteau [1], Clemens, Ehrenmüller and Person [2], Coulson and Perarnau [3] and Ma, Hou, and Gao [18].

Exact bounds for all values of $3 \leq r < n$ were obtained in [15].

Theorem 4 (Theorem 1.7 in [15]). *Let* $t = t(n) = \lfloor \frac{n-1}{2} \rfloor$, *and suppose* $3 \leq r < n$. *Let H be an r-graph. If (a)* $r \leq t$ *and* $\delta(H) \geq \binom{t}{r-1} + 1$ *or (b)* $r \geq n/2$ *and* $\delta(H) \geq r$, *then H contains a hamiltonian Berge cycle.*

Salia [19] proved an exact result of Pósa type extending Theorem 4 for $n > 2r$ to hypergraphs with "few" vertices of small degree. In [15], some bounds on the circumference of r-graphs with given minimum degree were obtained:

Theorem 5 ([15]). *Let n, k, and r be positive integers such that* $n \geq k$ *and* $\lfloor (n-1)/2 \rfloor \geq r \geq 3$. *Let H be an n-vertex, r-uniform hypergraph. If*

(a) $k \leq r + 1$ *and* $\delta(H) \geq k - 1$, *or*
(b) $r + 2 \leq k < \lfloor (n-1)/2 \rfloor + 2$ *and* $\delta(H) \geq \binom{k-2}{r-1} + 1$, *or*
(c) $k \geq \lfloor (n-1)/2 \rfloor + 2$ *and* $\delta(H) \geq \binom{\lfloor (n-1)/2 \rfloor}{r-1} + 1$,

then H contains a Berge cycle of length k or longer.

For an analog of Theorem 2, we define connectivity of a hypergraph with the help of its *incidence bipartite graph*:

Definition 1.2 Let H be a hypergraph. The **incidence graph** I_H **of** H is the bipartite graph with $V(I_H) = X \cup Y$ such that $X = V(H)$, $Y = E(H)$ and for $x \in X$, $y \in Y$, $xy \in E(I_H)$ if and only if the vertex x belongs to the edge y in H.

It is easy to see that if H is an r-graph with minimum degree $\delta(H)$, then each $x \in X$ and each $y \in Y$ satisfy $d_{I_H}(x) \geq \delta(H)$, $d_{I_H}(y) = r$. Moreover, there is a bijection between the set of Berge cycles of length c in H and the set of cycles of length $2c$ in I_H: a Berge cycle $v_1, e_1, \ldots, v_c, e_c, v_1$ can also be viewed as a cycle in I_H with the same sequence of vertices.

Using the notion of the incidence graph, we also define connectivity in hypergraphs.

Definition 1.3 A hypergraph H is k-**connected** if its incidence graph I_H is a k-connected graph.

Theorem 3 of Jackson applied to I_H of a 2-connected r-graph H yields the following approximation of an analog of Theorem 2 for $k \le r-1$:

Corollary 6 *Let n, k, r be positive integers with $2 \le k \le r-1$. If H is an n-vertex 2-connected r-graph H with $\delta(H) \ge k+1$, then H contains a Berge cycle of length at least* $\min\{2k, n, |E(H)|\}$.

On the other hand, for all $3 \le k \le r$, there are 2-connected r-graphs H_k with $\delta(H_k) \ge k-2$ that do not have Berge cycle of length at least $\min\{2k, |V(H_k)|, |E(H_k)|\}$. A series of such examples is as follows. For $m \ge 2$, let $V(H_k) = A_1 \cup \ldots \cup A_m \cup \{x, y\}$ where $A_i = \{a_{i,1}, \ldots, a_{i,r-1}\}$ for $1 \le i \le m$, and let $E(H_k) = E_1 \cup \ldots \cup E_m$ where for each $1 \le i \le m$ and $1 \le j \le k-1$, $E_i = \{e_{i,1}, \ldots, e_{i,k-1}\}$ and $e_{i,j} = (A_i - a_{i,j}) \cup \{x, y\}$. Each Berge cycle in H_k can contain edges from at most two E_is, and $|E_i| = k-1$ for all $1 \le i \le m$.

1.3 Our Results and Structure of the Paper

Our main result is the following.

Theorem 7 *Let n, k, r be positive integers with $3 \le r \le k-2 \le n-2$. If H is an n-vertex 2-connected r-graph with*

$$\delta(H) \ge \binom{k-1}{r-1} + 1, \tag{1}$$

then H contains a Berge cycle of length at least $\min\{2k, n\}$.

We point out that for 2-connected hypergraphs, the minimum degree required to guarantee a Berge cycle of length at least $2k$ is roughly of the order 2^{r-1} times smaller than the sharp bound guaranteed in Theorem 5(b). Furthermore, the bound $\delta(H) = \binom{k-1}{r-1} + 1$ is best possible as demonstrated by the following constructions.

Construction 1.1 *Let $q \ge 2$ be an integer and $4 \le r+1 \le k \le n/2$. For $n = q(k-2)+2$, let $H_1 = H_1(k)$ be the r-graph with $V(H_1) = \{x, y\} \cup V_1 \cup V_2 \cup \ldots \cup V_q$ where for all $1 \le i \le q$, $|V_i| = k-2$ and $V_i \cup \{x, y\}$ induces a clique. Any Berge cycle in H_1 has length at most $2(k-2)+2 = 2k-2$.*

Construction 1.2 *Let $4 \le r+1 \le k \le n/2$. Let $H_2 = H_2(k)$ be the r-graph with $V(H_2) = X \cup Y$ where $|X| = k-1$, $|Y| = n-(k-1)$, and $E(H_2)$ is the set of all hyperedges containing at most one vertex in Y. No Berge cycle can contain consecutive vertices in Y, so any Berge cycle has length at most $2k-2$.*

Observe that both H_1 and H_2 have minimum degree $\binom{k-1}{r-1}$. Moreover, H_2 is $(k-1)$-connected and can be defined for all $n \ge k$. Therefore, the bound in Theorem 7 cannot be further decreased by requiring higher connectivity.

Remark 1 Problems on conditions for the existence of long Berge paths and cycles (in particular, Turán-type analogs of the Erdős-Gallai Theorem) attracted recently considerable attention, see e.g., [6, 9–13] and references in them. These results yield some Dirac-type bounds, but the implied bounds are significantly weaker than the bound in Theorem 7.

Remark 2 The extremal hypergraph of the Turán-type problem in [10] (the maximum number of edges among all r-uniform, 2-connected hypergraphs with no Berge cycle of length at least $\min\{2k, n\}$) contains H_2 as a subhypergraph. But the extremal hypergraph H_1 has fewer edges.

We also present a bound for $k = 2$ that is better than given by Corollary 6:

Proposition 8 *Let $3 \le r < n$ be positive integers. Then every n-vertex 2-connected r-graph H contains a Berge cycle of length at least* $\min\{4, |E(H)|\}$.

A sharpness example is an r-graph $H_3 = H_3(r, s)$ with vertex set $\{v_1, v_2\} \cup \bigcup_{i=1}^{s} U_i$ where $|U_i| = r-1$ for $1 \le i \le s$ and edge set is $\bigcup_{i=1}^{s}\{e_{i,1}, e_{i,2}\}$ where $e_{i,j} = U_i \cup \{v_j\}$ for $1 \le i \le s$ and $1 \le j \le 2$. This r-graph is 2-connected for $s \ge 2$ and has no Berge cycles of length more than 4.

A related notion is the *codiameter* of a hypergraph H which is the maximum integer k such that for every two vertices $u, v \in V(H)$, H contains a Berge u, v-path of length at least k. (Recall that the length of a Berge path is the number of its edges.)

In graphs, having codiameter k is equivalent to the property that for any two vertices x, y, graph $G + xy$ has a cycle of length at least $k+1$ passing through edge xy. This property is well studied, see [5, 7, 8]. It was proved recently in [16] that the bound $\delta(H) \ge \binom{\lfloor n/2 \rfloor}{r-1} + 1$ guarantees the largest possible codiameter, $n - 1$.

As an application of our main theorem, we prove the following Dirac-type bound.

Corollary 9 *Let n, k, r be positive integers with $n/2 \ge k \ge r+2$ and $r \ge 3$. If H is an r-uniform, n-vertex, 2-connected hypergraph with*

$$\delta(H) \ge \binom{k-1}{r-1} + 1,$$

then the codiameter of H is at least k.

For $n = q(k-2) + 2$, the construction $H_1(k)$ shows that Corollary 9 is sharp: the longest Berge path from x to y contains $k - 1$ edges. We also note that 2-connectivity is necessary: for large n divisible by r, we may take r copies of $K_{n/r}^{(r)}$ and a single edge intersecting each clique in one vertex. This hypergraph has minimum degree $\binom{n/r-1}{r-1}$ (which does not depend on k) but there are pairs of vertices that are connected only by a one-edge Berge path.

1.4 Outline of the Paper

The structure of the paper is as follows. In Sect. 2 we present a simple proof of Proposition 8 and derive Corollary 9 from Theorem 7. In Sect. 3 we set up the proof of our main result, Theorem 7. We introduce notation and define so called *lollipops*. Each lollipop is roughly speaking a pair of a Berge cycle C and a Berge path (or a partial Berge path, defined in the next section) P such that P starts in C and extends outward. In particular, we define criteria for which we will choose an *optimal* lollipop (C, P).

In the subsequent five sections we consider all possible cases of best lollipops (C, P) and find a contradiction in each of them. In particular, in Sect. 4, we show that in an optimal lollipop, P has a positive length. In Sect. 5, inspired by Dirac's proof of Theorem 2, we show that the end vertex of the P cannot have too many neighbors in P. One of the key ingredients of the proof is a modification of a Dirac's lemma on paths in 2-connected graphs (Lemma 5.7). In Sects. 6 and 7, we show that P must be a Berge path and cannot be too long. Finally in Sect. 8, using the structure of (C, P) established in previous sections, we analyze how the neighborhoods of two vertices in P can interact and conclude that we must be able to construct a longer cycle than C.

We note that if $k \geq n/2$ then by Theorem 5, $\delta(H) \geq \binom{k-1}{r-1} + 1 \geq \binom{\lfloor (n-1)/2 \rfloor}{r-1} + 1$ implies that H contains a Berge cycle of length n. Thus when proving Theorem 7 we will assume

$$k < n/2 \; and \; \min\{2k, n\} = 2k. \tag{2}$$

2 Short Proofs

In this section, we present a proof of Proposition 8 and show how to derive Corollary 9 from our main result.

2.1 Proof of Proposition 8

Proof Suppose H is a counter-example to the proposition, i.e. for some $3 \leq r < n$, H is n-vertex 2-connected r-graph and each Berge cycle in H has length at most $\min\{3, |E(H)|-1\}$. Since H is 2-connected, by Corollary 6, $\delta(H) \leq 2$, so $\delta(H) = 2$.

Assume first, a pair $\{u_1, v_1\}$ of vertices in H is not in any edge. Since the incidence graph I_H of H is 2-connected, by Menger's Theorem it has a cycle $C = u_1, e_1, u_2, e_2, \dots, u_s, e_s, v_1, f_1, v_2, \dots, v_t, f_t, u_1$ containing u_1 and v_1. Since no edge contains both u_1 and v_1, the four edges e_1, e_s, f_1, f_t of H are distinct. Then C corresponds to a cycle in H of length at least 4, a contradiction. Thus,

$$\textit{for each pair}\{u, v\} \subset V(H) \textit{ there is an edge} e_{uv} \textit{ containing } u \textit{ and } v. \tag{3}$$

Since $\delta(H) = 2$, let $u \in V(H)$ with $d_H(u) = 2$ and e_1, e_2 be the edges containing u. Let $A_0 = e_1 \cap e_2$, $A_1 = e_2 \backslash e_1$ and $A_2 = e_1 \backslash e_2$.

By (3), $e_1 \cup e_2 = V(H)$. We claim that

$$\textit{some edge } e_0 \textit{ of } H \textit{ contains } A_1 \cup A_2. \tag{4}$$

Indeed, let $x_1 \in A_2$ and $x_2 \in A_1$. By (3), there is an edge e_3 containing x_1 and x_2. If e_3 omits some $y_1 \in A_2$ and some $y_2 \in A_1$, then again by (3), there is an edge e_4 containing y_1 and y_2, and so H has 4-cycle $x_1, e_1, y_1, e_4, y_2, e_2, x_2, e_3, x_1$, a contradiction. Thus we may assume $e_3 \supset A_1$ and $y_1 \in A_2 \backslash e_3$. Since $|A_2| = |A_1|$, there is $y_2 \in A_1 - x_2$. Again by (3), there is an edge e_4 containing y_1 and y_2, and so H has 4-cycle $x_1, e_1, y_1, e_4, y_2, e_2, x_2, e_3, x_1$. This proves (4).

So, for $0 \le i \le 2$, $e_i \supseteq V(H)\backslash A_i$. Since $|E(H)| \ge 4$, there is an edge $g \notin \{e_0, e_1, e_2\}$. If some two vertices of g are in the same A_i, say $u, v \in g \cap A_0$, then H has 4-cycle $u, g, v, e_1, x_1, e_0, x_2, e_2, u$, where $x_1 \in A_2$ and $x_2 \in A_1$. Otherwise, $r = 3$ and g has a vertex in each of A_0, A_1, A_2. Since $|V(H)| \ge 4$, some A_i has at least two vertices, say $|A_1| \ge 2$. For $0 \le i \le 2$, let $u_i \in g \cap A_i$. Let $v \in A_1 - u_1$. Then H has 4-cycle $u_0, g, u_1, e_2, v, e_0, u_2, e_1, u_0$. This contradiction finishes the proof. □

2.2 Proof of Corollary 9 on Codiameters

Proof Suppose

$$n/2 \ge k \ge r + 2 \ge 5, \tag{5}$$

and H is an r-uniform, n-vertex, 2-connected hypergraph with $\delta(H) \ge \binom{k-1}{r-1} + 1$. Then by Theorem 7, H contains a cycle $C = v_1, e_1, \ldots, v_c, e_c, v_1$ with $c \ge 2k$. Fix $u, v \in V(H)$. If $u, v \in V(C)$ then there exists a segment in C from u to v with at least $\lceil (c+2)/2 \rceil \ge k + 1$ vertices. This is a path of length at least k.

Otherwise, consider the incidence graph I_H which is 2-connected. There exist shortest disjoint (graph) paths P_1 and P_2 in I_H from $V(C) \cup E(C)$ to $\{u, v\}$, say $u \in P_1, v \in P_2$. If $u \in V(C)$, then we have $P_1 = u$ and similar for v. In H, P_1 and P_2 correspond to either Berge paths or partial Berge paths that end with u and v respectively. Let a_1, a_2 be the first elements of P_1 and P_2 respectively, and let Q be the longer of the two a_1, a_2-segments along C. If the two a_1, a_2-segments along C have equal length, we choose one arbitrarily.

If without loss of generality, $u \in V(C)$, then $|V(Q)| \ge \lceil (c+1)/2 \rceil \ge k + 1$. Appending P_2 to the end of Q gives a path of length at least $k+1$ from u to v. Finally, if $u, v \notin V(C)$, then $|V(Q)| \ge \lceil c/2 \rceil \ge k$, so $P_1 \cup Q \cup P_2$ is a u, v-path with at least $k + 2$ vertices. □

3 Setup and Simple Properties of Best Lollipops

In this section we present some hypergraph notation and define lollipops. We also derive a series of useful properties of optimal lollipops.

3.1 Notation and Setup

For a hypergraph H, and a vertex $v \in V(H)$,

$$N_H(v) = \{u \in V(H) : \text{there exists } e \in E(H) \text{ such that } \{u, v\} \subset e\}$$

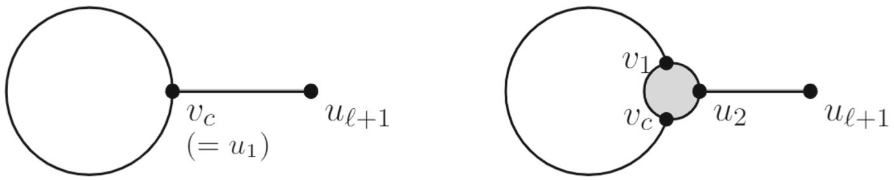


Fig. 1 An o-lollipop and a p-lollipop

is the H*-neighborhood* of v. The *closed* H*-neighborhood* of v is the set $N_H[v] = N_H(v) \cup \{v\}$.

When G is a subhypergraph of a hypergraph H and $u, v \in V(H)$, we say that u and v are G*-neighbors* if there exists an edge $e \in E(G)$ containing both u and v.

When we speak of an x, y-(Berge) path P and $a, b \in V(P)$, then $P[a, b]$ denotes the unique segment of P from a to b.

Let $r \geq 3$. We consider a counter-example H. Taking into account (2), $k < n/2$ and H is a 2-connected n-vertex r-uniform hypergraph satisfying (1) such that

$$H \textit{ does not contain a Berge cycle of length at least } 2k. \tag{6}$$

A *lollipop* (C, P) is a pair where C is a Berge cycle and P is a Berge path or a partial Berge path that satisfies one of the following:

- P is a Berge path starting with a vertex in C, $|V(C) \cap V(P)| = 1$, and $|E(C) \cap E(P)| = 0$. We call such a pair (C, P) an **ordinary lollipop** (or o-lollipop for short). See Fig. 1 (left).
- P is a partial Berge path starting with an edge in C, $|V(C) \cap V(P)| = 0$, and $|E(C) \cap E(P)| = 1$. We call such a pair (C, P) a **partial lollipop** (or p-lollipop for short). See Fig. 1 (right).

A lollipop (C, P) *is better* than a lollipop (C', P') if

(a) $|V(C)| > |V(C')|$, or
(b) Rule (a) does not distinguish (C, P) from (C', P'), and $|E(P)| > |E(P')|$; or
(c) Rules (a) and (b) do not distinguish (C, P) from (C', P'), and the total number of vertices of $V(P) - V(C)$ contained in the edges of C counted with multiplicities is larger than the total number of vertices of $V(P') - V(C')$ contained in the edges of C'; or
(d) Rules (a)–(c) do not distinguish (C, P) from (C', P'), and (C, P) is an o-lollipop while (C', P') is a p-lollipop; or
(e) Rules (a)–(d) do not distinguish (C, P) from (C', P'), and the number of edges in $E(P) - E(C)$ fully contained in $V(P) - V(C)$ is larger than the number of edges in $E(P') - E(C')$ fully contained in $V(P') - V(C')$.

The criteria (a)–(e) define a partial ordering on the (finite) set of lollipops, and hence we can choose a best lollipop (C, P). Say $C = v_1, e_1, \ldots, v_c, e_c, v_1$. If (C, P) is a o-lollipop then let $P = u_1, f_1, \ldots, f_\ell, u_{\ell+1}$, where $u_1 = v_c$. If (C, P) is a p-lollipop then let $P = f_1, u_2, f_2, \ldots, f_\ell, u_{\ell+1}$ where $f_1 = e_c$. With this notation, we have

$|E(P)| = \ell$, $|V(P)| = \ell + 1$ if P is a Berge path, and $|V(P)| = \ell$ if P is a partial Berge path. Assume $c < 2k = \min\{2k, n\}$.

Denote by H' the subhypergraph of H with $V(H') = V(H)$ and $E(H') = E(H) - E(C) - E(P)$. Define

$$H'' = \begin{cases} H' & \text{when } (C, P) \text{ is a p-lollipop,} \\ H' \cup \{f_1\} & \text{when } (C, P) \text{ is a o-lollipop.} \end{cases} \tag{7}$$

Since we consider mostly Berge paths and cycles, from now on we will refer to them simply as paths and cycles. We will differentiate graph paths and cycles when needed.

3.2 Simple Properties of Best Lollipops

In this subsection we consider best lollipops (C, P) and prove some basic claims to be used throughout the rest of the paper. The following claim immediately follows from the assumption (6) and $c < 2k$.

Claim 3.1 (a) *If $a_1 = e_i$ and $b_1 = e_j$ for some $i, j \in [c]$, then the longer of the two subpaths of C connecting $\{v_i, v_{i+1}\}$ with $\{v_j, v_{j+1}\}$ and using neither of e_i and e_j has at least $\lceil c/2 \rceil$ vertices. In particular, this path omits at most $k - 1$ vertices in C.*

(b) *If $a_1 = e_i$ and $b_1 = v_j$ for some $i, j \in [c]$, then the longer of the two subpaths of C connecting $\{v_i, v_{i+1}\}$ with v_j and not using e_i has at least $\lceil (c+1)/2 \rceil$ vertices. In particular, this path omits at most $k - 1$ vertices in C.*

(c) *If $a_1 = v_i$ and $b_1 = v_j$ for some $i, j \in [c]$, then the longer of the two subpaths of C connecting v_i with v_j has at least $\lceil (c+2)/2 \rceil$ vertices. In particular, this path omits at most $k - 2$ vertices in C.*

We call a path satisfying Claim 3.1 a *long a_1, b_1-segment of C*.

Claim 3.2 *Let (C, P) be a best lollipop. For each $1 \le i \le c$ and $2 \le m \le \ell + 1$, if some edge $g \notin E(C)$ contains $\{u_m, v_i\}$, then*

(a) *neither e_{i-1} nor e_i intersect $V(P) - u_1$, and*
(b) *no edge in H'' intersects both $V(P) - u_1$ and $\{v_{i-1}, v_{i+1}\}$ (indices count modulo c).*

In particular, the set $N_{H''}(V(P) - u_1) \cap V(C)$ does not contain two consecutive vertices of C.

Proof Let $g \notin E(C)$ contain $\{u_m, v_i\}$ such that if $g \in E(P)$, say $g = f_q$, then we may assume $u_m = u_{q+1}$. Suppose e_{i-1} contains u_j for some $2 \le j \le \ell + 1$. If either $j \ge m$ or $g \ne f_{m-1}$, then we may replace the segment v_{i-1}, e_{i-1}, v_i in C with $v_{i-1}, e_{i-1}, u_j, P[u_j, u_m], u_m, g, v_i$. Otherwise we replace the segment with $v_{i-1}, e_{i-1}, u_j, P[u_j, u_{m-1}], u_{m-1}, g, v_i$. We obtain a longer cycle, contradicting the choice of C. The case with $u_j \in e_i$ is symmetric. This proves (a).

Suppose now some $e \in E(H'')$ contains $\{u_j, v_{i-1}\}$ for some $2 \le j \le \ell+1$ (the case when $e \supset \{u_j, v_{i+1}\}$ is symmetric). If $e \neq g$, then similarly to before we may replace the segment v_{i-1}, e_{i-1}, v_i in C with $v_{i-1}, e, u_j, P[u_j, u_m], u_m, g, v_i$ or $v_{i-1}, e, u_j, P[u_j, u_{m-1}], u_{m-1}, g$ to get a longer cycle.

If $e = g$, then by (a), $e_{i-1} \cap (V(P) - u_1) = \emptyset$. Note that in this case $g \in E(H'')$. Let C' be obtained from C by replacing the edge e_{i-1} with g. If $g \neq f_1$, then we let $P' = P$, otherwise, by the definition (7) of H'', P is a path, and we define partial path $P' = f_1, u_2, f_2, \dots, u_{\ell+1}$. Then (C', P') is better than (C, P) by Rule (c) in the definition of better lollipops. □

Call a lollipop (C', P') *good* if $|E(C')| = c$ and $|E(P')| = \ell$. In particular, each best lollipop is a good lollipop.

Claim 3.3 *Suppose (C, P) is a good lollipop. Let $\widetilde{H}$ be the subhypergraph of H with $E(\widetilde{H}) = E(H) - E(C) - E(P)$.*

Then all $\widetilde{H}$-neighbors of $u_{\ell+1}$ are in $V(C) \cup V(P)$, and moreover

(1) *if (C, P) is an o-lollipop, then $u_{\ell+1}$ has no $\widetilde{H}$-neighbors in $\{v_1, v_2, \dots, v_\ell\} \cup \{v_{c-1}, v_{c-2}, \dots, v_{c-\ell}\}$, and $u_{\ell+1}$ is not in any edge in the set $\{e_1, e_2, \dots, e_{\ell-1}\} \cup \{e_c, e_{c-1}, \dots, e_{c-\ell}\}$,*
(2) *if (C, P) is a p-lollipop, then $u_{\ell+1}$ has no $\widetilde{H}$-neighbors in $\{v_1, v_2, \dots, v_\ell\} \cup \{v_c, v_{c-1}, \dots, v_{c-\ell+1}\}$, and $u_{\ell+1}$ is not in any edge in the set $\{e_1, \dots, e_{\ell-1}\} \cup \{e_{c-1}, \dots, e_{c-(\ell-1)}\}$.*

Proof Let $e \in E(\widetilde{H})$ contain $u_{\ell+1}$. Suppose first there is a vertex $y \in V(H) - (V(C) \cup V(P))$ such that $y \in e$. Let P' be the path obtained from P by adding edge e and vertex y to the end of P. Then (C, P') is a lollipop with $|V(P')| > |V(P)|$, a contradiction.

Now suppose e contains v_i for some $i \in \{1, \dots, \ell\}$. Then we can replace the segment $v_c, e_c, v_1, \dots, v_i$ from v_c to v_i in C with the path $v_c, e_c, P[e_c, u_{\ell+1}], u_{\ell+1}, e, v_i$ to obtain a cycle of length at least $c - (\ell-1) + \ell > c$, contradicting the choice of C.

The proof for $i \in \{c, \dots, c-\ell\}$ or $i \in \{c, \dots, c-\ell+1\}$ is very similar, but when (C, P) is a p-lollipop, we replace the segment $v_1, e_c, v_c, \dots, e_i, v_i$ instead with $v_1, e_c, P[e_c, u_{\ell+1}], u_{\ell+1}, e, v_i$.

Finally suppose (C, P) is an o-lollipop and for some $1 \le i \le \ell$, $u_{\ell+1} \in e_{i-1}$ (modulo c). The cycle obtained by replacing the segment from v_c to v_i with the path $v_c, P, u_{\ell+1}, e_{i-1}, v_i$ has length at least $c+1$, contradicting the choice of C. The argument for e_{c-i} and the argument in the case (C, P) is a p-lollipop and $e_{i-1} \neq e_c$ are similar. □

Claim 3.4 *Let (C, P) be a best lollipop.*

(A) *If $u_{\ell+1} \in f_m$ for some $1 \le m \le \ell-1$ and P' is obtained from P by replacing the subpath $u_m, f_m, u_{m+1}, \dots, u_{\ell+1}$ with the subpath $u_m, f_m, u_{\ell+1}, f_\ell, u_\ell, \dots, u_{m+1}$, then (C, P') also is a best lollipop.*
(B) *If some edge $g \in E(H')$ contains $V(P) - V(C)$ or is contained in $V(P) - V(C)$ and contains $\{u_{\ell+1}, u_m\}$ for some $1 \le m \le \ell-1$, and if P' is obtained from P by replacing the subpath $u_m, f_m, u_{m+1}, \dots, u_{\ell+1}$ with the subpath $u_m, g, u_{\ell+1}, f_\ell, u_\ell, \dots, u_{m+1}$, then (C, P') also is a best lollipop.*

Proof Let us check the definition of a best lollipop. Part (A) holds because the vertex set and edge set of $P' - V(C)$ are the same as those of $P - V(C)$.

In Part (B), $V(P') - V(C) = V(P) - V(C)$, and $E(P')$ is obtained from $E(P)$ by deleting f_m and adding g. But since g contains $V(P) - V(C)$ or is contained in $V(P) - V(C)$, (C, P) cannot be better than (C, P'). □

Claim 3.5 *For $2 \leq q \leq \ell$ and $1 \leq i, j \leq c$, the following hold:*

(1) *If $u_q \in e_i$ and $u_{\ell+1} \in e_j$ then $j = i$ or $|j - i| \geq (\ell + 1) - q + 1$.*
(2) *If there exists an edge $e \in E(H'')$ such that $\{v_i, u_q\} \subset e$, and if $u_{\ell+1} \in e_j$, then either $j > i$ and $j - i \geq (\ell + 1) - q + 1$, or $i > j$ and $i - j \geq (\ell + 1) - q + 2$.*
(3) *If there exist distinct edges $e, f \in E(H'')$ such that $\{v_i, u_q\} \subset e$ and $\{v_j, u_{\ell+1}\} \subset f$, then $j = i$ or $|j - i| \geq (\ell + 1) - q + 2$.*

Proof We will prove (1). If $j \neq i$, then we can replace the segment of C from e_i to e_j containing $|j - i|$ vertices with $e_j, u_{\ell+1}, P[u_{\ell+1}, u_q], u_q, e_i$ which contains $(\ell + 1) - q + 1$ vertices. The new cycle cannot be longer than C. The proofs for (2) and (3) are similar so we omit them. □

4 Nontrivial Paths in Best Lollipops

In this section, we show that the path or partial path P has length at least 2. In particular, since H is connected, and $|C| < n$, there is an edge intersecting both $V(C)$ and $V(H) - V(C)$. Thus $\ell \geq 1$. Below we show in fact $\ell \geq 2$ using the notion of *expanding* sets that can be used to modify C into a longer cycle.

Suppose $\ell = 1$ and u_2 is the unique vertex in $V(P) \backslash V(C)$. Say that a set $W \subseteq V(C)$ is *u_2-expanding* if for every distinct $v_j, v_{j'} \in W$, there is a $v_j, v_{j'}$-path $Q(v_j, v_{j'})$ whose all internal vertices are not in $V(C) \cup \{u_2\}$ and all edges are in $E(H) \backslash E(C)$. One example of a u_2-expanding set is $V(C) \cap g$ where g is any edge in $E(H) \backslash E(C)$. Another useful example is a set of the form $N_{H'}(w) \cap V(C)$ for a vertex $w \in V(H) - V(C) - u_2$.

Suppose W is a u_2-expanding set and $v_j, v_{j'} \in W$ where $j < j'$. If $u_2 \in e_j \cap e_{j'}$, then the cycle

$$\begin{aligned} &v_1, e_1, v_2, \ldots, e_{j-1}, v_j, Q(v_j, v_{j'}), v_{j'}, e_{j'-1}, v_{j'-1}, \ldots, v_{j+1}, e_j, \\ &u_2, e_{j'}, v_{j'+1}, e_{j'+1}, \ldots, e_{c-1}, v_c, e_c, v_1 \end{aligned}$$

is longer than C, a contradiction. A symmetric longer cycle can be found if $u_2 \in e_{j-1} \cap e_{j'-1}$. Thus

$$\begin{aligned} &u_2 \textit{ is contained in at most one edge of } \{e_j : v_j \in W\} \\ &\textit{and in at most one edge of } \{e_{j-1} : v_j \in W\}. \end{aligned} \tag{8}$$

Therefore,

$$\text{if the vertices of } W \text{ form on } C \text{ exactly } q \text{ intervals of consecutive vertices and } B \text{ is the set of edges in } C \text{ containing } u_2, \text{ then } |B| \leq c - |W| + 1 - q + 1. \text{ Moreover, if } q = 1, \text{ say } W = \{v_{j_1}, v_{j_1+1}, \ldots, v_{j_1+|W|-1}\} \text{ and } |B| = c - |W| + 1, \text{ then } B = E(C) \setminus \{e_{j_1}, e_{j_1+1}, \ldots, e_{j_1+|W|-2}\}. \tag{9}$$

Now we are ready to prove that $\ell \geq 2$.

Lemma 4.1 *Suppose $n/2 \geq k \geq r + 2 \geq 5$ and H is a 2-connected r-graph satisfying (1). Let (C, P) be a best lollipop. If $c < 2k$, then $\ell = |E(P)| \geq 2$.*

Proof Suppose $\ell = 1$. If there exists $e \in E(H) - E(C)$ containing at least 2 vertices $u, u' \notin V(C)$, then let P' be a shortest path or partial path from $V(C) \cup E(C)$ to $\{u, u'\}$ which avoids e. Such a P' exists because H is 2-connected. Without loss of generality, P' ends with u. Then $(C, (P', e, u'))$ is better than (C, P). It follows that

$$\text{for each } e \in E(H) - E(C), |e \cap V(C)| \geq r - 1. \tag{10}$$

Case 1: P is a path, say $P = v_c, f_1, u_2$. Recall that $H'' = H' \cup \{f_1\}$. Since $d_H(u_2) \geq 2$ and every edge containing u_2 intersects C, by the maximality of ℓ, $N_{H''}(u_2) \subseteq V(C)$. Let $A = N_{H''}(u_2)$, $a = |A|$, $B = \{e_i \in E(C) : u_2 \in e_i\}$ and $b = |B|$. By Claim 3.2, A does not intersect the set $\bigcup_{e_i \in B}\{v_i, v_{i+1}\}$ and no two vertices of A are consecutive on C. Therefore,

$$2k - 1 \geq c \geq 2a + b. \tag{11}$$

It follows that

$$1 + \binom{k-1}{r-1} \leq d_H(u_2) \leq \binom{a}{r-1} + b \leq \binom{a}{r-1} + c - 2a. \tag{12}$$

Case 1.1: $d_{H''}(u_2) \geq 2$. Then $a \geq r$. Since the RHS of (12) is monotonically increasing with a when $a \geq r - 1 \geq 2$, if $a \leq k - 2$, then $1 + \binom{k-1}{r-1} \leq \binom{k-2}{r-1} + c - 2k + 4$ and hence

$$\binom{k-2}{r-2} \leq c - (2k - 3) \leq 2,$$

a contradiction.

Suppose now $a = k - 1$. Then (12) yields $b \geq 1$, (11) yields $b \leq 1$, and in order to have equality, $c = 2k - 1$ and all r-tuples of vertices containing u_2 and contained in $A \cup \{u_2\}$ are edges of H''.

It is convenient in this case to rename the vertices in C so that $B = \{e_{2k-1}\}$ and $A = \{v_2, v_4, \ldots, v_{2k-2}\}$. Since $k = a + 1 \geq r + 1$, for every $1 \leq j \leq k - 1$ we can choose an edge $g_{2j} \in E(H'')$ containing u_2 and v_{2j} so that all g_{2j} are distinct. Since

$r \le k-2$, some vertex in $V(C) - A$ is not in e_{2k-1}, say $v_{2i+1} \notin e_{2k-1}$ for some $1 \le i \le k-2$.

Suppose $v_{2i+1} \in e_j$ for some $j \in [2k-2] - \{2i, 2i+1\}$. By symmetry, we may assume $j > 2i+1$. If j is even then the cycle

$$C_j = v_{2i+1}, e_j, v_{j+1}, e_{j+1}, v_{j+2}, \ldots, v_{2i}, g_{2i}, u_2, g_j, v_j, e_{j-1}, v_{j-1}, \ldots, v_{2i+1}$$

is longer than C. If j is odd then by the choice of v_{2i+1}, $j \neq 2k-1$, and the cycle

$$C'_j = v_{2i+1}, e_j, v_j, e_{j-1}, v_{j-1}, \ldots, v_{2i+2}, g_{2i+2}, u_2, g_{j+1}, \\ \times v_{j+1}, e_{j+1}, v_{j+2}, \ldots, v_{2i+1}$$

is longer than C, a contradiction.

Similarly, if for some odd $j \neq 2i+1$ there is an edge $h_j \in E(H'')$ containing v_{2i+1} and v_j, then we may assume $j > 2i+1$, and the cycle C''_j obtained from C'_j by replacing e_j with h_j is longer than C. Recalling that A is the set of vertices with even indices in C we obtain $N_{H-\{e_{2i},e_{2i+1}\}}(v_{2i+1}) \cap V(C) \subseteq A$. Since $|A| = k-1$ and $d_H(v_{2i+1}) \ge \binom{k-1}{r-1} + 1$, some edge $h \in E(H'') \cup \{e_{2i}, e_{2i+1}\}$ containing v_{2i+1} contains also a vertex $w \notin V(C)$. Since $v_{2i+1} \notin A \cup e_{2k-1}$, $w \neq u_2$. Consider the lollipop (C_1, P_1) where C_1 is obtained from C by replacing the subpath $v_{2i}, e_{2i}, v_{2i+1}, e_{2i+1}, v_{2i+2}$ with the subpath $v_{2i}, g_{2i}, u_2, g_{2i+2}, v_{2i+2}$, and $P_1 = v_{2i}, e_{2i}, v_{2i+1}, h, w$. This lollipop satisfies $|V(C_1)| = |V(C)|$ but $|E(P_1)| > |E(P)|$, contradicting the choice of (C, P).

Case 1.2: $d_{H''}(u_2) = 1$. Then $d_H(u_2) = 1 + b$, $A = f_1 \cap V(C)$ and $a = r-1$. By (11), $d_H(u_2) = 1 + b \le 1 + c - 2a = c - 2r + 3$. In particular,

$$1 + \binom{k-1}{2} \le 1 + \binom{k-1}{r-1} \le d_H(u_2) \le (2k-1) - 2r + 3 \le 2k-4, \tag{13}$$

and thus $k^2 - 7k + 12 \le 0$. For $k \ge r+2 \ge 5$, this is impossible.

Case 2: P is a partial path, say $P = e_c, u_2$. If there is an edge $h \in E(H')$ containing u_2, then by (10), h contains some $v_j \in V(C)$. So, the lollipop. (C, P') where $P' = v_j, h, u_2$ also is better than (C, P) by Rule (d), a contradiction. So, $d_H(u_2) = b$, where $B = \{e_i \in E(C) : u_2 \in e_i\}$ and $b = |B|$.

Suppose there exists $w \in V(H) - (V(C) \cup \{u_2\})$. Let us show that

$$d_{H'}(w) \le 1. \tag{14}$$

Indeed, suppose $g_1, g_2 \in E(H')$ and $w \in g_1 \cap g_2$. Let $W = V(C) \cap (g_1 \cup g_2)$. As observed before, this W is u_2-expanding. Since $g_2 \neq g_1$, by (10), $|W| \ge r$. Also, by Claim 3.2, vertices in g_2 could not be next to vertices in g_1 on C. Thus if $|W| = r$, then $|g_1 \cap g_2| = r-1$, and hence no two vertices of W are consecutive on C. In this case, by (9), $b \le c - |W| + 1 - q + 1$ where $q = |W| = r$. So, similarly to (13) we get

$$1 + \binom{k-1}{2} \le d_H(u_2) \le c - 2r + 2 \le 2k - 5,$$

which yields $k^2 - 7k + 14 \le 0$, an impossibility. Thus $|W| \ge r + 1$. But still since vertices in g_2 could not be next to vertices in g_1 on C, $q \ge 2$. So, we again get

$$1 + \binom{k-1}{2} \le d_H(u_2) \le c - (r+1) + 1 - 2 + 1 \le 2k - 5,$$

and come to a contradiction. This proves (14).

If $n = c + x$ and $|E(H)| = c + y$, then

$$k(c + x) = k \cdot n \le \sum\nolimits_{v \in V(H)} d_H(v) = r(c + y). \tag{15}$$

If $r \ge n/2$, then by Theorem 4, H has a Hamiltonian cycle; thus $n > 2r$. So, we conclude from (15) that

$$y \ge \frac{(r+2)(c+x)}{r} - c = \frac{2(c+x)}{r} + x = \frac{2n}{r} + x > 4 + x.$$

Since $d_{H'}(u_2) = 0$ and by (14), at most $x - 1$ edges in H' contain a vertex outside of $V(C)$. It follows that at least 6 edges of H' are contained in $V(C)$. If at least one of these edges is not an interval of consecutive vertices on C, (9) yields $b \le c - r$. Also if all of these edges form intervals on C, then the "Moreover" part of (9) yields $b \le c - r$. Hence,

$$1 + \binom{k-1}{2} \le d_H(u_2) = b \le c - r \le 2k - 4,$$

and thus $k^2 - 7k + 12 \le 0$. For $k \ge r + 2 \ge 5$, this is impossible. □

5 Vertex $u_{\ell+1}$ has Few Neighbors in P

In this section we show that $u_{\ell+1}$ cannot have too many H'-neighbors in P or be contained in too many edges in P. In particular, we will prove that it has at most $k-2$ such neighbors, and is in at most $k - 1$ such edges. The proof is a modification of Dirac's proof of Theorem 2 for 2-connected graphs. The interested reader may look at the relevant sections in [4] or [17] to see the proof idea in the simpler setting of graphs.

Let $S_1 = (N_{H'}(u_{\ell+1}) \cap V(P)) \cup \{u_\ell\}$, $S_2 = \{u_m \in V(P) : u_{\ell+1} \in f_m \text{ and } u_m \notin S_1\}$ and $S = S_1 \cup S_2$.

We will prove a series of claims. In each claim, we construct a cycle containing almost all of $S_1 \cup S_2$ and at least half the vertices in C. Thus if $S_1 \cup S_2$ or S_1 is too large (in particular, if $|S_1 \cup S_2| \geq k$ or $|S_1| \geq k-1$), we obtain a cycle that is longer than C.

We use $h'(a, b)$ to denote an edge in H' containing a and b if we know such an edge exists. By definition, if $u_m \in S_2$ then there is no edge in H' containing u_m and $u_{\ell+1}$. In this case, $h'(u_m, u_{\ell+1})$ denotes f_m.

If the smallest index i with $u_i \in S_1 \cup S_2$ is such that $u_i \in S_2$ then we denote this index by $i_1 - 1$, otherwise if $u_i \in S_1$ then we denote it by i_1. Let the other indices i such that $u_i \in S$ be $i_2, \ldots, i_\alpha$ in increasing order.

Index the vertices of S_1 by $j_1, j_2, \ldots, j_\beta$ in increasing order.

If (C, P) is an o-lollipop, then let $X = V(C) - v_c$, otherwise let $X = V(C)$. Set $Y = \{u_{i_1+1}, u_{i_1+2}, \ldots, u_{\ell+1}\}$ and $Z = \{u_{j_1+1}, u_{j_1+2}, \ldots, u_{\ell+1}\}$. Observe that $Z \subseteq Y$.

Claim 5.1 *If* $|S_1 \cup S_2| \geq k-1$ *then no edge in* H' *intersects both* X *and* Y.

Proof Suppose (C, P) is a lollipop and an edge in $E(H')$ intersects both X and Y. Among such edges, choose e containing u_i with the maximum possible i. Let i' be the largest index less than i such that $u_{i'} \in S_1 \cup S_2$.

By the definition of Y, $i > i_1$ and hence $i' \geq i_1 - 1$. Suppose a vertex in $X \cap e$ is v_j. Let Q be a long v_c, v_j-segment of C guaranteed by Claim 3.1. If $i = \ell + 1$, then consider the cycle

$$C_0 = v_c, Q, v_j, e, u_{\ell+1}, f_\ell, u_\ell, \ldots, f_1, v_c.$$

If (C, P) is an o-lollipop, then C_0 has at least $c - (k-2)$ vertices in C and at least k vertices in $S_1 \cup S_2 \cup \{u_{\ell+1}\}$, at most one of which is in C (namely $v_c = u_1$). So $|C_0| \geq c - (k-2) + (k-1) > c$, a contradiction. If (C, P) is a p-lollipop, then C_0 is guaranteed only $c - (k-1)$ vertices in C, but none of the at least k vertices in $S_1 \cup S_2 \cup \{u_{\ell+1}\}$ is in C. So $|C_0| \geq c - (k-1) + k > c$, again.

Thus, suppose $i \leq \ell$. Then $h'(u_{i'}, u_{\ell+1}) \neq e$ and hence

$$C_0' = v_c, Q, v_j, e, u_i, f_i, \ldots, u_\ell, f_\ell, u_{\ell+1}, h'(u_{\ell+1}, u_{i'}), f_{i'-1}, \ldots, f_1, v_c$$

is a cycle. Similarly to C_0, it has at least $c-(k-2)+(k-1) > c$ vertices when (C, P) is an o-lollipop, and at least $c - (k-1) + k$ vertices when (C, P) is a p-lollipop, a contradiction. □

Claim 5.2 *If* $|S_1 \cup S_2| \geq k$ *then no* f_m *with* $m \geq i_1$ *intersects* X*, and if* $|S_1| \geq k-1$*, then no* f_m *with* $m \geq j_1$ *intersects* X.

Proof Suppose $m \geq i_1$ and f_m contains some $v_j \in V(C)$. By symmetry, we may assume that $j \leq c/2$ when (C, P) is an o-lollipop and $j \leq (c+1)/2$ when (C, P) is a p-lollipop. Let Q be the path $v_c, e_{c-1}, v_{c-1}, \ldots, e_j, v_j$.

Suppose first that $|S_1 \cup S_2| \geq k$. If $m = i_1$ and the smallest index i with $u_i \in S_1 \cup S_2$ is such that $u_i \in S_1$, then let $i' = m$; otherwise, let i be the largest index less than m such that $u_{i'} \in S_1 \cup S_2$. Then

$$C_0'' = v_c, Q, v_j, f_m, u_{m+1}, f_{m+1}, \ldots, u_\ell, f_\ell, u_{\ell+1}, h'(u_{\ell+1}, u_{i'}), f_{i'-1}, \ldots f_1, v_c$$

is a cycle. It contains all vertices in $S_1 \cup S_2 \cup \{u_{\ell+1}\}$ apart from u_m. If $(S_1 \cup S_2) \cap V(C) \neq \emptyset$, then (C, P) must be an o-lollipop and $u_1 \in S_1 \cup S_2$. Hence C_0'' has at least $c-(j-1)+(k+1)-2 \geq (c-c/2)+k > c$ vertices, a contradiction. Otherwise, if $(S_1 \cup S_2) \cap V(C) = \emptyset$, then C_0'' has at least $c-(j-1)+(k+1)-1 \geq (c-(c+1)/2)+k+1 > c$ vertices.

Now suppose $|S_1| \geq k-1$. In this case, let i' be the largest index that is at most m such that $u_{i'} \in S_1$. Then the same cycle C_0'' as above contains all vertices in $S_1 \cup \{u_{\ell+1}\}$. Similarly, we get either $|C_0''| > c-(c/2-1)+k-1 > c$ or $|C_0''| > c-((c+1)/2-1)+k > c$. $\square$

Claim 5.3 *Suppose $|S_1 \cup S_2| \geq k$. If (C, P) is an o-lollipop, then no $e_j \in E(C)$ intersects Y. If (C, P) is a p-lollipop then no $e_j \in E(C)$ with $j \neq c$ intersects Y.*

Proof Suppose $e_j \in V(C)$ contains some $u_i \in Y$ where $j \neq c$ when (C, P) is a p-lollipop. By symmetry, we may assume that $j \leq c/2$. Let Q be the path $v_c, e_{c-1}, v_{c-1}, \ldots, e_{j+1}, v_{j+1}$.

Let i' be the largest index less than i such that $u_{i'} \in S_1 \cup S_2$. Consider the cycle

$$C_1 = v_c, Q, v_{j+1}, e_j, u_i, P[u_i, u_{\ell+1}], u_{\ell+1}, h'(u_{\ell+1}, u_{i'}), u_{i'}, P[u_{i'}, v_c], v_c.$$

It contains at least $c - c/2$ vertices in C and all vertices in $S_1 \cup S_2 \cup \{u_{\ell+1}\}$. Hence $|C_1| \geq c/2 + (k+1) - 1 > c$. $\square$

Claim 5.4 *Suppose $|S_1| \geq k-1$ and some edge $e_j \in E(C)$ contains some $u_i \in Z$. Then either (C, P) is an o-lollipop, $c = 2k-1$, $e_j = e_{k-1}$, $|S_1| = k-1$ and $j_1 = 1$, or (C, P) is a p-lollipop and $j = c$.*

Proof Suppose $e_j \in V(C)$ contains some $u_i \in Z$ where $j \neq c$ when (C, P) is a p-lollipop. As in the proof of Claim 5.3, we may assume that $j \leq c/2$, and if c is an o-lollipop, we may assume $j \leq (c-1)/2$. Let Q be the path $v_c, e_{c-1}, v_{c-1}, \ldots, e_{j+1}, v_{j+1}$.

Let i' be the largest index less than i such that $u_{i'} \in S_1$. Consider

$$C_1 = v_c, Q, v_{j+1}, e_j, u_i, P[u_i, u_{\ell+1}], u_{\ell+1}, h'(u_{\ell+1}, u_{i'}), u_i', P[u_i', v_c], v_c.$$

If (C, P) is a o-lollipop, then

$$|V(C_1)| \geq c - j + k - 1 \geq c - (c-1)/2 + k - 1 = c + \frac{2k-c-1}{2} \geq c \quad (16)$$

with equality only if $c = 2k-1$, $j = k-1$, $u_1 \in S_1$, and $|S_1| = k-1$.

If (C, P) is a p-lollipop, then $S_1 \cap V(C) = \emptyset$, so instead of $|V(C_1)| \geq c-j+k-1$ as in (16) we have $|V(C_1)| \geq c-j+k$ and conclude that $|V(C_1)| > c$. $\square$

Claims 5.1–5.4 together can be summarized as the following two corollaries.

Corollary 5.5 *Suppose $|S_1 \cup S_2| \geq k$. Then the only edges in H that may intersect both X and Y are $f_1, \ldots, f_{i_1-1}$.*

Corollary 5.6 *Suppose $|S_1| \geq k-1$ and an edge $g \in E(H)$ intersects X and Z. Then either $g \in \{f_1, \ldots, f_{j_1-1}\}$ or (C, P) is an o-lollipop, $g = e_{k-1}$, $c = 2k-1$, $|S_1| = k-1$, and $j_1 = 1$.*

Finally we will show that $|S_1|$ and $|S_1 \cup S_2|$ cannot be too large. For this, we use the notion of *aligned* paths in graphs introduced in [4] and apply Lemma 5.7 below to the incidence bigraph I_H of H.

Let P and P' be paths in a graph starting from the same vertex. We say P' is *aligned with* P if for all $u, v \in V(P) \cap V(P')$, if u appears before v in P then u also appears before v in P'.

Lemma 5.7 (Lemma 5 in [17]). *Let P be an x, y-path in a 2-connected graph G, and let $z \in V(P)$. Then there exists an x, z-path P_1 and an x, y-path P_2 such that (a) $V(P_1) \cap V(P_2) = \{x\}$ and (b) each of P_1 and P_2 is aligned with P.*

Lemma 5.8 *(A) $|S_1 \cup S_2| \leq k-1$, and (B) $|S_1| \leq k-2$.*

Proof Recall that by Lemma 4.1, $\ell \geq 2$. We first prove (A). Suppose towards contradiction that $|S_1 \cup S_2| \geq k$.

Case 1. The smallest index i with $u_i \in S_1 \cup S_2$ satisfies $u_i \in S_2$. By the definition of S_2 and i_1, $i = i_1 - 1$, and i is the unique index less than i_1 such that $u_{\ell+1} \in f_i$. In particular, $i_1 \geq 2$ and moreover if (C, P) is a p-lollipop, since the first vertex of $V(P)$ is u_2, $i_1 - 1 \geq 2$.

Consider the 2-connected incidence bipartite graph I_H of H and the (graph) path

$$P' = v_1, e_1, v_2, \ldots, v_c, f_1, \ldots, f_\ell, u_{\ell+1}$$

in I_H. We apply Lemma 5.7 to P' with $z = u_{i_1}$ to obtain two (graph) paths P_1 and P_2 satisfying (a) and (b) in I_H.

We modify P_i as follows: if $P_i = w_1, w_2, \ldots, w_{j_i}$, let q_i be the last index such that $w_{q_i} \in X' := \{v_1, e_1, \ldots, v_c, e_c\}$ and let p_i be the first index such that $w_{p_i} \in Y' := \{u_{i_1}, f_{i_1}, u_{i_1+1}, \ldots, f_\ell, u_{\ell+1}\}$.

If $w_{p_i} = u_s$ for some s, then set $P_i' = P_i[w_{q_i}, w_{p_i}]$. If $w_{p_i} = f_s$ for some s, then set $P_i' = P_i[w_{q_i}, w_{p_i}], u_{s+1}$.

Observe that P_1' and P_2' are Berge paths or partial Berge paths in H. Moreover, P_1' ends with vertex $z = u_{i_1}$ and contains no other elements of Y' since it is aligned with P'. It is possible that f_{i_1-1} is in P_1'.

If both P_1' and P_2' begin with v_1, then some P_i' avoids f_1 and first intersects the set $\{u_2, f_2, \ldots, u_{\ell+1}\}$ in I_H at some vertex w_j. Then replacing the segment v_1, e_c, v_c in C with the longer segment $v_1, P_i'[v_1, w_j], w_j, P[w_j, f_1], f_1, v_c$ yields a cycle in H that is longer than C, a contradiction. Therefore we may assume that P_1' and P_2' are vertex-disjoint and edge-disjoint in H.

Next we show that

$$\textit{no edge in } H' \textit{ containing } u_{\ell+1} \textit{ is in } P_1' \textit{ or } P_2'. \tag{17}$$

Indeed, suppose $h \in E(H')$ contains $u_{\ell+1}$. Then by the maximality of ℓ and Claim 5.1, $h \subset V(P)$. Therefore, by the definition of $i_1, \ldots, i_\alpha$, $h \subseteq \{u_{i_1}, \ldots, u_{i_\alpha}\}$. But such edges are not in $E(P_1') \cup E(P_2')$ by construction. This proves (17).

Observe that for $m \geq i_1$, if f_m is in P_2', then by the definition of P_2', it must be the last edge of P_2'.

Let a_1 and b_1 be the first elements of P_1' and P_2' respectively. Let Q be a long b_1, a_1-segment of C guaranteed by Claim 3.1 (recall that if P_1' is a path, then a_1 is the first vertex of P_1' and if P_1' is a partial path, then it is the first edge, and similar for b_1).

Next we show

$$f_{i_1-1} \notin E(P_2'). \tag{18}$$

Suppose $f_{i_1-1} \in E(P_2')$. Since P_2 is aligned with P', the segment $P_2'[b_1, f_{i_1-1}]$ does not intersect $P[u_{i_1}, u_{\ell+1}]$. Then

$$b_1, Q, a_1, P_1', u_{i_1}, P[u_{i_1}, u_{\ell+1}], u_{\ell+1}, f_{i_1-1}, P_2'[f_{i_1-1}, b_1], b_1$$

contains at least $c-(k-1)$ vertices in C and all vertices in $S_1 \cup S_2 \cup \{u_{\ell+1}\} - \{u_{i_1-1}\}$, i.e., it has at least $c-(k-1)+k+1-1 > c$ vertices. This proves (18).

Let u_g be the last vertex of P_2'.

Case 1.1. f_{g-1} is the last edge of P_2'. Since $u_g \in Y$, $g-1 \geq i_1$. Hence by Corollary 5.5, f_{g-1} does not intersect X. Then P_2' has at least two edges and at least one internal vertex, say z. By the definition of P_2' and the fact that $f_{g-1} \neq e_c$, $z \notin X \cup Y \cup \{u_{i_1}\}$.

Let g' be the largest index less than $g-1$ such that $u_{g'} \in S_1 \cup S_2$. If $g' \neq i_1 - 1$ (so $g' \geq i_1$), consider

$$C_1 = b_1, Q, a_1, P_1', u_{i_1}, P[u_{i_1}, u_{g'}], u_{g'}, h'(u_{g'}, u_{\ell+1}), u_{\ell+1}, P[u_{\ell+1}, u_g], u_g, P_2', b_1.$$

This cycle has at least $c-(k-1)$ vertices in C and $z \notin X \cup Y$. The cycle C_1 also may miss at most two vertices in $S_1 \cup S_2$ (namely u_{i_1-1} and u_{g-1}), and since we are in Case 1,

$$\big(S_1 \cup S_2 \cup \{u_{\ell+1}\} - \{u_{g-1}, u_{i_1-1}\}\big) \cap V(C) = \emptyset.$$

Therefore C_1 contains at least

$$\begin{aligned}&|V(Q)| + |V(P_2') - (V(C) \cup V(P))| + |S_1 \cup S_2 \cup \{u_{\ell+1}\} - \{u_{g-1}, u_{i_1-1}\}| \\ &\quad \geq c-(k-1)+1+k+1-2 > c\end{aligned}$$

vertices, a contradiction.

If $g' = i_1 - 1$, then by (18), $f_{g'} \notin E(P_2')$. First suppose $b_1 \neq v_c$. We let Q' be a long b_1, v_c-segment of C if (C, P) is an o-lollipop or a long b_1, e_c-segment of C if (C, P) is a p-lollipop (without loss of generality, this path ends with v_c), and take the cycle

$$C_2 = b_1, Q', v_c(=u_1), \ldots, u_{i_1-1}, f_{i_1-1}, u_{\ell+1}, P[u_{\ell+1}, u_g], u_g, P_2', b_1$$

which again omits only u_{g-1} from $S_1 \cup S_2$ (which is possibly in $V(C)$) and satisfies

$$|C_2| \geq c - (k-1) + 1 + k + 1 - 2 > c.$$

Suppose now that $b_1 = v_c$. If $f_{i_1-1} \in E(P_1')$, let

$$C_3 = v_c, Q, a_1, P_1'[a_1, f_{i_1-1}], f_{i_1-1}, u_{\ell+1}, P[u_{\ell+1}, u_g], u_g, P_2', v_c.$$

Recall that P_2' has an internal vertex $z \notin X \cup Y \cup \{u_{i_1}\}$. Also, all vertices in $S_1 \cup S_2 \cup \{u_{\ell+1}\} - \{u_{i_1-1}, u_{g-1}\}$ are in C_3 and none of them belongs to C. Therefore $|C_3| \geq c - (k-1) + 1 + (k+1) - 2 > c$.

Lastly, if $f_{i_1-1} \notin E(P_1')$, then the cycle

$$C_4 = v_c, Q, a_1, P_1', u_{i_1}, f_{i_1-1}, u_{\ell+1}, P[u_{\ell+1}, u_g], P_2', v_c$$

contains at least $c - (k-1) + k + 1 - 1 > c$ vertices.

Case 1.2. The last edge of P_2' is not f_{g-1}. Then we let g' be the largest index less than g such that $u_{g'} \in S_1 \cup S_2$. In this case, the cycle C_1 from the previous subcase can miss only u_{i_1-1} in $S_1 \cup S_2$ and contains at least $k-1$ vertices in $S_1 \cup S_2 - \{u_{i_1-1}\}$ which are disjoint from $V(C)$. We get

$$|C_1| \geq |V(Q)| + |S_1 \cup S_2 - \{u_{i_1-1}\}| + |\{u_{\ell+1}\}| \geq c - (k-1) + (k-1) + 1 > c.$$

This finishes Case 1.

Case 2. The smallest i with $u_i \in S_1 \cup S_2$ is such that $u_i \in S_1$. Recall that in this case $i = i_1$

and the other indices of the vertices in $S_1 \cup S_2$ are $i_2, \ldots, i_\alpha$ in increasing order.

Now, define P', P_1', P_2' and Q as in Case 1. Then we can repeat the final part of the proof of Case 1 with the simplification that the cycle C_1 omits at most the vertex u_{g-1} in $S_1 \cup S_2$ and this occurs only if f_{g-1} is the last edge of P_2'. As in Case 1.1, since $f_{g-1} \neq e_c$ and $g - 1 \geq i_1$, P_2' contains at least one vertex outside of $V(C) \cup Y \cup \{u_{i_1}\}$. Note also that it may be the case $u_{i_1} = u_1 \in V(C)$.

If no vertices of $S_1 \cup S_2$ are omitted from C_1, then

$$|C_1| \geq |V(Q)| + |S_1 \cup S_2| - |\{u_{i_1}\}| + |\{u_{\ell+1}\}| \geq c - (k-1) + k - 1 + 1 > c.$$

Otherwise, if $u_{g-1} \in S_1 \cup S_2$ was omitted from C_1, then

$$|C_1| \geq |V(Q)| + |S_1 \cup S_2| - 1 - |\{u_{i_1}\}| + |\{u_{\ell+1}\}| + |V(P_2') \\ -V(C) \cup Y \cup \{u_{i_1}\}| > c.$$

This proves Part (A).

Now we prove (B). Recall that $u_{j_1}, \ldots, u_{j_\beta}$ are the H'-neighbors of $u_{\ell+1}$ and suppose $\beta \geq k-1$. Corollary 5.6 asserts that apart from $f_1, \ldots, f_{i_1-1}$ only e_{k-1} may intersect both X and Z. First part of our proof is to show that e_{k-1} does not intersect Z.

Claim 5.9 *$f_i \subseteq V(P)$ for all $u_i \in N_{H'}(u_{\ell+1})$.*

Proof By Claim 5.2, if $u \in f_i - V(P)$, then $u \notin V(C) \cup V(P)$. If P is a path, then we can replace it with the longer path u_1, $P[u_1, u_i]$, u_i, $h'(u_i, u_{\ell+1})$, $u_{\ell+1}$, $P[u_{\ell+1}, f_i]$, f_i, u. Otherwise we replace P with the partial path f_1, $P[f_1, u_i]$, u_i, $h'(u_i, u_{\ell+1})$, $u_{\ell+1}$, $P[u_{\ell+1}, f_i]$, f_i, u. □

Claim 5.10 *$\ell \geq k$.*

Proof Suppose $\ell \leq k-1$. Since $|S_1| \geq k-1$, $\ell = k-1$ and each vertex in $V(P) - \{u_{\ell+1}\}$ including u_1 is in S_1.

If $u_\ell \notin N_{H'}(u_{\ell+1})$, then $|N_{H'}(u_{\ell+1})| \leq \ell - 1 = k-2$, $N_{H'}(u_{\ell+1}) = V(P) - \{u_\ell, u_{\ell+1}\}$, and by Claims 5.4 and 5.9, the only edges containing $u_{\ell+1}$ and not contained in $V(P)$ could be e_{k-1} and f_ℓ.

We get $d_H(u_{\ell+1}) \leq \binom{k-2}{r-1} + |E(P)| + 1 \leq \binom{k-2}{r-1} + k$. When $r \geq 4$, this is less than $\binom{k-1}{r-1} + 1 = \delta(H)$. If $r = 3$, then each edge f_i with $i < \ell$ containing $u_{\ell+1}$ satisfies $f_i = \{u_i, u_{i+1}, u_{\ell+1}\}$. Thus for $i \leq \ell - 2$, such an edge f_i is a subset of $N_{H'}[u_{\ell+1}]$ and is accounted for in the $\binom{|N_{H'}(u_{\ell+1})|}{r-1}$ term of $d(u_{\ell+1})$. Hence $d_H(u_{\ell+1}) \leq \binom{k-2}{r-1} + |\{f_{\ell-1}, f_\ell, e_{k-1}\}| \leq \binom{k-2}{r-1} + k - 2 \leq \binom{k-1}{r-1}$, a contradiction.

If $u_\ell \in N_{H'}(u_{\ell+1})$, then by Claim 5.9, $d_{H-E(C)}(u_{\ell+1}) \leq \binom{k-1}{r-1}$, and the only edge containing $u_{\ell+1}$ not contained in $V(P)$ could be e_{k-1}. Then in order to satisfy $d(u_{\ell+1}) \geq \delta(H)$, we need that $u_{\ell+1} \in e_{k-1}$ and every r-tuple contained in $V(P)$ and containing $u_{\ell+1}$ is an edge in H.

Hence we can reorder the vertices in $V(P) - u_1$ to make any vertex apart from u_1 the last vertex. The resulting path together with C is also a best lollipop. By Claims 5.1–5.9 and the above, either $d(u_i) < \binom{k-1}{r-1} + 1 \leq \delta(H)$ for some i leading to a contradiction, or each $u_i \in V(P)$ is contained in e_{k-1}. In the latter case, $r = |e_{k-1}| \geq 2 + \ell = k+1$, a contradiction. □

Claim 5.11 *$u_{\ell+1} \notin e_{k-1}$.*

Proof Suppose $u_{\ell+1} \in e_{k-1}$. By Corollary 5.6, this is possible only if (C, P) is an o-lollipop and $c = 2k-1$. By Claim 5.10, $\ell \geq k$ and so the cycle

$$C_0 = v_c, e_{c-1}, \ldots, v_k, e, u_{\ell+1}, f_\ell, u_\ell, \ldots, f_1, v_c$$

has at least $k + k = 2k$ vertices, a contradiction. □

Claim 5.12 *If (C, P) is an o-lollipop, then $e_{k-1} \cap Z = \emptyset$.*

Proof Suppose $u_i \in e_{k-1} \cap Z$. By Corollary 5.6, this is possible only if $c = 2k-1$, $|S_1| = k-1$ and $j_1 = 1$.

By Claim 5.11, $i < \ell + 1$. Let i' be the largest number less than i such that $u_{i'} \in N_{H'}(u_{\ell+1})$. Denote $I_1 = \{i'+1, i'+2, \ldots, i-1\}$ and $I_2 = \{1, \ldots, i'\} \cup \{i, i+1, \ldots, \ell\}$. By the choice of i', $I_1 \cap S_1 = \emptyset$.

Consider the cycle

$$C_1 = v_c, e_{c-1}, \ldots, v_k, e_{k-1}, u_i, f_i, \ldots, u_\ell, f_\ell, u_{\ell+1}, h'(u_{\ell+1}, u_{i'}), f_{i'-1}, \ldots, u_1.$$

We have $|C_1| \geq k + |S_1 - \{u_1\}| + |\{u_{\ell+1}\}| \geq k + k - 2 + 1 = 2k - 1$ with equality only if $S_1 = \{u_i : i \in I_2\}$ and $|S_1| = |I_2| = k - 1$. In particular, this means that the indices of the vertices in S_1 form two intervals, $\{1, \ldots, i'\}$ and $\{i, i+1, \ldots, \ell\}$, and the second of these intervals starts from i. This yields $e \cap V(P) = \{u_i\}$.

Since we proved $|S_1 \cup S_2| \leq k - 1 = |S_1|$, we must have

$$for\ each\ m \in I_1, u_{\ell+1} \notin f_m. \tag{19}$$

By Claim 5.9, for each $u_m \in N_{H'}(u_{\ell+1})$, $f_m \subseteq V(P)$. Suppose now that for some $m \in I_2 - \{i'\}$, and $i' < i'' < i$, $u_{\ell+1}, u_{i''} \in f_m$ and H' has an edge g containing $\{u_m, u_{m+1}\}$. In this case, if $m \geq i$, then the cycle

$$\begin{aligned} C_2 = v_c, e_{c-1}, \ldots, v_k, e_{k-1}, u_i, f_i, \ldots, u_m, g, u_{m+1}, f_{m+1}, \ldots, \\ u_\ell, f_\ell, u_{\ell+1}, f_m, u_{i''}, f_{i''-1}, \ldots, u_1 \end{aligned}$$

is longer than C, and if $1 \leq m \leq i'$, then the cycle

$$\begin{aligned} C_3 = v_c, e_{c-1}, \ldots, v_k, e_{k-1}, u_i, f_i, \ldots, u_\ell, f_\ell, u_{\ell+1}, f_m, u_{i''}, f_{i''-1}, \ldots, \\ f_{m+1}, u_{m+1}, g, u_m, \ldots, u_1 \end{aligned}$$

is longer than C. Therefore,

If $m \in I_2$, $i' < i'' < i$ and $\{u_{\ell+1}, u_{i''}\} \subset f_m$, then no edge in H' contains $\{u_m, u_{m+1}\}$. (20)

For $1 \leq m \leq \ell$, call the edge f_m *fitting* if $u_{\ell+1} \in f_m$ and $f_m \subseteq N_{H'}[u_{\ell+1}]$ and *non-fitting* if $u_{\ell+1} \in f_m$ and $f_m \nsubseteq N_{H'}[u_{\ell+1}]$. Let R denote the set of fitting edges and R' denote the set of non-fitting edges. By (19), if $f_m \in R$, then $m \in I_2$. By the definition of I_2, if $m_1 \in I_2 - \{i', \ell\}$, then $m_1 + 1 \in I_2$.

Case 1. $u_\ell \in N_{H'}(u_{\ell+1})$. By Claim 5.11, all edges containing $u_{\ell+1}$ must either be contained in $S_1 \cup \{u_{\ell+1}\}$ or be non-fitting edges. Since $\delta(H) \geq 1 + \binom{k-1}{r-1}$, this implies $R' \neq \emptyset$. Moreover, if there is a non-fitting edge $f_{m_1} \notin \{f_{i'}, f_\ell\}$, then by Claim 5.2 and (20),

$$\begin{aligned} &\text{the } \binom{k-3}{r-3} r\text{-tuples in the set } N_{H'}[u_{\ell+1}] \text{ containing } \{u_{m_1}, u_{m_1+1}, u_{\ell+1}\} \\ &\text{are not edges of } H'. \end{aligned} \tag{21}$$

The existence of such non-fitting $f_{m_1} \notin \{f_{i'}, f_\ell\}$ is not possible if $r = 3$ because in this case $f_{m_1} = \{u_{\ell+1}, u_{m_1}, u_{m_1+1}\} \subseteq N_{H'}[u_{\ell+1}]$. So we may suppose $r \geq 4$. By (21) we have

$$d(u_{\ell+1}) \leq \binom{|S_1|}{r-1} - \binom{k-3}{r-3} + |R'| \leq \binom{k-1}{r-1} - \binom{k-3}{r-3} + |R'|.$$

Thus to have $d(u_{\ell+1}) \geq 1 + \binom{k-1}{r-1}$, we need at least $1 + \binom{k-3}{r-3} \geq k-2$ non-fitting edges f_m for $m \in I_2$.

Since by the case, (20) and Claim 5.9, f_ℓ has no vertex outside of $N_{H'}[u_{\ell+1}]$ and hence is fitting, for each of the $k-2 \geq r-1 \geq 3$ values of $m \in I_2 - \{\ell\}$, f_m must be non-fitting. But then at least $1 + \binom{k-3}{r-3} \geq k-2$ of the r-tuples contained in $N_{H'}[u_{\ell+1}]$ and containing $u_{\ell+1}$ are not edges of H. So $d(u_{\ell+1}) \leq \binom{k-1}{r-1} - (k-2) + (k-2) < \delta(H)$.

Hence in order to have $d(u_{\ell+1}) \geq \binom{k-1}{r-1} + 1$ we may assume that the only non-fitting edge is $f_{i'}$ and moreover every r-subset of $N_{H'}[u_{\ell+1}]$ containing $u_{\ell+1}$ is an edge of H'. In particular, there is an edge $g \in E(H')$ containing $u_{\ell+1}, u_{i'}$ and u_{i+1}. Then

$$C_4 = v_c, e_{c-1}, \ldots, v_k, e_{k-1}, u_i, f_{i-1}, \ldots, u_{i'+1}, f_{i'}, u_{\ell+1},\\ f_\ell, \ldots, u_{i+1}, g, u_{i'}, f_{i'-1}, \ldots, f_1, v_c$$

is longer than C.

Case 2. $u_\ell \notin N_{H'}(u_{\ell+1})$. Then $|N_{H'}(u_{\ell+1})| \leq k-2$, and by Claim 5.11,

$$\begin{aligned} |R| + |R'| = d_P(u_{\ell+1}) &= d_H(u_{\ell+1}) - d_{H'}(u_{\ell+1}) \\ &\geq 1 + \binom{k-1}{r-1} - \left[\binom{k-2}{r-1} - |R|\right] \\ &= 1 + \binom{k-2}{r-2} + |R|. \end{aligned} \tag{22}$$

So, if $r \geq 4$, then $k \geq r+2 \geq 6$, and $|R'| \geq 1 + \binom{k-2}{2} = 1 + \frac{(k-2)(k-3)}{2} \geq 1 + \frac{3(k-2)}{2}$. But (19) yields $d_P(u_{\ell+1}) \leq |I_2| = k-1$, a contradiction. On the other hand, if $r = 3$, then similarly to Case 1, $R' \subseteq \{f_\ell, f_{i'}\}$, and hence (22) yields

$$2 \geq |R'| \geq 1 + \binom{k-2}{r-2} = k-1 \geq r+1 = 4,$$

a contradiction. □

We now complete the proof of (B). As in the proof of (A), we apply Lemma 5.7 to the same path

$$P' = v_1, e_1, v_2, \ldots, v_c, f_1, \ldots, f_\ell, u_{\ell+1}$$

in I_H with $z = u_{j_1}$. We obtain paths P_1 and P_2 and modify them to P_1' and P_2' with the same rules as in (A) but with $Z' = \{u_{j_1}, f_{j_1}, u_{j_1+1}, \ldots, u_{\ell+1}\}$ in place of Y'.

We again get that P_1' and P_2' are vertex-disjoint and edge-disjoint and (17) holds. Let Q be a long segment of C connecting P_1' and P_2' with at least $c-(k-1)$ vertices. Suppose the endpoints of Q are the vertices a_1 and b_1.

Let u_i be the last vertex of P_2', and let i' be the smallest index less than i such that $u_{i'} \in S_1$. Consider the cycle

$$C' = a_1, Q, b_1, P_2', u_i, P[u_i, u_{\ell+1}], u_{\ell+1}, h'(u_{\ell+1}, u_{i'}), u_{i'} P[u_{i'}, u_{j_1}], u_{j_1}, P_1', a_1$$

which contains all k vertices in $S_1 \cup \{u_{\ell+1}\}$. If none of these vertices is in C, then $|C'| \geq c - (k-1) + k > c$, a contradiction. If there is such a vertex, it could be only u_1, in which case (C, P) is an o-lollipop and $j_1 = 1$. Then by Corollary 5.6 and Claim 5.12, P_2' contains at least one vertex outside of $V(C \cup P)$. It follows that $|C'| \geq c - (k-1) + k - 1 + 1 > c$, a contradiction again. □

6 Partial Berge Paths in Best *p*-Lollipops are Long

In this section we concentrate on p-lollipops and show that the partial path P in them must be long (namely, $\ell \geq k$). We do this by showing that $u_{\ell+1}$ has no H'-neighbors inside of C, and hence P must be sufficiently long to contain all H'-neighbors of $u_{\ell+1}$. The main lemma of this section is the following.

Lemma 6.1 *If (C, P) is a p-lollipop then $|V(P)| = \ell \geq k$.*

Proof In Section 4 we showed that $\ell \geq 2$. Suppose towards contradiction that $2 \leq \ell \leq k-1$. We will first show that

$$\text{all } H'\text{-neighbors of } u_{\ell+1} \text{ are contained in } V(P). \tag{23}$$

By Claim 3.3, all H'-neighbors of $u_{\ell+1}$ are in $V(C) \cup V(P)$. If $u_{\ell+1} \in e \in E(H')$ and $v_i \in e$ for some $v_i \in V(C)$, we let $P' = v_i, e, u_{\ell+1}, P[u_{\ell+1}, u_2], u_2$. Observe that $V(P') - V(C) = V(P) - V(C)$, and (C, P') is better than (C, P) by Rule (d). This proves (23).

Next we show that

$$u_{\ell+1} \text{ is contained in at least } k \text{ edges in } E(C) \cup E(P). \tag{24}$$

By (23), $|N_{H'}(u_{\ell+1})| \leq |V(P) - \{u_{\ell+1}\}| \leq k-2$. Then the number of edges in $E(C) \cup E(P)$ containing $u_{\ell+1}$ must be at least

$$\delta(H) - \binom{|N_{H'}(u_{\ell+1})|}{r-1} \geq \binom{k-1}{r-1} + 1 - \binom{k-2}{r-1} \geq k-1,$$

with equality only if $r = 3$, $|N_{H'}(u_{\ell+1})| = k-2$ (and so $V(P) = N_{H'}[u_{\ell+1}]$), $u_{\ell+1}$ is contained in all $\binom{k-2}{r-1}$ possible H'-edges, and no edge of $E(C) \cup E(P)$ containing $u_{\ell+1}$ is a subset of $N_{H'}[u_{\ell+1}]$. If this is the case, then $e = \{u_2, u_3, u_{\ell+1}\} \in E(H')$, and we swap f_2 with e to get a partial path that is better than P by Rule (e). This proves (24).

Say $|V(P)| = \ell = k - a$ where $2 \leq k - a \leq k - 1$. Since $|E(P)| = k - a$, by (24), $u_{\ell+1}$ is contained in at least a edges in $E(C) - e_c$. By Claim 3.3(2), none of these edges is in the set $\{e_1, \ldots, e_{\ell-1}\} \cup \{e_{c-1}, \ldots, e_{c-(\ell-1)}\}$. Thus, $u_{\ell+1}$ is contained in at least a edges in $\{e_{k-a}, e_{k-a+1}, \ldots, e_{c-(k-a)}\}$. Moreover, $u_{\ell+1}$ is contained in exactly a such edges if and only if it is contained in all $k - a$ edges of P (in particular, $u_{\ell+1} \in e_c$).

Let e_i contain $u_{\ell+1}$ for some $i \neq c$. Consider the partial path $P' = e_i, u_{\ell+1}, f_\ell, \ldots, u_2$. Then $V(P') = V(P)$ and $E(P') - E(C) = E(P) - E(C)$. Thus (C, P') also is a best lollipop. So, as above we get that all H'-neighbors of u_2 are in $V(P)$, u_2 is contained in at least k edges of $E(P') \cup E(C) = E(P) \cup E(C)$, and at least a edges of $E(C) - \{e_i\}$ with equality only if $u_2 \in e_i$ by Claim 3.3. Moreover, each of these edges is of distance at least $k - a$ from e_i.

Let B_i be the set of edges of $E(C)$ containing u_i for $i \in \{2, \ell + 1\}$. Observe that $|B_i| \geq a + 1$. Let $t = |B_2 \cap B_{\ell+1}|$. If $t = 0$, let $e_\alpha, e_\beta, e_\gamma$ be edges such that $\alpha < \beta < \gamma$ (modulo c), $e_\alpha, e_\gamma \in B_{\ell+1}$, and $e_\beta \in B_2$. Then the segment from e_α to e_γ in C contains at least $2(k - a - 1)$ edges not in $B_2 \cup B_{\ell+1}$ by Claim 3.5. We get

$$2k - 1 \geq |E(C)| \geq |B_2| + |B_{\ell+1}| + 2(k-a-1) \geq 2(a+1) + 2(k-a-1) = 2k,$$

a contradiction.

Now suppose $1 \leq t \leq |B_2|$. Then surrounding each edge in $B_2 \cap B_{\ell+1}$ there are two intervals of $2(k - a - 1)$ edges that are disjoint from $B_2 \cup B_{\ell+1}$. Moreover if there exists $e_\alpha \in B_2 - B_{\ell+1}$ and $e_\beta \in B_{\ell+1} - B_2$, then each pair of vertices in $(B_2 \cap B_{\ell+1}) \cup \{e_\alpha, e_\beta\}$ has distance at least $k - a$. In this case, there are at least $t + 2$ intervals of $(k - a - 1)$ edges not in $B_2 \cup B_{\ell+1}$. Therefore

$$\begin{aligned} 2k - 1 \geq |E(C)| \geq |B_2 \cup B_{\ell+1}| + (t+2)(k-a-1) \geq 2(a+1) - t \\ +(t+2)(k-a-1) = t(k-a-2) + 2k \geq 2k, \end{aligned}$$

a contradiction. If $B_2 \subsetneq B_{\ell+1}$ or vice versa, then we have $t \geq |B_2| \geq a+1$. As before, for any $e_\beta \in B_{\ell+1} - B_2$, each pair of edges in $(B_2 \cap B_{\ell+1}) \cup \{e_\beta\}$ has distance at least $k - a$. So instead we get

$$\begin{aligned} 2k - 1 \geq |B_{\ell+1} - B_2| + t + (t+1)(k-a-1) \geq 1 + t \\ +(t+1)(k-a-1) = (t+1)(k-a) \geq (a+2)(k-a). \end{aligned}$$

But this does not hold when $a \geq 1$, $k \geq 3$, and $k - a \geq 2$.

The last case is $B_2 = B_{\ell+1}$. If $t \geq a + 2$, then $2k - 1 \geq t(k - a - 1) + t \geq (a + 2)(k - a)$, a contradiction again. So we consider the case where $t = |B_2| = |B_{\ell+1}| = a + 1$. Because $B_{\ell+1}$ must contain a edges within the at most $2a$ edges of $\{e_{k-a}, \ldots, e_{c-(k-a)}\}$ we must have $\ell = k - a = 2$ by Claim 3.5. Without loss of generality, we may assume that $B_2 = B_{\ell+1} = \{e_c, e_2, e_4, \ldots, e_{2k-4}\}$. We also have $r = |e_c| \geq |\{v_c, v_1, u_2, u_{\ell+1}\}| = 4$.

Suppose the edge $f_2 \in E(P)$ contains a vertex $v_i \in V(C)$. By Claim 3.2, e_{i-1}, e_i cannot contain $u_{\ell+1}$. So we must have that $c = 2k - 1$ and $i = 2k - 2$. Therefore f_2 contains at least $r - 1$ vertices outside of $V(C)$. As $\ell = 2$, $|(f_2 \cap V(P)) - V(C)| \leq 2$. So, since $r = |e_c| \geq 4$, there exists $u \in f_2$ with $u \notin V(C) \cup V(P)$.

By Claim 3.5, u cannot belong to e_i if $e_{i-1} \in B_2$ or $e_{i+1} \in B_2$. Hence $\{e_i \in E(C) : u \in e_i\} \subseteq B_2$. If an edge $e_i \in B_2$ contains all vertices in $f_2 - V(C)$, then $|e_i| \geq 2 + |f_2| - 1 = r + 1$, a contradiction. Therefore some $u \in f_2$ is contained in at most $(|B_2| - 1) + 1 = k - 1$ edges of $E(C) \cup E(P)$, and hence $d_{H'}(u) \geq \delta(H) - (k - 1) \geq 1$.

Say $u \in e \in E(H')$. If there exists $w \notin V(C)$ in e, then either $(C, e_c, u_2, f_2, u, e, w)$ or $(C, e_c, u_3, f_2, u, e, w)$ is a better lollipop. than (C, P). So e must contain $r-1$ vertices in $V(C)$. Without loss of generality, $v_i \in e$ and $e_i \in B_2$. Then replacing the segment v_i, e_i, v_{i+1} in C with $v_i, e, u, f_2, u_2, e_i, v_{i+1}$ yields a cycle longer than C. □

By applying Claim 3.3 and using that $\ell \geq k$, $c < 2k$, we obtain the following corollary.

Corollary 6.2 *If (C, P) is a p-lollipop, then the only edge of C that may contain $u_{\ell+1}$ is e_c.*

7 The Paths in Lollipops are Short

In this section we show that P cannot be too long (namely, $\ell \leq k-2$). Our first step will be to show that if $u_{\ell+1}$ has H'-neighbors in C and P is long, then we can find a better cycle than C. Then we apply Lemma 5.8 to analyze the case where all H'-neighbors of $u_{\ell+1}$ are in P. As a result of Lemma 6.1 and the lemma below, we obtain that (C, P) is an o-lollipop.

Lemma 7.1 *If $k \geq r+2 \geq 5$, then $\ell \leq k-2$.*

Proof Suppose $\ell \geq k-1$ and recall that by Lemma 6.1 we have equality only if (C, P) is an o-lollipop.

Case 1. Some $h \in E(H')$ contains $u_{\ell+1}$ and some $v_i \in V(C) - v_c$. By symmetry we may assume $i \leq c/2$ when (C, P) is an o-lollipop and $i \leq (c+1)/2$ when (C, P) is a p-lollipop. Consider the cycle

$$C_1 = v_i, C[v_i, v_c], v_c, P, u_{\ell+1}, h, v_i.$$

If (C, P) is an o-lollipop, then C_1 has at least $(c-(c-2)/2)+k-1 = c + \frac{2k-2-c+2}{2} > c$ vertices, a contradiction. If (C, P) is a p-lollipop, then C_1 has at least $(c-(c-1)/2)+k = c + \frac{2k-c+1}{2} > c$ vertices, a contradiction again. This finishes Case 1.

For $2 \leq m \leq \ell+1$, let $B_m = \{e_j \in E(C) : u_m \in e_j\}$ and $b_m = |B_m|$. By Claim 3.3 and Corollary 6.2,

$$\begin{aligned} &\textit{if } b_{\ell+1} > 0, \textit{then either} \ell = k-1, (C, P) \textit{is an } o-\textit{lollipop and} \\ &B_{\ell+1} = \{e_{k-1}\}, \textit{or} (C, P) \textit{is a } p-\textit{lollipop and } B_{\ell+1} = \{e_c\}. \end{aligned} \tag{25}$$

Let $F = \{f_m \in E(P) - \{e_c\} : u_{\ell+1} \in f_m\}$. By Lemma 5.8(A), $|F| \leq k-1$.

Case 2. $N_{H'}(u_{\ell+1}) \subset V(P)$.

By (25), Lemma 5.8(B) and the fact that $|F| \leq k-1$,

$$1 + \binom{k-1}{r-1} \leq d(u_{\ell+1}) \leq \binom{|N_{H'}(u_{\ell+1})|}{r-1} + |F| + b_{\ell+1} \leq \binom{k-2}{r-1} + (k-1) + 1. \tag{26}$$

For $r \geq 4$, regrouping, we get

$$k-1 \geq \binom{k-2}{r-2} \geq \binom{k-2}{2} = \frac{(k-2)(k-3)}{2},$$

yielding $k^2 - 7k + 8 \leq 0$, which is not true for $k \geq 6$.

This settles the case $r \geq 4$.

So suppose $r = 3$. In particular, if (C, P) is a p-lollipop, then $u_{\ell+1} \notin e_c = \{v_c, v_1, u_2\}$. Thus $b_{\ell+1} > 0$ only if (C, P) is an o-lollipop. If $|N_{H'}(u_{\ell+1})| \leq k-3$, then

$$\begin{aligned} d(u_{\ell+1}) \leq \binom{k-3}{2} + k = \binom{k-3}{2} + \binom{k-3}{1} + 3 = \binom{k-2}{2} \\ +3 \leq \binom{k-2}{2} + \binom{k-2}{1} = \binom{k-1}{2}, \end{aligned}$$

a contradiction.

Hence by Lemma 5.8(B), $|N_{H'}(u_{\ell+1})| = k-2$, $u_\ell \in N_{H'}(u_{\ell+1})$, and $|S_1 \cup S_2| \leq k-1$. If $|F| + b_{\ell+1} \leq k-2$, then the RHS of (26) is at most $\binom{k-1}{r-1}$; so suppose $|F| + b_{\ell+1} \geq k-1$.

By Claim 3.4(A), for every $f_m \in F \setminus \{f_\ell\}$ the lollipop (C, P_m) where P_m is obtained from P by replacing the subpath $u_m, f_m, u_{m+1}, \ldots, u_{\ell+1}$ with the subpath $u_m, f_m, u_{\ell+1}, f_\ell, u_\ell, \ldots, u_{m+1}$ also is a best lollipop. Since $|F| \geq k-1-b_{\ell+1} \geq (r+2)-1-1 = 3$ and e_{k-1} may contain only one vertex of P, for some $f_m \in F$, $u_{m+1} \notin e_{k-1}$ and hence by (25) u_{m+1} does not belong to any edge of C. So we may assume that $b_{\ell+1} = 0$. Then in view of (26), if $d(u_{\ell+1}) \geq 1 + \binom{k-1}{2}$, then

$$\begin{aligned} &|F| = k-1, \textit{ each } f_m \in F \textit{ is not contained in } N_{H'}[u_{\ell+1}], \textit{and any two} \\ &\quad \textit{vertices in } N_{H'}(u_{\ell+1}) \textit{ form an edge of } H' \textit{ together with } u_{\ell+1}. \end{aligned} \tag{27}$$

So, since $u_\ell \in N_{H'}[u_{\ell+1}]$, $f_\ell \not\subset N_{H'}[u_{\ell+1}]$, but there is $g \in E(H')$ such that $\{u_\ell, u_{\ell+1}\} \subset g$. Moreover, since $|N_{H'}(u_{\ell+1})| = k-2 \geq r = 3$, we can choose $g \subseteq N_{H'}(u_{\ell+1}) - \{u_1\}$ and

$$\textit{each vertex in } N_{H'}(u_{\ell+1}) \textit{ belongs to at least two edges of } H'. \tag{28}$$

Then for P' obtained from P by replacing f_ℓ with g, the pair (C, P') also is a best lollipop.

Suppose $f_\ell = \{u_\ell, u_{\ell+1}, u\}$. By (27), $u \notin N_{H'}[u_{\ell+1}]$. If $u \in V(C) - V(P)$, then we have Case 1 for (C, P'), a contradiction. If $u \notin V(C) \cup V(P)$, then we can extend P' by adding edge f_ℓ and vertex u. So, $u \in V(P) - N_{H'}[u_{\ell+1}]$. But then in view of (28), the size of $N_{H'}(u_{\ell+1})$ corresponding to (C, P') will be $k-1$ because of the new vertex u, a contradiction. □

Lemma 7.1 together with Lemma 6.1 yield

Corollary 7.2 *(C, P) is an o-lollipop.*

8 Finishing Proof of Theorem 7

In this section we complete the proof of Theorem 7. One notable part of this section is that we construct another optimal lollipop in which the vertex u_2 plays the role of $u_{\ell+1}$. We consider the H'-neighborhoods of both u_2 and $u_{\ell+1}$ as well as the edges in C containing these vertices, and we analyze how these sets can interact. We conclude that u_2 and $u_{\ell+1}$ cannot both have degree more than $\binom{k-1}{r-1}$ without creating a cycle longer than C.

By Lemmas 4.1 and 7.1, $2 \leq \ell \leq k-2$. Corollary 7.2 gives that (C, P) is an o-lollipop. The following lemma will be useful for bounding the size of $N_{H'}(u_{\ell+1})$.

Lemma 8.1 *Let $s+1 \geq b \geq 0$. Let $Q = v_0, v_1, \ldots, v_{s+1}$ be a graph path, and I be a non-empty independent subset of $\{v_1, \ldots, v_s\}$. If B is a set of b edges of Q such that no edge in B contains any vertex in I, then $|I| \leq \lceil \frac{s-b}{2} \rceil$.*

Moreover if $s-b$ is odd and $|I| = \frac{s-b+1}{2}$, then for every $1 \leq i \leq s$, $v_i \in I$, or $e_i \in B$, or $e_{i-1} \in B$, or $\{v_{i-1}, v_{i+1}\} \subseteq I$.

Proof The claim is trivial if $b = 0$ so assume $b > 0$. Iteratively contract all b edges of B, say $Q' = v'_0, v'_1, \ldots, v'_{s+1-b}$ is the new path obtained. Observe that since I was disjoint from the edges in B, after contraction I is still an independent set in Q' such that $I \subseteq \{v'_1, \ldots, v'_{s-b}\}$. Therefore $|I| \leq \lceil \frac{s-b}{2} \rceil$.

Now suppose $s-b$ is odd, $|I| = \lceil \frac{s-b}{2} \rceil$, and for some i, $e_i, e_{i-1} \notin B$. If without loss of generality $v_{i+1} \notin I$, then we contract the edge $v_i v_{i+1}$ and apply the result to the new path, I, and B to obtain $I \leq \lceil \frac{s-1-b}{2} \rceil < \lceil \frac{s-b}{2} \rceil$. □

For $i \in \{2, \ell+1\}$, let $A_i = N_{H''}(u_i) \cap V(C)$ and B_i be the set of edges in $E(C)$ containing u_i. Also, let $a_i = |A_i|$ and $b_i = |B_i|$. Let $F = \{f_m : u_{\ell+1} \in f_m\}$. We will heavily use the fact that

$$1 + \binom{k-1}{r-1} \leq d_H(u_{\ell+1}) = b_{\ell+1} + |F| + d_{H'}(u_{\ell+1}). \tag{29}$$

Claim 8.2 *If $\ell \geq 2$, then some edge $e \in E(H'')$ containing $u_{\ell+1}$ intersects C.*

Proof If the claim fails, then $|N_{H''}(u_{\ell+1})| \leq |V(P) - V(C)| = \ell - 1$ and $f_1 \notin F$. Hence using Claim 3.3,

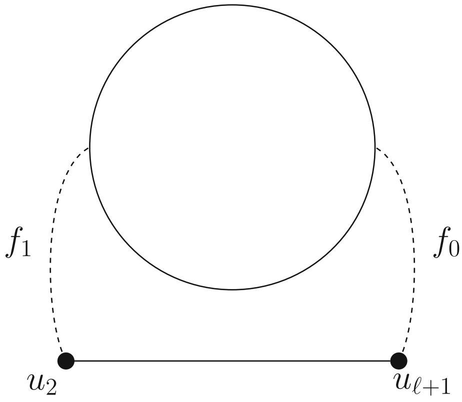
$$1+\binom{k-1}{r-1} \leq d_H(u_{\ell+1}) \leq \binom{|N_{H'}(u_{\ell+1})|}{r-1} + (c-2\ell) + \ell - 1 \leq \binom{\ell-1}{r-1} + c - \ell - 1. \tag{30}$$

Since $\ell \leq k-2$ and the function $h(\ell) := \binom{\ell-1}{r-1} + c - \ell - 1$ does not decrease for integers $\ell \geq r-1$, in the range $r-1 \leq \ell \leq k-2$, (30) gives

$$1 + \binom{k-1}{r-1} \leq \binom{(k-2)-1}{r-1} + (2k-1) - (k-2) - 1 = \binom{k-3}{r-1} + k,$$

which is not true for $k \geq r+2 \geq 5$.

Fig. 2 An o-lollipop (C, P) with edge f_0 containing $u_{\ell+1}$



Otherwise $\binom{\ell-1}{r-1} = 0$, so (30) yields $1 + \binom{k-1}{2} \leq (2k-1) - 2 - 1$, which is not true for $k \geq 5$. □

Fix an edge $f_0 \in E(H'')$ containing $u_{\ell+1}$ and some $v_j \in V(C)$ provided by Claim 8.2 (see Fig. 2). Possibly, $f_0 = f_1$. Consider path $P' = v_j, f_0, u_{\ell+1}, f_\ell, \ldots, f_2, u_2$. Since $V(P) - V(C) = V(P') - V(C)$ and $f_1 \not\subseteq V(P) - V(C)$, (C, P') is also a best lollipop. Thus many arguments we apply to $u_{\ell+1}$ will also apply symmetrically to u_2.

Let $F' = \{f_m : m \in \{0, 2, 3, \ldots, \ell\} and u_2 \in f_m\}$. If $r = 3$, then for $\ell \geq 3$, not all of f_1, f_2 and f_3 contain $\{u_2, u_{\ell+1}\}$. So, we may assume

$$if r = 3, then |F| \leq \max\{2, \ell - 1\}. \tag{31}$$

Claim 8.3 $A_{\ell+1} = A_2$.

Proof Suppose $A_{\ell+1} \neq A_2$. By symmetry, we may assume $A_2 - A_{\ell+1} \neq \emptyset$. We may rename the vertices in C and edges in H'' so that $v_c \in A_2 - A_{\ell+1}$ and f_1 contains v_c and u_2. This new lollipop (we still call it (C, P)) remains a best lollipop. So by Claim 3.3, $N_{H'}(u_{\ell+1}) \subseteq \{v_{\ell+1}, v_{\ell+2}, \ldots, v_{c-\ell-1}\} \cup \{v_c\}$. By Claim 3.3, $B_{\ell+1} \subseteq \{e_\ell, e_{\ell+1}, \ldots, e_{c-\ell-1}\}$. By Claim 3.2, if $e_i \in B_{\ell+1}$, then $v_i, v_{i+1} \notin N_{H'}(u_{\ell+1})$, and $N_{H'}(u_{\ell+1}) \cap V(C)$ does not contain two consecutive vertices of C. Hence, remembering that $v_c \notin A_{\ell+1}$, we apply Lemma 8.1 to the (graph) path $Q' = v_\ell, \ldots, v_{c-\ell}$, $I = A_{\ell+1}$ and $B' = B_{\ell+1}$, and get

$$a_{\ell+1} \leq \left\lceil \frac{(c - 2\ell - 1) - b_{\ell+1}}{2} \right\rceil = \left\lceil \frac{c - 1 - b_{\ell+1}}{2} \right\rceil - \ell \leq k - 1 - \ell - \left\lfloor \frac{b_{\ell+1}}{2} \right\rfloor. \tag{32}$$

Since $u_{\ell+1} \notin f_1$, $|F| \leq \ell - 1$. Since $N_{H'}(u_{\ell+1}) \subseteq A_{\ell+1} \cup V(P) - \{u_{\ell+1}\}$ and $|V(P) - V(C) - u_{\ell+1}| = \ell - 1$, $|N_{H'}(u_{\ell+1})| \leq a_{\ell+1} + \ell - 1$. Combining this with (32) and (29), we get

$$1+\binom{k-1}{r-1} \le b_{\ell+1}+(\ell-1)+\binom{k-2-\lfloor b_{\ell+1}/2\rfloor}{r-1}. \tag{33}$$

For fixed k, r, ℓ satisfying the theorem, the maximum of the sum $b_{\ell+1}+\binom{k-2-\lfloor b_{\ell+1}/2\rfloor}{r-1}$ over nonnegative $b_{\ell+1}$ such that $k-2-\lfloor b_{\ell+1}/2\rfloor \ge r-1$ is achieved at $b_{\ell+1}=1$. Hence (33) yields $1+\binom{k-1}{r-1} \le 1+(\ell-1)+\binom{k-2}{r-1}$, which in turn gives $\binom{k-2}{r-2} \le \ell-1 \le k-3$, a contradiction. □

In view of this claim, let $A = A_{\ell+1} = A_2$ and $a = |A|$. Since $v_c \in A$, instead of (32) we have

$$a \le 1+\left\lceil \frac{(c-2\ell-1)-b_{\ell+1}}{2} \right\rceil = 1+\left\lceil \frac{c-1-b_{\ell+1}}{2} \right\rceil - \ell \le k-\ell-\left\lfloor \frac{b_{\ell+1}}{2} \right\rfloor. \tag{34}$$

Claim 8.4 $|N_{H'}(u_{\ell+1})| = a+\ell-1$, *i.e.*, $N_{H'}[u_{\ell+1}] = A_{\ell+1} \cup V(P)$.

Proof If $|N_{H'}(u_{\ell+1})| \le a+\ell-2$, then by (29) and (34),

$$1+\binom{k-1}{r-1} \le b_{\ell+1}+|F|+\binom{|N_{H'}(u_{\ell+1})|}{r-1} \le b_{\ell+1}+|F|+\binom{k-2-\lfloor b_{\ell+1}/2\rfloor}{r-1}. \tag{35}$$

For fixed $k, r, \ell, |F|$ satisfying the theorem, the maximum of the RHS of (35) over suitable $b_{\ell+1}$ is achieved at $b_{\ell+1}=1$. Hence (35) yields $1+\binom{k-1}{r-1} \le 1+|F|+\binom{k-2}{r-1}$, i.e. $\binom{k-2}{r-2} \le |F|$. Since $\ell \le k-2$, for $r=3$ by (31), this gives $\binom{k-2}{1} \le \max\{2, \ell-1\} \le k-3$, an impossibility, and for $r \ge 4$ this yields $\binom{k-2}{2} \le \ell \le k-2$, which is not true for $k \ge r+2 \ge 6$. □

Claim 8.5 *For all $v_j, v_{j'} \in A$, either $j'=j$ or $|j'-j| > \ell$ (modulo c).*

Proof Suppose the claim fails. By symmetry, we may assume $v_c, v_j \in A$ and $1 \le j \le \ell$.

By Claim 3.5(3), if there exists $e, f \in E(H'')$ such that $\{v_c, u_2\} \subset e$ and $\{v_j, u_{\ell+1}\} \subset f$, then $f = e$. Thus the only edge of H'' containing v_c or v_j and $u_{\ell+1}$ is f_1. Hence $d_{H'}(u_{\ell+1}) \le 1+\binom{|N_{H'}(u_{\ell+1})|-2}{r-1}$. Therefore, by (34) instead of (33) we get

$$1+\binom{k-1}{r-1} \le b_{\ell+1}+\ell+1+\binom{k-3-\lfloor b_{\ell+1}/2\rfloor}{r-1}. \tag{36}$$

For fixed k, r, ℓ satisfying the theorem, the maximum of the sum $\binom{k-3-\lfloor b_{\ell+1}/2\rfloor}{r-1}+b_{\ell+1}$ over suitable $b_{\ell+1}$ is achieved at $b_{\ell+1}=1$. Hence (36) together with $\ell \le k-2$ yields $1+\binom{k-1}{r-1} \le 1+(\ell+1)+\binom{k-3}{r-1} \le k+\binom{k-3}{r-1}$, which is not true when $k \ge r+2 \ge 5$. □

Claim 8.6 *Each $f_i \in F$ is contained in $N_{H'}[u_{\ell+1}]$.*

Proof Assume there exists f_i not contained in $N_{H'}[u_{\ell+1}]$. Since $A_2 \supseteq f_1 \cap V(C)$, $A_2 \neq \emptyset$. So, by Claims 8.3 and 8.4, $d_{H'}(u_{\ell+1}) > 0$. Let $w \in f_i - N_{H'}[u_{\ell+1}]$. Also by Claim 8.4, $u_i \in N_{H'}(u_{\ell+1})$. Suppose first $i = 1$. Let $v_j \in A_{\ell+1}$. By Claim 8.4, there is an edge $h \in E(H')$ containing $\{u_{\ell+1}, v_j\}$. Then path $P_0 = v_j, h, u_{\ell+1}, f_\ell, u_\ell, \ldots, u_2, f_1, w$ is longer than P, a contradiction.

Suppose now $i \geq 2$. Let $g \in E(H')$ contain $\{u_{\ell+1}, u_i\}$. Let $P_1 = u_1, f_1, \ldots, u_i, g, u_{\ell+1}, f_\ell, u_\ell, \ldots, u_{i+1}$. Since $V(P_1) - V(C) = V(P) - V(C)$ and $f_i \not\subset V(P)$, the lollipop (C, P_1) is a best lollipop. If $w \notin V(C)$, then by appending to P_1 edge f_i and vertex w we get a better lollipop, a contradiction. So, $w \in V(C) - A$, say $w = v_j$. Let $P_2 = v_j, f_i, u_{i+1}, f_{i+1}, \ldots, u_{\ell+1}, g, u_i, f_{i-1}, \ldots, u_2$. Again, (C, P_2) is a best lollipop. Define H'_2 to be the hypergraph with $E(H'_2) = E(H) - E(C) - E(P_2)$, and $H''_2 = H'_2 + f_i$. Note that H'_2 and H''_2 play the role of H' and H'' respectively for the best lollipop (C, P_2). Moreover, define $A'_2 = N_{H''_2}(u_2) \cap V(C)$ (A'_2 plays the role of $A_{\ell+1}$). Then many of the claims we proved for (C, P) also apply to (C, P_2). Namely, $N_{H'_2}[u_2] = A'_2 \cup V(P_2) = A'_2 \cup V(P)$ by Claim 8.4. Since $v_j \in f_i$, we have $v_j \in A'_2$, so $v_j \in N_{H'_2}(u_2)$ and there exists some edge in $E(H) - E(C) - E(P_2)$ containing both u_2 and v_j. Since $E(H) - E(C) - E(P_2) \subseteq E(H'')$, This implies that $v_j \in A_2 = A$, a contradiction. $\square$

By Claim 8.6, instead of (29) we have

$$d(u_{\ell+1}) \leq \binom{|N_{H'}(u_{\ell+1})|}{r-1} + b_{\ell+1} \leq \binom{a+\ell-1}{r-1} + b_{\ell+1} \\ \leq \binom{k-1-\lfloor b_{\ell+1}/2 \rfloor}{r-1} + b_{\ell+1}. \tag{37}$$

Claim 8.7 $b_{\ell+1} = 1$ *and* $|N_{H'}(u_{\ell+1})| = k-1$.

Proof If $b_{\ell+1} = 0$, then by (37), $d(u_{\ell+1}) \leq \binom{k-1}{r-1} < \delta(H)$. On the other hand, if $b_{\ell+1} \geq 2$ then the maximum of the RHS of (37) is achieved at $b_{\ell+1} = 3$, and so is at most $\binom{k-1-1}{r-1} + 3 \leq \binom{k-2}{r-1} + k - 2 < \delta(H)$. This proves $b_{\ell+1} = 1$. In view of this, if $|N_{H'}(u_{\ell+1})| \leq k-2$, then $d(u_{\ell+1}) \leq \binom{k-2}{r-1} + 1 < \delta(H)$. $\square$

By Claims 8.4 and 8.7, we must have

$$a - 1 = |A - V(P)| = k - 1 - \ell = \left\lceil \frac{(2k-1) - 1 - b_{\ell+1}}{2} \right\rceil - \ell \\ = \left\lceil \frac{(2k-1) - 1 - 1}{2} \right\rceil - \ell.$$

We apply Lemma 8.1 to the (graph) path $v_1, v_2, \ldots, v_{c-1}$, with $I = A - V(P)$, $B = B_{\ell+1}$, and $c - 1 = 2k - 2$. In particular since the numerator $2k - 3$ is odd, the “equality" part of Lemma 8.1 holds. That is,

$$\text{for every } 1 \le i \le c-1,\, v_i \in A - V(P),\ \text{or } e_i \in B_{\ell+1},\ \text{or } e_{i-1} \in B_{\ell+1} \text{ or } \{v_{i-1}, v_{i+1}\} \in A - V(P). \tag{38}$$

We now complete the proof of Theorem 7 by showing that for some i, (38) does not hold.

Since $|V(P) - \{u_{\ell+1}\}| = \ell \le k-2$ and $|N_{H'}(u_{\ell+1})| = k-1$, A contains some $v_j \in V(C) - \{v_c\}$. By Claim 3.3, $j \in \{\ell+1, \dots, c-\ell-1\}$. By symmetry, we may assume $j < c - \ell - 1$.

We now show that (38) does not hold for $i = j+1$. By Claim 8.5, v_{j+1} and v_{j+2} are not in $A - V(P)$. By Claim 3.2(a), $e_j, e_{j+1} \notin B_{\ell+1}$. Thus (38) fails, completing the proof of Theorem 7.

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